## Abstract

This is a review of celestial holography. We see that, semiclassically, the subleading soft graviton theorem gives an enhancement of the Lorentz symmetry of scattering in 4D asymptotically flat gravity to Virasoro. As we can compute S-matrices on a basis of boost eigenstates, we can construct celestial amplitudes. We come to a conclusion that celestial symmetries and the constraints impose on celestial scattering.

**Keywords:** AFS, BMS

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# Contents

# Preface

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# Chapter 1

# Introduction

This is a review of celestial holography[A1].

# Chapter 2

# Bondi Sachs Formalism

## 2.1 Bondi sachs Metric

Retarded time: $u = t - r$ ; Advanced time: $v = t + r$

The Bondi-Sachs coordinates $x^a = (u, r, x^A)$ are based on a family of outgoing null hypersurfaces $u = constant$. The hypersurfaces $x^0 = u = constant$ are null, i.e. the normal co-vector $k_a = -\nabla_a u$ satisfies $g^{ab}(\nabla_a u)(\nabla_b u) = 0$, so that $g^{uA} = 0$. The coordinate $x^1 = r$, which varies along the null rays, is chosen to be an areal coordinate such that $det[g_{AB}] = r^4 \mathbf{q}$, where $\mathbf{q}(x^A)$ is the determinant of the unit sphere metric $q_{AB}$ associated with the angular coordinates $x^A$, e.g. $q_{AB} = diag(1, \sin^2\theta)$ for standard spherical coordinates $x_A = (\theta, \phi)$.

$$g_{rr} = g_{rA} = 0$$

Bondi-Sachs metric takes the form

$$g_{ab}dx^a dx^b = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{2.1}$$

where $g_{AB} = r^2 h_{AB}$ with $det[h_{AB}] = \mathbf{q}(x^A)$

so that the conformal 2-metric $h_{AB}$ has only two degrees of freedom. The determinant condition implies $h^{AB}\partial_r h_{AB} = h^{AB}\partial_u h_{AB} = 0$, where $h^{AC}h_{CB} = \delta^A_B$.

$D_A$ denotes the covariant derivative of the metric $h_{AB}$. And the non-zero contravarient components of the metric(1) are

$$g^{ur} = -e^{-2\beta}; g^{rr} = \frac{V}{r}e^{-2\beta}; g^{rA} = U^A e^{-2\beta}; g^{AB} = \frac{1}{r^2}h^{AB}$$

A suitable representation of $h_{AB}$ with two functions $\gamma(u, r, \theta, \phi)$ and $\delta(u, r, \theta, \phi)$ encoding to the $+$ and $\times$ polarization of gravitational waves is

$$h_{AB}dx^A dx^B = (e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2)\cosh 2\delta + 2\sin\theta \sinh 2\delta d\theta\phi$$

This differs from the original form of of Sachs metric by the transformation $\gamma \to (\gamma + \delta)/2$ and $\delta \to (\gamma - \delta)/2$, which gives a less natural description of gravitational waves in the weak field approximation. In the original axisymmetric Bondi metric with rotational symmetry in the $\phi$-direction, $\delta = U^\phi = 0$ and $\gamma = \gamma(u, , r, \theta)$ and the metric becomes

$$g_{ab}dx^a dx^b = (-\frac{V}{r}e^{2\beta} + r^2Ue^{2\gamma})du^2 - 2e^{2\beta}dudr + r^2Ue^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2) \tag{2.2}$$

where $U = U^\theta$

## 2.2 The electromagnetic analogue

Lets consider the Minkowski metric in outgoing null spherical coordinates $(u, r, x^A)$ corresponding to the flat space version of the Bondi-Sachs metric,

$$\eta_{ab}dx^a dx^b = -du^2 - dudr + r^2 q_{AB}dx^A dx^B \tag{2.3}$$

Assume that the charge-current sources of the electromagnetic field are enclosed by a 3-dimensional timelike worldtube $\Gamma$, with spherical cross-sections of radius $r = R$, such that the outgoing null cones $N_u$ from the vertices $r = 0$ intersect $\Gamma$ at proper time $u$ in spacelike spheres $S_u$, which are coordinatized by $x^A$.

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

$$A_A \to A_a + \nabla_a \chi$$

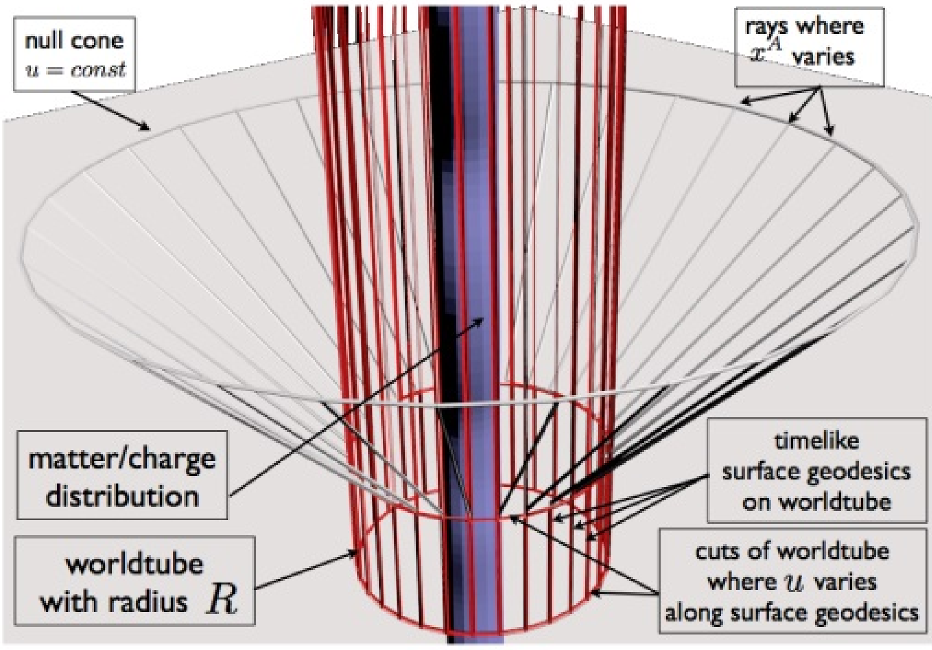


**Figure 2.1:** The charge current is bound inside the cylindrical worldtube.

$$\chi(u,r,x^A) = -\int_R^r A_r dr \tag{2.4}$$

leads to the null gauge $A_r = 0$, which is the analogue of the Bondi-Sachs coordinate condition $g_{rr} = g_{rA} = 0$. The remaining gauge freedom $\xi(u,x^A)$ may be used to set either

$$A_u|_\Gamma = 0 \text{ or } \lim_{r\to\infty} A_u(u,r,x^A) = 0$$

Hereafter, we implicity assume that the limit $r \to \infty$ is taken holding $u = const$ and $x^A = const$. There remains the freedom $A_B \to A_B + \nabla_B \chi(x^C)$.

The vacuum Maxwell equations $M^b := \nabla_a F^{ab} = 0$ imply the identity

$$0 = \nabla_b M^b = \partial_u M^u + \frac{1}{r^2}\partial_r(r^2 M^r) + \frac{1}{\sqrt{q}}\partial_C(\sqrt{q}M^C) \tag{2.5}$$

Now lets designate as the main equations the components of Maxwell's equations $M^u = 0$ and $M^A = 0$, and designate $M^r = 0$ as the supplementary condition. Then if the main equations are satisfied implies

$$0 = \partial_r(r^2 M^r) \tag{2.6}$$

so that the supplementary condition is satisfied everywhere if it is satisfied at some specified value of r, e.g. on $\Gamma$ or at $\mathcal{J}^+$.

The main equations separate into the Hypersurface Equation

$$M^u = 0 \Rightarrow \partial_r(r^2\partial_r Au) = \partial_r(\bar\partial_B A^B) \tag{2.7}$$

and the evolution equation

$$M^A = 0 \Rightarrow \partial_r\partial_u A_B = \frac{1}{2}\partial_r^2 A_B + \frac{1}{2r^2}\bar\partial^C(\bar\partial_C A_B - \bar\partial_B A_C) + \frac{1}{2}\partial_r\bar\partial_B A_u \tag{2.8}$$

where hereafter $\bar\partial^A$ denotes the covariant derivative with respect to the unit sphere metric $q_{AB}$, with $\bar\partial^A = q^{AB}\bar\partial_B$. The supplementary condition $M^r = 0$ takes the explicit form

$$\partial_u(r^2\partial_r A_u) = \partial^B(\partial_r A_B - \partial_u A_B + \bar\partial_B A_u) \tag{2.9}$$

A formal integration of the hypersurface equation yields

$$\partial_r A_u = \frac{Q(u,x^A) + \bar\partial_B A^B}{r^2} + \mathcal{O}(1/r^3) \tag{2.10}$$

where $Q(u, x^A)$ enters as a function of integration. In the null gauge with $A_r = 0$, the radial component of the electric field corresponds to $E_r = F_{ru} = \partial_r A_u$. Thus, using the divergence theorem to eliminate $\bar{\partial}_B A_B$, the total charge enclosed in a large sphere is

$$q(u) := \lim_{r\to\infty} \frac{1}{4\pi} \oint E_r r^2 \sin\theta d\phi = \frac{1}{4\pi} \oint Q(u, x^A) \sin\theta d\theta d\phi \tag{2.11}$$

This motivates calling $Q(u, x^A)$ the charge aspect. The integral of the supplementary condition (10) over a large sphere then gives the charge conservation law

$$\frac{dq(u)}{du} = 0 \tag{2.12}$$

### 2.2.1 Einstein equations and their Bondi-Sachs solution

The Einstein equations, in geometric units $G = c = 1$

$$E_{ab} := R_{ab} - \frac{1}{2} g_{ab} R^c_c - 8_{ab} = 0 \tag{2.13}$$

where $R_{ab}$ is the Ricci tensor, $R^c_c$ its trace and $T_{ab}$ the matter stress-energy tensor. Before expressing the Einstein equations in terms of the Bondi-Sachs metric variables (1), consider the consequence of the contracted Bianchi identities. Assuming the matter satisfies the divergence-free condition $\nabla_b T^b_a = 0$, the Bianchi identities imply

$$0 = \nabla_b E^b_a = \frac{1}{\sqrt{-g}} \partial_b(\sqrt{-g} E^b_a) + \frac{1}{2}(\partial_a g^{bc}) E_{bc} \tag{2.14}$$

In analogy to the electromagnetic case, this leads to the designation of the components of Einstein's equations, consisting of

$$E^u_a = 0; E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \tag{2.15}$$

as the main equations. Then if the main equations are satisfied, referring to the metric (1), $E^b_r = -e^{2\beta} E^{ab} = -e^{2\beta} g^{ba} E^u_a = 0$ and the $a = r$ component of the conversation condition (14) reduces to $(\partial_r g^{AB}) E_{AB} = -(2/r) g^{AB} E_{AB} - 0$ so that the component $g^{AB} E_{AB} = 0$ is trivially satisfied and it is assumed that the areal coordinate $r$ is non-singular.

The retarded time $u$ and angular components $x^A$ of the conservation condition (14) now reduce to

$$\partial_r(r^2 e^{2\beta} E^r_u) = 0; \partial_r(r^2 e^{2\beta} E^r_A) = 0 \tag{2.16}$$

so that the $E^r_u$ and $E^r_A$ equations are satisfied everywhere if they are satisfied on a finite worldtube $\Gamma$ or in the limit $r \to \infty$. Furthermore, if the null foliation consists of non-singular null cones, they are automatically satisfied due to regularity conditions at the vertex $r = 0$. These equations were called supplementary conditions by Bondi and Sachs. Evaluated in the limit $r \to \infty$ they are related to the asymptotic flux conservation laws for total energy and angular momentum. In particular, the equation $\lim_{r\to\infty}(r^2 E^r_u) = 0$ gives rise to the famous Bondi mass loss equation.

The main Einstein equations separate further into the Hypersurface equations:

$$E^u_a = 0 \tag{2.17}$$

and the Evolution equations:

$$E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0 \tag{2.18}$$

In terms of the metric variables (1) the hypersurface equations consist of one first order radial differential equation determining $\beta$ along the null rays,

$$E^u_r = 0 \Rightarrow \partial_r \beta = \frac{r}{16} h^{AC} h^{BD} (\partial_r h_{AB}) + 2^T_{rr} \tag{2.19}$$

two second order radial differential equations determining $U^A$,

$$E^u_A = 0 \Rightarrow \partial_r[r^4 e^{-2\beta} h_{AB}(\partial_r U^B)] = 2r^4 \partial_r(\frac{1}{r^2} D_A \beta) - r^2 h^{EF} D_E(\partial_r h_{AF}) + 16\pi r^2 T_{rA} \tag{2.20}$$

and a radial equation to determine $V$,

$$\begin{aligned} E^u_u = 0 \qquad \Rightarrow 2e^{-2\beta}(\partial_r V) = \quad & \mathcal{R} - 2h^{AB}[D_A D_B \beta + (D_A \beta)(D_B \beta)] \\ & + \frac{e^{-2\beta}}{r^2} D_A[\partial_r(r^4 U^A)] - \frac{1}{2} r^4 e^{-4\beta} h_{AB} (\partial_r U^A)(\partial_r U^B) \\ & + 8\pi[h^{AB} T_{AB} - r^2 T^a_a] \end{aligned} \tag{2.21}$$

where $D_A$ is the covariant derivative and $\mathcal{R}$ is the Ricci scalar with respect to the conformal 2-metric $h_{AB}$.

In the asymptotic inertial frame, often referred to as a Bondi frame, the metric approaches the Minkowski metric at null infinity, so that

$$\lim_{r\to\infty} \beta = \lim_{r\to\infty} U^A = 0; \lim_{r\to\infty} \frac{V}{r} = 1; \lim_{r\to\infty} h_{AB} = q_{AB} \tag{2.22}$$

The conformal 2-metric $h_{AB}$ on an initial null hypersurface $N_0, u = u_0$, which has the asymptotic $1/r$ expansion

$$h_{AB}(u_0, r, x^C) = q_{AB} + \frac{c_{AB}(u_0, x^E)}{r} + \frac{d_{AB}(u_0, x^E)}{r^2} + ....... \tag{2.23}$$

with $c^{AB} := q^{AD}q^{BE}c_{DE}$ and $d^{AB} := q^{AD}q^{BE}d_{DE}$. Furthermore, the derivative of the determinant condition $det(h_{AB}) = \mathbf{q}(x^C)$ requires

$$c_{AB}(u, x^C) := \lim_{r\to\infty} r(h_{AB} - q_{AB}) \tag{2.24}$$

Lets define a function

$$M(u_0, x^A) := -\frac{1}{2} \lim_{r\to\infty} [V(u_0, r, x^C) - r] \tag{2.25}$$

is called mass aspect.

After integrating above eqns we get

$$\beta(u_0, r, x^A) = -\frac{1}{32}\frac{c^{AB}c_{AB}}{r^2} + \mathcal{O}(r^{-3}) \tag{2.26}$$

$$V(u_0, r, x^A) = r - 2M(u_0, x^A) + O(r^{-1}) \tag{2.27}$$

Here $M$ is called the mass aspect since in the static, spherically symmetric case, where $h_{AB} = q_{AB}$, $\beta = U^A = 0$ and $M(u, x^A) = m$, the metric (1) reduces to the Eddington-Finkelstein metric for a Schwarzschild mass $m$.

The time-dependent Bondi mass $m(u)$ for an isolated system is

$$m(u) := \frac{1}{4\pi}\oint M(u, \theta, \phi) \sin\theta d\theta d\phi \tag{2.28}$$

Bondi mass loss formula

$$\frac{dm(u)}{du} := -\frac{1}{4\pi}\oint |N|^2 \sin\theta d\theta d\phi \tag{2.29}$$

## 2.3 Asymptotically flat

In the previous sections, flat Minkowski space in retarded coordinates near $\mathcal{I}^+$ was described by the metric

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{2.30}$$

We would now like to study gravitational theories in which the metric is asymptotic to, but not exactly equal to, the flat metric. We will work in Bondi coordinates $(u, r, z, \bar{z})$, and we abbreviate $\Theta^A = (z, \bar{z})$. In this gauge, the most general four dimensional metric takes the form

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}(d\Theta^A + \frac{1}{2}U^A du)(d\Theta^B + \frac{1}{2}U^B du) \tag{2.31}$$

where

$$\partial_r \det(\frac{g_{AB}}{r^2}) = 0 \tag{2.32}$$

Together with eq(2.32), we also set $g_{rr} = g_{rA} = 0$. This is the Bondi gauge.

Till now we have not imposed any condition for asymptotic flatness. To introduce asymptotic flatness for large $r$ where $(u, z, \bar{z})$ is fixed, we introduce some fall-off conditions.

For the natural choice made by Bondi, van der Burg, Metzner, and Sachs (BMS) 1 2, the large-r structure of the metric is constrained to be

$$\begin{aligned} ds^2 = \quad & -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ & + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC\bar{z}\bar{z}d\bar{z}^2 + D_z C_{zz}dudz + D_{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} \\ & + \frac{1}{r}(\frac{4}{3}(N_z + u\partial_r m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz}))dudz + c.c. + ... \end{aligned} \tag{2.33}$$

Equation eq(2.33) corresponds to the large-$r$ falloffs:

$$\begin{aligned} &g_{uu} = -1 + \mathcal{O}(\frac{1}{r}), \quad g_{ur} = -1 + \mathcal{O}(\frac{1}{r^2}), \quad g_{uz} = \mathcal{O}(1), \\ &g_{zz} = \mathcal{O}(r), \quad g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1), \quad g_{rr} = g_{rz} = 0 \end{aligned} \tag{2.34}$$

Here $m_B$ and $N_z$ are respectively known as Bondi mass aspect and Bondi angular momentum aspect.

The Bondi news tensor is

$$N_{zz} = \partial_u C_{zz} \tag{2.35}$$

It is the gravitational analog of the field strength $F_{uz} = \partial_u A_z$ in electrodynamics. Its square is proportional to the energy flux across $\mathcal{I}^+$.

We need further boundary conditions near boundaries $\mathcal{I}^+_\pm$ of $\mathcal{I}^+$. We can see that if $N_{zz}$ does not fall fast enough then the solution will have infinite energy.

## 2.4 Supertranslation

### 2.4.1 BMS analysis

The calculation for BMS group can be found in 1,2. For simplicity, we restrict consideration to diffeomorphisms that have the large-$r$ falloffs:

$$\xi^u, \xi^r \sim \mathcal{O}(1), \qquad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}(\frac{1}{r}) \tag{2.36}$$

Eq(2.36) is equivalent to the statement that the vector field is $\mathcal{O}(1)$ at large $r$ in an orthonormal frame, thereby eliminating boosts and rotations that grow with $r$ at infinity. The Lie derivative of the metric components at large $r$ are then

$$\begin{aligned}
\mathcal{L}_\xi g_{ur} &= -\partial_u \xi^u + \mathcal{O}(\frac{1}{r}), \\
\mathcal{L}_\xi g_{zr} &= r^2 \gamma_{z\bar{z}} \partial_r \xi^{\bar{z}} - \partial_z \xi^u + \mathcal{O}(\frac{1}{r}), \\
\mathcal{L}_{-} &= r\gamma_{z\bar{z}}[2\xi^r + rD_z \xi^z + rD_{\bar{z}} \xi^{\bar{z}}] + \mathcal{O}(1), \\
\mathcal{L}_\xi g_{uu} &= -2\partial_u \xi^u - 2\partial_u \xi^r + \mathcal{O}(\frac{1}{r})
\end{aligned} \tag{2.37}$$

Then requiring eq(2.31) and eq(2.34) we get that at large $r$

$$\xi = f\partial_u - \frac{1}{r}(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r + ... \tag{2.38}$$

where $f(z, \bar{z})$ is any function of $(z, \bar{z})$.

The action of supertranslations on the $\mathcal{I}^+$ data $m_B, \& C_{zz}$, and $N_{zz}$ can be determined by computing the Lie derivative of the appropriate component of the metric and then extracting the appropriate coefficient in the large-$r$ expansion:

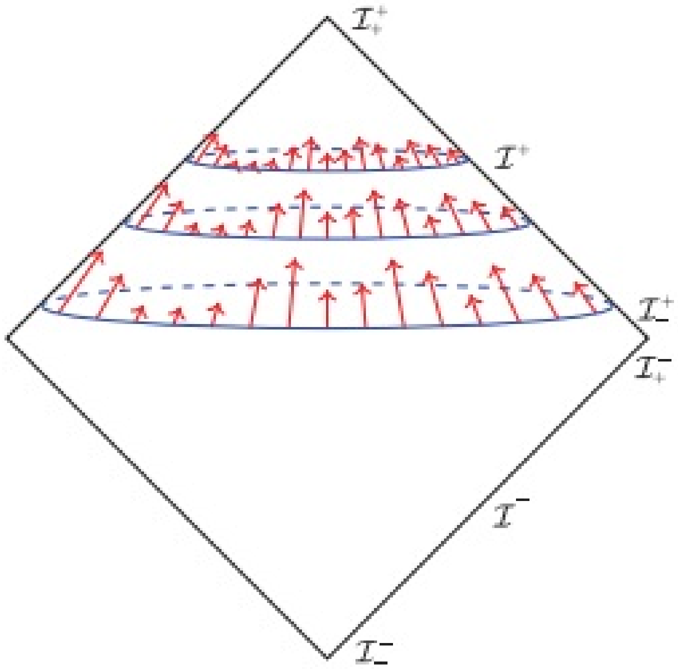


**Figure 2.2:** Under a supertranslation, retarded time u is shifted independently at every angle on $\mathcal{J}$[B1].

$$\begin{aligned}
\mathcal{L}_f N_{zz} &= f\partial_u N_{zz}, \\
\mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}[N^{zz}D_z^2 f + 2D_z N^{zz} D_z f + c.c.], \\
\mathcal{L}_f C_{zz} &= f\partial_u C_{zz} - 2D_z^2 f
\end{aligned} \tag{2.39}$$

Now we utilize the equations of motion. Lets assume that the geometry is governed by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T^M_{\mu\nu} \tag{2.40}$$

Since we are here interested in the structure of null infinity, we assume that $T^M_{\mu\nu}$ is a matter stress tensor corresponding to massless modes. Plugging in the explicit form of the metric eq(2.21) and expanding in large $r$, we find that the leading $uu$ component of Einstein's equations is

$$\partial_u m_B = \frac{1}{4}[D_z^2 N_{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}] - T_{uu} \tag{2.41}$$

where

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^M_{uu}] \tag{2.42}$$

There is analogous relationships at $\mathcal{I}^-$. The metric in advanced Bondi coordinates $(v, r, z, z^-)$ has the expansion

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}dv^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}2 + \ldots \tag{2.43}$$

where $m_B$ and $C_{zz}$ depend on $(v, z, \bar{z})$.

$$\mathcal{L}_f N_{zz} = f\partial_v N_{zz}, \qquad \mathcal{L}_f C_{zz} = f\partial_v C_{zz} - 2D_z^2 f \tag{2.44}$$

$$\begin{aligned}
\partial_v m_B &= \frac{1}{4}[D_z^2 N_{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}] + T_{vv}, \\
T_{vv} &= \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^M_{vv}]
\end{aligned} \tag{2.45}$$

### 2.4.2 Conserved charges

Now, we can get infinite number of supertranslation charges.

$$
\begin{aligned}
Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^\pm} d^2z\gamma_{z\bar{z}} f m_B, \\
Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^\pm} d^2z\gamma_{z\bar{z}} f m_B
\end{aligned} \tag{2.46}
$$

The matching condition implies the conservation law

$$
Q_f^+ = Q_f^- \tag{2.47}
$$

Integrating by parts, using the constraint equation, and assuming $m_B$ decays to zero in the far future, we can write

$$
\begin{aligned}
Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}} f[T_{uu} - \frac{1}{4}(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}})], \\
Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-} dvd^2z\gamma_{z\bar{z}} f[T_{vv} - \frac{1}{4}(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}})]
\end{aligned} \tag{2.48}
$$

Upon taking $f(z,\bar{z}) = \delta^2(z-w)$, eq(2.47) implies

$$
\frac{1}{4\pi G}\int_{\mathcal{I}^+} du\gamma_{z\bar{z}}[T_{uu} - \frac{1}{4}(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}})] = \frac{1}{4\pi G}\int_{\mathcal{I}^-} dv\gamma_{z\bar{z}}[T_{vv} - \frac{1}{4}(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}})] \tag{2.49}
$$

## 2.5 Superrotation

The first nontrivial corrections to the flat metric near $\mathcal{I}$ is due to $C_{zz}$ and $m_B$. The next order correction is due to $N_z$. We see in this section how matching condition gives rise to superrotation charges.

### 2.5.1 Conserved charges

From $uz$ component of Einstein's equations we get

$$\partial_u N_z = \frac{1}{4}\partial_z(D_z^2 C^{zz} - D_{\bar{z}} z^2 C^{\bar{z}\bar{z}} - u\partial_u\partial_z m_B - T_{uz} \tag{2.50}$$

where

$$T_{uz} \equiv 8\pi G \lim_{rto\infty}[r^2 T^M_{uz}] - \frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_z N^{zz} \tag{2.51}$$

Now from similar analysis as the superrotation case we can have superrotation charges for arbitrary $Y^z$ on the sphere.

$$Q_Y^+ = \frac{1}{8\pi G}\int_{\mathcal{I}^\pm} d^2z(Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = \frac{1}{8\pi G}\int_{\mathcal{I}^-_+} d^2z(Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = Q_Y^- \tag{2.52}$$

### 2.5.2 Symmetries

Previously we required that the components of the vector $\xi$ be bounded in orthonormal frame. But when this restriction is lifted, supertranslations generalize to superrotation.

Lorentz killing vectors are of the form

$$\xi_Y = (1+\frac{u}{2r})Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^z\partial_{\bar{z}} - \frac{1}{2}(u+r)D_z Y^z\partial_r + \frac{u}{2}D_z Y^z\partial_u + c.c... \tag{2.53}$$

where $(Y^z, Y^{\bar{z}})$ is a two dimensional vector field on $\mathcal{CS}^2$. At null infinity $\xi_Y$ simplifies to

$$\xi_Y|_{\mathcal{I}+} = Y^z\partial_z + \frac{u}{2}D_z Y^z\partial_u + c.c. \tag{2.54}$$

We can calculate

$$\mathcal{L}_Y g_{ur} = \mathcal{O}(\frac{1}{r^2}) \tag{2.55}$$

$$\mathcal{L}_Y g_{zr} = \mathcal{O}(\frac{1}{r}) \tag{2.56}$$

$$\mathcal{L}_Y g_{z\bar{z}} = \mathcal{O}(r) \tag{2.57}$$

$$\mathcal{L}_Y g_{uu} = \mathcal{O}(\frac{1}{r}) \tag{2.58}$$

$$\mathcal{L}_Y g_{\bar{z}\bar{z}} = 2r^2\gamma_{z\bar{z}}\partial_{\bar{z}}Y^z + \mathcal{O}(r) \tag{2.59}$$

### 2.5.3 Canonical formulation

Lie derivative of $C_{zz}$ and $N_{zz}$ with respect to Y[3]:

$$\delta_Y C_{zz} = \frac{u}{2}D\cdot YN_{zz} + Y\cdot DC_{zz} - \frac{1}{2}D\cdot YC_{zz} + 2D_zY^zC_{zz} - uD_z^3Y^z \tag{2.60}$$

$$\delta_Y N_{zz} = \frac{u}{2}D\cdot Y\partial_u N_{zz} + Y\cdot DN_{zz} + 2D_zY^zN_{zz} - D_z^3Y^z \tag{2.61}$$

The conserved superrotation charge

$$Q_Y^+ = \frac{1}{8\pi G}\int_{\mathcal{I}\pm} d^2z(Y_{\bar{z}}N_z + Y_zN_{\bar{z}}) \tag{2.62}$$

Integrating by parts and using eq(2.50) we get

$$\begin{aligned} Q_Y^+ &= Q_H^+ + Q_S^+, \\ Q_S^+ &= -\frac{1}{16\pi G}\int_{\mathcal{I}+} dud^2z[D_z^3Y^zuN_{\bar{z}}^z + D_{\bar{z}}^3Y^{\bar{z}}uN_z^{\bar{z}}], \\ Q_H^+ &= \frac{1}{8\pi G}\int_{\mathcal{I}+} dud^2z(Y_{\bar{z}}T_{uz} + Y_zT_{u\bar{z}} + u\partial_zY_{\bar{z}}T_{uu} + u\partial_{\bar{z}}Y_zT_{uu}) \end{aligned} \tag{2.63}$$

The soft charges are linear in the metric flutuation $C_{zz}$, while the hard charge is quadratic.

To compute commutators, we need [4]

$$[N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}(u',w,\bar{w})] = 16\pi Gi\gamma_{z\bar{z}}\delta^2(z-w)\delta(u-u') \tag{2.64}$$

It follows that

$$[Q_S^+, C_{zz}] = -iuD_z^3Y^z \tag{2.65}$$

$$[Q_H^+, C_{zz}] = \frac{iu}{2}D\cdot YN_{zz} + iY\cdot DC_{zz} - \frac{i}{2}D\cdot YC_{zz} + 2iD_zY^zC_{zz} \tag{2.66}$$

Putting this together, we conclude that

$$[Q_Y^+, \ldots] = i\delta_Y \tag{2.67}$$

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