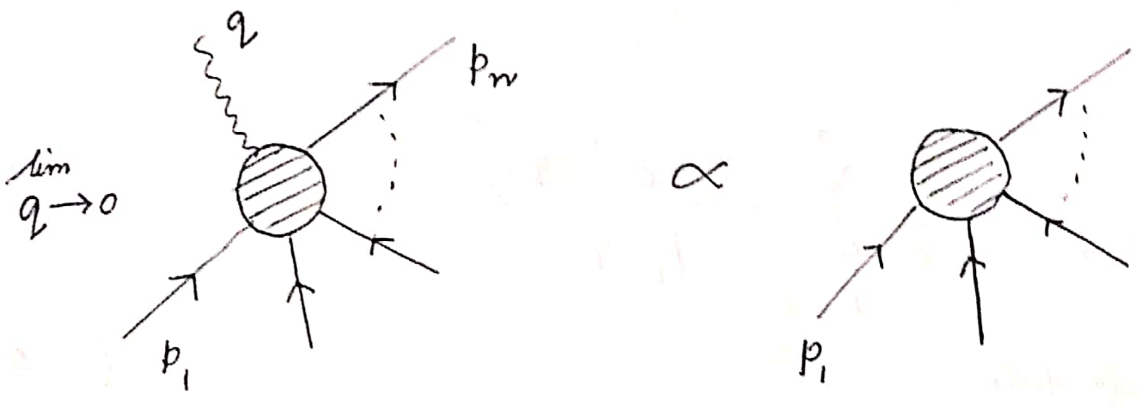
Lectures on Celestial Holography

Section 2 : Soft theorems and asymptotic symmetries

2.1 Soft theorems

→ Universal relationship obeyed by scattering amplitudes in any theory with massless particles.

$$\lim_{\omega \to 0} \mathcal{A}^{\pm}_{n+1}(q) = \left[S^{(0)\pm}_{n} + S^{(1)\pm}_{n} + \mathcal{O}(\omega) \right] \mathcal{A}_n \qquad (2.1)$$



For simplicity we have focused on tree-level scattering. In gravity and gauge theory, the scattering of high energy charged particles is accompanied by radiation. The radiation can be described as a collection of quanta (photons, gravitons) of different energies. When energy carried by one single quantum is small we get (2.1).

$\mathcal{A}_{n+1}$ = scattering amplitude of n general particles of four momentum $p_1, \ldots, p_n$ and one massless particle of four momentum $q = (\omega, \vec{q})$ and +ve ~~or~~ or −ve helicity

$\mathcal{A}_n$ = the same scattering amplitude in the absence of the massless particle.

In gravity

$$S_n^{(0)\pm} = \frac{\varkappa}{2} \sum_{k=1}^{n} \frac{\left(p_k \cdot \varepsilon^{\pm}(q)\right)^2}{p_k \cdot q}$$

leading soft factor

(2.2)

$$S_n^{(1)\pm} = -\frac{i\varkappa}{2} \sum_{k=1}^{n} \frac{\varepsilon^{\pm}(q) \cdot p_k}{p_k \cdot q}\, q \cdot J_k \cdot \varepsilon^{\pm}(q)$$

Subleading soft factor

where $\varkappa = \sqrt{32\pi G}$

In QED

$$\begin{cases} S_n^{(0)\pm} = \displaystyle\sum_{k=1}^{n} Q_k \frac{p_k \cdot \varepsilon^{\pm}(q)}{p_k \cdot q} \\ \\ S_n^{(1)\pm} = -i \displaystyle\sum_{k=1}^{n} Q_k \frac{q \cdot J_k \cdot \varepsilon^{\pm}(q)}{p_k \cdot q} \end{cases} \qquad (2.3)$$

G = Newton's constant

Q_k = charge of k-th particle

J_k = total angular momentum of particle k.

→ Polarizing tensor of graviton

$$\varepsilon^{\pm}_{\mu\nu}(q) = \varepsilon^{\pm}_{\mu}(q)\, \varepsilon^{\pm}_{\nu}(q) \qquad (2.4)$$

where $\varepsilon^{\pm}_{\mu}(q)$ is the polarization of a helicity-1 particle obeying

$$\varepsilon^{\pm}(q)\cdot q = 0 \quad ; \quad \varepsilon^{\pm}(q)\cdot\varepsilon^{\pm}(q) = 0 \; ;$$

$$\varepsilon^{\pm}(q)\cdot\varepsilon^{\mp}(q) = 1 \qquad (2.5)$$

Note: We pick a gauge in which the gravitation is transverse and traceless

$$q^{\mu} \varepsilon_{\mu\nu} = q^{\nu} \varepsilon_{\mu\nu} = \varepsilon^{\mu}{}_{\mu} = 0$$

For simplicity we work in units where

$$8\pi G = 1 \; ; \quad \kappa = \sqrt{32\pi G} = 2 \qquad (2.6)$$

2.2 Penrose diagrams of Minkowski Space

→ Minkowski metric

$$ds^2 = -dt^2 + d\vec{x}^2$$

$$= -dt^2 + dr^2 + r^2 d\Omega_2^2 \qquad (2.8)$$

where $$d\Omega_2^2 = d\theta^2 + (\sin\theta)^2 d\phi^2 \qquad (2.9)$$

metric on unit 2-sphere

→ Introduction of retarded and advanced coordinates

$$u = t - r \quad ; \quad v = t + r \qquad (2.10)$$

Let,

$$z = \cot\frac{\theta}{2} e^{i\phi} \quad ; \quad \bar{z} = \cot\frac{\theta}{2} e^{-i\phi} \qquad (2.11)$$

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$\cancel{-dt^2 + dr^2 = -(dt-dr)^2}$

$$-dt^2 + dr^2 = -(dt-dr)^2 - 2\,dt\,dr + 2dr^2$$

$$= -(dt-dr)^2 + 2dr\,(dt-dr)$$

$\cancel{-du^2 +}$

$$= -du^2 - 2\,du\,dr$$

Similarly, we can calculate

$$-dt^2 + dr^2 = -dv^2 + 2\,dv\,dr$$

$$z = \cot\frac{\theta}{2}\, e^{i\phi}$$

$$\Rightarrow dz = -\frac{\operatorname{cosec}^2\frac{\theta}{2}}{2}\, e^{i\phi}\, d\theta + i\cot\frac{\theta}{2}\, e^{i\phi}\, d\phi$$

$$\bar{z} = \cot\frac{\theta}{2}\, e^{-i\phi}$$

$$\Rightarrow d\bar{z} = -\frac{\operatorname{cosec}^2\frac{\theta}{2}}{2}\, e^{-i\phi}\, d\theta - i\cot\frac{\theta}{2}\, e^{-i\phi}\, d\phi$$

$$\cancel{dz\, d\bar{z} = \frac{\operatorname{cosec}^4\frac{\theta}{2}}{4}\, e^{2i\phi}\, d\theta^2 + \frac{\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}}{2}\, i\, d\theta\, d\phi - \frac{\operatorname{cosec}^2\frac{\theta}{2}\cot\frac{\theta}{2}}{2}\, d\theta\, d\phi}$$

$$dz\, d\bar{z} = \frac{\csc^4 \frac{\theta}{2}}{4}\, d\theta^2 + \cot^2 \frac{\theta}{2}\, d\phi^2$$

$$z\bar{z} = \cot^2 \frac{\theta}{2}$$

$\cancel{(1+z\bar{z})}$

$$\boxed{\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} = 2 \sin^4 \frac{\theta}{2}}$$

$$2\gamma_{z\bar{z}}\, dz\, d\bar{z} = d\theta^2 + 4 \sin^2\frac{\theta}{2} \cos^2\frac{\theta}{2}\, d\phi^2$$

$$= d\theta^2 + \sin^2\theta\, d\phi^2$$

In retarded coordinates $(u, r, z, \bar{z})$ the metric (2.8) becomes

$$ds^2 = -du^2 - 2du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z} \qquad (2.12)$$

In advanced coordinates $(v, r, z, \bar{z})$ the metric (2.8) becomes

$$ds^2 = -dv^2 + 2dv\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z} \qquad (2.13)$$

To understand the assymptotic structure of (2.8) we introduce coordinates (T, R)

$$u = \tan U \,, \quad v = \tan V \,, \quad T = U + V \,, \quad R = V - U$$

(2.14)

$$ds^2 = -du\,dv + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$

$$u = \tan U = \tan\left(\frac{T-R}{2}\right)$$

$$\Rightarrow du = \cancel{\frac{\sec^2}{2}} \; \frac{1}{2} \sec^2\left(\frac{T-R}{2}\right)(dT - dR)$$

$$v = \tan V = \tan\left(\frac{T+R}{2}\right)$$

$$\Rightarrow dv = \frac{1}{2} \sec^2\left(\frac{T+R}{2}\right)(dT + dR)$$

$$\therefore ds^2 = -\frac{1}{4} \sec^2\left(\frac{T-R}{2}\right) \sec^2\left(\frac{T+R}{2}\right)(dT^2 - dR^2)$$

$$+ 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$

$$r^2 = \left(\frac{v-u}{2}\right)^2$$

$$= \frac{1}{4} \cdot \frac{\sin^2 R}{\cos^2\left(\frac{T-R}{2}\right)\cos^2\left(\frac{T+R}{2}\right)}$$

(2.8) becomes

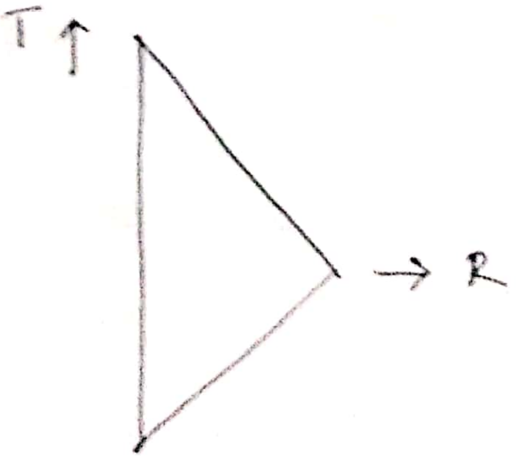
$$ds^2 = \Omega^2(T,R)\left(-dT^2 + dR^2 + 2\sin^2 R\, \gamma_{z\bar{z}}\, dz\, d\bar{z}\right)$$

$$\Omega^{-2}(T,R) = 4\cos^2\left[\tfrac{1}{2}(T-R)\right]\cos^2\left[\tfrac{1}{2}(T+R)\right] \qquad (2.15)$$

In original Minkowski space

$r > 0$; $0 < u < v < \infty$

So, $-\frac{\pi}{2} < U < V < \frac{\pi}{2}$ and $0 < R < \pi$



2.3 Asymptotically flat spacetime

→ Asymptotically flat spacetimes have the same ~~causal~~ casual structure as Minkowski space at infinity. ~~Any~~

An AFS admits an expansion in powers of r^{-1} around the Minkowski ~~space~~ metric near $\mathcal{I}^+$

$$
\begin{aligned}
ds^2 = & -du^2 - 2\,du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z} \\
& \frac{2m_B}{r} du^2 + r C_{zz}\, dz^2 + r C_{\bar{z}\bar{z}}\, d\bar{z}^2 \\
& + 2 g_{uz}\, du\, dz + 2 g_{u\bar{z}}\, du\, d\bar{z} + \cdots
\end{aligned}
\tag{2.16}
$$

Solving Einstein's eqn

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^M_{\mu\nu} \tag{2.17}$$

order by order in a large-r expansion one finds

$$
\begin{aligned}
g_{uz} = & \frac{1}{2} D^z C_{zz} + \frac{1}{6r} C_{zz} D_z C^{zz} \\
& + \frac{2}{3r} N_z + \mathcal{O}(r^{-2})
\end{aligned}
\tag{2.18}
$$

where $D_{\bar{z}}$ is the covariant derivative associated with $\gamma_{z\bar{z}}$.

m_B = Bondi Mass aspect

N_A = " Angular momentum aspect

$$N_{zz} = \partial_u C_{zz} \qquad (2.19)$$

Outgoing news tensor.

They are all f^n-s of $(u, z, \bar{z})$.

→ m_B, C_{zz}, N_z are not all independent

uu constraint gives

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T_{uu}^{M(2)}$$

$$- \frac{1}{4} N_{zz} N^{zz} \qquad (2.20)$$

uz constraint gives us

$$\partial_u N_z = \frac{1}{4} D_z \left(D_z^2 C^{zz} - D_z^2 C^{\bar{z}\bar{z}} \right)$$

$$- T^{M\,(2)}_{uz} + \partial_z m_B + \frac{1}{16} D_z \partial_u \left(C_{zz} C^{zz} \right)$$

$$- \frac{1}{4} \left(N^{zz} D_z C_{zz} + N_{zz} D_z C^{zz} \right)$$

$$- \frac{1}{4} D_z \left(C^{zz} N_{zz} - N^{zz} C_{zz} \right) \tag{2.21}$$

where $T^{M\,(2)}_{\mu\nu} = \lim_{r \to \infty} r^2 T^{M}_{\mu\nu}$