Ellis drainhole: The Ellis drainhole is the earliest known complete mathematical model of traversable wormhole. It is static, spherically symmetric solution of the Einstein vaccum field equations augmented by inclusion of a scalar field ϕ minimally coupled to the geometry of spacetime with coupling polarity opposite to the orthodox polarity (-ve instead of +ve).

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -2\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\phi^{,\kappa}\phi_{,\kappa} g_{\mu\nu}\right) \longrightarrow ①$$

where $\Box\phi = 0 \longrightarrow ②$

The Einstein's field eqns above ~~is simil~~ are equivalent to $R_{\mu\nu} = -2\phi_{,\mu}\phi_{,\nu} \longrightarrow ③$

$$\cancel{Tr(R_{\mu\nu})} \therefore R = R_{\mu}{}^{\mu} = -2\phi^{,\mu}\phi_{,\mu} \longrightarrow ④$$

Ellis drainhole line element: From now on we will use the Ricci tensor with ~~+ve~~ -ve polarity.

$$c^2 d\tau^2 = c^2 dt^2 - \left[d\rho - f(\rho)\, c\, dt\right]^2 - r^2(\rho)\, d\Omega^2 \longrightarrow ⑤$$

$$f^2(\rho) = 1 - e^{-(2m/n)\phi} \longrightarrow ⑥$$

$$r(\rho) = \sqrt{(\rho - m)^2 + a^2}\; e^{(m/n)\phi} \longrightarrow ⑦$$

$$\phi = \alpha(\rho) = \frac{n}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho - m}{a}\right)\right] \longrightarrow ⑧$$

Note: t and ρ are not temporal and radial Schwarzschild coordinates.

Schwarzschild coordinates are related to Ellis coordinates in the following way.

$$\left.\begin{aligned}
T &= t + \frac{1}{c}\int \frac{f(\rho)}{1-f^2(\rho)}\,d\rho \\
r &= r(\rho) \\
\theta &= \theta \\
\phi &= \phi
\end{aligned}\right\} \longrightarrow (9)$$

$$c^2 d\tau^2 = c^2 dt^2 - \left[d\rho + f(\rho)\, c\, dt\right]^2 - r^2 d\Omega^2$$

$$= c^2\left(1 - f^2(\rho)\right) dt^2 + 2 f(\rho)\, c\, dt\, d\rho - d\rho^2$$

$$- r^2 d\Omega^2$$

$$T = t + \frac{1}{c}\int \frac{f(\rho)}{1-f^2(\rho)}\,d\rho$$

$$\Rightarrow dT = dt + \frac{1}{c}\,\frac{f(\rho)}{1-f^2(\rho)}\,d\rho$$

$$\Rightarrow dT^2 = dt^2 + \frac{2}{c}\,\frac{f(\rho)}{1-f^2(\rho)}\,dt\,d\rho + \frac{1}{c^2}\,\frac{f^2(\rho)}{\left(1-f^2(\rho)\right)^2}\,d\rho^2$$

$$\Rightarrow c^2\left(1-f^2(\rho)\right) dT^2 = c^2\left(1-f^2(\rho)\right) dt^2 + 2 f(\rho)\, c\, dt\, d\rho$$

$$+ \frac{f^2(\rho)\, d\rho^2}{1-f^2(\rho)}$$

$$\cancel{\Rightarrow c^2\left(1-f^2(\rho)\right) dT^2}$$

$$\Rightarrow c^2(1-f^2(\rho))\,dt^2 + 2f(\rho)\,c\,dt\,d\rho - d\rho^2$$

$$= c^2(1-f^2(\rho))\,dT^2 - \frac{f^2(\rho)}{1-f^2(\rho)}\,d\rho^2 - d\rho^2$$

$$= c^2(1-f^2(\rho))\,dT^2 + \frac{-\cancel{f^2(\rho)} - 1 + \cancel{f^2(\rho)}}{1-f^2(\rho)}\,d\rho^2$$

$$= c^2(1-f^2(\rho))\,dT^2 - \frac{d\rho^2}{1-f^2(\rho)}$$

$$\therefore c^2 d\tau^2 = c^2(1-f^2(\rho))\,dT^2 - \frac{d\rho^2}{1-f^2(\rho)} - r^2\,d\Omega^2 \longrightarrow (10)$$

Note: For the case of Schwarzschild metric

$f(\rho) = \sqrt{\frac{2MG}{c^2\rho}}$ and $r = r(\rho) = \rho$.

Therefore ρ and r are equivallent for Schwarzschild metric.

For Ellis drainhole metric, $r(\rho) = \sqrt{(\rho-m)^2 + a^2}\; e^{(m/n)\phi}$

Derivation / Ansatz: Ellis's original intention was to replace the Schwarzschild metric with a nonsingular model for a graviating elementary particle. The Schwarzschild metric is found by inserting the most general spherically symmetric and static metric tensor into the vacuum field equations and solving away. In his effort to get rid of the singularity by adding a minimally coupled negative parity massless scalar field into the picture, Ellis inserted the same ansatz (expressed in Ellis's coordinates) into the field eqn given above, to derive the Ellis's drainhole.

In Ellis's coordinates, the most general spherically symmetric and static line element is

$$c^2 d\tau^2 = c^2 dt^2 - \left[d\rho^2 - f(\rho)\, c\, dt\right]^2 - r^2(\rho)\, d\Omega^2$$

Now, because of the spherical symmetry, the scalar ansatz has to be

$$\phi = \alpha(\rho) \longrightarrow (11)$$

$$g_{\mu\nu} = \begin{pmatrix} 1-f^2(\rho) & f(\rho) & 0 & 0 \\ f(\rho) & -1 & 0 & 0 \\ 0 & 0 & -r^2(\rho) & 0 \\ 0 & 0 & 0 & -r^2(\rho)\sin^2\theta \end{pmatrix}$$

$$\longrightarrow (12)$$

$$
g^{\mu\nu} = \begin{pmatrix} 1 & f(\rho) & 0 & 0 \\ f(\rho) & -\cancel{a}\left[1-f^2(\rho)\right] & 0 & 0 \\ 0 & 0 & -\dfrac{1}{r^2(\rho)} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{r^2(\rho)\sin^2\theta} \end{pmatrix} \longrightarrow (13)
$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

$$\Gamma^{0}_{00} = \frac{1}{2} g^{0\sigma}\left(\partial_0 g_{0\sigma} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}\right) = f^2 f'$$

$$\Gamma^{1}_{00} = \frac{1}{2} g^{1\sigma}\left(\partial_0 g_{0\sigma} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}\right) = ff'\left(f^2(\rho)-1\right)$$

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = -f^2 f' \;;\; \Gamma^{0}_{01} = \Gamma^{0}_{10} = -ff'$$

$$\Gamma^{2}_{33} = \cancel{\tfrac{1}{2}g} - \sin\theta\cos\theta \;;\; \Gamma^{1}_{33} = \left(f^2-1\right) rr' \sin^2\theta$$

$$\Gamma^{0}_{22} = frr' \;;\; \Gamma^{3}_{13} = \Gamma^{3}_{31} = \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{r'}{r}$$

$$\Gamma^{0}_{33} = frr'\sin^2\theta \;;\; \Gamma^{1}_{22} = \left(f^2-1\right) rr'$$

$$\Gamma^{3}_{32} = \Gamma^{3}_{23} = \cot\theta \;;\; \Gamma^{1}_{11} = ff' \;;\; \Gamma^{0}_{11} = f'$$

$$R_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho}$$

$$R_{00} = \partial_0 \Gamma^\rho_{0\rho} - \partial_\rho \Gamma^\rho_{00} + \Gamma^\sigma_{0\rho}\Gamma^\rho_{\sigma 0} - \Gamma^\sigma_{00}\Gamma^\rho_{\sigma\rho}$$

$$\cancel{= -\left(ff'\left(f^2(\rho)-1\right)\right)' + f^2 f' f^2 f' + 2(f^2 f')^2}$$

$$= -\partial_1 \Gamma^1_{00} + \Gamma^0_{0\rho}\Gamma^\rho_{00} + \Gamma^1_{0\rho}\Gamma^\rho_{10}$$

$$- \Gamma^0_{00}\Gamma^\rho_{0\rho} - \Gamma^1_{00}\Gamma^\rho_{1\rho}$$

$$\cancel{= -\left(ff'\left(f^2(\rho)-1\right)\right)' +}$$

$$= -\partial_1 \Gamma^1_{00} + \cancel{\Gamma^0_{00}\Gamma^0_{00}} + \cancel{\Gamma^0_{01}\Gamma^1_{00}}$$

$$+ \Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{01}\Gamma^1_{10} - \cancel{\Gamma^0_{00}\Gamma^0_{00}}$$

$$- \Gamma^0_{00}\Gamma^1_{01} - \cancel{\Gamma^1_{00}\Gamma^0_{10}} - \Gamma^1_{00}\Gamma^1_{11}$$

$$- \Gamma^1_{00}\Gamma^2_{12} - \Gamma^1_{00}\Gamma^3_{13}$$

$$= -\left(ff'\left(f^2(\rho)-1\right)\right)' - (f^2 f')^2\left(f^2(\rho)-1\right) + (f^2 f')^2$$

$$+ (f^2 f')^2 - ff'\left(f^2(\rho)-1\right) ff'$$

$$- ff'\left(f^2(\rho)-1\right)\frac{r'}{r} - ff'\left(f^2(\rho)-1\right)\frac{r'}{r}$$

$$= \cancel{2}(f')^2(1-f^2) + ff''(1-f^2) - ff' \cdot 2ff'$$

$$+ 2(ff')^2(1-f^2) + 2(f^2f')^2 + 2ff'(1-f^2)\frac{r'}{r}$$

$$= (f')^2(1-f^2) + ff''(1-f^2) - 2(ff')^2(1-f^2)$$

$$+ 2(ff')^2(1-f^2) + 2ff'(1-f^2)\frac{r'}{r}$$

$$= (1-f^2)\left[(f')^2 + ff'' - 2(\cancel{f}f')^2 + 2(f\cancel{f}')^2 + 2ff'\frac{r'}{r}\right]$$

$$= (1-f^2)\left[(f')^2 + ff'' + 2ff'\frac{r'}{r}\right]$$

$$= (1-f^2)\frac{(f'r)^2 + ff''r^2 + 2ff'r'r}{r^2}$$

$$\nabla^2(\ell(\rho))$$

$$\cancel{\left(\frac{r^2\ell'}{r^2}\right)'}$$

$$= (1-f^2)\frac{\cancel{2}(ff'r^2)'}{r^2}$$

$$= \frac{(r^2\ell')'}{r^2}$$

$$\cancel{r^2\ell'}$$

$$= (1-f^2)\nabla^2\left(\frac{f^2}{2}\right)$$

$$\therefore \boxed{R_{00} = (1-f^2)\nabla^2\left(\frac{f^2}{2}\right)} \longrightarrow (14)$$

$$R_{11} = \partial_1 \Gamma^{\rho}_{1\rho} - \partial_\rho \Gamma^{\rho}_{11} + \Gamma^{\sigma}_{1\rho}\Gamma^{\rho}_{\sigma 1} - \Gamma^{\sigma}_{11}\Gamma^{\rho}_{\sigma\rho}$$

$$= \partial_1 \Gamma^{0}_{10} + \cancel{\partial_1 \Gamma^{1}_{11}} + \partial_1 \Gamma^{2}_{12} + \partial_1 \Gamma^{3}_{13}$$

$$- \cancel{\partial_1 \Gamma^{1}_{11}} + \Gamma^{0}_{1\rho}\Gamma^{\rho}_{01} + \Gamma^{1}_{1\rho}\Gamma^{\rho}_{11}$$

$$+ \Gamma^{2}_{1\rho}\Gamma^{\rho}_{21} + \Gamma^{3}_{1\rho}\Gamma^{\rho}_{31} - \Gamma^{0}_{11}\Gamma^{\rho}_{0\rho}$$

$$- \Gamma^{1}_{11}\Gamma^{\rho}_{1\rho}$$

$$= \partial_1 \Gamma^{0}_{10} + \partial_1 \Gamma^{2}_{12} + \partial_1 \Gamma^{3}_{13} + \Gamma^{0}_{10}\Gamma^{0}_{01}$$

$$+ \Gamma^{0}_{11}\Gamma^{1}_{01} + \Gamma^{1}_{10}\Gamma^{0}_{11} + \cancel{\Gamma^{1}_{11}\Gamma^{1}_{11}}$$

$$\cancel{+ \Gamma^{1}_{12}\Gamma^{2}_{12} + \Gamma^{3}_{13}\Gamma^{3}_{13}} + \Gamma^{2}_{12}\Gamma^{2}_{21}$$

$$+ \Gamma^{3}_{13}\Gamma^{3}_{31} - \Gamma^{0}_{11}\Gamma^{0}_{00} - \cancel{\Gamma^{0}_{11}\Gamma^{1}_{01}}$$

$$- \Gamma^{1}_{11}\Gamma^{0}_{10} - \cancel{\Gamma^{1}_{11}\Gamma^{1}_{11}} - \Gamma^{1}_{11}\Gamma^{2}_{12}$$

$$- \Gamma^{1}_{11}\Gamma^{3}_{13}$$

$$\therefore \boxed{R_{11} = -\nabla^2\left(\frac{1}{2}f^2\right) + 2\frac{r''}{r}} \longrightarrow (15)$$

$$\therefore \boxed{R_{22} = \frac{1}{2}\left(\left[1-f^2\right](r^2)'\right)' - 1} \longrightarrow (16)$$

$$\therefore \boxed{R_{33} = -\sin^2\theta\left(\left(\frac{1}{2}\left(\left[1-f^2\right](r^2)'\right)' - 1\right)\right)} \longrightarrow (17)$$

$$\therefore \boxed{R_{01} = R_{10} = f\,\nabla^2\left(\frac{1}{2}f^2\right)} \longrightarrow (18)$$

$$\Box\phi = 0 \Rightarrow g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi = 0$$

$$\Rightarrow \cancel{g^{00}\nabla_0}\; g^{\mu\nu}\left(\partial_\mu\partial_\nu\phi - \Gamma^\sigma_{\mu\nu}\partial_\sigma\phi\right) = 0$$

$$\Rightarrow g^{00}\left(\partial_0\partial_0\phi - \Gamma^\sigma_{00}\partial_\sigma\phi\right) + g^{01}\left(\partial_0\partial_1\phi - \Gamma^\sigma_{01}\partial_\sigma\phi\right)$$

$$+ g^{10}\left(\partial_1\partial_0\phi - \Gamma^\sigma_{10}\partial_\sigma\phi\right) + g^{11}\left(\partial_1\partial_1\phi - \Gamma^\sigma_{11}\partial_\sigma\phi\right)$$

$$+ g^{22}\left(\partial_2\partial_2\phi - \Gamma^\sigma_{22}\partial_\sigma\phi\right) + g^{33}\left(\partial_3\partial_3\phi - \Gamma^\sigma_{33}\partial_\sigma\phi\right)$$

$$= 0$$

$$\Rightarrow \Box\alpha = -g^{00}\Gamma^1_{00}\partial_1\alpha - g^{01}\Gamma^1_{01}\partial_1\alpha$$

$$- g^{10}\Gamma^1_{10}\partial_1\alpha + g^{11}\partial_1\partial_1\alpha - g^{11}\Gamma^1_{11}\partial_1\alpha$$

$$- g^{22}\Gamma^1_{22}\partial_1\alpha - g^{33}\Gamma^1_{33}\partial_1\alpha$$

$$= 0$$

$$\Rightarrow -ff'(f^2-1)\alpha' + ff^2f'\alpha' + ff^2f'\alpha'$$

$$- [1-f^2]\alpha'' + [1-f^2]ff'\alpha'$$

$$+ \frac{1}{r^2}(f^2-1)rr'\alpha' + \frac{1}{r^2\cancel{\sin^2\theta}}(f^2-1)rr'\cancel{\sin^2\theta}\,\alpha'$$

$$= - [1-f^2]\alpha'' + \cancel{[ff'(f^2-1}$$

$$+ \left[-2ff'(f^2-1) + 2f^3f' + \frac{2}{r}(f^2-1)r'\right]\alpha'$$

$$= -(1-f^2)\alpha'' + \left(2ff' + \frac{2}{r}(f^2-1)r'\right)\alpha'$$

$$= \frac{1}{r^2}\left((f^2-1)r^2\alpha'\right)'$$

$$= 0$$

$$\Rightarrow \boxed{\left((f^2-1)r^2\alpha'\right)' = 0} \longrightarrow (19)$$

$$R_{00} = 2\partial_0\phi\,\partial_0\phi \Rightarrow \boxed{(1-f^2)\nabla^2\left(\frac{f^2}{2}\right) = 0} \longrightarrow (20)$$

$$R_{11} = 2\partial_1\phi\,\partial_1\phi \Rightarrow \boxed{-\nabla^2\left(\frac{f^2}{2}\right) + 2\frac{r''}{r} = (\alpha')^2} \longrightarrow (21)$$

$$R_{22} = 2\partial_2\phi\,\partial_2\phi \Rightarrow \boxed{\frac{1}{2}\left([1-f^2](r^2)'\right)' - 1 = 0} \longrightarrow (22)$$

$$R_{33} = 2\partial_3\phi\,\partial_3\phi \Rightarrow \boxed{-\sin^2\theta\left(\frac{1}{2}\left([1-f^2](r^2)'\right)' - 1 = 0\right.} \longrightarrow (23)$$

$$R_{01} = R_{10} = 2\partial_0\phi\,\partial_1\phi \Rightarrow \boxed{f\nabla^2\left(\frac{f^2}{2}\right) = 0} \longrightarrow (24)$$

From (19), —

$$\left((f^2-1)\, r^2 \alpha'\right)' = 0 \Rightarrow (f^2-1)\, r^2 \alpha' = n \quad \text{(Integrating)} \longrightarrow (25)$$

From (20), —

$$\left(r^2 \left(-\frac{f^2}{2}\right)'\right)' = 0 \Rightarrow r^2 \frac{(-f^2)'}{2} = m \quad \text{(Integrating)}$$

$$\Rightarrow r^2 (1-f^2)' = 2m \longrightarrow (26)$$

From (21), —

$$\frac{r''}{r} = (\alpha')^2$$

$$\left((1-f^2)\left(\frac{r^2}{2}\right)'\right)' = 1$$

$$\Rightarrow (1-f^2)\left(\frac{r^2}{2}\right)' = \rho - m \longrightarrow (27)$$

(Integrating)

From $(26)/2 + (27)$, —

$$\left(\frac{r^2}{2}\right)(1-f^2)' + (1-f^2)\left(\frac{r^2}{2}\right)' = \rho - m + m$$

$$\Rightarrow \left(\frac{r^2}{2}(1-f^2)\right)' = \rho$$

$$\Rightarrow \frac{r^2}{2}(1-f^2) = \frac{\rho^2}{2} + \frac{C}{2} \quad \text{(Integrating)}$$

$$\Rightarrow r^2(1-f^2) = \rho^2 + C \longrightarrow (28)$$

From (25), —

$$\alpha' = \frac{n}{(f^2-1)\, r^2} = -\frac{n}{\rho^2 + C} \longrightarrow (25.2)$$

From (27), —

$$(1-f^2)(r^2)' = 2(\rho - m)$$

$$\Rightarrow (1-f^2)\, r r' = (\rho - m) \longrightarrow (29)$$

$(28)/(29)$, —

$$\frac{r'}{r} = \frac{\rho - m}{\rho^2 + C} \longrightarrow (30)$$

From (27), —

$$(\alpha')^2 = \frac{r''}{r} = \frac{-(r')^2 + r r'' + (r')^2}{r^2} = +\left(\frac{r'}{r}\right)' + \left(\frac{r'}{r}\right)^2$$

$$= +\left(\frac{p-m}{p^2+c}\right)' + \left(\frac{p-m}{p^2+c}\right)^2$$

$$= + \frac{(p^2+c) - (p-m)2p + (p-m)^2}{(p^2+c)^2}$$

$$= + \frac{\cancel{p^2} + c - \cancel{2p^2} + \cancel{2mp} + \cancel{p^2} - \cancel{2mp} - m^2}{(p^2+c)^2}$$

$$\Rightarrow (\alpha')^2 = + \frac{c - m^2}{(p^2+c)^2} \longrightarrow (31)$$

From (25.2) and (31), —

$$-n = \sqrt{c - m^2} \Rightarrow c = -m^2 + n^2 \longrightarrow (32)$$

$$\therefore \alpha' = - \frac{n}{p^2 - m^2 + n^2} \longrightarrow (33)$$

From (30), —

$$\frac{r'}{r} = \frac{p-m}{p^2 - m^2 + n^2}$$

$$\cancel{\Rightarrow (\log r)' = \frac{p-m}{p^2+m^2-n^2}}$$

$$\cancel{\Rightarrow \log r = \int \frac{p-m}{p^2+m^2-n^2}}$$

$$\therefore \frac{r'}{r} - \frac{m}{n}\alpha' = \frac{\rho - m}{\rho^2 \mp m^2 + n^2} + \frac{m}{\rho^2 \mp m^2 + n^2}$$

$$= \frac{\rho}{\rho^2 \mp m^2 + n^2}$$

$$\Rightarrow \left(\ln r - \frac{m}{n}\alpha\right)' = \frac{\rho}{\rho^2 \mp m^2 + n^2}$$

$$\Rightarrow \ln r - \frac{m}{n}\alpha = \frac{1}{2}\ln\left|\rho^2 \mp m^2 + n^2\right| + \text{constant}$$

$$\Rightarrow \ln r = \frac{1}{2}\ln\left|\rho^2 \mp m^2 + n^2\right| + \frac{m}{n}\alpha$$

[Here we have absorbed the constant in α because as $\phi = (\alpha(\rho) + \text{const})$ does not change the situation as in Einstein's eqn derivatives of ϕ appears, not any ϕ term]

$$\Rightarrow r^2(\rho) = \left|\rho^2 \mp m^2 + n^2\right| e^{\frac{2m}{n}\alpha} \longrightarrow (34)$$

From (28),

$$r^2(1 - f^2) = \rho^2 \mp m^2 + n^2$$

$$\Rightarrow f^2(\rho) = 1 - \operatorname{sgn}\left(\rho^2 \mp m^2 + n^2\right) e^{-2\frac{m}{n}\alpha} \longrightarrow (35)$$

<u>Note</u>: The boundary condition for the field ϕ is, $\lim_{\rho\to\infty}\phi = \lim_{\rho\to\infty}\alpha(\rho) = 0$ resulting the spacetime to be asymptotically flat.

There are 3 different cases $n^2 < m^2$
$m^2 = n^2$
$n^2 > m^2$

For only one case we will get a drainhole. From the line element we can see that $r(\rho)$ gives the radius of the sequence of 2-spheres along the ρ coordinate needed to assemble the pure space hypersurface of the metric. We are looking for a drainhole, which is basically a wormhole with a gravity field that pushes test particle through the wormhole in one direction. So, to find the case that contains a drainhole, we must look for a radius function that is continuous, never zero, reaches one nonzero minimum and increases monotonically when moving in either direction away from the minimum. This corresponds to the double funnel ~~of~~ shape, characteristics of wormholes. In all three cases, the integration constant is selected in such a way so that $\lim_{\rho \to \infty} \alpha(\rho) = 0$

Case 1: $n^2 < m^2$; $a = \sqrt{m^2 - n^2}$

$$\alpha' = \frac{-n}{\rho^2 \mp m^2 + n^2}$$

$$\Rightarrow \alpha(\rho) = \int \alpha' \, d\rho = \int \frac{-n}{\rho^2 - a^2} \, d\rho = \frac{n}{2a} \ln\left(\frac{\rho + a}{\rho - a}\right)$$

$$r^2(\rho) = \left|\rho^2 - a^2\right| \left|\frac{\rho + a}{\rho - a}\right|^{\frac{m}{a}} = \frac{\left|\rho + a\right|^{\frac{m}{a} + 1}}{\left|\rho - a\right|^{\frac{m}{a} - 1}}$$

$$f^2(\rho) = 1 - \mathrm{sgn}\left(\left|\rho^2 - a^2\right| \left|\frac{\rho + a}{\rho - a}\right|^{\frac{m}{a}}\right)$$

$r^2(\rho)$ clearly possesses both zero point and a singularity.

Case 2 : $m^2 = n^2$

$$\alpha' = \frac{-n}{\rho^2}$$

$$\Rightarrow \alpha = \frac{n}{\rho}$$

$$r^2(\rho) = \rho^2 e^{\frac{2m}{\rho}}$$

$$f^2(\rho) = 1 - \text{sgn}\, \rho^2 e^{-\frac{2m}{\rho}} = 1 - e^{-\frac{2m}{\rho}}$$

$r^2(\rho)$ has singularity.

Case 3 : $n^2 > m^2$; $a = \sqrt{n^2 - m^2}$

$$\alpha' = -\frac{n}{\rho^2 + a^2}$$

$$\Rightarrow \alpha(\rho) = \frac{n}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right)\right]$$

$$r^2(\rho) = (\rho^2 + a^2)\, e^{\frac{2m}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right)\right]}$$

$$f^2(\rho) = 1 - e^{-\frac{2m}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right)\right]}$$

$r^2(\rho)$ is continuous, never zero, reaches one nonzero minimum at $\rho = m$ and increases monotonically when moving in either direction away from this minimum.

It is the case that contains drainhole.

$$T = t + \frac{1}{c} \int \frac{f(\rho)}{1 - f^2(\rho)} \, d\rho$$

So, T has temporal curve. When $m = 0$ then $f(\rho) = 0$

So, m is like gravity here. When $m = 0$, —

$$ds^2 = c^2 dT^2 - d\rho^2 - (\rho^2 + a^2) \, d\Omega^2$$

$$ds^2 = c^2 (1 - f^2(\rho)) \, dT^2 - \frac{d\rho^2}{(1 - f^2(\rho))} - r^2(\rho) \, d\Omega^2$$

$$= c^2 (1 - f^2(\rho)) \left(dT^2 - \frac{d\rho^2}{(1 - f^2(\rho))^2} \right) - r^2(\rho) \, d\Omega^2$$

$$= c^2 (1 - f^2(\rho)) \left(dT^2 - d\rho_*^2 \right) - r^2 \, d\Omega^2$$

where $d\rho_*^2 = \dfrac{d\rho^2}{(1 - f^2(\rho))^2}$

$$d\rho_*^2 = e^{\frac{4m}{a} \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right) \right]} \, d\rho^2$$

$$\Rightarrow d\rho_* = e^{\frac{2m}{a} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right) \right)} \, d\rho$$

$$ds^2 = e^{\frac{2m}{a}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right)\right)}(dT^2 - d\rho_*^2) - r^2 d\Omega^2$$

$$dU\,dV = dT^2 - d\rho_*^2$$
$$= (dT + d\rho_*)(dT - d\rho_*)$$

$$dU = dT + d\rho_* \qquad dV = dT - d\rho_*$$

$$\therefore \frac{\partial U}{\partial T} = 1 = \frac{\partial V}{\partial T}$$
$$\therefore \frac{\partial U}{\partial \rho} = e^{\frac{4m}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho}{a}\right)\right]} d\rho = -\frac{\partial V}{\partial \rho}$$

From this we can get U-V penrose diagram.

If we embed ~~the~~ $\left(dT^2 - \frac{d\rho^2}{1-f^2(\rho)}\right)$ part in 3-D space we get the catenoid shape.