VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.
  - The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

The axioms need to be satisfied to be a **vector space:**

- Commutivity:
  X+Y=Y+X
- Associativity:
  (X+Y)+Z=X+(Y+Z)
- Existence of negativity:
  X+(-X)=0
- Existence of Zero:
  X+0=X

- Associativity of Scalar multiplication:
  (ab)u=a(bu)
- Right hand distributive:
  k(u+v)=ku+kv
- Left hand distributive:
  (a+b)u=au+bu
- Law of Identity:
  1.u=u

Subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

Conditions

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Ex: The set of singular matrices is not a subspace of $M_{2\times2}$

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$$\therefore W_2 \text{ is not a subspace of } M_{2\times2}$$

Linear Dependence

Let a set of vectors S in a vector space V

$$S=\{v_1,v_2,\ldots,v_k\}$$
$$c_1v_1+c_2v_2+\ldots+c_kv_k=0$$

If the equations has only the trivial solution (i.e. not all zeros) then S is called linearly dependent

Example

Let

a = [1 2 3] **b** = [4 5 6] c= [5 7 9]

Vector **c** is a linear combination of vectors **a** and **b**, because **c** = **a** + **b**. Therefore, vectors **a**, **b**, and **c** is linearly dependent.

Let a set of vectors S in a vector space V

$$S=\{v_1,v_2,\ldots,v_k\}$$
$$c_1v_1+c_2v_2+\ldots+c_kv_k=0$$

If the equations has only the trivial solution ($c_1 = c_2 = \ldots = c_k = 0$) then S is called linearly independent

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Basis

A set of vectors in a vector space V is called a basis if the vectors are linearly independent and every vector in the vector space is a linear combination of this set.

Condition

Let B denotes a subset of a vector space V. Then, B is a basis if and only if

1. B is a minimal generating set of V

2. B is a maximal set of linearly independent vectors.

6-5. VECTOR SPACE ASSOCIATED WITH A GRAPH

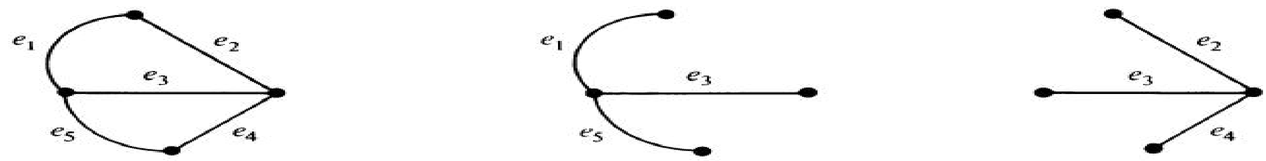
Let us consider the graph G in Fig. 6-5 with four vertices and five edges e_1, e_2, e_3, e_4, e_5. Any subset of these five edges (i.e., any subgraph g) of G can be represented by a 5-tuple:

$$\mathbf{X} = (x_1, x_2, x_3, x_4, x_5)$$

such that

$$x_i = 1 \qquad \text{if } e_i \text{ is in } g \text{ and}$$
$$x_i = 0 \qquad \text{if } e_i \text{ is } \textit{not} \text{ in } g.$$

For instance, the subgraph g_1 in Fig. 6-5 will be represented by (1, 0, 1, 0, 1).



Altogether there are 2^5 or 32 such 5-tuples possible, including the zero vector $\mathbf{0} = (0, 0, 0, 0, 0)$, which represents a null graph,† and (1, 1, 1, 1, 1), which is G itself.

It is not difficult to see that the ring-sum operation between two subgraphs corresponds to the modulo 2 addition between the two 5-tuples representing the two subgraphs. For example, consider two subgraphs

$$g_1 = \{e_1, e_3, e_5\} \qquad \text{represented by } (1, 0, 1, 0, 1), \text{ and}$$
$$g_2 = \{e_2, e_3, e_4\} \qquad \text{represented by } (0, 1, 1, 1, 0).$$

The ring sum

$$g_1 \oplus g_2 = \{e_1, e_2, e_4, e_5\} \qquad \text{represented by } (1, 1, 0, 1, 1),$$

6-6. BASIS VECTORS OF A GRAPH

Linear Dependence: A set of vectors $X_1, X_2, \ldots, X_r$ (over some field F) is said to be *linearly independent* if for scalars $c_1, c_2, \ldots, c_r$ in F the expression

$$c_1X_1 + c_2X_2 + \cdots + c_rX_r = 0$$

holds only if $c_1 = c_2 = \cdots = c_r = 0$. Otherwise, the set of vectors is said to be *linearly dependent*. For example, consider the set of three vectors, over the field of real numbers:

$$X_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

An arbitrary *linear combination* of these three vectors set to zero gives

$$c_1X_1 + c_2X_2 + c_3X_3 = \begin{pmatrix} c_1 \\ 4c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ 2c_2 \end{pmatrix} + \begin{pmatrix} 3c_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_3 \\ 4c_1 + c_2 \\ 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is, $2c_2 = 0$, $4c_1 + c_2 = 0$, and $c_1 + 3c_3 = 0$, which hold only if $c_1 = c_2 = c_3 = 0$. Thus the set of vectors $\{X_1, X_2, X_3\}$ is linearly independent.

On the other hand, consider another set of vectors (over the same field of real numbers):

$$X_4 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \qquad X_5 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad X_6 = \begin{pmatrix} .5 \\ 0 \\ 1 \end{pmatrix}.$$

Setting an arbitrary linear combination of these vectors to zero,

$$c_4X_4 + c_5X_5 + c_6X_6 = \begin{pmatrix} c_5 + .5c_6 \\ 2c_4 + 2c_5 \\ -2c_4 + c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

gives $c_4 = -c_5 = .5c_6 = \alpha$, where α can be any real number not necessarily zero. Therefore, the set $\{X_4, X_5, X_6\}$ is linearly dependent.

6-8. ORTHOGONAL VECTORS AND SPACES

Consider two vectors (4, 2) and (−3, 6) in a plane (which is also called a two-dimensional Euclidean space E_2), as shown in Fig. 6-8. These vectors are orthogonal because their dot product $4 \cdot (-3) + 2 \cdot 6 = 0$. Generalizing this notion to a k-dimensional vector space, we have the following definitions:

Dot Product: The *dot product* of two vectors X and Y in a vector space W is a scalar quantity defined as

$$\begin{aligned} X \cdot Y &= (x_1, x_2, \ldots, x_k) \cdot (y_1, y_2, \ldots, y_k) \\ &= x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_k \cdot y_k. \end{aligned}$$

Orthogonal Vectors: Two vectors are called orthogonal if their dot product is zero; and two subspaces are said to be *orthogonal to* each other if every vector in one is orthogonal to every vector in the other.

Returning to the vector space associated with a graph G, the dot product of two vectors, each representing a subgraph of G, is the modulo 2 sum of the products of the corresponding entries in the two vectors. For example, the dot product of the vectors representing subgraphs g_1 and g_2 in Fig. 6-5 is

$$\begin{aligned}(1, 0, 1, 0, 1)\cdot(0, 1, 1, 1, 0) &= 1\cdot 0 + 0\cdot 1 + 1\cdot 1 + 0\cdot 1 \\ &\quad + 1\cdot 0 \qquad (\text{mod } 2 \text{ sum}) \\ &= 0 + 0 + 1 + 0 + 0 \\ &= 1.\end{aligned}$$

10.4 Cut-Set Matrix

Let G be a graph with m edges and q cutsets. The cut-set matrix $C = [c_{ij}]_{q\times m}$ of G is a (0, 1)-matrix with

$$c_{ij} = \begin{cases} 1, & \text{if ith cutset contains jth edge}, \\ 0, & \text{otherwise}. \end{cases}$$

Example Consider the graphs shown in Figure 10. 7.

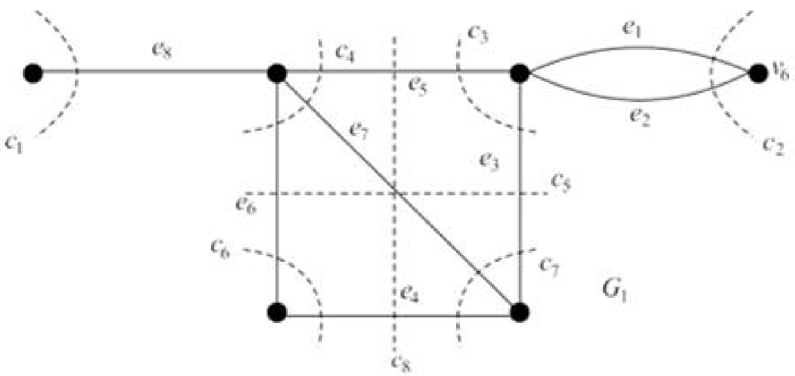


Fig. 10.7(a)

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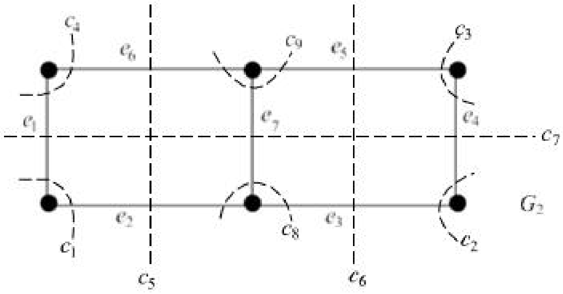

Fig. 10.7(b)

In the graph G_1, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

The cut-sets are $c_1 = \{e_8\}$, $c_2 = \{e_1, e_2\}$, $c_3 = \{e_3, e_5\}$, $c_4 = \{e_5, e_6, e_7\}$, $c_5 = \{e_3, e_6, e_7\}$, $c_6 = \{e_4, e_6\}$, $c_7 = \{e_3, e_4, e_7\}$ and $c_8 = \{e_4, e_5, e_7\}$.

The cut-sets for the graph G_2 are $c_1 = \{e_1, e_2\}$, $c_2 = \{e_3, e_4\}$, $c_3 = \{e_4, e_5\}$, $c_4 = \{e_1, e_6\}$, $c_5 = \{e_2, e_6\}$, $c_6 = \{e_3, e_5\}$, $c_7 = \{e_1, e_4, c_7\}$, $c_8 = \{e_2, e_3, e_7\}$ and $c_9 = \{e_5, e_6, e_7\}$.

Thus the cut-set matrices are given by

$$C(G_1) = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \; c \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ c_4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ c_5 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ c_6 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ c_7 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ c_8 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}, \text{ and}$$

$$C(G_2) = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline c_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ c_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ c_5 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ c_6 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ c_7 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ c_8 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ c_9 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}.$$

We have the following observations about the cut-set matrix $C(G)$ of a graph G.

1. The permutation of rows or columns in a cut-set matrix corresponds simply to re-naming of the cut-sets and edges respectively.
2. Each row in $C(G)$ is a cut-set vector.
3. A column with all zeros corresponds to an edge forming a self-loop.
4. Parallel edges form identical columns in the cut-set matrix.
5. In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. That is, for a non-separable graph G, $C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. For example, in the graph G_1 of Figure 10.7, the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of C, left after deleting rows c_1, c_2, c_5, c_8 and columns e_1, e_2, e_8.
6. It follows from observation 5, that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n - 1$.

The following result for connected graphs shows that cutset matrix, incidence matrix and the corresponding graph matrix have the same rank.

10.7 Path Matrix

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q\times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & if\, jth\ edge\ lies\ in\ the\ ith\ path, \\ 0, & otherwise. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

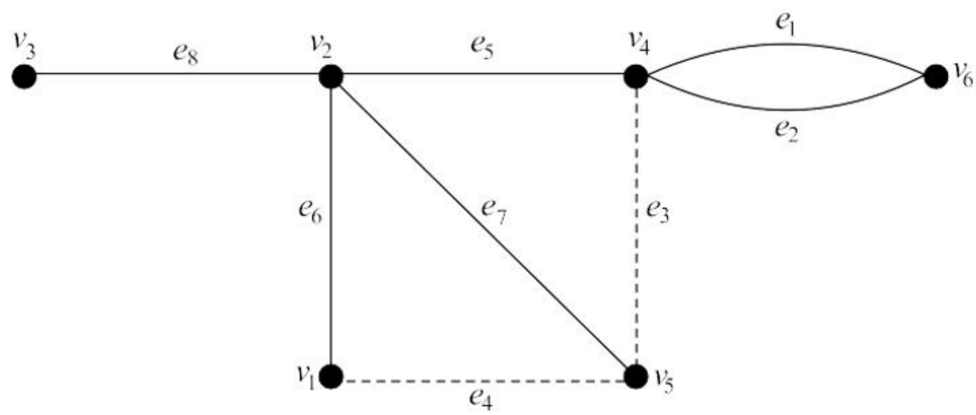

Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\},\ p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}.$$

We have the following observations about the path matrix.

1. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
2. A column of all ones corresponds to an edge that lies in every path between u and v.
3. There is no row with all zeros.
4. The ring sum of any two rows in $P(u, v)$ corresponds to a cycle or an edge-disjoint union of cycles.

The next result gives a relation between incidence and path matrix of a graph.