2. Quantum Field Theory in Minkowski Spacetime

2.1 Scalar field:

Let's consider a scalar field $\phi(t,x)$ defined in Minkowski space (n-dim), ~~only~~ satisfying

$$(\Box + m^2)\phi = 0 \qquad (2.1)$$

$$(t, \vec{x}) \equiv (x^0, \vec{x})$$

Lagrange density $\longrightarrow$ $\mathcal{L} = \frac{1}{2}\left(\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2\right)$ (2.2)

Action $\longrightarrow$ $S = \int \mathcal{L}\, d^n x$ (2.3)

We get (2.1) from $\delta S = 0$ (2.4)

One set of soln. of (2.1) is

$$u_k(t,\vec{x}) \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (2.5)$$

where $\omega \equiv (k^2 + m^2)^{1/2}$ (2.6)

$$k = |\vec{k}| = \left(\sum_{i=1}^{n-1} k_i^2\right)^{1/2} \qquad (2.7)$$

and the cartesian components of $\vec{k}$ can the values

$$-\infty < k_i < \infty \quad ; \quad i = 1, 2, \ldots\ldots, n-1$$

The modes ~~of~~ (2.5) are said to be positive frequency w.r.t. to t, being eigenfunctions of the operator $\frac{\partial}{\partial t}$:

$$\frac{\partial}{\partial t} u_k(t,\vec{x}) = -i\omega u_k(t,\vec{x}) \quad ; \omega > 0 \qquad (2.8)$$

Lets define the scalar product

$$(\phi_1, \phi_2) = -i \int \left\{ \phi_1(x)\left[\partial_t \phi_2^*(x)\right] - \left[\partial_t \phi_1(x)\right]\phi_2^*(x) \right\} d^{n-1}x$$

$$= -i \int \phi_1(x) \overleftrightarrow{\partial} \phi_2^*(x) \; d^{n-1}x \qquad (2.9)$$

where t denotes a spacelike hyperplane of simultaneity at instant t. Then the u_k modes (2.5) are orthogonal

$$(u_{\vec{k}}, u_{\vec{k}'}) = 0 \quad ; \vec{k} \neq \vec{k}' \qquad (2.10)$$

If we choose

$$u_{\vec{k}} = \left[2\omega(2\pi)^{n-1}\right]^{-1/2} e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad (2.11)$$

then the u_k functions are normalized in the scalar product (2.9):

$$(u_{\vec{k}}, u_{\vec{k}'}) = \delta^{(n-1)}(\vec{k} - \vec{k}') \qquad (2.12)$$

For many purposes it is more convenient to restrict the solutions $u_{\vec{k}}$ to the interior of a spacelike $(n-1)$-torus of side L (i.e., choose periodic boundary conditions). Then

$$u_{\vec{k}} = \left(2L^{n-1}\,\omega\right)^{-1/2} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (2.13)$$

where

$$k_i = \frac{2\pi j_i}{L}, \qquad j_i = 0, \pm 1, \pm 2, \ldots,$$

$$i = 1, \ldots, n-1$$

Thus $$\left(u_{\vec{k}},\, u_{\vec{k}'}\right) = \delta_{\vec{k}\vec{k}'} \qquad (2.14)$$

2.2. Quantization:

$$\left.\begin{aligned} &\left[\phi(t,\vec{x}), \phi(t,\vec{x}')\right] = 0 \\ &\left[\pi(t,\vec{x}), \pi(t,\vec{x}')\right] = 0 \\ &\left[\phi(t,\vec{x}), \pi(t,\vec{x}')\right] = i\,\delta^{(n-1)}(\vec{x}-\vec{x}') \end{aligned}\right\} \qquad (2.15)$$

where $$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi \qquad (2.16)$$

$$\phi(t,\vec{x}) = \sum_k \left[a_k u_k(t,\vec{x}) + a_k^{+} u_k^{*}(t,\vec{x})\right] \qquad (2.17)$$

The equal time commutation relations for ϕ and π are equivalent to

$$\left.\begin{aligned} [a_{\vec{k}}, a_{\vec{k}'}] &= 0 \\ [a^{+}_{\vec{k}}, a^{+}_{\vec{k}'}] &= 0 \\ [a_{\vec{k}}, a^{+}_{\vec{k}'}] &= \delta_{\vec{k}\vec{k}'} \end{aligned}\right\} \quad (2.18)$$

<u>Fock Space</u>:

$$a_{\vec{k}}|0\rangle = 0 \qquad \forall \vec{k} \qquad (2.19)$$

$$|1_{\vec{k}}\rangle = a^{+}_{\vec{k}}|0\rangle \qquad (2.20)$$

$$|1_{\vec{k}_1}, 1_{\vec{k}_2}, \ldots, 1_{\vec{k}_J}\rangle = a^{+}_{\vec{k}_1} a^{+}_{\vec{k}_2} \cdots a^{+}_{\vec{k}_J}|0\rangle \qquad (2.21)$$

if all $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_J$ are distinct.

If any $a^{+}_{\vec{k}}$ are repeated then

$$|{}^{1}n_{\vec{k}_1}, {}^{2}n_{\vec{k}_2}, \ldots, {}^{j}n_{\vec{k}_j}\rangle$$

$$= \left({}^{1}n!\,{}^{2}n! \cdots {}^{j}n!\right)^{-1/2} \left(a^{+}_{\vec{k}_1}\right)^{{}^{1}n} \left(a^{+}_{\vec{k}_2}\right)^{{}^{2}n} \cdots \left(a^{+}_{\vec{k}_j}\right)^{{}^{j}n} |0\rangle$$

(2.22)

2.3. Energy-momentum :

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi_{,\lambda}\phi_{,\delta} + \frac{1}{2}m^2\phi^2\eta_{\alpha\beta} \quad (2.26)$$

$$\text{Hamiltonian density} = T_{tt} = \frac{1}{2}\left[(\partial_t\phi)^2 + \sum_{i=1}^{n-1}(\partial_i\phi)^2 + m^2\phi^2\right] \quad (2.27)$$

$$\text{momentum density} = T_{ti} = \partial_t\phi\,\partial_i\phi\,, \quad i = 1,\ldots,n-1 \quad (2.28)$$

From (2.17) and (2.27) and integrating over all space,

$$H = \int_t T_{tt}\, d^{n-1}x = \frac{1}{2}\sum_{\vec{k}}\left(a^{+}_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^{+}_{\vec{k}}\right)\omega \quad (2.29)$$

From (2.17) and (2.28) and integrating over all space.

$$P_i = \int_t T_{ti}\, d^{n-1}x = \sum_{\vec{k}} a^{+}_{\vec{k}}a_{\vec{k}}\, k_i \quad (2.30)$$

From (2.18) and (2.29), —

$$H = \sum_{\vec{k}}\left(a^{+}_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\right)\omega \quad (2.31)$$

$$\left.\begin{aligned} N_{\vec{k}} &\equiv a^{+}_{\vec{k}}\, a_{\vec{k}} \\ N &\equiv \sum_{\vec{k}} N_{\vec{k}} \end{aligned}\right\} \tag{2.32}$$

$$[N, H] = [N, P_i] = 0 \tag{2.33}$$

From (2.19) and (2.24) one obtains

$$\langle 0 | N_{\vec{k}} | 0 \rangle = 0, \quad \forall \vec{k} \tag{2.34}$$

$$\langle {}^{1}n_{\vec{k}_1}, {}^{2}n_{\vec{k}_2}, \ldots\ldots, {}^{j}n_{\vec{k}_j} | N_{\vec{k}_i} | {}^{1}n_{\vec{k}_1}, {}^{2}n_{\vec{k}_2}, \ldots\ldots, {}^{j}n_{\vec{k}_j} \rangle \tag{2.35}$$

$$= {}^{i}n$$

$$\langle | N | \rangle = \sum_{i} {}^{i}n \tag{2.36}$$

2.4. Vacuum energy divergence :

$$\langle 0|\vec{P}|0\rangle = 0 \qquad (2.37)$$

$$\langle 0|H|0\rangle = \langle 0|0\rangle \sum_{\vec{k}} \frac{1}{2}\omega = \sum_{\vec{k}} \frac{1}{2}\omega \qquad (2.38)$$

Not only is the right hand side of (2.38) non-zero ; it is actually infinite

$$\sum_{\vec{k}}\left(\frac{1}{2}\omega\right) = \frac{1}{2}\left(\frac{L}{2\pi}\right)^{n-1} \int \omega \, d^{n-1}k$$

$$= \left(\frac{L^2}{4\pi}\right)^{\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \left(k^2+m^2\right)^{1/2} k^{n-2} \, dk \qquad (2.39)$$

$$:a_{\vec{k}} a^{\dagger}_{\vec{k}}: = a^{\dagger}_{\vec{k}} a_{\vec{k}} \qquad (2.40)$$

$$:H: = \sum_{\vec{k}} a^{\dagger}_{\vec{k}} a_{\vec{k}} \, \omega \qquad (2.41)$$

From (2.17) and (2.26),

$$\phi_{,\alpha} \phi_{,\beta} = \sum_{\vec{k}} \sum_{\vec{k}'} \left(a_{\vec{k}} \partial_\alpha u_{\vec{k}} + a^{\dagger}_{\vec{k}} \partial_\alpha u^*_{\vec{k}} \right) \left(a_{\vec{k}'} \partial_\beta u_{\vec{k}'} + a^{\dagger}_{\vec{k}'} \partial_\beta u^*_{\vec{k}'} \right)$$

From (2.19),

$$\langle 0 | a^{\dagger}_{\vec{k}} = 0 \qquad (2.42)$$

$$\langle 0 | a_{\vec{k}} a^{\dagger}_{\vec{k}'} | 0 \rangle = \delta_{\vec{k}\vec{k}'}$$

$$\langle 0 | \phi_{,\alpha} \phi_{,\beta} | 0 \rangle = \sum_{\vec{k}} u_{\vec{k},\alpha} u^*_{\vec{k},\beta} \qquad (2.43)$$

In general,

$$\langle 0 | T_{\alpha\beta} | 0 \rangle = \sum_{\vec{k}} T_{\alpha\beta} \left[u_{\vec{k}}, u^*_{\vec{k}} \right] \qquad (2.43)$$

Similarly,

$$\left\langle {}^{1}n_{\vec{k}_1}, {}^{2}n_{\vec{k}_2}, \ldots \right| T_{\alpha\beta} \left| {}^{1}n_{\vec{k}_1}, {}^{2}n_{\vec{k}_2}, \ldots \right\rangle$$

$$= \sum_{\vec{k}} T_{\alpha\beta}\left[u_{\vec{k}}, u^{*}_{\vec{k}}\right] + 2\sum_{i} {}^{i}n\, T_{\alpha\beta}\left[u_{\vec{k}_i}, u^{*}_{\vec{k}_i}\right] \qquad (2.44)$$

2.5. Dirac Spinor Field :

$$\mathcal{L} = \frac{1}{2} i\left(\bar{\psi}\gamma^{\alpha}\psi_{,\alpha} - \bar{\psi}_{,\alpha}\gamma^{\alpha}\psi\right) - m\bar{\psi}\psi \qquad (2.45)$$

2.6. Electromagnetic field

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \tag{2.54}$$

$$F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha} \tag{2.55}$$

$\delta S = \delta\left(\int \mathcal{L} \, d^n x \right) = 0$ yields

$$F^{\alpha\beta}{}_{,\beta} = 0 \tag{2.56}$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{2.57}$$

The lagrangian in (2.54) is invariant under local gauge transformation

$$A_\alpha \rightarrow A_\alpha^\Lambda = A_\alpha + \partial_\alpha \Lambda(x) \tag{2.58}$$

where $\Lambda(x)$ is an arbitrary differentiable scalar f^n.