Celestial Holography

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Outline

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Aim

- AdS/CFT provided a concrete realization of the holographic principle: a theory of gravity in an arbitrary number of dimensions should be dual to a quantum theory in one dimension less. A concrete realization of this duality in any but asymptotically negatively curved backgrounds remains an important open problem. The goal is to review some of the recent developments addressing this problem in asymptotically flat spacetimes (AFS)

Universal Relation

- universal relation obeyed by scattering amplitudes with massless particles (tree-level scattering)
- In gravity and gauge theory, the scattering of high-energy charged particles is accompanied by radiation which can be described as collection of quanta (photons, gravitons) of different energies

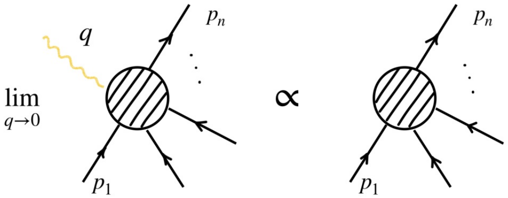


Figure 1: The soft limit relates an amplitude with a low-energy massless particle to the same amplitude without the massless particle.

Universal Reationship continued

$$\lim_{\omega \to 0} \mathcal{A}^{\pm}_{n+1}(q) = \left[S^{(0)\pm}_n + S^{(1)\pm}_n + \mathcal{O}(\omega) \right] \mathcal{A}_n.$$

$A_{n+1} =$ scattering amplitude of n generic particles of four-momenta $p_1, ..., p_n$ and one massless particle of four-momentum $q = (\omega, \vec{q})$ and positive or negative helicity.

$A_n =$ the same scattering amplitude in the absence of the massless particle.

Leading and Sub-leading term

In gravity,

$$S_n^{(0)\pm} = \frac{\kappa}{2}\sum_{k=1}^{n}\frac{\left(p_k\cdot\varepsilon^{\pm}(q)\right)^2}{p_k\cdot q}\ , \quad S_n^{(1)\pm} = -\frac{i\kappa}{2}\sum_{k=1}^{n}\frac{\varepsilon^{\pm}(q)\cdot p_k}{p_k\cdot q}q\cdot\mathcal{J}_k\cdot\varepsilon^{\pm}(q)\ , \quad \kappa=\sqrt{32\pi G}$$

In QED,

$$S_n^{(0)\pm} = \sum_{k=1}^{n}Q_k\frac{p_k\cdot\varepsilon^{\pm}(q)}{p_k\cdot q}\ , \quad S_n^{(1)\pm} = -i\sum_{k=1}^{n}Q_k\frac{q\cdot\mathcal{J}_k\cdot\varepsilon^{\pm}(q)}{p_k\cdot q}$$

$G =$ Newton's universal gravitational constant
$Q_k =$ the charge of the k-th particle
$J_k =$ total angular momentum of the k-th particle
For our convinience we set $8\pi G = 1$. Therefore, $\kappa = \sqrt{32\pi G} = 2$.

Soft theorem continued

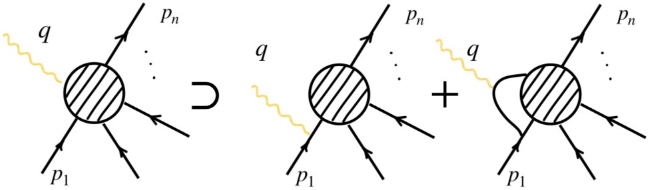


Figure 2: In the soft limit, the amplitude will include contributions from Feynman diagrams where the soft particle attaches to external and internal lines. Diagrams where the soft particle attaches to an internal line are subleading in the soft limit.

Soft theorem continued

The soft theorem captures the behavior of the scattering amplitude in an expansion around $\omega = 0$. The leading term has a simple pole at $\omega = 0$ which can be understood by considering the Feynman diagrams contributing to the scattering of n + 1 particles, as shown in figure 2. In particular, as $\lim \omega \to 0$, the leading order contribution comes from diagrams where the massless particle attaches to an external line. In this limit, an internal propagator goes on-shell and the amplitude develops a pole in q

$$\lim_{\omega \to 0} \mathcal{A}_{n+1}(q) = \left[\sum_{k=1}^{n} -i \frac{V_k(\varepsilon, p_k)}{2p_k \cdot q} + \mathcal{O}(\omega^0) \right] \mathcal{A}_n$$

Minkowski metric

The Minkowski metric in (3+1) spacetime takes the form

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit two-sphere.

Introducing new coordinates

Lets, introduce retarded and advanced coordinates

$$u = t - r; v = t + r$$

and coordinates$(z, \bar{z})$ related to the angular coordinates (θ, ϕ) by a stereographic projection

$$z = \cot\frac{\theta}{2}e^{\iota\varphi}; \bar{z} = \cot\frac{\theta}{2}e^{-\iota\varphi}$$

New form of the metric

In retarded coordinates $(u, r, z, \bar{z})$ the metric becomes

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$

In advanced coordinates $(v, r, z, \bar{z})$ the metric becomes

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$

where $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$

Introducing new coordinates

Lets introduce (T, R)

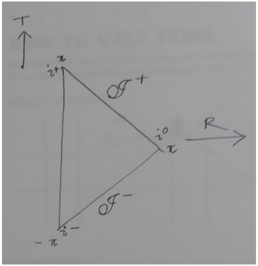
$$u = \tan U; v = \tan V; T = U + V; R = V - U$$

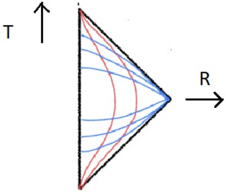
Then the metric takes the form

$$ds^2 = \Omega^2(T, R)(-dT^2 + dR^2 + 2\sin^2 R\gamma_{z\bar{z}}dzd\bar{z})$$

where $\Omega^{-2} = 4\cos^2\frac{T-R}{2}\cos^2\frac{T+R}{2}$
In the original coordinates, Minkowski space is covered by $r > 0, -\infty < u < v < \infty$, therefore the ranges of the new coordinates are $\frac{\pi}{2} < U < V < \frac{\pi}{2} and 0 < R < \pi$.

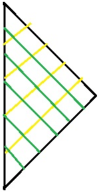
Penrose diagram of Minkowski space





const. r

const. t



(r-r_0)/t = const.

Solution space

Asymptotically flat metrics solving Einstein's equation are of the form

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

where

$$g_{AB} = r^2\bar{\gamma}_{AB} + rC_{AB} + D_{AB} + \frac{1}{4}\bar{\gamma}_{AB}C^C_D C^D_C + o(r^{-\epsilon})$$

The background metric is

$$\bar{\gamma}_{AB}dx^A dx^B = e^{2\varphi}(d\theta^2 + \sin\theta d\phi^2) = e^{2\tilde{\varphi}}d\zeta d\bar{\zeta},$$

$$\zeta = \cot\frac{\theta}{2}e^{i\phi}, \quad \tilde{\varphi} = \varphi - \varphi_0, \quad \varphi_0 = \ln P, \quad P = \frac{1}{2}(1+\zeta\bar{\zeta}).$$

We assume for simplicity that $\varphi, \tilde{\varphi}$ do not depend on u, $\varphi = \varphi(x^A)$. Indices on C_{AB}, D_{AB} are raised with the inverse of $\bar{\gamma}_{AB}$ and $C^A_A = 0 = D^A_A$. In addition $\partial_u D_{AB} = 0$ and the news tensor is $N_{AB} = \partial_u C_{AB}$.

$$\beta = -\frac{1}{32}r^{-2}C^A_BC^B_A - \frac{1}{12}r^{-3}C^A_BD^B_A + o(r^{-3-\epsilon})$$

$$g_{uA} = \frac{1}{2}\bar{D}_BC^B_A + \frac{2}{3}r^{-1}\Big[(\ln r + \frac{1}{3})\bar{D}_BD^B_A + \frac{1}{4}C_{AB}\bar{D}_CC^{CB} + N_A\Big] + o(r^{-1-\epsilon})$$

where $\bar{D}_A$ is the covariant derivative associated to $\bar{\gamma}_{AB}$ and $N_A(u, x^A)$ is the angular momentum aspect.

$$\frac{V}{r} = -\frac{1}{2}\bar{R} + r^{-1}2M + o(r^{-1-\epsilon})$$

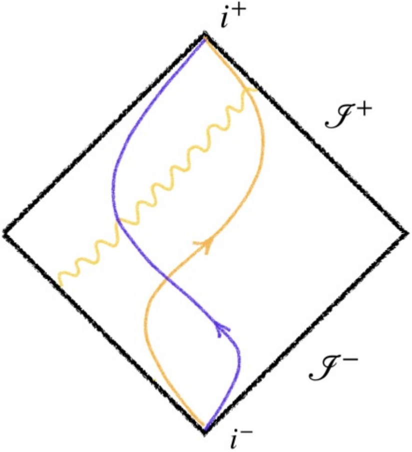
where $\bar{R}$ is the scalar curvature of $\bar{D}_A$, $\bar{R} = 2e^{-2\varphi - 2\bar{\Delta}\varphi}$ with $\bar{\Delta}$ the Laplacian for $\bar{\gamma}_{AB}$ and $M(u, x^A)$ is the mass aspect

Solution space continued

$$\partial_u M = -\frac{1}{8} N^A_B N^B_A + \frac{1}{8} \bar{\Delta} \bar{R} + \frac{1}{4} \bar{D}_A \bar{D}_C N^{CA}$$

$$\begin{aligned}\partial_u N_A = \partial_A M + \frac{1}{4} C^B_A \partial_B \bar{R} + \frac{1}{16} \partial_A \left[N^B_C C^C_B \right] - \frac{1}{4} \bar{D}_A C^C_B N^B_C \\ - \frac{1}{4} \bar{D}_B \left[C^B_C N^C_A - N^B_C C^C_A \right] - \frac{1}{4} \bar{D}_B \left[\bar{D}^B \bar{D}_C C^C_A - \bar{D}_A \bar{D}_C C^{BC} \right]\end{aligned}$$

Penrose diagram of Minkowski space



AFS

Asymptotically flat spacetimes have the same causal structure as Minkowski space at infinity. An asymptotically flat spacetime admits an expansion in powers of r^{-1} around the Minkowski metric near $\mathcal{J}^+$

$$
\begin{aligned}
ds^2 =& - du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \\
& - \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{uz}dudz + 2g_{u\bar{z}}dud\bar{z} + \cdots
\end{aligned}
$$

We are working in Bondi gauge defined by $\partial_r det(\frac{g_{AB}}{r^2}) = 0$ and $g_{rr} = g_{rA} = 0$ where A, B run over the transverse indices $z, \bar{z}$.

AFS continued

Solving the Einstein equation order by order in a large-r expansion one finds

$$g_{uz} = \frac{1}{2}D^z C_{zz} + \frac{1}{6r}C_{zz}D_z C^{zz} + \frac{2}{3r}N_z + \mathcal{O}(r^{-2})$$

where D_z is the covariant derivative associated with $\gamma_{z\bar{z}}$. Here m_B and N_z are the Bondi mass aspect and angular momentum aspect respectively. The outgoing news tensor is $N_{zz} = \partial_u C_{zz}$.

- These are all functions of $(u, z, \bar{z})$
- m_B, C_{zz}, N_z are not all independent

AFS continued

The uu constraint gives

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T_{uu}^{M(2)} - \frac{1}{4} N_{zz} N^{zz}$$

The uz constraint gives

$$\begin{aligned}\partial_u N_z =& \frac{1}{4} D_z \left(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}} \right) - T_{uz}^{M(2)} + \partial_z m_B + \frac{1}{16} D_z \partial_u \left(C_{zz} C^{zz} \right) \\ & - \frac{1}{4} \left(N^{zz} D_z C_{zz} + N_{zz} D_z C^{zz} \right) - \frac{1}{4} D_z \left(C^{zz} N_{zz} - N^{zz} C_{zz} \right).\end{aligned}$$

where we define $T_{\mu\nu}^{M(2)} = \lim_{r \to \infty} r^2 T_{\mu\nu}^{M}$

Reference

- G. Barnich and C. Troessaert, BMS charge algebra," JHEP 12 (2011) 105, arXiv:1106.0213 [hep-th].

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