Exploring Gauss codes on higher genus surfaces

Eric Och and Dylan Fridman

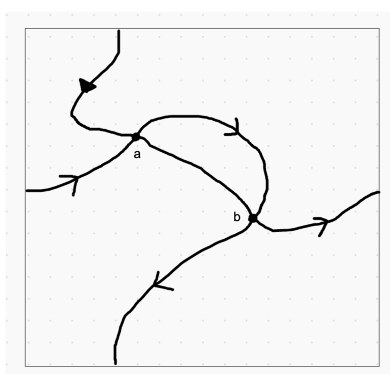
November 2020

Abstract

We prove that for any curve γ minimally embedded on a surface S, the four following properties are equivalent: 1) γ has an alternating diagram, 2) between every instance of a symbol in the Gauss code of γ there are an even number of letters, 3) γ is 2-*colorable*, and 4) γ crosses every fundamental loop of S an even number of times. We also discuss without much depth other ideas related to Gauss codes of curves on orientable higher genus surfaces.

1 Introduction

We began thinking the question of how to characterize the unsigned Gauss codes that are minimally embeddable on the torus. As an example the code $[abab]$ cannot be embedded on the plane regardless of the signing but can be embedded on the torus:



Thus, 1 is the minimum genus surface in which we can embedded a curve with Gauss code $[abab]$, so we say that $[abab]$ is minimally embeddable on the torus. There's an $O(n)$ solution for the problem of deciding whether a given unsigned Gauss code is minimally embeddable on the plane, but as far as we know there's no efficient algorithm for the torus. Clearly, there's an exponential time solution by computing the euler characteristic for every possible signing.

The planar solution checks for two conditions, the simplest one being that between the two instances of each letter in the unsigned Gauss code there should be an even number of letters. For example, $[abcabc]$ satisfies the *parity condition* whereas $[abab]$ doesn't.

Some of the progress done towards a linear time solution for surfaces different than the sphere assumes that the embedding of the curve is 2-*colorable*. It seemed reasonable to us that the problem is also more approachable whenever the *parity condition* holds, so we conjectured that there might be some kind of relationship between the 2-*colorability* and the *parity condition*. Most of our work is related to exploring this relationship.

2 Main results

2.1 Basic definitions

Definition 1 (Gauss code)**.** We define an unsigned Gauss code G as a sequence of length $2n$ in which each element of a set S with n elements occurs twice in G. We define a signed Gauss code as an unsigned Gauss code in which each instance of a symbol has a $+$ or $-$ associated to it, but no two instances of the same symbol have the same sign.

Remark. Every curve in which every intersection is a double crossing naturally corresponds to a family of unsigned Gauss codes and a family of signed Gauss codes, in which the symbols represent the crossings of the intersections, and the signing corresponds to the orientation of the crossing.

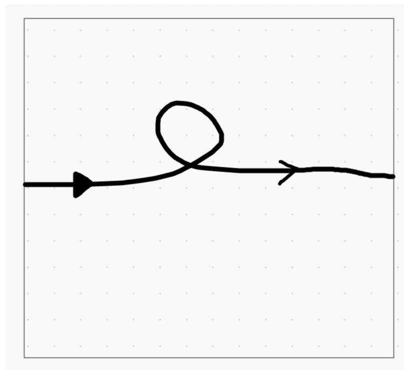
Definition 2 (Parity condition)**.** We say that a Gauss code G satisfies the *parity condition* if and only if (iff from now on) for all the elements s in S (the image of G), there are an even number of symbols between each occurrence of s in G. We'll also say that a curve satisfies the *parity condition* iff any of its corresponding Gauss codes satisfies the *parity condition*.

Definition 3 (2-colorable)**.** We say that a curve is 2-*colorable* iff the dual graph of the graph that corresponds to the curve is bipartite.

Definition 4 (Essential parity)**.** We say that a curve γ embedded in a surface S of genus g is *essentiality even* iff it goes through each edge of the $4g$-gon representing S an even number of times. If a curve is not *essentially even*, it's essentially odd.

2.2 Original conjecture

At first, we thought that perhaps the three properties defined previously (*parity condition*, *even essentiality* and 2-*colorability*) are equivalent. This doesn't actually hold:



The above curve satisfies the parity condition but is not *essentially even*. Also, clearly there's something wrong with the conjecture because the Gauss code doesn't have the information of how many times the curves wraps around: there are infinitely many curves, each corresponding to a different element of the fundamental group of the torus that have $[aa]$ as their Gauss code.

However, we have the unformalized intuition that if a Gauss code is minimally embeddable, then the information for the essential parity is contained within the code. This is because minimal embeddability implies that no side in the $4g$-gon representing the surface is left uncrossed by the curve, and glueing any additional loop will probably have to cross the curve (and will thus be registered on the Gauss code).

Thus, we ended reformulating the conjectured equivalence as follows:

Theorem 1. For any curve γ on a surface S such that the Gauss code of γ is minimally embeddable on S, γ satisfies the *parity condition* iff γ is essentially even iff γ is 2-*colorable*.

The proof is divided into several lemmas, given in the next subsections.

2.3 Minimally embeddable $\rightarrow$ cellularly embeddable

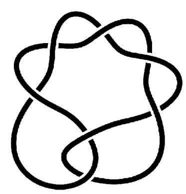
Lemma 1. Any curve minimally embedded on a surface S is also a cellular embedding on S.

Proof. Assume for contradiction that a minimally embedded curve γ on S is not celullarly embedded. Then, there is face containing a fundamental loop of S. We know that if we cut a fundamental loop of any surface we reduce its genus by 1, but we now get an embedding of γ onto T, a surface with a smaller genus than S, contradicting the minimality of the embedding. $\square$

2.4 Parity condition and the alternating diagram

Definition 5 (Alternating diagram)**.** An alternating diagram for a curve γ is an assignment of *under* or *over* to every crossing such that each intersection has one of its crossings going *under* and the other one *over*.

Example of an alternating diagram:



Lemma 2. Any curve satisfies the parity condition iff it has an alternating diagram.

Proof. If a curve satisfies the parity condition, we can simply alternate between over and under for each letter in the Gauss code, and given that between any pair of the same symbol there is an even number of letters, we will always assign opposites values to the positions with the same symbol. If a curve has an alternating diagram, then in the Gauss code we must have gone through an even number of letters between any pair of the same symbol in order for one instance of the symbol to be *under* and the other one to be be *over*. □

2.5 Parity condition implies 2-colorable

Lemma 3. Any curve γ minimally embedded on a surface S that satisfies the *parity condition* is 2-*colorable*.

Proof. We now know that the parity condition is equivalent to having an alternating diagram. We take the graph G representing γ and give every edge of G a direction: it goes from the undercrossing to the overcrossing. This is clearly well-defined because of the existence of an alternating diagram. Thanks to the minimal embeddability, we know that the graph is cellularly embeddable by Lemma 1, so the boundary of every face is a cycle of edges. We now note that the head of each dart is the tail of another one. This follows from the fact that any neighbouring edges in the boundary of a face correspond to different crossings, so if one points towards the node, the other one points away from it. Thus, every face has either a clockwise boundary or a counterclockwise boundary. Given that any two neighbouring faces share a dart, which in one face will be pointing counterclockwise and on the other one clockwise, we can assign a color to each face depending on the orientation of its boundary and conclude that γ is 2-*colorable*.

□

2.6 2-colorable → essentially even

Lemma 4. If a curve is 2-*colorable* then it is essentially even.

Proof. Suppose for contradiction that the curve γ is not essentially even. Then, the $4g$-gon representing γ has a side that is crossed an odd number of times. However, this implies that the number of faces bordering the side are also odd, and they are in a cycle. Thus, the dual of the graph representing the curve is not bipartite and the curve is not 2-*colorable*. □

With this, we can also conclude that if a curve γ is minimally embedded on a surface S, and satisfies the parity condition, then γ is essentially even. We are going to need this later on. Note that all we are left to do to finish the proof of Theorem 1 is showing that if γ is essentially even (and minimally embedded), then γ satisfies the parity condition.

2.7 Parity condition is preserved through homotopy

2.7.1 $0 \to 1$ and $1 \to 0$

It should be clear that the $0 \to 1$ and $1 \to 0$ moves preserve the parity condition because they simply add 2 identical symbols to a random part of the gauss code. The gap between these two symbols is 0 (even) and the other gaps either increase by 2 or do not change.

2.7.2 $3 \to 3$

The intital state of $3 \to 3$ move on a gauss code which fulfills the parity condition is either:

$$[x, y, S_1, y, z, S_2, z, x, S_3],$$
$$[x, y, S_1, y, z, S_2, x, z, S_3],$$
$$[x, y, S_1, z, y, S_2, z, x, S_3], \text{ or}$$
$$[x, y, S_1, z, y, S_2, x, z, S_3]$$

(where S_n is a string of even length). Performing the $3 \to 3$ move changes these gauss codes by swapping every pair of symbols between the S_n's. For example, $[x, y, S_1, y, z, S_2, z, x, S_3]$ transforms to $[y, x, S_1, z, y, S_2, x, z, S_3]$. All of these swaps preserve the *parity condition* for x, y, and z, and clearly it still holds for any other letter in S_n, thus it is preserved globally.

2.7.3 $0 \to 2$ and $2 \to 0$

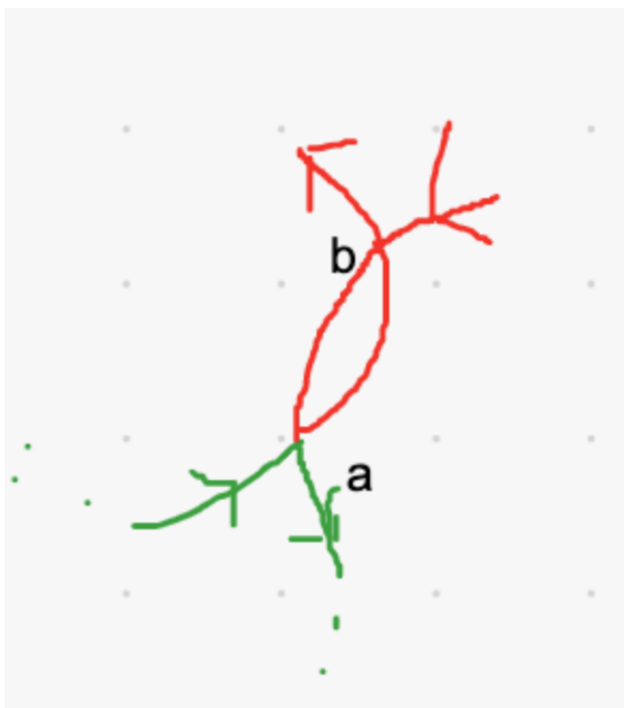
These moves only preserve the parity condition when the graph is essentially even, but since we have shown that the parity condition (along with minimally embedded) imply that something is essentially even, this can be assumed. Ultimately, this section will show that an embedding being essentially even implies that homotopy moves preserve the parity condition.

Let us show that $2 \to 0$ moves preserve parity conditions. Since every $2 \to 0$ simply deletes 2 pairs of adjacent letters in the Gauss code, it clearly preserves the parity condition.

Now, let us consider a $0 \to 2$ move that creates two new intersections a and b. The gaps between all the other letters (similarly to in the $0 \to 1$ move) are increased by 0, 2 or 4 as the new bigon is empty so there are no letters between a and b (within the bigon).

So all we have to show is that, with an essentially even curve, the gaps between the two copies of a and the gaps between the two copies of b are even. Since they are adjacent, the a-gaps and the b-gaps are the same parity. So WLOG, let us focus on the a gaps.

There are actually 2 a-gaps, corresponding to the gap 'outside' of the two 'a's and the gap 'inside' the two 'a's. They are coloured red and green respectively in the diagram below.

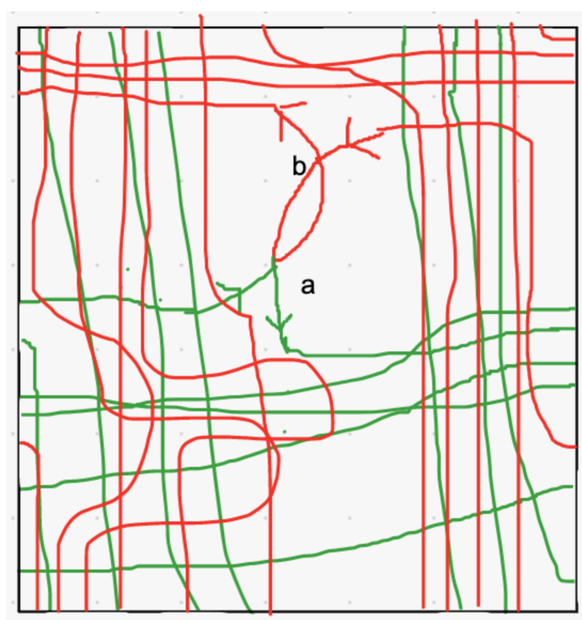


We imagine that the two red strands connect at some point, as do the two green strands. We aim to show that the number of intersections on either the red or the green strand is even (they must have the same parity as their sum plus 2 for the copies of a add to 2n for a gauss code with n distinct letters).

Not shown in the diagram are the numerous intersections that will define the parity of the lengths of these segments. These intersections fall into two sections. The first is self intersections of the red loop with itself or of the green loop with itself. These increase the length of those segments by 2 (as both copies of the letter corresponding to the self-intersection are in the same gap, green or red) Thus these do not impact the parity of the gap.

The other section is the intersections between the red and green loops. There must be an even number of these in total for the gaps to be of even parity.

Since the overall curve is essentially even, the number of essential loops of each 'type' must be even. What I mean by type is the number of paths from one edge to the opposite edge for each such pair of edges. On a torus there are 2 (2 pairs of sides on the square configuration space), on a double torus, 4 etc.. Since the green and red segments are both closed loops, they must contain whole numbers of each of these types of essential loops. For type t, let $2p_t$ be the number of essential loops (since it must be even) and g_t and r_t be the number of green and red loops of that type s.t. $g_t + r_t = 2p_t$. Let G_t and R_t be the sets of these loops. Then for each pair of types i and j, we are interested in the number of intersections betwen the segments of the curve that comprise G_i and R_j. By enumerating all of these we will find all the red green intersections in the curve.
Before we do this, we may need to perturb the curve and perform some $3 \to 3$ so that these sets can be seen clearly. We want the curve to look like it does in the diagram below



That is to say, the parts of the curve which are vertically essential and parts of the curve which are horizontally essential are distinct. This can be achieved by pushing the intersections up and down the curve without having them meet (which does not change the parity because it does not change the gauss code) and with $3 \to 3$ moves. It is okay for these vertical and horizontal essential curves to wiggle and intersect themselves and eachother (such as a few of the red vertical ones do in the bottom left corner as long as we break up all the diagonals into 1 horizontal and 1 vertical component.

This can aslo be done, albeit more complexly, on higher genus surfaces by breaking down each 'diagonal' into multiple segments which each go between opposite sides rather than non-opposite sides

Once we have done this, we can consider two pairs of sides. On the torus 2g-gon, there are only 2 pairs of sides, but on higher genus surfaces we will still enumerate it 2 pairs at a time.

Say we have sides i, i', j and j' where sides are opposite to their prime versions. Then we will have red lines going from i to i' (R_i), red lines from j to j' (R_j) and G_i and G_j.

We are only interested in the intersections of R_i and G_i, R_j and G_i, R_i and G_j and R_j and G_j. For the pairs R_i and G_i and R_j and G_j, the lines are running in the same direction. Any time a strand crosses another, it must eventually cross back over, otherwise it has made progress towards an essential cycle of the other type (ie if a vertical green line crosses a vertical red line to the left and never crosses back over, it is using part of the horizontal green line, so by our original set of transformations that crossing will not happen in this pairing). This idea is shown below, where if you ensure that each type of essential loop starts and ends in a small neighborhood, then all the red-green crossings must happen in pairs otherwise the red and green would start and end on different sides of each other

What that means is that there must be an even number of crossings in these pairings. In the other pairings, where the sets are 'perpendicular' to each other we have the situation as below.

In this section, again, there are trivial crossings which come in pairs where the wiggles in the lines can mean the a red essential line crosses a green essential line multiple time, however, since each red line must cross the green lines an odd number of times in order to go from one side of the octagon to the other, each red line has an odd number of crossings with each green line. For the sake of parity, we can consider these odd numbers of crossings to be 1. Then there are $r_i * g_j$ crossings here. Likewise for the other pairing there are $r_j * g_i$ crossings.

Total number of crossings for these 2 pairs of edges:

$$\begin{aligned}
&= r_i * g_j + r_j * g_i \\
&= r_i * (2p_j - r_j) + r_j * (2p_i - r_i) \\
&= 2p_j r_i + 2p_i r_j - 2r_j r_i \\
&= 2(p_j r_i + p_i r_j - r_j r_i)
\end{aligned}$$

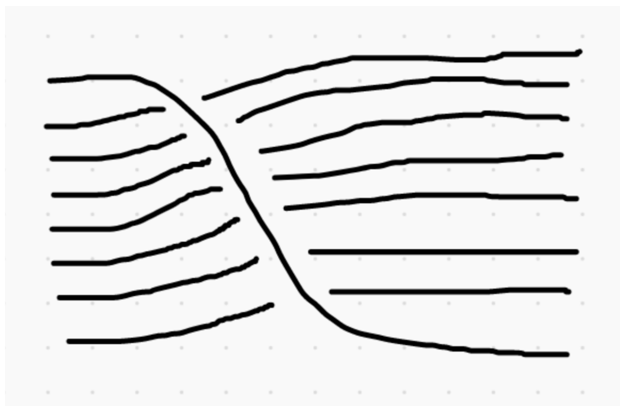
This number is even, so for each pair of sides, the number of intersections is even. Since every intersection happens between two lines which each belong to one of the types, the total number of red-green intersections (and thus the length of the gap between copies of a) is a sum of these even values and is thus even.

Thus the letters for the intersections created by a $0 \to 2$ move have even length gaps and so $0 \to 2$ moves preserve the parity condition.

2.8 Putting it all together

Now that we have shown that assuming that a curve is essentially even, the parity condition is preserved through homotopy, all we need to do is show that there exists some curve β homotopic to our original curve γ that satisfies the parity condition.

Consider each fundamental loop of our surface we can consider the following homotopic way of crossing the loop:



Let's take an arbitrary intersection. We want to show that it will go through an even number of other intersections before coming back. We note that the curve will intersect for every wrap above it, and then it will cross every wrap before coming back. Thus, the number of crossing will be $2k$ for some k, and thus even.

Now that we established that even essentiality implies the parity condition, we've finished proving Theorem 1.

2.9 2-colorability is preserved through homotopy

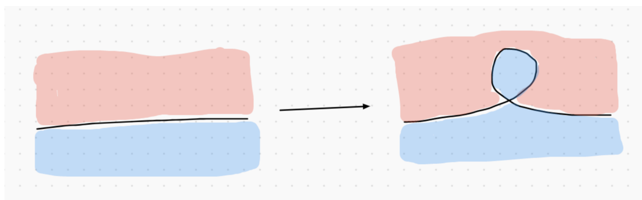
We have finished the proof of Theorem 1, but we'll give a direct visual proof that 2-colorability is preserved through homotopy, which may give a better intuition of the correspondence between 2-*colorability* and the *parity condition*.

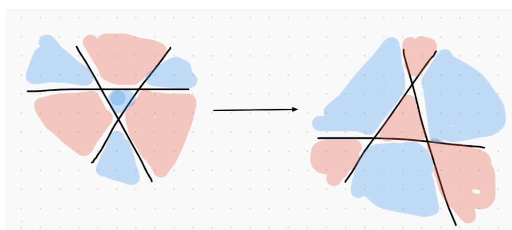
We know that any homotopy between curves can be decomposed into five moves (read [2]). We show that 2-*colorability* is preserved through these moves.

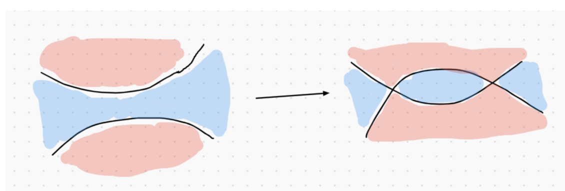
Lemma 5. *For any pair of homotopic curves γ_1 and γ_2, γ_1 is 2-colorable iff γ_2 is 2-colorable.*

Proof. The $1 \to 0$ move and the $0 \to 1$ move preserve 2-*colorability* because the face contained by an empty loop doesn't change the 2-*colorability*, we can always color it in the opposite color of the only face with which it shares boundary. The 2-*colorability* can be preserved through a $3 \to 3$ by switching the color in the inside of the triangle. Lastly, the faces created with a $2 \to 0$ move don't share an edge and thus can be colored in the same color than before the move. $\square$

Some drawings might prove helpful to understand the lemma:







3 Other interesting problems

3.1 Shortest minimally embeddable Gauss codes

One unresolved question we tackled is what's the minimum length that Gauss code needs to have to be minimally embeddable in an orientable surface of genus g. Note that there are two variants of the same problem, one considering signed Gauss codes and the other one considering unsigned ones. Let's denote by $\{A_n\}$ and $\{B_n\}$ the sequences representing the minimum number of intersections that a curve needs to have such that its Gauss code is minimally embeddable on a surface with genus n, unsigned and signed respectively.

Note that $A_n \geq B_n$ for all n, because if we have an unsigned Gauss code that can be embedded on a surface S, we can then take the signing of the embedding to get a signed Gauss code that can also be embedded on S.

We computed a few elements of both sequences (or bounds), along with some corresponding Gauss codes:

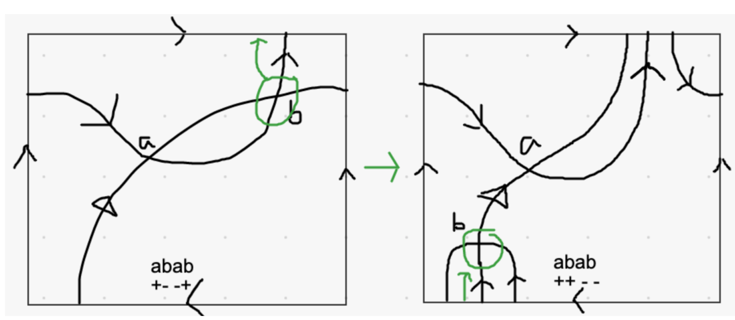
Genus	A_n	B_n	Unsigned example	Signed example
0	0	0	$\varnothing$	$\varnothing$
1	2	2	$abab$	$ABab$
2	5	3	$ababcdcede$	$ABacbC$
3	6	5	$abacdedbfcfe$	$ABabCDEced$
4	$11 \geq A_4 \geq 8$	7		$ABabCDcdEFegfG$

The upper bound for the genus 4 unsigned case was obtained from the Gauss code [$aghbakcdkghedbfcfoepop$], which cannot be realized on an orientable surface with genus smaller than 4.

We note that all of the examples with an even n satisfy the opposite of the *parity condition*: between every symbol there's always an odd number of symbols, this indicates that gauss codes which violate the parity condition can have higher minimum genera while staying shorter than their counterparts which fulfill the parity condition.

3.2 Classifying Curves on Non-Orientable Surfaces

The added factor of non-orientability introduces new types of intersections for signed gauss codes. The first thing to note is that the standard signing—where a '+' indicates that the other strand crosses this part of the curve from left to right and a '-' indicates that it crosses from right to left—does not work in non-orientable surfaces. This is because left and right are not consistently defined on non-orientable surfaces. It can be most clearly seen by simply moving the intersection over the boundary which is connected with a twist. The diagram below shows what happens when this is done:

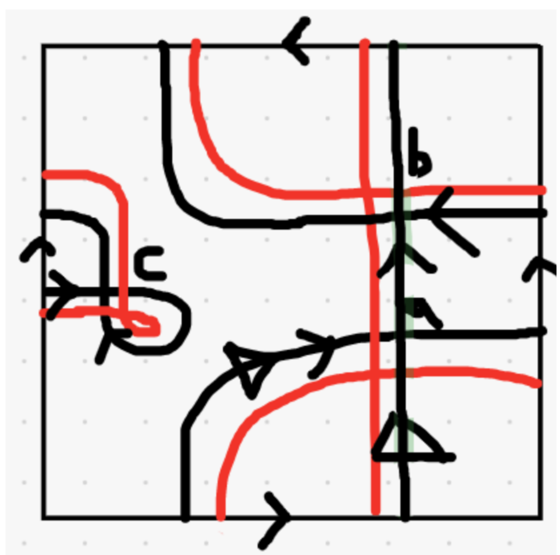


Each time you cross over an edge with a twist, left and right swap and so the signed gauss code changes. This means that our old definition of left and right are unsuitable.

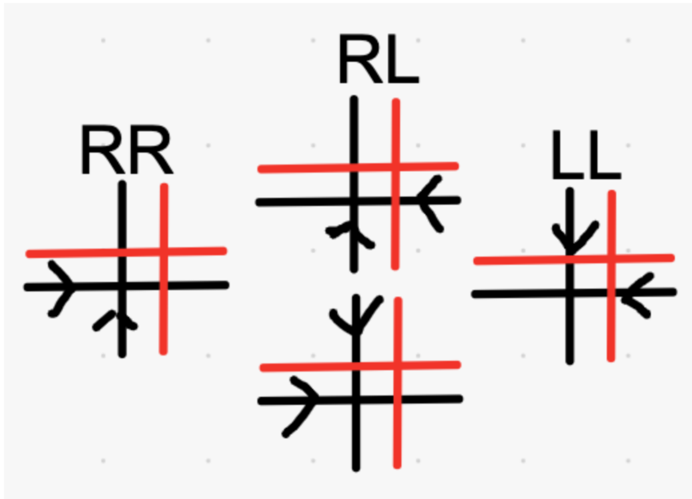
Instead, we will draw a red line on one side of the curve (which we will call left), effectively thickening it. Then, we can consider a crossing to go from left to right when it hits the red side of the thickened curve first, and then the black. Likewise if it first hits the black, then the red, we consider the crossing right to left. These classifications are consistent even if we move the intersection over an edge connected with a twist.

This, of course, restricts us to orientable curves, meaning those which complete a whole number of twists (rather than any half twists). This corresponds to curves which traverse the twisted edges of the configuration space an even number of times. Otherwise the red line would finish its first traversal of the black line on the opposite side to the one at which it started.

An orientable curve would then look like this:



Also in this diagram, we see that this new classification yields 3 different types of intersections, seen in intersection a, b and c. Both crossings in intersection a are from left to right and both in b are from right to left. c is actually the normal crossing we have on orientable surfaces where 1 crossing is left to right and the other is right to left. The diagram below shows that all possible crossings fit into these categories:



Any intersectiion can be rotated so that the red and black lines are in the orientations shown, and then the four different directions that each strand comprise all possible intersections. As we can see, these break down into three categories (both left, both right and 1 right 1 left). The two crossings in the middle are equivalent since the top one turns into the bottom one when it is moved over an edge connected with a twist.

One thing to note is that this system works exactly the same if we draw the line on the right (or simply consider it to be already on the right, such as if we had started drawing the line from the triangle to the left of intersection a). All this does is flips the 'sign' (- = right to left, + = left to right) of every symbol. For instance, the curve above has two 'signed' gauss codes depending on this choice, for simplicity, upper case letters correspond to + and lower case to -. With that in mind, the curve above is AbAcCb or aBaCcB.

Recall that any finite unsigned gauss code can be embedded on an orientable surface of finite genus by simply adding a handle for every connection. With this new signing, we are no longer restricted in our signed gauss codes to have 1 + and 1 - for each letter, as a result, we can embed any finite signed gauss code (with any signing at all) on a non-orientbale surface of finite genus by adding a handle and/or a crosscap for every connection.

4 Next steps

There were many questions that we didn't have enough time to explore fully. First of all, our original question: how to efficiently decide if an unsigned Gauss code is minimally embeddable on the torus. Also, we would like to see how the distribution among the genera look like for the a gauss code of length $2n$.

Also, it would be very interesting to investigate more the sequences $\{A_n\}$ and $\{A_n\}$ mentioned in the subsection *Shortest minimally embeddable Gauss codes*. Lastly, it would be helpful to read up on the current literature on the field.

5 Appendix

5.1 Python code

All the code written for this project can be found at `https://github.com/Raikhen/final-project-cs49`.

6 References

1. Lins, Sóstenes & Lima, Emerson & Silva, Valdenberg. (2003). An Affine Linear Solution for the 2-Face Colorable Gauss Code Problem in the Klein Bottle and a Quadratic System for Arbitrary Closed Surfaces.

2. Erickson, Jeff. (2020). Notes for CS 598: One-Dimensional Computational Topology. `http://jeffe.cs.illinois.edu/teaching/topology20/`.