3. Quantum Field theory in curved spacetime

3.1. Spacetime Structure :

The pseudo-Riemannian metric $g_{\mu\nu}$ associated with the line element

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \quad ; \quad \mu, \nu = 0, 1, \ldots\ldots, (n-1)$$

has signature $(n-2)$. Several coordinate patches with associated $g_{\mu\nu}$ may be needed to cover the entire mfd.

$$g \equiv |\det g_{\mu\nu}|$$

A conformal transform^n of the metric may be described by

$$g_{\mu\nu}(x) \longrightarrow \bar{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x). \qquad (3.1)$$

$\Omega(x)$ is cont. non-vanishing real ~~[illegible]~~ function.

$$\Gamma^\rho_{\mu\nu} \longrightarrow \bar{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \Omega^{-1}\left(\delta^\rho_\mu\, \Omega_{;\nu} + \delta^\rho_\nu\, \Omega_{;\mu} - g_{\mu\nu}\, g^{\rho\alpha}\, \Omega_{;\alpha}\right)$$

$$R^\nu_\mu \longrightarrow \bar{R}^\nu_\mu = \Omega^{-2} R^\nu_\mu - (n-2)\,\Omega^{-1}\left(\Omega^{-1}\right)_{;\mu\rho} g^{\rho\nu} + (n-2)^{-1}\,\Omega^{-n}\left(\Omega^{n-2}\right)_{;\rho\sigma} g^{\rho\sigma}\, \delta^\nu_\mu$$

$$R \longrightarrow \bar{R} = \Omega^{-2} R + 2(n-1)\Omega^{-3}\, \Omega_{;\mu\nu}\, g^{\mu\nu}$$
$$+ (n-1)(n-4)\,\Omega^{-4}\, \Omega_{;\mu}\, \Omega_{;\nu}\, g^{\mu\nu}$$

from these

$$\left[\Box + \frac{1}{4}(n-2) R/(n-1)\right]\phi \longrightarrow \left[\bar{\Box} + \frac{1}{4}(n-2)\bar{R}/(n-1)\right]\bar{\phi}$$
$$= \Omega^{-(n+2)/2}\left[\Box + \frac{1}{4}(n-2)R/(n-1)\right]\phi$$
(3.5)

where

$$\Box\phi = g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = (-g)^{-1/2}\partial_\mu\left[(-g)^{1/2} g^{\mu\nu}\partial_\nu \phi\right] \quad (3.6)$$

$$\bar{\phi}(x) \equiv \Omega^{(2-n)/2}(x)\, \phi(x) \quad (3.7)$$

2-dim Minkowski space:

$$ds^2 = dt^2 - dx^2 \tag{3.8}$$

Lets introduce new coordinates,

$$\left.\begin{aligned} u &= t - x \\ v &= t + x \end{aligned}\right\} \tag{3.9}$$

$$ds^2 = du\, dv \tag{3.10}$$

then in new coordinates

$$g_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.11}$$

Lets perform coordinate transformation

$$\left.\begin{aligned} u' &= 2\tan^{-1} u \\ v' &= 2\tan^{-1} v \end{aligned}\right\} \tag{3.12}$$

where $$-\pi \leq u', v' \leq \pi \tag{3.13}$$

So,

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{u'}{2}\right)\sec^2\left(\frac{v'}{2}\right) du'\, dv' \tag{3.14}$$

then

$$g_{\mu\nu}(u', v') = \frac{1}{8}\sec^2\left(\frac{u'}{2}\right)\sec^2\left(\frac{v'}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.15}$$

Now lets perform a conformal transform^{n} with

$$\Omega^2(x) = \left[\frac{1}{4} \sec^2\left(\frac{1}{2} u'\right) \sec^2\left(\frac{1}{2} v'\right)\right]^{-1}$$

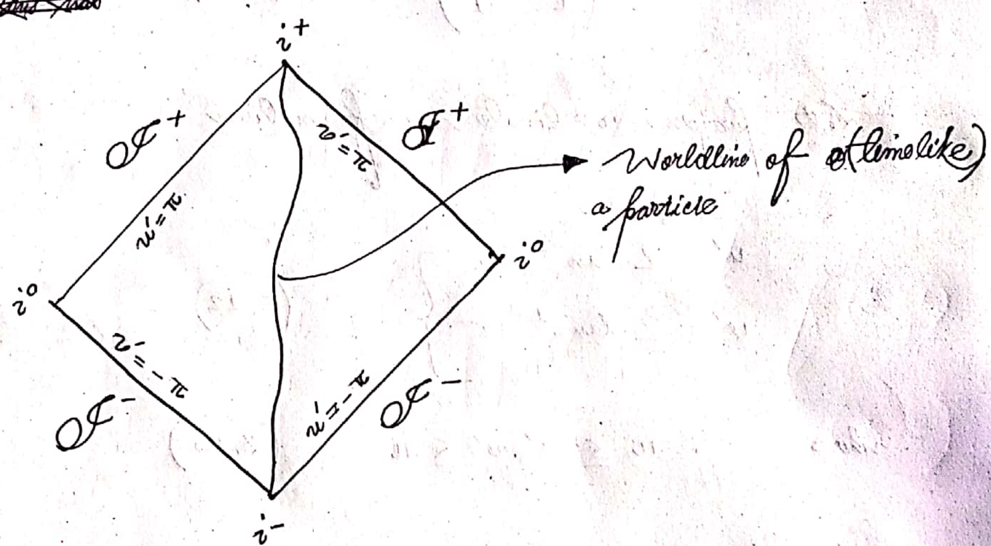
then,

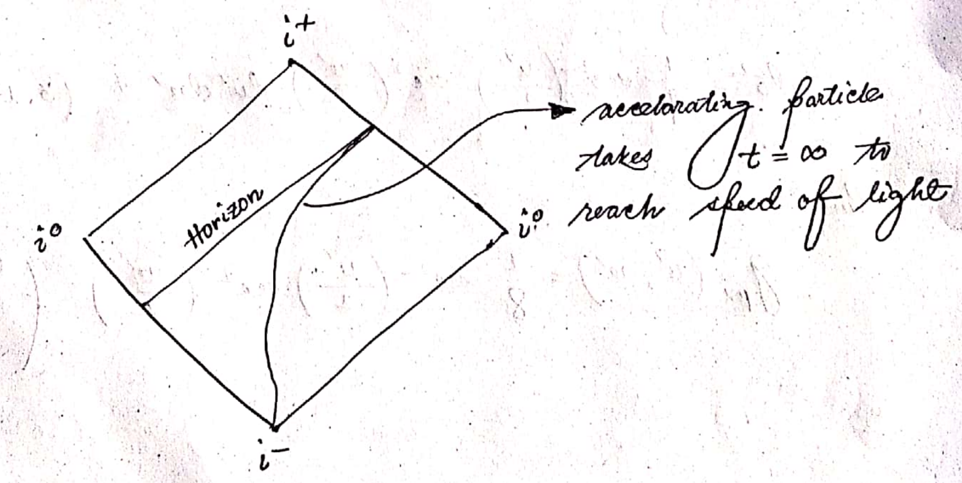
$$g_{\mu\nu}(u', v') \rightarrow \bar{g}_{\mu\nu}(u', v') = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (3.16)$$

and the conformally-related line element is given by

$$d\bar{s}^2 = du'dv' \qquad (3.17)$$

~~Thus Thus~~





4-dim Schwarzschild spacetime:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$
$$- r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{3.18}$$

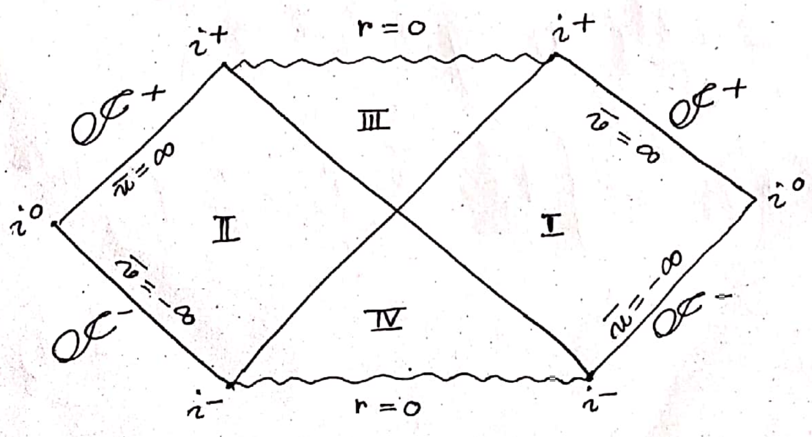
Transforming to Kruskal coordinates defined by

$$\left.\begin{aligned} \bar{u} &= -4Me^{-\frac{u}{4M}} \\ \bar{v} &= 4Me^{\frac{v}{4M}} \end{aligned}\right\} \tag{3.19}$$

where $u = t - r^*$, $v = t + r^*$ and

$$r^* = r + 2M \ln\left|\left(\frac{r}{2M}\right) - 1\right|$$

$$\therefore ds^2 = \left(2M/r\right) e^{-\frac{r}{2M}} d\bar{u}\, d\bar{v} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{3.20}$$



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Note: Frequently we will be restricted to spacetime with special geometrical symmetries. These can be described using K.V. ξ^μ which are soln-s of Killing's eqn:

$$\mathcal{L}_{\xi}\, g_{\mu\nu}(x) = 0 \qquad (3.21)$$

$\mathcal{L}_{\xi}$ → Lie derivative along vector field ξ^μ.

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (3.22)$$

We shall also be interested in the symmetries associated with conformal flatness, when the spacetime is conformal to Minkowski space. The geometry then admits a conformal k.v. field, which satisfies the conformal generalization of (3.21)

$$\mathcal{L}_{\xi}\, g_{\mu\nu} = \lambda(x)\, g_{\mu\nu}(x) \qquad (3.23)$$

where $\lambda(x)$ is some (non-singular, non-vanishing) scalar $f^{\underline{n}}$-s.

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3.2. Scalar field Quantization :

~~[illegible]~~ Lagrangian density $\longrightarrow$

$$\mathcal{L}(x) = \frac{1}{2}\left[-g(x)\right]^{\frac{1}{2}} \left\{ g^{\mu\nu}(x)\, \phi(x)_{,\mu}\, \phi(x)_{,\nu} - \left[m^2 + \xi R(x)\right] \phi^2(x) \right\} \quad (3.24)$$

where $\phi(x)$ is the scalar field and m the mass of the field quanta. The coupling between the scalar field and the gravitational field represented by the term $\xi R \phi^2$, where ξ is ~~[illegible]~~ a numerical factor and $R(x)$ is the Ricci scalar curvature, is included as the only possible local, scalar coupling of this sort with the correct dimensions.

$$\text{Action} = \int \mathcal{L}(x)\, d^n x \quad (3.25)$$

where n is the spacetime dimension.

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$$\left[\Box_x + m^2 + \xi R(x)\right]\phi(x) = 0 \tag{3.26}$$

Two values of ξ are of particular interest : the so called minimally coupled case where $\xi = 0$ and the conformally coupled case

$$\xi = \frac{1}{4}\left[(n-2)/(n-1)\right] \equiv \xi(n) \tag{3.27}$$

In this latter case if $m=0$ the action and hence the field eqn-s are invariant under conformal transform^n (3.1), if the field is assumed to transform as in (3.7) indeed, from (3.5) it is clear that if

$$\left[\Box + \frac{1}{4}(n-2)R/(n-1)\right]\phi = 0$$

then

$$\left[\bar{\Box} + \frac{1}{4}(n-2)\bar{R}/(n-1)\right]\bar{\phi} = 0$$

The scalar product (2.9) is generalized to

$$(\phi_1, \phi_2) = -i\int \phi_1(x)\overleftrightarrow{\partial_\mu}\,\phi_2^*(x)\left[-g(x)\right]^{1/2} d\Sigma^\mu \tag{3.28}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, with n^μ a future-directed unit vector orthogonal to the spacelike hypersurface Σ and $d\Sigma$ is the volume element in Σ. The hypersurface Σ is taken to be a Cauchy surface in the (globally hyperbolic) spacetime and one can show, using Gauss' theorem that the value of (ϕ_1, ϕ_2) is independent of Σ.

There exists a complete set of mode solutions $u_i(x)$ of (3.26) which are orthonormal in the product (3.28), i.e, satisfying

$$(u_i ; u_j) = \delta_{ij} \quad ; \quad (u_i^*, u_j^*) = -\delta_{ij} \quad ; \quad (u_i, u_j^*) = 0 \tag{3.29}$$

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right] \tag{3.30}$$

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Lets, consider, therefore a second complete orthonormal set of modes $\bar{u}_j(x)$. The field ϕ may be expanded in this set also

$$\phi(x) = \sum_j \left[\bar{a}_j \bar{u}_j(x) + \bar{a}_j^{\dagger} \bar{u}_j^{*}(x) \right] \tag{3.32}$$

$$\bar{a}_j |\bar{0}\rangle = 0 \quad \forall j \tag{3.33}$$

[a new Fock Space].

As both sets are are complete, the new modes $\bar{u}_j$ can be expanded in terms of the old:

$$\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^{*} \right) \tag{3.34}$$

Conversely,

$$u_i = \sum_j \left(\alpha_{ji}^{*} \bar{u}_j - \beta_{ji} \bar{u}_j^{*} \right) \tag{3.35}$$

These relations are known as Bogolubov transform^{n} (Bogolubov 1958). The matrices α_{ij}, β_{ij} are called Bogolubov coeff. and by using (3.34) and (3.29) they can be evaluated as

$$\alpha_{ij} = (\bar{u}_i, u_j) \quad ; \quad \beta_{ij} = -(\bar{u}_i, u_j^*) \qquad (3.36)$$

~~Eq~~ Equating the expansions (3.30) and (3.32) and making use of (3.34), (3.35) and the orthonormality of the modes (3.29) one obtains

$$a_i = \sum_j \left(\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger \right) \qquad (3.37)$$

$$\bar{a}_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \right) \qquad (3.38)$$

The Bogolubov coeff. posses the following properties

$$\sum_k \left(\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^* \right) = \delta_{ij} \qquad (3.39)$$

$$\sum_k \left(\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} \right) = 0 \qquad (3.40)$$

It follows immediatly from (3.37) that the two Fock space based on the two choices of modes u_i and $\bar{u}_j$ are different so long as $\beta_{ij} \neq 0$. For example $|\bar{0}\rangle$ will not be annihilated by a_i:

$$a_i|\bar{0}\rangle = \sum_j \beta^*_{ji}\,|\bar{1}_j\rangle \neq 0 \qquad (3.41)$$

in contrast to (3.33). In fact, the expectation value of the operator $N_i = a_i^\dagger a_i$ for the number of u_i-mode particles in the state $|\bar{0}\rangle$ is

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \qquad (3.42)$$

which is to say that the vacuum of the $\bar{u}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode.

If u_j are positive frequency modes w.r.t. some time like K.v. field ξ, satisfying

$$\mathcal{L}_\xi u_j = -i\omega u_j\,, \quad \omega > 0 \qquad (3.43)$$

(c.f. (2.8) which can be written $\mathcal{L}_{\partial_t} u_k = -i\omega u_k$) and $\bar{u}_k$ are a linear combination of u_j alone (not u_j^*), i.e. containing only +ve frequencies with respect to ξ, then

$\beta_{jk} = 0$. In that case $\bar{a}_p |0\rangle = 0$ as well as $a_i |0\rangle = 0$.

Thus, the two sets of modes u_j and $\bar{u}_p$ share a common vacuum state. If any $\beta_{jk} \neq 0$, then $\bar{u}_k$ will contain a mixture of +ve-(u_j) and -ve-(u_j^*) frequency modes, and particles will be present.

More generally the Fock Space based on $|0\rangle$ can be related to that based on $|\bar{0}\rangle$ using the completeness of Fock Space basis elements:

$$|{}^1n_{i_1}, {}^2n_{i_2}, \dots \rangle$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j_1, \dots j_k} |\bar{1}_{j_1}, \bar{1}_{j_2}, \dots \bar{1}_{j_k}\rangle \langle \bar{1}_{j_1}, \dots \bar{1}_{j_k} | {}^1n_{i_1}, \dots \rangle$$

(3.44)

In the notation used here we have, for example

$$|{}^1n_{i_1}\rangle = |1_{i_1}, 1_{i_1}, \dots 1_{i_1}\rangle / ({}^1n_i!)^{1/2}$$