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ESC 407
Computational Methods in Engineering Science
Exam 1

Start: 8:00 a.m. on Saturday, October 24
End: 4:00 p.m. on Monday, October 26, 2020

Problem 1	*Problem 2*	*Problem 3*	**Total**

By signing below, I pledge that:

- ✓ I will not consult any external resources for any reason. I will not use the internet for *anything*, materials from other classes or books or any notes you have taken in other classes, etc.
- ✓ I will not consult with or talk to any other person about the exam. I will not check my exam answers with any person.
- ✓ If I have a question or need a clarification about the exam, the only person I will ask is Prof. Gray, either on Zoom or by email.

Signature: ______________________________

If you do not sign this pledge and return it with your exam, you will receive zero credit on the exam.

General Instructions

- This exam is open book and open notes, which means you are permitted to use any materials handed out in class, your own notes from the course, and anything on our Canvas website.
- You may use MATLAB (or other Python or whatever) as a calculator or to perform repetitive calculations, but you must show all detailed work written by hand, either with pencil/pen and paper or on a tablet.
- Submit the exam in Canvas, with the problems in the correct order, before 4:00 p.m. on Monday, October 26.
- *Failure to show **all** of your work on any give problem will result in a zero on that problem!*

Problem 1:

Given:

$f(x) = xe^{x^2}$

$x_0 = 0$

Find:

- upper bound for error on $x \in [0, .4]$
- approximate integral over same bound
- Find upper bound for error on integral
- approximate derivative at .2 and find true error

Calculations:

Finding expansion around zero

$$f(x) = xe^{x^2}$$

$$f'(x) = (1)(e^{x^2}) + (x)(2x\,e^{x^2})$$

$$f'(x) = e^{x^2}(2x^2+1)$$

$$f''(x) = (2x\,e^{x^2})(2x^2+1) + e^{x^2}(4x)$$

$$f''(x) = 2e^{x^2}x(2x^2+3)$$

$$f''(x) = 2e^{x^2}x(2x^2+3)$$

$$f'''(x) = (2e^{x^2}x)(4x) + ((4x^2e^{x^2}) + (2e^{x^2}))(2x^2+3)$$

$$f'''(x) = 2e^{x^2}(4x^2 + 2x^2(2x^2+3) + 2x^2+3)$$

$$f'''(x) = 2e^{x^2}(4x^4 + 12x^2+3)$$

$$f''''(x) = 4xe^{x^2}(4x^4+12x^2+3) + 2e^{x^2}(16x^3+24x)$$

$$f''''(x) = 4xe^{x^2}(4x^4+20x^2+15)$$

$$P_4(x) = (0)e^{(0)^2} + e^{(0)^2}(2(0)^2+1)x$$
$$+ 2e^{(0)^2}(0)(2(0)^2+3)\frac{x^2}{2!}$$
$$+ 2e^{(0)^2}(4(0)^4+12(0)^2+3)\frac{x^3}{3!}$$
$$+ 4(0)e^{(0)^2}(4(0)^4+20(0)^2+15)\frac{x^4}{4!}$$

$$P_4(x) = 0 + x + x^3$$

$$P_4(x) = x + x^3$$

a) Finding upper bound for error [0, .4]

$$|f(x) - P_4(x)| = R_4(x) = \frac{f^5(\xi(x))}{5!}(x)^5$$

$$f''''(x) = 4xe^{x^2}(16x^3 + 40x) + (4e^{x^2} + 8x^2e^{x^2})(4x^4 + 20x^2 + 15)$$

$$f'''''(x) = 4e^{x^2}((16x^4 + 40x^2) + (1 + 2x^2)(4x^4 + 20x^2 + 15))$$

$$f'''''(x) = 4e^{x^2}(8x^6 + 60x^4 + 90x^2 + 15)$$

$$R_4(x) = \frac{4e^{\xi^2}(8\xi^6 + 60\xi^4 + 90\xi^2 + 15)}{120}(x)^5$$

x and ξ are both in the numerator, so to maximize R_4 we pick .4 for both.

$R_4(.4) = .0124$

So final error bound is

$$R_4 \leq .0124$$

b) approximating integrals

$$\int_0^{.4} x e^{x^2} \, dx \qquad u = x^2 \qquad du = 2x$$

$$\int_0^{.16} \frac{e^u}{2} \, du$$

$$\frac{1}{2} \left[e^u \right]_0^{.16}$$

$$.08676$$

$$\int_0^{.4} x + x^3 \, dx$$

$$\left[\frac{x^2}{2} + \frac{x^4}{4} \right]_0^{.4}$$

$$.0864$$

c) Finding upper bound on integral error [0, .4]

$$|f(x) - P_4(x)| = R_4(x) = \frac{f^5(\xi(x))}{5!}(x)^5$$

we are trying to find error between the integrals which would be

$$\left| \int_0^{.4} f(x)\,dx - \int_0^{.4} P_4(x)\,dx \right| = \int_0^{.4} R_4(x)\,dx$$

$$\text{error} = \int_0^{.4} \frac{f^5(\xi(x))}{5!}(x)^5\,dx$$

$$\int_0^{.4} \frac{4e^{\xi^2}(8\xi^6+60\xi^4+90\xi^2+15)}{120}(x)^5\,dx$$

$$\left[x^6\left(\frac{e^{\xi^2}}{12}+\frac{\xi^2 e^{\xi^2}}{2}+\frac{\xi^4 e^{\xi^2}}{3}+\frac{2\xi^6 e^{\xi^2}}{45}\right)\right]_0^{.4}$$

Maximize error, so we pick .4 for ξ since it is on the numerator

$$\text{error} \leq \int_0^{.4} R_n(x)\,dx \leq 8.2698 * 10^{-4}$$

d) approximate error of derivatives for true error

$$f'(.2) = e^{(.2)^2}(2(.2)^2+1)$$

$$f'(.2) = 1.124076$$

$$P_4'(.2) = 1 + 3(.2)^2 = 1.12$$

true error = true − approx

true error = .004076

Problem 2:

Given:

Bisection method

$\sqrt[3]{25}$

rel error 10^{-4}

Find:

- approximation of the root
- number of iterations to reach .000001

Calculations:

Finding function / bracket

need to approximate $\sqrt[3]{25}$

need to find x

$x = \sqrt[3]{25}$

$x - \sqrt[3]{25} = 0$

$x^3 - 25 = 0$

Using bisection to find x in this equation

Calculator shows $\sqrt[3]{25} = 2.92401$

bracket can be $[2, 3]$

a) Finding bisect approximation within 10^{-4}

$(2)^3 - 25 = -17$

$(3)^3 - 25 = 2$

bracket does enclose the root

Run 1

$$\frac{2+3}{2} = 2.5 \qquad (2.5)^3 - 25 = -9.375$$

Run 2

$$\frac{2.5+3}{2} = 2.75 \qquad (2.75)^3 - 25 = -4.203125$$

$$\text{rel error} = \left| \frac{2.75 - 2.5}{2.75} \right| = .09091$$

Run 3

$$\frac{2.75+3}{2} = 2.875 \quad (2.875)^3 - 25 = -1.2363$$

rel error = .04347

Run 4

$$\frac{2.875+3}{2} = 2.9375 \quad (2.9375)^3 - 25 = .34741$$

rel error = .02128

Run 5

$$\frac{2.875+2.9375}{2} = 2.90625 \quad (2.90625)^3 - 25 = -.4530$$

rel error = .010752

Run 6

$$\frac{2.90625 + 2.9375}{2} = 2.9219 \qquad (2.9219)^3 - 25 = -.0549$$

rel error = .005347

Run 7

$$\frac{2.9219 + 2.9375}{2} = 2.9297 \qquad (2.9297)^3 - 25 = .146$$

rel error = .002667

Run 8

$$\frac{2.9219 + 2.9297}{2} = 2.9258 \qquad (2.9258)^3 - 25 = .0453$$

rel error = .001335

Run 9

$$\frac{2.9219 + 2.9258}{2} = 2.92383$$

$(2.92385)^3 - 25 = -.0049$

rel error $= 6.68 * 10^{-4}$

Run 10

$$\frac{2.92385 + 2.9258}{2} = 2.924825$$

$(2.924825)^3 - 25 = .0207$

rel error $= 3.33889 * 10^{-4}$

Run 11

$$\frac{2.92385 + 2.924825}{2} = 2.9243375 \quad (2.9243375)^3 - 25 = .00766$$

rel error $= 1.6697 * 10^{-4}$

Run 12

$$\frac{2.92385 + 2.9243375}{2} = 2.92409375 \quad (2.92409375)^3 - 25 = .00195$$

rel error $= .0000834 < 10^{-4}$

12 iterations needed

b) Finding iterations to get relative error below .000001

Start with the theroem, p is root approx, n is # of iterations

$$|p_n - p| \le \frac{b-a}{2^n}$$

$b = 3 \quad a = 2$

Can replace $|p_n - p|$ with desired accuracy

$\alpha = .000001$

$$\alpha \le \frac{b-a}{2^n}$$

Finding n

$$2^n \geq \frac{b-a}{\alpha}$$

$$n \ln(2) \geq \ln\left(\frac{b-a}{\alpha}\right)$$

$$n \geq \frac{\ln\left(\frac{b-a}{\alpha}\right)}{\ln(2)}$$

$$n \geq 19.932$$

We must run at least 20 iterations to get an accuracy better than .000001

Problem 3

Given:

Object of mass m

Placed at height h

Force F_s

$$F_s = k_1 \delta + k_2 \delta^{7/4}$$

deflection δ

max deflection δ_m

$$\frac{4}{11} k_2 \delta_m^{11/4} + \frac{1}{2} k_1 \delta_m^2 - mg(\delta_m + h) = 0$$

Properties:

$k_1 = 40 N/m$

$k_2 = 10 N/m^{7/4}$

$m = 3 kg$

$h = .5 m$

$g = 9.81 m/s^2$

Find:

Use Newton's method to find δ_{m0} w/ rel error of 10^{-4}. Return root, rel err, and # of iterations

Then, use secant method with the iterations of Newtons.

Return root and rel err

a) Finding root w/ newton's method within 10^{-4} rel error

initial guess for δ_{m0} is 1.4

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad k = 0, 1, 2 \ldots\ldots$$

$$f(\delta_m) = \frac{4}{11} k_2 \delta_m^{11/4} + \frac{1}{2} K_1 \delta_m^{2} - mg(\delta_m + h) = 0$$

$$f'(\delta_m) = K_2 \delta_m^{7/4} + K_1 \delta_m - mg = 0$$

Run1:

$$\delta_{m1} = 1.4 - \frac{f(1.4)}{f'(1.4)} = 1.4 - \frac{-7.5438}{44.5887}$$

$$\delta_{m1} = 1.5692$$

$$\text{Rel err} = \left| \frac{1.5692 - 1.4}{1.5692} \right| = .1078$$

Run2:

$$\delta_{m2} = 1.5692 - \frac{f(1.5692)}{f'(1.5692)} = 1.5692 - \frac{.9045}{55.3375}$$

$$\delta_{m2} = 1.5529$$

$$\text{Rel err} = \left| \frac{1.5529 - 1.5692}{1.5529} \right| = .0105$$

Run 3!

$$\delta_{m_3} = 1.5529 - \frac{f(1.5529)}{f'(1.5529)} = 1.5529 - \frac{.0086}{54.2846}$$

$$\delta_{m_3} = 1.5527$$

$$\text{Relerr} = .0001217$$

Run 4!

$$\delta_{m_4} = 1.5527 - \frac{f(1.5527)}{f'(1.5527)} = 1.5527 - \frac{8.0976 * 10^{-7}}{54.2743}$$

$$\delta_{m_4} = 1.552699985$$

$$\text{Relerr} = 9.60899 * 10^{-9} < 10^{-4}$$

took 4 iterations

b) use secant method for 4 iterations

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \qquad k = 0, 1, 2 \ldots$$

$\delta_{m0} = .1 \quad \delta_{m1} = 2$

Run 1:

$$\delta_{m2} = 2 - \frac{f(2)(2 - .1)}{f(2) - f(.1)} = 2 - \frac{58.6861}{48.3389}$$

$\delta_{m2} = .7859$ $\qquad$ rel error $= \left| \dfrac{.7859 - 2}{.7859} \right| = 1.5447$

Run 2!

$$\delta_{m3} = .7859 - \frac{f(.7859)(.7859-2)}{f(.7859)-f(2)} = .7859 - \frac{28.67133}{-54.5036}$$

$\delta_{m3} = 1.3119$ rel error $= .40095$

Run 3:

$$\delta_{m4} = \delta_{m3} - \frac{f(\delta_{m3})(\delta_{m3}-\delta_{m2})}{f(\delta_{m3})-f(\delta_{m2})} = 1.7887$$

rel error $= .26652$

Run 4!

$$\delta_{m_5} = \delta_{m_4} - \frac{f(\delta_{m_4})(\delta_{m_4} - \delta_{m_3})}{f(\delta_{m_4}) - f(\delta_{m_3})} = 1.51899$$

rel error = .17756

Newton's method seems to converge much quicker than secant for this equation.