Solutions to Homework 5:

1. Your answer to question 1.

   Proof. We are going to do induction proof.
   Base Case: $n = 1$.

$$(2(1)+1)\cdot 3^{1-1} = 1 \cdot 3^1 \quad (1)$$
$$3 = 3 \quad (2)$$

   Thus, base case holds.
   Induction Step: Now, assume that $\sum_{i=1}^{k}(2i+1)\cdot 3^{i-1} = k\cdot 3^k$ for all $k \in \mathbb{N}$.

   Consider the following:

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i+1)\cdot 3^{i-1} &= \sum_{i=1}^{k}(2i+1)\cdot 3^{i-1} + (2(i+1)+1)\cdot 3^{(i+1)-1} && (3)\\
&= i\cdot 3^i + (2(i+1)+1)\cdot 3^{(i+1)-1} \text{ By IH} && (4)\\
&= i\cdot 3^i + (2i+3)\cdot 3^i && (5)\\
&= i\cdot 3^i + 2i\cdot 3^i + 3\cdot 3^i && (6)\\
&= 3i\cdot 3^i + 3\cdot 3^i && (7)\\
&= 3^i(3i+3) && (8)\\
&= 3^i\cdot 3(i+1) && (9)\\
&= 3^{i+1}(i+1) && (10)\\
& && (11)
\end{aligned}$$

   Thus, we have proven $\sum_{i=1}^{k+1}(2i+1)\cdot 3^{i-1} = 3^{i+1}(i+1)$ by the principle of mathematical induction. □

2. Your answer to question 2.

Proof. We are going to prove using induction.
Base Case:

(a) For $n = 1$
$a_1 = -3 = 5^1 - 2^{1+2} = 5 - 8 = -3$

(b) For $n = 2$
$a_2 = 9 = 5^2 - 2^{2+2} = 25 - 16 = 9$

Both cases hold.

Inductive Step: Assume that $a_k = 5^k - 2^{k+2}$ for all $n \geq 1$. We need to prove that $a_{k+1} = 5^{k+1} - 2^{(k+1)+2}$.

We can use the given recurrence relation to find a_{k+1} as required.

$$
\begin{aligned}
a_{k+1} &= 7a_k - 10a_{k-1} && (12)\\
&= 7(5^k - 2^{k+2}) - 10a_{k-1} && \text{By hypotheses.} \quad (13)\\
&= 7(5^k - 2^{k+2}) - 10(5^{k-1} - 2^{(k-1)+2}) && \text{By hypotheses for } a_{k-1}. \quad (14)\\
&= 5^k(7) - 2^{k+2}(7) - 5^{k-1}(10) + 2^{k+1}(10) && (15)\\
&= 5^{k+1} - 2^{(k+1)+2} && (16)
\end{aligned}
$$

This complete our proof. So, we have proven that $a_n = 5^n - 2^{n+2}$ for all $n \geq 1$. □

3. Your answer to question 3.
Explicit Formula: Consider the following:

$$f(x) = xe^x \tag{17}$$
$$f^1(x) = e^x + xe^x \tag{18}$$
$$f^2(x) = e^x + (e^x + xe^x) = 2e^x + xe^x \tag{19}$$
$$f^3(x) = e^x + e^x + (e^x + xe^x) = 3e^x + xe^x \tag{20}$$

Thus, we get the explicit function for $f^{(n)}(x)$ as follows.

$$f^{(n)}(x) = ne^x + xe^x \tag{21}$$

Now, we will prove that the explicit function above holds for all derivatives $f^{(n)}(x)$ for all $n \in \mathbb{N}$.

Proof. We are going to do induction proof.
Base Case: for $n = 1$, $f^{(1)}(x) = e^x + xe^x$. Thus, the formula for the derivative holds.
Inductive Step: Assume for all $k \in \mathbb{N}$ that $f^{(k)}(x) = ke^x + xe^x$.

Now, consider the assumption, $f^{(k)}(x) = ke^x + xe^x$. Now, let's find the $(k + 1)$-th derivative of $f(x)$, which is $f^{(k+1)}(x)$:

$$f^{(k+1)}(x) = (ke^x + xe^x)' \tag{22}$$
$$= (ke^x)' + (xe^x)' \tag{23}$$
$$= ke^x + e^x + xe^x \tag{24}$$
$$productrule = (k + 1) \cdot e^x + xe^x \tag{25}$$

Thus, $f^{(k+1)}(x) = (k+1) \cdot e^x + xe^x$. This mathches the formula for $(k+1)$-th derivative.
Therefore, we have proven by the principle of mathematical induction that $f^{(k)}(x) = ke^x + xe^x$ for all $k \in \mathbb{N}$. □

4. Your answer to question 4.

Proof. We are going to do induction proof.
Base Case: at $n = 1$, $3^{4 \cdot 1} + 9 = 90$. $10|90$ thus base case holds.
Inductive Step: Assume for all $k \in \mathbb{N}$ that $10 \mid 3^{4k} + 9$. Then consider the following:

$$\begin{align} 3^{4(k+1)} + 9 &= 3^{4k} \cdot 3^4 + 9 \tag{26} \\ &= 3^{4k} \cdot (80 + 1) + 9 \tag{27} \\ &= (3^{4k} \cdot 80 + 3^{4k}) + 9 \tag{28} \\ &= 3^{4k} \cdot 80 + 3^{4k}) + 9 \tag{29} \\ &= (3^{4k} \cdot 80) + (3^{4k} + 9) \tag{30} \end{align}$$

Consider the expression $3^{4(k+1)} + 9 = (3^{4k} \cdot 80) + (3^{4k} + 9)$. We know that, on the right hand side, the first term is divisible by 10 by inspection and the second term is divisible by 10 by the induction assumption. Thus, the induction step holds.

Therefore, the statement holds for natural numbers by induction. □

5. Your answer to question 5.

Proof. We are going to prove using induction.
Base Case:

(a) For $n = 1$, $a_1 = 3$, $2^1 = 2$, and $4^1 = 4$. Thus, $2 < 3 < 4$ is true.
(b) For $n = 2$, $a_2 = 12$, $2^2 = 4$, and $4^2 = 16$. Thus, $4 < 12 < 16$ is true.

Base cases hold.

Inductive Step: Assume that $2^k < a_k < 4^k$ for $k \geq 1$. We need to prove that this also hold for $2^{k+1} < a_{k+1} < 4^{k+1}$.

Consider the relation $a_n = 2a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. We will use the induction hypotheses to get a_{k+1}.

$$\begin{align} a_{k+1} &= 2a_k + a_{k-1} + a_{k-2} \tag{31} \\ &> 2(2^k) + 4^k + 4^{k-1} && \text{By Hypotheses} \tag{32} \\ &= 2^{k+1} + 4^k + 2^k && \text{Since } 4^{k-1} = 2^{2k-k} = 2^k. = 3(2^k) + 4^k \tag{33} \\ &< 3(4^k) + 4^k && \text{By Hypotheses} \tag{34} \\ &= 4^k(3 + 1) \tag{35} \\ &= 4^{k+1} \tag{36} \end{align}$$

We have shown that $2^{k+1} < a_{k+1} < 4^{k+1}$ as required. Thus, we have proven $2^n < a_n < 4^n$ for $n \geq 1$ using mathematical induction. □

6. Your answer to question 6.

Proof. We will be proving using mathematical induction.
Base Case: For $n = 1$, we have $a_1 = 4$ and $a_2 = a_1^2 - 2a_1 = 16 - 8 = 8$. Now, because $4 < 8$ the base case holds.

Inductive Step: Assume that $a_k < a_{k+1}$ for all $k \in \mathbb{N}$. We will show that this also holds for $a_{k+1} < a_{k+2}$.

Consider the following.

$$\begin{aligned} a_{k+2} &= a_{k+1}^2 - 2a_{k+1} && (37) \\ &= (a_k^2 - 2a_k)^2 - 2(a_k^2 - 2a_k) && (38) \\ &= a_k^4 - 4a_k^3 + 4a_k^2 - 2a_k^2 + 4a_k && (39) \\ &= a_k^4 - 4a_k^3 + 2a_k^2 + 4a_k && (40) \end{aligned}$$

Now, consider the difference of $a_{k+2} - a_k$:

$$\begin{aligned} a_{k+2} - a_k &= (a_k^4 - 4a_k^3 + 2a_k^2 + 4a_k) - a_k && (41) \\ &= a_k^4 - 4a_k^3 + 2a_k^2 + 3a_k && (42) \\ &= a_k(a_k^3 - 4a_k^2 + 2a_k + 3) && (43) \end{aligned}$$

Since $k \geq 1$, we know that $a_k > 0$. The inequality $a_k(a_k^3 - 4a_k^2 + 2a_k + 3) > 0$ depends on the sign of the expression inside the parentheses.

Now,

$$a_k^3 - 4a_k^2 + 2a_k + 3 = a_k^2(a_k - 4) + 3(a_k - 1) \qquad (44)$$

we see that both terms in the right hand side are positive (since a_k, $a_k - 1$ and $a_k - 4$ are positive). Therefore, the expression above is always positive as follows:

$$\begin{aligned} a_{k+2} - a_k &> 0 && (45) \\ \text{and we have} & && (46) \\ a_{k+2} &> a_k && (47) \end{aligned}$$

Therefore, we have proven by using mathematical induction that the sequence a_n for $n \in \mathbb{N}$ is increasing. □

7. Your answer to question 7.

   (a) Part (a). We can prove this by breaking up the sums and showing each term is greater than or equal to $\frac{1}{2n}$.

$$\frac{1}{n+1} \geq \frac{1}{2n} \qquad \text{Since } n+1 \leq 2n. \tag{48}$$

$$\frac{1}{n+2} \geq \frac{1}{2n} \tag{49}$$

$$... \tag{50}$$

$$\frac{1}{2n} \geq \frac{1}{2n} \tag{51}$$

   Now, summing all the inequality we have:

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2} \tag{52}$$

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} \geq \frac{1}{2} \tag{53}$$

   The inequality holds for any $n \in \mathbb{N}$.

   (b) Part (b). We will use mathematical induction.

   Proof. **Base Case:** $k = 1$. For $k = 1$ we have $\frac{1}{1} \geq \frac{1+2}{2}$. This is true as $1 \geq \frac{3}{2}$.
   Inductive Step: Assume that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^m} \geq \frac{m+2}{2}$ for all $m \in \mathbb{N}$.
   Consider the following.

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^m}\right) + \frac{1}{2^{m+1}} \geq \frac{m+2}{2} + \frac{1}{2^{m+1}} \qquad \text{By IH.} \tag{54}$$

$$\geq \frac{m+2}{2} + \frac{1}{2} \qquad \text{By Part (a).} \tag{55}$$

$$= \frac{m+2+1}{2} \tag{56}$$

$$= \frac{m+3}{2} \tag{57}$$

   Hence, we have proven this by using mathematical induction. □

8. Your answer to question 8.

Proof. We are going to do induction proof.
Base Case: at $n = 1$, $2^{1^2} > 1!$, $2 > 1$ thus the base case holds.

Inductive Step: Assume for all $k \in \mathbb{N}$ that $2^{k^2} > k!$.

We need to show that $2^{(k+1)^2} > (k+1)!$ for all $k \in \mathbb{N}$. Consider the following.

$$\begin{aligned}
2^{(k+1)^2} &= 2^{k^2+2k+1} && (58)\\
&= 2^{k^2} \cdot 2^{2k+1} && (59)\\
&> k! \cdot 2^{2k+1} && \text{By the induction hypotheses.} \quad (60)\\
&= (k!\ 2^k) \cdot 2^{k+1} && (61)\\
&> k!\ (k+1) && \text{Since } 2^{k+1} > k+1 \textbf{ (proven below)}. \quad (62)\\
&= (k+1)! && (63)
\end{aligned}$$

We have proven that $2^{(k+1)^2} > (k+1)!$ for all $k \in \mathbb{N}$ by using mathematical induction. □

Additional proof supporting above proof.

Proof that $2^n > n$ for all $n \in \mathbb{N}$. We are going to use induction.

Base Case: For $n = 1$, $2^1 = 2 > 1$. Thus, base case holds.

Inductive Step: Assume that $2^k > k$ for all $k \in \mathbb{N}$. We will prove that this also holds for $2^{k+1} > k+1$ for all $k \in \mathbb{N}$.

Consider the following.

$$\begin{aligned}
2^k &> k && \text{Induction Hypotheses.} \quad (64)\\
2 \cdot 2^k &> 2 \cdot k && \text{multiply both by 2.} \quad (65)\\
& && (66)
\end{aligned}$$

Observe that $2k > k+1$. This is true because for $k \geq 1$, $2k$ is always greater than $k+1$.

Now, we have the following.

$$\begin{aligned}
2^{k+1} > 2k &> k+1 && (67)\\
2^{k+1} &> k+1 && (68)
\end{aligned}$$

So, we have shown that $2^{k+1} > k+1$. This completes our proof. We have proven by mathematical induction that $2^n > n$ for all $n \in \mathbb{N}$.