Numerical Sol^n of Ordinary Differential Equations

Differential Equation: A physical situation that concerns with the rate of change of one quantity with respect to another gives rise to a differential equation.

Consider the first order ordinary differential eq^n

$$\frac{dy}{dx} = f(x,y) \quad \text{——— (1)}$$

with the initial condition

$$y(x_0) = y_0 \quad \text{——— (2)}$$

Scan or click here for more resources

MORE RESOURCES

Picard's Method :-

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{--- (1)}$$

with initial condition

$$y = y_0 \text{ at } x = x_0 \qquad \text{--- (2)}$$

The eqn (1) can be written as

$$dy = f(x, y)\, dx$$

Integrating between the limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad \text{--- (3)}$$

which is an integral eqn and can be solved by iteration. Now by Picard's method, for first approximation y_1, we replace y by y_0 in $f(x,y)$ in the R.H.S of eqn (3)

i.e. $$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \quad \text{--- (4)}$$

For second approximation y_2, we replace y by y_1 in $f(x,y)$ on the RHS of eqn (3)

~~Continue this process, we obtain~~

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \quad \text{--- (5)}$$

Continue this process, we obtain $y_3, y_4, \ldots y_{n-1}, y_n$

i.e. $$\boxed{y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx} \quad \text{with } y(x_0) = y_0$$

This process is to be stopped when the two values of y i.e. y_n and y_{n-1} are same to the desired degree of accuracy.

Ex1:- Using the Picard's method to obtain y for $x = 0.1$ to 0.3. Given the differential eqn

(3 decimal) $$\frac{dy}{dx} = 1 + xy \quad \text{with } y(0) = 1$$

Soln:- Here, $f(x,y) = 1 + xy$

and $x_0 = 0$ and $y_0 = 1$

First Approximation:-

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = 1 + \int_0^x (1 + xy_0)\, dx = 1 + \int_0^x (1 + x \cdot 1)\, dx$$

$$= 1 + \int_0^x (1 + x)\, dx$$

$$= 1 + x + \frac{x^2}{2} \qquad (1)$$

Second Approximation :-

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_2 = 1 + \int_0^x \{1 + x\left(1 + x + \frac{x^2}{2}\right) dx$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \quad \text{--- (2)}$$

Third Approximation :-

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= y_0 + \int_0^x (1 + x y_2)\,dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\} dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \quad \text{--- (3)}$$

Now at $x = 0.1$

$y_1 = 1.105$, $y_2 = 1.1053458$, $y_3 = 1.1053465$,

$\therefore \quad y = 1.105$

Now at $x = 0.2$

$y_1 = 1.22$, $y_2 = 1.2228667$, $y_3 = 1.2228894$

$\therefore \quad y = 1.223$

Now at $x = 0.3$

$y_1 = 1.345$, $y_2 = 1.35550125$, $y_3 = 1.3551897$

$\therefore \quad y = 1.355$

Now at $x = 0.4$

$y_1 = 1.48$, $y_2 = 1.5045333$, $y_3 = 1.5053013$, $y_4 = 1.5053186$

$\therefore \quad y = 1.505$

Now at $x = 0.5$

$y_1 = 1.625$, $y_2 = 1.6744792$, $y_3 = 1.6768881$,

$y = 1.67$

Example *Use Picard's method to obtain y for x = 0.1. Given that:*

$$\frac{dy}{dx} = 3x + y^2;\ y = 1 \text{ at } x = 0.$$

Sol. Here $f(x, y) = 3x + y^2,\ x_0 = 0,\ y_0 = 1$

First approximation, $$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx$$

$$= 1 + \int_0^x (3x + 1)\, dx$$

$$= 1 + x + \frac{3}{2}x^2$$

Second approximation, $$y^{(2)} = 1 + x + \frac{5}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5$$

Third approximation, $$y^{(3)} = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{1157}{1260}x^7 + \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

when $x = 0.1$, we have

$$y^{(1)} = 1.115,\quad y^{(2)} = 1.1264,\quad y^{(3)} = 1.12721$$

Thus, $y = 1.127$ when $x = 0.1$.

Example 4. *If* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$, *find the value of y at x = 0.1 using Picard's method. Given that* $y(0) = 1$.

Sol. First approximation,

$$y^{(1)} = y_0 + \int_0^x \frac{y_0 - x}{y_0 + x}\, dx = 1 + \int_0^x \left(\frac{1 - x}{1 + x}\right) dx$$

$$= 1 + \int_0^x \left(\frac{2}{1 + x} - 1\right) dx$$

$$= 1 - x + 2 \log (1 + x)$$

Second approximation,

$$y^{(2)} = 1 + x - 2 \int_0^x \frac{x\, dx}{1 + 2 \log (1 + x)}$$

which is difficult to integrate.

Thus, when, $x = 0.1$, $y^{(1)} = 1 - 0.1 + 2 \log (1.1) = 0.9828$

Here in this example, only I approximation can be obtained and so it gives the approximate value of y for $x = 0.1$.

Example *Solve* $\frac{dy}{dx} = 1 + xy$ *with* $x_0 = 2$, $y_0 = 0$ *using Picard's method of successive approximations.*

Sol. Here,

$$y^{(1)} = y_0 + \int_2^x f(x, y_0)\, dx = 0 + \int_2^x [1 + x(0)]\, dx = x - 2$$

$$y^{(2)} = 0 + \int_2^x \{1 + x(x-2)\}\, dx$$

$$= \left(x - x^2 + \frac{x^3}{3} \right)_2^x = -\frac{2}{3} + x - x^2 + \frac{x^3}{3}$$

And third approximation,

$$y^{(3)} = 0 + \int_2^x \{1 + x\, y^{(2)}\}\, dx$$

$$= -\frac{22}{15} + x - \frac{1}{3}x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{15}$$

which is the required solution.

EULER'S METHOD :-

Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \quad \text{--- (1)}$$

with the initial condition $y(x_0) = y_0$

Integrating (1) w.r.t x between x_0 and x_1, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\therefore \quad y_1 - y_0 = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\Rightarrow \quad y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx \quad \text{--- (2)}$$

Now, replacing $f(x,y)$ by the approximation $f(x_0,y_0)$ we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0,y_0)\,dx$$

$$y_1 = y_0 + f(x_0,y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h\,f(x_0,y_0) \qquad (\because x_1 - x_0 = \Delta x = h)$$

This is the formula for first approximation y_1 of y.

Similarly, second approximation y_2 is given by

$$y_2 = y_1 + h f(x_1, y_1)$$

In general $\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$

To get the better accuracy by this method take the step length h (very small.)

Euler's Modified Method:-

The modified Euler's method gives greater improvement in accuracy over the original Euler's method. To get the better accuracy by Euler's method, let the integral is approximated by

the mean value of $f(x_0, y_0)$ and $f(x_1, y_1)$

Hence

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \quad \text{--- (1)}$$

where $y_1^{(0)}$ is the value of y_1 obtained by Euler's method, i.e.

$$y_1^{(0)} = y_1 = y_0 + hf(x_0, y_0) \qquad (2)$$

hence

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(0)})\right]$$

Proceeding similarly, we ~~have as is~~ obtain the formula

$$y_{n+1}^{(k)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(k-1)})\right] \qquad (3)$$

where $y_{n+1}^{(k)}$ is the k^{th} approximation to y_{n+1} obtained by Euler's method, i.e.

$$y_{n+1}^{(k)} = y_n + hf(x_n, y_n)$$

where $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots$

eqn (3) is called the modified Euler's method.

The process of iteration is continued until two successive approximations, $y_{n+1}^{(k)}$ and $y_{n+1}^{(k+1)}$ coincide up to the desired accuracy.

EX :- Solve the eqn $\frac{dy}{dx} = x + |\sqrt{y}|$ with $y(0)=1$ for $0 \le x \le 0.6$ in steps of 0.2 using Euler's modified method.

Soln :-

here $h = 0.2$, $x_0 = 0$, $y_0 = 1$

$$f(x,y) = x + |\sqrt{y}|$$

By Euler's method

$$y_1^{(0)} = y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h[x_0 + |\sqrt{y_0}|]$$
$$= 1 + 0.2[0 + |\sqrt{1}|]$$
$$= 1 + 0.2$$
$$y_1^{(0)} = y_1 = 1.2$$

Let us apply Euler's ~~method~~ modified method to improve the value obtained by Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.2}{2}\left[(0 + |\sqrt{1}|) + (0.2 + |\sqrt{1.2}|)\right]$$

$$y_1^{(1)} = 1.2295$$

Let us apply Euler's ~~method~~ modified method to improve the value obtained by Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.2}{2}\left[(0 + |\sqrt{1}|) + (0.2 + |\sqrt{1.2}|)\right]$$

$$y_1^{(1)} = 1.2295$$

and

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

and

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 1 + \frac{0.2}{2}\left[(0 + |\sqrt{1}|) + (0.2 + |\sqrt{1.2295}|)\right]$$

$$y_1^{(2)} = 1.2309$$

and

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.2}{2}\left[(0 + |\sqrt{1}|) + (0.2 + |\sqrt{1.23091}|)\right]$$

$$y_1^{(3)} = 1.2309$$

Example 1. *Given* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *with y = 1 for x = 0. Find y approximately for x = 0.1 by Euler's method.*

Sol. We have

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x} \; ; x_0 = 0, y_0 = 1, h = 0.1$$

Hence the approximate value of y at $x = 0.1$ is given by

$$y_1 = y_0 + hf(x_0, y_0) \qquad | \text{ using } y_{n+1} = y_n + hf(x_n, y_n)$$

$$= 1 + (.1) + \left(\frac{1-0}{1+0}\right) = 1.1$$

Much better accuracy is obtained by breaking up the interval 0 to 0.1 into five steps. The approximate value of y at $x_A = .02$ is given by,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.02 + (.02)\left(\frac{1.02 - .02}{1.02 + .02}\right) = 1.0392$$

$$y_3 = 1.0392 + (.02)\left(\frac{1.0392 - .04}{1.0392 + .04}\right) = 1.0577$$

$$y_4 = 1.0577 + (.02)\left(\frac{1.0577 - .06}{1.0577 + .06}\right) = 1.0756$$

$$y_5 = 1.0756 + (.02)\left(\frac{1.0756 - .08}{1.0756 + .08}\right) = 1.0928$$

Hence $y = 1.0928$ when $x = 0.1$

TAYLOR'S METHOD

Consider the differential equation

$$\left.\begin{aligned} &\frac{dy}{dx} = f(x, y) \\ &y(x_0) = y_0. \end{aligned}\right\} \tag{9}$$

with the initial condition $y(x_0) = y_0$.

If $y(x)$ is the exact solution of (9) then $y(x)$ can be expanded into a Taylor's series about the point $x = x_0$ as

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \frac{(x - x_0)^3}{3!}\, y_0''' + \ldots\ldots \tag{10}$$

where dashes denote differentiation with respect to x.

putting $x = x_1 = x_0 + h$ in eqn (2), we get

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \text{---}$$

Here, $y_0', y_0'', y_0''', \text{---}$ can be found by using eqn (1) from id successive differentiations at $x = x_0$. The series in eqn (4) can be truncated at any stage if h is small.

After obtaining y_1, we can calculate $y_1', y_1'', y_1''' ---$ from eqn (1) at $x_1 = x_0 + h$

Now expanding $y(x)$ by Taylor's series about $x = x_1$ we get

$$y_2 = y_1 + \frac{h}{\lfloor 1} y_1' + \frac{h^2}{\lfloor 2} y_1'' + \frac{h^3}{\lfloor 3} y_1''' + ---$$

Continue, we get

$$y_n = y_{n-1} + \frac{h}{\lfloor 1} y_{n-1}' + \frac{h^2}{\lfloor 2} y_{n-1}'' + \frac{h^3}{\lfloor 3} y_{n-1}''' + ---$$

Example 2. *For the differential eqn.,* $\frac{dy}{dx} = -xy^2$*,* $y(0) = 2$*. Calculate* $y(0.2)$ *by Taylor's series method retaining four non-zero terms only.*

Sol. Here $x_0 = 0, y_0 = 2$ Also $y' = -xy^2$

Taylor's series for $y(x)$ is given by

$$y(x) = y_0 + xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \frac{x^4}{24} y_0^{(iv)} + \frac{x^5}{120} y_0^{(v)} + \ldots\ldots \quad (19)$$

The values of the derivatives $y_0', y_0'', \ldots\ldots$, etc. are obtained as follows:

$$y' = -xy^2 \qquad y_0' = -x_0 y_0^2 = 0$$

$$y'' = -y^2 - 2xyy' \qquad y_0'' = -2^2 - 0 = -4$$

$$y''' = -4yy' - 2xy'^2 - 2xyy'' \qquad y_0''' = 0$$

$$y^{(iv)} = -6y'^2 - 6y'y'' - 6xy'y'' - 2xyy''' \qquad y_0^{(iv)} = 48$$

$$y^{(v)} = -24y'y'' - 8yy''' - 6xy''^2 - 8xy'y''' - 2xyy^{(iv)} \qquad y_0^{(v)} = 0$$

$$y^{(vi)} = -40y'y''' - 30y''^2 - 10\,yy^{(iv)} - 20xy''y''' - 10xy'\,y^{(iv)} - 2xyy^{(v)}. \qquad y_0^{(vi)} = -1440$$

We stop here as we shall get four non-zero terms in the Taylor's series (19).

$$\therefore \qquad y(x) = 2 + \frac{x^2}{2}(-4) + \frac{x^4}{24}(48) + \frac{x^6}{720}(-1440) + \ldots\ldots$$

$$= 2 - 2x^2 + 2x^4 - 2x^6 + \ldots\ldots$$

$$\therefore \qquad y(0.2) = 2 - 2(0.2)^2 + 2(0.2)^4 - 2(0.2)^6 + \ldots\ldots$$

$$= 2 - 0.08 + 0.0032 - 0.000128 = 1.923072$$

$$\simeq 1.9231 \quad \text{correct up to four decimal places.}$$

RUNGE-KUTTA METHODS

More efficient methods in terms of accuracy were developed by two German Mathematicians **Carl Runge** (1856-1927) and **Wilhelm Kutta** (1867-1944). These methods are well-known as Runge-Kutta methods. They are distinguished by their orders in the sense that they agree with Taylor's series solution up to terms of h^r where r is the order of the method.

These methods do not demand prior computation of higher derivatives of $y(x)$ as in Taylor's method. In place of these derivatives, extra values of the given function $f(x, y)$ are used.

The fourth order Runge-Kutta method is used widely for finding the numerical solutions of linear or non-linear ordinary differential equations.

Runge-Kutta methods are referred to as single step methods. The major disadvantage of Runge-Kutta methods is that they use many more evaluations of the derivative $f(x, y)$ to obtain the same accuracy compared with multi-step methods. A class of methods known as Runge-Kutta methods combines the advantage of high order accuracy with the property of being one step.

First Order Runge-Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \; y(x_0) = y_0 \tag{1}$$

Euler's method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \tag{2}$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots \tag{3}$$

Comparing (2) and (3), it follows that Euler's method agrees with Taylor's series solution up to the term in h. Hence *Euler's method is the first order Runge-Kutta method*.

Second Order Runge-Kutta Method :-

Runge-Kutta method of second order is given by

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2) \; ; \; x_1 = x_0 + h$$

where $K_1 = hf(x_0, y_0)$

$K_2 = hf(x_0 + h, y_0 + K_1)$ similarly for other intervals.

Here, we can observe that Runge-Kutta method of second order is nothing but Euler's modified method.

Third Order Runge-Kutta Method :-

Runge-Kutta method of 3^{rd} order is given by

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3) \; ; \quad x_1 = x_0 + h$$

where,

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

and $$K_3 = hf(x_0 + h, y_0 + K')$$

where $$K' = hf(x_0 + h, y_0 + K_1)$$

Similarly for other intervals.

Fourth Order Runge-Kutta Method :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$

with the initial condition $y(x_0) = y_0$

Let h be the interval between equidistant values of x. Then the first increment in y is computed from the formulae

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where, and $x_1 = x_0 + h$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Similarly, the increment in y for the second interval is computed by

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

and $x_2 = x_1 + h$

where,

$$K_1 = hf(x_1, y_1)$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

and similarly for the next intervals.

Example 1. *Solve the equation* $\frac{dy}{dx} = x + y$ *with initial condition* $y(0) = 1$ *by Runge-Kutta rule, from* $x = 0$ *to* $x = 0.4$ *with* $h = 0.1$.

Sol. Here $f(x, y) = x + y, h = 0.1, x_0 = 0, y_0 = 1$

We have,

$$k_1 = hf(x_0, y_0) = 0.1\,(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ \ y_0 + \frac{k_1}{2}\right) = 0.1\,(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ \ y_0 + \frac{k_2}{2}\right) = 0.1105$$

$$k_4 = hf(x_0 + h,\ \ y_0 + k_3) = 0.12105$$

$$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.11034$$

Thus, $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta y = 1.11034$

Now for the second interval, we have

$$k_1 = hf(x_1, y_1) = 0.1\,(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.13263$$

$$k_4 = hf(x_1 + h,\ \ y_1 + k_3) = 0.14429$$

$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.132460$

Hence $\quad x_2 = 0.2 \quad$ and $\quad y_2 = y_1 + \Delta y = 1.11034 + 0.13246 = 1.24280$

Similarly, for finding y_3, we have

$$k_1 = hf(x_2, y_2) = 0.14428$$

$$k_2 = 0.15649$$

$$k_3 = 0.15710$$

$$k_4 = 0.16999$$

Repeating the above process

$\therefore \quad y_3 = 0.13997$

and for $\quad y_4 = y(0.4)$, we calculate

$$k_1 = 0.16997$$

$$k_2 = 0.18347$$

$$k_3 = 0.18414$$

$$k_4 = 0.19838$$

$\therefore \quad y_4 = 1.5836$