Numerical Differentiation

The method of obtaining the derivatives of a function using a numerical technique is known as numerical differentiation.

The choice of the formula is the same as incase of interpolation problems.

Numerical Differentiation

FORMULAE FOR DERIVATIVES

(1) **Newton's forward difference interpolation formula is**

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad (1)$$

where $$u = \frac{x-a}{h} \quad (2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots \quad (3)$$

Differentiating eqn. (2) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \quad (4)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\right] \quad (5)$$

Expression (5) provides the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Formula (5) becomes simple for tabulated values of x, in particular when $x = a$ and $u = 0$

Putting $u = 0$ in (5), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\right]} \qquad (6)$$

Differentiating eqn. (5) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right] \qquad (7)$$

Putting $u = 0$ in (7), we get

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right)} \qquad (8)$$

Newton's backward difference interpolation formula is

$$y = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \tag{10}$$

where $u = \dfrac{x - x_n}{h}$ (11)

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right)}$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right)}$$

Similarly, we get

$$\boxed{\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right)}$$

and so on.

Stirling's central difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \dots \tag{19}$$

where $$u = \frac{x-a}{h} \tag{20}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) - \dots\right]}$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \dots\right)}$$

Bessel's central difference interpolation formula is

$$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{(u+1)\,u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$+ \frac{(u+1)\,u(u-1)(u-2)\left(u - \frac{1}{2}\right)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(u+2)(u+1)\,u(u-1)(u-2)(u-3)}{6!}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots$$

where $u = \dfrac{x-a}{h}$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right]}$$

For unequally spaced values of the argument

(*i*) **Newton's divided difference formula is**

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)$$

$$(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)\,(x - x_2)(x - x_3)\,\Delta^4 f(x_0) + \ldots \quad (34)$$

$f'(x)$ is given by

$$f'(x) = \Delta f(x_0) + \{2x - (x_0 + x_1)\}\,\Delta^2 f(x_0) + \{3x^2 - 2x(x_0 + x_1 + x_2)$$

$$+ (x_0 x_1 + x_1 x_2 + x_2 x_0)\}\,\Delta^3 f(x_0) + \ldots \quad (35)$$

Example 2. *The table given below reveals the velocity 'v' of a body during the time 't' specified. Find its acceleration at t = 1.1.*

t:	*1.0*	*1.1*	*1.2*	*1.3*	*1.4*
v:	*43.1*	*47.7*	*52.1*	*56.4*	*60.8.*

Sol. The difference table is:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
1.1	47.7		– 0.2		
		$\boxed{4.4}$		0.1	
1.2	52.1		$\boxed{-0.1}$		0.1
		4.3		$\boxed{0.2}$	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

Let $\qquad a = 1.1,$

$\therefore \qquad v_0 = 47.7$ and $h = 0.1$

Acceleration at $t = 1.1$ is given by

$$\left[\frac{dv}{dt}\right]_{t=1.1} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right] = \frac{1}{0.1}\left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2)\right]$$

$$= 45.1667$$

Hence the required acceleration is **45.1667.**

Example 4. *The distance covered by an athlete for the 50 meter race is given in the following table:*

Time (sec):	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Distance (meter):	*0*	*2.5*	*8.5*	*15.5*	*24.5*	*36.5*	*50*

Determine the speed of the athlete at $t = 5$ sec., correct to two decimals.

t	s	∇s	$\nabla^2 s$	$\nabla^3 s$	$\nabla^4 s$	$\nabla^5 s$	$\nabla^6 s$
0	0						
		2.5					
1	2.5		3.5				
		6		– 2.5			
2	8.5		1		3.5		
		7		1		$\boxed{-3.5}$	
3	15.5		2		$\boxed{0}$		1
		9		$\boxed{1}$		– 2.5	
4	24.5		$\boxed{3}$		– 2.5		
		$\boxed{12}$		– 1.5			
5	36.5		1.5				
		13.5					
6	50						

Here we use Newton's Backward difference Formula for first derivative

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right)}$$

The speed of the athlete at $t = 5$ sec is given by

$$\left(\frac{ds}{dt}\right)_{t=5} = \frac{1}{h}\left[\nabla s_5 + \frac{1}{2}\nabla^2 s_5 + \frac{1}{3}\nabla^3 s_5 + \frac{1}{4}\nabla^4 s_5 + \frac{1}{5}\nabla^5 s_5\right]$$

$$= \frac{1}{1}\left[12 + \frac{1}{2}(3) + \frac{1}{3}(1) + \frac{1}{4}(0) + \frac{1}{5}(-3.5)\right]$$

$$= 13.1333 \approx 13.13 \text{ metre/sec.}$$

Example 5. *Find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $x = 6$*, given that*

x:	4.5	5.0	5.5	6.0	6.5	7.0	7.5
y:	9.69	12.90	16.71	21.18	26.37	32.34	39.15.

Sol. Here $a = 6.0 \quad \therefore \quad y_0 = 21.18$ and $h = 0.5$

The forward difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4.5	9.69				
		3.21			
5.0	12.9		0.60		
		3.81		0.06	
5.5	16.71		0.66		0
		4.47		0.06	
6.0	21.18		0.72		0
		$\boxed{5.19}$		0.06	
6.5	26.37		$\boxed{0.78}$		0
		5.97		$\boxed{0.06}$	
7.0	32.34		0.84		
		6.81			
7.5	39.15				

We know that

$$\left[\frac{dy}{dx}\right]_{x=6} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right)$$

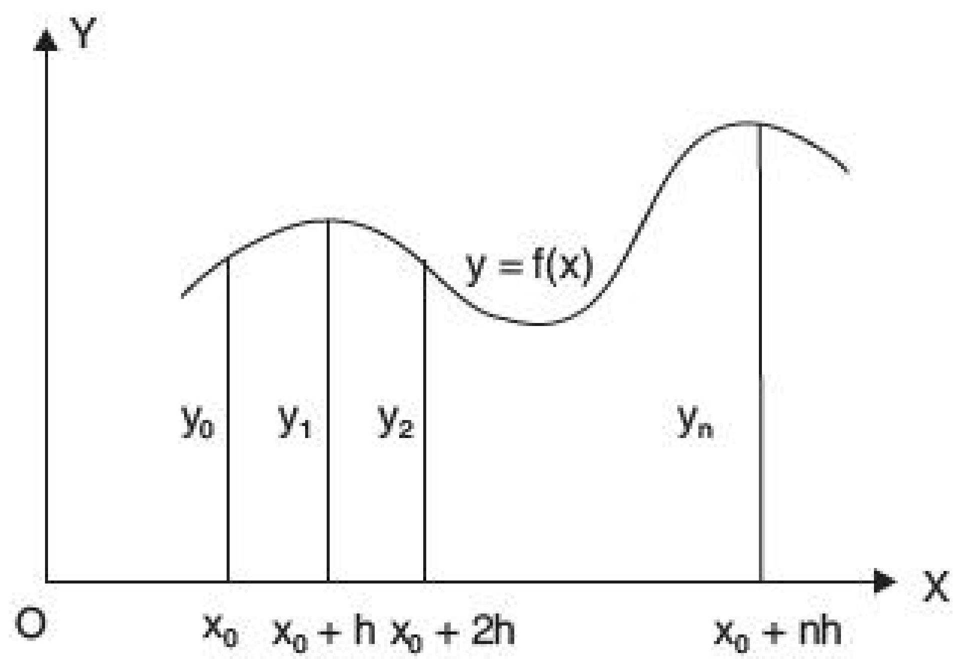
$$= \frac{1}{0.5}\left[5.19 - \frac{1}{2}(0.78) + \frac{1}{3}(0.06)\right] = 9.64$$

and $$\left[\frac{d^2y}{dx^2}\right]_{x=6} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right]$$

$$= \frac{1}{0.25}\ [0.78 - 0.06] = 4(0.72) = 2.88.$$

NUMERICAL INTEGRATION

Given a set of tabulated values of the integrand $f(x)$, determining the value of $\int_{x_0}^{x_n} f(x)\, dx$ is called numerical integration. The given interval of integration is subdivided into a large number of subintervals of equal width h and the function tabulated at the points of subdivision is replaced by any one of the interpolating polynomials like Newton-Gregory's, Stirling's, Bessel's over each of the subintervals and the integral is evaluated.



NEWTON-COTE'S QUADRATURE FORMULA

Let $I = \int_a^b y\,dx$, where y takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \dfrac{b-a}{n}$ so that

$$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \ldots\ldots,\ x_n = x_0 + nh = b.$$

$$\therefore \qquad I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Since any x is given by $x = x_0 + rh$ and $dx = h\,dr$

$$\therefore \qquad I = h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] dr$$

[by Newton's forward interpolation formula]

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 \right.$$

$$\left. + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \ldots\ldots\right]_0^n$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\right] \quad (49)$$

This is a **general quadrature formula** and is known as **Newton-Cote's quadrature formula.** A number of important deductions *viz.* Trapezoidal rule, Simpson's one-third and three-eighth rules, Weddle's rule can be immediately deduced by putting $n = 1, 2, 3,$ and 6, respectively, in formula (49).

$$I = \int_{x_0}^{x_0 + nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\right]$$

TRAPEZOIDAL RULE (n = 1)

Putting $n = 1$ in formula (49) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of an order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}\left[2y_0 + (y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly, for the next sub-interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2), \ldots\ldots, \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})\right]}$$

which is known as **Trapezoidal rule.** By increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

SIMPSON'S ONE-THIRD RULE (n = 2)

Putting $n = 2$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4), \;\ldots\ldots,$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]}$$

which is known as **Simpson's one-third rule.**

While using this formula, the given interval of integration must be divided into an even number of sub-intervals

SIMPSON'S THREE-EIGHTH RULE (n = 3)

Putting $n = 3$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3) as a polynomial of degree three so that differences of order higher than three vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\,[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}\,[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_3 + 3y_4 + 3y_5 + y_6], \ldots$$

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]}$$

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

Example 1. *Use Trapezoidal rule to evaluate* $\int_0^1 x^3\,dx$ *considering five sub-intervals.*

Sol. Dividing the interval $(0, 1)$ into 5 equal parts, each of width $h = \frac{1-0}{5}$ $= 0.2$, the values of $f(x) = x^3$ are given below:

x:	0	0.2	0.4	0.6	0.8	1.0
$f(x)$:	0	0.008	0.064	0.216	0.512	1.000
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule, we have

$$\int_0^1 x^3\,dx = \frac{h}{2}\,[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}\,[(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$$

$$= 0.1 \times 2.6 = 0.26.$$

Example 2. *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

Hence compute an approximate value of π in each case.

(*i*) *Simpson's* $\frac{1}{3}$ *rule taking* $h = \frac{1}{4}$

(*ii*) *Simpson's* $\frac{3}{8}$ *rule taking* $h = \frac{1}{6}$

Sol. (*i*) The values of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$ are given below:

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	1	$\frac{16}{17}$	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{12}\left[(1+0.5)+4\left\{\frac{16}{17}+.64\right\}+2(0.8)\right] = 0.785392156$$

Also $$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore \qquad \frac{\pi}{4} \simeq 0.785392156 \quad \Rightarrow \quad \pi \simeq 3.1415686$$

(ii) The values of $f(x) = \dfrac{1}{1+x^2}$ at $x = 0, \dfrac{1}{6}, \dfrac{2}{6}, \dfrac{3}{6}, \dfrac{4}{6}, \dfrac{5}{6}, 1$ are given below:

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$:	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\ [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right) + 3\left\{\frac{36}{37} + \frac{9}{10} + \frac{9}{13} + \frac{36}{61}\right\} + 2\left(\frac{4}{5}\right)\right]$$

$$= 0.785395862$$

Also, $$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

$$\therefore \qquad \frac{\pi}{4} = 0.785395862$$

$$\Rightarrow \qquad \pi = 3.141583$$

Example 10. *Evaluate* $\int_1^2 e^{-\frac{1}{2}x}\,dx$ *using four intervals.*

Sol. The table of values is:

x:	1	1.25	1.5	1.75	2
$y = e^{-x/2}$:	.60653	.53526	.47237	.41686	.36788
	y_0	y_1	y_2	y_3	y_4

Since we have four (even) subintervals here, we will use Simpson's $\frac{1}{3}$rd rule.

$$\therefore \qquad \int_1^2 e^{-\frac{1}{2}x}\,dx = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{.25}{3}\,[(.60653 + .36788) + 4(.53526) + .41686) + 2(.47237)]$$

$$= 0.4773025.$$

Example 5. *Evaluate* $\int_{0.6}^{2} y\,dx$, *where y is given by the following table:*

x:	*0.6*	*0.8*	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*
y:	*1.23*	*1.58*	*2.03*	*4.32*	*6.25*	*8.36*	*10.23*	*12.45.*

Sol. Here the number of subintervals is 7, which is neither even nor a multiple of 3. Also, this number is neither a multiple of 4 nor a multiple of 6, hence using Trapezoidal rule, we get

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\ [(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{0.2}{2}\ [(1.23 + 12.45) + 2(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)]$$

| Here $h = 0.2$

$$= 7.922.$$

Example 6. *Find* $\int_1^{11} f(x)\,dx$, *where f(x) is given by the following table, using a suitable integration formula.*

x:	1	2	3	4	5	6	7	8	9	10	11
f(x):	543	512	501	489	453	400	352	310	250	172	95

Sol. Since the number of subintervals is 10 (even) hence we shall use Simpson's $\frac{1}{3}$rd rule.

$$\int_1^{11} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{1}{3}\,[(543 + 95) + 4(512 + 489 + 400 + 310 + 172) + 2(501 + 453 + 352 + 250)]$$

$$= \frac{1}{3}\,[638 + 7532 + 3112] = 3760.67.$$

BOOLE'S RULE

Putting $n = 4$ in formula and neglecting all differences of order higher than four, we get

$$\int_{x_0}^{x_0+4h} f(x)\,dx = h\int_0^4 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0\right] dr$$

| By Newton's forward interpolation formula

$$= 4h\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!}\right]_0^4$$

$$= 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{3}{2}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right]$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly, $$\int_{x_0 + 4h}^{x_0 + 8h} f(x)\, dx = \frac{2h}{45} (7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\boxed{\int_{x_0}^{x_0 + nh} f(x)\, dx = \frac{2h}{45} [7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32\, y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots\ldots]}$$

This is known as **Boole's rule.**

While applying Boole's rule, the **number of sub-intervals should be taken as a multiple of 4.**

Example 13. *Evaluate* $\int_0^4 \frac{dx}{1+x^2}$ *using Boole's rule taking*

(*i*) $h = 1$ (*ii*) $h = 0.5$

Compare the results with the actual value and indicate the error in both.

Sol. (*i*) Dividing the given interval into 4 equal subintervals (*i.e.*, $h = 1$), the table is as follows:

x:	0	1	2	3	4
y:	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4

using Boole's rule,

$$\int_0^4 y\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2(1)}{45}\left[7(1) + 32\left(\frac{1}{2}\right) + 12\left(\frac{1}{5}\right) + 32\left(\frac{1}{10}\right) + 7\left(\frac{1}{17}\right)\right]$$

$$= 1.289412 \text{ (approx.)}$$

$$\therefore \qquad \int_0^4 \frac{dx}{1+x^2} = 1.289412.$$

(*ii*) Dividing the given interval into 8 equal subintervals (*i.e.*, $h = 0.5$), the table is as follows:

x:	0	.5	1	1.5	2	2.5	3	3.5	4
y:	1	0.8	0.5	$\frac{4}{13}$	.2	$\frac{4}{29}$	.1	$\frac{4}{53}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

using Boole's rule,

$$\int_0^4 y\,dx = \frac{2h}{45}[7(y_0)+32(y_1)+12(y_2)+32(y_3)+7(y_4)$$

$$+\ 7(y_4)+32(y_5)+12(y_6)+32(y_7)+7(y_8)]$$

$$= \frac{1}{45}\left[7(1)+32(.8)+12(.5)+32\left(\frac{4}{13}\right)+7(.2)+7(.2)\right.$$

$$\left.+32\left(\frac{4}{29}\right)+12(.1)+32\left(\frac{4}{53}\right)+7\left(\frac{1}{17}\right)\right]$$

$$= 1.326373$$

$$\therefore \qquad \int_0^4 \frac{dx}{1+x^2} = 1.326373$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^4 = \tan^{-1}(4) = 1.325818$$

$$\text{Error in result I} = \left(\frac{1.325818 - 1.289412}{1.325818}\right) \times 100 = 2.746\%$$

$$\text{Error in result II} = \left(\frac{1.325818 - 1.326373}{1.325818}\right) \times 100 = -0.0419\%.$$

Example 14. *A river is 80 m wide. The depth 'y' of the river at a distance 'x' from one bank is given by the following table:*

x:	0	10	20	30	40	50	60	70	80
y:	0	4	7	9	12	15	14	8	3

Find the approximate area of cross-section of the river using

(i) Boole's rule.

Sol. The required area of the cross-section of the river

$$= \int_0^{80} y\,dx$$

Here the number of sub intervals is 8.

(i) By Boole's rule,

$$\int_0^{80} y\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 + 7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{2\,(10)}{45}\,[7(0) + 32(4) + 12(7) + 32(9) + 7(12) + 7(12) + 32(15) + 12(14) + 32(8) + 7(3)]$$

$$= 708$$

Hence the required area of the cross-section of the river = 708 sq. m.

(ii) By Simpson's $\frac{1}{3}$rd rule

$$\int_0^{80} y\,dx = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}[(0 + 3) + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14)]$$

$$= 710$$

Example *Find the solution of* $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$ *which passes through (0, 1) in the interval (0, 0.5) such that the value of y is correct to three decimal places (use the whole interval as one interval only). Take h = 0.1.*

Sol. The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = 1 + xy; \quad y(0) = 1$$

i.e., $$y = y_0 = 1 \quad \text{at} \quad x = x_0 = 0$$

Here, $$y^{(1)} = 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y^{(3)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y^{(4)} = y^{(3)} + \frac{x^7}{105} + \frac{x^8}{384}$$

when $x = 0$, $y = 1.000$

$x = 0.1$, $y^{(1)} = 1.105$, $y^{(2)} = 1.1053$ ....

$\therefore$ $y = 1.105$ (correct up to 3 decimals)

$x = 0.2$, $y^{(1)} = 1.220$, $y^{(2)} = 1.223 = y^{(3)}$

$$\therefore \qquad y = 1.223 \qquad \text{(correct up to 3 decimals)}$$

$$x = 0.3, \quad y = 1.355 \quad \text{as} \quad y^{(2)} = 1.355 = y^{(3)}$$

$$x = 0.4, \quad y = 1.505 \qquad \text{(similarly)}$$

$$x = 0.5, \quad y = 1.677 \quad \text{as} \quad y^{(4)} = y^{(3)} = 1.677$$

Thus,

x	0	0.1	0.2	0.3	0.4	0.5
y	1.000	1.105	1.223	1.355	1.505	1.677

We have numerically solved the given differential eqn. for $x = 0$, .1, .2, .3, .4, and .5.