The Blackhole in three dimensional spacetime

The action is

$$I = \frac{1}{2\pi} \int \sqrt{-g}\left(R + 2\ell^{-2}\right) d^2x\, dt + B \qquad (1)$$

where B is a surface term and $-\Lambda = \ell^{-2}$

~~After extremizing I we get~~

$\cancel{ds^2 = -N^2}$

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2\left(N^\phi dt + d\phi\right)^2 \qquad (2)$$

where $N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}$

$$N^\phi(r) = -\frac{J^2}{2r^2}$$

Lets equate $N^2(r) = 0$ then, —

$$-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} = 0$$

$$\Rightarrow \frac{r^4}{\ell^2} - Mr^2 + \frac{J^2}{4} = 0$$

$$\Rightarrow r^2 = \frac{M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}}}{2/\ell^2}$$

$$\Rightarrow r_\pm = \ell\left[\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{M\ell}\right)^2}\right)\right]^{1/2}$$

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Of this, r_+ is the blackhole horizon. In order for the horizon to exist

$$M > 0 \text{ and } |J| \leq Ml \qquad (3)$$

In the extreme case $|J| = Ml$, both roots of $N^2 = 0$ coincide.

The vacuum state is obtained by making the blackhole disappear. That is by letting the horizon size go to zero. To do this lets take $M \longrightarrow 0$ and ~~from~~ due to (3) we have to have ~~also~~ $J \rightarrow 0$.

$$\lim_{M, J \to 0} N^2(r) = \lim_{M, J \to 0} \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right) = \frac{r^2}{l^2}$$

$$\lim_{M, J \to 0} N^{\phi}(r) = \lim_{M, J \to 0} \left(- \frac{J^2}{2r^2} \right) = 0$$

$$ds^2_{vac} = \lim_{M, J \to 0} ds^2 = - \frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2 \qquad (4)$$

When $M = -1$ and $J = 0$ then,

$$N^2(r) = 1 + \left(\frac{r}{\ell}\right)^2$$

$$N^\phi(r) = 0$$

$$\therefore ds^2 = -\left(1 + \left(\frac{r}{\ell}\right)^2\right) dt^2 + \left(1 + \left(\frac{r}{\ell}\right)^2\right)^{-1} dr^2 + r^2 d\phi^2$$

Anti de-sitter space again permissible.

$$R_{\mu\nu\lambda\rho} = -\ell^{-2}\left(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}\right)$$

$$\Rightarrow g^{\mu\lambda} R_{\mu\nu\lambda\rho} = -\ell^{-2}\left(g^{\mu\lambda} g_{\mu\lambda} g_{\nu\rho} - g^{\mu\lambda} g_{\nu\lambda} g_{\mu\rho}\right)$$

$$\Rightarrow R^{\lambda}{}_{\nu\lambda\rho} = -\ell^{-2}\left(\delta^{\mu}{}_{\mu} g_{\nu\rho} - \delta^{\mu}{}_{\nu} g_{\mu\rho}\right)$$

$$= -\ell^{-2}\left(3g_{\nu\rho} - g_{\nu\rho}\right)$$

$$\Rightarrow \boxed{R_{\nu\rho} = -2 g_{\nu\rho} \ell^{-2}}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\left(R + 2\ell^{-2}\right) = -2 g_{\mu\nu} \ell^{-2} - \frac{1}{2} g_{\mu\nu}\left(-2\delta^{\mu}{}_{\mu} + 2\right)\ell^{-2}$$

$$= 0$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R + 2l^{-2}\right) = 0$$

$$\Rightarrow g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} \left(R + 2l^{-2}\right) = 0$$

$$\Rightarrow R - \frac{\delta^{\mu}{}_{\mu}}{2} \left(R + 2l^{-2}\right) = 0$$

$$\Rightarrow R = -6l^{-2}$$

$$\Rightarrow g^{\mu\nu} R_{\mu\nu} = -2 g^{\mu\nu} g_{\mu\nu} l^{-2}$$

$$\Rightarrow R_{\mu\nu} = -2 g_{\mu\nu} l^{-2}$$

$\Rightarrow$

$$\cancel{\mathcal{L} = \frac{1}{2\pi} \left(R + 2l^{-2}\right) \sqrt{g}}$$

$$\cancel{\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}} = \frac{1}{2\pi} \frac{\partial R}{\partial \dot{g}_{ij}}}$$

Hamiltonian form of action,

$$I = \int \left[\pi^{ij} \dot{g}_{ij} - N^{\perp} \mathcal{H}_{\perp} - N^{i} \mathcal{H}_{i} \right] d^2x \, dt + B \qquad (2.4)$$

The surface term B will be discussed later. It differs from the B′ appearing in the Lagrangian form because the corresponding volume integrals differ by a surface term. The surface deformation generators $\mathcal{H}_{\perp}$, $\mathcal{H}_{i}$ are given by

$$\mathcal{H}_{\perp} = 2\pi g^{-1/2} \left(\pi^{ij} \pi_{ij} - \left(\pi^{i}_{i} \right)^2 \right) - (2\pi)^{-1} g^{1/2} \left(R + \frac{2}{\ell^2} \right) \qquad (2.5)$$

$$\mathcal{H}_{i} = -2 \pi^{j}_{i/j} \qquad (2.6)$$

Extremizing the hamiltonian action with respect to the lapse and shift function $N^{\perp}$, N^{i}, yields the constraint equation $\mathcal{H}_{\perp} = 0$ and $\mathcal{H}_{i} = 0$. ~~Extremizing~~ Extremization with respect to the spatial metric g_{ij} and its ~~conjugating~~ conjugate momentum π^{ij}, yields the purely spatial part of the second order field eqns (2.2), rewritten as a Hamiltonian system of first order in time.

2.2. Axially symmetric stationary field:

One may restrict the action principle to a class of fields that possess a rotational Killing vector ∂_ϕ and a timelike Killing vector ∂_t. If the radial coordinate is properly adjusted, the line element may be written as

$$ds^2 = -(N^\perp)^2(r)\,dt^2 + f^{-2}(r)\,dr^2 + r^2\left(N^\phi(r)\,dt + d\phi\right)^2$$

$$0 \leq \phi \leq 2\pi\,, \quad t_1 \leq t \leq t_2 \qquad (2.7)$$

The form of the momenta π^{ij} may be obtained from (2.7) through their relation

$$\pi^{ij} = -\left(\frac{1}{2}\right) g^{-\frac{1}{2}}\left(K^{ij} - K g^{ij}\right)$$

with the extrinsic curvature K_{ij}, which for a time independent metric, simply reads

$$2N^\perp K_{ij} = N_{i|j} + N_{j|i}$$

This gives as the only component of the momentum

$$\pi^{r}{}_{\phi} = \frac{\ell}{2\pi}\,p(r) \qquad (2.8)$$

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Now,

$$I = \int \left[\pi^{ij} \dot{g}_{ij} - N^{\perp} \mathcal{H}_{\perp} - N^{i} \mathcal{H}_{i} \right] d^2x\, dt + B$$

$$g_{ij} = \begin{pmatrix} f^{-2}(r) & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\mathcal{H}_{\perp} = 2\pi g^{-\frac{1}{2}} \left(\pi^{ij} \pi_{ij} - (\pi^{i}{}_{i})^2 \right) - (2\pi)^{-1} g^{1/2} \left(R + \frac{2}{\ell^2} \right)$$

$$= 2\pi \frac{f}{r} \cdot \frac{1}{r^2 f^2} \frac{1}{4\pi^2} p^2(r)$$

$$- \frac{1}{2\pi} \frac{r}{f} \left(R + \frac{2}{\ell^2} \right)$$

$$g^{\phi\phi} \pi^{r}{}_{\phi} = \pi^{r\phi} = \frac{1}{r^2} \cdot \frac{1}{2\pi} p(r)$$

$$g_{rr} \pi^{r}{}_{\phi} = \pi_{r\phi} = f^{-2}(r) \cdot \frac{1}{2\pi} p(r)$$

$$\pi^{r\phi} \pi_{r\phi} = \frac{1}{r^2 f^2} \frac{1}{4\pi^2} p^2(r)$$

$$g^{1/2} = \frac{r}{f}$$

$$\mathcal{H} = 2\pi f(r)\mathcal{H}_\perp = 2\ell^2 \frac{p^2}{r^3} + (f^2)' - 2\frac{r}{\ell^2} \qquad (2.10)$$

$$\mathcal{H}_i = \left(-2\pi^{j}{}_{i}/j\right) \cdots -2\ell p'(r)$$

$$\mathcal{H}_\phi = 2\pi\mathcal{H}_i = -2\ell p'(r) \qquad (2.11)$$

$$N(r) = f^{-1} N^\perp \qquad (2.12)$$

$$I = -\int \frac{1}{2\pi}\left[N(r)\,\mathcal{H}(r) + N^\phi \mathcal{H}_\phi\right] dt\, dr\, d\phi + B$$

$$= -(t_2 - t_1)\int \left[N(r)\,\mathcal{H}(r) + N^\phi \mathcal{H}_\phi\right] dr + B \qquad (2.9)$$

2.3. Solutions: To find solutions under the assumptions of time independence and axial symmetry, one must extremize the reduced action (2.9). Variation with respect to N and N^ϕ, yields that the generators $\mathcal{H}$ and $\mathcal{H}_\phi$ must vanish. These constraint equations are readily solved to give

$$p = -\frac{J}{2\ell}$$

$$f^2 = -M + \left(\frac{r}{\ell}\right)^2 + \frac{J^2}{4r^2} \qquad (2.13)$$

where M and J are two constants of integration, which will be identified below as the mass and angular momentum, respectively.

~~Variation of the action with respect to f^2 and p yields the equations~~

$$\mathcal{H} = 0$$

$$\Rightarrow \cancel{2\ell^2} \frac{J^2}{\cancel{4\ell^2}_{2}} \cdot \frac{1}{r^3} + (f^2)' - 2\frac{r}{\ell^2} = 0$$

$$\Rightarrow \boxed{f^2 = -M + \left(\frac{r}{\ell}\right)^2 + \frac{J^2}{4r^2}} \qquad (2.13)$$

$$\mathcal{H}_\phi = -2\ell p' = 0$$

$$\Rightarrow \boxed{p = -\frac{J}{2\ell}}$$

Variation of the action with respect to f^2 and p yields the equations

$$I = -(t_2 - t_1)\int dr \left(N(r)\mathcal{H}(r) + N^\phi \mathcal{H}_\phi\right) + B$$

$$= -(t_2 - t_1)\int dr \left(N(r)\left(2\ell^2 \frac{p^2}{r^3} + (f^2)' - 2\frac{r}{\ell^2}\right) + N^\phi (-2\ell p')\right) + B$$

Varying I with p.

$$\frac{d}{dr}\left(\frac{\partial L}{\partial p'}\right) - \frac{\partial L}{\partial p} = 0 \Rightarrow -2\ell (N^\phi)' - \frac{2\ell^2}{r^3} N(r)\, 2p = 0$$

$$\Rightarrow (N^\phi)' + \frac{2\ell p}{r^3} N(r = 0)$$

Varying I with f^2 :

$$\frac{d}{dr}\left(\frac{\partial L}{\partial (f^2)'}\right) - \frac{\partial L}{\partial (f^2)} = 0 \qquad (2.14)$$

$$\Rightarrow N' = 0$$

which determine N and N^ϕ as

$$\boxed{N = \cancel{N(\infty)} \text{ constant } = N(\infty)}$$

$$(N^\phi)' + \frac{2lp}{r^3} N = 0$$

$$\Rightarrow (N^\phi)' = -\frac{2lp}{r^3} N(\infty)$$

$$\Rightarrow N^\phi = \frac{lp}{r^2} N(\infty) + \text{constant}.$$

$$= -\frac{J}{2r^2} N(\infty) + \text{constant}.$$

When $r \to \infty$, $N^\phi = \text{constant} = N^\phi(\infty)$

$$\therefore \boxed{N^\phi = -\frac{J}{2r^2} N(\infty) + N^\phi(\infty)} \qquad (2.15)$$

The constants of integration $N(\infty)$ and $N^\phi(\infty)$ are part of specification of the coordinate system, which is not fully fixed by the form of the line element (2.7)

The blackhole arises from Anti-de Sitter space through identifications by means of a discrete subgroup of its isometry by $SO(2,2)$. This implies that Blackhole is a solution of the source-free Einstein equations everywhere, including $r=0$. As we shall also see, the type of "singularity" that is found at $r=0$ is generically one in the causal structure and not in the curvature, which is everywhere finite (and constant). It should be emphasized that this statement means that $r=0$. ~~As we shall also see~~ is not a conical singularity.

3.1. Anti-de Sitter Space in 2+1 dimensions:

3.1.1. Metric: Anti de-Sitter space can be defined

in terms of its embedding in a 4-dim flat space of signature $(--++)$.

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \qquad (3.1.)$$

through the eqn.

$$-v^2 - u^2 + x^2 + y^2 = -\ell^2 \qquad (3.2.)$$

A system ~~covering~~ of coordinates covering the whole of the manifold may be introduced by setting

$u = \ell \cosh\mu \sin\lambda \quad , \quad v = \ell \cosh\mu \cos\lambda \qquad (3.3)$

where $\ell \sinh\mu = \sqrt{x^2+y^2}$ and $0 \leq \mu < \infty$, $0 \leq \lambda \leq 2\pi$.

$$ds^2 = \ell^2 \left(-\cosh^2\mu \; d\lambda^2 + \frac{dx^2+dy^2}{\ell^2+x^2+y^2}\right)$$

~~Further as we take~~

an expression that can be further simplified by passing to polar coordinates in x-y plane

$$x = \ell \sinh\mu \cos\theta \quad , \quad y = \ell \sinh\mu \sin\theta \qquad (3.5)$$

which yields

$$ds^2 = \ell^2 \left[-\cosh^2\mu \; d\lambda^2 + d\mu^2 + \sinh^2\mu \; d\theta^2\right] \qquad (3.6)$$

for metric of anti-de Sitter Space.

If unwrapped λ is denoted by t/ℓ and if one sets $r = \ell \sinh\mu$.

$$r = \ell \sinh\mu \Rightarrow \sinh\mu = \frac{r}{\ell} \Rightarrow \cosh^2\mu = 1 + \left(\frac{r}{\ell}\right)^2$$

$$dr = \ell \cosh\mu \; d\mu$$

$$\Rightarrow \ell^2 d\mu^2 = \frac{dr^2}{\cosh^2\mu} = \frac{dr^2}{\left(1+\left(\frac{r}{\ell}\right)^2\right)}$$

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Then the metric takes the form

$$\cancel{ds^2 = -\left(\left(\frac{r}{l}\right)^2 + 1\right) dt^2}$$

$$ds^2 = -\left(\left(\frac{r}{l}\right)^2 + 1\right) dt^2 + \left(\left(\frac{r}{l}\right)^2 + 1\right)^{-1} dr^2 + r^2 d\theta^2 \qquad (3.7.)$$

which is the metric (2.7.) with $M = -1$ and $J = 0$ (and ϕ replaced with θ).

~~3.1.2 Isometries~~

~~By construction, the anti-de Sitter metric is invariant under SO(2,2). The Killing vectors are~~

$$\cancel{J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}}$$

~~where $x^a = (u, v, x, y)$.~~

$$ds^2 = -\left(\left(\frac{r}{l}\right)^2 + 1\right) dt^2 + \left(\left(\frac{r}{l}\right)^2 + 1\right)^{-1} dr^2 + r^2 d\theta^2$$

$$\lambda(r) = \left(\frac{r}{l}\right)^2 + 1$$

$$\therefore \; F(r) = e^{\frac{1}{\alpha}\int dr\, \lambda^{-1}(r)} = e^{\frac{l}{\alpha}\tan^{-1}\left(\frac{r}{l}\right)}$$

$$\int dr\, \lambda^{-1}(r) = \int \frac{dr}{\left(\frac{r}{l}\right)^2 + 1} = l \tan^{-1}\left(\frac{r}{l}\right)$$

Lets take, coordinates,

$$u = e^{t/\alpha} F(r) \; ; \; v = e^{-\frac{t}{\alpha}} F(r)$$

~~where $\alpha = \frac{1}{\lambda'(1)} \frac{F(r)}{F'(r)} = \frac{1}{\left(\left(\frac{r}{\ell}\right)^2 + 1\right)}$~~

$$A(u,v) = \frac{1}{2}\alpha^2 \lambda e^{-\frac{2}{\alpha}\int dr\, \lambda^{-1}}$$

$$= \frac{1}{2}\alpha^2 \left(\left(\frac{r}{\ell}\right)^2 + 1\right) e^{-\frac{2\ell}{\alpha}\tan^{-1}\left(\frac{r}{\ell}\right)}$$

$$g(u,v) = r^2$$

$$ds^2 = 2A\, du\, dv + r^2 d\theta^2$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & A & 0 \\ A & 0 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

3.1.2: Isometry:

By construction, the anti-de Sitter metric is invariant under $SO(2,2)$. ~~I~~ The killing vectors are

~~$ds^2 = -du^2 - dv^2 + dx^2 + dy^2$~~

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b} \qquad (3.8)$$

where $x^a = (u, v, x, y)$

The vector J_{01} generates "time displacement" $(J_{01} = \partial_\lambda)$, whereas J_{23} generates rotations in the $x-y$ plane $(J_{23} = \partial_\theta)$. The most general killing vector is given by

$$\frac{1}{2} w^{ab} J_{ab}, \qquad w^{ab} = -w^{ba} \qquad (3.10)$$

3.1.3. Poincaré Coordinates

The coordinates are defined by

$$z = \frac{l}{u+x} \; ; \; \beta = \frac{y}{u+x} \; ; \; \gamma = \frac{-v}{u+x}$$

are called Poincaré coordinates. They only cover part of the space, namely just one ~~coordinates are therefore~~ of the infinitely many regions where $u+x$ has a definite sign (see. Fig I.). These coordinates are therefore not well adapted to the study of global properties.

$$ds^2 = -du^2 \cancel{+ dz} - dv^2 + dx^2 + dy^2$$

$$z = \frac{\ell}{u+x} \Rightarrow dz = -\ell \frac{(du + dx)}{(u+x)^2} = -\frac{z^2}{\ell}(du + dx)$$

$$\beta = \frac{y}{u+x} \Rightarrow d\beta = \frac{(u+x)\,dy - y(du + dx)}{(u+x)^2}$$

$$\gamma = \frac{-v}{u+x} \Rightarrow d\gamma = \frac{-dv\,(u+x) + v\,(du + dx)}{(u+x)^2}$$

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2$$

$$d\beta + d\gamma = \frac{(du + dx)(v - y) + (u+x)(dy - dv)}{(u+x)^2}$$

$$d\beta - d\gamma = \frac{-(du + dx)(v + y) + (u+x)(dy + dv)}{(u+x)^2}$$

$$d\beta^2 - d\gamma^2 = \cancel{\frac{(du+dx)^2(y^2-v^2)+}{(u+x)^4}}$$

$$= \frac{(du + dx)(y^2 - v^2) + (u+x)^2(dy^2 - dv^2) + (du + dx)(dy + dv)(v - y)(u+x) - (du + dx)(dy - dv)(v + y)(u + x)}{(u+x)^4}$$

$$\therefore\ ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + d\beta^2 - d\gamma^2\right) \qquad (3.12)$$