2. Shock Wave: an example

Gravitational field of a massless particle in Minkowski space is described by the metric

$$ds^2 = -d\hat{u}\left(d\hat{v} + 4p\ln(\rho^2)\,\delta(\hat{u})\,d\hat{u}\right) + dx^2 + dy^2$$

where $\rho^2 = x^2 + y^2$

The particle moves in the $\hat{v}$ direction with momentum p. By calculating geodesics ~~with~~ which ~~crossing~~ cross the shock wave, which is located at $u=0$, we obtain the following two physical effects of such a shock wave (see Appendix A): geodesics have a discontinuity $\Delta\hat{v}$ at $u=0$ and are refracted in the transverse direction. The shift $\Delta\hat{v}$ is given by

$$\Delta\hat{v} = -\frac{4Gp}{c^3}\ln\frac{\rho_0^2}{\ell_{Pl}^2} \qquad (2a)$$

which, for a photon, is

$$\Delta\hat{v} = -\frac{4\ell_{Pl}^2\nu}{c}\ln\frac{\rho_0^2}{\ell_{Pl}^2} \qquad (2b)$$

Where $E = pc = h\nu$ and $\rho = \rho(u)$ and $\rho(0) = \rho_0$.

There is also a refraction effect described by

$$\cot\alpha + \cot\beta = \frac{4Gp}{c^3\rho_0} \qquad (3a)$$

For a proton, is

$$\cot\alpha + \cot\beta = \frac{4\ell_{Pl}^2\nu}{c\rho_0} \qquad (3b)$$

For $m \to 0$, —

$$\frac{\partial y}{\partial \hat{u}} = -\frac{2p}{y_0}\,\mathrm{sgn}\,\hat{u} - pA \;\Rightarrow\; \left.\frac{\partial y}{\partial \hat{u}}\right|_{\varepsilon} - \left.\frac{\partial y}{\partial \hat{u}}\right|_{-\varepsilon} = -\frac{4p}{y_0}$$

$$\Rightarrow y = \begin{cases} -\dfrac{2p}{y_0}\hat{u} - pA\hat{u} + \chi; & \hat{u} > 0 \\[2ex] \dfrac{2p}{y_0}\hat{u} - pA\hat{u} + \chi; & \hat{u} < 0 \end{cases}$$

~~Now, lets sol — $pA\hat{u} = y_0$~~

~~$y = \frac{2p}{y_0}\hat{u} + y$~~

$$\cancel{y = \begin{cases} -\dfrac{2p}{y_0}\tilde{u} + y_0; & \tilde{u} > 0 \\[2ex] \dfrac{2p}{y_0}\tilde{u} + y_0; & \tilde{u} < 0 \end{cases}}$$

$\therefore$ difference b/w the inclinations is $-\dfrac{4p}{y_0}$

General Result

Consider a solution of the vacuum Einstein field eqns of the form

$$d\hat{s}^2 = 2A(u,v)\, du\, dv + g(u,v)\, h_{ij}(x^i)\, dx^i dx^j \quad (4)$$

~~Under what conditions~~

$$A_{,v} = 0 = g_{,v}$$

$$\frac{A}{g}\Delta f - \frac{g_{,uv}}{g} f = 32\pi p A^2 \delta(\rho) \quad (5)$$

where $f = f(x^i)$ represents the shift in v, Δf is the laplacian of f wrt. the 2 metric h_{ij} and the resulting metric described by (B.2) or (B.4). Eqn(5) represents our main result. We now turn to specific examples.

For a plane wave due to a photon in Minkowski Space we have

$$ds^2 = -du\, dv + dx^2 + dy^2 \quad (6a)$$

and thus

$$A = -\frac{1}{2} \quad (6b)$$

$$g = 1$$

$$\therefore A_{,v} = g_{,v} = 0 \;;\; \Delta f = -16\pi p\, \delta(\rho) \quad (7)$$

Where $\rho^2 = x^2 + y^2$. The solution of this equation, unique up to sol^n of homogeneous ~~sol^n~~ eqn, is

$$f = -4p \ln \rho^2 \quad (8)$$

which agrees precisely with (2) and (A.26).

For a spherical wave in Minkowski space we write the metric in the form

$$d\hat{s}^2 = - du\, dv + \frac{1}{4}(v-u)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (10\ a)$$

so that

$$A = -\frac{1}{2}$$
$$g = r^2 = \frac{1}{4}(v - u^2) \qquad (10\ b)$$

But $g_{,v}\big|_{u=0} \neq 0$. Thus $\nexists$ spherical waves (of this form in Minkowski space).

__Kruskal - Szekeres coord.:__

$$d\hat{s}^2 = -\frac{32m^3}{r} e^{-\frac{r}{2m}} du\, dv + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (11\ a)$$

$$\therefore A = -\frac{16m^3}{r} e^{-r/2m} \ ; \ g = r^2 \qquad (11\ b)$$

$$\boxed{uv = -\left(\frac{r}{2m} - 1\right) e^{r/2m}} \qquad (11\ c)$$

$$\Rightarrow u = -\left(\frac{r_{,v}}{2m}\right) e^{r/2m} - \left(\frac{r}{2m} - 1\right)\frac{r_{,v}}{2m} e^{r/2m} = \frac{-r\, r_{,v}}{4m^2} e^{\frac{r}{2m}}$$

$$\Rightarrow r_{,v} \propto u$$

So that all v-derivatives of ρ are proportional to u. Thus, the conditions on the metric coeff. A and g are satisfied at $u=0$.

Furthermore, since $g_{uv} \equiv A$ the condition on f becomes

$$\Delta f - f = 32\pi p g A|_{u=0}\, \delta(\theta)$$
$$= -2\pi\kappa\,\delta(\theta) \qquad (12)$$

where $\kappa = 2^9 m^4 p\, e^{-\frac{2m}{r}}$ and where we have arranged the coordinates so that the photon is at $\theta = 0 = u$.