28/01/25

Computer Vision

- Recap: • $\min_{\theta \in \Theta} \mathbb{E}_q \left[\| \tilde{\mu}(x^{(0)}, x^{(t)}) - f_\mu(x^{(t)}, t; \theta) \|_2^2 \right]$ — ①

  $\hookrightarrow q(x^{(t-1)} | x^{(t)}, x^{(0)}) \sim \mathcal{N}(x^{(t-1)}; \tilde{\mu}(x^{(t)}, x^{(0)}), \tilde{\beta}_t I)$

  • $\tilde{\mu}(x^{(0)}, x^{(t)}) = \dfrac{\sqrt{\bar{\alpha}_{t-1}}}{(1-\bar{\alpha}_t)} \cdot \beta_t \cdot x^{(0)} + \dfrac{\sqrt{\alpha_t}(1-\bar{\alpha}_{t-1})}{(1-\bar{\alpha}_t)} x^{(t)}$ — ②

  • $p_\theta(x^{(t-1)} | x^{(t)}) \sim \mathcal{N}(x^{(t-1)}; f_\mu(x^{(t)}, t; \theta), \sigma_t^2 I)$ — ③

  $q(x^{(t)} | x^{(t-1)}) \sim \mathcal{N}(x^{(t)}; \sqrt{1-\beta_t} \cdot x^{(t-1)}, \beta_t I)$ — ④

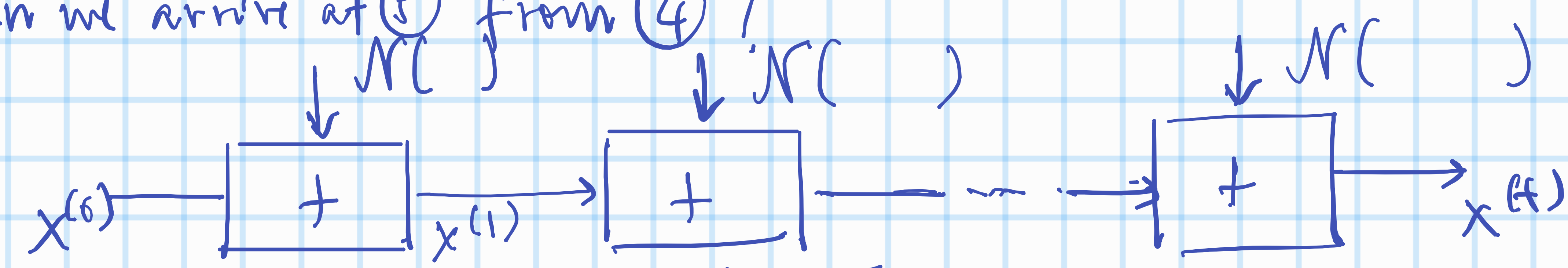
- Denoising model:

  $q(x^{(t)} | x^{(0)}) \sim \mathcal{N}(x^{(t)}; \sqrt{\bar{\alpha}_t} \cdot x^{(0)}, (1-\bar{\alpha}) I)$ — ⑤

  or $x^{(t)} = \sqrt{\bar{\alpha}_t}\, x^{(0)} + \sqrt{(1-\bar{\alpha}_t)} \cdot \epsilon \qquad \epsilon \sim \mathcal{N}(0, I)$, — ⑥

  where $\alpha_t = 1 - \beta_t$, $\bar{\alpha}_t = \prod_{s=1}^{t} \alpha_s$ — ⑦

Can we arrive at ⑤ from ④?



$= \sqrt{1-\beta_1} \cdot x^{(0)} + \sqrt{\beta_1} \cdot \epsilon$

Q. Show how $x^{(t)}$ relates to $x^{(0)}$

From ⑥, $x^{(0)} = \dfrac{1}{\sqrt{\bar{\alpha}_t}} \cdot \left(x^{(t)} - \sqrt{(1-\bar{\alpha}_t)} \cdot \epsilon \right)$ — ⑧

Plug ⑧ into ②

$\sqrt{\bar{\alpha}_t} = \sqrt{\alpha_1 \cdots \alpha_t}$; $\alpha_t = 1 - \beta_t$

$\sqrt{\bar{\alpha}_t} = \sqrt{\alpha_t \cdot \bar{\alpha}_{t-1}}$

Writing ② again,

$$\tilde{\mu}(x^{(t)}, x^{(0)}) = \frac{\sqrt{\bar{\alpha}_{t-1}}}{(1-\bar{\alpha}_t)} \beta_t \cdot x^{(0)} + \frac{\sqrt{\alpha_t}\,(1-\bar{\alpha}_{t-1})}{(1-\bar{\alpha}_t)} x^{(t)}$$

From ②,

$$\tilde{\mu}(x^{(t)}, x^{(0)}) = \frac{\sqrt{\cancel{\bar{\alpha}_{t-1}}}}{(1-\bar{\alpha}_t)} \beta_t \left(\frac{1}{\sqrt{\cancel{\bar{\alpha}_t}}\sqrt{\alpha_t}} \cdot (x^{(t)} - \sqrt{1-\bar{\alpha}_t}\,\epsilon) \right) +$$

$$\frac{\sqrt{\alpha_t}}{(1-\bar{\alpha}_t)} (1-\bar{\alpha}_{t-1}) \cdot x^{(t)}$$

$$= \frac{1}{(1-\bar{\alpha}_t)\cdot\sqrt{\alpha_t}} \left(\beta_t + \alpha_t (1-\bar{\alpha}_{t-1}) \right) x^{(t)}$$

$$- \frac{\beta_t}{\sqrt{\alpha_t}\sqrt{(1-\bar{\alpha}_t)}} \cdot \epsilon$$

$$= \frac{1}{\sqrt{\alpha_t}\cdot(1-\bar{\alpha}_t)} \cdot \left(\overbrace{\beta_t + \alpha_t}^{1} - \underbrace{\alpha_t \cdot \bar{\alpha}_{t-1}}_{\alpha_t \cdot \alpha_{t-1} \cdots \alpha_1} \right) \cdot x^{(t)}$$

$$= \frac{\cancel{(1-\bar{\alpha}_t)}}{\sqrt{\alpha_t}\cdot\cancel{(1-\bar{\alpha}_t)}} \cdot x^{(t)} - \frac{\beta_t}{\sqrt{\alpha_t}\sqrt{1-\bar{\alpha}_t}} \cdot \epsilon$$

$$\tilde{\mu}(x^{(t)}, x^{(0)}) = \frac{1}{\sqrt{\alpha_t}} \left(x^{(t)} - \frac{\beta_t}{\sqrt{1-\bar{\alpha}_t}} \cdot \epsilon \right) \quad - ⑨$$

From ⑨, our $f_\mu(\quad)$ effectively learns to predict the noise term in ⑨. In other words we have $f_\epsilon(x^{(t)}, \epsilon, t)$

If $f_\mu(x^{(t)}, x^{(0)}) = \frac{1}{\sqrt{\alpha_t}}\left(x^{(t)} - \frac{\beta_t}{\sqrt{1-\bar{\alpha}_t}} \cdot f_\epsilon(\qquad)\right)$, then

1 becomes

$$E_q\left[\frac{\beta_t^2}{\alpha_t(1-\bar{\alpha}_t)}\|\epsilon - f_\epsilon(x^{(t)}, \epsilon, t)\|^2\right] \quad - (10)$$