1 APPLICATIONS OF DERIVATIVES

Life is good only for two things - discovering mathematics and teaching mathematics.

– Siméon Poisson

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Each problem that I solved became a rule, which served afterwards to solve other problems.

– René Des Cartes

1.1 Introduction

We have defined the derivative of a function and studied several methods to find the derivative of a function.

In the introductory article in std. XI, semester II, we had introduced the notion of a derivative using the slope of a tangent to a curve intuitively. Now we will study this application and several other applications of a derivative such as rate of change of a quantity *w.r.t.* another quantity, finding approximate values of a function at some value in its domain, equations of tangents and normals to a curve at a point and the orthogonality of curves, increasing and decreasing functions and maximum and minimum values of a function. These mathematical concepts are used to apply differentiation to find optimum values in Physics, Economics, Social Science, Biology, Chemistry etc. **Des Cartes** and **Newton** explained creation, the shape and colour of rainbows using these ideas. Geophysicists use differential calculus when studying the structure of the earth's crust while searching for oil.

1.2 Rates of Change

Let $s = f(t)$ be the equation of rectilinear motion of a particle, where s represents displacement at time t (i.e. directed distance from origin). If the displacements at time t_1 and t_2 are respectively s_1 and s_2, its average velocity during time interval $t_2 - t_1$ is given by the ratio $\frac{s_2 - s_1}{t_2 - t_1}$. Let $\Delta s = s_2 - s_1$, $\Delta t = t_2 - t_1$ and average velocity $= \frac{\Delta s}{\Delta t}$.

As $t_2 \to t_1$, we get instantaneous velocity v of the particle at time t_1.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Thus rate of change of displacement $s = f(t)$ *w.r.t.* time t is the instantaneous velocity of the particle at time t.

Similarly for any function $y = f(x)$, $\frac{dy}{dx}$ is the rate of change of $y = f(x)$ *w.r.t.* x.

For another example if volume $V = f(r)$, r radius, $\frac{dV}{dr}$ is the rate of change of volume of a sphere *w.r.t.* radius.

For a 'smooth' continuous curve $y = f(x)$, let $P(x, f(x))$ and $Q(x + h, f(x + h))$ be two points on the curve. (Fig. 1.1)

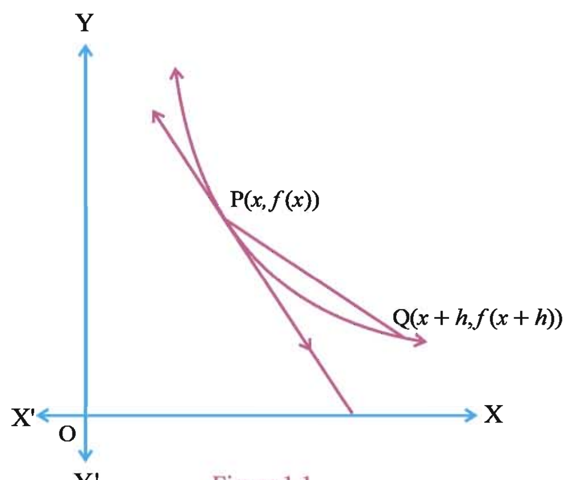

Figure 1.1

Slope of the secant $\overleftrightarrow{PQ} = \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$

As $h \to 0$, Q $\to$ P, P remaining on the curve. Since the curve is 'smooth and continuous',

slope of tangent at P $= \lim\limits_{Q \to P}$ (slope of $\overleftrightarrow{PQ}$)

$= \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

$= f'(x)$

$\therefore$ The slope of the tangent at P(x, $f(x)$) to the curve $y = f(x)$ is $f'(x)$.

In practice, we encounter many problems in which the rate *w.r.t.* time is required.
In these circumstances x, y etc. are functions of time t.

So by Chain rule $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$ will be useful to calculate such rates.

Example 1 : Find the rate of change of volume of a sphere *w.r.t.* radius. Find this rate when $r = 3$ *cm*.

Solution : For a sphere, $V = \frac{4}{3}\pi r^3$, where V is the volume and r is the radius of the sphere.

$\therefore \quad \dfrac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$\therefore \quad \left(\dfrac{dV}{dr}\right)_{r=3} = 4\pi \times 9 = 36\pi \ cm^3/cm$

$\therefore$ The rate of change of volume of a sphere, *w.r.t.* radius when the radius is 3, is $36\pi \ cm^3/cm$.

Example 2 : The rate of change of volume of a sphere *w.r.t.* time is $16\pi \ cm^3/sec$. Find the rate of change of its surface area *w.r.t.* time at the moment when the radius is 2 *cm*.

Solution : Volume of a sphere, $V = \frac{4}{3}\pi r^3$, where r is the radius

Volume changes *w.r.t.* time. So r and V are functions of time t.

$\therefore \quad \dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt} = \frac{4}{3}\pi \times 3r^2 \dfrac{dr}{dt}$

$= 4\pi r^2 \dfrac{dr}{dt}$

$\therefore \quad 16\pi = 4\pi r^2 \dfrac{dr}{dt}$ $\quad \left(\dfrac{dV}{dt} = 16\pi \ cm^3/sec\right)$

$\therefore \quad \dfrac{dr}{dt} = \dfrac{4}{r^2} \ cm/sec$

Now surface area of a sphere, $S = 4\pi r^2$

$\therefore \quad \dfrac{dS}{dt} = \dfrac{dS}{dr} \cdot \dfrac{dr}{dt}$

$= 8\pi r \dfrac{dr}{dt}$

$= 8\pi r \cdot \dfrac{4}{r^2}$

$= \dfrac{32\pi}{r} = 16\pi \ cm^2/sec$ $\quad (r = 2)$

$\therefore \quad \left(\dfrac{dS}{dt}\right)_{r=2} = \dfrac{32\pi}{2} = 16\pi \ cm^2/sec$

$\therefore$ The rate of change of surface area of the sphere is $16\pi \ cm^2/sec$, when $r = 2$ *cm*.

Example 3 : A stone is dropped into a quiet lake and circular ripples are formed. Circular wave fronts move at the speed of radius increasing at the rate of 5 *cm/sec*. How fast is the area increasing when the radius is 10 *cm* ?

Solution : Area of a circle, $A = \pi r^2$, where r is the radius.

$$\therefore \quad \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= 2\pi r \frac{dr}{dt}$$

Now $r = 10$ *cm* and $\frac{dr}{dt} = 5$ *cm/sec*

$$\therefore \quad \frac{dA}{dt} = 2\pi \times 10 \times 5 = 100\pi \ cm^2/sec.$$

$\therefore$ The area enclosed by the waves increases at the rate of 100π cm^2/sec.

> We say as x increases, y increases if and only if $\frac{dy}{dx} > 0$. We say as x increases, y decreases if and only if $\frac{dy}{dx} < 0$. Later on in this chapter, we will study the concept of an increasing (decreasing) function. If $\frac{dy}{dx} > 0$, then y is an increasing function of x and if $\frac{dy}{dx} < 0$, then y is a decreasing function of x.

Example 4 : Air is being pumped into a spherical balloon so that its volume increases at the rate 80 cm^3/sec. How fast is the radius of the balloon increasing when the diameter is 32 *cm* ?

Solution : Volume of a sphere, $V = \frac{4}{3}\pi r^3$, where r is its radius.

$$\therefore \quad \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now $\frac{dV}{dt} = 80 \ cm^3/sec, \ r = \frac{32}{2} = 16 \ cm$

$$\therefore \quad 80 = 4\pi \cdot 256 \frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = \frac{5}{64\pi} \ cm/sec$$

$\therefore$ The radius increases at the rate of $\frac{5}{64\pi}$ *cm/sec*

Example 5 : A ladder 5 *m* long is leaning against a wall. The bottom of the ladder is pulled away along the floor away from the wall at the rate 3 *cm/sec*. How fast is its height on the wall decreasing when the foot of the ladder is 4 *m* away from the wall ?

Solution : Let l be the length of the ladder. A is the end-point of the ladder on the wall. C is the point where the ladder touches the ground. $\overline{AB}$ is a part of the wall.

From the figure 1.2, $x^2 + y^2 = l^2$.

$$\therefore \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Now $l = 5 \ m, \ y = 4 \ m$

$$\therefore \quad x = \sqrt{l^2 - y^2}$$

$$= \sqrt{25 - 16}$$

$$= 3 \ m$$

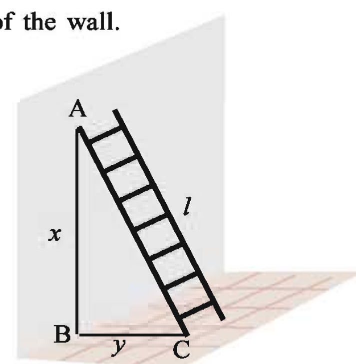

Figure 1.2

$\frac{dy}{dt} = 3\ cm/sec$ $\left(\frac{dy}{dt} > 0 \text{ as } y \text{ is increasing when } t \text{ increases}\right)$

$\therefore$ $\frac{dy}{dt} = 0.03\ m/sec$

$\therefore$ $3\frac{dx}{dt} + 4(0.03) = 0$

$\therefore$ $\frac{dx}{dt} = -0.04$ $\left(\frac{dx}{dt} < 0 \text{ as } x \text{ is decreasing when } t \text{ is increasing}\right)$

$\therefore$ The height of the ladder on the wall is decreasing at the rate of 4 *cm/sec*.

Example 6 : Find the point on the curve $y = x^3 + 7$, where the non-zero rate of change of y *w.r.t.* time is 3 times the rate of change of x *w.r.t.* time.

Solution : We have $y = x^3 + 7$.

Given $\frac{dy}{dt} = 3\frac{dx}{dt}$ **(i)**

Now $\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ **(ii)**

$\therefore$ From (i) and (ii) $3\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

$\therefore$ $x^2 = 1$ $\left(\frac{dx}{dt} \neq 0\right)$

$\therefore$ $x = 1$ or -1

$\therefore$ $y = 8$ or 6

$\therefore$ The required points on $y = x^3 + 7$ where the non-zero rate of change of y *w.r.t.* t is 3 times rate of change of x *w.r.t.* t are $(1, 8)$ and $(-1, 6)$.

Example 7 : On a national highway, a car is driven East at a speed of 60 *km/hr* and a staff bus is driven South at a speed of 50 *km/hr*. Both are headed for the intersection of the roads. The car is 600 *m* away and the bus is 800 *m* away from the intersection. Find the rate at which the car and the bus are approaching each other.

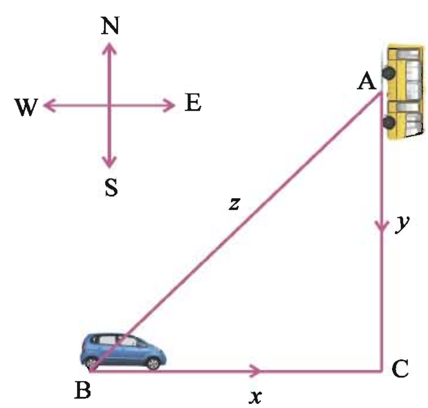

Figure 1.3

Solution : C is the intersection of the roads. B represents the position of the car and A represents the position of the bus at a time. Let BC = x, AC = y at a moment. The distance between the car and the bus is AB = z.

From figure 1.3, $x^2 + y^2 = z^2$.

$\frac{dx}{dt} = -60\ km/hr$, $\frac{dy}{dt} = -50\ km/hr$, negative as x and y are decreasing functions of time.

$x = 0.6\ km$ and $y = 0.8\ km$

$\therefore$ $z = \sqrt{(0.6)^2 + (0.8)^2} = 1\ km$

Now, $x^2 + y^2 = z^2$

$\therefore \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$\therefore \quad \frac{dz}{dt} = \frac{1}{z} \left(x \frac{dx}{dt} + y \frac{dy}{dt}\right)$

$= \frac{1}{1} (0.6 (-60) + 0.8(-50))$

$= -76 \ km/hr$

$\therefore$ The bus and the car are approaching each other at the rate of 76 *km/hr*

Example 8 : The total cost in rupees associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 10x + 10000$. Find the marginal cost, when 20 units are produced.

[Note : Marginal cost means the rate of change of total cost *w.r.t.* the output x.]

Solution : We have $C(x) = 0.005x^3 - 0.02x^2 + 10x + 10000$

$\therefore$ Marginal cost MC $= \frac{dC}{dx} = (0.005)3x^2 - (0.02)2x + 10$

$\therefore \quad \left(\frac{dC}{dx}\right)_{x = 20} = (0.005)1200 - (0.02)40 + 10$

$= 6 - 0.8 + 10$

$= 15.2$

$\therefore$ The required marginal cost is ₹ 15.2.

Example 9 : The total revenue in rupees received from the sale of x units is given by $R(x) = 10x^2 + 20x + 1500$. Find the marginal revenue when $x = 5$.

[Note : Marginal revenue means the rate of change of total revenue *w.r.t.* the number of units sold.]

Solution : We have $R(x) = 10x^2 + 20x + 1500$

$\therefore \quad \frac{dR}{dx} = 20x + 20$

$\therefore \quad \left(\frac{dR}{dx}\right)_{x = 5} = 100 + 20 = 120$

$\therefore$ The marginal revenue is ₹ 120.

Example 10 : The volume of a cube is increasing at the rate of 12 cm^3/sec. Find the rate at which the surface area is increasing, when the length of the edge of the cube is 10 *cm*.

Solution : Volume of a cube, $V = x^3$, where x is the length of an edge.

$\therefore \quad \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt}$

$= 3x^2 \frac{dx}{dt}$

But $\frac{dV}{dt} = 12 \ cm^3/sec$

$\therefore \quad 12 = 3x^2 \frac{dx}{dt}$

$\therefore \quad \frac{dx}{dt} = \frac{4}{x^2}$

Now surface area of the cube, S = $6x^2$

$\therefore \quad \frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt}$

$= 12x\frac{dx}{dt}$

$= 12x \times \frac{4}{x^2}$

$= \frac{48}{x}$

$\therefore \quad \left(\frac{dS}{dt}\right)_{x = 10} = \frac{48}{10}$

$\therefore \quad \frac{dS}{dt} = 4.8 \ cm^2/sec$

$\therefore$ The rate of increase of surface area is 4.8 cm^2/sec.

Example 11 : A water tank is in the shape of an inverted cone. The radius of the base is 4 *m* and the height is 6 *m*. The tank is being emptied for cleaning at the rate of 2 m^3/min. Find the rate at which the water level will be decreasing, when the water is 3 *m* deep.

Solution : Let the height of the water level at any instant be h and the radius of water cone be r.

Using similarity of triangles, $\frac{OA}{BC} = \frac{OD}{BD}$

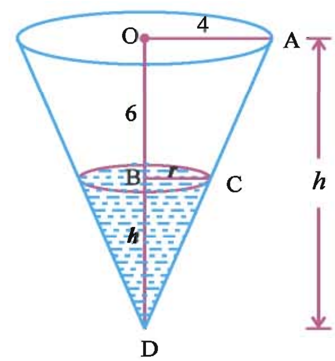

Figure 1.4

$\therefore \quad \frac{4}{r} = \frac{6}{h}$

$\therefore \quad \frac{r}{h} = \frac{2}{3}$

$\therefore \quad r = \frac{2h}{3}$

Now the volume of water at any time t is,

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{4h^2}{9}\right)h$

$= \frac{4\pi h^3}{27}$

$\therefore \quad \frac{dV}{dt} = \frac{4\pi}{27}\left(3h^2\frac{dh}{dt}\right)$

$\therefore \quad \frac{dV}{dt} = \frac{4\pi h^2}{9}\frac{dh}{dt}$

$\therefore \quad \frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}$

Now $\frac{dV}{dt} = -2 \ m^3/min$ **(Volume is decreasing)**

$\therefore \quad \frac{dh}{dt} = \frac{9}{4\pi h^2}(-2)$

$\therefore \quad \left(\frac{dh}{dt}\right)_{h = 3} = \frac{-9}{2\pi(9)}$

$= -\frac{1}{2\pi}$

$\therefore$ The height is decreasing at the rate $\frac{1}{2\pi}$ m/min.

Exercise 1.1

1. The surface area of a cube increases at the rate of 12 cm^2/sec. Find the rate at which its volume increases, when its edge has length 5 cm.
2. Find the rate of change of volume of a cone *w.r.t.* its radius, when the height is kept constant.
3. Find the rate of change of lateral surface area of a cone *w.r.t.* to its radius, when the height is kept constant.
4. The volume of a sphere increases at the rate 8 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.
5. The volume of a closed hemisphere increases at the rate of 4 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.
6. A cylinder is heated so that its radius remains twice of its height at any moment. Find the rate of increase of its volume, when the radius is 3 cm and the radius increases at the rate 2 cm/sec. Find the rate of increase of its total surface area also in this case.
7. A stone is dropped into a quiet lake and ripples move in circles with radius increasing at a speed 4 cm/sec. At the time when the radius of a circular wave is 10 cm, find the rate at which the area enclosed by the waves increases.
8. A rectangular plate is expanding. Its length x is increasing at the rate 1 cm/sec and its width y is decreasing at the rate 0.5 cm/sec. At the moment when $x = 4$ and $y = 3$, find the rate of change of (1) its area (2) its perimeter (3) its diagonal.
9. A ladder 7.5 m long leans against a wall. The ladder slides along the floor away from the wall at the rate of 3 cm/sec. How fast is the height of the ladder on the wall decreasing, when the foot of the wall is 6 m away from the wall ?
10. A concrete mixture is pouring on ground at the rate of 8 cm^3/sec to form a cone in such a way that the height of the cone is always $\frac{1}{4}$*th* of the radius at the time. Find the rate of increase of the height, when the radius is 8 cm.
11. The total cost in rupees associated with the production of x units is given by
$C(x) = 0.005x^3 - 0.004x^2 + 20x + 1000$. Find the marginal cost when $x = 10$.
12. The total revenue in rupees received from the sale of x units of a product is given by
$R(x) = 20x^2 + 15x + 50$. Find the marginal revenue when $x = 15$.
13. A man 2 m tall walks away at a rate of 4 m/min from source of light 6 m high from the ground. How fast is the length of his shadow changing ?
14. Area of a triangle is increasing at a rate of 4 cm^2/sec and its altitude is increasing at a rate of 2 cm/sec. At what rate is the length of the base of the triangle changing, when its altitude is 20 cm and area is 30 cm^2 ?
15. Two sides of a triangle have lengths 4 m and 5 m. The measure of the angle between them is increasing at a rate of 0.05 rad/sec. Find the rate at which the area of the triangle increases, when the angle between the sides (fixed) has measure $\frac{\pi}{3}$.

16. Two sides of a triangle have lengths 10 *m* and 15 *m*. The angle between them has the measure increasing at a rate of 0.01 *rad/sec*. How fast is the third side increasing when the angle between sides having lengths 10 *m* and 15 *m* (fixed) has measure $\frac{\pi}{3}$?

17. The radius of a spherical balloon increases at the rate of 0.3 *cm/sec*. Find the rate of increase of its surface area, when the radius is 5 *cm*.

18. If $y = 3x - x^3$ and x increases at the rate of 3 units per second, how fast is the slope of the curve changing when $x = 2$?

19. A particle moves on the curve $y = x^3$. Find the points on the curve at which the y-coordinate changes *w.r.t.* time thrice as fast as x-coordinate.

20. Find the points on the parabola $y^2 = 4x$ for which the rate of change of abscissa and ordinate is same.

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1.3 Increasing and Decreasing Functions

We have seen in the third semester that $f(x) = a^x$, $a \in R^+$, $x \in R$ is an increasing function of x for $a > 1$ i.e. as x increases, the value of $f(x)$ also increases. This was observed looking at the graph of $f(x) = a^x$. But this is not always possible or even convenient for all functions. Let us find a criterion for this.

Consider $f(x) = 2x + 3$, $x \in R$. Here obviously,

$$\begin{aligned} x_1 < x_2 &\Rightarrow 2x_1 < 2x_2 \\ &\Rightarrow 2x_1 + 3 < 2x_2 + 3 \\ &\Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in R \end{aligned}$$

Thus f is 'increasing' on R. We have observed *sine* is increasing in $\left(0, \frac{\pi}{2}\right)$.

Consider $f(x) = x^2$, $x \in R$ (Fig. 1.5)

In the first quadrant $f(x) = x^2$ increases with x and as x proceeds towards right of Y-axis, y-coordinate increases. But on the left of Y-axis, as x increases, y decreases.

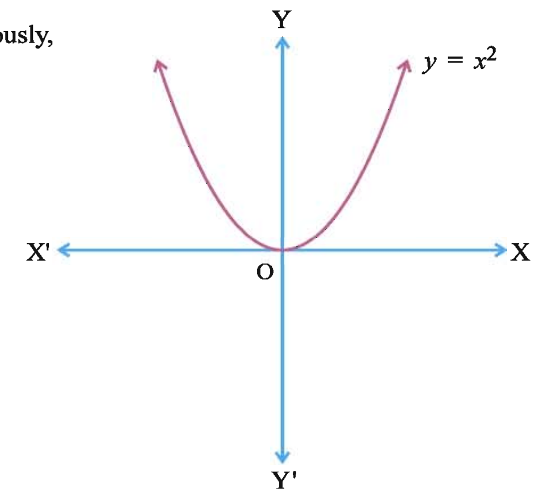


Figure 1.5

Now let us formally define this concept.

Definition : Let (a, b) be a subset of the domain of a function. We say,

(1) f is increasing on (a, b) (denoted by $f \uparrow$) if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \forall x_1, x_2 \in (a, b)$$

(2) f is strictly increasing on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in (a, b)$

(3) f is decreasing on (a, b) (denoted by $f \downarrow$) if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \forall x_1, x_2 \in (a, b)$

(4) f is strictly decreasing on (a, b) if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \forall x_1, x_2 \in (a, b)$

We say f is increasing (or decreasing or strictly increasing or strictly decreasing) on R or a subset of R which is a subset of its domain D, if f is increasing in every open interval (or decreasing or strictly increasing or strictly decreasing) which is a subset of R or of D as the case may be.

Consider following graphs :

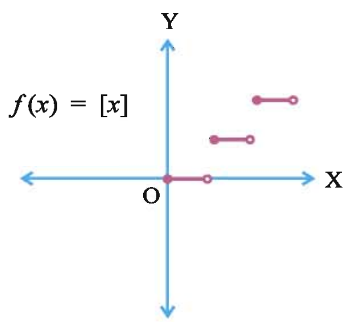


Figure 1.6

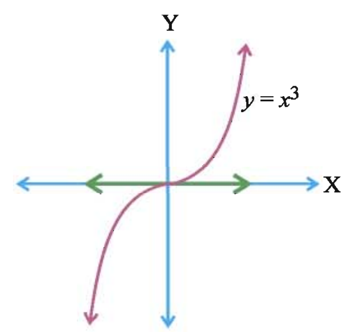


Figure 1.7

Figure 1.6 is the graph of the increasing function $f(x) = [x]$ in $[0, 1)$, $[1, 2)$... It is increasing on R.

Note : See that increasing actually means non-decreasing.

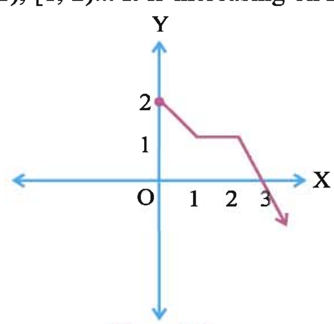


Figure 1.8

Figure 1.7 represents the graph of a strictly increasing function.

Figure 1.8 is the graph of $f(x) = \begin{cases} 2 - x & 0 \le x \le 1 \\ 1 & 1 < x < 2 \\ 3 - x & x \ge 2 \end{cases}$

Here f is decreasing for $x \ge 0$.

Note : f is constant, so f is non-increasing and non-decreasing for $1 < x < 2$.

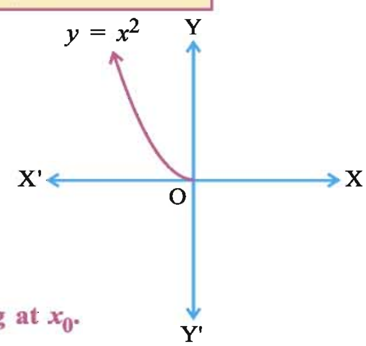


Figure 1.9

$f(x) = x^2$, $x < 0$ represents the graph of a decreasing function. (Fig. 1.9)

A function increasing or decreasing at a point :

Let f be defined on a domain containing an open interval I. Let $x_0 \in$ I and let some h, $h > 0$ be so small that $(x_0 - h, x_0 + h) \subset$ I.

If f is increasing in $(x_0 - h, x_0 + h)$, we say f is increasing at x_0.

If f is decreasing in $(x_0 - h, x_0 + h)$, we say f is decreasing at x_0.

If f is strictly increasing in $(x_0 - h, x_0 + h)$, we say f is strictly increasing at x_0.

If f is strictly decreasing in $(x_0 - h, x_0 + h)$, we say f is strictly decreasing at x_0.

If f is increasing for all $x_0 \in$ I (decreasing, strictly decreasing or strictly increasing), then we say f is increasing (decreasing, strictly decreasing or strictly increasing) on I.

Now we will find some criteria to determine the nature of a function whether increasing or decreasing.

Theorem 1.1 : If f is continuous on $[a, b]$ and differentiable in (a, b), then

(1) f is increasing on (a, b) if $f'(x) \geq 0$ $\quad \forall x \in (a, b)$

(2) f is decreasing on (a, b) if $f'(x) \leq 0$ $\quad \forall x \in (a, b)$

(3) f is strictly increasing on (a, b) if $f'(x) > 0$ $\quad \forall x \in (a, b)$

(4) f is strictly decreasing on (a, b) if $f'(x) < 0$ $\quad \forall x \in (a, b)$

(5) f is constant on (a, b) if $f'(x) = 0$ $\quad \forall x \in (a, b)$

Proof : Let $x_1 \in (a, b)$, $x_2 \in (a, b)$ and $x_1 < x_2$. Since f is continuous on $[a, b]$ and differentiable in (a, b), there exists $c \in (x_1, x_2) \subset (a, b)$ so that $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$.

(Mean value theorem)

(1) If $f'(x) \geq 0$, $\forall x \in (a, b)$, $f'(c) \geq 0$ as $c \in (x_1, x_2) \subset (a, b)$.

Since $x_1 < x_2$, $x_2 - x_1 > 0$

$\therefore$ $f'(c)(x_2 - x_1) \geq 0$

$\therefore$ $f(x_2) - f(x_1) \geq 0$

$\therefore$ $f(x_1) \leq f(x_2)$

$\therefore$ $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$, $\forall x_1, x_2 \in (a, b)$

$\therefore$ f is increasing on (a, b).

(2) If $f'(x) \leq 0$, $\forall x \in (a, b)$, $f'(c) \leq 0$.

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$, $\forall x_1, x_2 \in (a, b)$,

$\therefore$ f is decreasing on (a, b).

(3) If $f'(x) > 0$, $\forall x \in (a, b)$, $f'(c) > 0$.

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, $\forall x_1, x_2 \in (a, b)$,

$\therefore$ f is strictly increasing on (a, b).

(4) If $f'(x) < 0$, $\forall x \in (a, b)$, $f'(c) < 0$.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$, $\forall x_1, x_2 \in (a, b)$,

$\therefore$ f is strictly decreasing on (a, b).

(5) If $f'(x) = 0$, $\forall x \in (a, b)$, $f'(c) = 0$.

$f(x_2) - f(x_1) = 0$, $\forall x_1, x_2 \in (a, b)$

$\therefore$ $f(x_2) = f(x_1)$ $\quad \forall x_1, x_2 \in (a, b)$

$\therefore$ f is a constant function on (a, b).

Note : Do you remember how arbitrary constant was introduced in indefinite integration ?

In view of the remark preceding the theorem, f is increasing or decreasing on $[a, b]$ also according as $f'(x) \geq 0$ or $f'(x) \leq 0$ respectively in (a, b).

Similar remarks apply for strictly increasing and strictly decreasing functions.

Example 12 : Prove that *sine* function is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution : $\frac{d}{dx} \sin x = \cos x$

$\cos x > 0$, if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore$ *sine* function is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 13 : Prove that $f(x) = \left(\frac{1}{2}\right)^x$ is strictly decreasing on R.

Solution : $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

$\therefore$ $f'(x) = -2^{-x} \log 2 < 0$ as $\log_e 2 > 0$ and $2^{-x} > 0$.

$\therefore$ f is strictly decreasing on any interval $(a, b) \subset$ R.

$\therefore$ $f(x) = \left(\frac{1}{2}\right)^x$ is strictly decreasing on R.

Example 14 : Prove that $f(x) = \tan x$, $x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \mid k \in \text{Z}\right\}$ is strictly increasing in every quadrant.

Solution : $f(x) = \tan x$

$\therefore$ $f'(x) = sex^2 x > 0 \quad \forall x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \mid k \in \text{Z}\right\}$.

$\therefore$ $f(x) = \tan x$ is strictly increasing in all intervals like $\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right)$,... etc.

$\therefore$ $f(x) = \tan x$ is strictly increasing in all quadrants.

Example 15 : Prove that f: R $\rightarrow$ R, $f(x) = ax + b$ is strictly increasing for $a > 0$ and strictly decreasing for $a < 0$.

Solution : $f(x) = ax + b$

$\therefore$ $f'(x) = a$

$\therefore$ If $a > 0$, $f'(x) > 0$ and so f is strictly increasing on R.

$\therefore$ If $a < 0$, $f'(x) < 0$ and so f is strictly decreasing on R.

As an example $f(x) = 5x + 7$ is strictly $\uparrow$ and $f(x) = -2x + 3$ is strictly $\downarrow$.

Example 16 : Prove that $f(x) = x^3$, $x \in$ R is increasing on R.

Solution : $f'(x) = 3x^2 \geq 0$

$\therefore$ f is $\uparrow$ on any $(a, b) \subset$ R

$\therefore$ f is $\uparrow$ on R.

Example 17 : Prove that f: R $\rightarrow$ R, $f(x) = x^3 + 3x^2 + 5x$ is strictly increasing on R.

Solution : $f(x) = x^3 + 3x^2 + 5x$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 + 6x + 5 \\ &= 3x^2 + 6x + 3 + 2 \\ &= 3(x + 1)^2 + 2 > 0, \quad \forall x \in \text{R} \end{aligned}$$

$\therefore$ f is strictly increasing on R.

Example 18 : Find the intervals in which $f: R \to R$, $f(x) = x^2 - 6x + 15$ is strictly increasing or strictly decreasing.

Solution : $f(x) = x^2 - 6x + 15$

$\therefore \quad f'(x) = 2x - 6$

If $x < 3$, $2x < 6$ and $f'(x) < 0$.

$\therefore \quad f$ is strictly decreasing for $x \in (-\infty, 3)$.

If $x > 3$, $2x > 6$ and $f'(x) > 0$.

$\therefore \quad f$ is strictly increasing for $x \in (3, \infty)$.

Example 19 : Determine in which intervals the function $f: R \to R$, $f(x) = x^3 - 6x^2 - 36x + 2$ is increasing and where it is decreasing.

Solution : $f(x) = x^3 - 6x^2 - 36x + 2$

$$\begin{aligned} \therefore \quad f'(x) &= 3x^2 - 12x - 36 \\ &= 3(x^2 - 4x - 12) \\ &= 3(x - 6)(x + 2) \end{aligned}$$

(1) If $x < -2$, then $x < 6$

$\therefore \quad x + 2 < 0, x - 6 < 0$

$\therefore \quad f'(x) = 3(x - 6)(x + 2) > 0$

$\therefore \quad f$ is $\uparrow$ in $(-\infty, -2)$. **(Infact strictly $\uparrow$)**

(2) If $-2 < x < 6$, then $x + 2 > 0$, $x - 6 < 0$

$\therefore \quad f'(x) = 3(x - 6)(x + 2) < 0$

$\therefore \quad f$ is $\downarrow$ in $(-2, 6)$.

(3) If $x > 6$, then $x + 2 > 0$, $x - 6 > 0$

$\therefore \quad f'(x) > 0$

$\therefore \quad f$ is $\uparrow$ in $(6, \infty)$.

Example 20 : Determine where $f(x) = tan^{-1}(sinx + cosx)$, $x \in (0, \pi)$ is increasing and in which interval it is decreasing.

Solution : $f(x) = tan^{-1}(sinx + cosx)$

$$\begin{aligned} \therefore \quad f'(x) &= \frac{1}{1 + (sinx + cosx)^2} \times (cosx - sinx) \\ &= \frac{cosx - sinx}{1 + (sinx + cosx)^2} \end{aligned}$$

(1) If $x \in \left(0, \frac{\pi}{4}\right)$, then $cosx > sinx$

$\left(cosx \in \left(\frac{1}{\sqrt{2}}, 1\right) \text{ and } sinx \in \left(0, \frac{1}{\sqrt{2}}\right)\right)$

Also $1 + (sinx + cosx)^2 > 0$

$\therefore \quad f'(x) > 0$ for $x \in \left(0, \frac{\pi}{4}\right)$

$\therefore \quad f$ is increasing in $\left(0, \frac{\pi}{4}\right)$.

(2) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x < \sin x$. Thus, $\cos x - \sin x < 0$ and if $x \in \left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$, $\sin x > 0$

$\therefore \quad \cos x - \sin x < 0$. For $x = \frac{\pi}{2}$, $\cos x - \sin x = 0 - 1 = -1 < 0$

$\therefore \quad$ If $x \in \left(\frac{\pi}{4}, \pi\right)$, $f'(x) < 0$

$\therefore \quad f$ is decreasing in $\left(\frac{\pi}{4}, \pi\right)$.

Example 21 : Prove that $f(x) = x^{100} + \sin x - 1$ is increasing for $x \in (0, \pi)$.

Solution : $f(x) = x^{100} + \sin x - 1$

$\therefore \quad f'(x) = 100x^{99} + \cos x$

For $x \in \left(0, \frac{\pi}{2}\right)$, $x^{99} > 0$, $\cos x > 0$. So $f'(x) > 0$.

For $x = \frac{\pi}{2}$, $x^{99} > 0$, $\cos x = 0$. So $f'(x) > 0$.

If $x \in \left(\frac{\pi}{2}, \pi\right)$, $x^{99} > 1$ and $-1 < \cos x < 0$.

$\therefore \quad f'(x) > 0$.

$\therefore \quad f$ is (strictly) increasing in $(0, \pi)$.

Example 22 : Prove $f(x) = \log \sin x$ is increasing in $\left(0, \frac{\pi}{2}\right)$.

Solution : $f(x) = \log \sin x$

$\therefore \quad f'(x) = \frac{1}{\sin x} \times \cos x = \cot x > 0$ in $\left(0, \frac{\pi}{2}\right)$.

$\therefore \quad f$ is increasing in $\left(0, \frac{\pi}{2}\right)$.

Example 23 : Determine intervals in which $f(x) = \frac{x}{\log x}$, $x > 1$ is increasing and where it is decreasing.

Solution : $f(x) = \frac{x}{\log x}$

$$\therefore \quad f'(x) = \frac{\log x - x \cdot \frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$$

(1) $x < e$, then $\log x < \log e = 1$

$\therefore \quad \log x - 1 < 0$. Also $(\log x)^2 > 0$

$\therefore \quad f'(x) < 0$.

$\therefore \quad f$ is $\downarrow$ in $(1, e)$.

(2) If $x > e$, then $\log x > 1$. So $\log x - 1 > 0$ and $(\log x)^2 > 0$

$\therefore \quad f'(x) > 0$.

$\therefore \quad f$ is $\uparrow$ in (e, ∞).

Example 24 : Prove $f(x) = \frac{tan x}{x}$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

Solution : $f(x) = \frac{tan x}{x} = \frac{sin x}{x cos x}$

$$\therefore \quad f'(x) = \frac{x cos x \cdot cos x - sin x\,(cos x - x sin x)}{(x cos x)^2}$$

$$= \frac{x(cos^2 x + sin^2 x) - sin x cos x}{(x cos x)^2}$$

$$= \frac{x - sin x\, cos x}{(x cos x)^2}$$

Now, $0 < x < \frac{\pi}{2}$. So $0 < sin\,x < x$, $0 < cos\,x < 1$

$\therefore \quad 0 < sin x\; cos x < x$

$\therefore \quad x - sin x\; cos x > 0$. Also $(x cos\,x)^2 > 0$

$\therefore \quad f'(x) > 0$

$\therefore \quad f$ is $\uparrow$ in $\left(0, \frac{\pi}{2}\right)$.

Exercise 1.2

1. Prove that $cot : R - \{k\pi \mid k \in Z\} \rightarrow R$ is decreasing in all quadrants.
2. Prove that *cosine* is a decreasing function in $(0, \pi)$.
3. Prove that *sec* is an increasing function in $\left(0, \frac{\pi}{2}\right)$.
4. Prove that *cosec* is an increasing function in $\left(\frac{\pi}{2}, \pi\right)$.
5. Prove that $f(x) = a^x$ is $\uparrow$, if $a > 1$.
6. Prove that $f(x) = \log_e x$ is $\uparrow$, $x \in R^+$.
7. **Determine the intervals in which f is increasing and the intervals in which f is decreasing :**

(1) $f : R \rightarrow R$, $\quad f(x) = 3x + 7$

(2) $f : R \rightarrow R$, $\quad f(x) = 8 - 5x$

(3) $f : R \rightarrow R$, $\quad f(x) = x^2 - 2x + 5$

(4) $f : R \rightarrow R$, $\quad f(x) = 9 + 3x - x^2$

(5) $f : R \rightarrow R$, $\quad f(x) = x^3 + 3x + 10$

(6) $f : R \rightarrow R$, $\quad f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

(7) $f : (0, \pi) \rightarrow R$, $\quad f(x) = sin\,x + cos\,x$

(8) $f : R \rightarrow R$, $\quad f(x) = -2x^3 - 9x^2 - 12x + 1$

(9) $f : R \rightarrow R$, $\quad f(x) = (x + 1)^3 (x - 3)^3$

(10) $f : \left(0, \frac{\pi}{2}\right) \rightarrow R$, $\quad f(x) = \log\, cos x$

(11) $f : \left(\frac{\pi}{2}, \pi\right) \rightarrow R$, $\quad f(x) = \log |\cos x|$

(12) $f : R - \{0\} \rightarrow R$, $\quad f(x) = e^{\frac{1}{x}}$

8. Prove that if I is an open interval and $I \cap [-1, 1] = \emptyset$, then $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

9. Prove that $f(x) = x^3 - 3x^2 + 3x + 100$ is increasing on R.

10. Prove that $f(x) = x^{100} + \sin x - 1$ is increasing on (0, 1).

11. Find intervals in which $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is increasing and intervals in which it is decreasing.

12. Find in which intervals, $f : R \rightarrow R$, $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is decreasing and intervals in which it is increasing.

13. Prove $f(x) = x^x$, $x \in R^+$ is increasing if $x > \frac{1}{e}$ and decreasing if $0 < x < \frac{1}{e}$.

14. Decide the intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or intervals in which it is decreasing. $x \in \left(0, \frac{\pi}{2}\right)$.

15. Find the value of a for which the function $f(x) = ax^3 - 3(a + 2)x^2 + 9(a + 2)x - 1$ is decreasing for all $x \in R$.

16. Find the values of a for which $f(x) = ax^3 - 9ax^2 + 9x + 25$ is increasing on R.

17. Prove that $f(x) = (x - 1)e^x + 1$ is increasing for all $x > 0$.

18. Prove that $f(x) = x^2 - x \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

19. Prove $f : R \rightarrow R$, $f(x) = x^2$ is increasing for $x \in R^+$ and decreasing for $x < 0$ without using derivative test and using the definition only.

20. Prove $f : R \rightarrow R$, $f(x) = 2^x + 2^{-x}$ is increasing for $x \in (0, \infty)$ and decreasing for $x \in (-\infty, 0)$.

21. **Determine intervals in which following functions are strictly increasing or strictly decreasing :**

(1) $f : R \rightarrow R$, $\quad f(x) = x^3 - 6x^2 - 36x + 2$

(2) $f : R \rightarrow R$, $\quad f(x) = x^4 - 4x$

(3) $f : R \rightarrow R$, $\quad f(x) = (x - 1)(x - 2)^2$

(4) $f : R \rightarrow R$, $\quad f(x) = 2x^3 - 12x^2 + 18x + 15$

(5) $f : R^+ \rightarrow R$, $\quad f(x) = x\sqrt{x+1}$

(6) $f : R^+ \rightarrow R$, $\quad f(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

(7) $f : (0, \pi) \rightarrow R$, $\quad f(x) = 2x + \cot x$

(8) $f : R \rightarrow R$, $\quad f(x) = 2\cos x + \sin^2 x$

(9) $f : R \rightarrow R$, $\quad f(x) = \log(1 + x^2)$

(10) $f : R \to R$, $f(x) = x^6 + 192x + 10$

(11) $f : R \to R$, $f(x) = xe^x$

(12) $f : R \to R$, $f(x) = x^2e^x$

(13) $f : R^+ \to R^+$, $f(x) = \frac{\log x}{\sqrt{x}}$

(14) $f : R^+ \to R^+$, $f(x) = x \log x$

*

1.4 Applications to Geometry

(1) Tangents and Normals : We know that if $y = f(x)$ is a differentiable function in (a, b), $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$, $x_0 \in (a, b)$.

So a tangent to a curve $y = f(x)$ at $(x_0, f(x_0))$ is the line passing through (x_0, y_0) and having slope $f'(x_0)$, where $y_0 = f(x_0)$. If a tangent at (x_0, y_0) is vertical, it does not have a slope.

The equation of tangent at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = f'(x_0)(x - x_0)$, where the tangent is not vertical. If the tangent is a vertical line through (x_0, y_0), its equation is $x = x_0$.

Note : A tangent may intersect the curve again. The tangents $y = 1$ or $y = -1$ intersect the graph of $y = sinx$, $x \in R$ in infinitely many points. (Touch)

A normal to a curve $y = f(x)$ at (x_0, y_0) is a line perpendicular to the tangent at that point and passing through (x_0, y_0). If the tangent is not horizontal, $f'(x_0) \neq 0$. Then the slope of the normal at (x_0, y_0) is $-\frac{1}{f'(x_0)}$, since slopes m_1, m_2 of perpendicular lines satisfy $m_1 m_2 = -1$.

$\therefore$ The equation of the normal at (x_0, y_0) is $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$ $\quad (f'(x_0) \neq 0)$

If $f'(x_0) = 0$, the equation of the normal at (x_0, y_0) is $x = x_0$. If the tangent at (x_0, y_0) is vertical, the equation of the normal at (x_0, y_0) is $y = y_0$.

Example 25 : Find the slope of the tangent and the normal to $y = x^3 - 2x + 4$ at $(1, 3)$.

Solution : The equation of the curve is $y = x^3 - 2x + 4$.

$$\frac{dy}{dx} = 3x^2 - 2$$

$$\therefore \left(\frac{dy}{dx}\right)_{x = 1} = 1$$

$\therefore$ The slope of the tangent to $y = x^3 - 2x + 4$ at $(1, 3)$ is 1.

Since a normal at a point is perpendicular to the tangent at the point , its slope at $(1, 3)$ is -1.

$(m_1m_2 = -1)$

Example 26 : Find the equation of the tangent and the normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1).

Solution : The equation of the circle is $x^2 + y^2 = a^2$.

$\therefore \quad 2x + 2y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$, if $y \neq 0$

$\therefore$ The equation of the tangent at (x_1, y_1) is,

$y - y_1 = -\dfrac{x_1}{y_1}(x - x_1) \qquad (y_1 \neq 0)$

$\therefore \quad yy_1 - y_1^2 = -xx_1 + x_1^2$

$\therefore \quad xx_1 + yy_1 = x_1^2 + y_1^2$

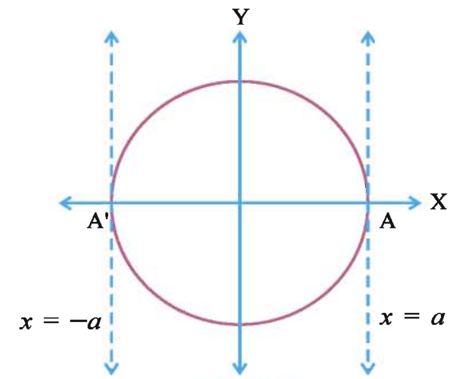


Figure 1.10

But $x_1^2 + y_1^2 = a^2$ as (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$.

$\therefore \quad xx_1 + yy_1 = a^2$ is the equation of tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$. $(y_1 \neq 0)$

Corresponding to $y_1 = 0$, A$(a, 0)$, A'$(-a, 0)$ are two points on the circle.

$\therefore$ The tangents at A and A' are vertical and have equations $x = a$ and $x = -a$ respectively.

Taking $(x_1, y_1) = (a, 0)$ or $(-a, 0)$ repectively in the equation $xx_1 + yy_1 = a^2$ also, we get

$xa + 0 = a^2$ i.e. $xa = a^2$ or $-xa = a^2$

$\therefore \quad x = a$ and $x = -a$ are tangents at A and A'. $(a \neq 0)$

$\therefore$ At all points (x_1, y_1) on $x^2 + y^2 = a^2$ the equation of tangent to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

A normal to $x^2 + y^2 = a^2$ is perpendicular to $xx_1 + yy_1 = a^2$ and passes through (x_1, y_1).

$\therefore$ Its equation is $xy_1 - yx_1 = x_1y_1 - y_1x_1 = 0$.

> A line perpendicular to $ax + by + c = 0$ and passing through (x_1, y_1) has equation
>
> $bx - ay = bx_1 - ay_1$.

$\therefore$ The equation of the normal to $x^2 + y^2 = a^2$ at (x_1, y_1) is $xy_1 - yx_1 = 0$ and it passes through the centre $(0, 0)$ of the circle.

$\therefore$ A radius (i.e. line containing radius) is always a normal to the circle.

Example 27 : Find the equation of the tangent and the normal to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at $x = a\cos^3\theta$, $y = a\sin^3\theta$. $\theta \in \left[0, \frac{\pi}{2}\right)$. $(a > 0)$

Solution : See that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$

$= a^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta)$

$= a^{\frac{2}{3}}$

$\therefore \quad (a\cos^3\theta, a\sin^3\theta)$ lies on $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Now $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \dfrac{-(a\sin^3\theta)^{\frac{1}{3}}}{-(a\cos^3\theta)^{\frac{1}{3}}} = -\tan\theta$

$\therefore$ The equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is $y - a\sin^3\theta = -\dfrac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

$\therefore \quad y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

$\therefore$ $x\sin\theta + y\cos\theta \;= a\sin\theta\cos\theta\,(\sin^2\theta + \cos^2\theta)$

$\qquad\qquad\qquad\qquad = a\sin\theta\cos\theta$

$\therefore$ The equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is

$$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$$

$\therefore$ The equation of the normal at $(a\cos^3\theta, a\sin^3\theta)$ is

$$\begin{aligned} x\cos\theta - y\sin\theta &= a\cos^3\theta\cos\theta - a\sin^3\theta\sin\theta \\ &= a(\cos^4\theta - a\sin^4\theta) \\ &= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \\ &= a\cos 2\theta \end{aligned}$$

$\therefore$ The equation of the normal at $(a\cos^3\theta, a\sin^3\theta)$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

Note : **Remember :** A line perpendicular to $ax + by + c = 0$ has equation $bx - ay = bx_1 - ay_1$, if it passes through (x_1, y_1).

Example 28 : Find the equation of the tangent and the normal to $y^2 = 4ax$ at $(at^2, 2at)$

Solution : The equation of the curve is $y^2 = 4ax$.

$\therefore$ $2y\dfrac{dy}{dx} = 4a$

$\therefore$ $2(2at)\dfrac{dy}{dx} = 4a$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{t}$, if $t \neq 0$

$\therefore$ The equation of the tangent at $(at^2, 2at)$ is

$y - 2at = \frac{1}{t}(x - at^2)$ $\qquad$ $(t \neq 0)$

$\therefore$ $ty - 2at^2 = x - at^2$

$\therefore$ $x - ty + at^2 = 0$ is the equation of the tangent to

$y^2 = 4ax$ at $(at^2, 2at)$ where $t \neq 0$

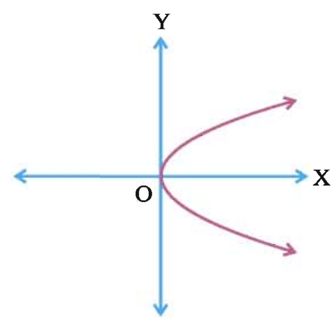


Figure 1.11

$\therefore$ The equation of normal at $(at^2, 2at)$ is $tx + y = t(at^2) + 2at$.

$\therefore$ $tx + y - 2at - at^3 = 0$ is the equation of the normal to $y^2 = 4ax$ at $(at^2, 2at)$. $\qquad (t \neq 0)$

Now if $t = 0$, the corresponding point on parabola is $(0, 0)$. The tangent at $(0, 0)$ is vertical and its equation is $x = 0$. Normal at $t = 0$ is perpendicular to $x = 0$ and passes through $(0, 0)$.

Hence its equation is $y = 0$.

Note : See that these equations can also be obtained from general equations by putting $t = 0$.

Example 29 : Find the equation of the tangent to $y = \sqrt{3x - 2}$ parallel to $4x - 2y + 5 = 0$.

Solution : The slope of the line $4x - 2y + 5 = 0$ is $m = -\frac{a}{b} = -\frac{4}{-2} = 2$.

$\therefore$ The slope of tangent to $y = \sqrt{3x-2}$ must be 2 as parallel lines have same slopes.

$\therefore \quad \frac{dy}{dx} = 2$

$\therefore$ Since $y = \sqrt{3x-2}$ is the equation of the curve,

$\therefore \quad \frac{dy}{dx} = \frac{1 \cdot 3}{2\sqrt{3x-2}} = 2$

$\therefore \quad 9 = 16(3x - 2)$

Let (x_0, y_0) be the point of contact.

Then $x_0 = \frac{1}{3}\left(\frac{9}{16}+2\right) = \frac{41}{48}$, $y_0 = \sqrt{3\times\frac{41}{48}-2}$

$$= \sqrt{\frac{41}{16}-2} = \frac{3}{4}$$

$\therefore$ The equation of tangent at $\left(\frac{41}{48}, \frac{3}{4}\right)$ is $y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$ **$(m = 2)$**

$\therefore \quad 24y - 18 = 48x - 41$

$\therefore \quad 48x - 24y = 23$ is the equation of the tangent to $y = \sqrt{3x-2}$ parallel to $4x - 2y + 5 = 0$.

[Verify that $48x - 24y = 23$ is parallel to $4x - 2y + 5 = 0$ and is not coincident with $4x - 2y + 5 = 0$.]

Example 30 : Find the points on $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to X-axis.

Solution : The equation of the curve is $x^2 + y^2 - 2x - 3 = 0$

$\therefore \quad 2x + 2y\frac{dy}{dx} - 2 = 0$ **(i)**

The tangent is parallel to X-axis. So its slope is zero.

$\therefore \quad \frac{dy}{dx} = 0$

$\therefore \quad 2x - 2 = 0$ **(using (i))**

$\therefore \quad x = 1$

Now, $x^2 + y^2 - 2x - 3 = 0$

$\therefore \quad 1 + y^2 - 2 - 3 = 0$ **$(x = 1)$**

$\therefore \quad y^2 = 4$

$\therefore \quad y = \pm 2$

$\therefore$ The tangents at (1, 2) and (1, −2) to the circle are $y = \pm 2$ and they are parallel to X-axis.

Example 31 : Find the point or points on $y = x^3 - 11x + 5$ at which the equation of the tangent is $y = x - 11$.

Solution : The equation is $y = x^3 - 11x + 5$.

$\therefore \quad \frac{dy}{dx} = 3x^2 - 11$ **(i)**

The slope of $y = x - 11$ is 1.

$\therefore$ The slope of the tangent is 1.

$\therefore \quad \frac{dy}{dx} = 1$

$\therefore \quad 3x^2 - 11 = 1$ **(using (i))**

$\therefore \quad 3x^2 = 12$

$\therefore \quad x^2 = 4$

$\therefore \quad x = \pm 2$

$\therefore$ If $x = 2$, $y = x^3 - 11x + 5 = -9$. If $x = -2$, $y = x^3 - 11x + 5 = 19$

$\therefore$ Point of contact may be $(2, -9)$ or $(-2, 19)$.

At $(2, -9)$, the equation of the tangent is $y + 9 = 1(x - 2)$ **(slope = 1)**

$\therefore \quad y = x - 11.$

$\therefore$ But the tangent at $(-2, 19)$ cannot have equation $y = x - 11$ as $(-2, 19)$ does not lie on $y = x - 11$.

$\therefore$ The tangent at $(2, -9)$ has equation $y = x - 11$.

Example 32 : Show that tangents to $y = 7x^3 + 11$ at $x = 2$ and at $x = -2$ are parallel.

Solution : The equation of the curve is $y = 7x^3 + 11$.

$\therefore \quad \frac{dy}{dx} = 21x^2 = 84$ at $x = \pm 2$

If $x = 2$, $y = 7x^3 + 11 = 67$. If $x = -2$, $y = -45$.

$\therefore$ The equations of tangents at $(2, 67)$ and $(-2, -45)$ are respectively $y - 67 = 84(x - 2)$ and $y + 45 = 84(x + 2)$. **($m = 84$)**

$\therefore$ $84x - y = 101$ and $84x - y + 123 = 0$ are equations of the tangents at $(2, 67)$ and $(-2, -45)$ respectively.

They are having same slopes and are distinct lines.

$\therefore$ They are parallel.

Example 33 : Find the equation of the normal to $x^2 = 4y$ passing through $(1, 2)$.

Solution : The equation of the curve is $x^2 = 4y$

$\therefore \quad 2x = 4\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2}$

$\therefore$ The slope of the normal at (x_0, y_0) is $-\frac{2}{x_0}$ **($x_0 \neq 0$)**

$\therefore$ The equation of the normal at (x_0, y_0) is $y - y_0 = -\frac{2}{x_0}(x - x_0)$ **(i)**

If it passes through $(1, 2)$, $2 - y_0 = -\frac{2}{x_0}(1 - x_0)$

$\therefore \quad x_0\left(2 - \frac{x_0^2}{4}\right) = -2 + 2x_0$ **($x_0^2 = 4y_0$)**

$\therefore \quad 8x_0 - x_0^3 = -8 + 8x_0$

$\therefore \quad x_0^3 = 8$

$\therefore \quad x_0 = 2, \ y_0 = \frac{x_0^2}{4} = 1$

$\therefore$ The equation of the normal at $(2, 1)$ is $y - 1 = -\frac{2}{2}(x - 2) = -x + 2$ **(using (i))**

$\therefore$ $x + y = 3$ is the equation of the normal to $x^2 = 4y$ passing through $(1, 2)$.

Note : (1) If $x_0 = 0$, then $y_0 = 0$. Normal at (x_0, y_0) is $x = 0$. It does not pass through (1, 2)

(2) Here the normal passes **through (1, 2)** and is not at (1, 2). It is proved to be a normal at (2, 1). **(1, 2) does not lie on $x^2 = 4y$.**

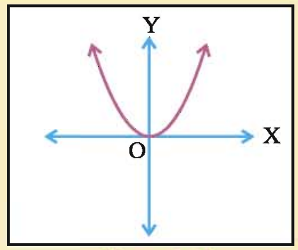


Figure 1.12

Example 34 : Prove that the sum of the intercepts (if they exist) on axes by any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is constant. $(c > 0)$. $(x \neq 0, y \neq 0)$

Solution : The equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}} \qquad (x \neq 0)$$

$\therefore$ The equation of the tangent at (x_1, y_1) is $y - y_1 = -\sqrt{\frac{y_1}{x_1}}\,(x - x_1)$

$$\therefore \quad \frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = -\frac{x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}} \qquad (x_1 \neq 0, y_1 \neq 0)$$

$$\therefore \quad \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1} = \sqrt{c} \qquad ((x_1, y_1) \text{ lies on } \sqrt{x} + \sqrt{y} = \sqrt{c})$$

$\therefore$ It intersects axes at $(\sqrt{x_1}\sqrt{c}, 0)$, $(0, \sqrt{y_1}\sqrt{c})$.

$\therefore$ The sum of the intercepts on axes is

$$\sqrt{x_1}\sqrt{c} + \sqrt{y_1}\sqrt{c} = \sqrt{c}\,(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{c}\sqrt{c} = c$$

$\therefore$ The sum of the intercepts of any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ on axes is constant.

Note : If $x_1 = 0$ or $y_1 = 0$, the points on the curve are $(0, c)$ or $(c, 0)$. The tangents at these points are respectively $x = 0$ and $y = 0$ and do not have both the intercepts.

Example 35 : Prove that any normal to $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from origin. $\theta \neq \frac{k\pi}{2}$, $k \in Z$

Solution : Since $x = a\cos\theta + a\theta\sin\theta$ and $y = a\sin\theta - a\theta\cos\theta$

$$\frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin\theta}{\cos\theta} \qquad (\cos\theta \neq 0)$$

$\therefore$ The slope of the normal at θ-point is $-\frac{\cos\theta}{\sin\theta}$. $\qquad (\sin\theta \neq 0)$

$\therefore$ The equation of the normal at θ-point is $(y - a sin\theta + a\theta\ cos\theta) = -\dfrac{cos\theta}{sin\theta}(x - a cos\theta - a\theta\ sin\theta)$

$\therefore$ $y sin\theta - a sin^2\theta + a\theta\ sin\theta\ cos\theta = -x cos\theta + a cos^2\theta + a\theta\ sin\theta\ cos\theta$

$\therefore$ $x cos\theta + y sin\theta = a(cos^2\theta + sin^2\theta) = a$

$\therefore$ $x cos\theta + y sin\theta = a$ is the equation of the normal at θ-point. $\left(\theta \neq \frac{k\pi}{2}\right)$

If its distance from origin is p, then $p = \dfrac{|c|}{\sqrt{a^2+b^2}}$

$= \dfrac{|-a|}{\sqrt{cos^2\theta + sin^2\theta}} = |a|$ which is a constant.

[What happens if $\theta = \frac{k\pi}{2}$?]

(2) Angle between two curves :

The measure of the angle between two curves is defined to be the measure of the angle between the tangents to them at their point of intersection.

A result : Let $y = f(x)$ and $y = g(x)$, $x \in (a, b)$, be equations of two curves and $f(x)$ and $g(x)$ are differentiable in (a, b). If they intersect at (x_0, y_0), $x_0 \in (a, b)$. The measure α of the angle between them is given by

$$tan\alpha = \left|\frac{f'(x_0) - g'(x_0)}{1 + f'(x_0)\, g'(x_0)}\right|$$

Explanation : We know if m_1 and m_2 are slopes of two lines, the measure α of the angle between them is given by

$$tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Also the slopes of tangents at (x_0, y_0) are $f'(x_0)$ and $g'(x_0)$.

So $m_1 = f'(x_0)$ and $m_2 = g'(x_0)$. Hence the result.

If $f'(x_0)\ g'(x_0) = -1$, $\alpha = \frac{\pi}{2}$ and we say the curves intersect orthogonally.

If $f'(x_0) = g'(x_0)$, the curves touch each other at (x_0, y_0).

Example 36 : Prove that $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect orthogonally at every point of intersection.

Solution : Let us first find the points of intersection.

$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72$ **(i)**

$4x^2 - 4y^2 = 20$ using $x^2 - y^2 = 5$. **(ii)**

Solving (i) and (ii), $13y^2 = 52$

$\therefore$ $y^2 = 4$. So $y = \pm 2$

$\therefore$ $x^2 - 4 = 5$ $(x^2 - y^2 = 5)$

$\therefore$ $x^2 = 9$. So, $x = \pm 3$

$\therefore$ The points of intersection are (3, 2), (3, $-$2), ($-$3, $-$2), ($-$3, 2).

For the first curve $2x - 2y\dfrac{dy}{dx} = 0$

The slope of the tangent to $x^2 - y^2 = 5$ denoted by m_1 is given by $m_1 = \dfrac{x}{y}$.

For the second curve $8x + 18y\dfrac{dy}{dx} = 0$.

$\therefore$ The slope of the tangent to $4x^2 + 9y^2 = 72$ at (x, y) denoted by m_2 is given by $m_2 = -\frac{4x}{9y}$

$\therefore$ $m_1 m_2 = -\frac{4x^2}{9y^2} = -\frac{36}{36} = -1$

$\therefore$ At all the points of intersection the curves (hyperbola and ellipse) intersect orthogonally.

Example 37 : Prove that $y = ax^3$, $x^2 + 3y^2 = b^2$ are orthogonal.

Solution : The slope of the tangent to $y = ax^3$ at (x, y) is denoted by m_1. So $m_1 = \frac{dy}{dx} = 3ax^2$.

$x^2 + 3y^2 = b^2$ implies $2x + 6y\frac{dy}{dx} = 0$

$\therefore$ The slope of the tangent to $x^2 + 3y^2 = b^2$ at (x, y) is denoted by m_2. So $m_2 = \frac{dy}{dx} = -\frac{x}{3y}$.

$\therefore$ $m_1 m_2 = (3ax^2)\left(-\frac{x}{3y}\right) = -\frac{ax^3}{y} = -1$ as at the point of intersection $y = ax^3$

$\therefore$ The curves intersect at right angles.

[The curves do intersect as substituting $y = ax^3$ in $x^2 + 3y^2 = b^2$, we get $x^2 + 3a^2b^6 = b^2$. This equation has a solution.]

Example 38 : Find the measure of the angle between $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$.

Solution : The equations of curves are $x^2 + y^2 - 4x - 1 = 0$, $x^2 + y^2 - 2y - 9 = 0$.

$\therefore$ At the point of intersection, $x^2 + y^2 = 4x + 1 = 2y + 9$.

$\therefore$ $4x - 2y = 8$

$\therefore$ $2x - y = 4$

$\therefore$ $y = 2x - 4$

$\therefore$ Substituting $y = 2x - 4$ in $x^2 + y^2 - 4x - 1 = 0$, $x^2 + (2x - 4)^2 - 4x - 1 = 0$

$\therefore$ $5x^2 - 20x + 15 = 0$

$\therefore$ $x^2 - 4x + 3 = 0$

$\therefore$ $x = 3$ or 1. So correspondingly $y = 2x - 4 = 2$ or -2

$\therefore$ The points of intersection of the circles are $(3, 2)$ and $(1, -2)$.

Now for $x^2 + y^2 - 4x - 1 = 0$, $2x + 2y\frac{dy}{dx} - 4 = 0$ **(i)**

and for $x^2 + y^2 - 2y - 9 = 0$, $2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0$. **(ii)**

(1) At (3, 2) : $6 + 4\frac{dy}{dx} - 4 = 0$, $6 + 4\frac{dy}{dx} - 2\frac{dy}{dx} = 0$ **(Using (i) and (ii))**

$\therefore$ For $x^2 + y^2 - 4x - 1 = 0$ slope of tangent $m_1 = -\frac{1}{2}$.

For $x^2 + y^2 - 2y - 9 = 0$ slope of tangent $m_2 = -3$.

$\therefore$ $\tan\alpha = \left|\frac{-\frac{1}{2} + 3}{1 + \frac{3}{2}}\right| = 1$ $\left(\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$

$\therefore$ $\alpha = \frac{\pi}{4}$

(2) At (1, −2) : $2 - 4\frac{dy}{dx} - 4 = 0$, $2 - 4\frac{dy}{dx} - 2\frac{dy}{dx} = 0$ **(Using (i) and (ii))**

$\therefore$ As before $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{3}$

$\therefore$ $\tan\alpha = \left|\frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}}\right| = 1$

$\therefore \quad \alpha = \frac{\pi}{4}$

$\therefore$ The circles intersect at both the points at an angle having measure $\frac{\pi}{4}$.

Example 39 : Where does the normal to $x^2 - xy + y^2 = 3$ at $(-1, 1)$ intersect the curve again ?

Solution : $x^2 - xy + y^2 = 3$ is the equation of the curve.

$\therefore \quad 2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

$\therefore$ At $(-1, 1)$, $-2 - \left(-\frac{dy}{dx} + 1\right) + 2\frac{dy}{dx} = 0$

$\therefore \quad 3\frac{dy}{dx} = 3$

$\therefore$ The slope of the tangent at $(-1, 1)$ is $\frac{dy}{dx} = 1$.

So the slope of the normal at $(-1, 1)$ is -1.

$\therefore$ The equation of the normal at $(-1, 1)$ is $y - 1 = -1(x + 1)$

$\therefore \quad x + y = 0$ is the equation of the normal at $(-1, 1)$.

To find the points of intersection, let us solve.

$x + y = 0$ and $x^2 - xy + y^2 = 3$

Substitution $y = -x$ in $x^2 - xy + y^2 = 3$,

$\therefore \quad 3x^2 = 3$

$\therefore \quad x = \pm 1$

Since $x = -y$, the point of intersection is $(1, -1)$ as $x \neq -1$.

The normal drawn at $(-1, 1)$ intersects the curve at $(1, -1)$.

[$(-1, 1)$ is the point at which normal is drawn. So it is the foot of the normal. Hence $x \neq -1$.]

Example 40 : Prove that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a^2 \neq b^2)$ and $xy = c^2$ cannot intersect orthogonally.

Solution : One of the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore$ The slope of the tangent to the curve, $m_1 = \frac{dy}{dx} = \frac{b^2x}{a^2y}$ **(Why $y \neq 0$)**

The other curve has equation $xy = c^2$

$\therefore \quad x\frac{dy}{dx} + y = 0$

$\therefore$ The slope of the tangent to the curve, $m_2 = -\frac{y}{x}$

$\therefore \quad m_1m_2 = \left(\frac{b^2x}{a^2y}\right)\left(-\frac{y}{x}\right) = -\frac{b^2}{a^2} \neq -1$ as $a^2 \neq b^2$.

$\therefore$ The curves (hyperbolas) cannot intersect at right angles.

Note : If $a^2 = b^2$, they intersect orthogonally. Hence rectangular hyperbolas $x^2 - y^2 = a^2$ and $xy = c^2$ intersect orthogonally. That they do intersect can be verified.

Exercise 1.3

1. Find the equation of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1).
2. Find the equation of the tangent to $y^2 = 4ax$ at (x_1, y_1).
3. Find the slope of the tangent to $y = x^3 + 5x + 2$ at (2, 20).
4. Find the slope of the normal to $y^2 = 4x$ at (1, 2).
5. Find the equation of the tangent to $y^2 = 16x$, which is parallel to the line $4x - y = 1$.
6. Find the equation of the normal to $y^2 = 8x$ perpendicular to the line $2x - y - 1 = 0$.
7. Prove that the curves $\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1$, $\frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1$ intersect orthogonally, if they intersect. $(\lambda_1 \neq \lambda_2)$
8. Prove that portion of any tangent to $x = a\cos^3\theta$, $y = a\sin^3\theta$ intercepted between axes has constant length.
9. Prove that $2x^2 + y^2 = 3$ and $y^2 = x$ intersect at right angles.
10. Prove that circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ are orthogonal.
11. (1) Find the equation of the tangent to $y = \sin x$ at $\left(\frac{\pi}{2}, 1\right)$.
    (2) Where does it intersect the curve again ?
12. Find equation of tangent to $x = \cos\theta$, $y = \sin\theta$ $\theta \in [0, 2\pi)$ at $\theta = \frac{\pi}{4}$.
13. Find equation of tangent to $y = 4x^3 - 2x^5$ passing through origin.
14. (2, 3) lies on $y^2 = ax^3 + b$. The slope of the tangent at (2, 3) is 4. Find a and b.
15. The slope of the tangent to $xy + ax + by = 2$ at (1, 1) is 2. Find a and b.
16. Find the equation of tangent to $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.
17. Prove that parabola $y^2 = x$ and hyperbola $xy = k$ intersect at right angles, if $8k^2 = 1$.
18. Where does the normal to $y = x - x^2$ at (1, 0) intersect the curve again ?
19. Find a, b if tangent to $y = ax^2 + bx$ at (1, 1) is $y = 3x - 2$.
20. Find the equation of tangent to $x^3 + y^3 = 6xy$ at (3, 3). At which point is the tangent horizontal or vertical ?
21. Prove $xy = c^2$, $c \neq 0$ and $x^2 - y^2 = k^2$, $k \neq 0$ intersect orthogonally. **(Compare : Example 40)**
22. Find the equation of the tangent to given curves at given point :

(1) $\frac{x^2}{16} - \frac{y^2}{9} = 1$	at $\left(-5, \frac{9}{4}\right)$
(2) $\frac{x^2}{9} + \frac{y^2}{36} = 1$	at $(-1, 4\sqrt{2})$
(3) $y^2 = x^3(2 - x)$	at (1, 1)
(4) $y^2 = 5x^4 - x^2$	at (1, 2)
(5) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$	at (3, 1)

23. Find points on $x^2y^2 + xy = 2$ where tangent has slope -1.

24. Find the measure of the angle between

(1) $y = x^2, y = (x - 2)^2$ (2) $x^2 - y^2 = 3, x^2 + y^2 - 4x + 3 = 0$

25. Find the equations of tangents to $y = cos(x + y)$ parallel to $x + 2y = 0$.

26. Find the equations of tangents to $y = \frac{1}{x-1}$, $x \neq 1$ parallel to the line $x + y + 7 = 0$.

27. Prove that $\frac{x}{a} + \frac{y}{b} = 2$ touches $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ for all $n \in \mathrm{N} - \{1\}$, the point of contact being (a, b).

28. X-axis touches $y = ax^3 + bx^2 + cx + 5$ at P$(-2, 0)$ and intersects Y-axis at Q. The slope of the tangent at Q is 3. Find a, b, c.

*

1.5 Approximation and Differentials

Error : We know that $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$, where f is a differentiable function in (a, b) and $x \in (a, b)$, $x + h \in (a, b)$.

$\therefore$ If h is 'very small',

$\frac{f(x+h) - f(x)}{h} = f'(x) + u(h)$ where $u(h)$ is a function of h and as $h \to 0$, $u(h) \to 0$.

$\therefore$ $f(x + h) - f(x) = f'(x)h + u(h)h$.

Let $f(x + h) - f(x) = \Delta f(x)$ and $h = (x + h) - x = \Delta x$.

$\therefore$ $\Delta f(x)$ is a 'small' change in $f(x)$ caused by a 'small' change Δx in x.

$\therefore$ **$\Delta f(x) = f'(x)\Delta x + u(\Delta x)\Delta x$**

$f'(x)\Delta x$ is called differential of $y = f(x)$ and is denoted by dy. Also $\Delta f(x) = \Delta y$.

$\Delta y = dy + u(\Delta x)\Delta x$

Since $u(\Delta x)\Delta x$ is very small and can be neglected, we say dy is an approximate value of Δy and we write $\Delta y \simeq dy$.

Also $dy = f'(x)\Delta x$ (i)

Moreover for the function $y = x$, $f'(x) = 1$.

$dx = 1 \cdot \Delta x$

$\therefore$ For the independent variable x, $\Delta x = dx$.

Thus from (i) $dy = f'(x)\Delta x = f'(x)dx$

$\therefore$ $f'(x) = \frac{(dy)}{(dx)}$

$\therefore$ $\frac{dy}{dx} = \frac{(dy)}{(dx)}$

On L.H.S. we have derivative of $y = f(x)$ and is not a ratio, but on R.H.S. We have a ratio $\frac{(dy)}{(dx)}$ of differential of y and differential of x.

Δy is also called an error in calculation of $f(x)$.

$\therefore \quad \Delta y \simeq dy = f'(x)\Delta x$.

Moreover $f(x + \Delta x) \simeq f(x) + f'(x)\Delta x$.

Geometrical Interpretation of Differential :

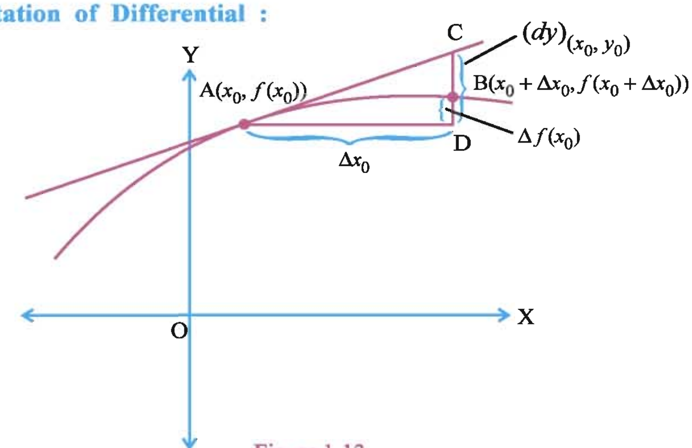


Figure 1.13

Let $A(x_0, f(x_0))$ be a point on the curve $y = f(x)$.

$B(x_0 + \Delta x_0, f(x_0 + \Delta x_0))$, is also on the curve. C is the point on the tangent at A to the curve $y = f(x)$ lying on the vertical line through B.

The equation of the tangent at A is $y - y_0 = f'(x_0)(x - x_0)$ **($f'(x_0)$ is slope of the tangent)**

At C, $x = x_0 + \Delta x_0$

$\therefore$ y-coordinate of C, $y = y_0 + (x_0 + \Delta x_0 - x_0)f'(x_0)$

$$= f(x_0) + f'(x_0)\Delta x_0$$

$$= f(x_0) + (dy)_{(x_0, y_0)}$$

$CD = y$-coordinate of $C - f(x_0) = (dy)_{(x_0, y_0)}$

$BD = f(x_0 + \Delta x_0) - f(x_0) = \Delta f(x_0) = \Delta y_0$

$\therefore \quad BC = |\Delta y_0 - (dy)_{(x_0, y_0)}|$

As B moves nearer and nearer to A on the curve, $BC \to 0$. Hence $dy \simeq \Delta y$.

Thus $f(x_0 + \Delta x_0) \simeq f(x_0) + f'(x_0)\Delta x_0$ is called the approximate value of $f(x)$ for $x = x_0 + \Delta x_0$ obtained by linear approximation using tangent to $y = f(x)$.

Example 41 : Obtain approximate value of $\sqrt{101}$ and $\sqrt{99}$ using differentiation.

Solution : Let $f(x) = \sqrt{x}$, $x \in R^+$

Let $x = 100$ and $x + \Delta x = 101$ **(We know $\sqrt{100}$.)**

$\therefore \quad \Delta x = 1$. $(\Delta x = x + \Delta x - x = 101 - 100)$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$$

Now $f(x + \Delta x) \simeq f(x) + f'(x)\Delta x$

$\therefore \quad f(101) \simeq f(100) + f'(100)\,\Delta x$

$= \sqrt{100} + (0.05)(1) = 10.05$

$\therefore$ An approximate value of $\sqrt{101}$ is 10.05.

For $\sqrt{99}$, let $x = 100$, $x + \Delta x = 99$, $\Delta x = -1$ **($\Delta x = 99 - 100 = -1$)**

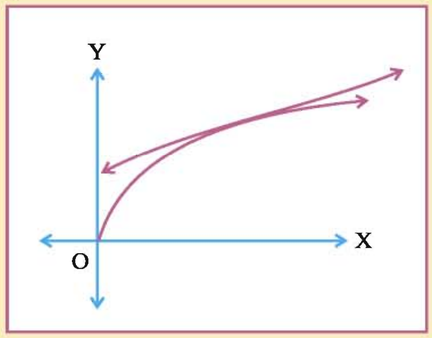
$\therefore \quad \sqrt{99} = f(99) \simeq f(100) + f'(100)\,\Delta x$

$= \sqrt{100} + (0.05)(-1)$

$= 10 - 0.05 = 9.95$

x	Approximate Value	Actual Value
$\sqrt{101}$	10.05	10.0498756....
$\sqrt{99}$	9.95	9.94987437....
$\sqrt{102}$	10.1	10.0995049....
$\sqrt{98}$	9.9	9.89949493....

We observe that as $\Delta x \to 0$, actual value approaches true value. Here the actual value is smaller than the approximate value, as the tangent lies above the graph of $y = \sqrt{x}$ or $y^2 = x$.



Example 42 : Find approximate value of $(65)^{\frac{1}{3}}$.

[**Note :** We will henceforth not use the phrase 'using differentiation' but it is implied.]

Solution : $f(x) = x^{\frac{1}{3}}$.

$x = 64$, $x + \Delta x = 65$. So, $\Delta x = 1$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3(64)^{\frac{2}{3}}} = \frac{1}{48}$. So $\Delta f(x) \simeq f'(x)\,\Delta x = \frac{1}{48}$

$\therefore \quad (65)^{\frac{1}{3}} = (64)^{\frac{1}{3}} + \Delta f(x) \simeq 4 + \frac{1}{48} = \frac{193}{48}$

Example 43 : Find $tan\ 46^{\circ}$.

Solution : Let $f(x) = tanx$ and $x = \frac{\pi}{4}$, $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$ **$\left(45^{\circ} = \frac{\pi}{4}^{R}\right)$**

$\therefore \quad \Delta x = 1 \cdot \frac{\pi}{180} = \frac{\pi}{180}^{R}$

$\therefore \quad f'(x) = sec^2 x = (\sqrt{2})^2 = 2$

$\therefore \quad \Delta f(x) \simeq f'(x)\,\Delta x = 2 \cdot \frac{\pi}{180} = \frac{\pi}{90}$

$\therefore \quad tan 46^o = tan 45^o + \Delta f(x)$

$$\simeq 1 + \frac{\pi}{90}$$

$\therefore$ An approximate value of $tan\ 46^o$ is $1 + \frac{\pi}{90}$.

Example 44 : Find approximate value of **(1)** $cos^{-1}\ (-0.49)$ **(2)** $sec^{-1}\ (-2.01)$

Solution : (1) Let $f(x) = cos^{-1}x$, $x = -0.5$, $\Delta x = 0.01$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-\frac{1}{4}}} = -\frac{2}{\sqrt{3}}, \quad \Delta f(x) \simeq f'(x)\,\Delta x = -\frac{1}{50\sqrt{3}}$$

$\therefore \quad cos^{-1}\ (-0.49) = cos^{-1}\ (-0.5) + \Delta f(x)$

$$\simeq \pi - cos^{-1}\ (0.5) - \frac{1}{50\sqrt{3}}$$

$$= \pi - \frac{\pi}{3} - \frac{1}{50\sqrt{3}}$$

$$= \frac{2\pi}{3} - \frac{1}{50\sqrt{3}}$$

Another method : Let $f(x) = cos^{-1}x$, $x = 0.5$, $\Delta x = -0.01$

$\therefore \quad cos^{-1}\ (-0.49) = \pi - cos^{-1}\ (0.49)$

$$\simeq \pi - (cos^{-1}\ (0.5) + f'(x)\,\Delta x)$$

$$= \pi - \frac{\pi}{3} - \left(-\frac{2}{\sqrt{3}}\right)(-0.01)$$

$$= \frac{2\pi}{3} - \frac{1}{50\sqrt{3}}$$

(2) Let $f(x) = sec^{-1}x$, $x = 2$, $\Delta x = 0.01$

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{2\sqrt{3}}, \quad \Delta f(x) \simeq f'(x)\,\Delta x = \frac{1}{200\sqrt{3}}$$

$\therefore \quad sec^{-1}\ (-2.01) = \pi - sec^{-1}\ (2.01)$

$$\simeq \pi - (sec^{-1}2 + f'(x)\,\Delta x)$$

$$= \pi - \left(\frac{\pi}{3} + \frac{1}{200\sqrt{3}}\right)$$

$$= \frac{2\pi}{3} - \frac{1}{200\sqrt{3}}$$

Example 45 : Find approximate value of **(1)** $\log_e 10.01$ **(2)** $\log_{10} 10.1$ **(3)** $\log_e(e + 0.1)$

($\log_{10}e = 0.4343$, $\log_e 10 = 2.3026$)

Solution : (1) Let $f(x) = \log_e x$

Let $x = 10$, $\Delta x = 0.01$, $f'(x) = \frac{1}{x} = \frac{1}{10} = 0.1$

$\therefore \quad \Delta f(x) \simeq f'(x)\,\Delta x = 0.001$

$\therefore \quad \log_e(10.01) \simeq \log_e 10 + f'(x)\,\Delta x$

$$= 2.3026 + 0.001$$

$$= 2.3036$$

(Actually $\log_e 10.01 = 2.30358459....$)

(2) Let $f(x) = \log_{10}x = \frac{\log_e x}{\log_e 10} = \log_e x \cdot \log_{10} e$

$= (0.4343) \log_e x$

Let $x = 10$, $\Delta x = 0.1$

$\therefore \quad f'(x) = \frac{0.4343}{x} = \frac{0.4343}{10} = 0.04343$

$\therefore \quad \Delta f'(x) \simeq f'(x)\, \Delta x = (0.04343)\,(0.1) = 0.004343$

$\therefore \quad \log_{10}(10.1) \simeq \log_{10}10 + f'(x)\, \Delta x$

$= 1.004343$

(Actually $\log_{10}(10.1) = 1.00432137....$)

(3) Let $f(x) = \log_e x$, $x = e$, $\Delta x = 0.1$

$\therefore \quad f'(x) = \frac{1}{x} = \frac{1}{e}, \quad \Delta f(x) \simeq f'(x)\, \Delta x = \frac{(0.1)}{(e)} = \frac{1}{10e}$

$\therefore \quad \log_e(e + 0.1) \simeq \log_e e + f'(x)\, \Delta x$

$= 1 + \frac{1}{10e} = 1.03678794$

(Actually it is 1.0367879441....)

Example 46 : If there is an error of x % in the measurement of radius of a sphere, what is the approximate error in the measurement of volume and surface area ?

Solution : There is x % error in the radius.

$\therefore \quad \Delta r = \frac{xr}{100}$.

Volume of a sphere, $V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$\therefore$ Error in volume $\Delta V \simeq \frac{dV}{dr}\, \Delta r$

$= 4\pi r^2 \cdot \frac{xr}{100}$

$= \frac{4}{3}\pi r^3 \cdot \frac{3x}{100} = \frac{3xV}{100}$

$\therefore$ There is approximately $3x$ % error in the volume.

Surface area $S = 4\pi r^2$

$\therefore \quad \frac{dS}{dr} = 8\pi r$

$\therefore$ Error in surface area $\Delta S \simeq \frac{dS}{dr}\, \Delta r$

$= 8\pi r \cdot \frac{xr}{100}$

$= 2(4\pi r^2)\, \frac{x}{100}$

$= \frac{2xS}{100}$

$\therefore$ There is approximately $2x$ % error in surface area.

Example 47 : The radius of a sphere is measured as 7 m with error of 0.02 m. What is the approximate error in the volume ?

Solution : For a sphere, volume $V = \frac{4}{3}\pi r^3$

$r = 7\ m,\ \Delta r = 0.02\ m$

$\therefore\ \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$$\therefore\ \Delta V \simeq \frac{dV}{dr}\Delta r$$
$$= 4\pi r^2 \cdot \Delta r$$
$$= 4\pi(49)(0.02)$$
$$= 3.92\ \pi\ m^3$$

$\therefore$ There is approximately 3.92 π m^3 error in the volume.

Example 48 : Find the approximate error in the surface area of a cube with edge x cm, when the edge is increased by 2 %.

Solution : $S = 6x^2,\ \Delta x = \frac{2x}{100}$

$\therefore\ \frac{dS}{dx} = 12x$

$$\therefore\ \Delta S \simeq \frac{dS}{dx} \cdot \Delta x$$
$$\therefore\ \Delta S \simeq 12x\ \Delta x$$
$$= 12x \cdot \frac{2x}{100}$$
$$= \frac{4(6x^2)}{100} = \frac{4S}{100}$$

$\therefore$ There is approximately 4 % increase in the surface area.

Example 49 : Prove that for a triangle inscribed in a circle of constant radius, sides change according to $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$ in usual notation, if da, db, dc are small.

Solution : We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ according to *sine* rule.

$a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$, R constant.

$\therefore\ \frac{da}{dA} = 2R\cos A,\ \frac{db}{dB} = 2R\cos B,\ \frac{dc}{dC} = 2R\cos C$

$\therefore\ da = \frac{da}{dA}\ \Delta A = 2R\cos A\ \Delta A$ etc.

$$\therefore\ \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(\Delta A + \Delta B + \Delta C)$$
$$= 2R(\Delta(A + B + C))$$
$$= 2R\ \Delta(\pi)$$
$$= 0$$
$$\therefore\ \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

Example 50 : When a circular plate is heated, its radius increases by 0.1 cm. Find the approximate increase in area, when the radius is 5 cm.

Solution : For a circle, area A = πr^2

$\therefore \quad \frac{dA}{dr} = 2\pi r$

$\therefore \quad \Delta A \simeq \frac{dA}{dr}\Delta r = 2\pi r\,\Delta r = 2\pi(5)(0.1)$

$\therefore \quad \Delta A \simeq \pi\ cm^2$

$\therefore$ There is $\pi\ cm^2$ increase in area approximately.

Example 51 : If $f(x) = cosx$, find the differential dy and evaluate dy when $x = \frac{\pi}{6}$ and $\Delta x = 0.01$.

Solution : $y = f(x) = cosx$

$\therefore \quad f'(x) = -sinx$. So $f'\left(\frac{\pi}{6}\right) = -sin\frac{\pi}{6} = -\frac{1}{2} = -0.5$

$\therefore \quad dy = f'(x)\,\Delta x = (-0.5)(0.01)$

$\therefore \quad dy = -0.005$

Example 52 : Prove that if h is very small, $sinh \simeq h$.

Solution : Let $f(x) = sinx,\ x = 0,\ x + \Delta x = h$

$\therefore \quad f'(x) = cosx,\ f'(0) = cos0 = 1$

$\therefore \quad f(x + \Delta x) \simeq f(x) + f'(x)\,\Delta x$

$\therefore \quad f(h) \simeq f(0) + f'(0)\,h$ $\qquad (h = \Delta x)$

$\therefore \quad sinh \simeq sin0 + cos0 \cdot h$

$\therefore \quad sinh \simeq h$, if h is small.

Exercise 1.4

Find approximate value (1 to 12) :

1. $\sqrt{0.37}$	2. $(0.999)^{\frac{1}{10}}$	3. $(80)^{\frac{1}{4}}$	4. $(255)^{\frac{1}{4}}$
5. $(399)^{\frac{1}{2}}$	6. $(32.1)^{\frac{1}{5}}$	7. $cos\,29^{\circ}$	8. $sin61^{\circ}$
9. $tan31^{\circ}$	10. $\log_e(100.1)$	11. $\log_{10}(10.01)$	12. $(16.1)^{\frac{1}{4}}$

13. If the radius of a cone is twice its height, find the approximate error in the calculation of its volume, when the radius is 10 cm and the error in the radius is 0.01 cm.

14. If there is an error in measuring its radius by Δr, what is the approximate error in the volume of a sphere?

15. Kinetic energy is given by $k = \frac{1}{2}mv^2$. For constant mass there is approximately 1 % increase in the energy. What increase in the velocity v which caused it ?

16. Area of a triangle is calculated using formula A = $\frac{1}{2}absinC$. If C = $\frac{\pi}{6}$ and there is an error in measuring C by x %, what is the percentage error in area approximately ? a, b are kept constant.

17. Find approximate value of $f(3.01)$ where $f(x) = x^3 - 2x^2 - 3x + 1$.

18. Find approximate value of $f(1.05)$ where $f(x) = 2x^2 - 3x + 5$.

19. Find the approximate increase in the volume of a cube when the length of its edge increases by 0.2 *cm* and its edge has length 10 *cm*.

20. Find the approximate increase in the total surface area of a cone when its height remains constant and the radius increases by 2 % at the time when its radius is 8 *cm* and the height is 6 *cm*.

21. Find approximate value of $cos\,\frac{11\pi}{36}$, knowing the value of $cos\,\frac{\pi}{3}$.

*

1.6 Maximum and Minimum Values

We have seen some applications of differential calculus. Now we will learn an important application of differential calculus to optimization problems.

We may wish to find maximum volume of a box, minimum cost of a can to contain fixed quantity of fruit juice or minimize the cost and maximize the profit etc.

Definition : A function f has an absolute or global maximum at c if $f(c) \geq f(x)$, $\forall x \in D_f$, $c \in D_f$ and a function has an absolute or global minimum at c if $f(c) \leq f(x)$, $\forall x \in D_f$, $c \in D_f$. The maximum and minimum values are also called the extreme values of f on D_f.

Definition : A function f defined on an interval I has a local maximum value at $c \in$ I, if for some $h > 0$, $(c - h, c + h) \subset$ I and $f(c) \geq f(x)$, $\forall x \in (c - h, c + h)$.

A function f defined on an interval I has a local minimum value at $c \in$ I, if for some $h > 0$, $(c - h, c + h) \subset$ I and $f(c) \leq f(x)$, $\forall x \in (c - h, c + h)$.

Note : If I is a closed interval local maximum or local minimum cannot occur at an end-point of the interval because of the condition $(c - h, c + h) \subset$ I.
However global maximum or global minimum may occur at an end-point.

$f(x) = sinx$, $x \in$ R takes global maximum 1 for $x = (4n + 1)\frac{\pi}{2}$, $n \in$ Z and global minimum -1 for $x = (4n + 3)\frac{\pi}{2}$, $n \in$ Z. Consider $f(x) = x^2$, $x \in$ R. Since $x^2 \geq 0$ $\forall x \in$ R, $f(0) = 0$ is global minimum as well as local minimum but f has no global maximum. But if the domain of f is restricted to $[-3, 5]$, say, it has a global maximum $f(5) = 25$.

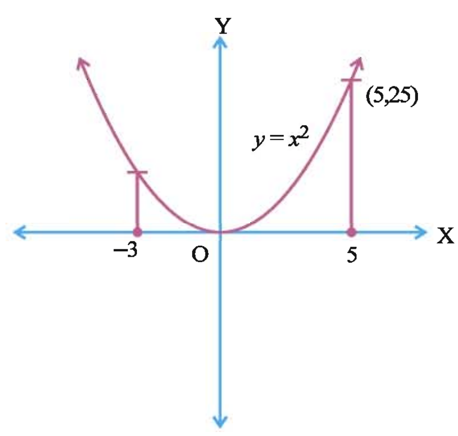


Figure 1.14

The function $f(x) = x^3$, $x \in$ R has no extreme value in R.

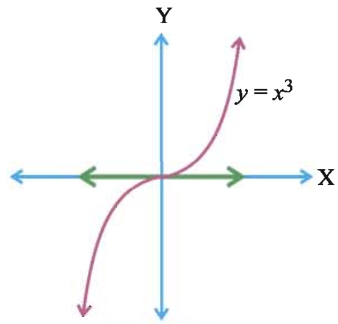


Figure 1.15

Look at following graph.

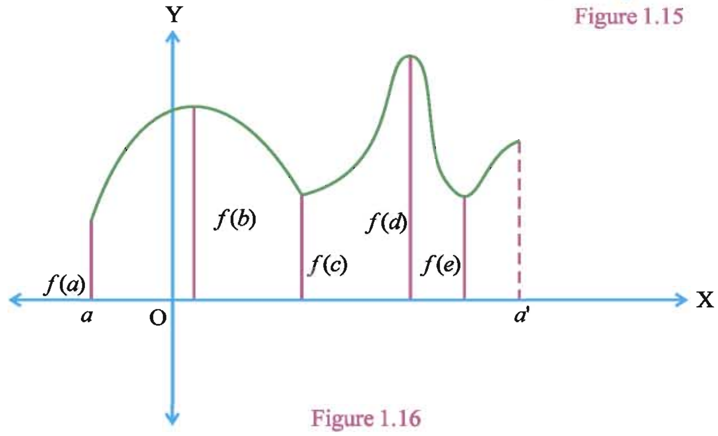


Figure 1.16

See that the global minimum occurs at $x = a$ in $[a, a']$ and the global maximum occurs at $x = d$. $f(b)$ is local maximum and $f(c)$ and $f(e)$ are local minimum values. Also global minimum occurs at an end-point of the interval but global maximum occurs at an interior point of the domain. Now we assume following result without giving proof.

The Extreme Value Theorem : If a function f is continuous on $[a, b]$, f attains its global maximum value at some $c \in [a, b]$ and global minimum value at some $d \in [a, b]$.

These are called extreme values of the function.

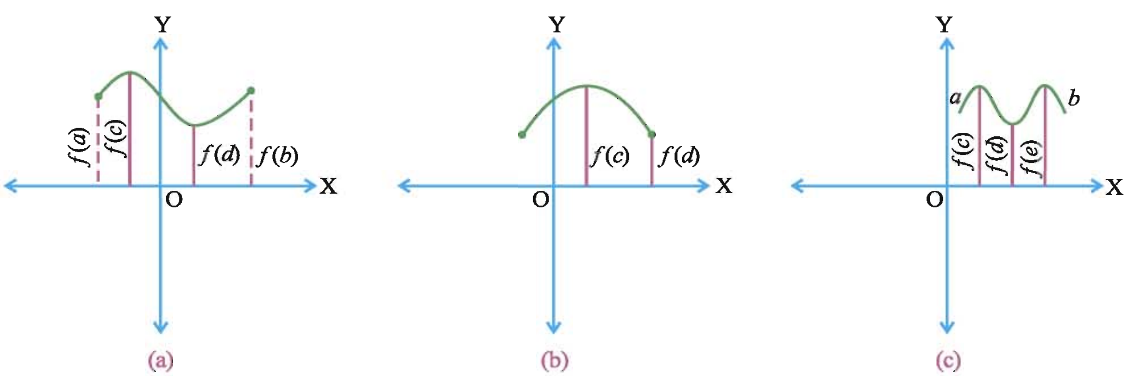


Figure 1.17

In figure 1.17(a) both maximum and minimum values of f occur at an interior point of $[a, b]$. In figure 1.17(b) the maximum occurs at $c \in (a, b)$ and minimum at $d = b$. In figure 1.17(c), there are two maxima (i.e. more than one).

Look at figure 1.18.

Here the domain of the function is [1, 4] but the function is discontinuous at $x = 2$. Its range is [0, 4). For no $x \in [1, 4]$, $f(x) = 4$. The function has no maximum.

Hence, we have kept the assumption that f is 'continuous' in the extreme value theorem.

But a discontinuous function could well have maximum and minimum value.

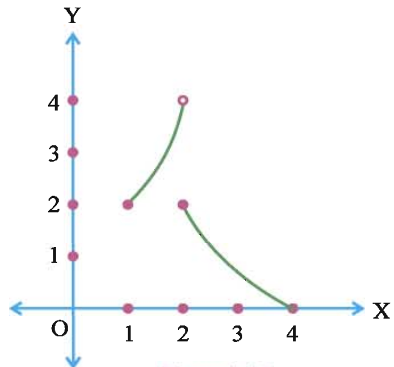


Figure 1.18

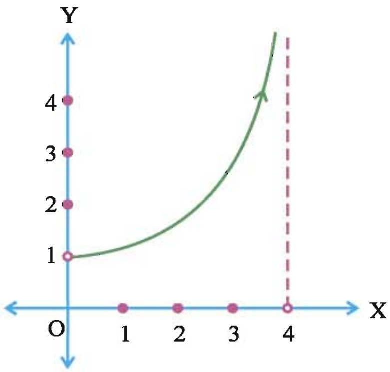


Figure 1.19

Look at figure 1.19.

The function is continuous on (0, 4), but it has neither maximum nor minimum value. The range is $(1, \infty)$. Hence, the condition 'closed interval' enters in the extreme value theorem.

$f(x) = x$ in (0, 2) has no maximum or minimum but $f(x) = x$ in [0, 2] has maximum $f(2) = 2$ and minimum $f(0) = 0$. For $f(x) = x$, let $x_1 \in (0, 2)$. Then $x_1 < \left(\frac{x_1 + 2}{2}\right) < 2$ as $x_1 < 2$.

See that in figure 1.19(a), we get $f\left(\frac{x_1 + 2}{2}\right) = \frac{x_1 + 2}{2}$ which is larger than $f(x_1)$, where $x_1 \in (0, 2)$. No $f(x)$ can be maximum. Similarly $f\left(\frac{x_1}{2}\right) < f(x_1)$, so $f(x)$ has no minimum value.

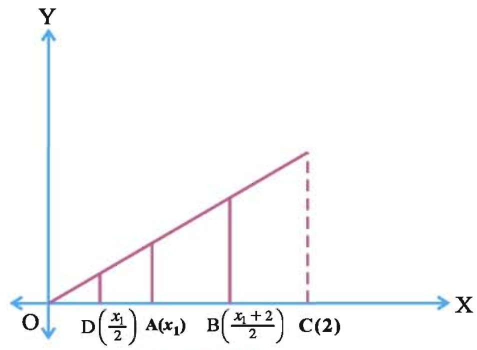


Figure 1.19(a)

Mid-point of $\overline{AC}$ is B and mid-point of $\overline{OA}$ is D. Thus we get a larger value $f\left(\frac{x_1 + 2}{2}\right)$ at B than any value $f(x_1)$ at A and a smallar value $f\left(\frac{x_1}{2}\right)$ at D than value at A.

$\therefore$ There is no maximum or no minimum.

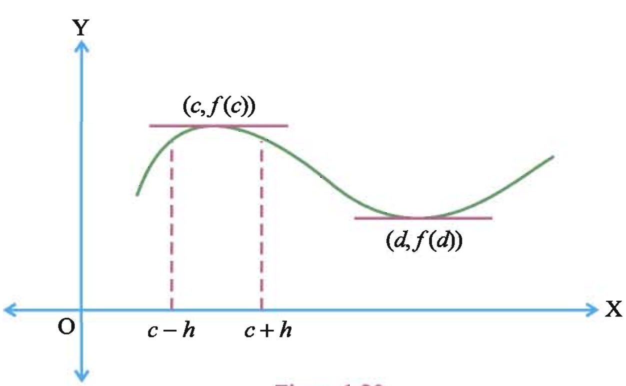


Figure 1.20

Look at the graph (figure 1.20). f has a local maximum at $x = c$. In $(c - h, c)$, f is increasing and therefore $f'(x) > 0$. In $(c, c + h)$, f is decreasing and so $f'(x) < 0$. As x takes values in $(c - h, c + h)$ and passes through c, $f'(x)$ changes from positive to negative. Also $f'(c) = 0$.

Similarly at $x = d$, f has a local minimum and f' changes sign from negative to positive and $f'(d) = 0$.

Thus we accept the following theorem without proof.

Theorem 1.2 (Fermat's Theorem) : If f has a local maximum or local minimum at c and if f is differentiable at c, then $f'(c) = 0$.

Although this is only a necessary condition and not sufficient. For $f(x) = x^3$, $f'(0) = 0$ but it does not have a maximum or minimum. Such a point where the graph crosses its horizontal tangent is called a point of inflexion. For $f(x) = x^3$, (0, 0) is a point of inflexion. At (0, 0) tangent is horizontal.

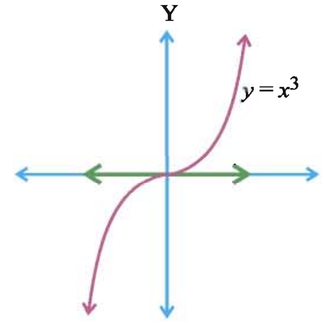


Figure 1.21

Fermat's theorem is named after **Pierre Fermat** (1601-1665). He was a French lawyer and mathematics was his hobby. He was one of the inventors of analytic geometry (the other being **Des Cartes**).

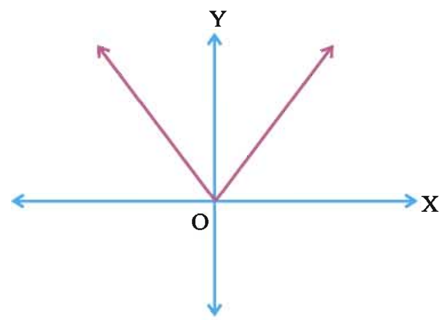


Figure 1.22

Also f may have an extreme value at $x = c$ and f may not be differentiable at c.

$f(x) = |x|$ has minimum at $x = 0$. $f(0) = |0| = 0$ is minimum value of $f(x) = |x|$ but f is not differentiable for $x = 0$.

Hence we define,

A Critical Number (Point) : A critical number (point) c of a function is a number $c \in D_f$ such that $f'(c) = 0$ or f is not differentiable at c.

Thus if f has a local maximum or local minimum at $x = c$, c is a critical number of f.

We now state following first derivative test from above discussion.

First Derivative Test : Let f be defined in an open interval $I = (a, b)$. $c \in I$ is a critical point of f and f is continuous at c.

(1) If there exists a positive number h such that $(c - h, c + h) \subset I$, $f'(x) > 0$ in $(c - h, c)$ and $f'(x) < 0$ in $(c, c + h)$, then f has a local maximum value at c.

(2) If there exists a positive number h such that $(c - h, c + h) \subset I$, $f'(x) < 0$ in $(c - h, c)$ and $f'(x) > 0$ in $(c, c + h)$, then f has a local minimum value at c.

(3) If $f'(x)$ does not change sign as x takes values in $(c - h, c + h)$ for any $h > 0$, f has neither maximum nor minimum value at $x = c$. Such a point is called a point of inflexion.

For some $h > 0$

$f'(x)$ changes from $+ve$ in $(c - h, c)$ to $-ve$ in $(c, c + h)$	$f(c)$ is a local maximum
$f'(x)$ changes from $-ve$ in $(c - h, c)$ to $+ve$ in $(c, c + h)$	$f(c)$ is a local minimum

Sometimes, it may not be easy to handle first derivative test. Then we may use following second derivative test.

Second Derivative Test : Let f be defined on an interval $I = [a, b]$. Let $c \in (a, b)$. Suppose $f''(c)$ exists. Then

(1) f has local maximum at $x = c$, if $f'(c) = 0, f''(c) < 0$.

(2) f has local minimum at $x = c$, if $f'(c) = 0, f''(c) > 0$.

(3) The test fails to give any conclusion if $f'(c) = f''(c) = 0$.

Note : $f''(c) < 0$, $f'(c) = 0$ means $f'(x)$ is decreasing at $x = c$ and since $f'(c) = 0$, $f'(x)$ changes from $+ve$ to $-ve$.

$\therefore$ $f(x)$ has a local maximum at $x = c$.

Similarly if $f''(c) > 0$, $f'(c) = 0$ we can conclude that $f(x)$ has a local minimum at $x = c$.

Example 53 : Find the critical points for $f(x) = x^{\frac{3}{5}}(4 - x)$, $x \in R^+ \cup \{0\}$.

Solution : $f(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$

$$\therefore \quad f'(x) = \frac{12}{5} x^{-\frac{2}{5}} - \frac{8}{5}x^{\frac{3}{5}}$$

$$= \frac{4}{5}\left(\frac{3}{x^{\frac{2}{5}}} - 2x^{\frac{3}{5}}\right)$$

$$= \frac{4}{5}\left(\frac{3 - 2x}{x^{\frac{2}{5}}}\right)$$

$\therefore$ $f'(x) = 0$, if $x = \frac{3}{2}$ and $f'(x)$ does not exist at $x = 0$ but $0 \in D_f$.

$\therefore$ The cricital points are 0 and $\frac{3}{2}$.

Example 54 : Find local maximum or minimum values of $f(x) = |x|$. $x \in R$

Solution : f is not differentiable at $x = 0$, $0 \in D_f$. So 0 is a critical point and the second derivative of f does not exist at $x = 0$.

$$\therefore \quad f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\therefore$ $f'(x) = 1$ if $x > 0$

and $f'(x) = -1$ if $x < 0$.

$\therefore$ $f'(x)$ changes from negative to positive as x passes through 0 and f is not differentiable at $x = 0$.

$\therefore$ $f'(x)$ changes from negative to positive as x changes from $(-h, 0)$ to $(0, h)$, $h > 0$.

$\therefore$ f has a local minimum value $f(0) = 0$ at $x = 0$. f has no maximum value.

Note : Obviously $f(x) = |x| \geq 0 \quad \forall x \in R$

$\therefore$ f has a local and global minimum at $x = 0$.

To find extreme values for a function defined on a closed interval $[a, b]$.

(1) Find local maximum and local minimum values of f.

(2) Find values of f at end-points.

The largest of the values obtained in (1) and (2) is global maximum and the smallest of the values obtained in (1) and (2) is the global minimum value of f.

Example 55 : Examine for maximum and minimum values : $f(x) = 3x^4 - 16x^3 + 18x^2$, $x \in [-1, 4]$

Solution : $f(x) = 3x^4 - 16x^3 + 18x^2$

$\therefore$ $f'(x) = 12x^3 - 48x^2 + 36x$

$= 12x(x^2 - 4x + 3)$

$= 12x(x - 3)(x - 1)$

$\therefore$ $f'(x) = 0 \Rightarrow x = 0, 1$ or 3.

$\therefore$ $f''(x) = 36x^2 - 96x + 36$

$\therefore$ $f''(0) = 36 > 0, f''(1) = -24 < 0, f''(3) = 72 > 0$

$\therefore$ $f(0)$ is local minimum and $f(0) = 0$ is local minimum.

f has local maximum at $x = 1$ and $f(1) = 5$ is local maximum.

f has local minimum at $x = 3$ and $f(3) = -27$ is local minimum.

Local maximum or minimum values can occur only at an interior point of $[-1, 4]$.

For global maxinum and minimum values, consider $f(-1)$ and $f(4)$.

$f(-1) = 37, \quad f(4) = 32$

$f(0) = 0, \quad f(1) = 5, \quad f(3) = -27, \quad f(-1) = 37, \quad f(4) = 32$

$\therefore$ $f(-1) = 37$ is global maximum occuring at an end-point.

$\therefore$ $f(3) = -27$ is global minimum and it occurs at an interior point $3 \in (-1, 4)$.

Example 56 : Find maximum and minimum values of the function $f(x) = x^3 - 12x + 1$, $x \in [-3, 5]$

Solution : $f(x) = x^3 - 12x + 1$

$\therefore$ $f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$

$\therefore$ $f'(x) = 0 \Rightarrow x = \pm 2$

$f''(x) = 6x$

$\therefore$ $f''(2) = 12 > 0$

$\therefore$ $f(2) = 8 - 24 + 1 = -15$ is local minimum value.

$\therefore$ $f''(-2) = -12 < 0$

$\therefore$ $f(-2) = -8 + 24 + 1 = 17$ is local maximum value.

Moreover, $f(-3) = -27 + 36 + 1 = 10, \quad f(5) = 125 - 60 + 1 = 66$

$f(2) = -15, f(-2) = 17$

$\therefore \quad f(5) = 66$ is global maximum and

$f(2) = -15$ is global minimum.

Example 57 : Find maximum and minimum values of the function $f(x) = 3x^5 - 5x^3 - 1$, $x \in [-2, 2]$

Solution : $f'(x) = 15x^4 - 15x^2$

$= 15x^2 (x^2 - 1)$

$= 15x^2 (x - 1)(x + 1)$

$\therefore \quad f'(x) = 0 \Rightarrow x = 0$ or $x = \pm 1$

$f''(x) = 60x^3 - 30x$

$f''(1) = 30 > 0$

$\therefore \quad f(1) = -3$ is local minimum value.

$f''(-1) = -30 < 0$

$\therefore \quad f(-1) = 1$ is local maximum value.

But $f''(0) = 0$

$\therefore$ Second derivative test fails.

$\therefore \quad f'(x) = 15x^2 (x - 1)(x + 1)$

$x^2 > 0$, if $x \neq 0$

If $-1 < x < 1$ then $x + 1 > 0$ and $x - 1 < 0$

$\therefore \quad f'(x) < 0$ for $-1 < x < 1$.

$\therefore \quad f'(x)$ does not change sign as x increases in $(-1, 1)$.

$\therefore$ 0 is a point of inflexion.

$\therefore \quad f(2) = 96 - 40 - 1 = 55$

$f(-2) = -96 + 40 - 1 = -57$. Also $f(1) = -3$, $f(-1) = 1$.

$\therefore \quad f(2) = 55$ is global maximum and $f(-2) = -57$ is global minimum value.

Example 58 : Determine maximum and minimum values of $f(x) = x - 2cosx$, $x \in [-\pi, \pi]$

Solution : $f(x) = x - 2cosx$

$\therefore \quad f'(x) = 1 + 2sinx$

$\therefore \quad f'(x) = 0 \Rightarrow sinx = -\frac{1}{2}$

$\therefore \quad x = -\frac{\pi}{6}, \frac{-5\pi}{6}$ $\qquad x \in (-\pi, \pi)$

Now $f''(x) = 2cosx$

$\therefore \quad f''\left(-\frac{\pi}{6}\right) = 2cos\left(-\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} > 0$

$\therefore \quad f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2 \times \frac{\sqrt{3}}{2} = -\frac{\pi}{6} - \sqrt{3}$

$\therefore \quad f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$ is local minimum value at $x = -\frac{\pi}{6}$.

$\therefore \quad f''\left(-\frac{5\pi}{6}\right) = 2cos\left(-\frac{5\pi}{6}\right) = 2cos\,\frac{5\pi}{6} = 2cos\left(\pi - \frac{\pi}{6}\right)$

$$= -2cos\frac{\pi}{6}$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0$$

$\therefore \quad f\left(-\frac{5\pi}{6}\right) = -\frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} - \frac{5\pi}{6}$ is local maximum value.

$f(\pi) = \pi - 2cos\pi = \pi + 2$

$f(-\pi) = -\pi - 2cos(-\pi) = -\pi - 2cos\pi = -\pi + 2$

$\therefore \quad f(\pi) = \pi + 2$ is global maximum value.

$\therefore \quad f\left(-\frac{\pi}{6}\right) = -\sqrt{3} - \frac{\pi}{6}$ is global minimum value.

Example 59 : Find maximum and minimum values of $f(x) = 4x + cotx$; $x \in (0, \pi)$

Solution : Now $f'(x) = 4 - cosec^2x = 0$

$\therefore \quad cosec^2x = 4$

$\therefore \quad sin^2x = \frac{1}{4}$

$\therefore \quad sinx = \frac{1}{2}$ $\qquad$ **$x \in (0, \pi)$**

$\therefore \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$f''(x) = -2cosecx\ (-cosecx\ cotx)$

$= 2cosec^2x\ cotx$

$\therefore \quad f''\left(\frac{\pi}{6}\right) = 2(4)\sqrt{3} > 0, \quad f''\left(\frac{5\pi}{6}\right) = -8\sqrt{3} < 0$

$\therefore \quad f\left(\frac{\pi}{6}\right) = \frac{2\pi}{3} + \sqrt{3}$ is local minimum value at $x = \frac{\pi}{6}$.

and $f\left(\frac{5\pi}{6}\right) = \frac{10\pi}{3} - \sqrt{3}$ is local maximum value at $x = \frac{5\pi}{6}$.

[Why no global maximum or global minimum ?]

Example 60 : Prove that out of all rectangles with given area, the square has minimum perimeter.

Solution : Let the given area be A and the lengths of the sides of the rectangle be x and y.

$\therefore \quad A = xy$

Now perimeter of the rectangle, $p = 2x + 2y$

$$= 2x + \frac{2A}{x}$$

Now $\frac{dp}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0$

$\therefore \quad x^2 = A$

$\therefore \quad x = \sqrt{A}$ $\qquad$ **(Since x is the side of a rectangle, $x > 0$)**

$\therefore \quad y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$

Since $x = y$ the rectangle becomes a square.

Also $\frac{d^2p}{dx^2} = 0 - 2A(-2x^{-3}) = \frac{4A}{x^3} > 0$

$\therefore$ The perimeter is minimum when the rectangle becomes a square.

Note : $(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4A$

$\therefore$ $(x + y)^2$ is minimum when $x = y$ as $(x - y)^2 \geq 0$ and minimum value of $(x - y)^2 = 0$ if $x = y$ and A is a constant.

$\therefore$ The perimeter of a square is minimum.

Example 61 : Find a point P on $y^2 = 8x$ nearest to A(10, 4) and also find minimum distance AP.

Solution : Parametric equations of a parabola are $(at^2, 2at)$.

Here $4a = 8$. So $a = 2$.

$\therefore$ A typical point on the parabola is $P(2t^2, 4t)$.

Now $AP^2 = (2t^2 - 10)^2 + (4t - 4)^2$

$= 4t^4 - 40t^2 + 100 + 16t^2 - 32t + 16$

Let $f(t) = 4t^4 - 24t^2 - 32t + 116$

$f'(t) = 16t^3 - 48t - 32$

$= 16(t^3 - 3t - 2)$

$= 16(t + 1)(t^2 - t - 2)$

$= 16(t + 1)^2(t - 2)$

$\therefore$ $f'(t) = 0 \Rightarrow t = -1$ or $t = 2$

Let $t \in (-1 - h, -1 + h)$ where $h > 0$. Let $t = -1 + t_1$

$\therefore$ $-1 - h < -1 + t_1 < -1 + h$ i.e. $-h < t_1 < h$

$\therefore$ $f'(t) = 16(t + 1)^2 (t - 2)$ $\qquad (t = -1 + t_1)$

$= 16t_1^2(-3 + t_1) > 0$ if $0 < t_1 < 3$

$\therefore$ $f'(t)$ does not change sign in $(-1 - h, -1 + h)$

$\therefore$ f has no maximum or minimum at $t = -1$.

$\therefore$ $f''(t) = 48t^2 - 48$

$\therefore$ $f''(2) = 192 - 48 = 144 > 0$

$\therefore$ $f(t)$ is minimum, if $t = 2$

$\therefore$ AP^2 is minimum for $t = 2$. For $t = 2$, P is (8, 8).

If P(8, 8), then $AP = \sqrt{(10-8)^2 + (8-4)^2}$

$= \sqrt{4+16}$

$= 2\sqrt{5}$ is minimum

$\therefore$ The point nearest to A(10, 4) and lying on $y^2 = 8x$ is P(8, 8).

Example 62 : Find the maximum area of a rectangle inscribed in a semi-circle of radius r.

Solution : Let us consider the semi-circle in upper half-plane of X-axis.

Let $A(x, y)$ be one vertex of the rectangle in the first quadrant. Obviously other vertices are $B(x, 0)$, $C(-x, 0)$ and $D(-x, y)$.

$\therefore$ AD = $2x$, AB = y

$\therefore$ The area of the rectangle $f(x) = 2xy$

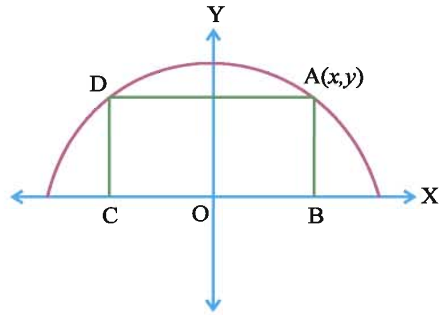


Figure 1.23

Also $x^2 + y^2 = r^2$

$\therefore \quad y = \sqrt{r^2 - x^2} \qquad (y > 0)$

$\therefore \quad f(x) = 2x\sqrt{r^2 - x^2}$

$\therefore \quad f'(x) = 2\sqrt{r^2 - x^2} + \dfrac{2x(-2x)}{2\sqrt{r^2 - x^2}}$

$= 2\sqrt{r^2 - x^2} - \dfrac{2x^2}{\sqrt{r^2 - x^2}}$

$= \dfrac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$

$\therefore \quad f'(x) = 0 \Rightarrow r^2 = 2x^2 \Rightarrow x = \dfrac{r}{\sqrt{2}}$

$\therefore \quad y = \sqrt{r^2 - x^2} = \sqrt{r^2 - \dfrac{r^2}{2}} = \dfrac{r}{\sqrt{2}}$

$\therefore \quad x = y = \dfrac{r}{\sqrt{2}}$

$\therefore$ The rectangle inscribed in a semi-circle is a square.

$$f''(x) = 2\left[(r^2 - 2x^2)\left(-\frac{1}{2}\right)(r^2 - x^2)^{-\frac{3}{2}}(-2x) + \frac{(-4x)}{\sqrt{r^2 - x^2}}\right]$$

$$f''\left(\frac{r}{\sqrt{2}}\right) = \frac{-8 \times \frac{r}{\sqrt{2}}}{\frac{r}{\sqrt{2}}} = -8 < 0$$

$\therefore$ Area is maximum for a square and maximum area is A $= 2xy = 2 \cdot \dfrac{r}{\sqrt{2}} \cdot \dfrac{r}{\sqrt{2}} = r^2$.

Note : (1) A $= 2xy$

Now $x^2 + y^2 = (x - y)^2 + 2xy$

$= (x - y)^2 + A$

$\therefore$ A $= r^2 - (x - y)^2$ is maximum if $(x - y)^2$ minimum. But $(x - y)^2 \geq 0$.

$\therefore$ $(x - y)^2$ has minimum value 0 when $x = y$. Hence maximum A $= r^2$.

(2) Let $x = r\cos\theta$, $y = r\sin\theta$ **(Parametric equations of $x^2 + y^2 = r^2$)**

$\therefore$ A $= 2xy = 2r^2\sin\theta\cos\theta = r^2\sin 2\theta$

$\therefore$ A is maximum when $\theta = \frac{\pi}{4}$ as $\sin 2\theta = 1$ is maximum for $\theta = \frac{\pi}{4}$.

$\therefore$ Maximum area $= r^2$

Example 63 : A cylinder is inscribed in a sphere of radius R. Prove that its volume is maximum if its height is $\frac{2R}{\sqrt{3}}$.

Solution : Let the radius and the height of the cylinder be r and h respectively.

Then $R^2 = r^2 + \frac{h^2}{4}$

Volume of the cylinder, $V = \pi r^2 h$

$\therefore \quad V = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$= \pi R^2 h - \frac{\pi}{4}h^3$

$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$

$\therefore \quad \frac{dV}{dh} = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$

Also $\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = \frac{-3\pi h}{2} = -\sqrt{3}\pi R < 0$

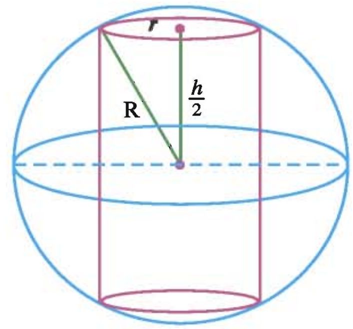


Figure 1.24

$\therefore$ The volume of the cylinder is maximum if $h = \frac{2R}{\sqrt{3}}$.

Maximum volume is $\pi r^2 h = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$= \pi\left(R^2 - \frac{R^2}{3}\right)\frac{2R}{\sqrt{3}}$

$= \frac{4\pi R^3}{3\sqrt{3}}$

Example 64 : A cylindrical can is to be made to hold 1 l oil. Find its radius and height to minimize the cost.

Solution : The cost of making the can is minimum, if the metal used to make the can is minimum.

Its total surface area S is given by $S = 2\pi r^2 + 2\pi rh$

Now the volume $V = \pi r^2 h$ and 1 *litre* is 1000 cm^3.

$\therefore \quad V = \pi r^2 h = 1000$

$\therefore \quad h = \frac{1000}{\pi r^2}$

$\therefore \quad S = 2\pi r^2 + 2\pi r \times \frac{1000}{\pi r^2}$

$= 2\pi r^2 + \frac{2000}{r}$

$\therefore \quad \frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow r^3 = \frac{500}{\pi}$

$\therefore \quad r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$

$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3} > 0$

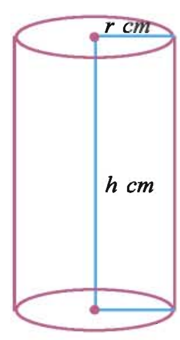


Figure 1.25

$\therefore$ Surface area and hence the cost is minimum if $r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$ cm and

$h = \frac{1000(\pi)^{\frac{2}{3}}}{\pi(500)^{\frac{2}{3}}} = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$ $cm = 2r$.

Thus the height of the cylinder should equal its diameter for minimum cost.

Example 65 : Find the point on the line $y = 2x - 3$ nearest to origin.

Solution : Let M = $(x, 2x - 3)$ be any point on the given line.

$$OM^2 = x^2 + (2x-3)^2$$
$$= 5x^2 - 12x + 9$$

Let $f(x) = 5x^2 - 12x + 9$

$\therefore \quad f'(x) = 10x - 12 = 0 \Rightarrow x = \frac{6}{5}$

Also $f''(x) = 10 > 0$

$\therefore$ Distance OM is minimum if $x = \frac{6}{5}$, $y = 2\left(\frac{6}{5}\right) - 3 = -\frac{3}{5}$

$$M = \left(\frac{6}{5}, -\frac{3}{5}\right)$$

$$OM = \sqrt{\frac{36}{25} + \frac{9}{25}} = \sqrt{\frac{45}{25}} = \frac{3\sqrt{5}}{5} = \frac{3}{\sqrt{5}}$$

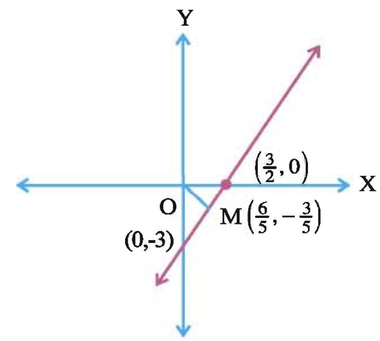


Figure 1.26

Note : $p = \frac{|c|}{\sqrt{a^2+b^2}} = \left|\frac{0+0-3}{\sqrt{4+1}}\right| = \frac{3}{\sqrt{5}}$

$\therefore$ OM is perpendicular distance of origin from $y = 2x - 3$ and M is the foot of perpendicular.

Exercise 1.5

Find the maximum and minimum values of following functions (1 to 15) :

1. $f(x) = 5 - 3x + 5x^2 - x^3 \qquad x \in R$
2. $f(x) = x^4 - 6x^2 \qquad x \in R$
3. $f(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}} \qquad x \in R^+$
4. $f(x) = 2\cos x + \sin^2 x \qquad x \in R$
5. $f(x) = \log_e(1 + x^2) \qquad x \in R$
6. $f(x) = xe^{-x} \qquad x \in [0, 2]$
7. $f(x) = \frac{\log_e x}{x} \qquad x \in [1, 3]$
8. $f(x) = \sqrt{16 - x^2} \qquad |x| \le 4$
9. $f(x) = \frac{x}{x+1} \qquad x \in [1, 2]$
10. $f(x) = \sin x + \cos x \qquad x \in [0, 2\pi]$
11. $f(x) = \frac{\cos x}{\sin x + 2} \qquad x \in [0, 2\pi]$
12. $f(x) = x\sqrt{1-x} \qquad 0 < x < 1$

13. $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 125 \qquad x \in [0, 3]$

14. $f(x) = sin2x \qquad x \in [0, 2\pi]$

15. $f(x) = 2x^3 - 24x + 107 \qquad x \in [1, 3]$

16. A window is in the form of a rectangle surmounted by semicircular opening. The total perimeter of the window is 10 *m*. Find dimensions of the window for maximum air flow through the window.

17. Prove that the right circular cone of maximum volume inscribed in a sphere of radius r has altitude $\frac{4r}{3}$.

18. Find two positive numbers whose sum is 16 and the sum of cubes of them is minimum.

19. Find positive numbers x, y for which $x + y = 35$ and the product x^2y^5 is maximum.

20. Show that the semi-vertical angle of the cone having maximum volume and given slant height l is $tan^{-1}\sqrt{2}$.

21. A open box with a square base is to be made. Its total surface area is c^2, a constant. Prove that its maximum volume is $\frac{c^3}{6\sqrt{3}}$.

22. Find a point on circle $x^2 + y^2 = 25$ whose distance from (12, 9) is minimum. Find also the point for which it is maximum. Explain geometrically.

23. Sum of circumference of a circle and perimeter of a square is constant. Prove that the sum of their areas is minimum when the ratio of the radius of the circle to a side of the square is 1:2.

24. An open tank with a square base is to be made to hold 4000 litres of water. What are the dimensions to make the cost minimum ?

25. $f(x) = x^3 + 3ax^2 + 3bx + c$ has a maximum at $x = -1$ and minimum zero at $x = 1$. Find a, b and c.

26. If a right triangle has hypotenuse having length 10 *cm*, what would be its largest area ?

*

Miscellaneous Examples :

Example 66 : Suppose we do not know formula for $g(x)$. But $g'(x) = \sqrt{x^2 + 12}$, $\forall x \in$ R. Also $g(2) = 4$. Find approximate value of $g(1.95)$.

Solution : Here $x = 2$. $\Delta x = 1.95 - 2 = -0.05$

$$g(x + \Delta x) \simeq g(x) + g'(x)\,\Delta x$$

$$\therefore \quad g(1.95) \simeq g(2) + g'(2)(-0.05)$$

$$= 4 - (0.05)4$$

$$= 4 - 0.2$$

$$= 3.8$$

Example 67 : Find the common tangents of $y = 1 + x^2$ and $y = -1 - x^2$. Also find their points of contact.

Solution : Let $\overleftrightarrow{PQ}$ touch $y = 1 + x^2$ at P and $y = -1 - x^2$ at Q. Let P have x-coordinate a.

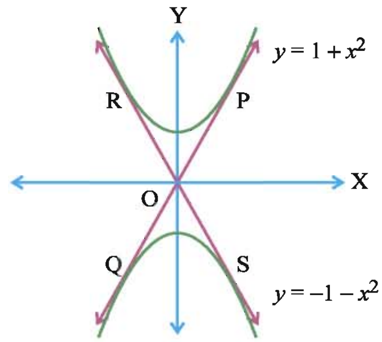


Figure 1.27

$\therefore$ $P(a, 1 + a^2)$, $Q = (-a, -(1 + a^2))$

Slope of $\overleftrightarrow{PQ} = \dfrac{1 + a^2 - (-(1 + a^2))}{a - (-a)}$

$= \dfrac{2(1 + a^2)}{2a} = \dfrac{1 + a^2}{a}$

Also $y = 1 + x^2 \Rightarrow \dfrac{dy}{dx} = 2x$

$\therefore$ Slope of tangent at P $= 2a$.

$\therefore$ $\dfrac{1 + a^2}{a} = 2a$ **($\overleftrightarrow{PQ}$ is a tangent)**

$\therefore$ $1 + a^2 = 2a^2$

$\therefore$ $a^2 = 1$

$\therefore$ $a = \pm 1$

$\therefore$ $P = (1, 2)$, $Q = (-1, -2)$

Similarly, $R = (-1, 2)$, $S(1, -2)$

The equation of $\overleftrightarrow{PQ}$ is $y - 2 = 2(x - 1)$

$\therefore$ $y - 2 = 2x - 2$

$\therefore$ $2x - y = 0$

Similarly, the equation of $\overleftrightarrow{RS}$ is $2x + y = 0$.

$\therefore$ The equations of common tangent are $2x - y = 0$ and $2x + y = 0$.

Example 68 : The position of a particle is given by $s = f(t) = t^3 - 6t^2 + 9t$, s is in meters, t is in seconds.

(1) Find the instantaneous velocity, when $t = 2$.

(2) When is the particle at rest ?

(3) Find the distance travelled in first 5 seconds.

Solution : $\dfrac{ds}{dt} = f'(t) = 3t^2 - 12t + 9$

(1) $[f'(t)]_{t = 2} = 12 - 24 + 9 = -3 m/sec$

(2) When the particle is at rest, its velocity at that time is zero.

$\therefore$ $3t^2 - 12t + 9$

$\therefore$ $t^2 - 4t + 3 = 0$

$\therefore$ $t = 1$ or 3

$\therefore$ The particle is at rest at $t = 1$ and $t = 3$.

(3) $f'(t) = 3(t - 1)(t - 3)$

$\therefore$ For $t < 1$ and $t > 3$, $f'(t) > 0$, and $f(t)$ is increasing and $f(t)$ is decreasing for $t \in (1, 3)$. The motion is divided into 3 parts $(0, 1)$, $(1, 3)$, $(3, 5)$.

$\therefore$ Total distance covered is $s_1 + s_2 + s_3$, where

$s_1 = |f(1) - f(0)| = 4$, $s_2 = |f(3) - f(1)| = |0 - 4| = 4$

$s_3 = |f(5) - f(3)| = 20$

$\therefore$ Total distance covered is $20 + 4 + 4 = 28$ m.

Note : $|f(5) - f(0)| = 20$ is not the total distance covered.

Example 69 : An exhibition is to be arranged in a rectangular ground. A fencing of 80 m is done on three sides of the plot and the fourth side is not to be covered by fencing. What should be the dimensions of the ground to cover maximum area ?

Solution : We have $2x + y = 80$

$A = xy = x(80 - 2x) = 80x - 2x^2$

$\therefore$ $\frac{dA}{dx} = 0 \Rightarrow 80 - 4x = 0 \Rightarrow x = 20$

$\therefore$ $\frac{d^2A}{dx^2} = -4 < 0$

$\therefore$ Largest area is covered if the length is

$y = 80 - 2x = 80 - 40 = 40$ m

and the breadth is $x = 20$ m.

$\therefore$ Maximum area covered is $40 \times 20 = 800$ m^2

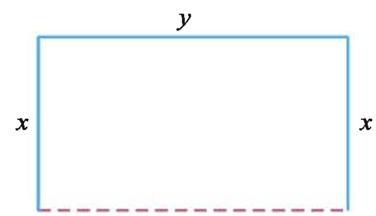


Figure 1.28

Only for information :

C(x) is the cost of producing x units. C(x) is the cost function.

C'(x) is the marginal cost.

$c(x) = \frac{C(x)}{x}$ **is the cost per unit. $c(x)$ is average cost function.**

$c'(x) = \frac{xC'(x) - C(x)}{x^2}$

For minimum of average cost $c'(x) = 0$.

$\therefore$ $xC'(x) = C(x)$

$\therefore$ $C'(x) = \frac{C(x)}{x} = c(x)$

If the average cost is minimum, marginal cost = average cost.

If the profit is maximum, marginal revenue $\frac{dR}{dx}$ = marginal cost $\frac{dC}{dx}$ and R"(x) < C"(x).

If $p(x)$ is the sale price per unit, if x units are sold, p is called demand function.

The total revenue is R(x) = $xp(x)$.

R(x) is called revenue function. R'(x) is marginal revenue function.

If P(x) is the profit function.

$P(x) = R(x) - C(x)$

For maximum profit P'(x) = 0

$\therefore$ $R'(x) = C'(x)$

$\therefore$ **Marginal revenue = marginal cost for maximum profit.**

Also $P''(x) = R''(x) - C''(x) < 0$

$\therefore$ $R''(x) < C''(x)$ **for maximum profit.**

Example : A company estimates that the cost of producing x ball-pens is $C(x) = 3000 + 2x + 0.001x^2$.

(1) Find the cost, average cost and marginal cost of producing 1000 ball-pens.

(2) At what production level, will the average cost be minimum and what is that minimum average cost ?

Solution : (1) The average cost function is $c(x) = \frac{C(x)}{x}$

$$= \frac{3000 + 2x + 0.001x^2}{x}$$

$$= \frac{3000}{x} + 2 + 0.001x$$

Also marginal cost function is $C'(x) = 2 + 0.002x$

$\therefore$ For production of 1000 ball-pens, $C(1000) = 3000 + 2000 + \frac{1}{1000} \times (1000)^2$

$= ₹\ 6000$

$\therefore$ $c(x) = \frac{6000}{1000} = ₹\ 6$ per ball-pen.

$C'(x) = 2 + \frac{2}{1000} \times 1000 = ₹\ 4$

(2) For minimum average cost :

Marginal cost = Average cost

$C'(x) = c(x)$

$\therefore$ $2 + 0.002x = \frac{3000}{x} + 2 + 0.001x$

$\therefore$ $0.001x = \frac{3000}{x}$

$\therefore$ $x^2 = 3000 \times 1000$

$\therefore$ $x = \sqrt{3 \times 10^6} = \sqrt{3} \times 10^3 = 1730$

$\therefore$ Hence, 1730 ball-pens should be manufactured for minimum average cost.

Minimum average cost $= c(1730) = \frac{3000}{1730} + 2 + (0.001)(1730)$

$= \frac{300}{173} + 2 + 1.73$

$= 1.73 + 2 + 1.73$

$= ₹\ 5.46$

Example 70 : Find the point on $xy = 8$, nearest to P(3, 0) having integer coordinates and the minimum distance. **$(x > 0)$**

Solution : Let the required point on $xy = 8$ be $Q\left(x, \frac{8}{x}\right)$

$\therefore$ $PQ^2 = (x - 3)^2 + \frac{64}{x^2}$

Let $f(x) = (x - 3)^2 + \frac{64}{x^2}$

$f'(x) = 2(x - 3) - \frac{128}{x^3} = 0 \Rightarrow x - 3 = \frac{64}{x^3}$

$\therefore$ $x^4 - 3x^3 - 64 = 0$

$\therefore$ $(x - 4)(x^3 + x^2 + 4x + 16) = 0$

$\therefore \quad x = 4$ (Verify that $x^3 + x^2 + 4x + 16 = 0$ has no integer solution !)

$\therefore \quad f''(x) = 2 - \frac{(128)(-3)}{x^4}$

$\therefore \quad f''(4) = 2 + \frac{(128)(3)}{256} = \frac{7}{2} > 0$

$\therefore \quad f(x)$ is minimum for $x = 4$.

$\therefore$ The point nearest to P(3, 0) and lying on $xy = 8$ is Q(4, 2).

PQ $= \sqrt{1+4} = \sqrt{5}$

Example 71 : Find a point on $y^2 = 2x$ nearest to (1, 4) and the minimum distance.

Solution : For $y^2 = 2x = 4ax$, $a = \frac{1}{2}$

$\therefore$ Let Q(1, 4) and P$\left(\frac{1}{2}t^2, t\right)$ be any point on parabola.

$$\therefore \quad PQ^2 = \left(\frac{1}{2}t^2 - 1\right)^2 + (t-4)^2$$

$$= \frac{1}{4}t^4 - t^2 + 1 + t^2 - 8t + 16$$

$$= \frac{1}{4}t^4 - 8t + 17$$

Let $f(t) = \frac{1}{4}t^4 - 8t + 17$

$f'(t) = 0 \Rightarrow t^3 - 8 = 0 \Rightarrow t = 2$

$f''(t) = 3t^2 = 12 > 0$

$\therefore \quad f(t)$ is minimum, if $t = 2$

$\therefore$ P(2, 2), Q(1, 4)

$\therefore$ PQ $= \sqrt{1+4} = \sqrt{5}$ is the minimum distance.

Example 72 : A rectangular sheet of tin 45 $cm \times$ 24 cm is to be made into an open box by cutting off squares of the same size from each corner and folding up. Find the side of the square cut off from each corner for maximum volume of the box.

Solution : Let x cm be the side of the square removed from each corner.

$\therefore$ Length and breadth of the box are $(45 - 2x)$ cm and $(24 - 2x)$ cm. The height is x cm.

The volume V $= (45 - 2x)(24 - 2x)x$

$= 4x^3 - 138x^2 + 1080x$

$\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 276x + 1080 = 0 \Rightarrow x^2 - 23x + 90 = 0$

$\therefore \quad x = 18$ or 5

But if $x = 18$, breadth $24 - 2x = 24 - 36 < 0$

$\therefore \quad x \neq 18$ and so $x = 5$

The length of the side of square removed is 5 cm.

$\frac{d^2V}{dx^2} = 24x - 276 = 120 - 276 < 0$

$\therefore$ V is maximum if $x = 5$ cm

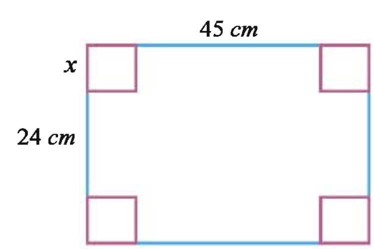


Figure 1.29

Exercise 1

1. Water is dripping out from conical funnel at the rate of 5 cm^3/sec. Slant height of the cone formed by water is 4 cm. Semi-vertical angle of the cone is $\frac{\pi}{6}$. Find the rate at which the slant height decreases.

2. Height of a kite is fixed at 40 m. The length of the string is 50 m at a moment. Velocity of the kite in horizontal direction is 25 m/sec at that time. Find the rate of slackening of the string at that time.

3. Altitude of a triangle increases at 2 cm/min. Its area increases at the rate 5 cm^2/min. Find the rate of change of length of base when the altitude is 10 cm and the area is 100 cm^2.

4. Find the intervals in which $f(x) = 2x^3 - 3x^2 - 36x + 25$ is (1) strictly increasing (2) strictly decreasing.

5. Find the intervals in which $f(x) = (x + 1)^3(x - 3)^3$ is (1) strictly increasing (2) strictly decreasing.

6. Prove $x^{101} + sinx - 1$ is increasing for $|x| > 1$.

7. Find the intervals where $f(x) = x^4 + 32x$ is increasing or decreasing. $x \in R$

8. Find the intervals in which $f(x) = x^2e^{-x}$ is increasing or decreasing. $x \in R$

9. Prove that curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each-other.

10. Find the equation of tangent to $y = be^{-\frac{x}{a}}$ where it intersects Y-axis.

11. Find the measure of the angle between $y^2 = 4ax$ and $x^2 = 4ay$.

12. Prove that $y = 6x^3 + 15x + 10$ has no tangent with slope 12.

13. Find points on the ellipse $x^2 + 2y^2 = 9$ at which tangent has slope $\frac{1}{4}$.

14. Find maximum and minimum values of $f(x) = x - 2sinx \quad x \in [0, 2\pi]$

15. Find maximum and minimum values of $f(x) = 1 - e^{-x} \quad x \geq 0$

16. Find maximum and minimum values of $f(x) = x^2 + \frac{2}{x} \quad x \neq 0$

17. Find where $f(x) = 4x - tanx$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is increasing or decreasing and find its maximum and minimum values.

18. Where does $f(x) = x + \sqrt{1-x}$, $0 < x < 1$ increase or decrease ? Find its maximum and minimum values.

19. Determine critical points for $f(x) = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$, $x \in [0, 6]$ and determine where the function is increasing or decreasing. Find also maximum and minimum values.

20. Find the maximum and minimum values of $f(x) = sin^4x + cos^4x$. $x \in \left[0, \frac{\pi}{2}\right]$.

21. Show that $f(x) = \left(\frac{1}{x}\right)^x$ has local maximum at $x = \frac{1}{e}$.

22. Show that out of all rectangles with given area a square has minimum perimeter.

23. Show that out of all rectangles inscribed in a circle, the square has maximum area.

24. Prove that the area of a right angled triangle with given hypotenuse is maximum, if the triangle is isoceles.

25. A point on the hypotenuse of a right triangle is at distances a and b from the sides making right angle. (a, b constant). Prove that the hypotenuse has minimum length $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

26. Show that the semi-vertical angle of a right circular cone with given surface and maximum volume is $sin^{-1}\frac{1}{3}$.

27. Find the measure of the angle between curves, if they intersect :

(1) $xy = 6$, $x^2y = 12$

(2) $y = x^2$, $x^2 + y^2 = 20$

(3) $2y^2 = x^3$, $y^2 = 32x$, $(x, y) \neq (0, 0)$

(4) $y^2 = 4ax$, $x^2 = 4by$

(5) $y^2 = 8x$, $x^2 = 27y$

(6) $x^2 + y^2 = 2x$, $y^2 = x$

28. (1) Prove $x^2 = 4y$, $x^2 + 4y = 8$ intersect orthogonally at (2, 1) and (–2, 1).

(2) Prove $x^2 = y$ and $x^3 + 6y = 7$ intersect at right angles at (1, 1).

29. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) The side of an equilateral triangle expands at the rate of $\sqrt{3}$ *cm/sec*. When the side is 12 *cm*, the rate of increase of its area is .... ☐

(a) 12 cm^2/sec (b) 18 cm^2/sec (c) $3\sqrt{3}$ cm^2/sec (d) 10 cm^2/sec

(2) The distance s moved by a particle in time t is given by $s = t^3 - 6t^2 + 6t + 8$. When the acceleration is zero, the velocity is .... ☐

(a) 5 *cm/sec* (b) 2 *cm/sec* (c) 6 *cm/sec* (d) –6 *cm/sec*

(3) The volume of a sphere is increasing at the rate of π cm^3/sec. The rate at which the radius is increasing is ......, when the radius is 3 *cm*. ☐

(a) $\frac{1}{36}$ *cm/sec* (b) 36 *cm/sec* (c) 9 *cm/sec* (d) 27 *cm/sec*

(4) There is 4 % error in measuring the period of a simple pendulum. The approximate percentage error in length is .... (Hint : $T = 2\pi\sqrt{\frac{l}{g}}$) ☐

(a) 4 % (b) 8 % (c) 2 % (d) 6 %

(5) Approximate value of $(31)^{\frac{1}{5}}$ is .... ☐

(a) 2.01 (b) 2.1 (c) 2.0125 (d) 1.9875

(6) The height and radius of a cylinder are equal. An error of 2 % is made in measuring height. The approximate percentage error in volume is ...... ☐

(a) 6 % (b) 4 % (c) 3 % (d) 2 %

(7) The tangent to $(at^2, 2at)$ is perpendicular to X-axis at ☐

(a) $(4a, 4a)$ (b) $(a, 2a)$ (c) (0, 0) (d) $(a, -2a)$

(8) The line $y = mx + 1$ touches $y^2 = 4x$, if $m =$ ......

(a) 0 (b) 1 (c) -1 (d) 2

(9) The equation of normal to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is ......

(a) $2x + y = 0$ (b) $y = 1$ (c) $x = 0$ (d) $x = y$

(10) $f(x) = x^x$ decreases in ......

(a) $(0, e)$ (b) $\left(0, \frac{1}{e}\right)$ (c) $(0, 1)$ (d) $(0, \infty)$

(11) $f(x) = 2|x - 2| + 3|x - 4|$ is ...... in $(2, 4)$.

(a) decreasing (b) increasing (c) constant (d) cannot be decided

(12) $f(x) = x^7 + 5x^3 + 125$ is ....

(a) decreasing in $(0, \infty)$ (b) decreasing in $(-\infty, 0)$

(c) increasing on R (d) neither increasing nor decreasing in R

(13) The local maximum value of $f(x) = x + \frac{1}{x}$ is ......

(a) 2 (b) -2 (c) 4 (d) -4

(14) The local minimum value of $\frac{x}{\log x}$ is ......

(a) -1 (b) 0 (c) $\frac{1}{e}$ (d) e

(15) If $\log_e 4 = 1.3868$, then approximate value of $\log_e 4.01 =$ ......

(a) 1.3867 (b) 1.3869 (c) 1.3879 (d) 1.3893

(16) The circumference of a circle is 20 *cm* and there is an error of 0.02 *cm* in its measurement. The approximate percentage error in area is ......

(a) 0.02 (b) 0.2 (c) π (d) $\frac{1}{\pi}$

(17) If the line $y = x$ touches the curve $y = x^2 + bx + c$ at $(1, 1)$, then ......

(a) $b = 1, c = 2$ (b) $b = -1, c = 1$ (c) $b = 1, c = 1$ (d) $b = 0, c = 1$

(18) $y = ae^x$, $y = be^{-x}$ intersect at right angles if ...... $(a \neq 0, b \neq 0)$

(a) $a = \frac{1}{b}$ (b) $a = b$ (c) $a = -\frac{1}{b}$ (d) $a + b = 0$

(19) Tangent to $y = 5x^5 + 10x + 15$...

(a) is always vertical

(b) is always horizontal

(c) makes acute angle with the positive X-axis

(d) makes obtuse angle with the positive X-axis

(20) $f(x) = 2x + cot^{-1}x - \log | x + \sqrt{1+x^2} |$ is ......

(a) decreasing on $(-\infty, 0)$ (b) decreasing on $(0, \infty)$

(c) constant (d) increasing on R

(21) The sum of two non-zero numbers in 12. The minimum sum of their reciprocals is ......

(a) $\frac{1}{10}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

(22) The local minimum value of $f(x) = x^2 + 4x + 5$ is ......

(a) 2 (b) 4 (c) 1 (d) -1

(23) The maximum value of $f(x) = 5cosx + 12sinx$ is ......

(a) 13 (b) 12 (c) 5 (d) 17

(24) The minimum value of $f(x) = 3cosx + 4sinx$ is ......

(a) 7 (b) 5 (c) -5 (d) 4

(25) $f(x) = x \log x$ has minimum value...

(a) 1 (b) 0 (c) e (d) $-\frac{1}{e}$

(26) $f(x) = \sqrt{3}\,cosx + sinx,\ x \in \left[0, \frac{\pi}{2}\right]$ is maximum for $x =$ ......

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) 0

(27) $f(x) = (x - a)^2 + (x - b)^2 + (x - c)^2$ has minimum value at $x =$ ......

(a) $\sqrt[3]{abc}$ (b) $a + b + c$ (c) $\frac{a+b+c}{3}$ (d) 0

(28) $f(x) = (x + 2)\, e^{-x}$ is increasing in ....

(a) $(-\infty, -1)$ (b) $(-1, -\infty)$ (c) $(2, \infty)$ (d) R^+

(29) The measure of the angle of intersection between $y^2 = x$ and $x^2 = y$ other than one at (0, 0) is ......

(a) $tan^{-1}\frac{4}{3}$ (b) $tan^{-1}\frac{3}{4}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

(30) The point where normal to $y = x^2 - 2x + 3$ is parallel to Y-axis is ......

(a) (0, 3) (b) $(-1, 2)$ (c) (1, 2) (d) (3, 6)

(31) The slope of normal to $(3t^2 + 1, t^3 - 1)$ at $t = 1$ is ......

(a) $\frac{1}{2}$ (b) -2 (c) 2 (d) $-\frac{1}{2}$

(32) The equation of normal to $3x^2 - y^2 = 8$ at $(2, -2)$ is ......

(a) $x + 2y = -2$ (b) $x - 3y = 8$ (c) $3x + y = 4$ (d) $x + y = 0$

(33) The angle made by the tangent with the $+ve$ direction of X-axis to $x = e^t cost$, $y = e^t sint$ at $t = \frac{\pi}{4}$ is ......

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{3}$

(34) The equation of tangent to $y = cosx$ at (0, 1) is ......

(a) $x = 0$ (b) $y = 0$ (c) $x = 1$ (d) $y = 1$

(35) The equation of normal to $y = sinx$ at $\left(\frac{\pi}{2}, 1\right)$ is ......

(a) $x = 1$ (b) $x = 0$ (c) $y = \frac{\pi}{2}$ (d) $x = \frac{\pi}{2}$

(36) At ...... on circle $x^2 + y^2 - 2x - 3 = 0$, the tangent is horizontal.

(a) $(0, \pm\sqrt{3})$ (b) $(2, \pm\sqrt{3})$ (c) $(1, 2), (1, -2)$ (d) $(3, 0)$

(37) The point on $y^2 = x$ where tangent makes angle of measure $\frac{\pi}{4}$ with the positive X-axis is ....

(a) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (b) (2, 1) (c) (0, 0) (d) (−1, 1)

(38) A cone with its height equal to the diameter of the base is expanding in volume at the rate of 50 cm^3/sec. If the base has area 1 m^2, the radius is increasing at the rate ......

(a) 0.0025 cm/sec (b) 0.25 cm/sec (c) 1 cm/sec (d) 4 cm/sec

(39) The rate of increase of $f(x) = x^3 - 5x^2 + 5x + 25$ is twice the rate of increase of x for $x =$ ....

(a) $-3, -\frac{1}{3}$ (b) $3, \frac{1}{3}$ (c) $-3, \frac{1}{3}$ (d) $3, -\frac{1}{3}$

(40) The radius of a cone increases at the rate of 4 cm/sec and the altitude is decreasing at the rate of 3 cm/sec. When the radius is 3 cm and altitude is 4 cm, the rate of change of lateral surface is ....

(a) 30π cm^2/sec (b) 10 cm^2/sec (c) 20π cm^2/sec (d) 22π cm^2/sec

(41) The rate of change of surface area of a sphere *w.r.t.* radius is ....

(a) 8π (diameter) (b) 3π (diameter) (c) 4π (radius) (d) 8π (radius)

(42) The rate of change of volume of a cylinder *w.r.t.* radius whose radius is equal to its height is ....

(a) 4 (area of base) (b) 3 (area of base) (c) 2 (area of base) (d) (area of base)

(43) $f(x) = tan^{-1} x - x$ is ....

(a) increasing on R (b) decreasing on R (c) increasing on R^+ (d) increasing on $(-\infty, 0)$

(44) $f(x) = tan\, x - x$, $x \in R - \{(2k - 1)\frac{\pi}{2} \mid k \in Z\}$ is ....

(a) increasing on its domain (b) decreasing on its domain

(c) increasing on $\left(0, \frac{\pi}{2}\right)$ (d) decreasing on $\left(0, \frac{\pi}{2}\right)$

(45) $f(x) = 2x - tan^{-1} x - \log | x + \sqrt{1+x^2} |$ is .... ($x \in$ R).

(a) increasing on R (b) decreasing on R

(c) has a minimum at $x = 1$ (d) has a maximum at $x = 1$

(46) If ......, then $f(x) = x^2 - kx + 20$ is strictly increasing on [0, 3].

(a) $k < 0$ (b) $0 < k < 1$ (c) $1 < k < 2$ (d) $2 < k < 3$

(47) $f(x) = | x - 1 | + | x - 2 |$ is increasing if ....

(a) $x > 2$ (b) $x < 1$ (c) $x < 0$ (d) $x < -2$

(48) Normal to $9y^2 = x^3$ at ...... makes equal intercepts on axes.

(a) $\left(-4, -\frac{8}{3}\right)$ (b) $\left(4, \pm\frac{8}{3}\right)$ (c) $\left(\pm 4, \frac{8}{3}\right)$ (d) $\left(8, \frac{8}{3}\right)$

(49) $y = mx + 4$ touches $y^2 = 8x$, if $m =$ ....

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) -2

(50) The measure of the angle between the curves $y = 2sin^2 x$ and $y = cos\, 2x$ at $x = \frac{\pi}{6}$ is ....

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

(51) The normal to $x^2 = 4y$ passing through (1, 2) has equation ....

(a) $2x = y$ (b) $x + y - 3 = 0$ (c) $2x + 3y - 8 = 0$ (d) $x - y + 1 = 0$

(52) The local minimum value of $x^2 + \frac{16}{x}$ ($x \neq 0$) is ....

(a) 12 (b) 22 (c) -12 (d) 2

(53) The minimum value of $secx$, $x \in \left[\frac{2\pi}{3}, \pi\right]$ is ....

(a) 1 (b) -2 (c) 2 (d) π

(54) The maximum value of $cosecx$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ is ....

(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

(55) If f is decreasing in $[a, b]$, its minimum and maximum values are respectively ...... and ....

(a) $f(a)$ and $f(b)$ (b) $f(b)$ and $f(a)$

(c) $f\left(\frac{a+b}{2}\right)$ and $f(a)$ (d) $f(b)$ and $f\left(\frac{a+b}{2}\right)$

Summary

We have studied the following points in this chapter :

1. Derivative as a rate measurer.
2. Increasing and decreasing functions.
3. Applications to Geometry : Tangents and normals
4. Angle between two curves.
5. Differentials and approximate values.
6. Maximum and minimum values.
7. Application to optimization problems and practical applications.

●

RAMANUJAN

He was born on 22nd of December 1887 in a small village of Tanjore district, Madras.

He failed in English in Intermediate, so his formal studies were stopped but his self-study of mathematics continued.

He sent a set of 120 theorems to Professor Hardy of Cambridge. As a result he invited Ramanujan to England.

Ramanujan showed that any big number can be written as sum of not more than four prime numbers.

He showed that how to divide the number into two or more squares or cubes.

When Mr Littlewood came to see Ramanujan in taxi number 1729, Ramanujan said that 1729 is the smallest number which can be written in the form of sum of cubes of two numbers in two ways,

i.e. $1729 = 9^3 + 10^3 = 1^3 + 12^3$

since then the number 1729 is called Ramanujan's number.