

To find value of  $\nabla f(0,1)$  at point  $(0,1)$

Given,

$$f(x,y) = x^2y + y^2$$

Now,

$$f_x(x,y) = \frac{\partial}{\partial x} (x^2y + y^2) = 2xy + (-1)(y^2) = 2xy - y^2$$

$$f_y(x,y) = x^2 + 2y$$

$$f_{xx}(x,y) = 2y$$

$$f_{yy}(x,y) = 2$$

$$f_{xy}(x,y) = 2x$$

at point  $(1,3)$

$$(f(1,3) = 1^2 \times 3 + 3^2 = 12$$

$$f_x(1,3) = 2 \times 1 \times 3 = 6$$

$$f_y(1,3) = 1^2 + 2 \times 3 = 7$$

~~$$f_{xx}(1,3) = 2 \times 3 = 6$$~~

$$f_{yy}(1,3) = 2$$

$$f_{xy}(1,3) = 2 \times 1 = 2$$

polynomial

Now, 1st degree Taylor of  $f(x,y)$  near  
at point  $(1,3)$

$$\begin{aligned} \partial + BL(x,y) &= yf(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) \\ &= 12 + 6(x-1) + 7(y-3) \\ &= 12 + 6x - 6 + 7y - 21 \\ &= 6x + 7y - 15 \end{aligned}$$

$\therefore L(x,y) = 6x + 7y - 15$   
~~is a polynomial of  $f(x,y)$~~

Again, 2nd degree Taylor polynomial

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(1,3)}{2}(x-1)^2 + \frac{f_{xy}(1,3)(x-1)(y-3)}{2} \\ &\quad + \frac{f_{yy}(1,3)}{2}(y-3)^2 \\ &= 6x + 7y - 15 + \frac{6}{2}(x-1)^2 + \frac{2(x-1)(y-3)}{2} \\ &\quad + \frac{2}{2}(y-3)^2 \\ &= 6x + 7y - 15 + 3(x-1)^2 + 2(x-1)(y-3) \\ &\quad + (y-3)^2 \end{aligned}$$

$$\begin{aligned}
 &= 6x + 7y - 15 + 3(x^2 - 2x + 1) + 2(xy - 3x + y + 3) \\
 &\quad + y^2 - 6y + 9
 \end{aligned}$$

$$\begin{aligned}
 &= 6x + 7y - 15 + 3x^2 - 6x + 3 + 2xy - 6x + 2y + 6 \\
 &\quad + y^2 - 6y + 9 \\
 &= \cancel{6x} - \cancel{6x} - \cancel{6x} + \cancel{7y} + \cancel{3} + \cancel{2xy} + \cancel{y^2} + \cancel{3} - \cancel{6y} + \cancel{6} \\
 &= 3x^2 + y^2 - 6x - y + 2xy + 3
 \end{aligned}$$

$$\therefore Q(x, y) = 3x^2 + y^2 - 6x - y + 2xy + 3 = (S.V.) \text{ J. } \text{ (Ans)}$$

Ques: 02

(S.V) failing to meet

$$\text{Given, } f(x, y) = \frac{(x^2 + y^2 + 1)}{(x^2 + y^2 + 1)} + (S.V.) \text{ J. } \text{ (S.V) }$$

$$f(x, y) = \underline{\ln(x^2 + y^2 + 1)}$$

$$\text{Now, } f_x(x, y) = \frac{(x^2 + y^2 + 1) \cdot 2x}{x^2 + y^2 + 1} + \frac{2x}{x^2 + y^2 + 1}$$

$$\begin{aligned}
 f_y(x, y) &= \frac{1 \cdot 2y}{x^2 + y^2 + 1} = \frac{2y}{x^2 + y^2 + 1} \\
 &= \frac{2y}{(x^2 + y^2 + 1) + 2x}
 \end{aligned}$$

$$f_{xx}(x,y) = \frac{(x^2+y^2+1)\cdot 2 - 2x \cdot (2x+0+0)}{(x^2+y^2+1)^2} = (0,0)$$

$$= \frac{2x^2 + 2y^2 + 2 - 4x^2}{(x^2+y^2+1)^2}$$

$$= \frac{2y^2 - 2x^2 + 2}{(x^2+y^2+1)^2}$$

$$f_{xy}(x,y) = \frac{2x}{(x^2+y^2+1)^2} (0+2y+0)$$

$$= -\frac{4xy}{(x^2+y^2+1)^2} \quad (0,0) \text{ failing to}$$

$$f_{yy}(x,y) = \frac{(x^2+y^2+1) \cdot 2 - 2y(0+2y+0)}{(x^2+y^2+1)^2} = (0,0)$$

$$= \frac{2x^2 + 2y^2 + 2 - 4y^2}{(x^2+y^2+1)^2}$$

$$= \frac{2x^2 - 2y^2 + 2}{(x^2+y^2+1)^2}$$

Now,

$$f(0,0) = \ln(0+0+1) = \frac{f_{xx}(0,0)}{-2(1+e^x+e^y)} = (1/e)$$

$$f_x(0,0) = 0$$

$$f_y(0,0) = 0$$

$$f_{xy}(0,0) = 0$$

$$f_{xx}(0,0) = 2$$

$$f_{yy}(0,0) = 2$$

$\therefore$  1st degree Taylor polynomial  $f(x,y)$  near  
at point  $(0,0)$

$$L(x,y) = f(0,0) + \frac{f_{xx}(0,0)(x-0) + f_y(0,0)(y-0)}{(1+e^x+e^y)} = 0 + 0 + 0 = 0$$

$$= 0$$

$$\therefore L(x,y) = 0$$

Again, 2nd degree Taylor polynomial of  $f(x,y)$   
near at point  $(0,0)$

$$\begin{aligned} Q(x,y) &= L(x,y) + \frac{f_{xx}(0,0)}{2} (x-0)^2 + f_{xy}(0,0) (x-0)(y-0) \\ &\quad + \frac{f_{yy}(0,0)}{2} (y-0)^2 \\ &= 0 + \frac{2}{2} x^2 + 0 \times xy + \frac{2}{2} y^2 \\ &= x^2 + y^2 \end{aligned}$$

To find the value of  $f(x,y)$  near  $(0,0)$

$$\therefore Q(x,y) = x^2 + y^2 \quad \text{(Ans)}$$

Ques no: 07

Given,  $(x+1)(y+1)(0,0) \neq (0,0) \neq (y+1)$

$$f(x,y) = xe^y + 1$$

Now,

$$f_x(x,y) = e^y$$

$$f_y(x,y) = xe^y + f_{xy}(x,y) = e^y$$

$$f_{xx}(x,y) = 0$$

$$f_{yy}(x,y) = xe^{2y} + 1 = (y+1)$$

Want to minimize  $f(x, y)$   $\Rightarrow$   $f(1, 0) = 1 \times e^0 + 1 = 2$

$\therefore f_x(1, 0) = e^0 = 1$  (i.e) tending to zero.

$$f_{xy}(1, 0) = \frac{e^0}{x-y} = \frac{1}{1-0} = 1$$

$$f_y(1, 0) = 1 \times e^0 = 1$$

$$f_{xx}(1, 0) = 0$$

$$f_{yy}(1, 0) = 1 \times e^0 = 1$$

$$f_{xy}(1, 0) = e^0 = 1$$

Now, 1st degree Taylor polynomial of  $f(x, y)$  near at point  $(1, 0)$

$$L(u, y) = f(1, 0) + f_x(1, 0)(u-1) + f_y(1, 0)(y-0)$$

$$= 2 + 1 \cdot (u-1) + 1 \times y$$

$$= 2 + u - 1 + y$$

$$\therefore L(u, y) = 1 + u + y$$

$$\therefore L(x, y) = 1 + x + y$$

$$\therefore L(x, y) = 1 + x + y$$

Again, <sup>begin</sup> ~~2nd~~ degree Taylor polynomial of  $f(x,y)$

~~out to coefficient to obtain abbr~~  
near at point  $(1,0)$

$$Q(x,y) \text{ to a function } f_{xx}(1,0) + f_{xy}(1,0)(x-1) + f_{yy}(1,0)(y-0) \\ Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{yy}(1,0)(y-0)^2 \quad (1)$$

$$(x^2)(x^2) - (x^2)(x^2) \frac{f_{yy}(1,0)}{2} (y-0)^2 \quad (2)$$

fring ~~need to fit~~ before  $f_{yy}(1,0)(y-0)^2 + \frac{1}{2} y^2$   
local  $= 1 + xy + \frac{0}{2}(x-1)^2 + 1(x-1)(y-0) + \frac{1}{2} y^2$

fring ~~leaving off to~~ to  $L(x,y)$   $\Rightarrow$  (1)

$$\text{ext to } \text{rep} \rightarrow \text{local } 1 + xy + \frac{1}{2} y^2$$

$$\text{local } = 1 + x + xy + \frac{y^2}{2}$$

fring ~~leaving off to~~  $\Rightarrow$  (1)

$$\text{local } \therefore Q(x,y) = 1 + x + xy + \frac{y^2}{2} \quad (\text{Ans})$$

fring ~~leaving off to~~  $\Rightarrow$  ext to  $L(x,y)$

abbr  $\Rightarrow$  ext to  $L(x,y)$

- fring ~~leaving off to~~  $\Rightarrow$  (1)

local  $\Rightarrow$  abbr  $\Rightarrow$  fring

multiplied no cancellation local  $\Rightarrow$  (2)

cancel and multiply up

Qn: no: 3

Given,  $x + y = \frac{1}{xy} \Rightarrow xy - 1 = (x+y) \text{ ext}$

$$f(x,y) = x^2 + y^2 + \frac{2}{xy} - 1 = (x+y) \text{ ext}$$

$$f_x(x,y) = 2x + \frac{2}{y} \left(-\frac{1}{x^2}\right) = 2x - \frac{2}{xy^2}$$

$$f_y(x,y) = 2y - \frac{2}{x^2y} = 0 = (x+y) \text{ ext}$$

Now, for critical points,  $x+y = (x+y) \text{ ext}$

$$(x+y) f_x(x,y) = 0 \quad \text{and} \quad 2x - \frac{2}{xy^2} = 0$$

$$(x+y) - 2x - \frac{2}{xy^2} = 0 \quad \text{ext} \quad 2y - \frac{2}{xy^2} = 0$$

$$\Rightarrow x - \frac{1}{ny^2} = 0 \quad \Rightarrow y = \frac{1}{ny^2}$$

$$\Rightarrow n = \frac{1}{y^3}$$

$$0 \leq y = \frac{1}{n} = \frac{1}{y^3}$$

extremes  $\Rightarrow y = \frac{1}{n} = \frac{1}{y^3} \quad (x+y) \text{ ext}$

Thus, the critical points are  $(1,1)$ ;  $(-1,-1)$

minimising

$$\frac{1}{xy} + \frac{2}{xy^2} = \frac{1}{y^2} + \frac{2}{y^2} = \frac{1+2}{y^2} = \frac{3}{y^2} \text{ ext.}$$

$$f_{xx}(x,y) = 2 - \frac{2}{y} (-2) \frac{1}{x^3} = 2 + \frac{4}{x^3 y}$$

$$f_{yy}(x,y) = 2 - \frac{2}{x} (-2) \frac{1}{y^3} = 2 + \frac{4}{x y^3}$$

$$f_{xy}(x,y) = 0 - \frac{2}{x^2} (-2) \frac{1}{y^2} = \frac{4}{x^2 y^2}$$

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$$

$$= \left(2 + \frac{4}{x^3 y}\right) \left(2 + \frac{4}{x y^3}\right) - \left(\frac{4}{x^2 y^2}\right)^2$$

at point  $(1,1)$

$$D(1,1) = \left(2 + 4\right) \left(2 + \frac{4}{1}\right) - \left(\frac{4}{1 \times 1}\right)^2$$

$$= 6 \times 6 - 4$$

$$= 36 - 4 = 32 > 0$$

$$f_{xx}(1,1) = 2 + 4 = 6 > 0$$

Therefore  ~~$f(x,y)$~~  has a relative minimum at point  $(1,1)$

$$\sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Again at point  $(-1, -1)$

$$D(-1, -1) = f \left\{ 2 + \frac{4}{(-1)^3 (-1)} \right\} \left\{ 2 + \frac{4}{(-1)(-1)^3} \right\} + \left\{ \frac{2}{(-1)^2 (-1)} \right\}$$

$$\begin{aligned} &= (2+4)(2+4) = 4 \quad \text{left} \\ &= 6 \times 6 - 4 = 32 > 0 = \text{right} \end{aligned}$$

$$f_{xx}(-1, -1) = (2+4) = 6 > 0 \quad \text{, d.t.}$$

~~f(x, y)~~ has a relative minimum at point  $(1, 1)$  (Any),  $\frac{\partial^2 f}{\partial x^2} > 0$  by

(4D)  $\rightarrow$  finding the maxima

$$\left\{ \begin{array}{l} f(x,y) = xy + \left\{ \frac{2}{x} + \frac{4}{y} + 2 \right\} \\ f_x(x,y) = y - \frac{2}{x^2} \end{array} \right. \Rightarrow (1,1)$$

$$f_x(x,y) = y - \frac{2}{x^2} \stackrel{(4D)}{\Rightarrow} (1,1)$$

$$f_y(x,y) = x - \frac{4}{y^2}, \quad p = 2x^2$$

$$\text{let, } y - \frac{2}{x^2} = 0$$

$$\text{to minimize } y = \frac{2}{x^2} \quad \text{--- (i)}$$

$$\text{and } x - \frac{4}{y^2} = 0 \quad \text{from (iv)}.$$

$$\Rightarrow x = \frac{4}{y^2} \quad \text{(4D) taking}$$

$$\Rightarrow xy^2 = 4 \quad \text{--- (ii)}$$

$$(i) \times y - (ii) \times x$$

$$xy^2 = 2y$$

$$(i) \times y - (ii) \times x$$

$$2y - 4x = 0$$

$$\Rightarrow y = 2x$$

$$y = \frac{2}{x^2} = \frac{2}{1^2}$$

Now from eq (i)

$$x^2 \cdot 2x = 2$$

$$\Rightarrow x^3 = 1 \quad \therefore x = 1$$

$$\therefore y = 2$$

stationary point (1,2)

$$f_{xx}(u,y) = -2(-2) \frac{1}{x^3} = \frac{4}{x^3}$$

$$f_{yy}(u,y) = -4(-2) \frac{1}{y^3} = \frac{8}{y^3}$$

$$f_{xy}(u,y) = 1 + 0 = 1$$

$$D(u,y) = f_{xx}(u,y) \times f_{yy}(u,y) - \left( \frac{f_{xy}(u,y)}{y^3} \right)^2 = (uv)^2 - (uv)^2 = 0$$

$$= \frac{4}{x^3} \times \frac{8}{y^3} - 1^2 = 0 = \text{Exp. well}$$

at point  $(1,2)$

$$D(1,2) = \frac{4}{1^3} \times \frac{8}{2^3} - 1^2 = 3 > 0$$

$$\text{and } f_{xx}(1,2) = \frac{4}{1^3} = 4 > 0$$

$\therefore$  a relative minimum at point

Thus,  $f(u,y)$  has

$(1,2)$  (Am)

~~$$f(1,2) = 1 \times 2 + \frac{2}{2} + \frac{4}{2}$$~~

~~$$= 2 + 1 + 2 = 5$$~~

$\therefore (1,2)$  is a local maximum point

Ques

Given,

$$f(x,y) = x^4 + y^4 - 4xy + 1 \quad = \quad (x,y)_{\text{ext}}$$

$$f_x(x,y) = 4x^3 - 4y \quad \stackrel{\text{L} = 0}{=} \quad (x,y)_{\text{ext}}$$

$$f_y(x,y) = 4y^3 - 4x \quad \stackrel{(x,y)_{\text{ext}} \text{ & } f_{xy}(x,y) \neq 0}{=} \quad (x,y)_{\text{ext}}$$

$$\text{Now, } 4x^3 - 4y = 0 \quad \stackrel{\text{L} = 0}{=} \quad \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} =$$

$$\Rightarrow x^3 - y = 0 \quad \text{--- (1)}$$

$$\therefore y = x^3 \quad \text{--- (1)}$$

$$\text{and } 4y^3 - 4x = 0 \quad \stackrel{x = y^3}{=} \quad (x,y)_{\text{ext}}$$

$$y^3 - x = 0 \quad \text{--- (2)}$$

$$\Rightarrow (x^3)^3 - x = 0$$

$$x^9 - x = 0 \quad = \quad (x,y)_{\text{ext}}, \text{ bro}$$

$$\Rightarrow x(x^8 - 1) = 0$$

$$x = 0, \quad x^{8-1} = 0$$

$$(0,0) \quad (x,y)_{\text{ext}}$$

$$\therefore x = 0, \pm 1$$

$$\text{then } y = 0$$

$$x = 0, \text{ then } y = 0 \quad (0,0)$$

$$x = 1, \text{ then } y = 1 \quad (1,1)$$

$$x = -1, \text{ then } y = -1 \quad (-1,-1)$$

stationary point  $(0,0)$ , ~~(0,0)~~  $(1,1)$  &  $(-1,-1)$

Again,

$$f_{xx}(x,y) = 12x^2$$

and (x,y) is a local maxima

$$f_{yy}(x,y) = 12y^2$$

$$f_{xy}(x,y) = -4$$

$$\text{Now, } D(x,y) = f_{xx}(x,y) \times f_{yy}(x,y) - \{f_{xy}(x,y)\}^2$$

$$= 12x^2 \times 12y^2 - (-4)^2$$

$$= 144x^2y^2 - 16$$

$\rightarrow$  taking to origin

at point  $(0,0)$

$$D(0,0) = 144(0)^2(0)^2 - 16 = -16 < 0$$

saddle point (Ay)

therefore,  $(0,0)$  is a saddle point

at point  $(1,1)$   $\therefore D = (1,1)_{xx}$

local minima and  $= 12(1)^2 \times 12(1)^2 - (-4)^2$

$$D(1,1) = 144 - 16$$

$\rightarrow$  taking to origin  $\therefore 144 - 16 = 128 > 0$

$\therefore D(1,1) = 128 > 0$  (global minima)

$$f_{xx}(1,1) = 12(1)^2 = 12 > 0$$

$\therefore f_{xx}(1,1) > 0$

therefore,  $f(x,y)$  has a local minimum value at point  $(1,1) = (x_0, y_0)$

Therefore, the minimum value  $P = f(1,1) = (x_0, y_0)$

$$f(1,1) = -14(1+1)^4 + 4(1)(1) + 1 = (x_0, y_0)$$

$$\begin{aligned} &= -14 \cdot 2^4 + 4 + 1 \\ &= -14 \cdot 16 + 4 + 1 \\ &= -224 + 4 + 1 \\ &= -219 \end{aligned}$$

Again at point  $(-1,-1)$

$$D(-1,-1) = 144 \cdot (-1)^4 - 16 = 144 - 16 = 128$$

$$\begin{aligned} D(-1,-1) &= 144 - 16 \\ &= 128 \end{aligned}$$

$$f_{xx}(-1,-1) = 12(-1)^4 = 12$$

Therefore,  $f(x,y)$  has another local minimum value  $P$  at point  $(-1,-1)$

Therefore, the minimum value,

$$\begin{aligned} P &= f(-1,-1) = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1 \\ &= 1 + 1 - 4 + 1 \\ &= -1 \end{aligned}$$

(10)

(A)

The given function,

$$f(x,y) = x^2y^3 - 4y$$

and given vector,  $\vec{v} = 2\hat{i} - 5\hat{j}$

thus, the unit vector,  $\hat{v} = \frac{2\hat{i} - 5\hat{j}}{\sqrt{2^2 + (-5)^2}} = \frac{2\hat{i} - 5\hat{j}}{\sqrt{4+25}}$

$$\hat{v} = \left( \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right)$$

Now, the partial derivatives of  $f$  are

$$f_x(x,y) = 2xy^3, f_y(x,y) = 3x^2y^2 - 4$$

at point  $(2, -1)$

$$f_x(2, -1) = 2 \times 2 \times (-1)^3 = -4$$

$$f_y(2, -1) = 3 \times 2^2 \times (-1)^2 - 4 = 12 - 4 = 8$$

Therefore, the directional derivatives of  $f(x,y)$  at

point  $(2, -1)$  in the direction of vector

$$\vec{v} = 2\hat{i} - 5\hat{j}$$

$$\begin{aligned} D_u f(2, -1) &= -4 \times \frac{2}{\sqrt{29}} + 8 \times \left( -\frac{5}{\sqrt{29}} \right) \\ &= -\frac{8}{\sqrt{29}} - \frac{40}{\sqrt{29}} = \frac{-48}{\sqrt{29}} \quad (\text{Ans}) \end{aligned}$$

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(a)

Given,

$$f(x,y,z) = x^2y + y^2z$$

Maxima & minima w.r.t.

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial z} = 0$$

$$f_x(x,y,z) = 2xy + 0 = 2xy$$

$$f_y(x,y,z) = x^2 + 2yz$$

$$f_z(x,y,z) = 0$$

at point  $(1, 2, 3)$

$$\text{gradient of } f(x,y,z) = f_x(1, 2, 3)\hat{i} + f_y(1, 2, 3)\hat{j} + f_z(1, 2, 3)\hat{k}$$

$$P = 2x^2\hat{i} + (1^2 + 2x2x3)\hat{j} + 2^2\hat{k}$$

$$S = P - SI = P - (1-2)\hat{i} - 4\hat{i} + 13\hat{j} + 4\hat{k}$$

$$\text{to find } \nabla f(1, 2, 3)$$

orthogonal to surface at  $(1, 2, 3)$

$\nabla f(1, 2, 3)$  along

$$\hat{e}_3 = \hat{k}$$

$$(1-2)x\hat{x} + \frac{2}{2}\hat{y} + 2\hat{z} = (1-2)x\hat{x}$$

$$\frac{\partial f}{\partial x} = 2x - \frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}$$

(b)

(2)

$$\text{Given, } f(x,y,z) = x^2y + y^2z$$

$$f_x(x,y,z) = 2xy \quad \text{and} \quad f_y(x,y,z) = x^2 + 2yz$$

$$f_y(x,y,z) = x^2 + 2yz$$

$$f_z(x,y,z) = y^2$$

unit vector of the given vector,  $\hat{v} = \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{2^2 + (-1)^2 + 2^2}}$

$$+ (2\hat{s}_x) \cdot \frac{\hat{i}}{|\hat{v}|} + (2\hat{s}_y) \cdot \frac{\hat{j}}{|\hat{v}|} + (2\hat{s}_z) \cdot \frac{\hat{k}}{|\hat{v}|} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Thus, the directional derivatives of  $f(x,y,z)$  at  $(1,2,3)$

(i.e., in the direction of  $\hat{v} = 2\hat{i} - \hat{j} + 2\hat{k}$ )

$$D_u f(1,2,3) = f_x(1,2,3) \times \frac{2}{3} + f_y(1,2,3) \left(-\frac{1}{3}\right) + f_z(1,2,3) \left(\frac{2}{3}\right)$$

$$= 2 \times 1 \times 2 \times \frac{2}{3} + (1^2 + 2 \times 2 \times 3) \left(-\frac{1}{3}\right) +$$

$$2^2 \times \frac{2}{3}$$

$$= \frac{8}{3} - \frac{13}{3} + \frac{8}{3} = \frac{8 - 13 + 8}{3} = \frac{3}{3} = 1$$

(5)

(J)

Given,

$$\vec{F} = (x^y + 3 \cos 3z) \hat{i} + (y^x + 3 \sin 3z) \hat{j} + (e^{xy} + 3 \cos 3z) \hat{k}$$

$$\vec{F} = yz e^{xy} \hat{i} + xz e^{xy} \hat{j} + (e^{xy} + 3 \cos 3z) \hat{k}$$

Therefore, the divergence of  $\vec{F}$ ,  $\text{div } \vec{F}$

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = (\vec{F})_s$$

$$= \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left\{ yz e^{xy} \hat{i} + xz e^{xy} \hat{j} + (e^{xy} + 3 \cos 3z) \hat{k} \right\}$$

$$= \frac{\partial}{\partial x} (yz e^{xy}) + \frac{\partial}{\partial y} (xz e^{xy}) +$$

$$+ \frac{\partial}{\partial z} (e^{xy} + 3 \cos 3z)$$

$$(1) \quad yz e^{xy} \cdot y + xz e^{xy} \cdot x + (0 - 3 \sin 3z \cdot 3)$$

$$(2) \quad y^2 z e^{xy} + x^2 z e^{xy} - 9 \sin 3z$$

$$\therefore \text{div } \vec{F} = y^2 z e^{xy} + x^2 z e^{xy} - 9 \sin 3z \quad (\text{Ans})$$

$$\frac{x^2 - 2x + 2}{x^2} = \frac{x^2}{x^2} + \frac{-2x}{x^2} + \frac{2}{x^2} =$$

$$\begin{aligned}
 \text{curl } \vec{F} &= \nabla \times \vec{F} - \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \vec{F} \\
 &= \left| \begin{array}{ccc}
 \hat{i} & \hat{j} & \hat{k} \\
 \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
 y^2 e^{xy} & xz e^{xy} & e^{xy} + 3 \cos 3z
 \end{array} \right| = \vec{G} \\
 &= \hat{i} \left( e^{xy} \cdot x + 0 - x e^{xy} \right) - \hat{j} \left( e^{xy} \cdot y + 0 - y e^{xy} \right) \\
 &\quad + \hat{k} \left( xz \cdot e^{xy} \cdot y + e^{xy} \cdot z - y^2 e^{xy} \cdot x \right. \\
 &\quad \left. + 3 \sin 3z - e^{xy} \cdot z \right) \\
 &= \hat{i} \times 0 - \hat{j} \times 0 + \hat{k} \times 0 + \vec{G} \\
 &= 0 \quad (\text{Ans})
 \end{aligned}$$

(6)

Given,

$$\vec{F} = xy\hat{i} + y\sin z \hat{j} + (y \cos x) \hat{k}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (xy\hat{i} + y\sin z \hat{j} + (y \cos x) \hat{k})$$

$$= \frac{\partial}{\partial x} (xy^2) + \frac{\partial}{\partial y} (y\sin z) + \frac{\partial}{\partial z} (y \cos x) =$$

$$= y^2 + \sin z + 0$$

$$= y^2 + \sin z$$

$$\nabla \cdot (\vec{F} \times \vec{G}) = \vec{\nabla} \cdot (\vec{F} \times \vec{G}) + i(\vec{G} \cdot \vec{S} + \vec{A} \cdot \vec{P}) = 0$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & y \cos x \\ xy^2 & -y^2z & \left( \frac{\partial}{\partial y} + i \frac{\partial}{\partial z} \right) \end{vmatrix} =$$

$$(if \hat{i}(\cos x - y \cos z) - \hat{j}(-y \sin x - xy) + (xy + y \sin x) \hat{k} =$$

$$+ \hat{k} \left( \frac{0 - 2x}{y^2} + \frac{0 + 0}{y^2} - 2x \hat{k} \right)$$

$$\therefore \text{curl } \vec{F} = (\cos x - y \cos z) \hat{i} + (xy + y \sin x) \hat{j} + \left( \frac{0 - 2x}{y^2} \right) \hat{k}$$

(12)

Ex 12

Given,  $\vec{F} = ((3x + 2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} + (z - 7x)\hat{k}$

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} \\ &= \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left( (3x + 2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} + (z - 7x)\hat{k} \right) \end{aligned}$$

$$= \frac{\partial}{\partial x} (3x + 2z^2) + \frac{\partial}{\partial y} \left( \frac{x^3y^2}{z} \right) + \frac{\partial}{\partial z} (z - 7x)$$

$$= 3 + 0 + \frac{x^3}{z} (2z) \quad \text{Ans}$$

$$\therefore \operatorname{div} \vec{F} = \frac{2x^3y}{z} + 2$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+2z^2 & \frac{x^3y^2}{z} & 7x-8z \end{vmatrix}$$

$$= \hat{i} \left( 0 - x^3 y \cdot \frac{-1}{z^2} \right) - \hat{j} (7 - 0 - 0 - 4z)$$

$$+ \hat{k} \left( \frac{y^2}{z} \cdot 3x^2 - 0 \right)$$

$$= \frac{x^3 y^2}{z^2} \hat{i} - (7 - 4z) \hat{j} + \frac{3x^2 y^2}{z} \hat{k}$$

$$\operatorname{curl} \vec{F} = \frac{x^3 y^2}{z^2} \hat{i} + (4z - 7) \hat{j} + \frac{3x^2 y^2}{z} \hat{k}$$

(Ans)

(8)

$$S = xy + 2yz + 2xz$$

Let, Length of the box =  $2x$  (meter)

Breadth of the box =  $y$  (meter)

Height of the box =  $z$  (meter)

(i) Height  $z = \frac{12 - 2xy}{2x + 2y}$

(ii) Surface area  $= 2xy + 2yz + 2xz$

$$(5+4)(2x+y) \Rightarrow 10x + 5y + 8xy + 4y = 12$$

$$\Rightarrow 12 - 2xy = 12 - 2xy$$

$$\Rightarrow z = \frac{12 - 2xy}{2x + 2y}$$

$$\text{Volume, } V = 2xy^2$$

$$V = xy \cdot \frac{12 - 2xy}{2x + 2y}$$

$$\Rightarrow V = \frac{12xy - 2x^2y^2}{2x + 2y}$$

$$\Rightarrow V = \frac{(2x+2y)(12y - 2xy^2) - (12xy - 2x^2y^2)(2+0)}{(2x+2y)^2}$$

$$\Rightarrow V_x = \frac{(2x+2y)(12y - 2xy^2) - (12xy - 2x^2y^2)(2+0)}{(2x+2y)^2}$$

$$\Rightarrow V_x = \frac{24xy - 4x^2y^2 + 24y^2 - 4xy^3 - 24xy + 2x^2y^2}{(2x+2y)^2}$$

$$\Rightarrow V_x = \frac{24y^2 - 4xy^3 - 2x^2y^2}{(2x+2y)^2}$$

$$\therefore V_x = \frac{24y^2 - 4xy^3 - 2x^2y^2}{(2x+2y)^2}$$

$$\text{Now, } V_x = 0$$

(8)

$$\Rightarrow \frac{24y^2 - 4xy^3 - 2x^2y^2}{(2x+2y)^2} = 0 \quad \text{to find } \dots$$

for  $y$   
to obtain  
 $x^2y^2 + 2xy^3 - 12y^2 = 0$

$$\Rightarrow 24y^2 - 4xy^3 - 2x^2y^2 = 0 \quad \text{to find } \dots$$

$$\Rightarrow x^2y^2 + 2xy^3 - 12y^2 = 0 \quad \text{from (1)}$$

Again,

$$V_y = \frac{(2x+2y)(42x^2 - 2xy) - (12xy - x^2y^2)(0+2)}{(2x+2y)^2}$$

$$= \frac{-24x^2 - 4x^3y + 24xy - 4x^2y - 24xy + 2x^2y^2}{(2x+2y)^2}$$

$$= \frac{24x^2 - 2x^3y - 4x^2y^2}{(2x+2y)^2}$$

$$= \frac{24x^2 - 2x^3y}{(2x+2y)^2}$$

$$\text{for } V_y = 0$$

$$\Rightarrow \frac{24x^2 - 2x^3y}{(2x+2y)^2} = 0 \quad \text{--- (II)}$$

$$\Rightarrow x^2y^2 + 2x^3y - 12x^2 = 0$$

$$\Rightarrow (x^2 + 2x^3y) = xV$$

$$(I) xx^2 - (II) xy^2$$

$$x^4y^2 + 2x^3y^3 - 12x^2y^2 = 0$$

$$x^2y^4 + 2x^3y^3 - 12xy^2 = 0$$

$$x^4y^2 - x^2y^4 = 0$$

$$\Rightarrow x^2y^2(x^2 - y^2) = 0$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow x^2 = y^2$$

$$\therefore x = y$$

From eq (1)

$$y^4 + 2y^4 - 12y^2 = 0$$

$$\Rightarrow 3y^4 - 12y^2 = 0$$

$$\Rightarrow 3y^2(y^2 - 4) = 0$$

$$\Rightarrow y^2(y^2 - 4) = 0$$

(since  $y$  is the breath of box)

$$\Rightarrow y^2 - 4 = 0$$

$$\Rightarrow y = \sqrt{4}$$

$$\Rightarrow y = 2$$

$$\therefore x = 2$$

Again,

$$V_{xx}(x,y) = \frac{(2x+2y)^2(0 - 4y^3 - 4xy^2) - (24y^2 - 4y^3 - 2xy^2)2(2x+2y).2}{(2x+2y)^4}$$
$$\therefore V_{xx}(2,2) = \frac{(4+4)^2(-4 \times 8 - 4 \times 2 \times 4) - (24 \times 4 - 4 \times 2 \times 8 - 2 \times 4 \times 4)}{4(4+4)}$$
$$= \frac{64 \times (-64) - 0}{8^4}$$
$$= \frac{-64 \times 64}{8^4}$$
$$= -\frac{1}{1}$$
$$V_{yy}(x,y) = \frac{(2x+2y)^2(0 - 4y^3 - 4xy^2) - (24x^2 - 2xy^2 - 4x^3y)2(2x+2y).2}{(2x+2y)^4}$$
$$V_{yy}(2,2) = \frac{(4+4)^2(-4 \times 8 - 4 \times 2 \times 4) - (24 \times 4 - 2 \times 4 \times 4 - 4 \times 8 \times 2)4(4+4)}{(4+4)^4}$$
$$= \frac{64 \times (-64) - 0}{8^4}$$

$$= \frac{-64 \times 64}{8^4}$$

$$0 < \frac{\sigma}{\mu} =$$

$$\text{Now } \frac{\partial}{\partial x} \text{ diff to another term } \frac{\partial^2 L}{\partial x^2} = (2, 2)_{xx} \text{ V diag}$$

$$\text{Now } \frac{\partial}{\partial x} \frac{\partial}{\partial y} \text{ diff to another term } \frac{\partial^2 L}{\partial x \partial y} = (2, 2)_{xy}$$

$$V_{xy}(n, y) = \frac{(2n+2)^2(48y^2 - 12xy^2 - 4x^2y)}{(2n+2)^4}$$

$$\therefore V_{xy}(2, 2) = \frac{(4+4)^2(48 \times 2^2 - 12 \times 2 \times 4 - 4 \times 4 \times 2)}{(4+4)^4}$$

$$= \frac{64 \times (-32)}{8^4}$$

$$\text{Ex } \rho = \frac{1}{2}$$

$$\text{Now, } D(n, y) = V_{xx}(n, y) V_{yy}(n, y) - \left\{ V_{xy}(n, y) \right\}^2$$

$$\therefore D(2, 2) = V_{xx}(2, 2) \cdot V_{yy}(2, 2) - \left\{ V_{xy}(2, 2) \right\}^2$$

$$= (-1)(-1) - \left(-\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4} > 0$$

$$\text{Again } V_{xx}(2,2) = -1 < 0$$

Therefore, the ~~box~~ volume of the box will

be maximum when lengths  $x = 2$  and

$$\text{breath } y = 2$$

$$\text{Thus, } z = \frac{12 - xy}{2x + 2y}$$

$$= \frac{12 - 2 \times 2}{2 \times 2 + 2 \times 2}$$

$$= \frac{8}{8}$$

$$\therefore z = 1$$

$$\therefore \text{The maximum volume, } V_{\max} = 2 \times 2 \times 1 = 4 \text{ m}^3$$

(Any)

(Hence)  $\therefore V = (8, 0) \text{ (Ans)}$