QUANTUM FIELD THEORY IN A GRAVITATIONAL SHOCK WAVE BACKGROUND

Ctirad KLIMČÍK [1]
International School for Advanced Studies (SISSA/ISAS), Strada Costiera 11, I-34014 Trieste, Italy

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A scalar massless non-interacting quantum field theory on an arbitrary gravitational shock wave background is exactly solved. S-matrix and expectation values of the energy-momentum tensor are computed for an arbitrarily polarized sourceless gravitational shock wave and for a homogeneous infinite planar shell shock wave, all performed in any number of space-time dimensions. Expectation values of the energy density in scattering states exhibit a singularity which lies exactly at the location of the curvature singularity found in the infinite shell collision.

1. Introduction

In 1971 Aichelburg and Sexl [1] found an exact solution of the Einstein equations in four dimensions corresponding to a gravitational field generated by a massless particle of given energy. This metric, having the form of an axisymmetric plane-front shock wave, was studied in detail by Dray and 't Hooft [2]. They also constructed a shock wave metric corresponding to a planar shell of null matter with constant energy density along its wavefront and found the exact solutions describing a head-on collision of two such waves [3]. The structure of the singularities obtained was analogous to that found earlier [4,5] in collisions of sourceless shock waves.

Recently Ferrari, Pendenza and Veneziano [6] computed null geodesics in axisymmetric shock wave metrics in D dimensions. For a shock wave of finite (and also infinite) size and constant energy density ρ on its wavefront they found a focusing of null geodesics with a focal "distance" given by [6,7]

$$u_{\mathrm{F}} \stackrel{\mathrm{df}}{=} t_{\mathrm{F}} - z_{\mathrm{F}} = (D-2)(8\pi G\rho)^{-1} \qquad (1)$$

(see also ref. [8] about focusing of null geodesics in gravitational plane wave metrics).

It is natural to believe [6] that focusing of geodesics is an important sign of the generation of singularities in general. Hence it is interesting to know whether we may expect some focusing phenomena at the level of classical or even quantum field theory. Such information would be not only more complete since the geodesical picture can be reobtained in some appropriate limit, but also in a sense more relevant because the dynamics of general relativity is described by the Einstein "wave" equations. It may also happen that one finds no energy density singularity at the quantum level (the indicated "geodesical" singularity would be smeared due to wave or quantum effects). We shall see that we do obtain the singularity of the expectation value of the energy–momentum tensor in the scattering states, describing a massless particle impinging on the infinite planar shell shock wave (=infinite-size wave with constant energy density along its wavefront). This singularity does not survive for the finite-size shock wave suggesting that a generation of the curvature singularity in collision of the finite-size waves may be avoided. It would be, however, too hasty to say that the curvature singularity should disappear. In fact, a large but still finite energy density on the scale of the Schwarzschild radius for given energy might create a singularity as well.

Another reason to study the focusing phenomenon at the quantum level is the necessity of some quantitative expression for the energy density, if one wants to study, for instance, a backreaction on the metric. The singularity of the energy density thus suggests the

[1] On leave from Nuclear Centre, Charles University, V Holešovičkách 2, CZ-18000 Prague 8, Czechoslovakia.

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appearance of a curvature singularity via the Einstein equations.

In section 2 we solve exactly the generally covariant Klein–Gordon equation in an arbitrary shock wave background and give a general formula for the S-matrix. In particular, for the case of the Aichelburg–Sexl (= AS) metric in four dimensions one recovers the S-matrix found by 't Hooft [9] justifying thus his nice argument based on the behaviour of null geodesics in this metric (see refs. [9,2,10]).

In section 3 we derive a formula for the expectation values of the energy–momentum tensor in the scattering states. Due to special properties of the background, one does not encounter the usual problem with the renormalization of this quantity. We compute explicitly those expectation values for a large class of shock waves containing, among others, an arbitrarily polarized sourceless shock wave and also an infinite planar shell, all performed in any number of space-time dimensions.

Section 4 is devoted to a study of the focusing phenomena. The interesting structure of real and virtual focal points is observed in the case of the sourceless wave. The singularity of energy density at the location given by (1) obtained in the infinite shell case is discussed. Then we give an argument showing the smearing of the singularity for the case of the finite-size shock wave.

At the end we draw conclusions and discuss also briefly some motivations, stemming from the results obtained and concerning the smearing of gravitational singularities due to string effects.

2. General formalism

We call the gravitation shock wave any metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + f(\tilde{x})\delta(u)\mathrm{d}u^2 + \sum_i \mathrm{d}\tilde{x}_i^2 \,, \tag{2}$$

where $u \equiv t-z$, $v \equiv t+z$ and $\tilde{x}$ denotes $D-2$ transverse coordinates. The metric (2) corresponds to the shock wave moving along the positive z-axis with the wavefront $u=0$. Following refs. [2,6] one finds the only nonvanishing component of the Ricci tensor to be

$$R_{uu} = -\tfrac{1}{2}\tilde{\Delta}f(\tilde{x})\delta(u) \,, \tag{3}$$

where $\tilde{\Delta}$ is the laplacian in the transverse coordinates. Thus, unless the function f is "too singular" [#1], the metric (2) is always the solution of the Einstein equations with the source given by

$$\mathcal{T}_{uu} = \rho(\tilde{x})\delta(u) \,, \tag{4}$$

where

$$\rho(\tilde{x}) = -(16\pi G)^{-1}\tilde{\Delta}f(\tilde{x}) \,, \tag{5}$$

and all other components of $\mathcal{T}_{\mu\nu}$ being zero. (G is the gravitational constant.)

Here we note only that, given the desired energy profile $\rho(x)$ of the shock wave, we still do not have a unique solution of (5) unless we fix some supplementary conditions on f.

On both sides of the wavefront the spacetime is flat, therefore we can easily apply a general strategy of building a quantum field theory on a curved background (for a review see ref. [11]). We need to find in-modes of the generally covariant Klein–Gordon equation

$$-\frac{\partial^2}{\partial u \partial v}\varphi - \delta(u)f(\tilde{x})\frac{\partial^2}{\partial v^2}\varphi + \tfrac{1}{4}\tilde{\Delta}\varphi = 0 \,. \tag{6}$$

The in-modes must look like the free ones

$$u_{\mathrm{Free}\,k_-,\tilde{k}}(\tilde{x}, u, v)$$

$$= N_k \exp[\mathrm{i}(-k_- v - k_+ u + \tilde{k}\tilde{x})] \tag{7}$$

for $u<0$. Here the $k_{(D-1)}$, $\tilde{k}$ are the components of the $(D-1)$-momentum

$$k_\pm \overset{\mathrm{df}}{=} \tfrac{1}{2}(k_0 \pm k_{D-1})$$

and

$$N_k \overset{\mathrm{df}}{=} [(2\pi)^{D-1} 2k_-]^{-1/2} \,. \tag{8}$$

The normalization factor N_k ensures the usual normalization of the modes and also the annihilation and the creation operators with respect to the measure $\mathrm{d}k_-\,\widetilde{\mathrm{d}k}$ (we note that $\mathrm{d}k_-\,\widetilde{\mathrm{d}k} = (k_-/k_0)\mathrm{d}k_{D-1}\,\widetilde{\mathrm{d}k}$). If we know the looks of the in-modes in the out-region ($u>0$) it is easy to decompose them in terms of out-

[#1] We do not need for our purposes to specify the allowed singularities of the function f. We note, however, that f need not be, in general, regular. This happens, for instance, for the AS metric.

modes, to compute the corresponding Bogoliubov transformation between the $a_{\rm in}$, $a^+_{\rm in}$ and the $a_{\rm out}$, $a^+_{\rm out}$ operators and consequently the S-matrix elements.

We look for the in-solutions of the form

$$u_{k\,\rm in}(u,v,\tilde{x}) = N_k \exp(-ik_-v)\psi_{\tilde{k}\,\rm in}(u,\tilde{x}) \ . \tag{9}$$

From (6) we see that the functions $\psi_{\tilde{k}\,\rm in}$ fulfil the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial u}\psi_{\tilde{k}\,\rm in} = [-\tilde{\Delta}/4k_- - f(\tilde{x})\delta(u)k_-]\psi_{\tilde{k}\,\rm in} \ . \tag{10}$$

Hence for continuing the (known) solution from the in- to the out-region we need to know the kernel $G(\tilde{x}'', u'', \tilde{x}', u')$ of eq. (10). We see, however, that the interaction term is instantaneous in the "time" $u=0$, therefore the evolution for $u<0$ and for $u>0$ respectively is trivially governed by the free hamiltonian. We are interested, consequently, only in the quantity $G(\tilde{x}'', 0^+, \tilde{x}', 0^-)$.

We have

$$G(\tilde{x}'', u'', \tilde{x}', u') = \int\limits_{\substack{\tilde{q}(u')=\tilde{x}'\\ \tilde{q}(u'')=\tilde{x}''}} \mathscr{D}\tilde{q}(u)$$

$$\times\exp\left(\mathrm{i}\int\limits_{u'}^{u''} \mathrm{d}u[k_-\tilde{q}^2 + f(\tilde{q})k_-\delta(u)]\right)$$

$$= \int \mathscr{D}\tilde{q}(u)\exp[\mathrm{i}k_- f(\tilde{q}(0))]\exp\{\mathrm{i}S_{\rm Free}[\tilde{q}(u)]\}$$

$$= c\int \mathrm{d}\tilde{x}\exp[\mathrm{i}k_- f(\tilde{x})]$$

$$\times G_{\rm Free}(\tilde{x}'', u'', \tilde{x}, 0)G_{\rm Free}(\tilde{x}, 0, \tilde{x}', u') \ , \tag{11}$$

where $c=1$ in order to obtain a correct expression for the $f=0$ case.

Thus

$$G(\tilde{x}'', 0^+, \tilde{x}', 0^-)$$

$$= \delta(\tilde{x}'' - \tilde{x}')\exp[\mathrm{i}k_- f(\tilde{x}')] \ , \tag{12}$$

and

$$\psi_{\tilde{k}\,\rm in}(0^+, \tilde{x}) = \exp\{\mathrm{i}[\tilde{k}\tilde{x} + k_- f(\tilde{x})]\} \ . \tag{13}$$

Though elegant this path integral argument may seem to be somewhat formal, therefore we add another derivation. Regularizing the δ-function in (10) by an expression

$$\delta_\varepsilon(u) = \frac{1}{2\varepsilon}[\theta(u+\varepsilon) - \theta(u-\varepsilon)] \ , \tag{14}$$

we have

$$\mathrm{i}\frac{\partial}{\partial u}\psi_{\tilde{k}\,\rm in} = \hat{H}_\varepsilon(u)\psi_{\tilde{k}\,\rm in}$$

$$\stackrel{\rm df}{=} [-\tilde{\Delta}/4k_- - f(\tilde{x})\delta_\varepsilon(u)k_-]\psi_{\tilde{k}\,\rm in} \ . \tag{15}$$

A solution of (15) must be a continuous function, since the right-hand side is bounded. The only non-trivial propagation occurs in the interval $-\varepsilon < u < \varepsilon$. One has

$$\psi_{\tilde{k}\,\rm in}(+\varepsilon, \tilde{x})$$

$$= \int \mathrm{d}\tilde{q}\langle\tilde{x}|\exp[-\mathrm{i}\hat{H}_\varepsilon(0)(2\varepsilon)]|\tilde{q}\rangle\psi_{\tilde{k}\,\rm in}(-\varepsilon, \tilde{q}) \ , \tag{16}$$

since in this interval the hamiltonian is time independent. Performing now a limit $\varepsilon\to 0$ we have immediately (13).

From (6), (7), (10) we see that the energy $k_- + k_+$ remains positive for all out-modes in the decomposition of the in-mode, therefore no particle production is seen.

Now we look for a function (or a distribution) $\phi(k_-, \tilde{k}, l_-, \tilde{l})$ with the property (for $u>0$)

$$\int \mathrm{d}l_-\,\tilde{\mathrm{d}}l\,\phi(k,l)u_{l_-,\tilde{l}\,\rm out} = u_{k_-,\tilde{k}\,\rm in} \ , \tag{17}$$

and hence

$$a_{l_-,\tilde{l}\,\rm out} = \int \mathrm{d}k_-\,\tilde{\mathrm{d}k}\,\phi(k,l)a_{k_-,\tilde{k}\,\rm in} \ . \tag{18}$$

Clearly,

$$\phi(k,l) = \delta(k_- - l_-)\tilde{\phi}(k,l) \ . \tag{19}$$

Thus

$$\int \mathrm{d}\tilde{l}\,\tilde{\phi}(k,l)\exp(\mathrm{i}\tilde{l}\tilde{x}) = \exp[\mathrm{i}\tilde{k}\tilde{x} + \mathrm{i}k_- f(\tilde{x})] \ , \tag{20}$$

and

$$\tilde{\phi}(k,l) = \frac{1}{(2\pi)^{D-2}} \int d\tilde{x} \exp\{i[(\tilde{k}-\tilde{l})\tilde{x}+k_- f(\tilde{x})]\} . \quad (21)$$

Knowing $\tilde{\phi}(k,l)$, it is easy to compute the S-matrix elements. For example

$$\langle 0 | a_{l_{D-1},\tilde{l}\,\text{out}} a^+_{k_{D-1},\tilde{k}\,\text{in}} | 0 \rangle = (k_-/\sqrt{k_0 l_0})\tilde{\phi}(k,l)\delta(k_- - l_-) , \quad (22)$$

where the factor $k_-/\sqrt{k_0 l_0}$ takes care of a transition from the light-cone formalism to the usual one.

We note that for the AS metric in four dimensions, e.g.,

$$f(\tilde{x}) = -4P\ln(\tilde{x}/x_0)^2 , \quad (23)$$

the result of ref. [9] can be easily recovered (x_0 is an irrelevant scale and P a parameter corresponding to the energy of the massless particle generating the wave).

Thus we solved the field theory on the shock wave background reducing the problem to mere integration. Now we turn our attention to several interesting particular cases.

3. The energy–momentum tensor

We wish to compute expectation values of the energy–momentum tensor between scattering states. Preparing the system in some in-state with the sharp value $k_-\tilde{k}$ we are interested in the mean value of the energy–momentum density in the out-region. Due to the nice property of shock wave backgrounds, namely, they are essentially glued from two flat pieces, we need not care for the usual problems connected with the computation of the stress-tensor in a general curved background for a review, see ref. [11]). Indeed, in our case the energy–momentum tensor, as a local quantity, must be outside of the wavefront of the same form as in the Minkowski spacetime. Thus

$$T_{\mu\nu}(x)_{\text{outside}} = :\varphi_{,\mu}\varphi_{,\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}\varphi_{,\sigma}\varphi_{,\rho}: , \quad (24)$$

where

$$\varphi(x) = \int dk_- \, d\tilde{k} \, (a_{k\,\text{in(out)}} u_{k\,\text{in(out)}} + a^+_{k\,\text{in(out)}} u^*_{k\,\text{in(out)}}) , \quad (25)$$

where the double dots denote the usual normal ordering operation (which in our case is the same with respect to in- as to out-operators).

We have

$$\langle 0 | a_{k\,\text{in}} T_{\mu\nu}(x) a^+_{k\,\text{in}} | 0 \rangle_{\text{outside}} = (\delta^\sigma_\mu \delta^\rho_\nu - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}) \times (u_{k\,\text{in},\sigma} u^*_{k\,\text{in},\rho} + u_{k\,\text{in},\rho} u^*_{k\,\text{in},\sigma}) . \quad (26)$$

Knowing the fundamental quantity $\tilde{\phi}(k,l)$ and using (14) we can compute $u_{k,\,\text{in}}$ and hence the expectation values of the energy–momentum tensor.

Consider now a particular shock wave (2) with

$$f_A(\tilde{x}) = A_{ij}\tilde{x}_i\tilde{x}_j , \quad (27)$$

where A is some symmetric matrix. We shall always work in the transverse coordinate system, in which the matrix A is diagonal. Without loss of generality A can be considered non-singular, since otherwise the integration in (21) would trivially give rise to δ-functions in the transverse momenta corresponding to the zero eigenvalues.

If $\text{Tr}\,A=0$, we have an arbitrary polarized sourceless wave (see ref. [5]). The $A_{ij}=-a\delta_{ij}$ case corresponds to an infinite planar shell of null matter with a constant energy density along the wavefront

$$\rho = (D-2)a/16\pi G \quad (28)$$

(modulo some sourceless shock wave, see the remark after eq. (5)). The basic quantity $\phi(k,l)$ is easily computed to give

$$\tilde{\phi}_A(k,l) = \frac{1}{(2\pi)^{D-2}} \int d\tilde{x} \exp\{i[(\tilde{k}-\tilde{l})\tilde{x}+l_- a_j \tilde{x}_j^2]\} = (4\pi l_-)^{-(D-2)/2}\sqrt{\det iA^{-1}} \times \exp[-ia_j^{-1}(\tilde{k}-\tilde{l})_j^2/4l_-] , \quad (29)$$

where

$$A_{ij} = a_i\delta_{ij} . \quad (30)$$

Thus for $u>0$

$$u_{k\,\mathrm{in}}(\tilde{x},u,v)$$

$$=N_k \exp(-\mathrm{i}k_- v)(4\pi k_-)^{(2-D)/2}\sqrt{\det \mathrm{i}A^{-1}}$$

$$\times \int \mathrm{d}\tilde{l} \exp\Big[-\frac{\mathrm{i}\tilde{l}_j^2}{4k_-}(u-u_{\mathrm{F}_j})$$

$$-\mathrm{i}\sum_j \left(\frac{\tilde{k}_j u_{\mathrm{F}_j}}{2k_-}-\tilde{x}_j\right)\tilde{l}_j+\mathrm{i}\frac{u_{\mathrm{F}_j}\tilde{k}_j^2}{4k_-}\Big], \qquad (31)$$

where $a_i^{-1}=u_{\mathrm{F}_i}$.

The integral (31) is gaussian and one sees directly a singularity structure of the in-mode. Unless $u=u_{\mathrm{F}_j}$ for some j the integration gives

$$u_{k\,\mathrm{in}}(\tilde{x},u,v)$$

$$=N_k \exp(-\mathrm{i}k_- v)\left[\sqrt{\det \mathrm{i}A^{-1}}/\sqrt{\det \mathrm{i}R(u)}\right]$$

$$\times \exp(\mathrm{i}u_{\mathrm{F}_j}\tilde{k}_j^2/4k_-)$$

$$\times \exp\left[\mathrm{i}k_- \sum_j \frac{1}{u-u_{\mathrm{F}_j}}\left(\tilde{x}_j-u_{\mathrm{F}_j}\frac{\tilde{k}_j}{2k_-}\right)^2\right], \qquad (32)$$

where $R(u)_{ij}=(u-u_{\mathrm{F}_j})\delta_{ij}$.

If $u=u_{\mathrm{F}_j}$ for some j, the integration is again trivial, nevertheless the result is somewhat cumbersome, therefore we do not list it here in the general case.

One gets typically a product of $\delta(\tilde{x}_j-\tilde{k}_j u_{\mathrm{F}_j}/2k_-)$ (no summing) and of expressions of the kind (32). An interesting case is that in which one has full degeneracy, e.g. $u_{\mathrm{F}_j}=u_{\mathrm{F}}$ for all j. Then for $u=u_{\mathrm{F}}$ we get

$$u_{k\,\mathrm{in}}(\tilde{x},u_{\mathrm{F}},v)$$

$$=N_k \exp(-\mathrm{i}k_- v)(k_-/\pi)^{(2-D)/2}\sqrt{\det \mathrm{i}A^{-1}}$$

$$\times \exp\left[\mathrm{i}\frac{u_{\mathrm{F}}\tilde{k}^2}{4k_-}\delta\left(\tilde{x}-\frac{\tilde{k}u_{\mathrm{F}}}{2k_-}\right)\right]. \qquad (33)$$

Thus we obtained a "focusing" of the in-mode on the line $u=u_{\mathrm{F}}$, $\tilde{x}=\tilde{k}u_{\mathrm{F}}/2k_-$ and v arbitrary.

Now using (26) the computation of the expectation values of the energy–momentum tensor is straightforward though a bit tedious. We give explicitly the mean value of the energy density in the head-on scattering state ($\tilde{k}=0$) since we shall need it for further discussion. Unless $u=u_{\mathrm{F}_j}$ for some j one gets

$$\langle 0|a_{k_-,\tilde{k}=0\,\mathrm{in}} T_{00}(x) a^+_{k_-,\tilde{k}=0\,\mathrm{in}}|0\rangle$$

$$=2N_k^2|\det R(u)A|^{-1}$$

$$\times\left[\tfrac{1}{4}\mathrm{Tr}^2 R^{-1}(u)+k_-^2\left(\sum_j \frac{\tilde{x}_j^2}{(u-u_{\mathrm{F}_j})^2}+1\right)^2\right]. \qquad (34)$$

We note that for $A\to 0$ (34) gives $2N_k^2 k_-^2$, which is the free field result.

4. Discussion

We start the discussion with a remark that not all poles $u=u_{\mathrm{F}_j}$ are "physical", since the formula (34) is valid only for $u>0$. If some $u_{\mathrm{F}_j}<0$ one has a virtual focusing. In the classical language it corresponds to the focusing of some family of geodesics analytically continued from the out-region. As an example of such a situation one can take the sourceless wave for which $\mathrm{Tr}\,A=-\sum u_{\mathrm{F}_j}=0$ as follows from (5) and (27).

In the case of the infinite planar shell

$$f(\tilde{x})=-\tilde{x}^2/u_{\mathrm{F}},$$

$$u_{\mathrm{F}}=(D-2)(8\pi G\rho)^{-1}, \qquad (35)$$

formula (34) gives

$$\langle k_-,0|T_{00}(x)|k_-,0\rangle$$

$$=2N_k^2\frac{u_{\mathrm{F}}^{D-2}}{|u-u_{\mathrm{F}}|^{D-2}}$$

$$\times\left[\frac{(D-2)^2}{4(u-u_{\mathrm{F}})^2}+k_-^2\left(\frac{\tilde{x}^2}{(u-u_{\mathrm{F}})^2}+1\right)^2\right] \qquad (36)$$

for $u\neq u_{\mathrm{F}}$, $u>0$. The ρ in (35) denotes the energy density of the source null matter on the wavefront (not to be confused with the energy density of the quantum field).

Using (31) and (26) one has for $u=u_{\mathrm{F}}$

$$\langle k_-,0|T_{00}(x)|k_-,0\rangle=0 \quad \text{for } \tilde{x}\neq 0,$$

$$=\infty \quad \text{for } \tilde{x}=0. \qquad (37)$$

This result is obtained taking care of the usual damping ε-regulator needed for computing the gaussian integrals. The singularity for $\tilde{x}=0$ is an ill-behaved expression due to the multiplication of distributions.

For later discussion it is worthwhile to review the behaviour of null geodesics in the shock wave (35) [6]. One chooses u as an affine parameter and, for the family G of geodesics perpendicular to the wavefront, one gets:

$$r(u)=b-b(u/u_F)\theta(u)\,, \tag{38a}$$

$$v(u)=-(b^2/u_F)\theta(u) + (b^2/u_F^2)u\,\theta(u)+a\,, \tag{38b}$$

$$\varphi(u)=\varphi_0\,. \tag{38c}$$

Here b is the impact parameter, r is the transverse radial coordinate, φ the set of angular coordinates, a marks the "initial position" on the v axis and $\theta(u)$ the usual step function.

Now keep a fixed and vary b. From (38) follows that such a subfamily of geodesics gets focused at

$$u=u_F, \quad v=a, \quad \tilde{x}=0\,. \tag{39}$$

Varying also a one spreads the focal point (39) over the v-axis. Moreover, no geodesic from the G crosses the points $u=u_F$, $\tilde{x}\neq 0$, but one sees an "accumulation" of geodesics with large impact parameter b near the plane $u=u_F$. We observe that the described geometrical optics picture corresponds very well to the quantum results (36) and (37).

The energy density given by eqs. (36), (37) obviously depends on the state in which we calculate the mean value. We picked up the translation invariant scattering state being interested in modelling null matter in some translational invariant state, impinging perpendicularly on the shock wave. The family of geodesics G accomplishes this goal at the level of geometrical optics. One observes that both approaches give essentially the same singularity structure in the sense described above.

We stress, however, that the family G is not, strictly speaking, a classical limit of the scattering state $|k_-, 0\rangle$ in the sense that the expectation value of the quantum field in this state would look locally like a plane wave with an eikonal giving rise to the family G (for a discussion of the classical limit and the geometrical optics limit, see, for instance, refs. [12,13]). The reason why both pictures are analogous relies on the important role played in our analysis by the Schrödinger equation (10). From (32) and (33) it is easy to see that the "time" dependent potential term in (10) has an amusing property, namely it changes in the course of evolution the state with a sharp value of momentum into a state with a sharp value of the coordinate at the "time" $u=u_F$. One has for $u<0$

$$\psi_{\tilde{k}\,\mathrm{in}}(u,\tilde{x})=\exp[-\mathrm{i}(\tilde{k}^2/2m)u+\mathrm{i}\tilde{k}\tilde{x}] \overset{\mathrm{df}}{=} \exp[\mathrm{i}S(\tilde{x},u,\tilde{k})]\,, \tag{40}$$

and for $u>0$

$$\psi_{\tilde{k}\,\mathrm{in}}(u,\tilde{x})=A\exp\left(\mathrm{i}\frac{m}{2}\frac{[\tilde{x}-(\tilde{k}/m)u_F]^2}{u-u_F}\right) \overset{\mathrm{df}}{=} A\exp[\mathrm{i}S'(\tilde{x},u,\tilde{q})]\,, \tag{41}$$

where we put $m=2k_-$, $\tilde{q}=(\tilde{k}/m)u_F$ and A does not depend on $\tilde{x}$.

We observe that the functions $S(\tilde{x}, u, \tilde{k})$ and $S'(\tilde{x}, u, \tilde{q})$ are both the full integrals of the Hamilton–Jacobi equation (HJE) for a free particle with mass m. Having some full integral of the HJE $\Sigma(\tilde{x}, u, \tilde{\alpha})$, it is easy to find trajectories by expressing $\tilde{x}$ as a function of u, $\tilde{\alpha}$, and $\tilde{\beta}$ from the equation (see ref. [14])

$$\frac{\partial\Sigma}{\partial\tilde{\alpha}}(\tilde{x},u,\tilde{\alpha})=\tilde{\beta}\,. \tag{42}$$

The $\tilde{\beta}$ is obviously a canonically conjugated variable of the $\tilde{\alpha}$.

Hence giving the full integral of the HJE, fixing $\tilde{\alpha}$ and varying the canonically conjugate variable $\tilde{\beta}$, the family of the classical trajectories is defined. Thus one may say loosely, at least in our special case, that a quantum eigenstate of the observable $\tilde{\alpha}$ "contains" all classical states with $\tilde{\alpha}$ fixed and $\tilde{\beta}$ varying.

If, in particular, one considers the full integrals S, S' (from eqs. (40), (41)) for $\tilde{k}=0$ (and, consequently, $\tilde{q}=0$) one finds that the corresponding family of classical trajectories is precisely the family (38a) with varying impact parameter b and ϕ_0. Thus the behaviour of the phase of the in-mode $\psi_{\tilde{k}\,\mathrm{in}}$ is dictated by the behaviour of the beam of all classical trajectories with the incident momentum $\tilde{k}$.

In the same spirit the "geodesical content" of the state $|k_-, 0\rangle$ would be the full family G with varying positions in both v- and x-axis, namely both a and b, ϕ_0 in (38). In this sense, the scattering state $|k_-, 0\rangle$ corresponds to the family G, therefore our results may be said to be the quantum version of the geodesical focusing obtained before.

The analysis which we have just performed, sheds some light also on the case of the finite-size shock wave ($\rho=0$ for $|\tilde{x}|>R$). For $u>0$ the trajectories with large transverse momenta are not present in the family G since the trajectories with $b>R$ are only slightly deflected (slightly because they still "feel" the null matter in the domain $|\tilde{x}|<R$). Therefore, the corresponding quantum state cannot be a true position state in which, as we have seen, all momenta must be present (for $u>0$ the momenta here play the role of the parameter $\tilde{\beta}$ in eq. (42)). From the Heisenberg uncertainty principle one expects a spreading of the focal point over the scale h/Λ, where Λ is the momentum cut-off.

5. Conclusions and outlook

The implications of our results are seen mainly at the conceptual level. The formalism presented can constitute a step towards the physics of gravitational singularities at the quantum level. We knew previously about the singularity in the collision of two infinite planar shells [3]. Taking into account the backreaction on the metric, our results show that the singularity should survive in the mixture of quantum and classical language which we have used. A next natural step would be to study the full collision problem at the quantum level. Thus we arrive at the string picture, since it is believed that a consistent theory of quantum gravity is just the string theory. As a short digression we add several remarks concerning the subject.

As a consistent theory of quantum gravity the string theory, of course, should provide us in general with some information about the nature of curvature singularities which, hopefully, might occur only as a consequence of neglecting the string scale and should be smeared in a complete theory. Unfortunately one encounters many fundamental as well as technical obstacles in following such a programme. Therefore it is very important to have an example of a classical singularity which can be relatively easily controlled from the technical point of view and which occurs in a process admitting a treatment within the present days' framework of string theory. The scattering singularity discussed above may be a good candidate because scattering processes, at least in some kinematic domain, can be addressed in the string picture [15–17] (we mention also that some preliminary considerations concerning the problem of curvature singularities from the point of view of strings can be found in refs. [15,6]).

What we said suggests that, as the next step towards the full string collision problem, it would be interesting to study the string theory on the shock wave background, as in fact we intend to do soon. Although this approach would constitute itself only an approximation of the full collision (which is probably extremely difficult to compute), it may shed further light on the problem and may, as well, possess the advantages of our field theory treatment, namely technical simplicity and the exact control of the model.

Finally, the subject of field theory on the shock wave background deserves several words by itself. As we have seen, one can hardly overestimate the properties of this kind of background from the technical point of view. Solving the theory amounts, essentially, to matching two flat pieces of the manifold. Still, nontrivial, physically interesting phenomena occur, as we have seen.

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