4.5 Quantum Field Theory in Curved Spacetime

Lets consider, 2D ~~metric~~ Minkowski Space with metric (3.8) and (3.10)

$$ds^2 = d\bar{u}\, d\bar{v} = dt^2 - dx^2 \quad (4.66)$$

Under coord. transformation,

$$t = a^{-1} e^{a\xi} \sinh a\eta \quad (4.67)$$

$$x = a^{-1} e^{a\xi} \cosh a\eta \quad (4.68)$$

where $a = \text{const} > 0$ and $-\infty < \eta, \xi < \infty$ or equivalently

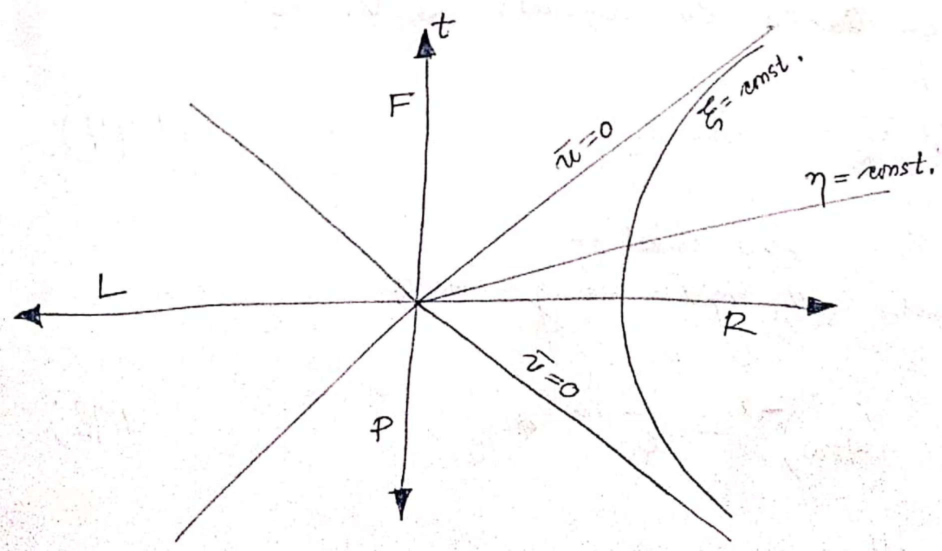
$$\bar{u} = -a^{-1} e^{-au} \quad (4.69)$$

$$\bar{v} = a^{-1} e^{av} \quad (4.70)$$

where $u = \eta - \xi$ and $v = \eta + \xi$

So (4.66) becomes

$$ds^2 = e^{2a\xi}\, du\, dv = e^{2a\xi} (d\eta^2 - d\xi^2) \quad (4.71)$$



The coordinates (η, ξ) cover only a quadrant of Minkowski Space, (i.e. $x > |t|$) shown in the figure. Lines of constant η are straight ($x \propto t$) and lines of constant ξ are hyperbolae (according to the eqn $x^2 - t^2 = a^{-2} e^{2a\xi} = \text{const.}$)

(4.72)

Lets, consider the quantization of massless ~~particle~~ scalar field ϕ in two dimensional Minkowski Spacetime. The wave eqn

$$\Box\phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi \equiv \frac{\partial^2 \phi}{\partial \bar{u}\, \partial \bar{v}} = 0 \qquad (4.75)$$

possesses standard orthonormal mode solns

$$\bar{u}_k = (4\pi\omega)^{-1/2} e^{ikx - i\omega t} \qquad (4.76)$$

(i.e. (2.11) with $n = 2$) where $\omega = |k| > 0$ and $-\infty < k < \infty$. These modes are +ve frequency with respect to the timelike K.V. ∂_t.

$$\partial_t \bar{u}_k = -i\omega \bar{u}_k \qquad (4.77)$$

The modes with $k > 0$ consist of right-moving waves

$$(4\pi\omega)^{-1/2} e^{-i\omega \bar{u}} \qquad (4.78)$$

along the rays $\bar{u} = \text{const.}$

~~while~~ while for $k<0$ one has left moving waves along $\bar{v} = \text{const.}$

$$(4\pi\omega)^{-1/2} e^{-i\omega\bar{v}} \tag{4.79}$$

In Rindler ~~Space~~ region R and L one may the system ~~as~~ (Fock Space) based on u_k. The metric (4.71) is conformal to the whole of Minkowski space for under the conformal transformⁿ $g_{\mu\nu} \to e^{-2a\xi} g_{\mu\nu}$. ~~reduces~~ (4.71) reduces to $d\eta^2 - d\xi^2$ with $-\infty < \eta, \xi < \infty$.

Now, the wave eqn. is conformally invariant. So, we can write it in Rindler coords. as

$$e^{2a\xi} \Box\phi = \left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\xi^2}\right)\phi \equiv \frac{\partial^2\phi}{\partial u\,\partial v} = 0 \tag{4.80}$$

For which the existing mode solns

$$u_k = (4\pi\omega)^{-1/2} e^{ik\xi \pm i\omega\eta} \tag{4.81}$$

$$\omega = |k| > 0 \quad , \quad -\infty < k < \infty$$

The upper sign in (4.81) applies in ~~rg~~ region L, the lower sign in region R. The presence of (sign) change can either be regarded as due to the 'time' reversal in L, or due to the fact right moving wave in R moves towards increasing value of ξ, while in L it moves towards decreasing value of ξ. In any case, the modes (4.81) are those which satisfy the normalization condition (3.29). They are +ve frequency w.r.t. the timelike Killing vector $+\partial_\eta$ in R and $-\partial_\eta$ in L, satisfying

$$\partial_{K.C.}\, u_K = -i\omega\, u_K \tag{4.82}$$

Lets define

$$
{}^{R}u_K = \begin{cases} (4\pi\omega)^{-1/2}\, e^{ik\xi - i\omega\eta}, & \text{in } R \\ 0, & \text{in } L \end{cases} \tag{4.83}
$$

$$
{}^{L}u_K = \begin{cases} (4\pi\omega)^{-1/2}\, e^{ik\xi + i\omega\eta}, & \text{in } L \\ 0, & \text{in } R \end{cases} \tag{4.84}
$$

$$\phi = \sum_{k=-\infty}^{\infty} \left(a_k \bar{u}_k + a_k^{\dagger} \bar{u}_k^{*} \right) \tag{4.85}$$

or

$$\phi = \sum_{k=-\infty}^{\infty} \left(b_k^{(1)}\, {}^L u_k + b_k^{(1)\dagger}\, {}^L u_k^{*} + b_k^{(2)}\, {}^R u_k + b_k^{(2)\dagger}\, {}^R u_k^{*} \right) \tag{4.86}$$

$$\bar{a}_k |0_M\rangle = 0$$
$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0 \tag{4.88}$$

Modes in (4.81) are not analytic as the f^ns ${}^R u_k$ do not go smoothly over the ${}^L u_k$ as while passing to L from R.

Now, although ${}^{L}u_k$ and ${}^{R}u_k$ are now analytic, the two (un-normalized) combination

$$ {}^{R}u_k + e^{-\pi\omega/a}\, {}^{L}u^{*}_{-k} \qquad (4.89)$$

$$ {}^{R}u^{*}_{-k} + e^{+\pi\omega/a}\, {}^{L}u_k \qquad (4.90)$$

are analytic and bounded, both for all real $\bar{u}, \bar{v}$ and everywhere in the lower half ~~of~~ complex $\bar{u}$ and $\bar{v}$ planes.

Because the modes (4.89) and (4.90) share the same +ve frequency, analytic properties of Minkowski modes $\bar{u}_k$, they must also share common vacuum state $|0_M\rangle$.

So, instead of (4.85) we can expand ϕ in terms of (4.89) and (4.90) as

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$$\phi = \sum_{k=-\infty}^{\infty} \left[2\sinh\left(\frac{\pi w}{a}\right)\right]^{-1/2} \left[d_k^{(1)} \left(e^{\frac{\pi w}{2a}}\, {}^{R}u_k + e^{-\frac{\pi w}{2a}}\, {}^{L}u^{*}_{-k} \right) + d_k^{(2)} \left(e^{-\frac{\pi w}{2a}}\, {}^{R}u^{*}_{-k} + e^{\frac{\pi w}{2a}}\, {}^{L}u_k \right) \right] + h.c.$$

where $d^{(1)}_k |0_M\rangle = d^{(2)}_k |0_M\rangle = 0$ (4.94)

When we calculate $(\phi, {}^R u_k)$, $(\phi, {}^L u_k)$ where ϕ is given by (4.86) and (4.93), and compare them, we obtain

$$b^{(1)}_k = \left[2\sinh\left(\frac{\pi w}{2a}\right)\right]^{-1/2}\left[e^{\frac{\pi w}{2a}} d^{(2)}_k + e^{-\frac{\pi w}{2a}} d^{(1)+}_{-k}\right]$$ (4.95)

$$b^{(2)}_k = \left[2\sinh\left(\frac{\pi w}{2a}\right)\right]^{-1/2}\left[e^{\frac{\pi w}{2a}} d^{(1)}_k + e^{-\frac{\pi w}{2a}} d^{(2)+}_{-k}\right]$$ (4.96)

$$\langle 0_M | b^{(1,2)+} b_k^{(1,2)} | 0_M\rangle = e^{-\frac{\pi w}{a}} \Big/ 2\sinh\left(\frac{\pi w}{a}\right)$$

$$= \left(e^{2\pi w/a} - 1\right)^{-1}$$

$$\langle 0_M | b_k^{(1)\dagger} b_k^{(1)} | 0_M \rangle = \langle 0_M | \left[2 \sinh\left(\frac{\pi W}{a}\right) \right]^{-1} \Big[\cancel{(\quad)} d_k^{(2)\dagger} d_k^{(2)}$$

$$+ \cancel{(\quad)} d_k^{(2)\dagger} d_{-k}^{(1)\dagger} + \left(e^{-\frac{\pi W}{a}} \right) d_{-k}^{(1)} d_{-k}^{(1)\dagger}$$

$$+ \cancel{(\quad)\, d_{-k}^{(1)}} d_k^{(2)} \Big] | 0_M \rangle$$

$$= \frac{e^{-\frac{\pi W}{a}}}{2 \sinh\left(\frac{\pi W}{a}\right)}$$

Similarly,

$$\langle 0_M | b_k^{(2)\dagger} b_k^{(2)} | 0_M \rangle = \frac{e^{-\pi W / a}}{2 \sinh\left(\frac{\pi W}{a}\right)}$$

Upon normalizing (4.89) and (4.90) we respectively get, —

$$\left[2\sinh\left(\frac{\pi w}{a}\right)\right]^{-1/2}\left(e^{\frac{\pi w}{2a}} R_{w_k} + e^{-\frac{\pi w}{2a}} L^{*}_{w_{-k}}\right)$$

and

$$\left[2\sinh\left(\frac{\pi w}{a}\right)\right]^{-1/2}\left(e^{-\frac{\pi w}{2a}} R^{*}_{w_{-k}} + e^{\frac{\pi w}{2a}} L_{w_k}\right)$$