2.1. Large-N action and equation of motion:

Classical action of RST Model

$$S_{cl} = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[e^{-2\phi}\left(R+4(\nabla\phi)^2+4\lambda^2\right)-\frac{N}{24}\phi R\right] + S_{CFT}$$

$$S_{CFT} = -\sum_{k=1}^{N}\frac{1}{4\pi}\int d^2x\sqrt{-g}\,(\nabla f_k)^2 \quad \cdots\cdots (2.1)$$

λ is chosen to be 1. $\quad \lambda = 1 \quad \cdots\cdots (2.2)$

In the quantum effective action, the conformal anomaly contributes a term

$$S_{anom} = -\frac{N}{96\pi}\int d^2x\sqrt{-g}\,R\,\Box^{-1}R \quad \cdots\cdots (2.3)$$

The large-N limit is $N \longrightarrow \infty$ with $Ne^{2\phi}$ held fixed.

It described by the sum,

$$S_{eff} = S_{cl} + S_{anom} + S_{CFT} \quad \cdots\cdots (2.4)$$

$ds^2 = -e^{2\rho}dx^+dx^-$ where $x^{\pm} = x^0 \pm x^1$

$$R_{\pm} = -2\partial_+\partial_-\rho$$

$$g_{\mu\nu} = \begin{bmatrix} 0 & -\frac{e^{2\rho}}{2} \\ -\frac{e^{2\rho}}{2} & 0 \end{bmatrix} \qquad g^{\mu\nu} = \begin{bmatrix} 0 & -2e^{-2\rho} \\ -2e^{-2\rho} & 0 \end{bmatrix}$$

$$\sqrt{-g} = \frac{e^{2\rho}}{2}$$

$$R = R_{\mu\nu} g^{\mu\nu} = 2R_{+-} g^{+-} = 2(-2\partial_+ \partial_- \rho)(-2e^{-2\rho})$$

$$\Rightarrow \boxed{R = 8 e^{-2\rho} \partial_+ \partial_- \rho}$$

$$(\nabla \phi)^2 = 2g^{+-} \partial_+ \phi \, \partial_- \phi = 2(-2e^{-2\rho}) \partial_+ \phi \, \partial_- \phi$$

$$\Rightarrow (\nabla \phi)^2 = -4e^{-2\rho} \partial_+ \phi \, \partial_- \phi$$

$$\Box^{-1} R = F \qquad \text{[Lets say]}$$

$$\Rightarrow \cancel{8}\ \Box F = R$$

$$\Rightarrow 2g^{+-} \partial_+ \partial_- F = 8e^{-2\rho} \partial_+ \partial_- \rho$$

$$\Rightarrow -4e^{-2\rho} \partial_+ \partial_- F = 8e^{-2\rho} \partial_+ \partial_- \rho$$

$$\Rightarrow \boxed{F = -2\rho}$$

$$\therefore\ S_{eff} = \frac{1}{2\pi}\int d^2x\ \frac{e^{2\rho}}{2}\left[e^{-2\phi}\left(8e^{-2\rho}\,\partial_+\partial_-\rho - 16e^{-2\rho}\,\partial_+\partial_-\phi + 4\right) - \frac{N}{24}\,\phi\cdot 8e^{-2\rho}\,\partial_+\partial_-\rho\right]$$

$$-\frac{N}{96\pi}\int d^2x\ \frac{e^{2\rho}}{2}\ 8e^{-2\rho}\,\partial_+\partial_-\rho\,(-2\rho)$$

$$-\sum_{k=1}^{N}\frac{1}{4\pi}\int d^2x\ \frac{e^{2\rho}}{2}\ 2\left(-2e^{-2\rho}\right)\partial_+f_k\,\partial_-f_k$$

$$= \frac{1}{2\pi}\int d^2x\left[e^{-2\phi}\left(4\partial_+\partial_-\rho - 8\partial_+\phi\,\partial_-\phi + 2e^{2\rho}\right) + \frac{N}{6}(\rho-\phi)\,\partial_+\partial_-\rho\right] + S_{CFT} \qquad \text{-----}(2.5)$$

where $$S_{CFT} = \frac{1}{2\pi}\sum_{k=1}^{N}\int d^2x\ \partial_+f_k\,\partial_-f_k$$

Lets assume,

$$\Omega = \frac{12}{N}e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4}\ln\frac{48}{N}$$

$$\chi = \frac{12}{N}e^{-2\phi} + \rho - \frac{\phi}{2} + \frac{1}{4}\ln\frac{3}{N} \qquad \text{-----}(2.6)$$

The resulting action

$$S_{eff} = \frac{N}{12\pi}\int d^2x\left(\partial_+\Omega\,\partial_-\Omega - \partial_+\chi\,\partial_-\chi + e^{2\chi-2\Omega}\right) + S_{CFT} \qquad \text{-----}(2.7)$$

$$\partial_\pm \Omega = -\frac{24}{N} e^{-2\phi} \partial_\pm \phi + \frac{1}{2} \partial_\pm \phi$$

$$\partial_\pm \chi = -\frac{24}{N} e^{-2\phi} \partial_\pm \phi + \partial_\pm \rho - \frac{1}{2} \partial_\pm \phi$$

$$\partial_+ \Omega \, \partial_- \Omega = \left(\frac{24}{N} e^{-2\phi}\right)^2 \partial_+ \phi \, \partial_- \phi + \frac{1}{4} \partial_+ \phi \, \partial_- \phi - \frac{24}{N} e^{-2\phi} \partial_+ \phi \, \partial_- \phi$$

$$\begin{aligned}
\partial_+ \chi \, \partial_- \chi = {} & \left(\frac{24}{N} e^{-2\phi}\right)^2 \partial_+ \phi \, \partial_- \phi + \partial_+ \rho \, \partial_- \rho + \frac{1}{4} \partial_+ \phi \, \partial_- \phi \\
& - \frac{24}{N} e^{-2\phi} \left(\partial_+ \phi \, \partial_- \rho + \partial_- \phi \, \partial_+ \rho\right) \\
& - \frac{1}{2} \left(\partial_+ \phi \, \partial_- \rho + \partial_- \phi \, \partial_+ \rho\right) \\
& + \frac{24}{N} e^{-2\phi} \partial_+ \phi \, \partial_- \phi
\end{aligned}$$

$$e^{2\chi - 2\Omega} = e^{2\rho - 2\phi + \ln \frac{12}{N}} = \frac{12}{N} e^{2\rho - 2\phi}$$

$$\partial_+ \Omega \, \partial_- \Omega - \partial_+ \chi \, \partial_- \chi + e^{2\chi - 2\Omega}$$

$$\begin{aligned}
= {} & -\frac{48}{N} e^{-2\phi} \partial_+ \phi \, \partial_- \phi - \partial_+ \rho \, \partial_- \rho + \frac{24}{N} e^{-2\phi} \left(\partial_+ \phi \, \partial_- \rho + \partial_- \phi \, \partial_+ \rho\right) \\
& + \frac{1}{2} \left(\partial_+ \phi \, \partial_- \rho + \partial_- \phi \, \partial_+ \rho\right) + \frac{12}{N} e^{2\rho - 2\phi}
\end{aligned}$$

$$= \underbrace{\frac{6}{N}\left(4\,\partial_+\rho\,\partial_-\rho - 8\,\partial_+\phi\,\partial_-\phi + 2e^{2\rho}\right)e^{-2\phi}}_{\xi}$$

$$- \frac{24}{N}e^{-2\phi}\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho + \partial_+\partial_-\rho\right)$$

$$- \left(\underbrace{\tfrac{1}{2}\partial_+\partial_-(\rho^2)}_{\text{upon integration the contribution is 0.}} - \rho\,\partial_+\partial_-\rho\right) + \frac{1}{2}\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho\right)$$

$$= \xi - \frac{24}{N}e^{-2\phi}\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho + \partial_+\partial_-\rho\right)$$

$$+ (\rho - \phi)\,\partial_+\partial_-\rho + \frac{1}{2}\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho + 2\phi\,\partial_+\partial_-\rho\right)$$

$$\iint e^{-2\phi}\,\partial_+\partial_-\rho\; dx^+ dx^-$$

$$= \int \cancel{\oint}\left(e^{-2\phi}\,\partial_-\rho + 2\int e^{-2\phi}\,\partial_+\phi\,\partial_-\rho\right) dx^-$$

$$= \int\left(e^{-2\phi}\,\partial_+\rho + 2\int e^{-2\phi}\,\partial_-\phi\,\partial_+\rho\right) dx^+$$

$$= \frac{1}{2}\int e^{-2\phi}\left(\partial_+\rho\, dx^+ + \partial_-\rho\, dx^-\right) + \int e^{-2\phi}\left(\partial_-\phi\,\partial_+\rho\, dx^+ + \partial_+\phi\,\partial_-\rho\, dx^-\right)$$

$$= -\iint e^{-2\phi}\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\phi\right) dx^+ dx^-$$

$$\Rightarrow \iint e^{-2\phi}\left(\partial_+\partial_-\rho + \partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho\right) dx^+ dx^-$$

$$= 0$$

$$2\iint \phi\,\partial_+\partial_-\rho\; dx^+ dx^-$$

$$= -\iint\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho\right) dx^+ dx^-$$

$$\Rightarrow \iint\left(\partial_+\phi\,\partial_-\rho + \partial_-\phi\,\partial_+\rho + 2\phi\,\partial_+\partial_-\rho\right) dx^+ dx^-$$

$$= 0$$

$$\therefore\ \partial_+\Omega\,\partial_-\Omega - \partial_+\chi\,\partial_-\chi + e^{2\chi - 2\Omega} = \xi + (\rho - \phi)\,\partial_+\partial_-\rho$$

then scales with an overall factor of N in our large-N limit where Ω, χ are fixed.

$$\Omega = \frac{12}{N} e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4} \ln \frac{48}{N}$$

$$\frac{\partial \Omega}{\partial \phi} = -\frac{24}{N} e^{-2\phi} + \frac{1}{2} = 0$$

$$\Rightarrow e^{-2\phi} = \frac{48}{N}$$

$$\boxed{\Omega \geq \frac{1}{4}} \quad \text{.......} (2.8)$$

Applying EL eqn to S_{eff}, —

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Omega)} \right) - \frac{\partial \mathcal{L}}{\partial \Omega} = 0$$

$$\Rightarrow \partial_+ (\partial_- \Omega) + \partial_- \partial_+ \Omega + 2 e^{2\chi - 2\Omega} = 0$$

$$\Rightarrow \partial_+ \partial_- \Omega = - e^{2\chi - 2\Omega}$$

Apply EL eqn to S_{eff}, —

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \chi)} \right) - \frac{\partial \mathcal{L}}{\partial \chi} = 0$$

$$\cancel{\Rightarrow \partial_+ \left(\frac{\partial \mathcal{L}}{\partial} \right.}$$

$$\Rightarrow - 2 \partial_+ \partial_- \chi - 2 e^{2\chi - 2\Omega} = 0$$

$$\Rightarrow \partial_+ \partial_- \chi = - e^{2\chi - 2\Omega}$$

$\therefore$ EOM for S_{eff} is

$$\boxed{\partial_+ \partial_- \Omega = \partial_+ \partial_- \chi = - e^{2\chi - 2\Omega}}$$

$\because \; \partial_+ \partial_- \chi = \partial_+ \partial_- \Omega$

$\Rightarrow \partial_- \chi = \partial_- \Omega + \alpha^-(x^-)$

$\Rightarrow \boxed{\chi = \Omega + \alpha^+(x^+) + \alpha^-(x^-)}$

The residual on-shell diffeomorphism symmetry of conformal gauge allows us to choose coordinates in which

$$\boxed{\chi = \Omega} \quad \text{-------} \quad (2.9)$$

For reasons which will become aparent we refer to these as Kruskal coordinates and denote the non-scalar quantities χ and ρ in Kruskal gauge as χ_K and ρ_K.

$$\chi_K = \Omega$$

$$\Rightarrow \frac{12}{N} e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4} \ln \frac{48}{N} = \frac{12}{N} e^{-2\phi} + \rho_K - \frac{\phi}{2} + \frac{1}{4} \ln \frac{3}{N}$$

$$\Rightarrow \boxed{\phi = \rho_K - \frac{1}{2} \ln \frac{N}{12}} \quad \text{------} \quad (2.10.i)$$

$$\Omega = \frac{12}{N} e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4} \ln \frac{48}{N}$$

$$= \frac{12}{N} e^{-2\rho_K - \ln \frac{12}{N}} + \frac{\rho_K}{2} - \frac{1}{4} \ln \frac{N}{12} - \frac{1}{4} \frac{48}{N}$$

$$\boxed{\Omega = e^{-2\rho_K} + \frac{1}{2}\left(\rho_K - \ln 2\right)} \quad (2.10.ii)$$

The equation of motion in this gauge is simply

$$\boxed{\partial_+ \partial_- \Omega = -e^{2\chi_K - 2\Omega} = -1} \quad \text{-------} (2.11)$$

$\because \ \chi_K = \Omega$

There are also constraint equations, which are given by varying (2.1) with respect to $g_{\pm\pm}$ before fixing to conformal gauge:

$$\delta_g S_{cl} = \frac{1}{2\pi}\int d^2x \left(\delta\sqrt{-g}\,\psi + \sqrt{-g}\left(\delta_g R + 4\delta_g (\nabla\phi)^2\right)e^{-2\phi} - \sqrt{-g}\,\frac{N}{24}\phi\,\delta_g R\right)$$

$$- \frac{1}{4\pi}\sum_{k=1}^{N}\int d^2x \left(\delta_g\sqrt{-g}\,(\nabla f_k)^2 + \sqrt{-g}\,\delta_g g^{\mu\nu}\,\partial_\mu f_k\,\partial_\nu f_k\right)$$

$$= \frac{1}{2\pi}\int d^2x \Big(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,\psi + \sqrt{-g}\big(\delta g^{\mu\nu} R_{\mu\nu}$$

$$+ g^{\mu\nu}\,\delta R_{\mu\nu} + 4\delta g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\big)e^{-2\phi}$$

$$- \sqrt{-g}\,\frac{N}{24}\phi\left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\right)\Big)$$

$$- \frac{1}{4\pi}\sum_{k=1}^{N}\int d^2x \left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}(\nabla f_k)^2 + \sqrt{-g}\,\delta g^{\mu\nu}\,\partial_\mu f_k\,\partial_\nu f_k\right)$$

$$= 0$$

$$\Rightarrow -\frac{1}{2} g_{\mu\nu}\psi + e^{-2\phi}\left(R_{\mu\nu} + 4\partial_\mu\phi\,\partial_\nu\phi\right)$$

$$- \frac{N}{24}\phi R_{\mu\nu} - \frac{1}{2}\sum_{k=1}^{N}\left(-\frac{1}{2}g_{\mu\nu}(\nabla f_k)^2 + \partial_\mu f_k\,\partial_\nu f_k\right) = 0$$

$$\Rightarrow 4e^{-2\phi}\,\partial_\pm\phi\,\partial_\pm\phi - \frac{1}{2}\sum \partial_\pm f_k\,\partial_\pm f_k = 0$$

$$\Rightarrow e^{-2\phi}\,\partial_\pm\phi\,\partial_\pm\phi = \frac{1}{8}\sum \partial_\pm f_k\,\partial_\pm f_k$$

$$\Omega = \frac{12}{N} e^{-2\phi} + \frac{\phi}{2} - \frac{1}{4} \ln \frac{48}{N}$$

$$\Rightarrow \quad \partial_\pm \Omega = -\frac{24}{N} e^{-2\phi} \partial_\pm \phi + \frac{1}{2} \partial_\pm \phi$$

$$\Rightarrow \quad \partial_\pm^2 \Omega = \frac{48}{N} e^{-2\phi} \partial_\pm \phi \, \partial_\pm \phi - \frac{24}{N} e^{-2\phi} \partial_\pm^2 \phi + \frac{1}{2} \partial_\pm^2 \phi$$

$$= \frac{6}{N} \sum_{k=1}^{N} \partial_\pm f_k \, \partial_\pm f_k - \frac{24}{N} e^{-2\phi} \partial_\pm^2 \phi + \frac{1}{2} \partial_\pm^2 \phi$$

$$\Rightarrow \boxed{\partial_\pm^2 \Omega = -T^f_{\pm\pm} - t_\pm} \quad \ldots\ldots (2.12)$$

where $$T^f_{\pm\pm} = -\frac{6}{N} \sum_{k=1}^{N} \partial_\pm f_k \, \partial_\pm f_k \quad \ldots\ldots (2.13)$$

is the matter stress tensor rescaled by a factor of $\frac{12\pi}{N}$ for notational convenience. $t_\pm$ depends on the choice of coordinates prescribed ~~by~~ for normal ordering. If we choose $x'^+(x^+)$ instead of x^+, t^+ shifts by the Schwarzian

$$t'_+ = (\partial'_+ x^+)^2 t_+ + \sqrt{\partial'_+ x} \, \partial_+^2 \sqrt{\partial_+ x'^+} \quad \ldots\ldots (2.14)$$

Note: $\mathcal{L}_{SCF} = \left(\sum (\nabla f_k)^2\right)\left(-\frac{1}{4\pi}\right)$

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\, g_{\mu\nu}$$

$$T_{\pm\pm} = -\frac{1}{4\pi}\cdot 2\sum \partial_\pm f_k\, \partial_\pm f_k$$

$$= -\frac{1}{2\pi}\sum \partial_\pm f_k\, \partial_\pm f_k$$

when rescaled with $\frac{12\pi}{N}$, $T_{\pm\pm} = -\frac{6}{N}\sum \partial_\pm f_k\, \partial_\pm f_k$