

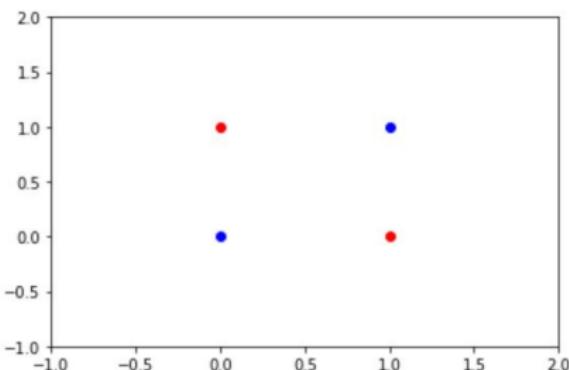
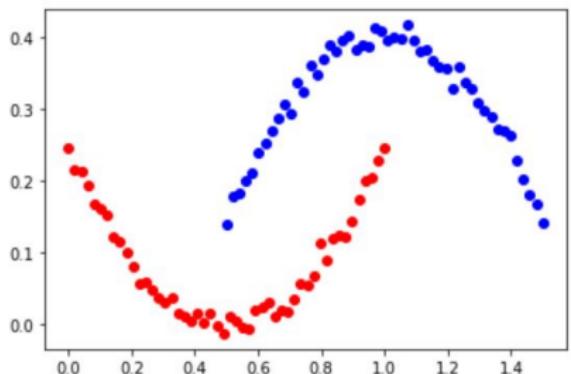
Deep Learning

2 Network of Perceptrons

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Linear Classifiers: Shortcomings

- Lower capacity: data has to be linearly separable
- Some times no hyper-plane can separate the data (e.g. XOR data)



Pre-processing

- ① Sometimes, data specific pre-processing makes the data linearly separable

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- ② Consider the xor case

$$\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$$

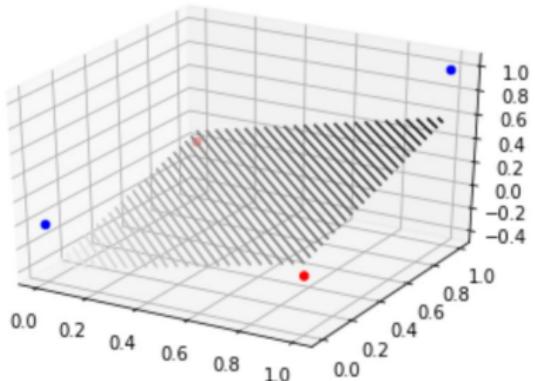
Pre-processing

- ① Sometimes, data specific pre-processing makes the data linearly separable

- ② Consider the xor case

$$\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$$

- ③ Consider the perceptron in the new space $f(\mathbf{x}) = \sigma(\mathbf{w}^T \phi(\mathbf{x}) + b)$



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Pre-processing

- ① (Recap: Polynomial regression): increasing the degree → increase the model capacity
- ② (Recap: Bias-Variance decomposition): to reduce the bias error, we increased the model capacity
- ③ Feature design (or pre-processing) may also be another way to reduce the capacity without affecting (or improving) the bias

Consider the XOR function

x_1	x_2	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Consider the XOR function

- If we attempt to realize XOR function with a single perceptron

x_1	x_2	XOR	
0	0	0	$w_0 < 0$
0	1	1	$w_2 + w_0 \geq 0$
1	0	1	$w_1 + w_0 \geq 0$
1	1	0	$w_1 + w_2 + w_0 < 0$

Consider the XOR function

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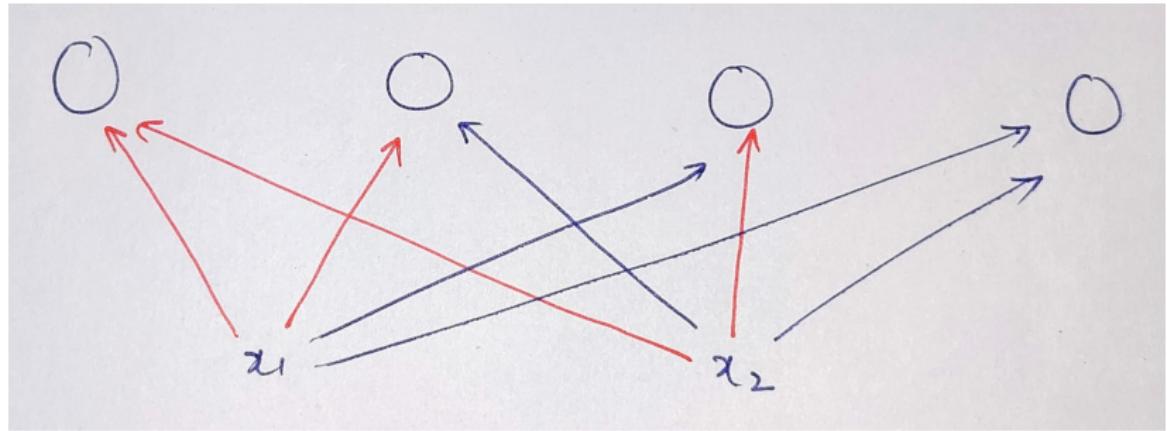
x_1	x_2	XOR	
0	0	0	$w_0 < 0$
0	1	1	$w_2 + w_0 \geq 0 \Rightarrow w_2 \geq -w_0$
1	0	1	$w_1 + w_0 \geq 0 \Rightarrow w_1 \geq -w_0$
1	1	0	$w_1 + w_2 + w_0 < 0 \Rightarrow w_1 + w_2 < -w_0$

Consider the XOR function

- Clearly, a single perception cannot represent the XOR function!

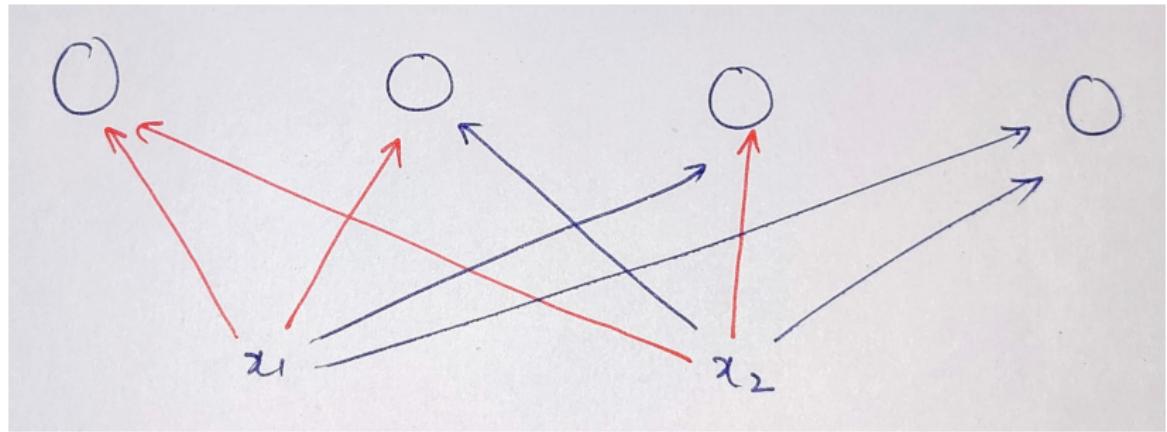
Let's see if multiple perceptions can do this

- Consider 4 perceptions



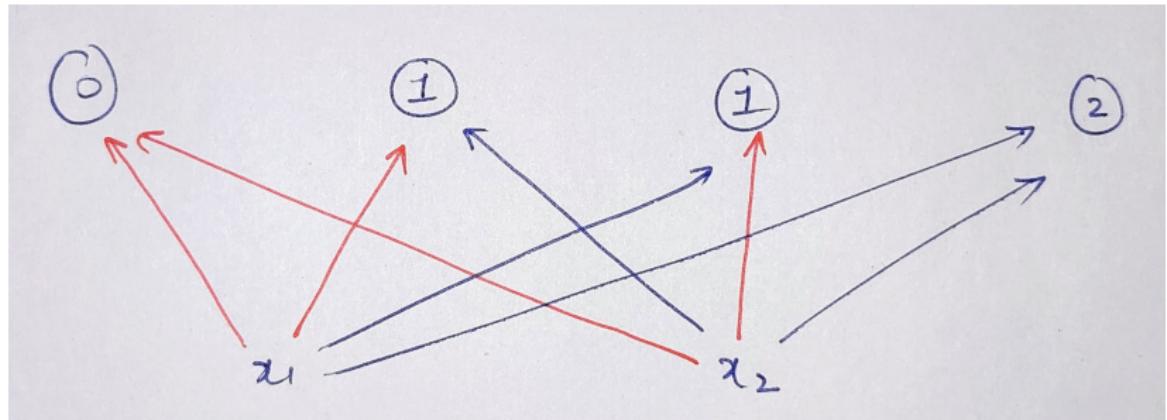
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- Consider 4 perceptions
- $\rightarrow = -1$ and $\rightarrow = +1$



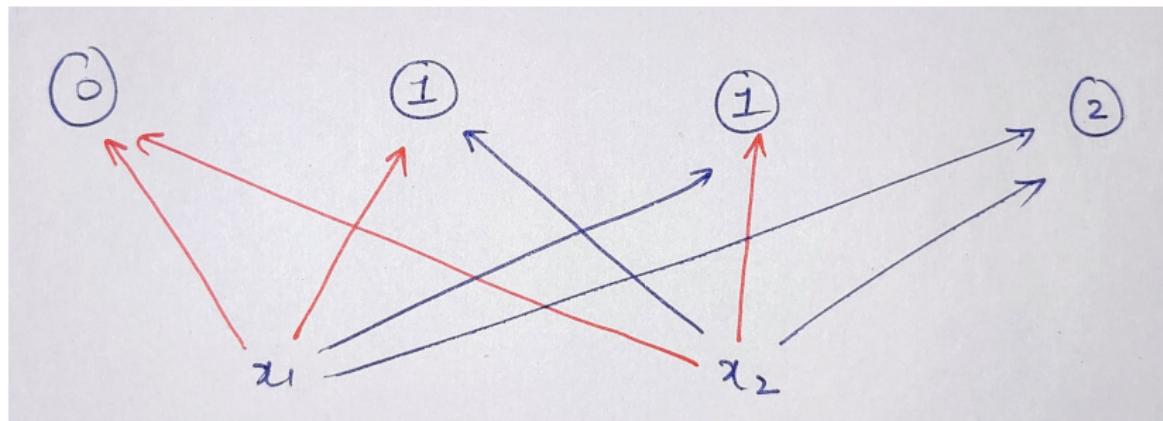
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- Let's have these thresholds



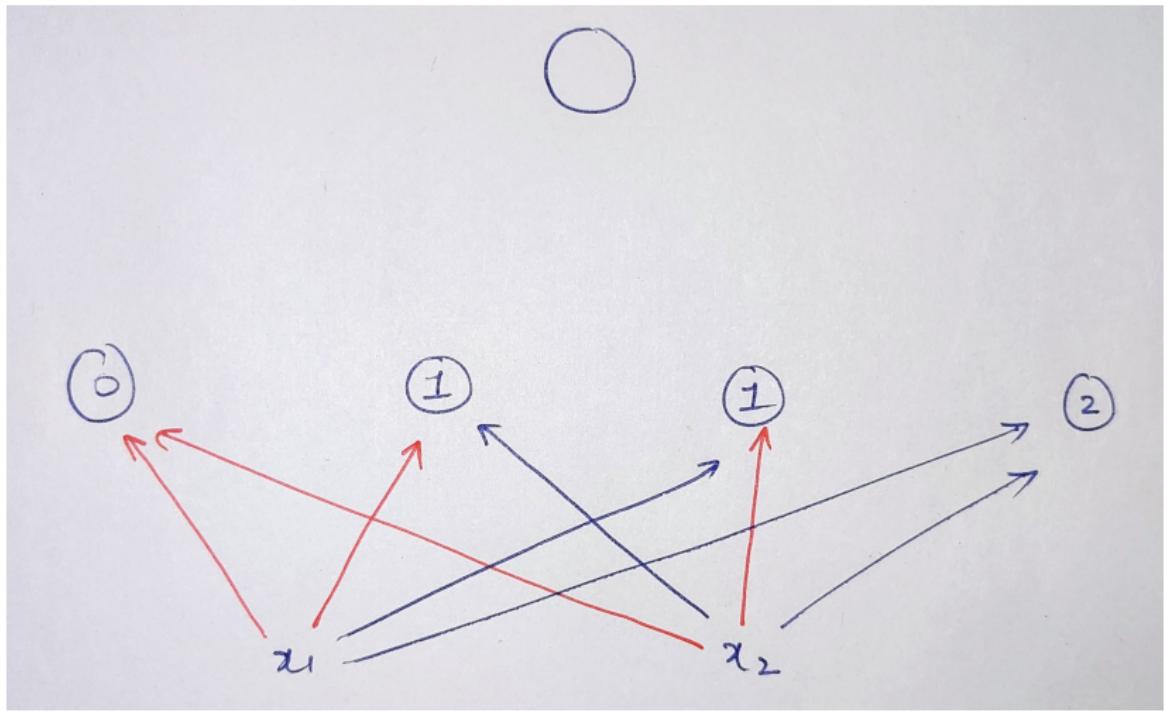
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- Let's have these thresholds
- Notice, each of them fire for exactly one specific input pattern



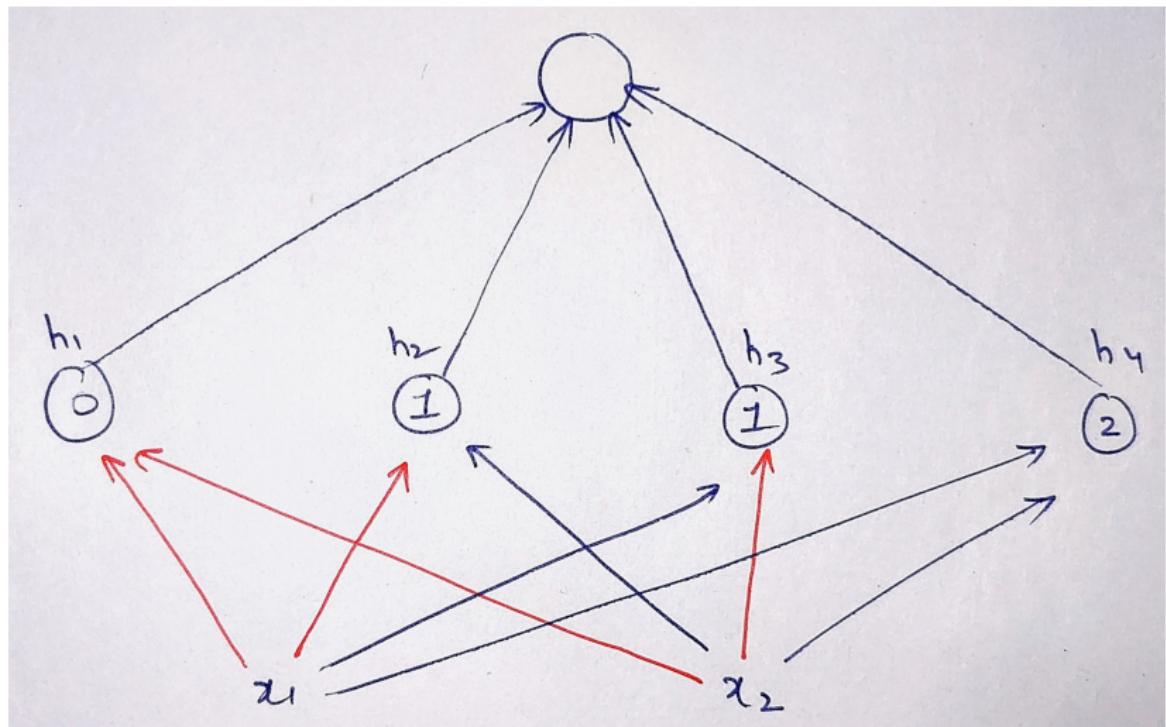
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- Let's now add another perceptron



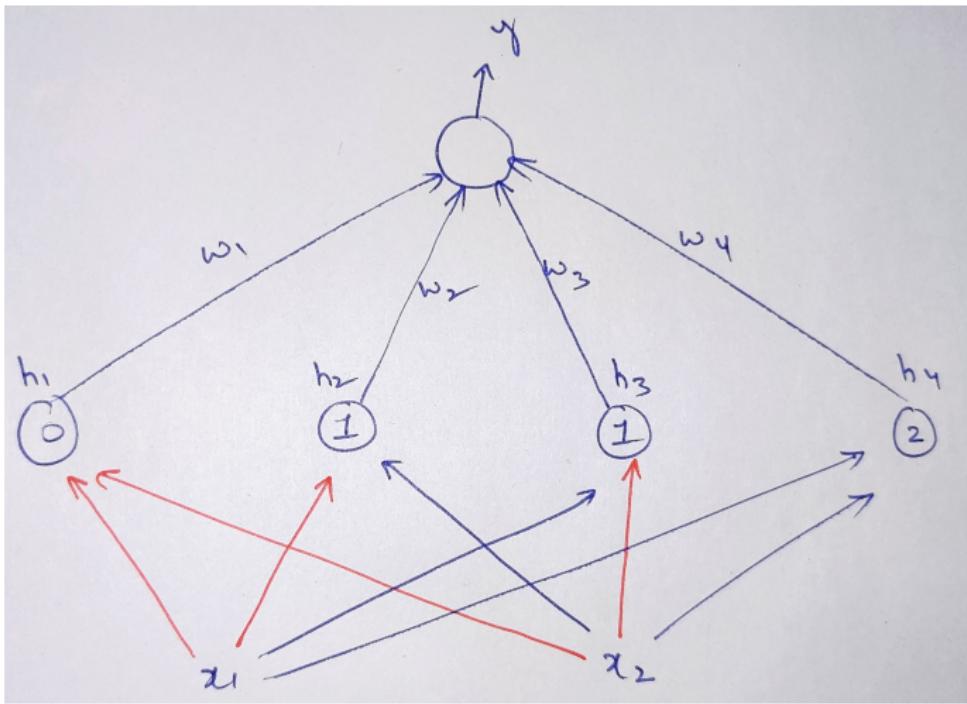
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- Connect the previous (hidden) ones and call it the output perceptron



Let's see if multiple perceptions can do this

- See if we can find a set of weights (W_i) to represent the XOR function



Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0					
0	1					
1	0					
1	1					

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
0	1					
1	0					
1	1					

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
0	1	0	1	0	0	
1	0	0	0	1	0	
1	1	0	0	0	1	

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	w_1
0	1	0	1	0	0	w_2
1	0	0	0	1	0	w_3
1	1	0	0	0	1	w_4

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	$w_1 + w_0$
0	1	0	1	0	0	$w_2 + w_0$
1	0	0	0	1	0	$w_3 + w_0$
1	1	0	0	0	1	$w_4 + w_0$

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y	$x \oplus 2$
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

$w_1 < -w_0$ $w_2 \geq -w_0$ $w_3 \geq -w_0$ $w_4 < -w_0$

Let's see if multiple perceptions can do this

- Clearly possible to find such weights → represent the XOR function!

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

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What about other 2-input Boolean functions?

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- Possible to represent!
- Leads to finding a different set of non-contradicting weights

What if there are more inputs?

- Can do the same with 2^n perceptions in the hidden layer and 1 in the output layer!

What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!

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What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!
- Note that $2^n + 1$ is a sufficient but not necessary
- **Caveat:** the size of the hidden layer grows exponentially!

Network of Perceptrons

- Generally referred to as MLP (Multi-Layered Network of Perceptrons)

Moving on from Boolean functions

- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$

Moving on from Boolean functions

- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$
- Can MLPs represent such functions?

Threshold-ing is very harsh!

- ① Perceptron's o/p is discontinuous!

$$\sigma(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ -1 & \text{else} \end{cases}$$



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$$\sigma(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ -1 & \text{else} \end{cases}$$



- ② Think of inputs -0.0001 and 0

Enough of Boolean functions!

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- ① Many real world problems have non-binary outputs
- ② Perceptron only gives two outputs!
- ③ Sigmoid neuron

$$f(\mathbf{x}) = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$

