Tamralipta Mahavidyalaya

Summer Internship

A Reading Project on Quantum Mechanics

Submitted by

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Certificate

This is to certify that the reading project in the field of **Quantum Mechanics** submitted by **Mr. Raikhik Das, Registration No. 20181123, IISER Pune**, as a part of Summer Internship has been performed at **Tamralipta Mahavidyalaya** and is a bonafide record of the work done by him under my guidance and supervision **from June 3, 2019 to July 18, 2019**.

10.08.19

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Date: 10.08.2019

Place: Tamluk

Acknowledgement

I would like to express my sincere gratitude towards **Dr. Sujit kumar Bera** for providing me this opportunity to do this project.

I am deeply indebted to my project mentor **Dr. Sujit Kumar Bera, department of physics, Tamralipta Mahavidyalaya**, for his valuable guidance, fruitful discussions, constant support and encouragement during the project work.

MIT OCW Lecture Notes on Quantum Physics by Prof. Barton Zwiebach have helped me a lot . I have also received help from the books: **Quantum Mechanics by Leonard I Schiff** and **Introduction to Quantum Mechanics by David J Griffiths**. And also without the help of **Wikipedia** this project would not have been possible.

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Loss of determinity and superposition:

The most vivid difference between classical mechanics and quantum mechanics is that in quantum mechanics the determinity is lost where as it remains intact in classical mechanics . Lets say that particle has a measurable quantity which can be two valued in case of the particle. We cannot claim before experiment that the particle will show which value . We see that for the same kind of particle We can get any one of the values which can not be determined before measuring the quantity. It can be said that a particle is in the superposition of all the characteristics.

When we consider superposition of states of *two* particles we can get the remarkable phenomenon called *quantum mechanical entanglement*. Entangled states of two particles are those in which we can't speak separately of the state of each particle. The particles are bound together in a common state in which they are *entangled* with each other.

Superposition:

Lets consider now two states $|A\rangle$ and $|B\rangle$. Assume, in addition, that when measuring some property Q in the state $|A\rangle$ the answer is always a, and when measuring the same property Q in the state $|B\rangle$ the answer is always b. Suppose now that our physical state $|\Psi\rangle$ is the superposition

$$|\Psi\rangle \;=\; \alpha|A\rangle + \beta|B\rangle\,, \qquad \alpha, \beta \in \mathbb{C}.$$

The probabilities to obtain a or b

$$\text{Probability}(a) \sim |\alpha|^2, \qquad \text{Probability}(b) \sim |\beta|^2.$$

Since the only two possibilities are to measure a or b, the actual probabilities must sum to one and therefore they are given by

$$a) \;=\; \frac{|\alpha|}{|\alpha|^2 + |\beta|^2} \qquad\qquad b) \;=\; \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2}$$

If we obtain the value a, immediate repeated measurements would still give a, so the state after the measurement must be $|A\rangle$ The same happens for b, so we have

After measuring a the state becomes $|\Psi\rangle = |A\rangle$

After measuring b the state becomes $|\Psi\rangle = |B\rangle$

Entanglement:

Let us consider two non-interacting particles. Particle 1 could be in any of the states

$$\{|u_1\rangle, |u_2\rangle, \ldots\}$$

while particle 2 could be in any of the states

$$\{|v_1\rangle, |v_2\rangle, ...\}$$

It may seem reasonable to conclude that the state of the full system, including particle 1 and particle 2 would be specified by stating the state of particle 1 and the state of particle 2. If that would be the case the possible states would be written as

$$|u_i\rangle \otimes |v_j\rangle, \quad i, j \in \mathbb{N}$$

for some specific choice of *i* and *j* that specify the state of particle one and particle two, respectively. Here we have used the symbol ⊗, which means *tensor* product, to combine the two states into a single state for the whole system. We will study ⊗ later, but for the time being we can think of it as a kind of product that distributes over addition and obeys simple rules, as follows

$$\begin{aligned}(\alpha_1|u_1\rangle + \alpha_2|u_2\rangle) \otimes (\beta_1|v_1\rangle + \beta_2|v_2\rangle) \;=\; & \alpha_1\beta_1|u_1\rangle \otimes |v_1\rangle + \alpha_1\beta_2|u_1\rangle \otimes |v_2\rangle \\ & + \alpha_2\beta_1|u_2\rangle \otimes |v_1\rangle + \alpha_2\beta_2|u_2\rangle \otimes |v_2\rangle\end{aligned}$$

The numbers can be moved across the ⊗ but the order of the states must be preserved. The state on the left-hand side –expanded out on the right-hand side– is still of the type where we combine a state of the first particle ($\alpha_1|u_1\rangle + \alpha_2|u_2\rangle$) with a state of the second particle ($\beta_1|v_1\rangle + \beta_2|v_2\rangle$).

However, we can construct more intriguing superpositions. Lets consider the following one

$$|u_1\rangle \otimes |v_1\rangle + |u_2\rangle \otimes |v_2\rangle \, .$$

A state of two particles is said to be **entangled** if it cannot be written in the factorized form (···)⊗(···) which allows us to describe the state by simply stating the state of each particle

$\alpha_1\beta_1 = 1,\qquad \alpha_1\beta_2 = 0,\qquad \alpha_2\beta_1 = 0,\qquad \alpha_2\beta_2 = 1.$

Wave particle duality:

Photoelectric effect and black-body radition are two famous evidences that photons have not only the wave but also particle character. On the other hand diffraction of electrons is one of evidence of electron, a particle having wave character. Infact we can show that every particle and every wave has both particke and wave character. This is called wave particle duality.

Where the wavelength and the momentum can be related by the equation,

$$\lambda = \frac{h}{p} \qquad \text{(for matter waves)}$$

Some results from de Broglie wavelength and Galilean transformation:

$$\Psi(\mathbf{x}, t) \in \mathbb{C}$$

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k,$$

$$x' = x - vt, \quad t' = t\,. \qquad\qquad p' = p - mv.$$

$$\lambda' = \frac{h}{p'} = \frac{h}{p - mv} \neq \lambda,$$

$$\phi = kx - \omega t = k(x - \tfrac{\omega}{k}t) = \frac{2\pi}{\lambda}(x - Vt) = \frac{2\pi x}{\lambda} - \frac{2\pi V}{\lambda}t,$$

$$\phi'(x', t') = \phi(x, t)$$

$$\phi'(x', t') = \frac{2\pi}{\lambda}(x - Vt) = \frac{2\pi}{\lambda}(x' + vt' - Vt') = \frac{2\pi}{\lambda}x' - \frac{2\pi(V - v)}{\lambda}t'.$$

$$\lambda' = \lambda \qquad\qquad \omega' = \frac{2\pi}{\lambda}(V - v) = \frac{2\pi V}{\lambda}\left(1 - \frac{v}{V}\right) = \omega\left(1 - \frac{v}{V}\right).$$

$$\Psi(x, t) \neq \Psi'(x', t')$$

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{p}{m} = v$$

$$p = \hbar k, \quad E = \hbar\omega$$

Phase and group velocity:

$$\psi(x, t) = \int dk\; \Phi(k) e^{i(kx - \omega(k)t)}.$$

$$\varphi(k) = kx - \omega(k)t \qquad\qquad \left.\frac{d\varphi}{dk}\right|_{k_0} = x - \left.\frac{d\omega}{dk}\right|_{k_0} t = 0$$

$$x = \left.\frac{d\omega}{dk}\right|_{k_0} t \qquad\qquad v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$$

$$\psi(x,0) = \int dk\ \Phi(k) e^{ikx}$$

With the help of Taylor expansion around $k=k_0$,

$$\omega(k) = \omega(k_0) + (k-k_0) \left.\frac{d\omega}{dk}\right|_{k_0} + \mathcal{O}\left((k-k_0)^2\right)$$

Neglecting the term related to $(k-k_0)^2$,

$$\psi(x,t) = \int dk\ \Phi(k)\, e^{ikx}\, e^{-i\omega(k_0)t} e^{-i(k-k_0)\left.\frac{d\omega}{dk}\right|_{k_0} t} .$$

$$\begin{aligned}\psi(x,t) &= e^{-i\omega(k_0)t + ik_0 \left.\frac{d\omega}{dk}\right|_{k_0} t} \int dk\ \Phi(k) e^{ikx} e^{-ik\left.\frac{d\omega}{dk}\right|_{k_0} t} \\ &= e^{-i\omega(k_0)t + ik_0 \left.\frac{d\omega}{dk}\right|_{k_0} t} \int dk\ \Phi(k) e^{ik\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t\right)} .\end{aligned}$$

$$\psi(x,t) = e^{-i\omega(k_0)t + ik_0 \left.\frac{d\omega}{dk}\right|_{k_0} t}\ \psi\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t\right)$$

$$|\psi(x,t)| = \left|\psi\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t, 0\right)\right|$$

$$x - \left.\frac{d\omega}{dk}\right|_{k_0} t = x_0 \quad \rightarrow \quad x = x_0 + \left.\frac{d\omega}{dk}\right|_{k_0} t$$

Wave function of a free particle:

$$\boxed{\text{Free particle wavefunction :} \quad \Psi(x,t) \;=\; e^{i(kx-\omega t)}\,,}$$

representing a particle with

$$p = \hbar k\,, \quad \text{and} \quad E = \hbar\omega\,.$$

In three dimensions the corresponding wavefunction would be

$$\boxed{\text{Free particle wavefunction :} \quad \Psi(\mathbf{x},t) \;=\; e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,,}$$

representing a particle with

$$\mathbf{p} = \hbar\,\mathbf{k}\,, \quad \text{and} \quad E = \hbar\omega\,.$$

Equations for a wave function:

$$\hat{p} \;\equiv\; \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad\qquad \hat{p}\,\Psi \;=\; p\,\Psi$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) \;=\; i\hbar(-i\omega)\Psi(x,t) \;=\; \hbar\omega\,\Psi(x,t) \;=\; E\,\Psi(x,t)$$

$$\hat{E} \;\equiv\; \frac{\hat{p}^2}{2m} \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$i\hbar\,\frac{\partial}{\partial t}\Psi(x,t) \;=\; -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

$$i\hbar\,\frac{\partial}{\partial t}\Psi(x,t) \;=\; \hat{E}\,\Psi(x,t)$$

When ,

$$\Psi(x,t) \;=\; e^{i(k_1 x-\omega_1 t)} + e^{i(k_2 x-\omega_2 t)}$$

Then momentum is not definite.

$$\hat{p}\,\Psi(x,t) \;=\; \hbar k_1\; e^{i(k_1 x-\omega_1 t)} + \hbar k_2\; e^{i(k_2 x-\omega_2 t)}$$

Schrodinger equation for a particle in potential:

$$E = \frac{p^2}{2m} + V(x,t) \qquad \hat{E} = \frac{\hat{p}^2}{2m} + V(x,t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\Big)\Psi(x,t)$$

$$\hat{H} \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\,\Psi(x,t)$$

$$\hat{x}f(x) \equiv xf(x) \qquad \hat{x}^k f(x) \equiv x^k f(x)$$

$$[\,\hat{x}\,,\,\hat{p}\,] = i\hbar \qquad [\,\hat{x}_i\,,\hat{p}_j\,] = i\hbar\,\delta_{ij}$$

Interpreting the wave function:

$$dP = |\Psi(\mathbf{x},t)|^2\, d^3\mathbf{x} \qquad \int_{\text{all space}} d^3\mathbf{x}\,|\Psi(\mathbf{x},t)|^2 = 1$$

Normalization:

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\,\Psi(x,t)\,dx = 1$$

Wave function as probability amplitude:

$$\rho(x,t) \equiv \Psi^*(x,t)\Psi(x,t) = |\Psi(x,t)|^2\,.$$

$$\mathcal{N}(t) \equiv \int \rho(x,t)dx \qquad \mathcal{N}(t_0) = 1$$

$$\frac{d\mathcal{N}(t)}{dt} = 0$$

$$\begin{aligned}\frac{d\mathcal{N}(t)}{dt} &= \int_{-\infty}^{\infty} \frac{\partial \rho(x,t)}{\partial t} dx \\ &= \int_{-\infty}^{\infty} \left(\frac{\partial \Psi^*}{\partial t} \Psi(x,t) + \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial t} \right) dx\end{aligned}$$

$$\begin{aligned}\frac{d\mathcal{N}(t)}{dt} &= \int_{-\infty}^{\infty} \left(\frac{i}{\hbar} (\hat{H}\Psi)^* \Psi - \frac{i}{\hbar} \Psi^* (\hat{H}\Psi) \right) dx \\ &= \frac{i}{\hbar} \Big(\int_{-\infty}^{\infty} (\hat{H}\Psi)^* \Psi dx - \int_{-\infty}^{\infty} \Psi^* (\hat{H}\Psi) dx \Big)\end{aligned}$$

$$\int_{-\infty}^{\infty} (\hat{H}\Psi)^* \Psi = \int_{-\infty}^{\infty} \Psi^* (\hat{H}\Psi)$$

Hermitian operator: $\displaystyle \int_{-\infty}^{\infty} (\hat{H}\Psi_1)^* \Psi_2 = \int_{-\infty}^{\infty} \Psi_1^* (\hat{H}\Psi_2)$

The probability current:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{i}{\hbar} ((\hat{H}\Psi)^* \Psi - \Psi^* (\hat{H}\Psi)) \\ &= \frac{i}{\hbar} \left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \Psi^*}{\partial x^2} \Psi - \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \right) + V(x,t)\Psi^*\Psi - \Psi^* V(x,t)\Psi \right]\end{aligned}$$

$$\frac{i}{\hbar} ((\hat{H}\Psi)^* \Psi - \Psi^* (\hat{H}\Psi)) = \frac{\hbar}{2im} \left(\frac{\partial^2 \Psi^*}{\partial x^2} \Psi - \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \right)$$

$$\begin{aligned}\frac{i}{\hbar} ((\hat{H}\Psi)^* \Psi - \Psi^* (\hat{H}\Psi)) &= \frac{\partial}{\partial x} \left[\frac{\hbar}{2im} \left(\frac{\partial \Psi^*}{\partial x} \Psi - \Psi^* \frac{\partial \Psi}{\partial x} \right) \right] \\ &= -\frac{\partial}{\partial x} \left[\frac{\hbar}{2im} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] \\ &= -\frac{\partial}{\partial x} \left[\frac{\hbar}{2im} 2i \, \mathrm{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) \right] \\ &= -\frac{\partial}{\partial x} \left[\frac{\hbar}{m} \mathrm{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) \right]\end{aligned}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\hbar}{m} \mathrm{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) \right] = 0 \,.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \qquad J(x,t) \equiv \frac{\hbar}{m} \mathrm{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right)$$

$$J = \frac{\hbar}{2im}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right)$$

$$\frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial J(x,t)}{\partial x} dt = -J(b,t) + J(a,t)$$

Wave packets and uncertainty:

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k) e^{ikx} dk.$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$$

Heisenberg uncertainty product: $\Delta x \, \Delta p \geq \frac{\hbar}{2}$

Wave packet shape changes:

The general solution of Schrodinger equation is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k) e^{i(kx - \omega(k)t)} dk$$

Under the assumption that $\Phi(k)$ peaks around some value $k = k_0$ we expanded the frequency $\omega(k)$ in a Taylor expansion around $k = k_0$. Keeping terms up to and including $(k - k_0)^2$ we have

$$\omega(k) = \omega(k_0) + (k - k_0) \left.\frac{d\omega}{dk}\right|_{k_0} + \frac{1}{2}(k - k_0)^2 \left.\frac{d^2\omega}{dk^2}\right|_{k_0} .$$

The second term played a role in the determination of the group velocity and the next term, with second derivatives of ω is responsible for the shape distorsion that occurs as time goes by. The derivatives are promptly evaluated,

$$\frac{d\omega}{dk} = \frac{dE}{dp} = \frac{p}{m} = \frac{\hbar k}{m}, \qquad \frac{d^2\omega}{dk^2} = \frac{\hbar}{m}.$$

$$e^{-i\omega(k)t} = e^{\cdots \, -i\frac{1}{2}(k-k_0)^2 \frac{\hbar}{m} t}$$

Assume we start with the packet at $t = 0$ and evolve in time to $t > 0$. This phase will be ignorable as long as its magnitude is significantly less than one:

$$(k - k_0)^2 \frac{\hbar}{m} t \ll 1.$$

Momentum space:

According to Fourior theorem ,

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(k) e^{ikx} dk \,,$$
$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx} dx$$

With the help of Parseval's theorem we can write,

$$\int_{-\infty}^{\infty} dx \, |\Psi(x)|^2 = \int_{-\infty}^{\infty} dk \, |\Phi(k)|^2$$

Now instead of k wec want to get the relations in terms of p.

So, lets assume,

$$\widetilde{\Phi}(p) = \Phi(k)$$

Now we can write,

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \widetilde{\Phi}(p) e^{ipx/\hbar} dp \,,$$
$$\widetilde{\Phi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ipx/\hbar} dx$$

Now if we substitute ,

$$\widetilde{\Phi}(p) \to \Phi(p)\sqrt{\hbar}$$

Then,

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p) \, e^{ipx/\hbar} \, dp \,,$$
$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x) e^{-ipx/\hbar} dx$$

According to Parseval's theorem,

$$\int_{-\infty}^{\infty} dx\, |\Psi(x)|^2 = \int_{-\infty}^{\infty} dp\, |\Phi(p)|^2$$

$|\Phi(p)|^2 dp$ is the probability to find the particle with momentum in the range $(p, p + dp)$

Here the function of p behaves very much as the wave function of x.

So , we say $\Phi(p)$ as the wavefunction in **momentum space** p.

Expectation values and their time dependence:

Expectation value $\langle \hat{Q} \rangle$ of any operator $\hat{Q}$:

$$\langle \hat{Q} \rangle = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\, \hat{Q} \Psi(x,t)$$

$$\begin{aligned}
i\hbar \frac{d}{dt} \langle Q \rangle &= i\hbar \frac{d}{dt} \int_{-\infty}^{\infty} d^3x\, \Psi^*(x,t) \hat{Q} \Psi(x,t) \\
&= i\hbar \int_{-\infty}^{\infty} d^3x \left(\frac{\partial \Psi^*}{\partial t} \hat{Q} \Psi + \Psi^* \hat{Q} \frac{\partial \Psi}{\partial t} \right) \\
&= i\hbar \int_{-\infty}^{\infty} d^3x \left(\frac{i}{\hbar} (\hat{H}\Psi)^* \hat{Q} \Psi - \frac{i}{\hbar} \Psi^* \hat{Q} (\hat{H}\Psi) \right) \\
&= \int_{-\infty}^{\infty} d^3x \left(\Psi^* \hat{Q} \hat{H} \Psi - (\hat{H}\Psi^*) \hat{Q} \Psi \right) \\
&= \int_{-\infty}^{\infty} d^3x\, \Psi^* \left[\hat{Q}, \hat{H} \right] \Psi
\end{aligned}$$

$$\boxed{i\hbar \frac{d}{dt} \langle \hat{Q} \rangle = \left\langle [\hat{Q}, \hat{H}] \right\rangle .}$$

Uncertainty:

The uncertainty $\Delta \hat{Q}_\Psi$ of an operator in a state Ψ is a real number whose square is given by

$$(\Delta Q)^2_\Psi = \langle Q^2 \rangle_\Psi - (\langle Q \rangle_\Psi)^2.$$

$$\Delta \hat{Q}_\Psi = 0 \iff \Psi \text{ is an eigenstate of } \hat{Q}$$

Stationary staes:

If the wave function of a particle can be write as the multiple of explicit functions of time and position then it is said that the particle is in a stationary states.

$$\Psi(x\,,t) \;=\; g(t)\,\psi(x)$$

After applying the Schrodinger equation ,

$$\left(i\hbar\frac{dg(t)}{dt}\right)\psi(x) = g(t)\,\hat{H}\psi(x)$$

$$i\hbar\,\frac{1}{g(t)}\frac{dg(t)}{dt} = \frac{1}{\psi(x)}\hat{H}\psi(x)$$

Here we can see both the right and left sides of the equation are respectively explicit of time and position. This means that both sides of the equation is equal to a constant term. Now we know that the Hamiltonian has the dimension of the energy . So we equate both sides with E. And the results we get can be summarized as following .

$$\text{Stationary state:}\quad \Psi(x,t) = e^{-iEt/\hbar}\,\psi(x)\,,\quad \text{with}\quad E\in\mathbb{R}\;\text{ and }\hat{H}\psi = E\psi$$

$$\hat{H}\Psi(x,t) = E\Psi(x,t)$$

Free particle on a circle:

Lets assume that the circumference of the circle is L. Then ,

$$x \sim x + L$$

$$\psi(x+L) = \psi(x)$$

After applying the Schrodinger equation and boundary conditions the results we get are following.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\,\psi\,.$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad \text{, where } k^2 \equiv \frac{2mE}{\hbar} \geq 0$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi\,.$$

$$\psi(x) \sim e^{ikx}.$$

$$e^{ik(x+L)} = e^{ikx} \quad \to \quad e^{ikL} = 1 \quad \to \quad kL = 2\pi n\,, \quad n \in \mathbb{Z}\,.$$

$$k_n \equiv \frac{2\pi n}{L}\,, \qquad n \in \mathbb{Z}.$$

$$\psi_n(x) = Ne^{ik_n x}$$

$$1 = \int_0^L \psi_n^*(x)\psi_n(x)dx = \int_0^L N^2 dx = N^2 L \quad \to \quad N = \frac{1}{\sqrt{L}}$$

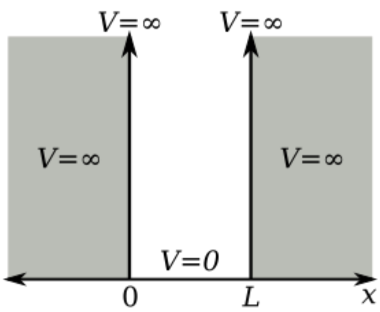
$$\psi_n(x) = \frac{1}{\sqrt{L}}e^{ik_n x} = \frac{1}{\sqrt{L}}e^{\frac{2\pi i n x}{L}}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 4\pi^2 n^2}{2mL^2} = \frac{2\pi^2\hbar^2 n^2}{mL^2}$$

Infinite square well:

Now , we have a potential which is as following

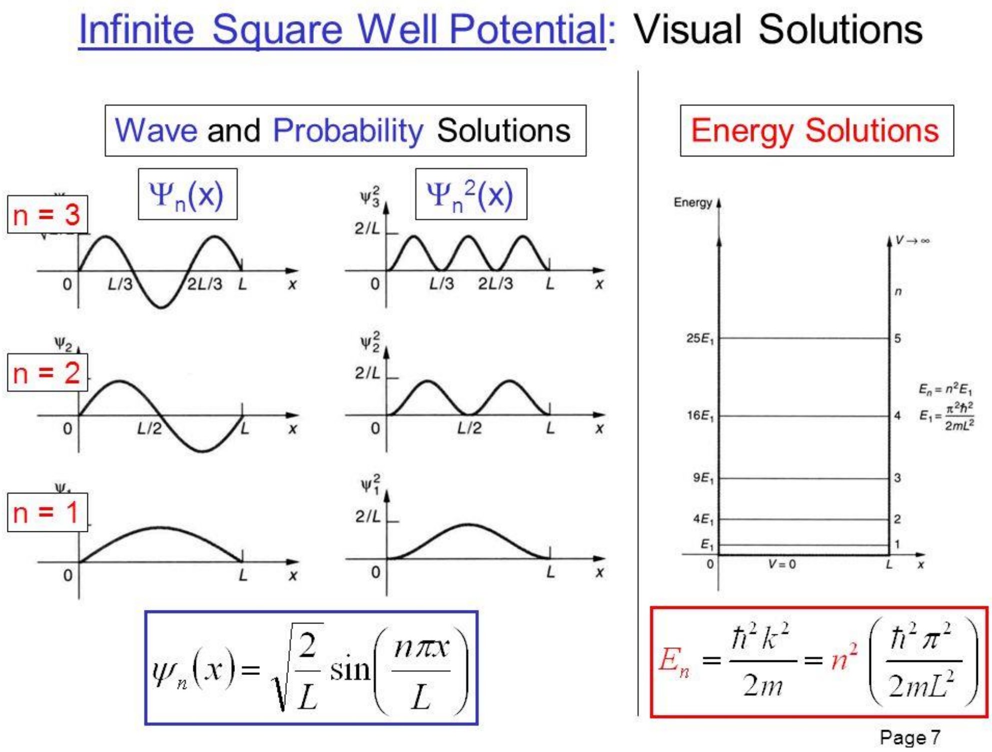
$$V(x) = \begin{cases} 0\,, & 0 < x < a\,, \\ \infty & x \leq 0\,,\, x \geq 0 \end{cases}$$



$$\psi(x) = 0 \quad \text{for } \ x < 0 \text{ and for } \ x > a$$

After applying the Schrodinger equation and the boundary conditions the results we get are summarized below .

$$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}, \quad n = 1, 2, \cdots$$

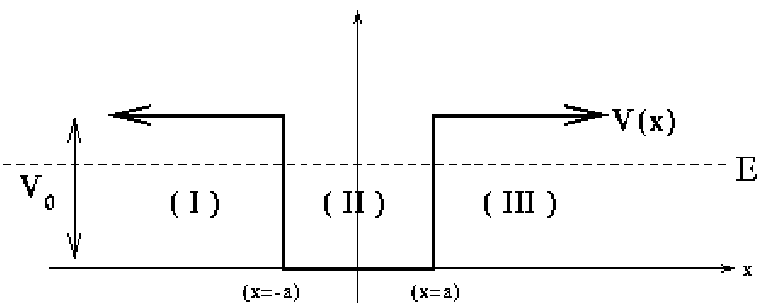


Here , L=a

Finite square well:

We have a potential as following

$$V(x) = \begin{cases} -V_0, & \text{for} \quad |x| \le a, \quad V_0 > 0 \\ 0, & \text{for} \quad |x| \ge a. \end{cases}$$



After applying the Schrodinger equation and some other mathematical calculations the results we get are summarized below .

$$-V_0 < E < 0\,.$$

$$\tilde{E} \;=\; E - (-V_0) = V_0 - |E| > 0$$

$$\frac{d^2\psi}{dx^2} \;=\; -\frac{2m}{\hbar^2}\,(E - V(x))\,\psi = \alpha(x)\psi$$

When ,

$$|x| < a$$

$$\frac{d^2\psi}{dx^2} \;=\; -\frac{2m}{\hbar^2}(E - (-V_0))\psi = -\frac{2m}{\hbar^2}(V_0 - |E|)\psi$$

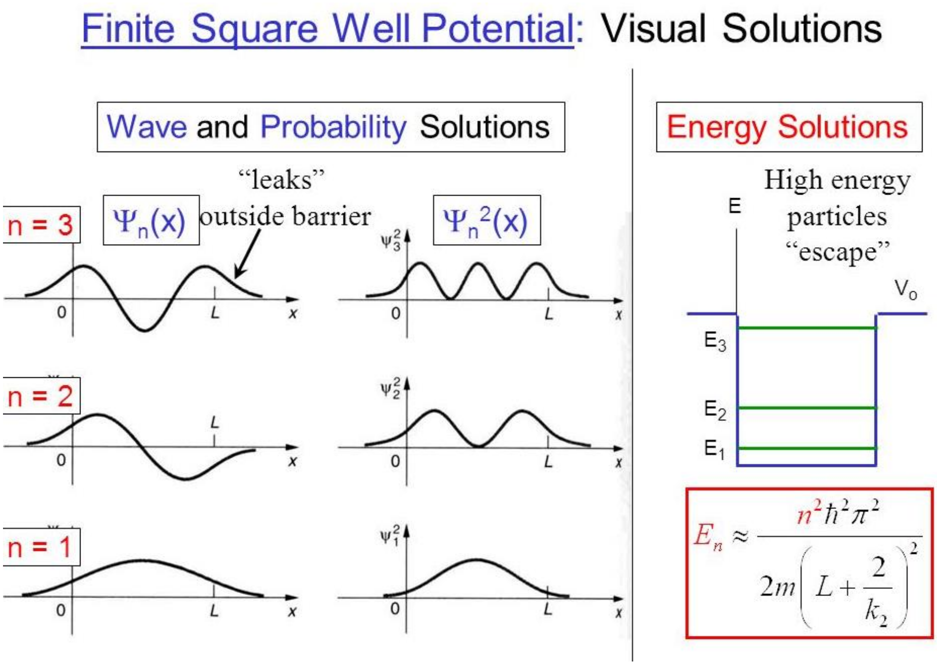
$$k^2 \;\equiv\; \frac{2m}{\hbar^2}(V_0 - |E|) > 0\,, \quad k > 0$$

When wave function is even then ,

$$\psi(x) = \cos kx\,, \quad |x| < a$$

When wave function is odd then ,

$$\psi(x) \;=\; \sin kx\,, \quad |x| < a$$



Here , L=a

When ,

$$|x| > a$$

$$\psi'' \;=\; -\frac{2m}{\hbar^2}\,(E - 0)\,\psi \;=\; \frac{2m|E|}{\hbar^2}\,\psi$$

$$\kappa^2 \;=\; \frac{2m|E|}{\hbar^2}\,, \quad \kappa > 0$$

$$\psi(x) \;=\; A\,e^{-\kappa|x|},\ |x| > a$$

$$k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}$$

Now for our convenience we introduce some unit free quantities which are given below.

$$\eta \equiv ka > 0\,,$$
$$\xi \equiv \kappa a > 0\,,$$
$$z_0^2 \equiv \frac{2mV_0 a^2}{\hbar^2}$$

Harmonic oscillator:

The Hamiltonian of a harmonic oscillator is as follows ,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2\hat{x}^2, \quad \text{where } [\hat{x}, \hat{p}] = i\hbar \quad , \quad V(x) = \tfrac{1}{2}\, m\omega^2 x^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\varphi(x) = E\varphi(x)$$

Now we set ,

$$x = a\,u, \qquad u \text{ unit free}, \quad [a] = L$$

$$\frac{\hbar^2}{ma^2} = m\omega^2 a^2 \quad \rightarrow \quad a^2 = \frac{\hbar}{m\omega}$$

Now after plugging the value of x=au ,

$$-\frac{\hbar^2}{2ma^2}\frac{d^2\varphi(u)}{du^2} + \frac{1}{2}m\omega^2 a^2 u^2\varphi(u) = E\varphi(u)$$

$$-\frac{d^2\varphi(u)}{du^2} + u^2\varphi(u) = \mathcal{E}\varphi(u), \quad \text{where} \quad \mathcal{E} \equiv \frac{2E}{\hbar\omega}\,, \quad E = \tfrac{1}{2}\hbar\omega\,\mathcal{E}$$

Some of the derived results are summarized below .

$$\mathcal{E} = 2n + 1\,, \quad n = 0, 1, 2, \ldots$$

$$h_n(u) \;=\; a_n u^n \;+ a_{n-2} u^{n-2} + \cdots \;, \quad n = 0, 1, 2, \ldots$$

$$\varphi_n(u) = h_n(u) e^{-u^2/2}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$