Gauss elimination method. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad ...(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad ...(3)$$

Here the second equation is the *pivotal equation* and b_2' is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Example 2. *Solve the following system of equations by Gauss-Elimination method.*

$$\left.\begin{aligned} 2x+y+z&=10 \\ 3x+2y+3z&=18 \\ x+4y+9z&=16 \end{aligned}\right\} \quad \ldots(1)$$

Sol. To eliminate x from the second equation of the system (1), we multiply the first equation by $\frac{3}{2}$ and subtract it from the second equation and obtain.

$$y+3z=6$$

Similarly, to eliminate x from the third equations of the system (1), we multiply the first euqaiton by $\frac{1}{2}$ and subtract it from the third equation and obtain.

$$7y+17z=22$$

Now, the system of equation (1), becomes

$$\left.\begin{aligned} 2x+y+z&=10 \\ y+3z&=6 \\ 7y+17z&=22 \end{aligned}\right\} \quad \ldots(2)$$

Now, to eliminate y from the third equation of the system (2), we multiply the second equation by 7 and subtract it from the third equation of the system (2) and obtain

$$4z=20$$

Thus, the system of equation (2) becomes

$$2x - y + z = 10;\ y + 3z = 6;\ 4z = 20 \quad \ldots(3)$$

Back substitution gives the solution.

$$z = 5,\ y = -9 \quad \text{and } x = 7. \quad \textbf{Ans.}$$

Example 3.19. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.*

(*Mumbai, B. Tech., 2005*)

Sol. We have

		Check sum	
	$x + 4y - z = -5$	-1	...(*i*)
	$x + y - 6z = -12$	-16	...(*ii*)
	$3x - y - z = 4$	5	...(*iii*)

Step I. To eliminate x, operate (*ii*) – (*i*) and (*iii*) – 3(*i*) :

	Check sum	
$-3y - 5z = -7$	-15	...(*iv*)
$-13y + 2z = 19$	8	...(*v*)

Step II. To eliminate y, operate $(v) - \frac{13}{3}(iv)$:

	Check sum	
$\frac{71}{3}z = \frac{148}{3}$	73	...(*vi*)

Step III. By back-substitution, we get

From (*vi*) : $$z = \frac{148}{71} = 2.0845$$

From (*iv*) : $$y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$$

From (*i*) : $$x = -5 - 4\left(-\frac{81}{71}\right) + \left(\frac{148}{71}\right) = \frac{117}{71} = 1.6479$$

Hence, $x = 1.6479$, $y = -1.1408$, $z = 2.0845$.

ITERATIVE METHODS OF SOLUTION

In this method each eqn of the system must possess one large coefficient and the large coefficient must be attached to a differiant unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficients matrix. When this condition is satisfied, the system will be solvable by the iterative method. The system,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$

In iterative method, we describe the following two methods

(1) Jacobi's method (2) Gauss-Seidel method

(1) Jacobi's iteration method. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \qquad \ldots(1)$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve for x, y, z respectively. Then the system can be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \qquad \ldots(2)$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Substituting these on the right sides of (2), the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Substituting the values x_1, y_1, z_1 on the right sides of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is repeated till the difference between two consecutive approximations is negligible.

Example 6. *Solve the following system of equations by Jacobi iteration method.*

$$3x + 4y + 15z = 54.8,\ x + 12y + 3z = 39.66 \text{ and } 10x + y - 2z = 7.74.$$

Sol. The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations, as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$10x + y - 2z = 7.74$$

$$x + 12y + 3z = 39.66$$

$$3x + 4y + 15z = 54.8$$

Now, we write the equations in the form

$$\left.\begin{aligned} x &= \frac{1}{10}(7.74 - y + 2z) \\ y &= \frac{1}{12}(39.66 - x - 3z) \\ z &= \frac{1}{15}(54.8 - 3x - 4y) \end{aligned}\right\} \qquad ...(1)$$

We start from an approximation $= x_0 = y_0 = z_0 = 0$

Substituting these on RHS of (1), we get

First approximation:

$$x_1 = \frac{1}{10}[7.74 - 0 + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12}[39.66 - 0 - 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15}[54.8 - 3(0) - 4(0)] = 3.6533333$$

Second approximation:

$$x_2 = \frac{1}{10}[7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12}[39.66 - 0.744 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15}[54.8 - 3(0.744) - 4(1.1383333)] = 3.1949778$$

Third approximation:

$$x_3 = \frac{1}{10}[7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12}[39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15}[54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

Fourth approximation:

$$x_4 = \frac{1}{10}[7.74 - 2.3903528 + 2(2.7545889)] = 1.0858825$$

$$y_4 = \frac{1}{12}[39.66 - 1.1802789 - 3(2.7545889)] = 2.5179962$$

$$z_4 = \frac{1}{15}[54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Fifth approximation:

$$x_5 = \frac{1}{10}\left[7.74 - 2.5179962 + 2(2.7798501)\right] = 1.0781704$$

$$y_5 = \frac{1}{10}\left[39.66 - 1.0858825 - 3(2.7798501)\right] = 2.5195473$$

$$z_5 = \frac{1}{15}\left[54.8 - 3(1.0858825) - 4(2.5179962)\right] = 2.7646912$$

Sixth approximation:

$$x_6 = \frac{1}{10}[7.74 - 2.5195473 + 2(2.7646912)] = 1.0749835$$

$$y_6 = \frac{1}{12}[39.66 - 1.0781704 - 3(2.7646912)] = 2.5239797$$

$$z_6 = \frac{1}{15}[54.8 - 3(1.0781704) - 4(2.5195473)] = 2.76582$$

Seventh approximation:

$$x_7 = \frac{1}{10}[7.74 - 2.5239797 + 2(2.76582)] = 1.074766$$

$$y_7 = \frac{1}{12}[39.66 - 1.0749835 - 3(2.76582)] = 2.523963$$

$$z_7 = \frac{1}{15}[54.8 - 3(1.0749835)] - 4(2.5239797) = 2.7652754$$

From the sixth and seventh approximations:

$x = 1.075$, $y = 2.524$ and $z = 2.765$ correct to three decimals. Ans

Solve, by Jacobi's iteration method, the equations

$20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. (Bhopal, B.E., 2009)

Sol. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \ldots(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85, \quad y_1 = -\frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Substituting these values on the right sides of the equations (*i*), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the 5th and 6th iterations being practically the same, we can stop. Hence the solution is

$$x = 1, y = -1, z = 1.$$

(2) **Gauss-Seidal iteration method.** This is a modification of Jacobi's method. As before the system of equations :

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \text{...(1)}$$

is written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \qquad \text{...(2)}$$

Here also we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively which may each be taken as zero. Substituting $y = y_0$, $z = z_0$ in the first of the equations (2), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2), we have

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2), we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

and so on *i.e.* as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

Example 8. *Solve the following system of equations using Gauss-Seidel method:*

$$10x + y + 2z = 44$$

$$2x + 10y + z = 51$$

$$x + 2y + 10z = 61$$

Sol. Given system of equations can be written as:

$$x = \frac{1}{10}(44 - y - 2z)$$

$$y = \frac{1}{10}(51 - 2x - z)$$

$$z = \frac{1}{10}(61 - x - 2y)$$

If we start by assuming $y_0 = 0 = z_0$ then, we obtain

$$x_1 = \frac{1}{10}(44 - 0 - 0) = 4.4$$

Now we susbtitute $x_1 = 4.4$ and $z_0 = 0$ for y_1 and we obtain

$$y_1 = \frac{1}{10}(51 - 8.8 - 0) = 4.22$$

Similarly, we obtain $z_1 = \frac{1}{10}(61 - 4.4 - 2 \times 4.22) = 4.816$

Now for **second approximation,** we obtain

$$x_2 = 4.0154$$

$$y_2 = 3.0148$$

$$z_2 = 5.0955$$

Third approximation is given by

$$x_3 = 3.0794$$

$$y_3 = 3.9746$$

$$z_3 = 4.9971$$

Similarly, if we proceed up to eighth approximation, then, we obtain

$$x_8 = 3.00$$

$$y_8 = 4.00$$

$$z_8 = 5.00$$

Example 3.29. *Apply Gauss-Seidal iteration method to solve the equations of Example 3.26.*

(V.T.U., B. Tech., 2006)

Sol. We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z) \quad \ldots(i)$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \ldots(ii)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \ldots(iii)$$

First iteration

Putting $y = y_0$, $z = z_0$ in (*i*), we get $x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

Putting $x = x_1$, $z = z_0$ in (*ii*), we have $y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

Putting $x = x_1$, $y = y_1$ in (*iii*), we obtain $z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

Second iteration

Putting $y = y_1$, $z = z_1$ in (*i*), we get $x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

Putting $x = x_2$, $z = z_1$ in (*ii*), we obtain $y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

Putting x= x2, y = y2in (iii) we get z2 = 1/20(25-2x2+3y2) = 0.9998

Third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1$, $y = -1$, $z = 1$.

ILL-CONDITIONED SYSTEM OF EQUATIONS

A system of equations A X = B is said to be ill-conditioned or unstable if it is highly sensitive to small changes in A and B *i.e.*, small change in A or B causes a large change in the solution of the system. On the other hand if small changes in A and B give small changes in the solution the system is said to be stable, or, well conditioned. Thus in a ill-conditioned system, even the small round off errors effect the solutions very badly. Unfortunately it is quite difficult to recognize an ill-conditioned system.

Example 3. *Show that the following system of linear equations is ill-conditioned.*

$$7x + 10y = 1$$

$$5x + 7y = 0.7$$

Sol. On solving the given equations we get $x = 0$ and $y = 0.1$.

Now, we make slight changes in the given system of equations. The new system become

$$7x + 10y = 1.01$$

$$5x + 7y = 0.69$$

Here we get $x = -0.17$ and $y = 0.22$.

Hence the given system is ill-conditioned.

Example 3.34. *Establish whether the system $1.01x + 2y = 2.01$; $x + 2y = 2$ is well-conditioned or not ?*

Sol. Its solution is $x = 1$ and $y = 0.5$.

Now consider the system $x + 2.01y = 2.04$ and $x + 2y = 2$ which has the solution $x = -6$ and $y = 4$.

Hence the system is ill-conditioned.

(4) Gauss-Jordan method. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately ***reducing the system to a diagonal matrix form*** *i.e. each equation involving only one unknown. From these equations, the unknowns x, y, z can be obtained readily.*

Example 3.22. *Apply Gauss-Jordan method to solve the equations*

$x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$. (V.T.U., B.E., 2009)

Sol. We have

$$x + y + z = 9 \quad \text{...}(i)$$

$$2x - 3y + 4z = 13 \quad \text{...}(ii)$$

$$3x + 4y + 5z = 40 \quad \text{...}(iii)$$

Step I. To eliminate x from (ii) and (iii), operate $(ii) - 2(i)$ and $(iii) - 3(i)$:

$$x + y + z = 9 \quad \text{...}(iv)$$

$$-5y + 2z = -5 \quad \text{...}(v)$$

$$y + 2z = 13 \quad \text{...}(vi)$$

Step II. To eliminate y from (iv) and (vi), operate $(iv) + \frac{1}{5}(v)$ and $(vi) + \frac{1}{5}(v)$:

$$x + \frac{7}{5}z = 8 \quad \text{...}(vii)$$

$$-5y + 2z = -5 \quad \text{...}(viii)$$

$$\frac{12}{5}z = 12 \quad \text{...}(ix)$$

Step III. To eliminate z from (vii) and $(viii)$, operate $(vii) - \frac{7}{12}$ (ix) and $(viii) - \frac{5}{6}$ (ix) :

$$x = 1$$

$$-5y = -15$$

$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1$, $y = 3$, $z = 5$.

Q1. $2x + 3y - z = 5$

$4x + 4y - 3z = 3$

$2x - 3y + 2z = 2$

Ans:- $x=1, y=2, z=3$

Q2. $x + 2y + 3z - u = 10$

$2x + 3y - 3z - u = 1$

$2x - y + 2z + 3u = 7$

$3x + 2y - 4z + 3u = 2$

Ans:- $x=1, y=2, z=2, u=1$

Q3. G.J

$5x - y + z = 10$

$2x+4y=12$

$x+y+5z=-1$

Start with the solution (2,3,0)

Ans:- $x=2.556, y=1.725, z = -1.055$

Using Gauss-Seidal Method

Q1.

$$8x - 3y + 2z = 20$$
$$6x + 3y + 12z = 35$$
$$4x + 11y - z = 33$$

Ans:

$x=3.0167$

$y=1.9858$

$z=0.9118$

Q2.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$