Appendix - B

$$d\hat{s}^2 = 2A(u,v)\,du\,dv + g(u,v)\,h_{ij}(x^j)\,dx^i dx^j \qquad (B.1)$$

This metric is assumed to satisfy Einstein's vaccum eqns. We introduced a shock wave by keeping (B.1.) for $u<0$ but replacing v by $v+f(x_i)$ for $u>0$

$$ds^2 = 2A(u, v+\theta f)\,du\,(dv + \theta f_i\,dx^i) + g(u, v+\theta f)\,h_{ij}\,dx^i dx^j \qquad (B.2)$$

where $\theta = \theta(u)$ is the usual step function. Changing to coordinates $(\hat{u}, \hat{v}, \hat{x}^i)$ defined by

$$\left.\begin{aligned} \hat{u} &= u \\ \hat{v} &= v+\theta f \\ \hat{x}^i &= x^i \end{aligned}\right\} \qquad (B.3)$$

we ~~first~~ obtain

$$ds^2 = 2A(\hat{u},\hat{v})\,d\hat{u}\,(d\hat{v} - \delta(\hat{u}) f\,d\hat{u}) + g(\hat{u},\hat{v})\,h_{ij}\,d\hat{x}^i d\hat{x}^j \qquad (B.4)$$

We note that the metric ds^2 given by (B.2) and (B.4) is in fact continuous, i.e. $\exists$ coordinates $(\bar{u}, \bar{v}, \bar{x}^i)$ s.t. the metric coeff. are cont. A possible choice is ~~simply~~ given implicitly by

$$\begin{aligned} \hat{u} &= \bar{u} \\ \hat{v} &= \bar{v} + \theta\bar{f} - \tfrac{1}{2}\bar{u}\theta^2 \frac{\hat{A}}{\hat{g}} h^{mn} \bar{f}_{,m}\bar{f}_{,n} \qquad (B.5) \\ \hat{x}^i &= \bar{x}^i - u\theta \frac{\hat{A}}{\hat{g}} h^{im} \bar{f}_{,m} \end{aligned}$$

where $\bar{f} = f(\bar{x}^i)$; $\hat{g} = g(\hat{u}, \hat{v})$ (B.6)
$\hat{A} = A(\hat{u}, \hat{v})$; $h^{ij} = h^{ij}(x^i)$

$$R_{\hat{u}\hat{i}} = -\frac{\hat{A}_{,v}}{\hat{A}} f_{,i}\,\delta$$

$$R_{\hat{v}\hat{i}} = 0$$

$$R_{\hat{i}\hat{j}} = R^{(2)}_{ij} - h_{ij}\left[\frac{\hat{g}_{,\hat{u}\hat{v}}}{\hat{A}} + \frac{\hat{g}_{\hat{v}\hat{v}}}{\hat{A}} f\delta\right]$$

$\vdots$ (B.7)

Ignoring δ^2 terms we transform to (u, v, x^i) coordinates and insert the vaccum equations (obtained by setting $f=0$) to get

$$R_{\hat{v}\hat{v}} = R_{vv} = 0$$

$$R_{vi} = R_{\hat{v}\hat{i}} = 0$$

$$R_{ij} = R_{\hat{i}\hat{j}} = -h_{ij}\frac{\hat{g}_{,\hat{v}\hat{v}}}{\hat{A}} f\delta$$

$$R_{ui} = R_{\hat{u}\hat{i}} + R_{\hat{v}\hat{v}}\,\theta f_{,i}$$

$$= f_{,i}\delta\left[-\frac{\hat{A}_{,\hat{v}}}{\hat{A}} + \theta f\left(\frac{\hat{A}^2_{,\hat{v}}}{\hat{A}^2} - \frac{\hat{A}_{,\hat{v}\hat{v}}}{\hat{A}} - \frac{\hat{g}_{,\hat{v}}\hat{A}_{,\hat{v}}}{\hat{g}\hat{A}}\right)\right]$$

$\vdots$

(B.8)

The stress-energy tensor for a massless particle located at the origin $\rho=0$ of the (x^i) 2-surface and at $u=0$ is

$$T^{ab} = 4p\,\delta(\rho)\,\delta(u)\,\delta^a_v\,\delta^b_v \qquad (B.9)$$

where p is the momentum of the particle. Thus, the only non-zero component is

$$T_{uu} = 4p\hat{A}^{2}\delta(\rho)\,\delta(u) \qquad (B.10)$$

Inserting (B.8) and (B.10) into the Einstein field equations, partially integrating the δ' term, noting that e.g. $\hat{A}_{,\hat{v}}(\hat{u}=0)$ $\Leftrightarrow A_{,v}(u=0)=0$ yield eq. (5).

$$R_{uu} - \frac{1}{2}g_{uu}R = 8\pi G T_{uu}$$

$$\Rightarrow \frac{\hat{A}}{\hat{g}}\Delta f\,\delta(u) + \frac{1}{4}\cancel{R} = 32\pi p A^2 G\,\delta(\rho)\delta(u)$$

$$\boxed{A_{,u}(u=0) = A_{,v}(u=0) = g_{,u}(u=0) = g_{,v}(u=0)}$$

$$\Rightarrow -\frac{2\hat{g}_{,uv}}{\hat{g}}f\,\delta(u) - \frac{\hat{g}_{,v}}{\hat{g}}f\,\delta'$$

$$\Rightarrow \frac{\hat{A}}{\hat{g}}\Delta f - \frac{\hat{g}_{,uv}}{\hat{g}}f = 32\pi p A^2 G\,\delta(\rho)$$