1. $f(x,y) = x^2y + y^2$

$\therefore f_x(x,y) = 2xy$, $f_y(x,y) = x^2 + 2y$

At point $(1,3)$,

$f_x(1,3) = 2.1.3 = 6$, $f_y(1,3) = 1^2 + 2.3 = 7$

$\therefore L(x,y) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$

$= 1^2.3 + 3^2 + 6(x-1) + 7(y-3)$

$= 12 + 6x - 6 + 7y - 21$

$= 6x + 7y - 15$

Again,

$f_{xx}(x,y) = 2y$, $f_{xy}(x,y) = 2x$, $f_{yy}(x,y) = 2$

$\therefore f_{xx}(1,3) = 2.3 = 6$, $f_{xy}(1,3) = 2.1 = 2$, $f_{yy}(1,3) = 2$

$\therefore Q(x,y) = L(x,y) + \frac{f_{xx}(1,3)}{2}(x-1)^2 + f_{xy}(1,3)(x-1)(y-3) + \frac{f_{yy}(1,3)}{2}(y-3)^2$

$= 6x + 7y - 15 + \frac{6}{2}(x^2 - 2x + 1) + 2(xy - 3x - y + 3) + \frac{2}{2}(y^2 - 6y + 9)$

$= 6x + 7y - 15 + 3x^2 - 6x + 3 + 2xy - 6x - 2y + 6 + y^2 - 6y + 9$

$= 3x^2 + y^2 + 2xy - 6x - y + 3$

2. $f(x,y) = \ln(x^2 + y^2 + 1)$

$f_x(x,y) = \frac{1}{x^2+y^2+1} \cdot 2x = \frac{2x}{x^2+y^2+1}$, $f_y(x,y) = \frac{1}{x^2+y^2+1} \cdot 2y = \frac{2y}{x^2+y^2+1}$

$f_x(0,0) = \frac{0}{0+0+1} = 0$, $f_y(0,0) = \frac{0}{0+0+1} = 0$, $f(0,0) = \ln(0+0+1) = 0$

$\therefore L(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y$

$= 0 + 0.x + 0.y$

$= 0$.

$f_{xx}(0,0) = 0$, $f_{xy}(0,0) = 0$, $f_{yy}(0,0) = 0$

$\therefore Q(x,y) = L(x,y) + \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)\,xy + \frac{f_{yy}(0,0)}{2}y^2$

$= 0 + 0 + 0 + 0$

$= 0$.

3. $f(x,y) = x^2+y^2+\frac{2}{xy}$

$f_x = 2x + \frac{2}{y}\cdot\left(-\frac{1}{x^2}\right) = 2x - \frac{2}{x^2y}$, $f_y = 2y - \frac{2}{xy^2}$

$f_{xx} = 2 - \frac{2}{y}\cdot\left(-\frac{2}{x^3}\right) = 2 + \frac{4}{x^3y}$

$f_{xy} = \left(-\frac{2}{x^2}\right)\left(-\frac{1}{y^2}\right) = \frac{2}{x^2y^2} = f_{yx}$

$f_{yy} = 2 - \frac{2}{x}\cdot\left(-\frac{2}{y^3}\right) = 2 + \frac{4}{xy^3}$

For maximum and minimum,

$f_x = 0$,

$\Rightarrow 2x - \frac{2}{x^2y} = 0$

$\Rightarrow \frac{2}{y^3} - \frac{2}{(y^3)^2y} = 0$

$\Rightarrow \frac{2}{y^3} - \frac{2}{y^7} = 0$

$\Rightarrow \frac{2y^4-2}{y^7} = 0$

$\Rightarrow y^4-1=0$

$\Rightarrow y^4=1$

$y = -1, 1$

$x = -1, 1$

$f_y = 0$

$\Rightarrow 2y - \frac{2}{xy^2} = 0$

$\Rightarrow \frac{2xy^3-2}{xy^2} = 0$

$\Rightarrow xy^3-1 = 0$

$\Rightarrow x = \frac{1}{y^3}$

Now, $D = f_{xx}f_{yy} - (f_{xy})^2$

$= \left(2+\frac{4}{x^3y}\right)\left(2+\frac{4}{xy^3}\right) - \left(\frac{2}{x^2y^2}\right)^2$

At $(-1,-1)$, $D = \left\{2+\frac{4}{(-1)^3(-1)}\right\}\left\{2+\frac{4}{(-1)(-1)^3}\right\} - \left\{\frac{2}{(-1)^2(-1)^2}\right\}^2$

$= (2+4)(2+4) - (2)^2$

$= 6\cdot6-4$

$= 32>0$

$f_{xx} = 2 + \frac{4}{(-1)^3(-1)} = 2+4 = 6>0$

so $(-1,-1)$ is a local minimum.

At (1);

$$D = \left(2+\frac{4}{1^3\cdot 1}\right)\left(2+\frac{4}{1\cdot 1^3}\right) - \left(\frac{2}{1^2\cdot 1^2}\right)^2$$

$= 6 \quad 6 \quad 4$

$= 32 > 0$

$f_{xx} = 2 + \frac{4}{1^3\cdot 1} = 6 > 0$

$\therefore$ (1,1) is also local minimum.

As there is no $D<0$ condition, so no saddle point.

4. $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

$f_x = y - \frac{2}{x^2}$, $f_y = x - \frac{4}{y^2}$

$f_{xx} = (-2)\left(-\frac{2}{x^3}\right) = \frac{4}{x^3}$,

$f_{xy} = 1$

$f_{yx} = 1$

$f_{yy} = (-4)\left(-\frac{2}{y^3}\right) = \frac{8}{y^3}$

Now, $f_x = 0$

$\Rightarrow y - \frac{2}{x^2} = 0$

$\Rightarrow y - \frac{2}{\frac{16}{y^4}} = 0$

$\Rightarrow y - \frac{2y^4}{16} = 0$

$\Rightarrow 8y - y^4 = 0$

$\Rightarrow y^4 = 8y$

$f_y = 0$

$\Rightarrow x - \frac{4}{y^2} = 0$

$\therefore x = \frac{4}{y^2}$

$\Rightarrow y^3 = 8$

$\therefore y = 2$

$\therefore x = 1$

Now, $D = \frac{4}{x^3}\cdot\frac{8}{y^3} - 1^2$

$= \frac{32}{x^3y^3} - 1$

At (1,2) point,

$$D = \frac{32}{1^3 \cdot 2^3} - 1$$

$$= 4-1$$

$$= 3 > 0$$

$$f_{xx}(1,2) = \frac{4}{1^7} = 4 > 0$$

$\therefore$ (1,2) is a local minimum.

As there is no $D<0$ condition, so no saddle point.

5. $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left\{yze^{xy}\hat{i} + xze^{xy}\hat{j} + (e^{xy}+3\cos 3z)\hat{k}\right\}$$

$$= yz\cdot e^{xy}\cdot y + xze^{xy}\cdot x + (0 - 3\sin 3z\cdot 3)$$

$$= y^2ze^{xy} + x^2ze^{xy} - 9\sin 3z$$

Curl $\vec{F} = \vec{\nabla}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xy} & xze^{xy} & e^{xy}+3\cos 3z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(e^{xy}+3\cos 3z) - \frac{\partial}{\partial z}(xze^{xy})\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(e^{xy}+3\cos 3z) - \frac{\partial}{\partial z}(yze^{xy})\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(xze^{xy}) - \frac{\partial}{\partial y}(yze^{xy})\right\}$$

$$= \hat{i}(xe^{xy} - x^2ze^{xy}) - \hat{j}(ye^{xy} - ye^{xy}) + \hat{k}\left[z\{x\cdot e^{xy}\cdot y + e^{xy}\} - z\{y\cdot e^{xy}\cdot x + e^{xy}\}\right]$$

$$= xe^{xy}(1-xz)\hat{i} - 0\hat{j} + z(\cancel{xe^{xy}}\cdot y + \cancel{e^{xy}} - \cancel{xe^{xy}}\cdot y - \cancel{e^{xy}})\hat{k}$$

$$= xe^{xy}(1-xz)\hat{i}$$

6. $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$

$= yz + \sin z + 0$

$= yz + \sin z$

$\operatorname{Curl}\vec{F} = \vec{\nabla}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y\sin z & y\cos x \end{vmatrix}$$

$= \hat{i}\left(\cos x - y\cos z\right) - \hat{j}\left(-y\sin x - xy\right) + \hat{k}\left(0 - xz\right)$

$= \left(\cos x - y\cos z\right)\hat{i} + y\left(\sin x + x\right)\hat{j} - xz\,\hat{k}$