Large Gauge Symmetry in Massless QED

$$Q_\varepsilon^+ = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon * F = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z\, \gamma_{z\bar z}\, F^{(2)}_{ru}\, \varepsilon$$

$$= \frac{1}{e^2}\int_{\mathcal{I}^-_+} \cancel{z}\, d^2z\, \gamma_{z\bar z}\, \varepsilon\, F^{(2)}_{rv}$$

$$\varepsilon(z,\bar z)\Big|_{\mathcal{I}^+_-} = \varepsilon(z,\bar z)\Big|_{\mathcal{I}^-_+}$$

$$\partial_u \varepsilon = 0 \qquad \partial_v \varepsilon = 0 \qquad \text{[Lets assume]}$$

$$\partial_u F^{(2)}_{ru} + D^z F^{(0)}_{uz} + D^{\bar z} F^{(0)}_{u\bar z} = \cancel{z}\; e^2 j_u^{(2)}$$

where D_z is covariant derivative on 2-sphere

$$Q_\varepsilon^+ = -\frac{1}{e^2}\int_{\mathcal{I}^+} du\, d^2z \left(\partial_z \varepsilon\, F^{(0)}_{u\bar z} + \partial_{\bar z}\varepsilon\, F^{(0)}_{uz}\right) \Big\}\, Q_S^+$$

$$+ \int_{\mathcal{I}^+} du\, d^2z\, \gamma_{z\bar z}\, j_u^{(2)}\, \varepsilon \Big\}\, Q_H^+$$

~~Massless charges only~~ $\mathcal{E}$ does not depend on u.

$$Q_s^+ \sim \int_{-\infty}^{\infty} F^{(0)}_{uz}\, e^{i\omega u}\, du \longrightarrow$$ this term creates and annihilates photons with energy ω.

$F^{(0)}_{uz}$ is transverse at $\mathcal{I}^+$.

When $\omega \to 0$ then $$\int_{-\infty}^{\infty} F^{(0)}_{uz}\, e^{i\omega u}\, du \longrightarrow \int_{-\infty}^{\infty} F^{(0)}_{uz}\, du$$

So from this term we get soft photon.

$\int_{-\infty}^{\infty} F^{(0)}_{uz}\, du \sim Q_s^+$. This is why Q_s^+ is called soft charge.

$$\int_{-\infty}^{\infty} F^{(0)}_{uz}\, du = N_z = e^2 \partial_z N$$

$$\partial_{\bar z} N_z - \partial_z N_{\bar z} = \int_{-\infty}^{\infty} du \left[\partial_{\bar z} F^{(0)}_{uz} - \partial_z F^{(0)}_{u\bar z} \right]$$

$$= \int_{-\infty}^{\infty} du\, \partial_u F^{(0)}_{z\bar z} \qquad \text{[Bianchi identity]}$$

$$= F^{(0)}_{z\bar z} \Big|_{\mathcal{I}^-_+}^{\mathcal{I}^+_+} \overset{\text{assume}}{=} 0$$

* Assumption: No asymptotic magnetic monopoles.

N can generally be complex f^n. Real according to the assumption of absence of asymptotic magnetic monopoles $N^- = 0$ here.

If we put the gauge condition $A^{(0)}_u = 0$,

$$N_z = A^{(0)}_z \Big|_{\mathcal{J}^+_+} - A^{(0)}_z \Big|_{\mathcal{J}^+_-}$$

If we want the energy to be finite then the difference between the gauge at $\mathcal{J}^+_+$ and $\mathcal{J}^+_-$ should be pure gauge.

$$\Gamma\{q^i, p_j\} = X^I \qquad \Omega = \frac{1}{2}\Omega_{IJ}\, dx^I \wedge dx^J$$

$$[A, B] = i\Omega^{IJ}\partial_I A\, \partial_J B$$

$$\Omega_{\mathcal{I}^+} = -\frac{1}{e^2}\int du\, d^2z \left(\delta F^{(0)}_{uz} \wedge \delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}} \wedge \delta A^{(0)}_{z}\right)$$

$$A^{(0)}_z = \hat{A}_z + e^2 \partial_z C \quad \text{where} \quad \partial_z C = \frac{1}{2e^2}\left[A^{(0)}_z\Big|_{\mathcal{I}^+_+} + A_z\Big|_{\mathcal{I}^+_-}\right]$$

depends on $(u, z, \bar{z})$

depends on $(u, z, \bar{z})$ and can Fourier transformed

$C(z, \bar{z})$ is real f^n as we are assuming magnetic monopole is zero at the boundary.

$$\cancel{\Omega_{\mathcal{I}^+} = -\frac{1}{2e^2}\int du\, d^2z\, \partial_u \delta \hat{A}_{\bar{z}} \wedge \delta \hat{A}_z}$$

$$\cancel{+2e^2 \int d^2z\, \partial_z}$$

$$\left[\partial_u \hat{A}_z (u, z, \bar{z}), \hat{A}_{\bar{w}} (u', w, \bar{w})\right] = -\frac{ie^2}{4}\delta(u-u')\,\delta^{(2)}(z-w)$$

$$\left[\hat{A}_z (u, z, \bar{z}), \hat{A}_{\bar{w}} (u', w, \bar{w})\right] = -\frac{ie^2}{4}\theta(u-u')\,\delta^{(2)}(z-w)$$

where $\theta(u-u') = \frac{1}{\pi i}\int \frac{d\omega}{\omega} e^{i\omega(u-u')}$

$$\left[C(z, \bar{z}), N(w, \bar{w})\right] = -\frac{i}{4\pi e^2}\log|z-w|^2$$

$$\left[Q^{+}_{\varepsilon}, A^{(0)}_{z}(u,z,\bar{z})\right] = i\partial_{z}\varepsilon(z,\bar{z}); \quad \left[Q^{+}_{\varepsilon}, N\right] = 0$$

$$\left[Q^{+}_{\varepsilon}, \hat{A}_{z}\right] = 0$$

$$\left[Q^{+}_{\varepsilon}, \phi(z,\bar{z})\right] = i\varepsilon(z,\bar{z})$$

$$\left[Q^{-}_{\varepsilon}, A^{(0)}_{z}(v,z,\bar{z})\right] = i\partial_{z}\varepsilon(z,\bar{z})$$

$$\left[j^{(2)}_{u}(u',w,\bar{w}), \phi_{k}(u,z,\bar{z})\right] = -Q_{k}\phi_{k}\gamma^{z\bar{z}}\delta^{(2)}(z-w)\,\delta(u-u')$$

$$\left[\int_{\mathcal{I}^{+}} \varepsilon * j, \phi_{k}(u,z,\bar{z})\right] = -Q_{k}\varepsilon\phi_{k} = i\delta_{\varepsilon}\phi_{k}$$

$$\left[Q^{+}_{\varepsilon}, \phi_{k}\right] = i\delta_{\varepsilon}\phi_{k}$$

Ward identity :

Quantum scattering amplitude $\rightarrow$ $\langle out | S | in \rangle$

Charge conservation $\rightarrow$ $\langle out | (Q^{+}_{\varepsilon}S - SQ^{-}_{\varepsilon}) | in \rangle = 0$

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$$\mathcal{G}_{\varepsilon}^{-}|in\rangle = -2\int d^2z\, \partial_z \varepsilon\, \partial_{\bar z} N^{-}|in\rangle + \sum_{k=1}^{m} \mathcal{G}_k^{in}\, \varepsilon(z_k^{in}, \bar z_k^{in})|in\rangle$$

$$\langle out|\mathcal{G}_{\varepsilon}^{+} = 2\int d^2z\, \partial_z \partial_{\bar z} \varepsilon\, \langle out|N$$

$$+ \sum_{k=1}^{m} \mathcal{G}_k^{out}\, \varepsilon(z_k^{out}, \bar z_k^{out})\langle out|$$

Hence, the Ward identity can be written as

$$2\int d^2z\, \partial_z \partial_{\bar z} \varepsilon\, \langle out|N(z,\bar z)S - SN^{-}(z,\bar z)|in\rangle$$

$$= \left[\sum_{k=1}^{m} \mathcal{G}_k^{in}\, \varepsilon(z_k^{in}, \bar z_k^{in}) - \sum_{k=1}^{m} \mathcal{G}_k^{out}\, \varepsilon(z_k^{out}, \bar z_k^{out})\right]$$

$$\langle out|S|in\rangle$$

Special case: $\quad \mathcal{E}(w, \overline{w}) = \dfrac{1}{z - w}$

Using $\quad \partial_{\bar{z}} \dfrac{1}{z-w} = 2\pi \delta^{(2)}(z - w)$

Now, the Ward identity can be written as

$$4\pi \langle \text{out} | \partial_z N S - S \partial_z N^- | \text{in} \rangle$$

$$= \left[\sum_{k=1}^{m} \frac{Q_k^{in}}{z - z_k^{in}} - \sum_{k=1}^{m} \frac{Q_k^{out}}{z - z_k^{out}} \right] \langle \text{out} | S | \text{in} \rangle$$

Mode Expansion:

Minkowski Space: $\quad ds^2 = -dt^2 + d\vec{x} \cdot d\vec{x}$

Near $\mathcal{I}^+$, ~~##~~ A_ν has the on-shell outgoing plane wave mode expansion

$$A_\nu(x) = c \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega} \Big[\varepsilon^{*\alpha}_\nu(\vec{q})\, a^{out}_\alpha(\vec{q})\, e^{i\hat{q}\cdot x}$$

$$+ \varepsilon^\alpha_\nu(\vec{q})\, a^{out}_\alpha(\vec{q})^\dagger\, e^{i\hat{q}\cdot x} \Big]$$

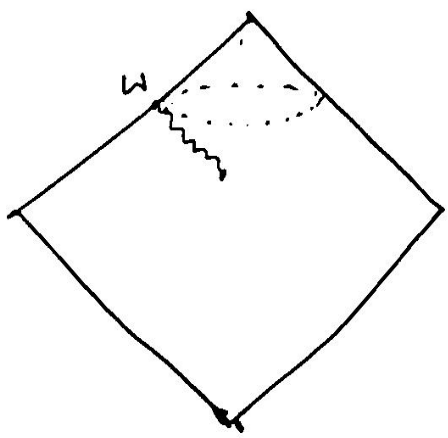
where $q^2 = 0$, the two polarization vectors satisfy a normalization condition

$$\varepsilon^\nu_\alpha\, \varepsilon^*_{\beta\nu} = \delta_{\alpha\beta}$$

$$\left[a^{out}_\alpha(\vec{q}),\, a^{out}_\beta(\vec{q})^\dagger \right] = \delta_{\alpha\beta} (2\pi)^3 (2\omega_q)\, \delta^{(3)}(\vec{q} - \vec{q}\,')$$

Minkowski Space in $(u, r, z, \bar{z})$:

$$ds^2 = -du^2 - 2du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$



The transformation to retarded from Cartesian coord. was given before. A null vector q^μ, satisfying $q^\mu q_\mu = 0$, is labeled by a point on the sphere, up to its overall magnitude. Hence, there is a natural map from null vectors q^μ to points $(z, \bar{z})$ on the sphere toward which the null vector is directed. We can write this as

$$q^\mu = \frac{\omega}{1+z\bar{z}} \left(1+z\bar{z},\ z+\bar{z},\ -i(z-\bar{z}),\ 1-z\bar{z}\right)$$

$$= \left(\omega, q^1, q^2, q^3\right)$$

As an example of this, let us suppose that z is taken to be the north pole at $z=0$. Then we find $q^\mu = \omega\,(1, 0, 0, 1)$ (i.e. a null vector pointing along the x^3-axis). We may further choose the ~~pt~~ polarization vectors orthogonal to q^μ as

$$\varepsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z})$$

$$\varepsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(z, 1, i, -z)$$

$$q_\mu \varepsilon^{\pm\mu}(\vec{q}) = 0$$

$$\varepsilon^\mu_\alpha \varepsilon^*_{\beta\mu} = \delta_{\alpha\beta}$$

Now let us consider the $\mathscr{I}^+$ field $A^{(0)}_z(u, z, \bar{z})$. By def.

$$A^{(0)}_z(u, z, \bar{z}) = \lim_{r\to\infty} A_z(u, r, z, \bar{z})$$

$$A^{(0)}_z(u, z, \bar{z}) = -\frac{i}{8\pi^2}\frac{\sqrt{2}\, e}{1+z\bar{z}}\int_0^\infty d\omega \left[a^{out}_+(\omega\hat{x})e^{-i\omega u} - a^{out}_-(\omega\hat{x})e^{i\omega u}\right];$$

$$\hat{x} = \hat{x}(z, \bar{z})$$

The Ward identity involves $\partial_z N$, so we need to determine its mode expansion. To be precise about the zero-momentum limit, define

$$\partial_z N = \frac{1}{2e^2} \lim_{\omega \to 0^+} \int_{-\infty}^{\infty} du \left(e^{i\omega u} + e^{-i\omega u} \right) F^{(0)}_{uz}$$

This definition ensures $\partial_z \partial_{\bar{z}} N$ is Hermitian.

$$\partial_z N = -\frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega \to 0^+} \Big[\omega a_+^{out}(\omega \hat{x}) + \omega a_-^{out}(\omega \hat{x})^\dagger \Big]$$

There is a familiar for $\partial_z N^-$:

$$\partial_z N^- = -\frac{1}{8\pi e} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega \to 0^+} \Big[\omega a_+^{in}(\omega \hat{x}) + \omega a_-^{in}(\omega \hat{x})^\dagger \Big]$$

The Ward identity ~~it follows~~ can be reexpressed as

$$\lim_{\omega\to 0}\left[\omega \langle out | \left(a_+^{out}(\omega\hat{x})\, S - S\, a_+^{in}(\omega\hat{x})\right) | in \rangle\right]$$

$$= \sqrt{2}\, e\,(1+z\bar{z})\left[\sum_{k=1}^{n}\frac{Q_k^{out}}{z-z_k^{out}} - \sum_{k=1}^{n}\frac{Q_k^{in}}{z-z_k^{in}}\right]$$

$$\langle out | S | in \rangle$$

Soft Theorem:

$$\lim_{\omega\to 0}\left[\omega \langle out | a_+^{out}(\vec{q})\, S | in \rangle\right]$$

$$= e \lim_{\omega\to 0}\left[\sum_{k=1}^{m}\frac{\omega Q_k^{out}\, p_k^{out}\cdot\varepsilon^{+}}{p_k^{out}\cdot q} - \sum_{k=1}^{n}\frac{\omega Q_k^{in}\, p_k^{in}\cdot\varepsilon^{+}}{p_k^{in}\cdot q}\right]$$

$$\langle out | S | in \rangle$$

$$= -\lim_{\omega\to 0}\left[\omega \langle out | S\, a_-^{in\,\dagger}(\vec{q}) | in \rangle\right]$$

where $q^\mu = (\omega, \vec{q})$ is the momentum of the soft photon, and we are taking the in-state and out-state in plane wave bases:

$$|in\rangle = |p_1^{in}, \ldots \ldots p_n^{in}\rangle$$

$$\langle out| = \langle p_1^{out}, \ldots \ldots, p_n^{out}|$$

The equality of the matrix elements involving in and out soft photons is a consequence of CPT invariance. In the limit where $\omega \to 0$ (the soft limit), the ratios appearing on the RHS are finite and depend only on the dir^{n} of $\vec{q}$, but not its magnitude. When one writes the formula without this factor of ω, there is a pole as $\omega \to 0$, which is called the "soft photon" or "Weinberg pole".

* <u>Note</u>: The ~~electromag~~ QED. case is analogous to AFS. "ε" in QED is analogous to "f" in AFS (mentioned in previous notes).

ε generates the gauge transformations in QED.

~~so~~ f " " supertranslation in AFS case.

(i.e. $u \rightarrow u + f(z, \bar{z})$). ~~Theorem there~~

From this analogy we can also get soft graviton theorem.