Theory Project
Semester 7

Supervisor: Suneeta Vardarajan
Student: Raikhik Das

1 Introduction

The standard Einstein-Maxwell equations in 2+1 spacetime dimensions, with a negative cosmological constant, admit a black hole solution. The 2+1 black hole -characterized by mass, angular momentum and charge, defined by flux integrals at infinity- has similarities to its 3+1 counterpart. But as it is easy to handle less dimensions, we often work with the 2+1 dimensional version.

2 Prerequisite calculations

2.1 One parameter subgroup of $SO(2,2)$

Description of the problem:

Our aim is to provide a complete classification of the inequivalent one-parameter subgroups of SO(2, 2). Two one-parameter subgroups g(t) and h(t), tϵ **R**, are said to be equivalent if and only if they are conjugate in $SO(2,2)$, i.e., $g(t) = k^1 h(t) k, k \epsilon SO(2,2)$

By an $SO(2,2)$ rotation of the coordinate axes in $\mathbf{R}^4$, one can then map $g(t)$ on $h(t)$. Since one-parameter subgroups are obtained by exponentiating infinitesimal transformations, the task at hand amounts to classifying the elements of the Lie algebra $SO(2,2)$ up to conjugation. Now, the elements of $SO(2,2)$ are described by antisymmetric tensors $\omega_{ab} = -\omega_{ba}$ in $\mathbf{R}^4$.

If one conjugates the infinitesimal transformation $R_{ab} = \delta_{ab} + \epsilon_{ab}$ by $k \in SO(2,2)$, ($k^T \eta k = \eta$, $\eta = diag(- - ++)$, one finds that the antisymmetric matrix $\omega \equiv (\omega_{ab})$ transforms as $\omega \to \omega = k^T \omega k$, $k \in SO(2,2)$

Hence we have to classify antisymmetric tensors under this equivalence relation.

Strategy:

Jordan - Chevalley decomposition: Any linear operator M can be uniquely decomposed as the sum of a semi-simple (diagonalizable over the complex numbers) linear operator S and a nilpotent operator N that commute

$$M = S + N$$

$$[S, N] = 0$$

with $N^q = 0$ for some q and $S = L^1(\text{diagonal matrix})L$, for some L.

The eigenvalues of S coincide with those of M and provide an intrinsic characterization of S. When the eigenvalues of S are non-degenerate, the nilpotent operator N is identically zero and M is thus completely characterized (up to similarity) by its eigenvalues. However, if some eigenvalues are repeated, N may be non-zero and M cannot be reconstructed from the knowledge of its eigenvalues: one needs also information about its nilpotent part (the dimensions of the irreducible invariant subspaces).

We shall construct the sought-for invariant classification of elements of $SO(2,2)$ by means of the Jordan - Chevalley decomposition of the operator $_{ab}$.Since $\eta^{ab} = \delta_{ab}$ for $SO(2,2)$, the operator $\iota\omega^a_b$ is, in general, not hermitian. Accordingly, it may possess a non-trivial nilpotent part when its eigenvalues are degenerate. The classification of the possible ω^a_b is analogous to the invariant classification of the electromagnetic field in Minkowski space and is also reminiscent of the Petrov classification of the Weyl tensor in General Relativity.

Because the matrix ab is real and antisymmetric, there are restrictions on its eigenvalues. These constraints are contained in the following elementary

Lemmas.

Lemma 1: If λ is an eigenvalue of ω_{ab} then $-\lambda$ is also an eigenvalue of ω_{ab}.

Lemma 2: If λ is an eigenvalue, then λ^* is also an eigenvalue.

The eigenvalues involve only two independent real numbers, whose knowledge is equivalent to knowing the two Casimir invariants.

$$I_1 =_{ab} \omega^{ab} \ ; \ I_2 = \frac{1}{2}\epsilon^{abcd}\omega_{ab}\omega_{cd}$$

Note: If $SO(2,2)$ is replaced by $SO(4)$, $\iota\omega^a_b$ becomes hermitian and therefore diagonalizable. Hence, there is no nilpotent part and $\iota\omega^a_b$ is completely characterized by its eigenvalues and thus by I_1 and I_2.

Types of antisymmetric tensors:

Type I_a:

3 Action Principle, Equations of Motion and their Solutions

3.1 Action Principle

Lets take the action to be

$$I = \frac{1}{2\pi}\int \sqrt{-g}[R + 2l^{-2}]d^2xdt + B' \tag{1}$$

where B' is a surface term and the radius l is related to the cosmological constant by $-\Lambda = l^{-2}$. [For our convenience numerical value of $(16\pi G)^{-1}$ in front of the action is taken to be $(2\pi)^{-1}$, i.e we se thw value of G, having the dimension of inverse energy, equal to $\frac{1}{8}$].

Extremization of the action with respect to the metric $g_{\mu\nu}(x,t)$ yields the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R + 2l^{-2}) = 0 \tag{2}$$

which, in a three dimensional spacetime, determine the full Riemann tensor as

$$R_{\mu\nu\lambda\rho} = -l^{-2}(g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho}) \tag{3}$$

describing a symmetric space of constant negative curvature.

One may pass to the hamiltonian form of eqn(1) which reads

$$I = \int [\pi^{ij}\dot{g}_{ij} - N^{\perp}H_{\perp} - N^{i}H_{i}]d^2x dt + B \tag{4}$$

The surface term B will be discussed below. It differs from the B' appearing in the lagrangian from because the corresponding volume integrals differs by a surface term. The surface deformation generators $H_{\perp}, H_i$ are given by

$$H_{\perp} = 2\pi g^{-\frac{1}{2}}(\pi^{ij}\pi_{ij} - (\pi_i^i)^2) - (2\pi)^{-1} g^{\frac{1}{2}}(R + \frac{2}{l^2}) \tag{5}$$

$$H_i = -2\pi^{j}_{i/j} \tag{6}$$

Extremizing the hamiltonian action with respect to the the lapse and shift functions $N_{\perp}, N_i$, yields the constraint equations H $=0$ and Hi $=0$ which are the , and , i components of (2.2). Extremization with respect to the spatial metric g_{ij} and its conjugate momentum π_{ij}, yields the purely spatial part of the second order field equations (2), rewritten as a hamiltonian system of first order in time.

3.2 Axially symmetric stationary field:

One may restrict the action principle to a class of fields that possess a rotational Killing vector $\frac{\partial}{\partial\phi}$ and a timelike Killing vector $\frac{\partial}{\partial t}$. If the radial coordinate is properly adjusted, the line element may be written as

$$ds^2 = -(N^{\perp})^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(N^{\phi}(r)dt + d\phi)^2 \tag{7}$$

where $0 \leq \phi < 2\pi$, $t_1 \leq t \leq t_2$.