3.2.2. Singularity in causal structure:

There are some Killing vectors that do fulfill (3.15) everywhere in anti-de Sitter space, for example $\frac{\partial}{\partial \theta}$, where θ is the angular coordinate appearing in (3.6).

However, the Killing vectors appearing in the identifications that give rise to the black hole are timelike or null in some regions. These regions must be cut out from anti-de Sitter space to make the identifications permissible. The resulting space, which we denote (adS) is invariantly under (3.13) because the norm of a Killing vector is constant along its orbits. Hence, the quotient can still be taken.

The space (adS)' is geodesically incomplete since one can find geodesics that go from $\xi \cdot \xi > 0$ to $\xi \cdot \xi < 0$. From the point of view of (adS)' - i.e., prior to the identifications - it is quite unnatural to remove the regions where $\xi \cdot \xi$ is not positive. However, once the identificationts are made, the frontier of region $\xi \cdot \xi > 0$, i.e., Explicit form of the identification:

$$\xi = \frac{r_+}{\ell} J_{12} - \frac{r_-}{\ell} J_{03} - J_{13} + J_{23} \qquad (3.17)$$

We compare this with $\xi = \frac{1}{2}\omega^{ab} J_{ab}$

and thus we get the real eigenvalues $\pm \frac{r_\pm}{\ell}$.

Hence, $I_1 = \omega_{ab}\,\omega^{ab} = -\frac{2}{\ell^2}\left(r_+^2 + r_-^2\right) = -2M$

$$I_2 = \frac{1}{2}\epsilon^{abcd}\,\omega_{ab}\,\omega_{cd} = -\frac{4}{\ell^2}\, r_+ r_- = -2\frac{|J|}{\ell}.$$

(3.18)

According to Appendix A. the ξ in (3.17) is

» of type I_b where $r_+ \neq r_-$

» of type II_a when $r_+ = r_- \neq 0$

» of type III^+ when $r_+ = r_- = 0$

To prove that the identifications by $e^{2\pi k \xi}$ yield the blackhole metric, we start by considering the non-extreme case $r_+^2 - r_-^2 > 0$. In that case, by performing an $SO(2,2)$ transformation, one can eliminate the last term in (3.17) and replace ξ by the simpler expression

$$\xi' = \frac{r_+}{e} J_{12} - \frac{r_-}{e} J_{03}$$

From (3.9) and (3.11) we get

$$-\xi = \frac{r_+}{e}\left(z\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial\beta} + \gamma\frac{\partial}{\partial\gamma}\right) - \frac{r_-}{e}\left(\beta\frac{\partial}{\partial\gamma} + \gamma\frac{\partial}{\partial\beta}\right)$$

$$+\frac{\partial}{\partial\beta} \qquad (3.20)$$

The norm of ξ' is given by

$$\xi' = \frac{r_+}{\ell}\begin{bmatrix} 0 \\ x \\ u \\ 0 \end{bmatrix} - \frac{r_-}{\ell}\begin{bmatrix} y \\ 0 \\ 0 \\ v \end{bmatrix} = \begin{bmatrix} -\frac{r_-}{\ell} y \\ \frac{r_+}{\ell} x \\ \frac{r_+}{\ell} u \\ -\frac{r_-}{\ell} v \end{bmatrix}$$

$$\xi' \cdot \xi' = \eta_{\alpha\beta}\, \xi'^{\alpha} \xi'^{\beta}$$

$$= \xi'^{T} \eta\, \xi$$

$$= -\left(\frac{r_-}{\ell}\right)^2 y^2 - \left(\frac{r_+}{\ell}\right)^2 x^2 + \left(\frac{r_+}{\ell}\right)^2 u^2 + \left(\frac{r_-}{\ell}\right)^2 v^2$$

$$\therefore\ \xi' \cdot \xi' = \left(\frac{r_+}{\ell}\right)^2 \left(u^2 - x^2\right) + \left(\frac{r_-}{\ell}\right)^2 \left(v^2 - y^2\right) \qquad (3.23)$$

Using (3.2)

$$\xi' \cdot \xi' = \frac{r_+^2 - r_-^2}{\ell^2}\left(u^2 - x^2\right) + r_-^2 \qquad (3.24)$$

According to the allowed region, where $\xi' \cdot \xi' > 0$ is

$$-\frac{r_-^2 \ell^2}{r_+^2 - r_-^2} < u^2 - x^2 < \infty \qquad (3.25)$$

The region $\xi' \cdot \xi' > 0$ can be divided in an ~~ist~~ infinite number of regions of three different types bounded by the null surfaces $u^2 - x^2 = 0$ or $v^2 - y^2 = l^2 - (u^2 - x^2) = 0$.

<u>Regions of type I</u> : Smallest connected regions with $u^2 - x^2 > l^2$ and y and v ~~at~~ of definite sign. This regions have no ~~restrictions~~ intersection with $y = 0$ this would violate $u^2 - x^2 = l^2 + y^2 - v^2 > l^2$.

These regions are called outer regions. The norm of the Killing vector fulfills $r_+^2 < \xi' \cdot \xi' < \infty$.

<u>Regions of type II</u> : Smallest connected regions with $0 < u^2 - x^2 < l^2$ and u and v of definite sign. These regions are called "intermediate ~~signs~~ regions". The norm of the Killing vector fulfills $r_-^2 < \xi' \cdot \xi' < r_+^2$.

$$\xi' \cdot \xi' = \frac{r_+^2 - r_-^2}{l^2} (u^2 - x^2) + r_-^2$$

$$\Rightarrow \quad r_-^2 < \xi' \cdot \xi' < r_+^2$$

Scanned with CamScanner

Regions of type III: Smallest connected regions with

$-\frac{r_-^2 \ell^2}{r_+^2 - r_-^2} < u^2 - x^2 < 0$ and x and v of definite sign.

These ~~regions~~ regions are called "inner regions" and only exists for $r_- \neq 0$. They do not intersect the $x = 0$ plane.

The norm of the Killing vectors fulfills $0 < \xi' \cdot \xi' < r_-^2$

$$\xi' \cdot \xi' = \frac{r_+^2 - r_-^2}{\ell^2} (u^2 - x^2) + r_-^2$$

$$\Rightarrow \quad 0 < \xi' \cdot \xi' < r_-^2$$

Scanned with CamScanner

Region I: $r_+ < r$:

$$\begin{aligned}
u &= \sqrt{A(r)}\ \cosh \tilde{\phi}(t,\phi) \\
x &= \sqrt{A(r)}\ \sinh \tilde{\phi}(t,\phi) \\
y &= \sqrt{B(r)}\ \cosh \tilde{t}(t,\phi) \\
v &= \sqrt{B(r)}\ \cosh \tilde{t}(t,\phi)
\end{aligned} \qquad (3.26)$$

Region II: $r_- < r < r_+$:

$$\begin{aligned}
u &= \sqrt{A(r)}\ \cosh \tilde{\phi}(t,\phi) \\
x &= \sqrt{A(r)}\ \sinh \tilde{\phi}(t,\phi) \\
y &= -\sqrt{B(r)}\ \cosh \tilde{t}(t,\phi) \\
v &= -\sqrt{-B(r)}\ \sinh \tilde{t}(t,\phi)
\end{aligned} \qquad (3.27)$$

Region III: $0 < r < r_-$:

$$\begin{aligned}
u &= \sqrt{-A(r)}\ \overset{\sinh}{\cancel{\cosh}}\ \tilde{\phi}(t,\phi) \\
x &= \sqrt{-A(r)}\ \overset{\cosh}{\cancel{\sinh}}\ \tilde{\phi}(t,\phi) \\
y &= -\sqrt{-B(r)}\ \sinh \tilde{t}(t,\phi) \\
v &= -\sqrt{-B(r)}\ \cancel{\sinh}\ \cosh \tilde{t}(t,\phi)
\end{aligned} \qquad (3.28)$$

where,

$$\left.\begin{aligned} A(r) &= \ell^2 \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right) \\ B(r) &= \ell^2 \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right) \end{aligned}\right\} \tag{3.29}$$

$$\left.\begin{aligned} \tilde{t} &= \frac{1}{\ell} \left(\frac{r_+ t}{\ell} - r_- \phi \right) \\ \tilde{\phi} &= \frac{1}{\ell} \left(\frac{r_- t}{\ell} + r_+ \phi \right) \end{aligned}\right\}$$

Then in coordinates t, r, ϕ the metric becomes

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2 (N^\phi \, dt + d\phi)^2 \tag{3.30}$$

with $-\infty < t < \infty$, $-\infty < \phi < \infty$ i.e., it is the black hole metric but with ϕ a non-periodic coordinates. The killing vector ξ' reads

$$\xi' = \frac{\partial}{\partial \phi}$$

By making the identification

$$\phi \to \phi + 2k\pi$$

one gets the blackhole spacetime as claimed above.

3.2.4. Extreme Case:

$$\boxed{r_+ = r_-} \longrightarrow \text{Extreme case}$$

When ϕ is not identified, it describes a portion of anti-de Sitter space for any value of $r_+^2 - r_-^2 > 0$, hence it also in a limit $r_+ - r_- \longrightarrow 0$. Similarly $\frac{\partial}{\partial \phi}$ is k.v. for any value of r_- and r_+.

$$I_1 = -\frac{2(r_+^2 + r_-^2)}{\ell^2} \quad ; \quad I_2 = -\frac{4r_+r_-}{\ell^2}$$

$$\lim_{r_+ \to r_-} I_1 = \lim_{r_+ \to r_-} I_2$$

$$\boxed{u + x > 0, \quad u + x < 0, \quad u - x > 0, \quad u - x < 0} \longrightarrow \text{4 patches.}$$

$$\beta = \frac{1}{2}\left(\frac{T}{\ell} + \phi + e^{2r_+\phi} - \frac{1}{2r_+}\right)$$

$$\gamma = \frac{1}{2}\left(\frac{T}{\ell} + \phi - e^{2r_+\phi} + \frac{1}{2r_+}\right)$$

$$z = \frac{1}{2r_+}\left(r^2 - r_+^2\right) e^{r_+\phi}$$

where $T' = 2t - \frac{l^2 r_+}{r^2 - r_+^2}$

$$dT' = 2dt + \frac{2r_+ l^2 r\, dr}{(r^2 - r_+^2)^2}$$

3.2.5. Absence of closed time like curve :

Causality property reads

$$(N^\perp)^2\left(\frac{dt}{d\lambda}\right)^2 = (N^\perp)^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(N^\phi \frac{dt}{d\lambda} + \frac{d\phi}{d\lambda}\right)^2 \leq 0$$

3.2.6 Blackhole has only 2 k.v.-s :

$$\exp(2\pi\xi)\,\eta\,[\exp(2\pi\xi)]^{-1} = \eta$$

$$\Rightarrow [\exp(2\pi\xi), \eta] = 0$$

From appendix A., —

$\xi = S + n$ → nilpotent part.

S → semi simple part with eigenvalues

$\therefore$ Semi Simple part of $\exp(2\pi\xi)$ is $\exp(2\pi s)$

$\therefore$ nilpotent part $= \exp(2\pi\xi) - \exp(2\pi s)$

$$= \exp(2\pi s)\exp(2\pi n) - \exp(2\pi s)$$

$$= \exp(2\pi s)\left(\exp(2\pi n) - 1\right)$$

$\therefore [s, \eta] = 0$

$\therefore [n, \eta] = 0$

$\therefore [\xi, \eta] = 0$

4. Global Structure :

4.1. Kruskal coordinates :

$$ds^2 = -\left(N^{\perp}\right)^2 dt^2 + \left(N^{\perp}\right)^{-2} dr^2 + r^2\left(N^{\phi} dt + d\phi\right)^2 \qquad (4.1)$$

$$ds^2 = \Omega^2\left(du^2 - dv^2\right) + r^2\left(N^{\phi} dt + d\phi\right)^2 \qquad (4.2)$$

4.2 Penrose Diagrams : $(r_+ \neq r_-)$

$$U + V = \tan\left(\frac{p+q}{2}\right) \quad ; \quad U - V = \tan\left(\frac{p-q}{2}\right) \qquad (4.10)$$

Scanned with CamScanner

4.3. Extreme Cases $M = 0$ and $M = \frac{|J|}{l}$

4.3.1. $M = 0$

$$ds^2 = \left(\frac{r}{l}\right)^2 dt^2 + \left(\frac{r}{l}\right)^{-2} dr^2 + r^2 d\phi^2$$

$$\left.\begin{aligned} u &= \frac{t}{l} - \frac{l}{r} \\ v &= -\frac{t}{l} - \frac{l}{r} \end{aligned}\right\} \quad (4.12)$$

$$ds^2 = r^2 du\, dv + r^2 d\phi^2 \qquad (4.13)$$

$UV = 2\tan$ [illegible]

$$ds^2 = l^2 \frac{dp^2 - dq^2}{\sin^2 p} + r^2 d\phi^2 \qquad (4.16)$$