Asymptotically flat spacetime

Supervisor: Dr. Sachin Jain
Student: Raikhik Das

1 Bondi-Sachs Formalism

1.1 Bondi-Sachs metric

Retarded time: $u = t - r$; Advanced time: $v = t + r$

The Bondi-Sachs coordinates $x^a = (u, r, x^A)$ are based on a family of outgoing null hypersurfaces $u = constant$. The hypersurfaces $x^0 = u = constant$ are null, i.e. the normal co-vector $k_a = -\nabla_a u$ satisfies $g^{ab}(\nabla_a u)(\nabla_b u) = 0$, so that $g^{uA} = 0$. The coordinate $x^1 = r$, which varies along the null rays, is chosen to be an areal coordinate such that $det[g_{AB}] = r^4\mathbf{q}$, where $\mathbf{q}(x^A)$ is the determinant of the unit sphere metric q_{AB} associated with the angular coordinates x^A, e.g. $q_{AB} = diag(1, \sin^2\theta)$ for standard spherical coordinates $x_A = (\theta, \phi)$.

$$g_{rr} = g_{rA} = 0$$

Bondi-Sachs metric takes the form

$$g_{ab}dx^a dx^b = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \quad (1)$$

where $g_{AB} = r^2 h_{AB}$ with $det[h_{AB}] = \mathbf{q}(x^A)$

so that the conformal 2-metric h_{AB} has only two degrees of freedom. The determinant condition implies $h^{AB}\partial_r h_{AB} = h^{AB}\partial_u h_{AB} = 0$, where $h^{AC}h_{CB} = \delta^A_B$.

D_A denotes the covariant derivative of the metric h_{AB}. And the non-zero contravarient components of the metric(1) are

$$g^{ur} = -e^{-2\beta}; g^{rr} = \frac{V}{r}e^{-2\beta}; g^{rA} = U^A e^{-2\beta}; g^{AB} = \frac{1}{r^2}h^{AB}$$

A suitable representation of h_{AB} with two functions $\gamma(u, r, \theta, \phi)$ and $\delta(u, r, \theta, \phi)$ encoding to the $+$ and $\times$ polarization of gravitational waves is

$$h_{AB}dx^A dx^B = (e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2)\cosh 2\delta + 2\sin\theta \sinh 2\delta d\theta\phi$$

This differs from the original form of of Sachs metric by the transformation $\gamma \to (\gamma+\delta)/2$ and $\delta \to (\gamma-\delta)/2$, which gives a less natural description of gravitational waves in the weak field approximation. In the original axisymmetric Bondi metric with rotational symmetry in the ϕ-direction, $\delta = U^\phi = 0$ and $\gamma = \gamma(u, , r, \theta)$ and the metric becomes

$$g_{ab}dx^a dx^b = (-\frac{V}{r}e^{2\beta}+r^2Ue^{2\gamma})du^2-2e^{2\beta}dudr+r^2Ue^{2\gamma}dud\theta+r^2(e^{2\gamma}d\theta^2+e^{-2\gamma}\sin^2\theta d\phi^2) \tag{2}$$

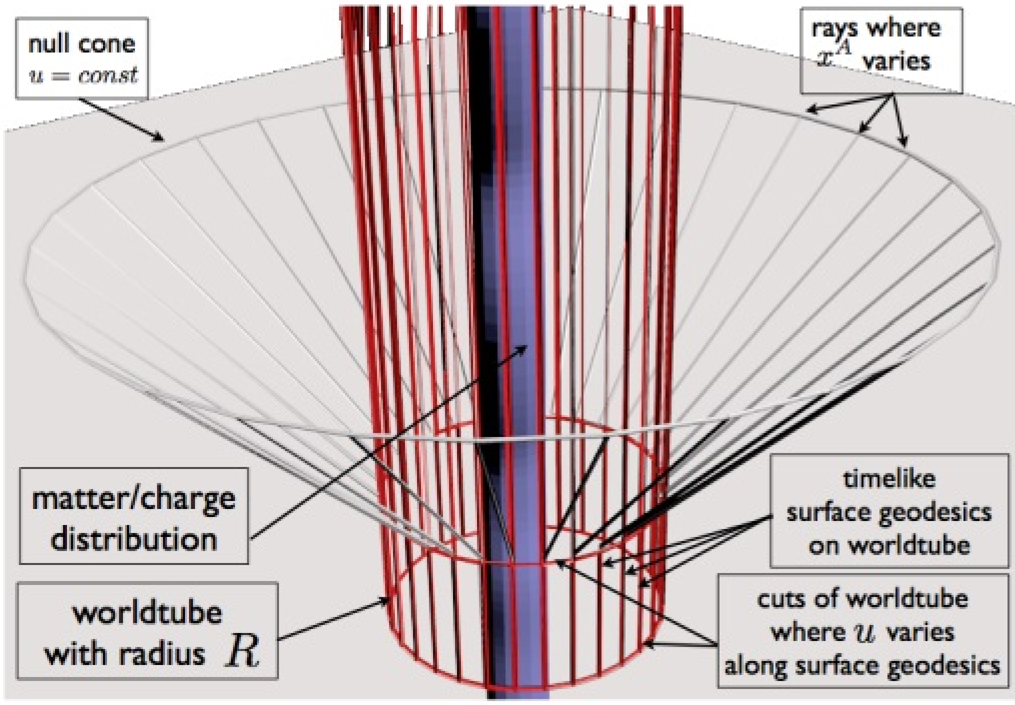
where $U = U^\theta$

1.2 The electromagnetic analogue

Lets consider the Minkowski metric in outgoing null spherical coordinates (u, r, x^A) corresponding to the flat space version of the Bondi-Sachs metric,

$$\eta_{ab}dx^a dx^b = -du^2 - dudr + r^2 q_{AB}dx^A dx^B \tag{3}$$

Assume that the charge-current sources of the electromagnetic field are enclosed by a 3-dimensional timelike worldtube Γ, with spherical cross-sections of radius $r = R$, such that the outgoing null cones N_u from the vertices $r = 0$ intersect Γ at proper time u in spacelike spheres S_u, which are coordinatized by x^A.


null cone
$u = const$
rays where
x^A varies
matter/charge
distribution
timelike
surface geodesics
on worldtube
worldtube
with radius R
cuts of worldtube
where u varies
along surface geodesics

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$
$$A_A \to A_a + \nabla_a \chi$$
$$\chi(u, r, x^A) = -\int_R^r A_r dr \tag{4}$$

leads to the null gauge $A_r = 0$, which is the analogue of the Bondi-Sachs coordinate condition $g_{rr} = g_{rA} = 0$. The remaining gauge freedom $\xi(u, x^A)$ may be used to set either

$$A_u|_\Gamma = 0 \text{ or } \lim_{r\to\infty} A_u(u, r, x^A) = 0$$

Hereafter, we implicity assume that the limit $r \to \infty$ is taken holding $u = const$ and $x^A = const$. There remains the freedom $A_B \to A_B + \nabla_B \chi(x^C)$.

The vacuum Maxwell equations $M^b := \nabla_a F^{ab} = 0$ imply the identity

$$0 = \nabla_b M^b = \partial_u M^u + \frac{1}{r^2}\partial_r(r^2 M^r) + \frac{1}{\sqrt{q}}\partial_C(\sqrt{q} M^C) \tag{5}$$

Now lets designate as the main equations the components of Maxwell's equations $M^u = 0$ and $M^A = 0$, and designate $M^r = 0$ as the supplementary condition. Then if the main equations are satisfied implies

$$0 = \partial_r(r^2 M^r) \tag{6}$$

so that the supplementary condition is satisfied everywhere if it is satisfied at some specified value of r, e.g. on Γ or at $\mathcal{I}^+$.

The main equations separate into the Hypersurface Equation

$$M^u = 0 \Rightarrow \partial_r(r^2 \partial_r Au) = \partial_r(\bar{\partial}_B A^B) \tag{7}$$

and the evolution equation

$$M^A = 0 \Rightarrow \partial_r \partial_u A_B = \frac{1}{2}\partial_r^2 A_B + \frac{1}{2r^2}\bar{\partial}^C(\bar{\partial}_C A_B - \bar{\partial}_B A_C) + \frac{1}{2}\partial_r \bar{\partial}_B A_u \tag{8}$$

where hereafter $\bar{\partial}^A$ denotes the covariant derivative with respect to the unit sphere metric q_{AB}, with $\bar{\partial}^A = q^{AB}\bar{\partial}_B$. The supplementary condition $M^r = 0$ takes the explicit form

$$\partial_u(r^2\partial_r A_u) = \partial^B(\partial_r A_B - \partial_u A_B + \bar{\partial}_B A_u) \tag{9}$$

A formal integration of the hypersurface equation yields

$$\partial_r A_u = \frac{Q(u,x^A) + \bar{\partial}_B A^B}{r^2} + \mathcal{O}(1/r^3) \tag{10}$$

where $Q(u,x^A)$ enters as a function of integration. In the null gauge with $A_r = 0$, the radial component of the electric field corresponds to $E_r = F_{ru} = \partial_r A_u$. Thus, using the divergence theorem to eliminate $\bar{\partial}_B A_B$, the total charge enclosed in a large sphere is

$$q(u) := \lim_{r\to\infty} \frac{1}{4\pi}\oint E_r r^2 \sin\theta d\phi = \frac{1}{4\pi}\oint Q(u,x^A)\sin\theta d\theta d\phi \tag{11}$$

This motivates calling $Q(u,x^A)$ the charge aspect. The integral of the supplementary condition (10) over a large sphere then gives the charge conservation law

$$\frac{dq(u)}{du} = 0 \tag{12}$$

1.3 Einstein equations and their Bondi-Sachs solution

The Einstein equations, in geometric units $G = c = 1$

$$E_{ab} := R_{ab} - \frac{1}{2}g_{ab}R^c_c - 8_{ab} = 0 \tag{13}$$

where R_{ab} is the Ricci tensor, R^c_c its trace and T_{ab} the matter stress-energy tensor. Before expressing the Einstein equations in terms of the Bondi-Sachs metric variables (1), consider the consequence of the contracted Bianchi identities. Assuming the matter satisfies the divergence-free condition $\nabla_b T^b_a = 0$, the Bianchi identities imply

$$0 = \nabla_b E^b_a = \frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}E^b_a) + \frac{1}{2}(\partial_a g^{bc})E_{bc} \tag{14}$$

In analogy to the electromagnetic case, this leads to the designation of the components of Einstein's equations, consisting of

$$E_a^u = 0; E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \tag{15}$$

as the main equations. Then if the main equations are satisfied, referring to the metric (1), $E_r^b = -e^{2\beta} E^{ab} = -e^{2\beta} g^{ba} E_a^u = 0$ and the $a = r$ component of the conversation condition (14) reduces to $(\partial_r g^{AB}) E_{AB} = -(2/r) g^{AB} E_{AB} - 0$ so that the component $g^{AB} E_{AB} = 0$ is trivially satisfied and it is assumed that the areal coordinate r is non-singular.

The retarded time u and angular components x^A of the conservation condition (14) now reduce to

$$\partial_r (r^2 e^{2\beta} E_u^r) = 0; \partial_r (r^2 e^{2\beta} E_A^r) = 0 \tag{16}$$

so that the E_u^r and E_A^r equations are satisfied everywhere if they are satisfied on a finite worldtube Γ or in the limit $r \to \infty$. Furthermore, if the null foliation consists of non-singular null cones, they are automatically satisfied due to regularity conditions at the vertex $r = 0$. These equations were called supplementary conditions by Bondi and Sachs. Evaluated in the limit $r \to \infty$ they are related to the asymptotic flux conservation laws for total energy and angular momentum. In particular, the equation $\lim_{r \to \infty} (r^2 E_u^r) = 0$ gives rise to the famous Bondi mass loss equation.

The main Einstein equations separate further into the Hypersurface equations:

$$E_a^u = 0 \tag{17}$$

and the Evolution equations:

$$E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \tag{18}$$

In terms of the metric variables (1) the hypersurface equations consist of one first order radial differential equation determining β along the null rays,

$$E_r^u = 0 \Rightarrow \partial_r \beta = \frac{r}{16} h^{AC} h^{BD} (\partial_r h_{AB}) + 2^T_{rr} \tag{19}$$

two second order radial differential equations determining U^A,

$$E^u_A = 0 \Rightarrow \partial_r[r^4 e^{-2\beta} h_{AB}(\partial_r U^B)] = 2r^4\partial_r(\frac{1}{r^2}D_A\beta) - r^2 h^{EF} D_E(\partial_r h_{AF}) + 16\pi r^2 T_{rA} \tag{20}$$

and a radial equation to determine V,

$$\begin{aligned} E^u_u = 0 \Rightarrow \quad & 2e^{2\beta}(\partial_r V) = \mathcal{R} - 2h^{AB}[D_A D_B\beta + (D_A\beta)(D_B\beta)] \\ & + \frac{e^{2\beta}}{r^2} D_A[\partial_r(r^4 U^A)] - \frac{1}{2} r^4 e^{4\beta} h_{AB}(\partial_r U^A)(\partial_r U^B) \\ & + 8\pi[h^{AB} T_{AB} r^2 T^a_a] \end{aligned} \tag{21}$$

where D_A is the covariant derivative and $\mathcal{R}$ is the Ricci scalar with respect to the conformal 2-metric h_{AB}.

In the asymptotic inertial frame, often referred to as a Bondi frame, the metric approaches the Minkowski metric at null infinity, so that

$$\lim_{r\to\infty} \beta = \lim_{r\to\infty} U^A = 0; \lim_{r\to\infty} \frac{V}{r} = 1; \lim_{r\to\infty} h_{AB} = q_{AB} \tag{22}$$

The conformal 2-metric h_{AB} on an initial null hypersurface $N_0, u = u_0$, which has the asymptotic $1/r$ expansion

$$h_{AB}(u_0, r, x^C) = q_{AB} + \frac{c_{AB}(u_0, x^E)}{r} + \frac{d_{AB}(u_0, x^E)}{r^2} + ... \tag{23}$$

with $c^{AB} := q^{AD}q^{BE}c_{DE}$ and $d^{AB} := q^{AD}q^{BE}d_{DE}$. Furthermore, the derivative of the determinant condition $det(h_{AB}) = \mathbf{q}(x^C)$ requires

$$q^{AB}c_{AB} = 0\ , \quad q^{AB}d_{AB} = \frac{1}{2}c^{AB}c_{AB}\ , \quad q^{AB}\partial_u c_{AB} = 0$$
$$q^{AB}\partial_u d_{AB} - c^{AB}\partial_u c_{AB} = 0.$$

$$c_{AB}(u, x^C) := \lim_{r\to\infty} r(h_{AB} - q_{AB}) \tag{24}$$

Lets define a function

$$M(u_0, x^A) := -\frac{1}{2}\lim_{r\to\infty}[V(u_0, r, x^C)r] \tag{25}$$

is called mass aspect.

After integrating above eqns we get

$$\beta(u_0, r, x^A) = -\frac{1}{32}\frac{c^{AB}c_{AB}}{r^2} + \mathcal{O}(r^{-3}) \tag{26}$$

$$V(u_0, r, x^A) = r - 2M(u_0, x^A) + O(r^1) \tag{27}$$

Here M is called the mass aspect since in the static, spherically symmetric case, where $h_{AB} = q_{AB}$, $\beta = U^A = 0$ and $M(u, x^A) = m$, the metric (1) reduces to the Eddington-Finkelstein metric for a Schwarzschild mass m.

The time-dependent Bondi mass $m(u)$ for an isolated system is

$$m(u) := \frac{1}{4\pi}\oint M(u, \theta, \phi)\sin\theta d\theta d\phi \tag{28}$$

Bondi mass loss formula

$$\frac{dm(u)}{du} := -\frac{1}{4\pi}\oint |N|^2 \sin\theta d\theta d\phi \tag{29}$$