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GT--4

Graph coloring is nothing but a simple way of labelling graph components such as vertices, edges, and regions under some constraints. In a graph, no two adjacent vertices, adjacent edges, or adjacent regions are colored with minimum number of colors. This number is called the **chromatic number** and the graph is called a **properly colored graph**.

Graph Coloring

While graph coloring, the constraints that are set on the graph are colors, order of coloring, the way of assigning color, etc. A coloring is given to a vertex or a particular region. Thus, the vertices or regions having same colors form independent sets.

Vertex Coloring

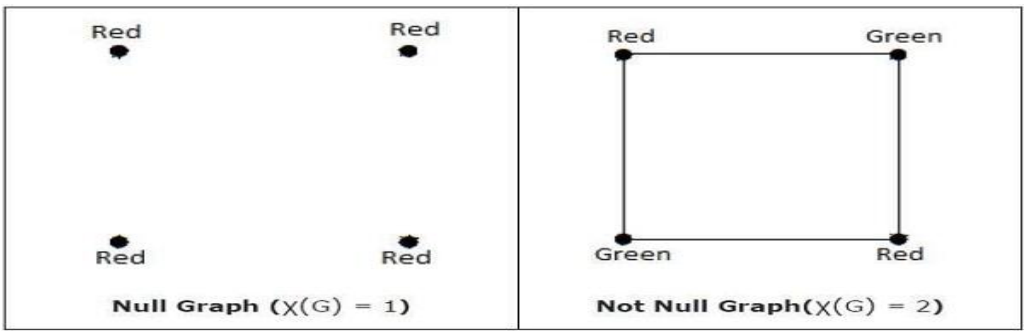
Vertex coloring is an assignment of colors to the vertices of a graph 'G' such that no two adjacent vertices have the same color. Simply put, no two vertices of an edge should be of the same color.

Chromatic Number

The minimum number of colors required for vertex coloring of graph 'G' is called as the chromatic number of G, denoted by X(G).

$\chi(G) = 1$ if and only if 'G' is a null graph. If 'G' is not a null graph, then $\chi(G) \geq 2$.

Example



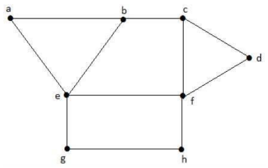
Note – A graph 'G' is said to be n-coverable if there is a vertex coloring that uses at most n colors, i.e., X(G) ≤ n.

Region Coloring

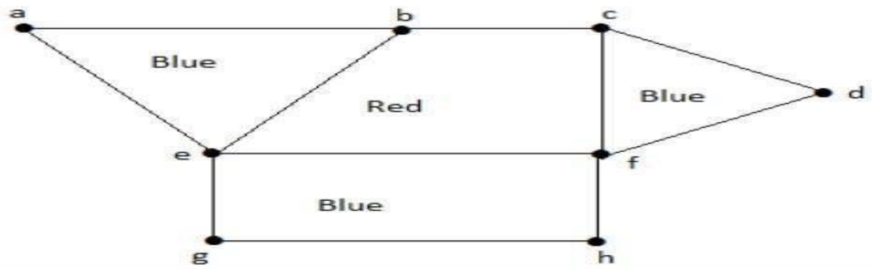
Region coloring is an assignment of colors to the regions of a planar graph such that no two adjacent regions have the same color. Two regions are said to be adjacent if they have a common edge.

Example

Take a look at the following graph. The regions 'aeb' and 'befc' are adjacent, as there is a common edge 'be' between those two regions.



Similarly, the other regions are also coloured based on the adjacency. This graph is coloured as follows –

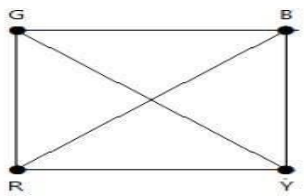


Example

The chromatic number of Kn is

- n
- n–1
- [n/2]
- [n/2]

Consider this example with K4.



In the complete graph, each vertex is adjacent to remaining (n – 1) vertices. Hence, each vertex requires a new color. Hence the chromatic number of Kn = n.

Applications of Graph Coloring

Graph coloring is one of the most important concepts in graph theory. It is used in many real-time applications of computer science such as –

- Clustering
- Data mining
- Image capturing
- Image segmentation
- Networking
- Resource allocation
- Processes scheduling

Note : # Chromatic number define as the least no of colors needed for coloring the graph .

and types of chromatic number are:

1) Cycle graph

2) planar graphs

3) Complete graphs

4) Bipartite Graphs:

5) Trees

Coverings

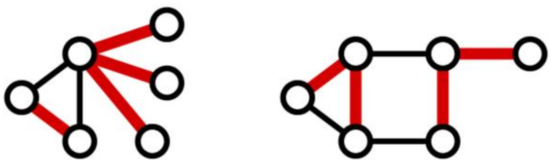
A graph covering of a graph G is a sub-graph of G which contains either all the vertices or all the edges corresponding to some other graph.

A sub-graph which contains all the vertices is called a line/edge covering. A sub-graph which contains all the edges is called a vertex covering.

1. Edge Covering

A set of edges which covers all the vertices of a graph G, is called a **line cover or edge cover** of G.
Edge covering does not exist if and only if G has an isolated vertex.
Edge covering of graph G with n vertices has at least n/2 edges.

Example

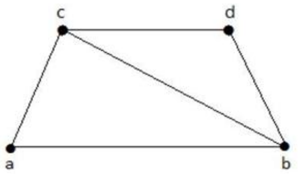


In the above graph, the red edges represent the edges in the edge cover of the graph.

Minimal Line covering

A line covering M of a graph G is said to be minimal line cover **if no edge can be deleted from M**.
Or minimal edge cover is an edge cover of graph G that is **not a proper subset of any other edge cover**.
No minimal line covering contains a cycle.

Example



From the above graph, the sub-graph having edge covering are:

$M_1 = \{\{a, b\}, \{c, d\}\}$

$M_2 = \{\{a, d\}, \{b, c\}\}$

$M_3 = \{\{a, b\}, \{b, c\}, \{b, d\}\}$

$M_4 = \{\{a, b\}, \{b, c\}, \{c, d\}\}$

Here, M_1, M_2, M_3 are minimal line coverings, but M_4 is not because we can delete $\{b, c\}$.

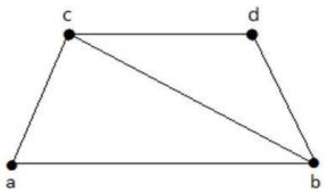
Minimum Line Covering

A **minimal line covering** with minimum number of edges is called a minimum line covering of graph G. It is also called **smallest minimal line covering**.

Every minimum edge cover is a minimal edge cove, but the converse does not necessarily exist.

The number of edges in a minimum line covering in G is called the line covering number of G and it is denoted by α_1.

Example



From the above graph, the sub-graph having edge covering are:

$M_1 = \{\{a, b\}, \{c, d\}\}$

$M_2 = \{\{a, d\}, \{b, c\}\}$

$M_3 = \{\{a, b\}, \{b, c\}, \{b, d\}\}$

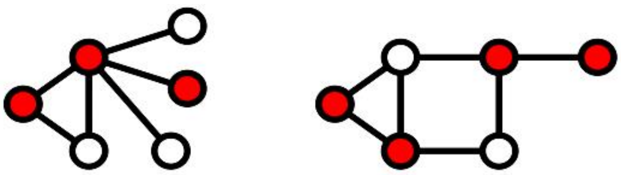
$M_4 = \{\{a, b\}, \{b, c\}, \{c, d\}\}$

In the above example, M_1 and M_2 are the minimum edge covering of G and α_1 =

2. Vertex Covering

A set of vertices which covers all the nodes/vertices of a graph G, is called a **vertex cover** for G.

Example

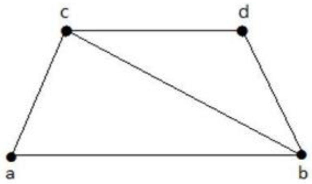


In the above example, each red marked vertex is the vertex cover of graph. Here, the set of all red vertices in each graph touches every edge in the graph.

Minimal Vertex Covering

A vertex M of graph G is said to be minimal vertex covering if no vertex can be deleted from M.

Example



The sub- graphs that can be derived from the above graph are:

$M_1 = \{b, c\}$

$M_2 = \{a, b, c\}$

$M_3 = \{b, c, d\}$

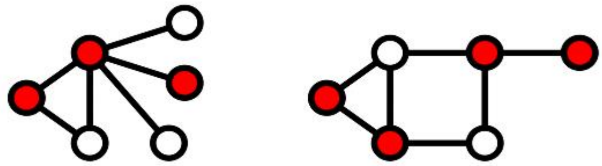
Here, M_1 and M_2 are minimal vertex coverings, but in M_3 vertex 'd' can be deleted.

Minimum Vertex Covering

A minimal vertex covering is called when minimum number of vertices are covered in a graph G. It is also called smallest minimal vertex covering.

The **number of vertices** in a minimum vertex covering in a graph G is called the vertex covering number of G and it is denoted by α_2.

Example 1

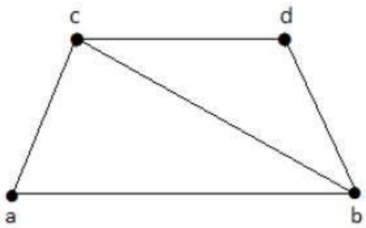


In the above graphs, the vertices in the minimum vertex covered are red.

α_2 = 3 for first graph.

And α_2 = 4 for the second graph.

Example 2



The sub- graphs that can be derived from the above graph are:

$M_1 = \{b, c\}$

$M_2 = \{a, b, c\}$

$M_3 = \{b, c, d\}$

Here, M1 is a minimum vertex cover of G, as it has only two vertices. Therefore, $\alpha_2 = 2$.

Properties of chromatic numbers (observations)

- A graph consisting of only isolated vertices is 1-chromatic.
- Every tree with two or more vertices is 2-chromatic.
- A graph with two or more vertices is at least 2-chromatic.
- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- A complete graph consisting of n vertices is n-chromatic

CHROMATIC POLYNOMIAL

A given graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G.

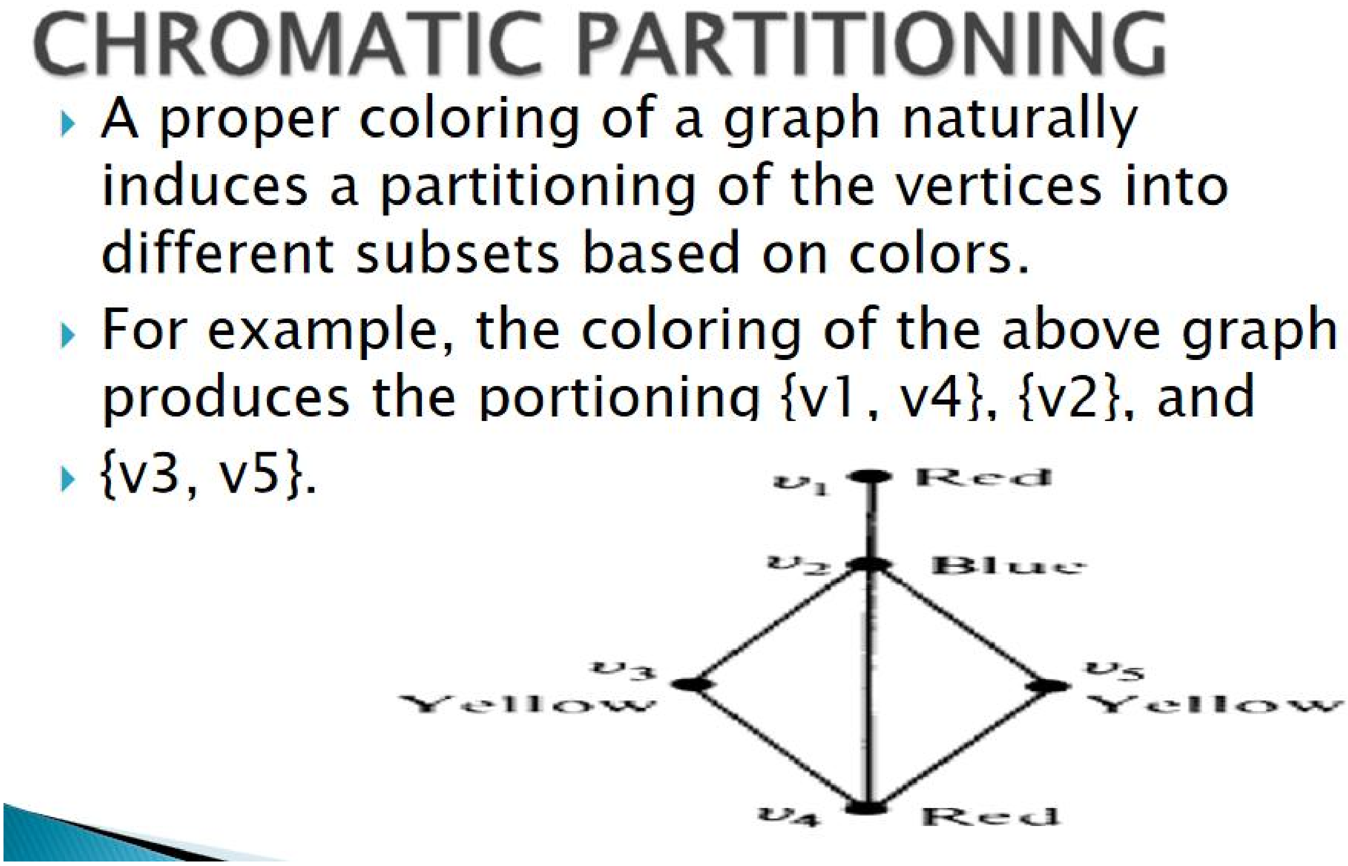
The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly colouring the graph, using λ of fewer colours. Let C_i be the different ways of properly colouring G using exactly i different colours. Since i colours can be chosen out of λ colours in $\binom{\lambda}{i}$ different ways, there are $c_i \binom{\lambda}{i}$ different ways of properly colouring G using exactly i colours out of λ colours.

$$P_n(\lambda) = \sum_{i=1}^{n} C_i \binom{\lambda}{i}$$

$$= C_1 \frac{\lambda}{1!} + C_2 \frac{\lambda(\lambda-1)}{2!} + C_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \ldots\ldots$$

$$\ldots + C_n \frac{\lambda(\lambda-1)(\lambda-2)\ldots\ldots(\lambda-n+1)}{n!}$$

Note : Please Do practice of above topic because that is numerical topic.



A matching graph is a subgraph of a graph where there are no edges adjacent to each other. Simply, there should not be any common vertex between any two edges.

Matching

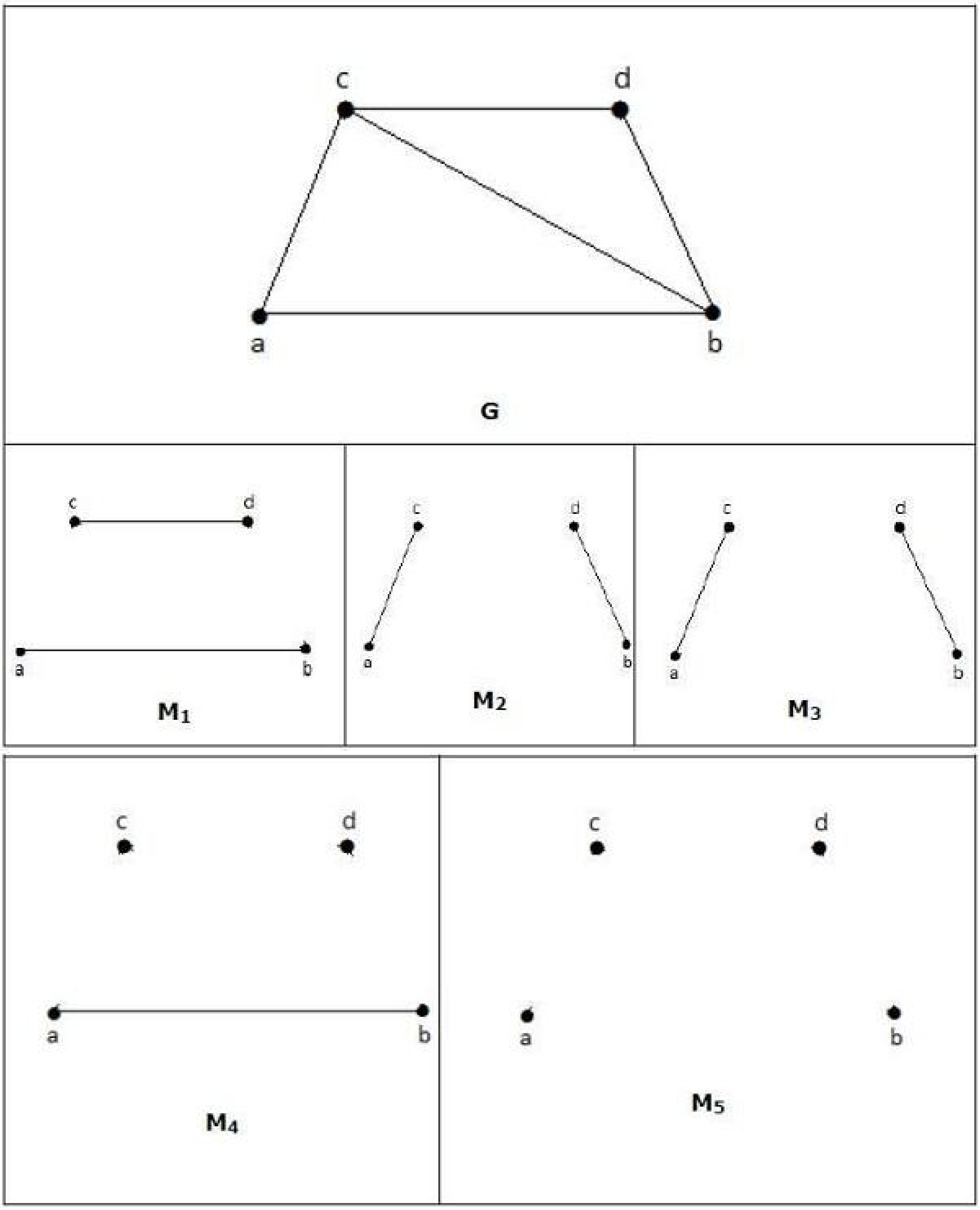
Let 'G' = (V, E) be a graph. A subgraph is called a matching M(G), ***if each vertex of G is incident with at most one edge in M****, i.e.,*

deg(V) ≤ 1 ∀ V ∈ G

which means in the matching graph M(G), the vertices should have a degree of 1 or 0, where the edges should be incident from the graph G.

Notation – *M(G)*

Example



In a matching,

if deg(V) = 1, then (V) is said to be matched

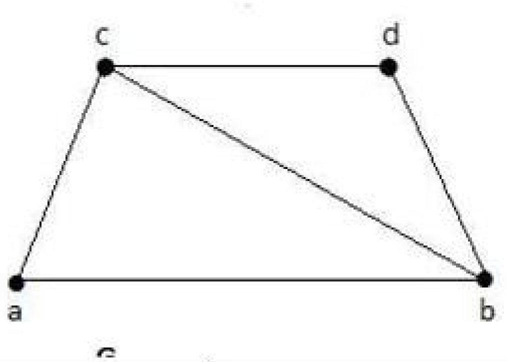
if deg(V) = 0, then (V) is not matched.

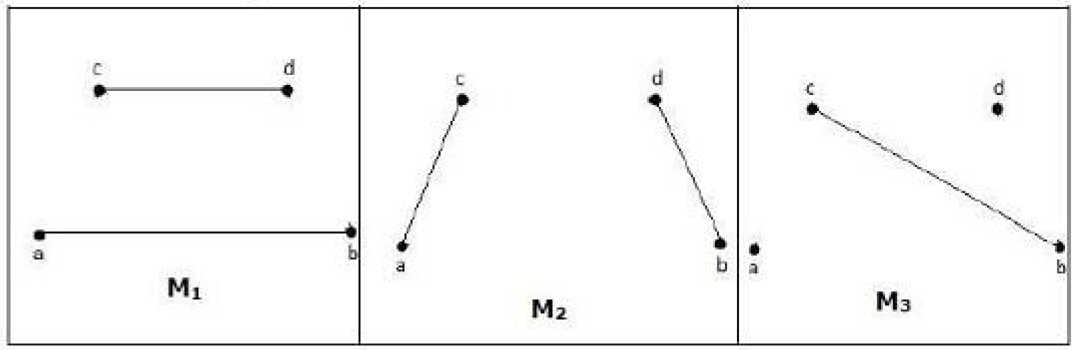
In a matching, no two edges are adjacent. It is because if any two edges are adjacent, then the degree of the vertex which is joining those two edges will have a degree of 2 which violates the matching rule.

Maximal Matching

A matching M of graph 'G' is said to maximal **if no other edges of 'G' can be added to M**.

Example





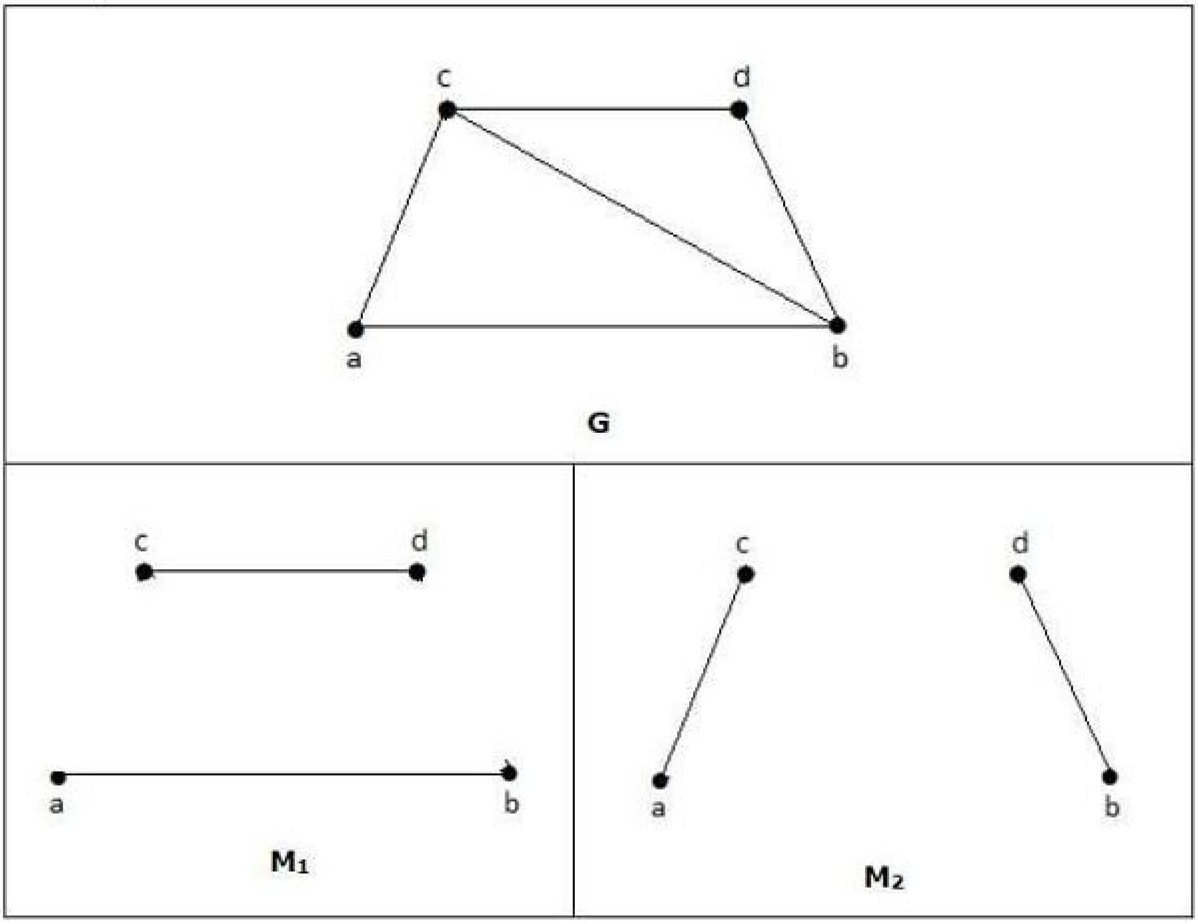
M1, M2, M3 from the above graph are the maximal matching of G.

Maximum Matching

It is also known as largest maximal matching. Maximum matching is defined as the maximal matching with maximum number of edges.

The number of edges in the maximum matching of 'G' is called its **matching number**.

Example



For a graph given in the above example, M1 and M2 are the maximum matching of 'G' and its matching number is 2. Hence by using the graph G, we can form only the subgraphs with only 2 edges maximum. Hence we have the matching number as two.

Perfect Matching

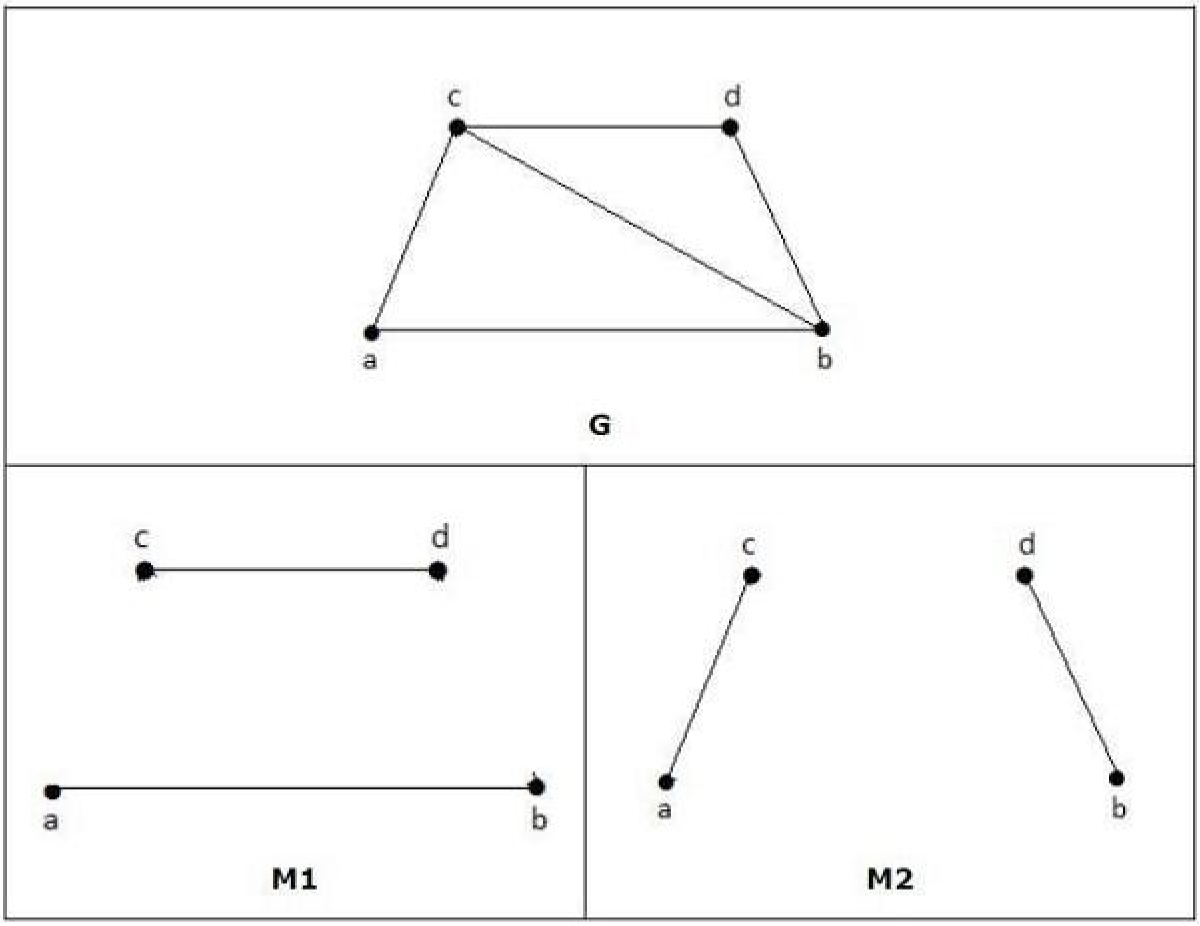
A matching (M) of graph (G) is said to be a perfect match, if every vertex of graph g (G) is incident to exactly one edge of the matching (M), i.e.,

$deg(V) = 1 \ \forall \ V$

The degree of each and every vertex in the subgraph should have a degree of 1.

Example

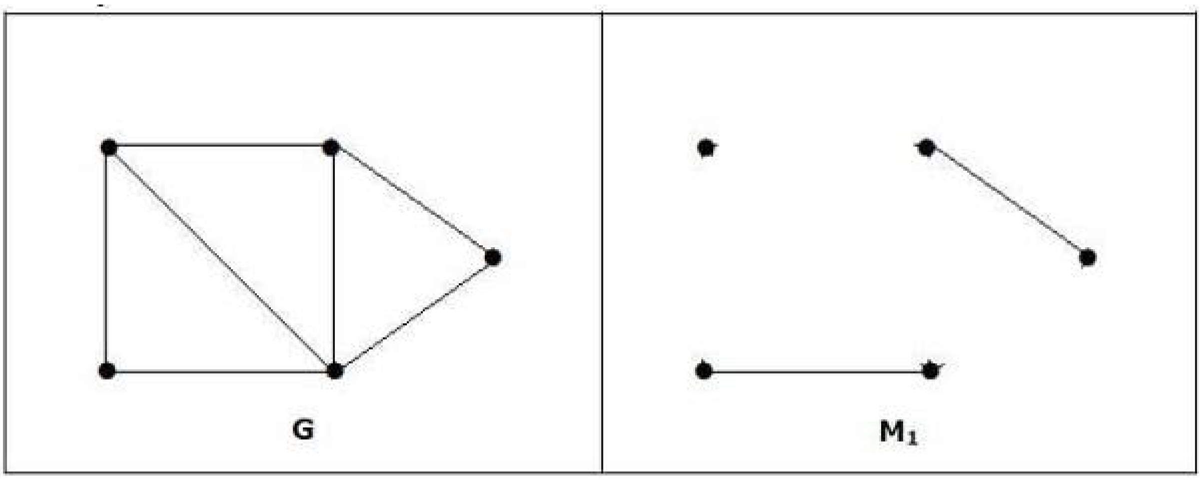
In the following graphs, M1 and M2 are examples of perfect matching of G.



Note – Every perfect matching of graph is also a maximum matching of graph, because there is no chance of adding one more edge in a perfect matching graph.

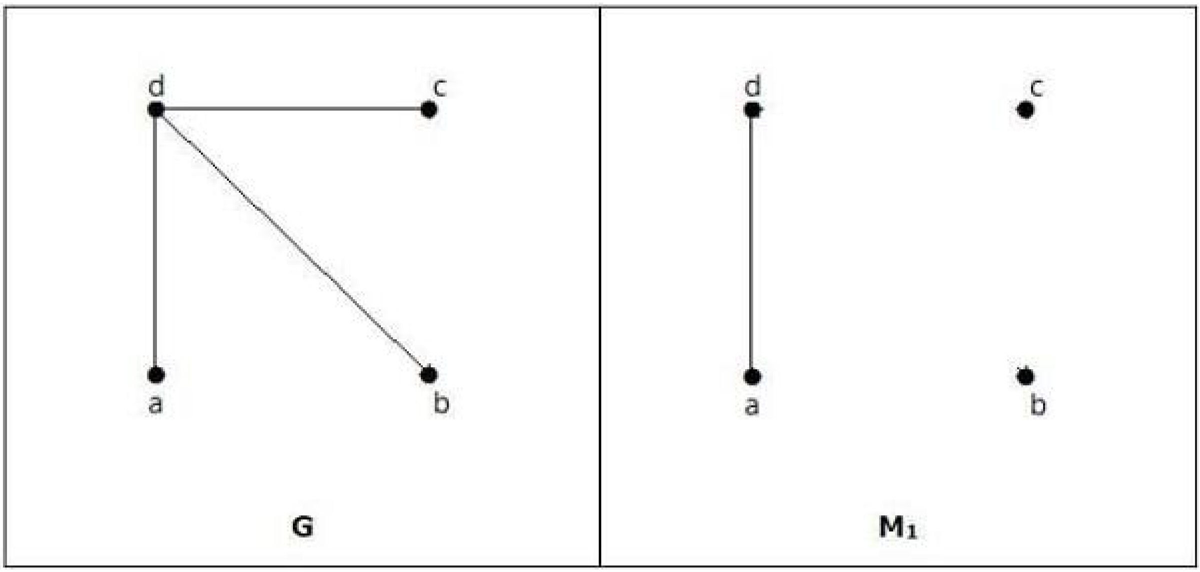
A maximum matching of graph need not be perfect. If a graph 'G' has a perfect match, then the number of vertices $|V(G)|$ is even. If it is odd, then the last vertex pairs with the other vertex, and finally there remains a single vertex which cannot be paired with any other vertex for which the degree is zero. It clearly violates the perfect matching principle.

Example



Note – The converse of the above statement need not be true. If G has even number of vertices, then M1 need not be perfect.

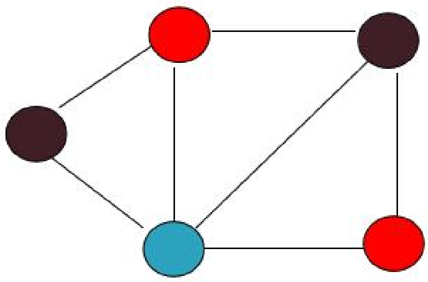
Example



It is matching, but it is not a perfect match, even though it has even number of vertices.

Coloring Planar graphs

- **Definition**: A graph is planar if it can be drawn in a plane without edge-crossings.



- **The four color theorem:** For every planar graph, the chromatic number is ≤ 4.
- Was posed as a conjecture in the 1850's. Finally proved in 1976(Appel and Haken) by the aid of computers.

The Four-Color Theorem

- The four color theorem states that any planar map can be colored with at most four colors.
- In graph terminology, this means that using at most four colors, any planar graph can have its nodes colored such that no two adjacent nodes have the same color.
- Four-color conjecture – Francis Guthrie, 1852 (F.G.)
- Many incomplete proofs (Kempe).
- 5-color theorem proved in 1890 (Heawood)
- 4-color theorem finally proved in 1977 (Appel, Haken)
  - First major computer-based proof

- Every planar graph has a chromatic number of four or less.
- Every triangular planar graph has a chromatic number of four or less.
- The regions of every planar, regular graph of degree three can be colored properly with four colors.
- **4-Colour Theorem:**
- If G is a planar graph, then $\chi(G) \leq 4$. By the following theorem, each planar graph can be decomposed into two bipartite graphs.
- Let G = (V, E) be a 4-chromatic graph, $\chi(G) \leq 4$.
- Then the edges of G can be partitioned into two subsets E_1 and E_2 such that (V, E_1) and (V, E_2) are both bipartite.