



Department of Mathematics and Natural Sciences

MAT 110

**Final Examination**

SUMMER 2021

**SET: 12**

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1. Given that  $f(x) = \frac{2}{1-x}$ , find  $f^{(n)}(0)$  using Maclaurin series.
2. Find points  $P$  and  $Q$  on the parabola  $y = 1 - x^2$  so that the triangle  $ABC$  formed by the  $x$ -axis and the tangent lines at  $P$  and  $Q$  is an equilateral triangle (see figure 1).

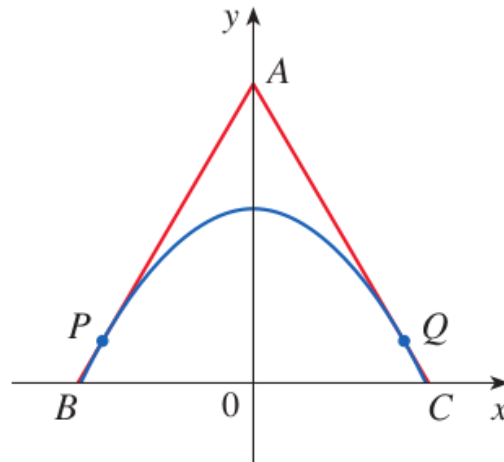


Figure 1: Figure for question 1

3. Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Find  $f_x(x, y)$ ,  $f_y(x, y)$  at all points. Leave your answer as a piecewise function.

4. If the length of the diagonal of a rectangular box must be  $L$ , what is the largest possible volume?
5. If a vector function  $\mathbf{F}$  depends on both space coordinates  $(x, y, z)$  and time  $t$ , show that  $d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t}$ .
6. Given the parameter  $\tau$ , constants  $\alpha$  and  $c$ , show that the parametric equations:

$$X(\tau) = \frac{c^2}{\alpha} \left( \cosh \left( \frac{\alpha\tau}{c} \right) - 1 \right), T(\tau) = \frac{c}{\alpha} \sinh \left( \frac{\alpha\tau}{c} \right)$$

describe a hyperbola on the  $X - T$  plane. This is hyperbolic motion in special relativity.

7. In three-dimensional cylindrical coordinates, sketch the surfaces of constant  $\rho$ , constant  $\varphi$  and constant  $z$ . Add a brief explanation.

Answer - 2

Given,

$$f(x) = \frac{2}{1-x}$$

$$f'(x) = \frac{d}{dx} \{ 2(1-x)^{-1} \}$$

$$= -2(1-x)^{-2} (-1)$$
$$= \frac{2}{(1-x)^2}$$

$$\therefore f'(0) = \frac{2}{(1-0)^2} = 2$$

$$f''(x) = \frac{d}{dx} \{ 2(1-x)^{-2} \}$$

$$= -4(1-x)^{-3} (-1)$$

$$= \frac{4}{(1-x)^3}$$

$$\therefore f''(0) = \frac{4}{(1-0)^3} = 4$$

$$\therefore f'''(0) = \frac{12}{(1-0)^4} = 12$$

and so on.

Therefore using Maclaurin series we have

$$f^{(n)}(0) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= 2 + 2x + \frac{4x^2}{2!} + \frac{12}{3!}x^3 + \dots$$

$$\therefore \underline{f^{(n)}(0) = 2 + 2x^2}$$

$$f^{(n)}(0) = 2 + 2x + 2x^2 + 2x^3 + \dots$$

(Ans)

Answer - 2

The equation of the parabola is  $y = 1 - x^2$

Now differentiating we get

$$\frac{d}{dx}(y) = \frac{d}{dx}(1 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = -2x$$

At P we have the tangent AB.

Now,  $\angle B = 60^\circ$   $\triangle ABC$  is equilateral triangle

$$\therefore \text{slope of AB} = \tan 60^\circ = \sqrt{3}$$

$$\therefore -2x_1 = \sqrt{3}$$

$$\Rightarrow x_1 = -\frac{\sqrt{3}}{2}$$

$$\therefore y_1 = 1 - x_1^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore P = \left( -\frac{\sqrt{3}}{2}, \frac{1}{4} \right), \quad Q = \left( \frac{\sqrt{3}}{2}, \frac{1}{4} \right)$$

$$\frac{s}{x-1} = (x)^0$$

$$\left\{ \frac{s}{(x-1)^2} \right\} \frac{b}{x^b} = (x)^1$$

$$(x-1)^{-2} \cdot \frac{s}{(x-1)^2} = (x)^1$$

$$\frac{s}{(x-1)^4} = (x)^1$$

$$\frac{s}{(x-1)^4} = (x)^1$$

$$\left\{ \frac{s}{(x-1)^4} \right\} \frac{b}{x^b} = (x)^1$$

$$(x-1)^{-4} \cdot \frac{s}{(x-1)^4} = (x)^1$$

$$\frac{s}{(x-1)^8} = (x)^1$$

$$\frac{s}{(x-1)^8} = (x)^1$$

Answer-3

Given,

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & x \neq 0 \\ 0, & x = 0 \end{cases}$$

when,  $x \neq 0$

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

$$\begin{aligned} \therefore f_n(x, y) &= \frac{(x^2 + y^2)y - xy(2x + 0)}{(x^2 + y^2)^2} \\ &= \frac{x^2y + y^3 - 2x^2y}{(x^2 + y^2)^2} \end{aligned}$$

$$\therefore f_n(x, y) = \frac{y^3 - x^2y}{(x^2 + y^2)^2} \quad (\text{Ans})$$

again,

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

$$\therefore f_y(x, y) = \frac{(x^2 + y^2) \cdot 1 - xy(0 + 2y)}{(x^2 + y^2)^2}$$

Ans



Answer 5

$$\vec{F} \equiv \vec{F}(x, y, z, t)$$

$$\therefore \frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial x} \frac{dx}{dt} + \frac{\partial \vec{F}}{\partial y} \frac{dy}{dt} + \frac{\partial \vec{F}}{\partial z} \frac{dz}{dt} + \frac{\partial \vec{F}}{\partial t}$$

$$\Rightarrow \frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial x} v_x + \frac{\partial \vec{F}}{\partial y} v_y + \frac{\partial \vec{F}}{\partial z} v_z + \frac{\partial \vec{F}}{\partial t}$$

[Multiplying both sides by dt]

$$\Rightarrow d\vec{F} = d\hat{n} \frac{\partial \vec{F}}{\partial \hat{n}} + d\hat{y} \frac{\partial \vec{F}}{\partial \hat{y}} + d\hat{z} \frac{\partial \vec{F}}{\partial \hat{z}} + \frac{\partial \vec{F}}{\partial t} dt$$

$$\Rightarrow d\vec{F} = d\hat{x} \hat{i} \left( \hat{i} \frac{\partial}{\partial \hat{x}} \vec{F} \right) + d\hat{y} \hat{j} \left( \hat{j} \frac{\partial}{\partial \hat{y}} \vec{F} \right) + d\hat{z} \hat{k} \left( \hat{k} \frac{\partial}{\partial \hat{z}} \vec{F} \right) + \frac{\partial \vec{F}}{\partial t} dt$$

$$\Rightarrow d\vec{F} = d\hat{n} \hat{i} \left[ \hat{i} \frac{\partial}{\partial \hat{x}} + \hat{j} \frac{\partial}{\partial \hat{y}} + \hat{k} \frac{\partial}{\partial \hat{z}} \right] \vec{F} + d\hat{y} \hat{j} \left[ \hat{i} \frac{\partial}{\partial \hat{x}} + \hat{j} \frac{\partial}{\partial \hat{y}} + \hat{k} \frac{\partial}{\partial \hat{z}} \right] \vec{F} + \frac{\partial \vec{F}}{\partial t} dt$$

$$= dx \hat{i} \cdot \nabla \vec{F} + dy \hat{j} \cdot \nabla \vec{F} +$$

$$dz \hat{k} \cdot \nabla \vec{F} + \frac{\partial \vec{F}}{\partial t} dt$$

$$= dx \hat{i} + dy \hat{j} + dz \hat{k} \cdot \nabla \vec{F} +$$

$$\frac{\partial \vec{F}}{\partial t} dt$$

$$= d\vec{r} \cdot \nabla \vec{F} + \frac{\partial \vec{F}}{\partial t} dt$$

Answer - 6

Given,

$$h(x) = \frac{c^2}{a} \left( \cosh \left( \frac{ax}{c} \right) - 1 \right)$$

$$T(x) = \frac{c}{a} \sinh \left( \frac{ax}{c} \right)$$

$$\text{Now } cT(x) = \frac{c^2}{a} \sinh \left( \frac{ax}{c} \right)$$

This gives

$$\left( h - \frac{c^2}{a} \right)^2 - c^2 T^2 = \frac{c^4}{a^2}$$

$\therefore$  it is a hyperbola. (Ans)