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DIGRAPH DEFINITION

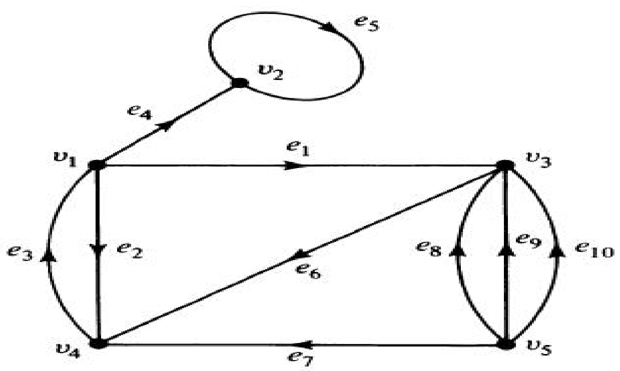
GT-5

A digraph D consists of a finite set V of points and a collection of ordered pairs of distinct points.

A directed graph or a digraph G consists of a set of vertices V = {v1, v2, …..}, a set of edge E = {e1, e2, …..} and a mapping ψ that maps every edge onto some ordered pair of vertices (vi, vj).

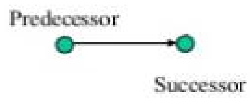
Loop and multiple edges in directed graph 1.4.3

- In a graph, a ***loop*** is an edge whose endpoints are equal.
- ***Multiple edges*** are edges having the same ordered pair of endpoints.
- A digraph is ***simple*** if each ordered pair is the head and tail of the most one edge; one loop may be present at each vertex.



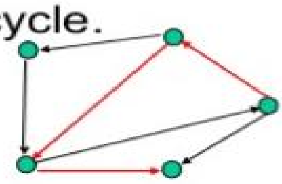
Loop and multiple edges in directed graph 1.4.3

- In the simple digraph, we write uv for an edge with tail u and head v.
  - If there is an edge form u to v, then v is a ***successor*** of u, and u is a ***predecessor*** of v.
  - We write $u \rightarrow v$ for "there is an edge from u to v".



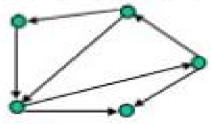
Path and Cycle in Digraph 1.4.6

- A digraph is a *path* if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering.
- A *cycle* is defined similarly using an ordering of the vertices on the cycle.

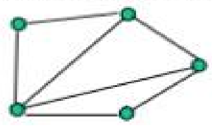


Underlying graph 1.4.9

- The *underlying graph* of a digraph D:
  - the graph G obtained by treating the edges of D as unordered pairs;
  - the vertex set and edges set remain the same, and the endpoints of an edge are the same in G as in D,
  - but in G they become an unordered pair.



A digraph



The underlying Graph

- A **digraph** is a graph whose edges are all directed
- Applications

  one-way streets
  flights
  task scheduling

IN-DEGREE AND OUT-DEGREE:-

If v is a vertex of a digraph D, the number of edges incident out of v is called the out-degree of v and the number of edges incident into v is called the in-degree of v. The out-degree of v is denoted by $d+(v)$ and the in-degree of v is denoted by $d-(v)$.

ISOLATED VERTEX

If v is a vertex of a digraph D then v is called an isolated vertex of D if $d+(v) = d-(v) = 0$.

PENDANT VERTEX

If v is a vertex of a digraph D then v is called a pendant vertex of D if $d+(v) + d-(v) = 1$.

SOURCE

If v is a vertex of a digraph D then v is called a source of D if $d-(v) = 0$.

SINK

If v is a vertex of a digraph D then v is called a sink of D if $d+(v) = 0$.

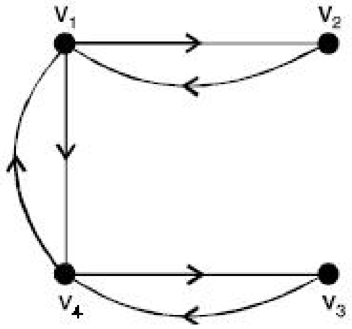
Some types of directed graph

- Simple directed graph
- Symmetric digraph
- Asymmetric or Anti symmetric
- Complete digraph
- Pseudo symmetric digraph

Simple directed graph

Simple directed graphs are directed graphs that have no loops (arrows that connect vertices to themselves) and no multiple arrows with same source and target nodes. As already introduced, in case of multiple arrows the entity is usually addressed as directed multigraph. Some authors describe digraphs with loops as loop-digraphs.
Simple Digraphs A digraphs that has no self-loop or parallel edges is called a simple digraph.

Symmetric directed graphs are directed graphs where all edges are bidirected (that is, for every arrow that belongs to the digraph, the corresponding inversed arrow also belongs to it).



Anti symmetric

- Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or anti symmetric.
- Balanced Digraphs
- A digraph D is said to be a balanced digraph or an isograph if $d+(v) = d-(v)$ for every vertex v of D.

Regular Digraph

A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

Complete directed graphs

Complete directed graphs are simple directed graphs where each pair of vertices is joined by a symmetric pair of directed arrows (it is equivalent to an undirected complete graph with the edges replaced by pairs of inverse arrows). It follows that a complete digraph is symmetric.

CONNECTED DIGRAPHS

Strongly Connected

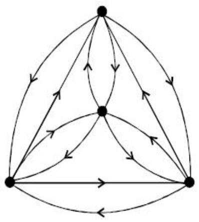
A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.

Weakly Connected

A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected.

A **complete symmetric digraph** is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex, and a complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

A **complete asymmetric digraph** of n vertices contains n(n - 1)/2 edges, but a complete symmetric digraph of n vertices contains n(n - 1) edges. A complete asymmetric digraph is also called a tournament or a complete tournament (the reason for this term will be made clear).

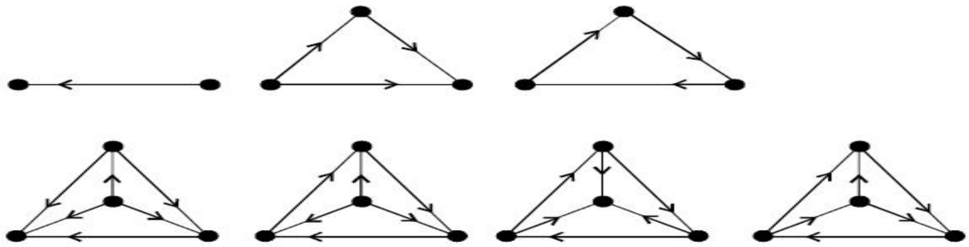


Pseudo symmetric digraph

If in-degree of a vertices in a digraph is equal to out-degree then digraph is said to be Pseudo symmetric **digraph or isograph or balanced .**

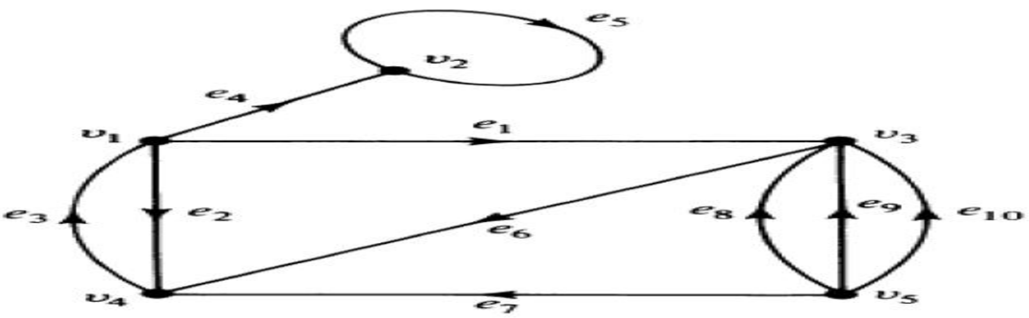
TOURNAMENTS:-

A tournament is an oriented complete graph.



DIRECTED PATHS

- A path in a directed graph is called Directed path.
- **Strongly connected digraph**: A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.
- **Weakly connected digraph:** A digraph G is said to be weakly connected if its corresponding undirected graph is connected. But G is not strongly connected.



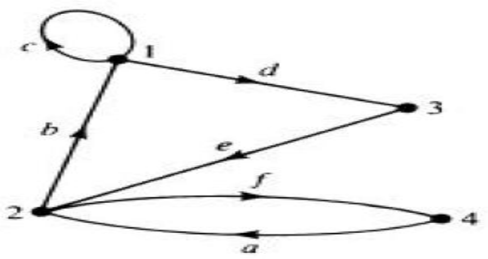
$v_5\ e_8\ v_3\ e_6\ v_4\ e_3\ v_1$ is a directed path from v_5 to v_1.
Whereas $v_5\ e_7\ v_4\ e_6\ v_3\ e_1\ v_1$ is a semi-path from v_5 to v.

Connectedness

- A digraph is said to be disconnected if it is not even weak. A digraph is said to be strictly weak if it is weak, but not unilateral.
- It is strictly unilateral, if it is unilateral but not strong. Two vertices of a digraph D are said to be
- 0-connected if there is no semi path joining them,
- 1-connected if there is a semi path joining them, but there is no u-v path or v-u path,
- 2-connected if there is a u-v or a v-u path, but not both
- 3-connected if there is u-v path and a v-u path.

EULER GRAPHS

In a digraph G, a closed directed walk which traverses every edge of G exactly once is called a directed Euler line. A digraph containing a directed Euler line is called an **Euler digraphs**



It contains directed Euler line **a b c d e f.**

HAND SHAKING DILEMMA

In a digraph D,the sum of the out-degree of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.

- **Directed walk**-A directed walk or a directed trail in D is a finite sequence whose terms are alternately vertices and edges in D such that each edge is incident out of the vertex preceeding it in the sequence and incident into the vertex following it.
- **Directed path**
- An open directed walk in which no vertex is repeated is called a directed path.
- **Directed circuit**
- A closed directed walk in which no vertices, except the initial and final vertices are repeated is called a directed circuit or a directed cycle.
- **Length**
- The number of edges present in a directed walk, directed path, directed circuit is called its length.
- **Semi-Walk**
- A semi-walk in a digraph D is a walk in the underlying graph of D, but is not a directed walk in D. A walk in D can mean either a directed walk or a semi-walk in D.
- **Semi-path**
- A semi-path in a digraph D is a path in the underlying graph of D, but is not a directed path in D. A path in D can mean either a directed path or a semi-path in D.
- **Semi-circuit**
- A semi-circuit in a digraph D is a circuit in the underlying graph of D, but is not a directed circuit in D. A circuit in D can mean either a directed circuit or a semi-circuit in D.

9-6. TREES WITH DIRECTED EDGES

A tree (for undirected graphs) was defined as a connected graph without any circuit. The basic concept as well as the term "tree" remains the same for digraphs. A *tree* is a connected digraph that has no circuit—neither a directed circuit nor a semicircuit. A tree of n vertices contains $n - 1$ directed edges and has properties similar to those with undirected edges. Trees with directed edges are of great importance in many applications, such as electrical network analysis, game theory, theory of languages, computer programming, and counting problems, to name a few.

In addition to being trees in the undirected sense, trees in digraphs have additional properties and variations resulting from the relative orientations of the edges. One such particularly useful type of rooted tree with directed edges is called an arborescence and is defined as follows:

Arborescence: A digraph G is said to be an arborescence if

1. G contains no circuit—neither directed nor semicircuit.
2. In G there is precisely one vertex v of zero in-degree.

ARBORESCENCE

- A digraph G is said to be an arborescence if
- (i) G contains no circuit, neither directed nor semi circuit.
- (ii) In G there is precisely one vertex v of zero in-degree.

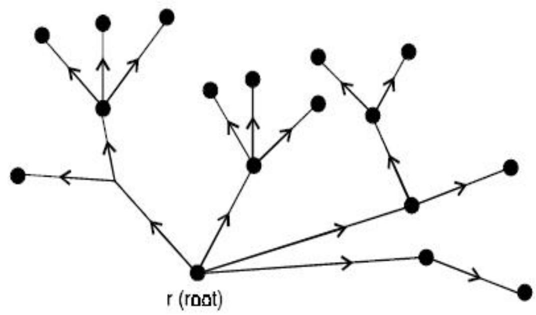


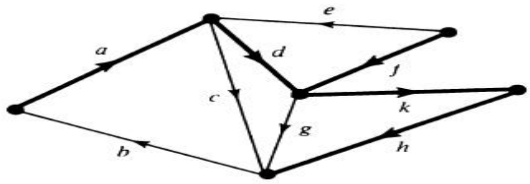
Fig. 8.13. Arborescence.

9-7. FUNDAMENTAL CIRCUITS IN DIGRAPHS

The edges of a connected digraph not included in a specified spanning tree T are also called chords with respect to T. Just as in the case of undirected graphs, every chord c_i when added to the spanning tree T produces a fundamental circuit (which may be a directed circuit or a semicircuit).

A cut-set in a connected digraph G (just as in an undirected graph) induces a partitioning of the vertices of G into two disjoint subsets V_1 and V_2 such that the cut-set consists of all those edges that have one end vertex in V_1 and the other in V_2. All edges in the cut-set may be directed from V_1 to V_2, or vice versa, or some edges may be directed from V_1 to V_2 and others from V_2 to V_1.†

The concepts of spanning trees, fundamental circuits, and fundamental cut-sets are illustrated in Fig. 9-15. A spanning tree is shown in heavy lines. Observe that some of the fundamental circuits are directed circuits and others are semicircuits. The five fundamental cut-sets, each corresponding to an edge in the spanning tree, are also shown.



Rank $r = 5$ Spanning tree T: $\{a, d, f, h, k\}$

Nullity $\mu = 4$ Chord-set with respect to T: $\{b, c, e, g\}$

Fundamental circuits with respect to T	$d\ f\ e$	(semicircuit)
	$d\ k\ h\ c$	(semicircuit)
	$k\ h\ g$	(semicircuit)
	$a\ d\ k\ h\ b$	(directed circuit)

Fundamental cut-sets with respect to T	$a\ b$
	$b\ c\ d\ e$
	$e\ f$
	$b\ c\ g\ k$
	$b\ c\ g\ h$

INCIDENCE MATRIX OF A DIGRAPH

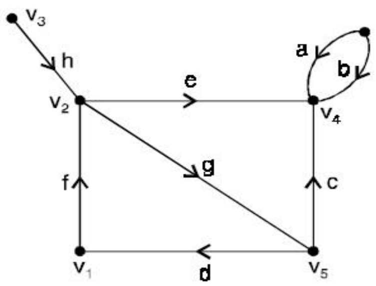
The incidence matrix of a digraph with n vertices, e edges and no self-loops in an n by n matrix $A = [a_{ij}]$ whose rows correspond to vertices and columns correspond to edges such that

$a_{ij} = 1$, if j^{th} edge is incident out of i^{th} vertex

$= -1$, if j^{th} edge is incident into i^{th} vertex

$= 0$, if j^{th} edge is not incident on i^{th} vertex.

For example, A digraph and its incidence matrix are shown in Fig. 8.17.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

CIRCUIT MATRIX OF A DIGRAPH

Let G be a digraph with e edges and q circuits. An arbitrary orientation is assigned to each of the q circuits. Then a circuit matrix B = [bij] of the digraph G is a q by e matrix defined as

$b_{ij} = 1$, if i^{th} circuit includes j^{th} edge, and the orientations of the edge and circuit coincide

$= -1$, if i^{th} circuit includes j^{th} edge, but the orientations of the two are opposite

$= 0$, if i^{th} circuit does not include the j^{th} edge.

For example, a circuit matrix of the digraph in Fig. 8.17 is

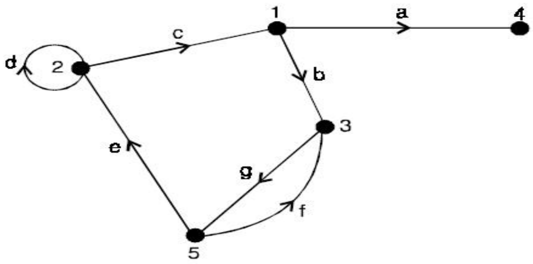
$$\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

ADJACENCY MATRIX OF A DIGRAPH :

Let G be a digraph with n vertices, containing no parallel edges. Then the adjacency matrix $X = [x_{ij}]$ of the digraph G is an n by n (0, 1) matrix whose element.

$$x_{ij} = 1, \text{ if there is an edge directed from } i^{\text{th}} \text{ vertex to } j^{\text{th}} \text{ vertex}$$
$$= 0, \text{ otherwise}$$

For example, a digraph and its adjacency matrix are shown in Fig. 8.18.



$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problem 8.2. *Find the in-degrees and out-degrees of the vertices of the digraphs shown in Fig. 8.22 below. Also, verify the handshaking dilemma.*

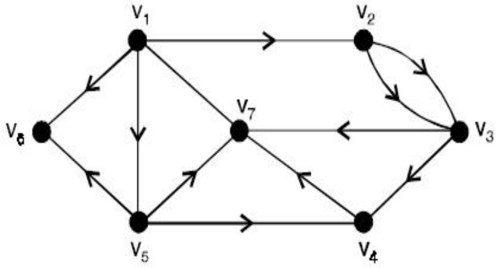


Fig. 8.22.

Solution. The given digraph has 7 vertices and 12 edges. The out-degree of a vertex is got by counting the number of edges that go out of the vertex and the in-degree of a vertex is got by counting the number of edges that end at the vertex. Thus, we obtain the following data.

Vertex	*Out-degree*	*In-degree*
v_1	4	0
v_2	2	1
v_3	2	2
v_4	1	2
v_5	3	1
v_6	0	2
v_7	0	4

This table gives the out-degrees and in-degrees of all vertices. We note that v_1 is a source and v_6 and v_7 are sinks.

Also, check that

$$\text{sum of out-degrees} = \text{sum of in-degrees}$$
$$= 12 = \text{No. of edges.}$$

Enumeration

- Cayley's (1857) classic paper, a great deal of work has been done on enumeration (also called counting) of different types of graphs.
- TYPES OF ENUMERATION
- **Type** 1. Counting the number of different graphs (or digraphs) of a particular kind.
- For **example**, all connected, simple graphs with eight vertices and two circuits.

- **Type** 2. Counting the number of subgraphs of a particular type in a given graph G, such as the number of edge-disjoint paths of length k between vertices a and b in G.

LABELED GRAPHS

All of the labeled graphs with three points are shown in Figure 6.1 below. We see that the 4 different graphs with 3 points become 8 different labeled graphs. To obtain the number of labeled graphs with P points, we need only observe that each of the $\binom{P}{2}$ possible lines is either present or absent.

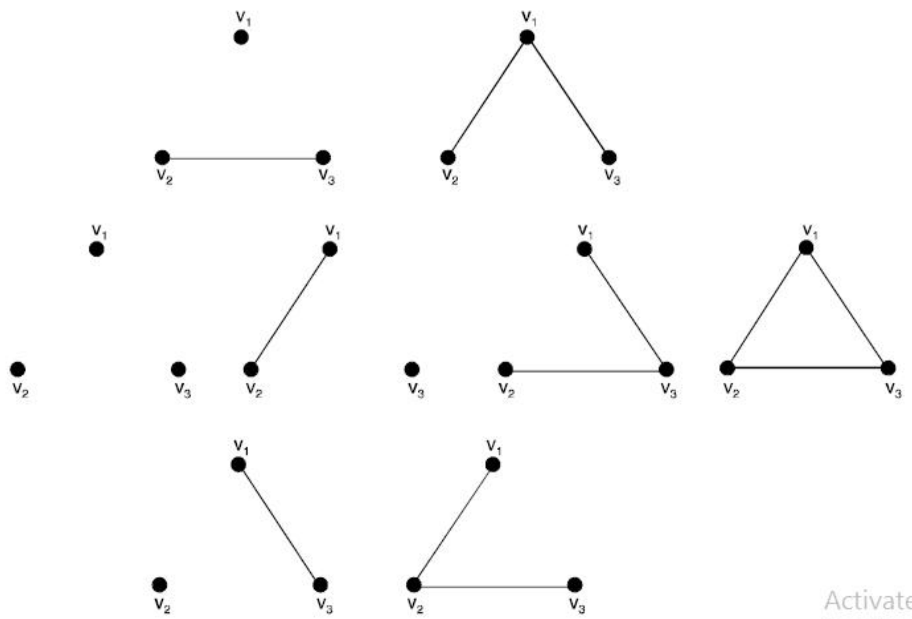


Fig. 6.1. The labeled graphs with 3 points.

COUNTING LABELED TREES

Expression $\binom{\frac{n(n-1)}{2}}{e}$ can be used to obtain the number of simple labeled graphs of n vertices and $n-1$ edges. Some of these are going to be trees and others will be unconnected graphs with circuits.

Problem 8.5. *Prove that a complete symmetric digraph of n vertices contains n(n – 1) edges and a complete asymmetric digraph of n vertices contains* $\frac{n(n-1)}{2}$ *edges.*

Solution. In a complete asymmetric digraph, there is exactly one edge between every pair of vertices.

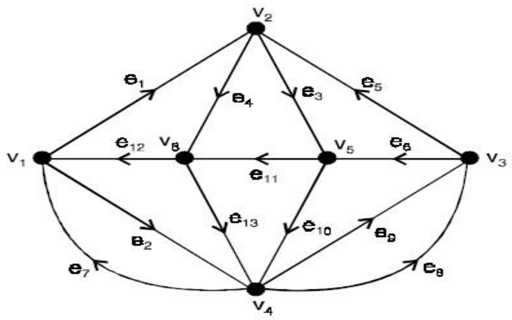
Therefore, the number of edges in such a digraph is precisely equal to the number of pairs of vertices. The number of pairs of vertices that can be chosen from n vertices is ${}^{n}C_2 = \frac{1}{2}n(n-1)$.

Thus, a complete asymmetric digraph with n vertices has exactly $\frac{1}{2}n(n-1)$ edges.

In a complete symmetric digraph there exist two edges with opposite directions between every pair of vertices.

Therefore, the number of edges in such a digraph with n vertices is $2 \times \frac{1}{2}n(n-1) = n(n-1)$.

Show that the digraph shown in Figure below, is an Euler digraph. Indicate a directed Euler line in it



Solution. By examining the given digraph, we find that

$$d^-(v_1) = 2 = d^+(v_1),\ d^-(v_2) = 2 = d^+(v_2)$$
$$d^-(v_3) = 2 = d^+(v_3),\ d^-(v_4) = 3 = d^+(v_3)$$
$$d^-(v_5) = 2 = d^+(v_5),\ d^-(v_6) = 2 = d^+(v_6).$$

Thus, for every vertex the in-degree is equal to the out-degree.

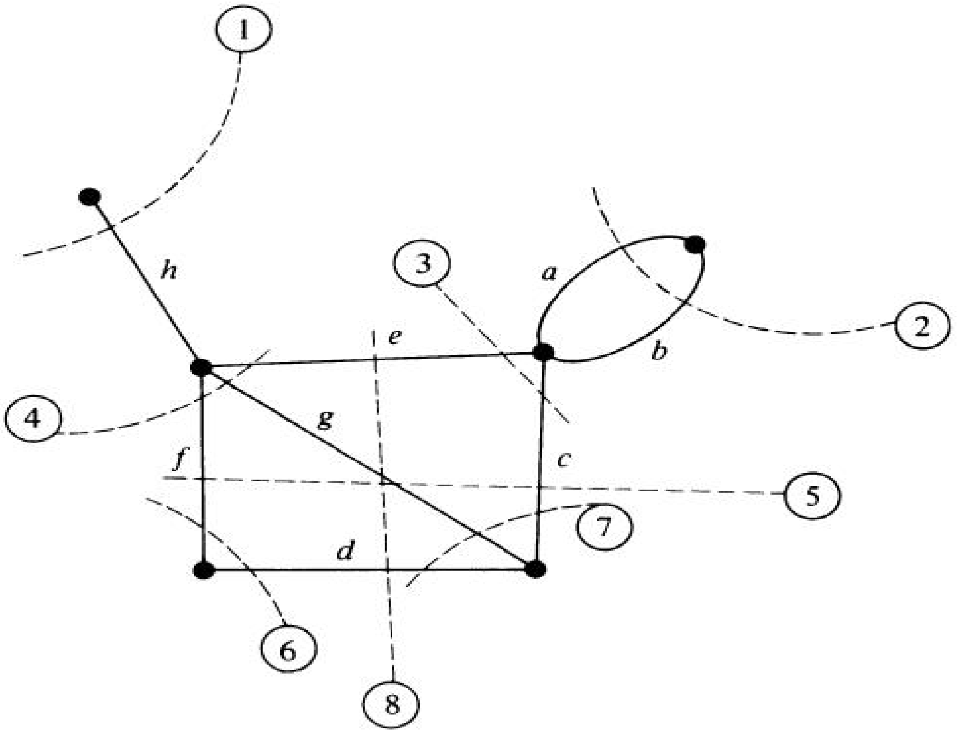
Therefore the digraph is an Euler digraph.

By starting at v_1, we can obtain the following closed directed walk that includes all the thirteen edges :

$v_1\, e_1\, v_2\, e_4\, v_6\, e_{12}\, v_1\, e_2\, v_4\, e_8\, v_3\, e_6\, v_5\, e_{11}\, v_6\, e_{13}\, v_4\, e_9\, v_3\, e_5\, v_2\, e_3\, v_5\, e_3\, v_5\, e_{10}\, v_4\, e_7\, v_1$

This is a directed Euler line in the given digraph.

Find cut set matrix



$$
C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$