Appendix - A

The exact exterior solution for a mass m at rest is the well known Schwarzschild metric, which in isotropic coordinates is given by

$$ds^2 = -\frac{(1-B)^2}{(1+B)^2} dT^2 + (1+B)^4 (dx^2 + dy^2 + dZ^2)$$

Where $B = \frac{m}{2R}$

Now, if $m << R$ then $A \sim 0$ and

$$ds^2 \simeq -(1-4B)\, dT^2 + (1+4B)(dx^2 + dy^2 + dZ^2) \qquad (A.1)$$

Lets, boost the rest frame with respect to coordinates (t, x, y, z) via

(A.2)

$$T = t \cosh\beta - z \sinh\beta$$

$$Z = -t \sinh\beta + z \cosh\beta$$

$$-dT^2 + dZ^2 = -dt^2 + dz^2$$

$$\Rightarrow dT^2 - dt^2 = dZ^2 - dz^2$$

$$\therefore ds^2 = -(1-4B)\, dT^2 + (1+4B)(dx^2 + dy^2 + dZ^2)$$

$$= -(1-4B)\, dt^2 + (1+4B)(dx^2 + dy^2 + dz^2) + 4B(dT^2 + dZ^2)$$

$$-4B(dt^2 + dz^2)$$

And lets set, $m = 2pe^{-\beta}$ for $p = \text{constant} > 0$ $\quad (A.3)$

Lets introduce null coordinates

$$u = t - z$$
$$v = t + z \qquad (A.4)$$

The momentum of the particle is

$$p^a = m\left[(\cosh\beta)\,\delta^a_t + (\sinh\beta)\,\delta^a_z\right] \qquad (A.5)$$

and thus

$$\lim_{\beta \to \infty} p^a = p\left(\delta^a_t + \delta^a_z\right) = 2p\,\delta^a_v \qquad (A.6)$$

in the limit the particle is massless and moves (at the speed of light) in the v-direction; p is its momentum and is kept finite.

$$t = \frac{u+v}{2}$$

$$R^2 = x^2 + y^2 + Z^2 \qquad z = \frac{v-u}{2}$$

$$Z = -t\sinh\beta + z\cosh\beta$$

$$= -t\,\frac{e^{\beta} - e^{-\beta}}{2} + z\,\frac{e^{\beta} + e^{-\beta}}{2}$$

$$= -\frac{u+v}{2}\,\frac{e^{\beta} - e^{-\beta}}{2} + \frac{v-u}{2}\,\frac{e^{\beta} + e^{-\beta}}{2}$$

$$= \frac{u}{4}\left(-e^{\beta} + \cancel{e^{-\beta}} - e^{\beta} - \cancel{e^{-\beta}}\right) + \frac{v}{4}\left(\cancel{e^{\beta}} + e^{-\beta} + \cancel{e^{\beta}} + e^{-\beta}\right)$$

$$\Rightarrow Z = \frac{u}{4} \cdot (-2) \frac{2p}{m} + \frac{v}{4} \cdot 2 \cdot \frac{m}{2p}$$

$$\Rightarrow Z = -\frac{p}{m} u + \frac{m}{4p} v$$

Similarly, $T = \frac{p}{m} u + \frac{m}{4p} v$

$$\therefore R^2 = x^2 + y^2 + \left(\frac{p}{m} u - \frac{m}{4p} v\right)^2 \qquad (A.8)$$

$$\therefore ds^2 = -(1-4B)\, dT^2 + (1+4B)(dx^2 + dy^2 + dZ^2)$$

$$= (1+4B)(-dT^2 + dZ^2 + dx^2 + dy^2) + 8A\, dT^2$$

$$= (1+4B)(-dt^2 + dz^2 + dx^2 + dy^2) + 8A\, dT^2$$

$$= (1+4B)(-du\,dv + dx^2 + dy^2) + 8B\left(\frac{p}{m} du + \frac{m}{4p} dv\right)^2 \qquad (A.7)$$

$$\therefore \lim_{\substack{m \to 0 \\ (u \neq 0,\, v,\, x,\, y) \text{ fixed}}} ds^2 = -du\left(dv - 4p \frac{du}{|u|}\right) + dx^2 + dy^2 \qquad (A.9)$$

This metric is flat although the coordinate v' satisfying

$$dv' = dv - \frac{4p\, du}{|u|} \qquad (A.10)$$

suffers a discontinuity at $u = 0$.

Lets introduce, —

$$\hat{u} = u + \frac{m^2 Z \ln(2R)}{pR} \; ; \; \hat{v} = v + \frac{4pZ \ln(2R)}{R} \qquad (A.11)$$

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$$\therefore R^2 = x^2 + y^2 + \left(\frac{p}{m}\hat{u} - \frac{m}{4p}\hat{v}\right)^2 \qquad (A.12)$$

$$\therefore R = \sqrt{x^2 + y^2 + \left(\frac{p}{m}u - \frac{m}{4p}v\right)^2}$$

$$\therefore \lim_{m \to 0} \frac{1}{R} = \frac{m}{p|u|} \qquad \boxed{u \neq 0}$$

$$\therefore \lim_{m \to 0} \frac{Z}{R} = \frac{p}{m}u \cdot \frac{m}{p|u|} = \frac{u}{|u|} = \theta$$

$$\therefore \hat{u} = u + \frac{m^2 Z \ln(2R)}{pR}$$

$$\Rightarrow d\hat{u} = du + \frac{m^2}{p} d\left(\frac{u}{|u|} \ln\left(\frac{p}{m}|u|\right)\right)$$

$$= du + \cancel{\frac{m^2}{p}}^{\to 0} \theta \frac{d(|u|)}{|u|}$$

$$= du \cancel{+ \ldots}$$

$$\therefore \hat{v} = v + \frac{4pZ\ln(2R)}{R}$$

$$\Rightarrow d\hat{v} = dv + 4p\frac{du}{|u|} \cancel{(\ldots)}$$

$$\lim_{m\to 0} ds^2 = -du\, dv + dx^2 + dy^2 \qquad (A.13)$$

(u, v, x, y) fixed

The timelike geodesics of metric (1) are given by,—

$$\dot{T} = E\left(1 + \frac{2m}{R}\right)$$

$$y\dot{Z} - Z\dot{y} = L\left(1 - \frac{2m}{R}\right)$$

$$\dot{y}^2 + \dot{Z}^2 = -M^2\left(1 - \frac{2m}{R}\right) + E^2 \qquad (A.14)$$

E = energy
L = angular momentum
M = mass of the test particle

~~where~~ the dot denotes derivatives w.r.t. affine parameter λ.

Now, we consider only null geodesics. (we set $M = O(m^2)$)

~~Expand~~ Expanding y, Z, T in powers of m and considering only the terms linear in m, we have,—

$$y = y_0 + m y_1$$
$$Z = Z_0 + m Z_1 \qquad (A.15)$$
$$T = T_0 + m T_1$$

Then from (A.14) we get,—

$$\dot{T}_0 = E$$

$$\dot{T}_1 = \frac{2E}{R_0}$$

$$y_0 \dot{Z}_0 - Z_0 \dot{y}_0 = L$$

$$y_0 \dot{Z}_1 - Z_1 \dot{y}_0 + y_1 \dot{Z}_0 - Z_0 \dot{y}_1 = -\frac{2L}{R_0}$$

$$\dot{y}_0^2 + \dot{Z}_0^2 = E^2$$

$$\dot{y}_0 \dot{y}_1 + \dot{Z}_0 \dot{Z}_1 = 0$$

where $R_0^2 = y_0^2 + Z_0^2$

(A.16)

We know, —

$$u = \frac{m}{2p}(T - Z) \qquad (A.17)$$

$$v = \frac{2p}{m}(T + Z)$$

We need $\dot{v}$ (and thus $\dot{v}$) to remain finite in the limit as $m \to 0$.

So, $(\dot{T}_0 + \dot{Z}_0) = 0$

$$\Rightarrow \dot{Z}_0 = -\dot{T}_0 = -E \qquad (A.18)$$

From (A.16)

$$\therefore \dot{y}_0 = 0$$

$$y_0 = -\frac{L}{E} \qquad (A.19)$$

$$\therefore \dot{Z}_1 = 0 \qquad (A.20)$$

$$\therefore \dot{u} = \frac{m}{2p}\left(\dot{T}_0 + m\dot{T}_1 - \dot{Z}_0 - m\dot{Z}_1\right)$$

$$= \frac{mE}{p} + \frac{m^2 E}{pR_0}$$

$$(A.21)$$

$$\therefore \dot{v} = \frac{2p}{m}\left(\dot{T}_0 + m\dot{T}_1 + \dot{Z}_0 + m\dot{Z}_1\right)$$

$$= 4p\frac{E}{R_0}$$

$$\therefore \frac{d}{d\lambda}\left(\ln\left(Z_0 + R_0\right)\right) = \frac{\dot{Z}_0 + \dot{R}_0}{Z_0 + R_0} = \cancel{\frac{\dot{Z}_0 + \frac{\dot{y}_0 + \dot{Z}_0}{R_0}}{Z_0 + R_0}}$$

$$= \frac{\dot{Z}_0 + \frac{Z_0 \dot{Z}_0}{R_0}}{Z_0 + R_0}$$

$$= \frac{\dot{Z}_0}{R_0}$$

$$\Rightarrow \frac{d}{d\lambda}\left(\ln\left(Z_0 + R_0\right)\right) = -\frac{E}{R_0}$$

$\therefore$ (A.21) can be integrated to give,—

$$u = \frac{mE}{p}\lambda - \frac{m^2}{p}\ln\left(Z_0 + R_0\right)$$
$$v = -4p\ln\left(Z_0 + R_0\right) \qquad \text{(A.22)}$$

where we have ignored an irrelevant integration constant and thus,—

$$\hat{u} = \frac{mE}{p}\lambda + \frac{m^2}{p}\left[\frac{Z_0 \ln(2R_0)}{R_0} - \ln\left(Z_0 + R_0\right)\right]$$

$$\hat{v} = 4p\left[\frac{Z_0 \ln(2R_0)}{R_0} - \ln\left(Z_0 + R_0\right)\right]$$

We now separate into a near region N and a far region F as follows:

$$\left.\begin{aligned} N &= \left\{ |\lambda| < \frac{1}{\sqrt{m}} \right\}; \\ F &= \left\{ \sqrt{m} \leq m|\lambda| < \infty \right\} \end{aligned}\right\} \quad (A.21)$$

$\therefore \lim_{\lambda \to -\infty} \hat{v} = 0$

$\therefore \lim_{\lambda \to \infty} \hat{v}$

$= -4p \ln\left[2R_0 (R_0 + Z_0)\right]$

$= -4p \ln y_0^2$

$\therefore \lim_{\lambda \to \pm\infty} \hat{u} = \frac{mE}{p}\lambda$

(A.25)

$\lim_{\lambda \to -\infty} Z_0 = \lim_{\lambda \to -\infty} R_0$

$\lim_{\lambda \to \infty} Z_0 = -\lim_{\lambda \to \infty} R_0$

When $\lambda \to \pm\infty$

$Z_0 = -E\lambda \Rightarrow p = \frac{-mZ_0}{\hat{u}}$

Thus the total shift in $\hat{v}$ is given by

$$\Delta\hat{v} = -4p \ln y_0^2 \qquad (A.26)$$

In the limit $m \to 0$, λ is infinite everywhere in F and $\hat{u}$ is zero in N, whereas $\hat{u}$ is a good affine parameter in F along the geodesic.

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The shift (A. 26) thus occurs, for small m, "essentially" only in N! Thus, in the limit as m goes to zero, the shift (A.26) occurs at $\hat{u}=0$ and represents a finite disconuity in $\hat{v}$ along null geodesics! This can also be seen by calculating

$$\lim_{\lambda \to \pm\infty} \dot{\hat{v}} = 0$$

thus showing that in the limit as m goes to zero $\hat{v}$ is constant in F, i.e for nonzero $\hat{u}$. This is just a reflection of the fact that, in the limit, F is flat. This is shown in fig. 2.

On inserting eqns. (A. 19) and (A. 20) in (A. 16), —

$$y_0 \dot{Z}_1 - Z_1 \dot{y}_0 + y_1 \dot{Z}_0 - \dot{Z}_0 \dot{y}_1 = - \frac{2L}{R_0}$$

$$\Rightarrow y_1 \dot{Z}_0 - \cancel{Z_0 y_1}\, Z_0 \dot{y}_1 = - \frac{2L}{R_0} \qquad (A.28)$$

The homogeneous ~~solution~~ equation for (A.28) has solution

$$y_1^h = A Z_0 \qquad (A.29)$$

for any constant A; it remains to find a perfect solution.

Multiplying eq. (A.28) by $\dot{Z}_0$ yields

$$E^2 y_1 - R_0 \dot{R}_0 \dot{y}_1 = \frac{2LE}{R_0} \qquad (A.30)$$

$$\boxed{\because Z_0^2 + y_0^2 = R_0^2}$$

$$R_0^2 = Z_0^2 + y_0^2$$

$$\Rightarrow R_0 \dot{R}_0 = Z_0 \dot{Z}_0$$

$$\Rightarrow \dot{R}_0^2 = \frac{Z_0^2 E^2}{R_0^2} = E^2\left(1 - \frac{y_0^2}{R_0^2}\right) \qquad \cancel{(A.30)} \quad (A.31)$$

$$\left[\because Z_0^2 = R_0^2 - y_0^2\right]$$

Lets suggest an ansatz for y_1 as a power series in R_0.
We thus obtain the particlar solution

$$y_1^p = \frac{2L}{y_0^2 E} R_0 = -\frac{2R_0}{y_0} \qquad (A.32)$$

The general solution to eqn. (A.28) is thus

$$y_1 = -\frac{2R_0}{y_0} + A Z_0 \qquad (A.33)$$

and therefore

$$y = -\frac{L}{E} + m\left[-\frac{2R_0}{y_0} + A Z_0\right] \qquad (A.34)$$

We are interested in the behaviour of y in far field F for m small. We obtain

$$\lim_{\substack{m \to 0 \\ \hat{u} \neq 0}} \frac{\partial y}{\partial \hat{u}}$$

$$\cancel{= \lim_{\substack{m \to 0 \\ \hat{u} \neq 0}} \frac{\partial}{\partial \hat{u}} \left[-\frac{Lmp}{\lambda \hat{u}} \right.}$$

$$Z_0 = -E\lambda$$

$$y_0 = -\frac{L}{E}$$

$$R_0 = \sqrt{E^2\lambda^2 + \frac{L^2}{E^2}}$$

$$\lim_{m \to 0} \hat{u} = \frac{mE}{\lambda} p$$

$$\frac{R_0}{y_0} = \sqrt{1 + \frac{Z_0^2}{y_0^2}} = \sqrt{1 + \frac{E^4\lambda^2}{L^2}}$$

$$\lim_{\substack{m\to 0\\ \hat{v}\neq 0}} y = -\frac{L}{E} - \frac{2p}{y_0}\frac{R_0}{Z_0}\hat{v} - pA\hat{v}$$

by using the fact when $m \to 0$ then $p = -\frac{mZ_0}{\hat{v}}$

further $\lim_{m\to 0} y = -\frac{L}{E} - \frac{2p}{y_0}\,\mathrm{sgn}\,\hat{v}\cdot\hat{v} - pA\hat{v}$

$$\lim_{\substack{m\to 0\\ \hat{v}\neq 0}} \frac{\partial y}{\partial \hat{v}} = -\frac{2p}{y_0}\,\mathrm{sgn}\,\hat{v} - pA$$

This behaviour is illustrated in fig 3. In general we have

$$\cot\alpha + \cot\beta = \frac{4p}{y_0}$$

From appendix B, —

$$\lim_{m\to 0} ds^2 = -d\hat{u}\left(d\hat{v} + 4p\ln y_0^2\,\delta(\hat{u})\,d\hat{u}\right) + dx^2 + dy^2$$

$$= -du\left(dv + \frac{4p\,du}{u}\left(1-2\theta(u)\right) + 4p\ln y_0\,\delta(u)\,du\right) + dx^2 + dy^2$$

For $u=0$, this metric gives us (A.13) and (A.9).