Qu: 01

Given,

$$f(x,y) = x^2y + y^2$$

Now,

$f_x(x,y) = 2xy$

$f_y(x,y) = x^2 + 2y$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = 2$

$f_{xy}(x,y) = 2x$

at point $(1,3)$

$f(1,3) = 1^2 \times 3 + 3^2 = 12$

$f_x(1,3) = 2 \times 1 \times 3 = 6$

$f_y(1,3) = 1^2 + 2 \times 3 = 7$

$f_{xx}(1,3) = 2 \times 3 = 6$

$f_{yy}(1,3) = 2$

$f_{xy}(1,3) = 2 \times 1 = 2$

Now, 1st degree Taylor polynomal of $f(x,y)$ near at point $(1,3)$

$$L(x,y) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$$

$$= 12 + 6(x-1) + 7(y-3)$$

$$= 12 + 6x - 6 + 7y - 21$$

$$= 6x + 7y - 15$$

$\therefore L(x,y) = 6x + 7y - 15$ polynomial of $f(x,y)$

Again, 2nd degree Taylor polynomial near at point $(1,3)$

$$Q(x,y) = L(x,y) + \frac{f_{xx}(1,3)}{2}(x-1)^2 + f_{xy}(1,3)(x-1)(y-3) + \frac{f_{yy}(1,3)}{2}(y-3)^2$$

$$= 6x + 7y - 15 + \frac{6}{2}(x-1)^2 + 2(x-1)(y-3) + \frac{2}{2}(y-3)^2$$

$$= 6x + 7y - 15 + 3(x-1)^2 + 2(x-1)(y-3) + (y-3)^2$$

$= 6x + 7y - 15 + 3(x^2 - 2x + 1) + 2(xy - 3x - y + 3)$
$+ y^2 - 6y + 9$

$= 6x + 7y - 15 + 3x^2 - 6x + 3 + 2xy - 6x - 2y + 6$
$+ y^2 - 6y + 9$

$\cancel{= 3x^2 - 6x - y + 2xy + 0 \; y^2 + 3}$

$= 3x^2 + y^2 - 6x - y + 2xy + 3$

$\therefore Q(x,y) = 3x^2 + y^2 - 6x - y + 2xy + 3$ (Ans)

Qu: 02

Given,

$$f(x,y) = \ln(x^2+y^2+1)$$

Now,

$$f_x(x,y) = \frac{1 \cdot 2x}{x^2+y^2+1} = \frac{2x}{x^2+y^2+1}$$

$$f_y(x,y) = \frac{1 \cdot 2y}{x^2+y^2+1} = \frac{2y}{x^2+y^2+1}$$

$$f_{xx}(x,y) = \frac{(x^2+y^2+1).2 - 2x.(2x+0+0)}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2+2y^2+2-4x^2}{(x^2+y^2+1)^2}$$

$$= \frac{2y^2-2x^2+2}{(x^2+y^2+1)^2}$$

$$f_{xy}(x,y) = -\frac{2x}{(x^2+y^2+1)^2}(0+2y+0)$$

$$= -\frac{4xy}{(x^2+y^2+1)^2}$$

$$f_{yy}(x,y) = \frac{(x^2+y^2+1)2 - 2y(0+2y+0)}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2+2y^2+2-4y^2}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

Now,

$f(0,0) = \ln(0+0+1) = \ln 1 = 0$

$f_x(0,0) = 0$

$f_y(0,0) = 0$

$f_{xy}(0,0) = 0$

$f_{xx}(0,0) = 2$

$f_{yy}(0,0) = 2$

$\therefore$ 1st degree Taylor polynomial of $f(x,y)$ near at point $(0,0)$

$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$

$= 0 + 0 + 0$

$= 0$

$\therefore L(x,y) = 0$

Again, 2nd degree Taylor polynomial of $f(x,y)$ near at point $(0,0)$

$$Q(x,y) = L(x,y) + \frac{f_{xx}(0,0)}{2}(x-0)^2 + f_{xy}(0,0)(x-0)(y-0) + \frac{f_{yy}(0,0)}{2}(y-0)^2$$

$$= 0 + \frac{2}{2}x^2 + 0 \times xy + \frac{2}{2}y^2$$

$$= x^2+y^2$$

$\therefore\ Q(x,y) = x^2+y^2$ (Ans)

<u>Qu: no: 07</u>

Given, $f(x,y) = xe^y + 1$

Now,

$$f_x(x,y) = e^y$$

$$f_y(x,y) = xe^y \qquad f_{xy}(x,y) = e^y$$

$$f_{xx}(x,y) = 0$$

$$f_{yy}(x,y) = xe^y$$

$\therefore\ f(1,0) = 1 \times e^{0} + 1 = 2$

$f_x(1,0) = e^{0} = 1$

$f_y(1,0) = 1 . \times e^{0} = 1$

$f_{xx}(1,0) = 0$

$f_{yy}(1,0) = 1 \times e^{0} = 1$

$f_{xy}(1,0) = e^{0} = 1$

Now, 1st degree Taylor polynomial of $f(x,y)$ near at point $(1,0)$

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$= 2 + 1 . (x-1) + 1 \times y$$

$$\cancel{= 2 + xe^{y} - e^{y} + y}$$

$$= 2 + x - 1 + y$$

$$= 1 + x + y$$

$$\therefore L(x,y) = 1 + x + y$$

Again, 2nd degree Taylor polynomial of $f(x,y)$ near at point $(1,0)$

$$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0) + \frac{f_{yy}(1,0)}{2}(y-0)^2$$

$$= 1+x+y + \frac{0}{2}(x-1)^2 + 1(x-1)(y-0) + \frac{1}{2}y^2$$

$$= 1+x+y+xy-y+\frac{1}{2}y^2$$

$$= 1+x+xy+\frac{y^2}{2}$$

$$\therefore Q(x,y) = 1+x+xy+\frac{y^2}{2}$$

(Ans)

Qu: no: 3

Given,

$$f(x,y) = x^2 + y^2 + \frac{2}{xy}$$

$$f_x(x,y) = 2x + \frac{2}{y}\left(-\frac{1}{x^2}\right) = 2x - \frac{2}{x^2 y}$$

$$f_y(x,y) = 2y - \frac{2}{xy^2}$$

Now, for critical point

$f_x(x,y) = 0$ and $2y - \frac{2}{xy^2} = 0$

$\Rightarrow 2x - \frac{2}{x^2 y} = 0$ $\quad\Rightarrow y = \frac{1}{xy^2}$

$\Rightarrow x - \frac{1}{x^2 y} = 0$ $\quad\Rightarrow y^3 = \frac{1}{x}$

$\Rightarrow x = \frac{1}{x^2 y}$ $\quad\Rightarrow x = \frac{1}{y^3}$

$\Rightarrow x^3 = \frac{1}{y}$

$y = \frac{1}{x^3}$

Thus, the ~~stationary~~ ~~critical~~ critical points are $(1,1)$; $(-1,-1)$

$$f_{xx}(x,y) = 2 - \frac{2}{y}(-2)\frac{1}{x^3} = 2 + \frac{4}{x^3 y}$$

$$f_{yy}(x,y) = 2 - \frac{2}{x}(-2)\frac{1}{y^3} = 2 + \frac{4}{x y^3}$$

$$f_{xy}(x,y) = 0 - \frac{2}{x^2}(-1)\frac{1}{y^2} = \frac{2}{x^2 y^2}$$

$$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$$

$$= \left(2 + \frac{4}{x^3 y}\right)\left(2 + \frac{4}{x y^3}\right) - \left(\frac{2}{x^2 y^2}\right)^2$$

at point (1,1)

$$D(1,1) = (2+4)\left(2 + \frac{4}{1}\right) - \left(\frac{2}{1\times1}\right)^2$$

$$= 6 \times 6 - 4$$

$$= 36 - 4 = 32 > 0$$

$$f_{xx}(1,1) = 2 + 4 = 6 > 0$$

Therefore, $f(x,y)$ has a relative minimum at point (1,1)

$f \ldots \cancel{x^2+y^2+2 = 1^2+1^2+2}$

Again at point $(-1,-1)$

$$D(-1,-1) = \left\{2 + \frac{4}{(-1)^3(-1)}\right\}\left\{2 + \frac{4}{(-1)(-1)^3}\right\} - \left\{\frac{2}{(-1)^2(-1)^2}\right\}^2$$

$$= (2+4)(2+4) - 4$$

$$= 6 \times 6 - 4 = 32 > 0$$

$$f_{xx}(-1,-1) = (2+4) > 0$$

$f(x,y)$ has a relative minimum at point $(1,1)$ (Ans)

(4)

$$f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$$

$$f_x(x,y) = y - \frac{2}{x^2}$$

$$f_y(x,y) = x - \frac{4}{y^2}$$

let, $y - \frac{2}{x^2} = 0$

$\Rightarrow y = \frac{2}{x^2}$ —— (I)

and $\Rightarrow x^2 y = 2$ —— (I)

$x - \frac{4}{y^2} = 0$

$\Rightarrow x = \frac{4}{y^2}$

$\Rightarrow xy^2 = 4$ —— (II)

(I) $\times y$ − (II) $\times x$

$$x^2y^2 = 2y$$

$$\underset{(-)}{x^2y^2} = \underset{(-)}{4x}$$

$$2y - 4x = 0$$

$$\Rightarrow y = 2x$$

Now from eq (I)

$x^2 \cdot 2x = 2$

$\Rightarrow x^3 = 1$

$\therefore x = 1$

$y = \frac{2}{x^2} = \frac{2}{1^2}$

$\therefore y = 2$

$\therefore$ stationary point $(1,2)$

$$f_{xx}(x,y) = -2(-2)\frac{1}{x^3} = \frac{4}{x^3}$$

$$f_{yy}(x,y) = -4(-2)\frac{1}{y^3} = \frac{8}{y^3}$$

$$f_{xy}(x,y) = 1 + 0 = 1$$

$$D(x,y) = f_{xx}(x,y) \times f_{yy}(x,y) - (f_{xy}(x,y))^2$$

$$= \frac{4}{x^3} \times \frac{8}{y^3} - 1^2$$

at point $(1,2)$

$$D(1,2) = \frac{4}{1^3} \times \frac{8}{2^3} - 1$$

$$= 4 - 1 = 3 > 0$$

and $f_{xx}(1,2) = \frac{4}{1^3} = 4 > 0$

Thus, $f(x,y)$ has a relative minimum at point $(1,2)$ (Ans)

$$\cancel{f(1,2) = 1\times 2 + \frac{2}{1} + \frac{4}{2}}$$

$$\cancel{= 2 + 2 + 2 = 6}$$ (Ans)

Qu: 9

Given,

$f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x(x,y) = 4x^3 - 4y$

$f_y(x,y) = 4y^3 - 4x$

Now, $4x^3 - 4y = 0$

$\Rightarrow x^3 - y = 0$ — (1)

$\therefore y = x^3$ — (1)

and $4y^3 - 4x = 0$

$y^3 - x = 0$ — (2)

$\Rightarrow (x^3)^3 - x = 0$

$x^9 - x = 0$

$\Rightarrow x(x^8 - 1) = 0$

$x = 0, \quad x^8 - 1 = 0$

$x = \pm 1$

$\therefore x = 0, 1$

$x = 0$, then $y = 0$

$x = 1$, then $y = 1$

$x = -1$, then, $y = -1$

stationary point $(0,0)$, $(1,1)$ & $(-1,-1)$

$\therefore$

Again,

$f_{xx}(x,y) = 12x^2$

$f_{yy}(x,y) = 12y^2$

$f_{xy}(x,y) = -4$

Now, $D(x,y) = f_{xx}(x,y) \times f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$

$= 12x^2 \times 12y^2 - (-4)^2$

$= 144x^2y^2 - 16$

at point $(0,0)$

$D(0,0) = 0 - 16 = -16 < 0$

therefore, $(0,0)$ is a saddle point (Ans)

at point $(1,1)$

$D(1,1) = 12(1)^2 \times 12(1)^2 - (-4)^2$

$= 144 - 16$

$= 128 > 0$

$f_{xx}(1,1) = 12 \times (1)^2 = 12 > 0$

therefore, $f(x,y)$ has a local minimum value at point $(1,1)$

Therefore, the minimum value

$$f(1,1) = 1^4 + 1^4 - 4(1)(1) + 1$$
$$= 1 + 1 - 4 + 1$$
$$= -1$$

Again at point $(-1,-1)$

$$D(-1,-1) = 144(-1)^2(-1)^2 - 16$$
$$= 144 - 16$$
$$= 128$$

$$f_{xx}(-1,-1) = 12(-1)^2 = 12$$

Therefore, $f(x,y)$ has another local minimum value at point $(-1,-1)$

Therefore, the minimum value,

$$f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1$$
$$= 1 + 1 - 4 + 1$$
$$= -1 \quad \text{(Ans)}$$

(10)

The given function,

$f(x,y) = x^2y^3 - 4y$

and given vector, $\vec{v} = 2\hat{i} - 5\hat{j}$

thus, the unit vector, $\hat{v} = \frac{2\hat{i} - 5\hat{j}}{\sqrt{2^2 + (-5)^2}} = \frac{2\hat{i} - 5\hat{j}}{\sqrt{4+25}}$

$= \frac{2\hat{i} - 5\hat{j}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{i} - \frac{5}{\sqrt{29}}\hat{j}$

Now, the partial derivatives of f are

$f_x(x,y) = 2xy^3$; $f_y(x,y) = 3x^2y^2 - 4$

at point (2, -1)

$f_x(2,-1) = 2 \times 2 \times (-1)^3 = -4$

$f_y(2,-1) = 3 \times 2^2 (-1)^2 - 4 = 12 - 4 = 8$

Therefore, the directional derivatives of $f(x,y)$ at point (2, -1) in the direction of vector $\vec{v} = 2\hat{i} - 5\hat{j}$

$D_u f(2,-1) = -4 \times \frac{2}{\sqrt{29}} + 8 \times \left(-\frac{5}{\sqrt{29}}\right)$

$= -\frac{8}{\sqrt{29}} - \frac{40}{\sqrt{29}} = \frac{-48}{\sqrt{29}}$ (Ans)

11

(a)

Given, $f(x,y,z) = x^2y + y^2z$

$f_x(x,y,z) = 2xy + 0 = 2xy$

$f_y(x,y,z) = x^2 + 2yz$

$f_z(x,y,z) = y^2$

at point $(1,2,3)$

gradient of $f(x,y,z)$

$$\nabla f(1,2,3) = f_x(1,2,3)\hat{i} + f_y(1,2,3)\hat{j} + f_z(1,2,3)\hat{k}$$

$$= 2\times1\times2\,\hat{i} + (1^2 + 2\times2\times3)\hat{j} + 2^2\hat{k}$$

$$= 4\hat{i} + 13\hat{j} + 4\hat{k}$$

$\therefore \nabla f(1,2,3)$

(b)

Given, $f(x,y,z) = x^2y + y^2z$

$f_x(x,y,z) = 2xy$

$f_y(x,y,z) = x^2 + 2yz$

$f_z(x,y,z) = y^2$

unit vector of the given vector, $\hat{v} = \dfrac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{2^2 + (-1)^2 + 2^2}}$

$= \dfrac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{9}}$

$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$

Thus, the directional derivatives of $f(x,y,z)$ at $(1,2,3)$ in the direction of $\bar{v} = 2\hat{i} - \hat{j} + 2\hat{k}$

$D_u f(1,2,3) = f_x(1,2,3) \times \frac{2}{3} + f_y(1,2,3)\left(-\frac{1}{3}\right) + f_z(1,2,3)\left(\frac{2}{3}\right)$

$= 2 \times 1 \times 2 \times \frac{2}{3} + (1^2 + 2 \times 2 \times 3)\left(-\frac{1}{3}\right) + 2^2 \times \frac{2}{3}$

$= \frac{8}{3} - \frac{13}{3} + \frac{8}{3} = \frac{8 - 13 + 8}{3} = \frac{3}{3} = 1$

(Ans)

(5)

Given,

$$\vec{F} = yz\,e^{xy}\,\hat{i} + xz\,e^{xy}\,\hat{j} + (e^{xy} + 3\cos 3z)\,\hat{k}$$

Therefore, the divergence of $\vec{F}$,

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left\{yz\,e^{xy}\,\hat{i} + xz\,e^{xy}\,\hat{j} + (e^{xy} + 3\cos 3z)\,\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(yz\,e^{xy}) + \frac{\partial}{\partial y}(xz\,e^{xy}) + \frac{\partial}{\partial z}(e^{xy} + 3\cos 3z)$$

$$= yz\,e^{xy}\cdot y + xz\,e^{xy}\cdot x + (0 - 3\sin 3z\cdot 3)$$

$$\therefore \text{div}\vec{F} = y^2z\,e^{xy} + x^2z\,e^{xy} - 9\sin 3z$$

(Ans)

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xy} & xze^{xy} & e^{xy} + 3\cos 3z \end{vmatrix}$$

$$= \hat{i}\left(e^{xy}.x + 0 - xe^{xy}\right) - \hat{j}\left(e^{xy}.y + 0 - ye^{xy}\right)$$

$$+ \hat{k}\left(zx.e^{xy}.y + e^{xy}.z - yze^{xy}.x - e^{xy}z\right)$$

$$= \hat{i}\times 0 - \hat{j}\times 0 + \hat{k}\times 0$$

$$= 0 \quad \text{(Ans)}$$

(6)

Given,

$$\vec{F} = xyz\,\hat{i} + y\sin z\,\hat{j} + (y\cos x)\,\hat{k}$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(xyz\,\hat{i} + y\sin z\,\hat{j} + y\cos x\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(y\sin z) + \frac{\partial}{\partial z}(y\cos x)$$

$$= yz + \sin z + 0$$

$$= yz + \sin z$$

$$\text{curl}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y\sin z & y\cos x \end{vmatrix}$$

$$= \hat{i}(\cos x - y\cos z) - \hat{j}(-y\sin x - xy) + \hat{k}(0 - zx)$$

$$\therefore \text{Curl}\,\vec{F} = (\cos x - y\cos z)\hat{i} + (xy + y\sin x)\hat{j} - zx\hat{k} \quad \text{(Ans)}$$

(12)

Given, $\vec{F} = (3x + 2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} - (z - 7x)\hat{k}$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left\{(3x+2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} - (z-7x)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(3x+2z^2) + \frac{\partial}{\partial y}\left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z}(-z+7x)$$

$$= 3 + 0 + \frac{x^3}{z}(2y) - 1$$

$$\therefore \text{div}\,\vec{F} = \frac{2x^3y}{z} + 2$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+2z^2 & \frac{x^3y^2}{z} & 7x-z \end{vmatrix}$$

$$= \hat{i}\left(0 - x^3y^2 \cdot \frac{-1}{z^2}\right) - \hat{j}\left(7-0-0-4z\right) + \hat{k}\left(\frac{y^2}{z} \cdot 3x^2 - 0\right)$$

$$= \frac{x^3y^2}{z^2}\hat{i} - (7-4z)\hat{j} + \frac{3x^2y^2}{z}\hat{k}$$

$$\therefore \text{curl}\,\vec{F} = \frac{x^3y^2}{z^2}\hat{i} + (4z-7)\hat{j} + \frac{3x^2y^2}{z}\hat{k}$$

(Ans)

(8)

Let,

Length of the box $= x$ (meter)

Breath of the box $= y$ (meter)

Height of the box $= z$ (meter)

Surface area $= xy + 2yz + 2zx$

$$\Rightarrow xy + 2yz + 2zx = 12$$

$$\Rightarrow 2z(y+x) = 12 - xy$$

$$\Rightarrow z = \frac{12 - xy}{2x+2y}$$

Volume, $V = xyz$

$$\Rightarrow V = xy \cdot \frac{12 - xy}{2x+2y}$$

$$\Rightarrow V = \frac{12xy - x^2y^2}{2x+2y}$$

$$\Rightarrow V_x = \frac{(2x+2y)(12y - 2xy^2) - (12xy - x^2y^2)(2+0)}{(2x+2y)^2}$$

$$\Rightarrow V_x = \frac{24xy - 4x^2y^2 + 24y^2 - 4xy^3 - 24xy + 2x^2y^2}{(2x+2y)^2}$$

$$\therefore V_x = \frac{24y^2 - 4xy^3 - 2x^2y^2}{(2x+2y)^2}$$

(8)

Now, $V_x = 0$

$$\Rightarrow \frac{24y^2 - 4xy^3 - 2x^2y^2}{(2x+2y)^2} = 0$$

$$\Rightarrow 24y^2 - 4xy^3 - 2x^2y^2 = 0$$

$$\Rightarrow x^2y^2 + 2xy^3 - 12y^2 = 0 \quad \text{--- (i)}$$

Again,

$$V_y = \frac{(2x+2y)(12x - 2x^2y) - (12xy - x^2y^2)(0+2)}{(2x+2y)^2}$$

$$= \frac{24x^2 - 4x^3y + 24xy - 4x^2y^2 - 24xy + 2x^2y^2}{(2x+2y)^2}$$

$$= \frac{24x^2 - 2x^2y^2 - 4x^3y}{(2x+2y)^2}$$

for $V_y = 0$

$$\Rightarrow \frac{24x^2 - 2x^2y^2 - 4x^3y}{(2x+2y)^2} = 0$$

$$\Rightarrow x^2y^2 + 2x^3y - 12x^2 = 0 \quad \text{--- (ii)}$$

$(I) \times x^2 - (II) \times y^2$

$$x^4y^2 + 2x^3y^3 - 12x^2y^2 = 0$$
$$x^2y^4 + 2x^3y^3 - 12x^2y^2 = 0$$
$$(-) \quad (-) \quad (+)$$

$$x^4y^2 - x^2y^4 = 0$$
$$\Rightarrow x^2y^2(x^2-y^2) = 0$$
$$\Rightarrow x^2-y^2=0$$
$$\Rightarrow x^2=y^2$$
$$\therefore x = y$$

From eq (I)

$$y^4 + 2y^4 - 12y^2 = 0$$
$$\Rightarrow 3y^4 - 12y^2 = 0$$
$$\Rightarrow 3y^2(y^2-4) = 0$$
$$\Rightarrow y^2(y^2-4) = 0$$

$y \neq 0$ (since y is the breath of box)

$$\Rightarrow y^2-4=0$$
$$\Rightarrow y = \sqrt{4}$$
$$y = 2$$
$$\therefore x = 2$$

Again,

$$V_{xx}(x,x) = \frac{(2x+2y)^2(0-4y^3-4xy^2)-(24y^2-4xy^3-2x^2y^2)\,2(2x+2y).2}{(2x+2y)^4}$$

$$\therefore V_{xx}(2,2) = \frac{(4+4)^2(-4\times 8-4\times 2\times 4)-(24\times 4-4\times 2\times 8-2\times 4\times 4).4(4+4)}{(4+4)^4}$$

$$= \frac{64\times(-64)-0}{8^4}$$

$$= \frac{-64\times 64}{8^4}$$

$$= -1$$

$$V_{yy}(x,y) = \frac{(2x+2y)^2(0-4x^3-4x^2y)-(24x^2-2x^2y^2-4x^3y)\,2(2x+2y).2}{(2x+2y)^4}$$

$$V_{yy}(2,2) = \frac{(4+4)^2(-4\times 8-4\times 4\times 2)-(24\times 4-2\times 4\times 4-4\times 8\times 2)4(4+4)}{(4+4)^4}$$

$$= \frac{64\times(-64)-0}{8^4}$$

$$= \frac{-64 \times 64}{8^4}$$

$$= -1$$

$$V_{xy}(x,y) = \frac{(2x+2y)^2(48y-12xy^2-4x^2y)-(24y^2-4xy^3-2x^2y^2)\cdot 2(2x+2y)\cdot 2}{(2x+2y)^4}$$

$$\therefore V_{xy}(2,2) = \frac{(4+4)^2(48\times 2-12\times 2\times 4-4\times 4\times 2)-(24\times 4-4\times 2\times 8-2\times 4\times 4)\cdot 4(4+4)}{(4+4)^4}$$

$$= \frac{64\times(-32)}{8^4}$$

$$= -\frac{1}{2}$$

Now, $D(x,y) = V_{xx}(x,y)\, V_{yy}(x,y) - \{V_{xy}(x,y)\}^2$

$$\therefore D(2,2) = V_{xx}(2,2) \cdot V_{yy}(2,2) - \{V_{xy}(2,2)\}^2$$

$$= (-1)(-1) - \left(-\tfrac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$= \frac{3}{4} > 0$

Again $V'_{xx}(2,2) = -1 < 0$

$\therefore$ Therefore, the ~~box~~ volume of the box will be maximum when length $x = 2$ and breath $y = 2$

Thus, $z = \frac{12 - xy}{2x + 2y}$

$= \frac{12 - 2\times2}{2\times2 + 2\times2}$

$= \frac{8}{8}$

$\therefore z = 1$

$\therefore$ The maximum volume, $V_{max} = 2\times2\times1$

$= 4 \, m^3$

(Ans)