Combi I*— Contests and Sets

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This text is part of a series that aims to vulgarize ideas and techniques found in top country camps' problem sets, without being directly stated, and so might be out of reach of lambda students.

1 Appetizer

Problem 1.1. (IMC 2002). Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students. Hint (given after 20 mins)[1]

2 Motivation

The most common approach by trainees is usually to work the cases of the max number of problems solved by a student and build from there by pigeonholing on that student's unsloved problems. And although this approach yields a solution, it's not the most effective way to view this type of problems, and might (will) lead you to loose a lot of time trying to conceive it.

A second approach, and a very powerful and popular one, is to try and translate the problem into a question about sets and try to count them, this will save you the pain of juggling each student's answers.

We are essentially provided with a pile of students, each student can be viewed as a pile of non solved problems, and we are asked to find two students such that their union of piles is the empty set. We have $\binom{200}{2} = 19900$ pairs available, on the other side, each problem was not solved by at most 80 students, so each problem can belong to at most $\binom{80}{2} = 3160$ pairs. With us having 6 problems, the max number of problems we could find in our sets is $6 * \cdot 3160 = 18960$. And so we're guarenteed to find muliple empty sets, since the number of pairs exceeds this number.

The intuition behind working with sets of non solved problems comes from the fact that we have a max number of unsolved problems (since each problem

*Inspired by Yufei Zhao's AwsomeMath 2007 handouts.

[1] In how many ways can you count the total number of solved problems?

was solved by at least 120 students) and so it's easier to bound this quantity. Counting the pairs comes from the problem statement.

Let's try applying this technique on another problem:

Problem 1.2. (China TST 1992) Sixteen students took part in a math competition where every problem was a multiple choice question with four choices. After the contest, it is found that any two students had at most one answer in common. Determine the maximum number of questions.

This technique is essentially counting the same quantity in two ways to bound it, and is part of an essential *savoir-faire :* finding the right set visualization and pigeonholing it . . .

3 Practice Problems

Problem 2.1 (Useful for constructions)

We repartition N participants into k groups each with n_i participants, we say a pair is *intik* if the two participants are in the same group, what's the minimum number of *intik* pairs we can have? In what configuration ?

Problem 2.2 (IMO 1998/2)[2]

In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}$$

Problem 2.3 (IMO 2005/6)[3]

In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there are at least 2 contestants who each solved exactly 5 problems each.

Problem 2.4 (IMO 2020/4)

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

[2] Source will be hidden until we present a solution to get them to do their best.
[3] idem