Energy Conditions :

In the actual universe the energy momentum tensor will be made up of contributions from a large number of different matter fields. It would therefore be impossibly complicated to describe the exact energy momentum tensor even if one knew the precise form of the contribution of each field and the equations of motion governing it. In fact, one has little idea of the behaviour of matter under extreme conditions of density and pressure. Thus it might ~~to~~ seem that one has little hope of predicting the occurence of singularities in the universe from the Einstein equations ~~as~~ as one does not know the right-hand of the equations. However there are certain inequalities which is physically reasonable to assume for the energy momentum tensor.

Energy conditions:

Name	Statement	Conditions
Weak	$T_{\alpha\beta} v^{\alpha} v^{\beta} \geq 0$	$\rho \geq 0 ;\ \rho + p_i > 0$
Null	$T_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0$	$\rho + p_i \geq 0$
Strong	$\left(T_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta}\right) v^{\alpha} v^{\beta} \geq 0$	$\rho + \sum_i p_i \geq 0$, $\rho + p_i \geq 0$
Dominant	$-T^{\alpha}{}_{\beta} v^{\beta}$ future directed	$\rho \geq 0,\ \rho \geq \lvert p_i \rvert$

Lets assume,—

$$T^{\alpha\beta} = \rho\, \hat{e}^{\alpha}_{0} \hat{e}^{\beta}_{0} + p_1 \hat{e}^{\alpha}_{1} \hat{e}^{\beta}_{1} + p_2 \hat{e}^{\alpha}_{2} \hat{e}^{\beta}_{2} + p_3 e^{\alpha}_{3} e^{\beta}_{3} \longrightarrow (1)$$

Since the vectors $\hat{e}^{\alpha}_{\mu}$ form an orthonormal basis, they satisfy the relation,—

$$g_{\alpha\beta}\, \hat{e}^{\alpha}_{\mu} \hat{e}^{\beta}_{\nu} = \eta_{\mu\nu} \longrightarrow (2)$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Eqn (1) and (2) imply that the quantities ρ (energy density) and p_i (principal pressure) are eigenvalues of the stress-energy tensor, and $\hat{e}^{\alpha}_{\mu}$ are the normalized ~~eigenvectors~~ eigenvectors.

The inverse metric can neatly be expressed in terms of the basis vectors. It is easy to check that the relation

$$g^{\alpha\beta} = \eta^{\mu\nu} \hat{e}^{\alpha}_{\mu} \hat{e}^{\beta}_{\nu} \longrightarrow (3)$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the inverse of $\eta_{\mu\nu}$.

This eqn is called completeness relations.

If the stress energy tensor is that of a perfect fluid, then $p_1 = p_2 = p_3 \equiv p$.

$$T^{\alpha\beta} = \rho \hat{e}^{\alpha}_{0} \hat{e}^{\beta}_{0} + p\left(\hat{e}^{\alpha}_{1} \hat{e}^{\beta}_{1} + \hat{e}^{\alpha}_{2} \hat{e}^{\beta}_{2} + \hat{e}^{\alpha}_{3} \hat{e}^{\beta}_{3}\right)$$

$$= \rho \hat{e}^{\alpha}_{0} \hat{e}^{\beta}_{0} + p\left(g^{\alpha\beta} + \hat{e}^{\alpha}_{0} \hat{e}^{\beta}_{0}\right)$$

$$= (\rho + p) \hat{e}^{\alpha}_{0} \hat{e}^{\beta}_{0} + p g^{\alpha\beta}$$

~~Lets assume a time like vec~~

Lets assume a timelike vector such that

$$\left.\begin{aligned} u^\alpha &= \gamma\left(\hat{e}^\alpha_0 + a\hat{e}^\alpha_1 + b\hat{e}^\alpha_2 + c\hat{e}^\alpha_2\right), \\ \gamma &= \left(1 - a^2 - b^2 - c^2\right)^{1/2} \end{aligned}\right\} \longrightarrow (4)$$

where a, b, c are arbitrary functions of the coordinates, restricted by $a^2 + b^2 + c^2 < 1$.

Now, lets assume a future directed null vector such that

$$k^\alpha = \hat{e}^\alpha_0 + a'\hat{e}^\alpha_1 + b'\hat{e}^\alpha_2 + c'\hat{e}^\alpha_3 \longrightarrow (5)$$

where a', b', c' are arbitrary function of coordinates.

$a'^2 + b'^2 + c'^2 = 1$.

Weak energy condition: The weak energy condition states the energy density of any matter distribution by any observer in spacetime, must be non-negative.

Because an observer with four-velocity u^α

measures the energy density to be $T_{\alpha\beta} v^{\alpha} v^{\beta} \geq 0$ ⟶ (6)

for any future-directed time like vector v^{α}. To put this in concrete form we substitute eqn (1) and (4) which gives us

$$\cancel{\rho + a^2 p_1 + b^2 \rho}$$

$$\rho + a^2 p_1 + b^2 p_2 + c^2 p_3 \geq 0$$

as a, b, c are aribitrary we can set $a = b = c = 0$, And this gives us $\rho \geq 0$.

Alternatively we can choose $b = c = 0$, then, —

$$\rho + a^2 p_1 \geq 0$$

Now $a^2 < 1$. $\therefore\ 0 \leq \rho + a^2 p_1 \leq \rho + p_1$

$\therefore\ \rho + p_1 > 0$ Similar expression holds for p_2 and p_3.

Therefore, weak energy condition implies, —

$$\rho \geq 0 \ ; \ \rho + p_i > 0$$

<u>Null energy condition</u> : The null energy condition makes the same statement as the weak form, except that v^{α} is replaced by an aribitrary, future-directed null vector k^{α}. Thus,

$$T_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0 \longrightarrow (8)$$

is the statement of the null energy condition. Substituting eqn (1) and (5) gives

$$\rho + a'^2 p_1 + b'^2 p_2 + c'^2 p_3 \geq 0$$

Choosing $b' = c' = 0$ enforces $a' = 1$, and we obtain $\rho + p_1 \geq 0$, with similar expressions holding for p_2 and p_3. The null energy condition therefore implies

$$\rho + p_i \geq 0 \longrightarrow (9)$$

<u>Strong Energy condition</u> : The statement of the strong energy condition is

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) v^\alpha v^\beta \geq 0 \longrightarrow (10)$$

$$\Rightarrow \cancel{T_{\alpha\beta} - \frac{1}{2} T} \quad T_{\alpha\beta} v^\alpha v^\beta \geq -\frac{1}{2} T$$

where ~~whe~~ v^α is any future-directed, normalized, timelike vector. Because $T_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} = \frac{R_{\alpha\beta}}{8\pi}$

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\Rightarrow R_{\alpha\beta} g^{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} g^{\alpha\beta} = 8\pi T_{\alpha\beta} g^{\alpha\beta}$$

$$\Rightarrow R - 2R = 8\pi T$$

$$\Rightarrow T = -\frac{R}{8\pi}$$

$$T_{\alpha\beta} = \frac{R_{\alpha\beta}}{8\pi} - \frac{R g_{\alpha\beta}}{8\pi \times 2}$$

$$\Rightarrow T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} = \frac{R_{\alpha\beta}}{8\pi}$$

Substituting eqn (1) and (4) gives, —

$$\gamma^2 (\rho + a^2 p_1 + b^2 p_2 + c^2 p_3) \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3)$$

$$T^{\alpha\beta} = \frac{1}{4}\left(\rho\, \cancel{\xi}\, \eta_{00} g^{\alpha\beta} + p_1 \eta_{11} g^{\alpha\beta} + p_2 \eta_{22} g^{\alpha\beta} + p_3 \eta_{33} g^{\alpha\beta}\right)$$

$$\Rightarrow T^{\alpha\beta} g_{\alpha\beta} = \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta} (-\rho + p_1 + p_2 + p_3)$$

$$\Rightarrow T = (-\rho + p_1 + p_2 + p_3)$$

Choosing $a = b = c = 0$ enforces $\gamma = 1$ and we get

$$p_1 + p_2 + p_3 \geq 0 \Rightarrow \sum_i p_i \geq 0$$

Alternatively, choosing $b = c = 0$ implies $\gamma^2 = \frac{1}{(1-a^2)}$

$$\frac{1}{(1-a^2)} (\rho + a^2 p_1) \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3)$$

$$\Rightarrow 2\rho + 2a^2 p_1 \geq \rho - p_1 - p_2 - p_3 - a^2 \rho + a^2 p_1 + a^2 p_2 + a^2 p_3$$

$$\Rightarrow \cancel{\rho(1+a^2)}\; \rho + p_1 + p_2 + p_3 \geq a^2 (p_2 + p_3 - \rho - p_1)$$

$0 < a^2 < 1$

$\therefore \; \rho + P_1 \geq 0$

Similar condition apply for P_2 and P_3. Therefore the strong energy condition implies, —

$$\rho + P_1 + P_2 + P_3 \geq 0 \quad ; \quad \rho + P_i \geq 0 \longrightarrow (11)$$

<u>Dominant Energy condition</u> : The dominant energy condition embodies the notion that matter should ~~be~~ flow along timelike or null world lines. It's precise statement is that if v^α is an arbitrary, future-directed, timelike vector field then

$-T^\alpha{}_\beta v^\beta$ is a future-directed, timelike ~~and~~ or null vector field. $\longrightarrow (12)$

The quantity $-T^\alpha{}_\beta v^\beta$ is the matter's momentum density as measured by an observer with four velocity v^α, and this is required to be timelike or null. Substituting (1) and (4) and demanding that $-T^\alpha{}_\beta v^\beta$ not to be spacelike gives, —

$$\rho^2 - a^2 P_1^2 - b^2 P_2 - c^2 P_3 \geq 0$$

choosing $a = b = c = 0$ we get $\rho^2 \geq 0$ and demanding that $-T^{\alpha}{}_{\beta} v^{\beta}$ is future directed selects the positive branch : $\rho \geq 0$

Alternatively, choosing $b = c = 0$ gives $\rho^2 \geq a^2 p_1^2$.

Now $a^2 < 1$. So $\rho \geq |p_1|$ Similarly $\rho \geq |p_i|$