Lemma 1 : If λ is an eigenvalue of ω_{ab} , then $-\lambda$ is also an eigenvalue of ω_{ab}.

Proof : $(\omega_{ab} - \lambda \eta_{ab}) e^{b} = 0$

characteristic eqn :

$$\det(\omega - \lambda\eta) = 0$$

$$\Rightarrow \det\left[(\omega - \lambda\eta)^{T}\right] = 0$$

$$\Rightarrow \det(-\omega - \lambda\eta) = 0$$

$$\Rightarrow \det(\omega + \lambda\eta) = 0$$

$\therefore$ $-\lambda$ is also a root of the characteristic eqn.

Lemma 2 : If λ is an eigenvalue, then λ^{*} is also an eigenvalue.

$$\det(\omega - \lambda\eta) = 0$$

$$\Rightarrow \det\left[(\omega - \lambda\eta)^{+}\right] = 0$$

$$\Rightarrow \det(-\omega - \lambda^{*}\eta) = 0$$

$$\Rightarrow \det(\omega + \lambda^{*}\eta) = 0$$

$$\Rightarrow \det(\omega - \lambda^{*}\eta) = 0$$

4.2.1. Types of eigenvalues: It follows from these theorems that the ~~four~~ four eigenvalues of ω are of the following four possible types

1. $\lambda, -\lambda, \lambda^*, -\lambda^*$; $\lambda = a + ib$; $a \neq 0 \neq b$

2. $\lambda_1 = \lambda_1^*, -\lambda_1, \lambda_2 = \lambda_2^*$ (λ_1 and λ_2 real)

3. $\lambda_1, -\lambda_1 = \lambda^*, \lambda_2, -\lambda_2 = \lambda_2^*$ (λ_1 and λ_2 are purely imaginary)

4. $\lambda_1 = \lambda_1^*, -\lambda_1, \lambda_2, -\lambda_2 = \lambda_2^*$ (λ_1 is real and λ_2 is purely imaginary)

In each case, the eigenvalues involve only two independent real ~~num~~ numbers, whose knowledge is equivalent to knowing the two Casimir ~~Invariance~~ Invariants.

$$I_1 = \omega^{ab}\omega_{ab} \quad ; \quad I_2 = \frac{1}{2}\epsilon^{abcd}\omega_{ab}\omega_{cd}$$

Multiple roots can occur only in the following circumstances:

- Case (2) and Case (3), when $\lambda_1 = \lambda_2$ (or $-\lambda_2$). If $\lambda_1 \neq 0$, then λ_1 and $-\lambda_1$ are distinct roots. If $\lambda_1 = 0$, then 0 is a quadruple root; or

- Case (2), (3) or (4), when one of the roots vanishes.

<u>Lemma 3</u>: Let v^a and u^a be eigenvectors of $w^a{}_b$ with respective eigenvalues λ and μ

$$w^a{}_b v^b = \lambda v^a \quad ; \quad w^a{}_b u^b = \mu u^a$$

Then $v_a u^a = 0$ unless $\lambda + \mu = 0$. In particular, if $\lambda \neq 0$, then v^a is a null vector.

<u>Proof</u>: One has $u_a w^a{}_b v^b = \lambda u_a v^a$

$$\Rightarrow -\mu u^a v_a = \lambda u_a v^a$$

$$\Rightarrow (\lambda + \mu) u_a v^a = 0$$

$$w^a{}_b u^b = \mu u^a$$

$$\Rightarrow (w^a{}_b u^b)^T = (\mu u^a)^T$$

$$\Rightarrow -(u^b)^T w^a{}_b = \mu (u^a)^T \Rightarrow$$

A.3. Type I_a

$$\omega_{ab} l^b = \lambda l_a$$

$$\omega_{ab} m^b = -\lambda m_b$$

$$\omega_{ab} l^{*b} = \lambda^* l^*_a$$

$$\omega_{ab} m^{*b} = -\lambda^* m^*_b$$

The only scalar products that can be different from zero are $l^a m_a$ and $l^{*a} m^*_a$. They cannot vanish since the metric would then be degenerate. By scaling m_a if necessary one can assume $l^a m_a = 1$. One then has also $l^{*a} m^*_a = 1$. The metric is given by

$$\eta_{ab} = l_a m_b + l^*_a m^*_b + [a \longleftrightarrow b]$$

Since $\left(\eta_{ab} - l_a m_b - l^*_a m^*_b - [a \longleftrightarrow b]\right) v^b = 0$

whenever v^a equals l^a, m^a, l^{*a}, m^{*b}.

The tensor is given by

$$\omega^{ab} = \lambda\left(l_a m_b - l_b m_a\right) + \lambda^*\left(l^*_a m^*_b - l^*_b m^*_a\right)$$

$$\omega^a{}_b = \lambda l^a l_a - \lambda m^b m_b + \lambda^* l^{*a} l^*_a - \lambda^* m^{*a} m^*_a$$

$$\omega^{ab} = \eta^{bc}\omega^a{}_c$$

Our goal is to achieve a ~~cono~~ canonical expression for $\omega^a{}_b$ over the real numbers. Therefore we decompose the ~~two~~ vectors l_a and m_a into their real and imaginary components

$$l_a = u_a + i v_a \quad ; \quad m_a = n_a + i q_a$$