Power of a Point*— Problem list

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1 Power of a point

Problem 1.1. Let AD be the A-altitude in triangle $\triangle ABC$. If H is a point on AD, prove that H is the orthocenter of $\triangle ABC$ if and only if $BD \cdot CD = AD \cdot DH$.

Problem 1.2. Let BT be the altitude and H be the intersection point of the altitudes of triangle ABC. Point N is symmetric to H with respect to BC. The circumcircle of $\triangle ATN$ intersects BC at points F and K. Prove that $FB = BK$.

Problem 1.3. (AIME I 2020/15)*. Let ABC be an acute triangle with circumcircle (ω) and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects (ω) at points X and Y with $HA = 3, HX = 2, HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

2 Radical Axis

Problem 2.1. (Orthocenter). Prove that the three altitudes of a triangle are concurrent by creating a radical center.

Problem 2.2. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects segment BD.

Problem 2.3. (APMO 2020/1). Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

*For further reading, I highly recommend Albert Zhu's *Power of a Point* handout, which was a major inspiration for this one.

Problem 2.4. Let A, B, C, D be points in the plane such that CD is a line perpendicular to AB and $C, D \notin AB$. If E and F are the orthocenters of $\triangle ABC$ and $\triangle ABD$, prove that the radical axis of the circles with diameters CE and DF is AB.

Problem 2.5. In $\triangle ABC$, let I be the incenter and D, E, F be the touch points of the incircle with BC, CA, AB, respectively. If AI meets (ABC) again at M and K, Q are the points where the tangent to (ABC) at M meets AC, AB, respectively, show that D lies on the radical axis of (EIK) and (FIQ).

3 Point Circles

Problem 3.1. (Circumcenter) Prove that the perpendicular bisectors of a triangle ABC meet at a single point.

Problem 3.2. (Polish MO 2018/5). An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle escribed on $\triangle ABC$ crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.

4 Addictional Problems

Problem 4.1. (Website 207). Let P be a point outside of ω, the tangents to ω from P touch it in points A and B. Let (l) be a line between [PA]and[PB] and Q a point on (l) such that $Q \notin (PO)$. The two tangents from Q touch ω in C and D. let $E = (CD) \cap (l)$. Prove that $\angle EPQ$ is a right angle.

Problem 4.2. (TSTST 2017/1). Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90°$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and E. Prove that $PR \perp OH$.