

On gravitational shock waves in curved spacetimes

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Abstract

Some years ago Dray and 't Hooft found the necessary and sufficient conditions to introduce a gravitational shock wave in a particular class of vacuum solutions to Einstein's equations. We extend this work to cover cases where non-vanishing matter fields and a cosmological constant are present. The sources of gravitational waves are massless particles moving along a null surface such as a horizon in the case of black holes. After we discuss the general case we give many explicit examples. Among them are the d -dimensional charged black hole (that includes the 4-dimensional Reissner–Nordström and the d -dimensional Schwarzschild solution as subcases), the 4-dimensional De Sitter and anti-De Sitter spaces (and the Schwarzschild–De Sitter black hole), the 3-dimensional anti-De Sitter black hole, as well as backgrounds with a covariantly constant null Killing vector. We also address the analogous problem for string-inspired gravitational solutions and give a few examples.

1. Introduction

In an S -matrix approach to black hole physics [1] one has to take into account the interactions between Hawking particles emitted from the black hole. However, in order to do that one would also like to know how the presence of particles around a black hole affects the black hole itself (the black hole reacts back). It has been convincingly argued [1] that the gravitational interaction between particles at Planckian energies (see Refs. [2–5]) dominates any other type of interaction and therefore one needs to know what gravitational effects the particles have on the original black hole geometry.

For the case of a massless particle moving along the horizon of a Schwarzschild black hole the result of this elementary classical black hole back reaction has been found by

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Dray and 't Hooft [6]. What they found generalized the gravitational shock wave due to a massless particle moving in flat Minkowski space [7]. In fact, Ref. [6] contains the necessary and sufficient conditions for being able to introduce a gravitational shock wave via a coordinate shift in more general *vacuum* solutions to Einstein's equations.²

The purpose of this paper is to extend the results of Ref. [6] to the case when matter fields and even a non-vanishing cosmological constant are present, since many interesting gravitational solutions belong in this category. The organization of this paper and some of the important results are the following: In Section 2 we address the problem of how a quite general class of solutions to Einstein's equations in the *presence of matter and cosmological constant* (in d dimensions) back reacts to a massless particle moving along a null surface. We also examine the geodesics of particles moving in these geometries and show that discontinuity and refraction effects take place as they try to cross the null surface. Our treatment, as the one in Ref. [6], is non-perturbative but exact. In Section 3 we consider the case of the d -dimensional Reissner–Nordström charged black hole [12]. After we give the general result we concentrate to the 4-dimensional case which physically is the most interesting one. We find that as we approach the extremal case of equal mass and charge the effect the massless particle has become gigantic in magnitude. In Section 4 we consider firstly the De Sitter and anti-De Sitter constant curvature spaces in four dimensions. In the former case we find that the discontinuity in geodesics we have already mentioned depends in a rather unexpected way on the angular distance from the position of the massless particle. We also consider the case of the Schwarzschild–De Sitter black hole which interpolates between the De Sitter space and the Schwarzschild black hole. We close this section with the 3-dimensional anti-De Sitter space which, after a discrete identification, can be interpreted as a 3-dimensional black hole [13]. In Section 5 we consider the analogous problem for string-inspired gravitational solutions. In particular for a 4-dimensional electrically and magnetically charged dilatonic black hole [14,15] and the background corresponding to the conformal field theory $SL(2, \mathbb{R})/\mathbb{R} \otimes \mathbb{R}^2$. We end the paper with concluding remarks and discussion in Section 6. Mainly in order not to interrupt the flow of the paper with too many mathematical details we have written Appendices A, C and D. In Appendix B we consider the case where the background geometry has a covariantly constant null Killing vector [16] (these geometries do not belong to the class considered already in Section 2).

2. General results

Let us consider the d -dimensional spacetime described by the metric

$$ds^2 = 2 A(u, v) du dv + g(u, v) h_{ij}(x) dx^i dx^j, \quad (2.1)$$

² The solution of Ref. [6] can be formulated as two Schwarzschild black holes of *equal masses* glued together at the horizon. For a spherical shell of massless matter moving along $u = u_0 \neq 0$ the solution [8] represents two Schwarzschild black holes of *unequal masses* glued together at $u = u_0$. Various gravitational shock waves obtained by infinitely boosting [7,6] known solutions have been found (see for instance Refs. [3,9–11]).

with $(i, j = 1, 2, \dots, d-2)$. Let us also assume that there exist some matter fields with non-vanishing components of the energy-momentum tensor given by

$$T = 2T_{uv}(u, v, x) du dv + T_{uu}(u, v, x) du^2 + T_{vv}(u, v, x) dv^2 + T_{ij}(u, v, x) dx^i dx^j. \quad (2.2)$$

Notice that this form of energy-momentum tensor is consistent with the Ricci tensor for (2.1) as given by (A.2) for $f = 0$.

We consider a massless particle located at $u = 0$ and moving with the speed of light in the v -direction and we want to find out what its effect is on the geometry described by (2.1). Similarly to Ref. [6], our ansatz will be that for $u < 0$ the spacetime is described by (2.1) and for $u > 0$ by (2.1) but with v shifted as $v \rightarrow v + f(x)$, where $f(x)$ is a function to be determined. Therefore the resulting spacetime metric and energy-momentum tensor are

$$ds^2 = 2A(u, v + \Theta f) du (dv + \Theta f_{,i} dx^i) + g(u, v + \Theta f) h_{ij}(x) dx^i dx^j, \quad (2.3)$$

where $\Theta = \Theta(u)$ is the Heaviside step function and³

$$T = 2T_{uv}(u, v + \Theta f, x) du (dv + \Theta f_{,i} dx^i) + T_{uu}(u, v + \Theta f, x) du^2 + T_{vv}(u, v + \Theta f, x) (dv + \Theta f_{,i} dx^i)^2 + T_{ij}(u, v + \Theta f, x) dx^i dx^j. \quad (2.4)$$

In order to compute various tensors it would be easier to transform to the new coordinates

$$\hat{u} = u, \quad \hat{v} = v + f(x)\Theta(u), \quad \hat{x}^i = x^i, \quad (2.5)$$

in which the metric (2.3) and the energy-momentum tensor take the form

$$ds^2 = 2\hat{A} d\hat{u} d\hat{v} + \hat{F} d\hat{u}^2 + \hat{g} \hat{h}_{ij} d\hat{x}^i d\hat{x}^j, \\ \hat{F} = F(\hat{u}, \hat{v}, \hat{x}) = -2\hat{A} \hat{f} \hat{\delta}, \quad (2.6)$$

and

$$T = 2(\hat{T}_{\hat{u}\hat{v}} - \hat{T}_{\hat{v}\hat{v}} \hat{f} \hat{\delta}) d\hat{u} d\hat{v} + (\hat{T}_{\hat{u}\hat{u}} + \hat{T}_{\hat{v}\hat{v}} \hat{f}^2 \hat{\delta}^2 - 2\hat{T}_{\hat{u}\hat{v}} \hat{f} \hat{\delta}) d\hat{u}^2 + \hat{T}_{\hat{v}\hat{v}} d\hat{v}^2 + \hat{T}_{ij} d\hat{x}^i d\hat{x}^j, \quad (2.7)$$

where the hats indicate that the corresponding quantities are evaluated at \hat{u} , \hat{v} , \hat{x}^i and where $\hat{\delta} = \delta(\hat{u})$ is the δ -function.

The various metric and field components must satisfy Einstein's equations in the presence of matter, which in d dimensions read (throughout this paper $c = \hbar = G = 1$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \Rightarrow R_{\mu\nu} = -8\pi \left(T_{\mu\nu} - \frac{1}{d-2} g_{\mu\nu} T_{\lambda}^{\lambda} \right) \equiv -8\pi \tilde{T}_{\mu\nu}. \quad (2.8)$$

³ For spherical sourceless shock waves in Minkowski space obtained with a different ansatz than (2.3) see Ref. [17].

Obviously, $\tilde{T} = \tilde{T}_{\mu\nu} dx^\mu dx^\nu$ is also of the form (2.2). We would like to find solutions of (2.8) with metric tensor of the form (2.6) and an energy–momentum tensor given by (2.7) plus the contribution of the energy–momentum tensor for a massless particle located at the origin of the transverse x -space and at $u = 0$ and moving with the speed of light in the v -direction

$$T^p = T_{uu}^p du^2 = \hat{T}_{\hat{u}\hat{u}}^p d\hat{u}^2 = -4p \hat{A}^2 \hat{\delta}^{(d-2)}(\hat{x}) \hat{\delta}(\hat{u}) d\hat{u}^2, \quad (2.9)$$

where p is the momentum of the particle. All the relevant tensor components appearing in (2.8) are given in Appendix A. To simplify the notation we will also drop the hats over the symbols keeping in mind however the transformation (2.5). Assuming that the parts of the equations (2.8) that do not involve the function f are satisfied, one finds by examining the linear in $f\delta$ terms that at $u = 0$ the additional conditions

$$g_{,v} = A_{,v} = T_{vv} = 0, \\ \Delta_{h_{ij}} f - \frac{d-2}{2} \frac{g_{,uv}}{A} f = 32\pi p g A \delta^{(d-2)}(x) \quad (2.10)$$

must also be satisfied, where the Laplacian is defined as $\Delta_{h_{ij}} = 1/\sqrt{h} \partial_i \sqrt{h} h^{ij} \partial_j$. In order to cast the differential equation in (2.10) into the given form we used the fact that at $u = 0$

$$\frac{A_{,uv}}{A} = -\frac{d-2}{2} \frac{g_{,uv}}{g} + 8\pi \tilde{T}_{uv}. \quad (2.11)$$

This equality follows from the $(\mu, \nu) = (u, v)$ components of (2.8) computed at $u = 0$ and for $f = 0$. Notice also that the differential equation in (2.10) does not explicitly depend on the components of the energy–momentum tensor of the matter fields. Its dependence on these fields is only implicit through the functions A, g that are determined from the f -independent part of Einstein's equations.

Next we examine the $f^2\delta^2$ type terms. This is important because such terms should also vanish (in a distribution sense), otherwise our considerations are perturbative in powers of f . Using the first line in (2.10) it is easy to see that the coefficient of $f^2\delta^2$ in the R_{uu} component of the Ricci tensor in (A.2) has terms of order $O(u)$ and $O(u^2)$ (in all of our examples such terms are of order $O(u^2)$ since there is a functional dependence only on the product uv). Remembering that we are really considering all the quantities involving δ -functions as distributions to be integrated over smooth functions we find that in this case such an integral vanishes. Moreover, because $T_{vv} = 0$ at $u = 0$ we have that $T_{vv} = O(u)$ (at least). Therefore we can take the terms in (2.8) involving $f^2\delta^2$ to zero. A last crucial remark is that because we have assumed that f is a function of the x^i 's only, the potential v -dependence in (2.10) should drop out for a consistent solution to exist. Mathematically that implies that the coefficient of the order $O(u)$ term in the expansion of $g(u, v)$ in powers of u should be a linear function of v . In fact this is the case in all the examples we explicitly work out.

For the 4-dimensional case in the absence of matter fields the condition (2.10) was given in Ref. [6]. It is remarkable that (2.10) has essentially the same form in the vacuum and in the presence of matter except that in the latter case the additional

condition $T_{uv} = 0$ at $u = 0$ should be imposed as well. Let us also emphasize that the differential equation in (2.10) is nothing but the Green function equation (of course with $\delta^{(d-2)}(x)$ replaced by $\delta^{(d-2)}(x - x')$). Thus instead of a point massless particle we could easily find the result if we had an extended source with density $\rho(x)$, i.e. $f(x) = \int \rho(x') f(x, x') dx'$. In this way we can consider spherical and planar shells of matter as in Refs. [8,18] or charged particles, cosmic strings, monopoles etc. (for the flat space case see Ref. [3]).

The following remark is now in order. According to the work of Ref. [4] in the 4-dimensional case in the equation for $f(x)$ in (2.10) half (in our normalization) the 2-dimensional curvature $R^{(2)}$ constructed out of the metric h_{ij} should be in place of the factor $g_{,uv}/A$. In our case using (2.8) for $(\mu, \nu) = (i, j)$, the fact that $g_{,v} = 0$ at $u = 0$ and (A.2) we compute that at $u = 0$

$$\frac{d-2}{2} \frac{g_{,uv}}{A} = \frac{1}{2} R^{(d-2)} + 4\pi \tilde{T}_{ij} h^{ij}. \quad (2.12)$$

Thus we obtain the result of Ref. [4], and in fact generalized in any number of spacetime dimensions, if $\tilde{T}_{ij} h^{ij} = 0$ or, after using (2.8), (2.2), if $T_{uv} = 0$. The stronger conditions $T_{ij} = 0$ and $T_{\lambda}{}^{\lambda} = 0$, which imply $\tilde{T}_{ij} = 0$ and in our case $T_{uv} = 0$ were imposed in Ref. [4]. However, in many interesting cases, such as the Reissner–Nordström charged black hole, the component T_{uv} is non-vanishing as we shall see.

Although the explicit form of the metric, once the function $f(x)$ is obtained by solving (2.10), is given by (2.6) it would be helpful to examine the geodesic equations in order to obtain a clearer understanding of how the original geometry (2.1) is affected by the presence of a massless particle moving in the v -direction at $u = 0$. It is shown in Appendix C that as the geodesic trajectory crosses the null surface $u = 0$ there is a shift in its v -component given by

$$\Delta v \equiv v|_{u=0^+} - v|_{u=0^-} = f(x), \quad (2.13)$$

as well as a refraction effect in the transverse x -plane expressed by the “refraction function”

$$R^i(x) \equiv \left. \frac{dx^i}{du} \right|_{u=0^-} - \left. \frac{dx^i}{du} \right|_{u=0^+} = \frac{A}{g} \Big|_{u=0} f_{,j} h^{ji} \quad (2.14)$$

that essentially measures how much the angle that the trajectory forms with the $u = 0$ surface changes as we cross this surface. In other words, a generic trajectory when it crosses $u = 0$ suffers the discontinuity (2.13) in its v -component, with the u and x^i components being continuous at that surface, and moreover its x^i components change direction along u according to (2.14). Although in both phenomena $f(x)$ determines the functional dependence there is however a qualitative difference. In the case of (2.13) the discontinuity equals $f(x)$ whereas in (2.14) the directional derivatives of $f(x)$ play the important role. Therefore there might be points x^i where there is a discontinuity in the trajectory but no refraction effect and vice versa. In fact this will be the case in some of our examples considered in the following sections.

3. The d -dimensional Reissner–Nordström charged black hole

With this section we shall start applying the general formalism that we developed in the previous one by considering the case of a massless particle moving in the outer horizon of a Reissner–Nordström charged black hole in d dimensions. Since the latter is a spherically symmetric, asymptotically flat solution of (2.8), our results will be a natural extension of those obtained by Dray and 't Hooft for the case of the 4-dimensional Schwarzschild black hole. The metric for the Reissner–Nordström charged black hole solution in d dimensions is [12]

$$ds^2 = -\lambda(r) dt^2 + \lambda^{-1}(r) dr^2 + r^2 d\Omega_{(d-2)}^2, \\ \lambda(r) = 1 - \frac{2C}{r^{d-3}} + \frac{E^2}{r^{2(d-3)}}, \quad (3.1)$$

where C, E are constants depending on the dimension of spacetime d and related to the mass and the charge of the black hole (see Ref. [12]). The energy–momentum tensor of the electromagnetic field is given by

$$T_{\mu\nu} = \frac{d}{8\pi(d-2)} \left(F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{d} g_{\mu\nu} F^2 \right), \quad F_{tr} = \sqrt{\frac{1}{2}(d-2)(d-3)} \frac{E}{r^{d-2}}, \quad (3.2)$$

where $\mu, \nu = t, r, i$ and F_{tr} denotes the only non-vanishing component of the electromagnetic field 2-form. It is possible to bring (3.1) into the form (2.1) by means of the following transformation:

$$u = e^{t/\alpha} F(r), \quad v = e^{-t/\alpha} F(r), \quad \alpha = \frac{2r_+}{d-3} \left(1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right)^{-1}, \quad (3.3)$$

where $r_{\pm}^{d-3} = C \pm \sqrt{C^2 - E^2}$, with $r_+ > r_-$, are the outer and inner horizons where the function $\lambda(r)$ in (3.1) vanishes.⁴ It turns out that

$$F(r) = \exp \left(\frac{1}{\alpha} \int dr \lambda^{-1}(r) \right) = \kappa (r - r_+)^{1/2} + \dots \quad (3.4)$$

where κ is a constant and the dots stand for higher order terms in the $(r - r_+)$ expansion.⁵ We also find that

$$A(u, v) = \frac{1}{2} \alpha^2 \lambda(r) \exp \left(-\frac{2}{\alpha} \int dr \lambda^{-1}(r) \right), \quad g(u, v) = r^2, \quad (3.6)$$

⁴ Then since the only non-vanishing components of the electromagnetic field would be F_{uv} , the corresponding energy–momentum tensor (3.2) would be of the form (2.2) with $T_{uu} = T_{vv} = 0$. Nevertheless, its precise form will not be needed for our purposes as we have already explained.

⁵ The explicit expression for κ is quite straightforward to write down in any number of dimensions. However, the result is complicated and not very enlightening. For $d = 4$ the expression is simpler,

$$\kappa = (r_+ - r_-)^{-(r_-^2/r_+^2)/2} e^{(r_+ - r_-)/(2r_+)}. \quad (3.5)$$

and that at $u = 0$ the conditions $g_{,v} = A_{,v} = 0$ of (2.10) are indeed satisfied. Using the values of the following functions at $u = 0$:

$$A = \frac{2r_+}{d-3} \kappa^{-2} \left(1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right)^{-1}, \quad g = r_+^2, \quad g_{,uv} = 2r_+ \kappa^{-2}, \quad (3.7)$$

we find from (2.10) that the shift function satisfies

$$\Delta_{(d-2)} f - a(d, r_+, r_-) f = 2\pi b(d, r_+, r_-) \delta^{(d-2)}(x), \quad (3.8)$$

where $\Delta_{(d-2)}$ is the Laplacian on the unit $(d-2)$ -sphere and $a(d, r_+, r_-)$, $b(d, r_+, r_-)$ are constants defined as

$$\begin{aligned} a(d, r_+, r_-) &\equiv \frac{1}{2} (d-2)(d-3) \left(1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right), \\ b(d, r_+, r_-) &\equiv 32pr_+^3 (d-3)^{-1} \kappa^{-2} \left(1 - \left(\frac{r_-}{r_+} \right)^{d-3} \right)^{-1}. \end{aligned} \quad (3.9)$$

Physically the most interesting is the 4-dimensional case. Then (3.8) takes the form of (D.1) with

$$\begin{aligned} c &= 1 - \frac{r_-}{r_+} \quad (0 < c \leq 1), \\ k &= 32pr_+^4 (r_+ - r_-)^{-(r_+^2 - r_-^2)/r_+^2} e^{-(r_+ - r_-)/r_+}. \end{aligned} \quad (3.10)$$

Obviously, if the black hole is uncharged (Schwarzschild), i.e. $r_- = 0$, we recover the result of Dray and 't Hooft [6]. The solution of the equation is given in (D.2).⁶ For $\frac{1}{4} \leq c \leq 1$ (and in fact for all $c \geq \frac{1}{4}$) an integral representation (proportional to a hypergeometric series) of the solution can be found using

$$\int_0^\infty ds e^{-(l+\frac{1}{2})s} \cos(\sqrt{c-1/4}s) = \frac{l+\frac{1}{2}}{l(l+1)+c}, \quad (3.11)$$

and the generating function for Legendre polynomials

$$\sum_{l=0}^\infty t^l P_l(\cos \theta) = (1 - 2 \cos \theta t + t^2)^{-1/2}. \quad (3.12)$$

We find (consult Ref. [19])

$$f(\theta; c) = -\frac{k}{\sqrt{2}} \int_0^\infty ds \cos(\sqrt{c-1/4}s) \frac{1}{\sqrt{\cosh s - \cos \theta}}$$

⁶ The solution of (3.8) for the higher than $d = 4$ dimensional cases can easily be given in terms of generalized spherical functions (see Eq. (D.12) in Appendix D).

$$= -\frac{k\pi}{2 \cosh(\sqrt{c-1/4}\pi)} F\left(\frac{1}{2} - i\sqrt{c-1/4}, \frac{1}{2} + i\sqrt{c-1/4}, 1, \cos^2 \frac{\theta}{2}\right). \quad (3.13)$$

For $0 < c \leq \frac{1}{4}$ we should replace $\sqrt{c-1/4}$ by $i\sqrt{1/4-c}$ with parallel replacement of the corresponding trigonometric functions by hyperbolic ones and vice versa. Notice that the solution blows up at the point of the unit 2-sphere where the particle was placed, i.e. at $\theta = 0$. Moreover, it is everywhere negative and for fixed c it is a monotonically increasing function of $\theta \in [0, \pi]$ approaching a constant at $\theta = \pi$. For fixed θ it also monotonically increases as a function of $c \in (0, 1]$. The “refraction function” is

$$R(\theta; c) = \left(\frac{A}{g}\right)_{u=0} \partial_\theta f(\theta; c). \quad (3.14)$$

As a function of θ it monotonically decreases from plus infinity at $\theta = 0$ to zero at $\theta = \pi$.

It is useful to examine what happens in the region close to the two poles of the sphere where the behavior is extreme.⁷ Close to the northern pole at $\theta = 0$ we have

$$f(\theta; c) \simeq \begin{cases} -k K_0(\sqrt{c}\theta) & \text{as } \theta \ll 1 \\ k \ln(\sqrt{c}\theta) & \text{as } \theta \lll 1, \end{cases} \quad (3.15)$$

whereas close to the southern pole at $\theta = \pi$

$$f(\theta; c) \simeq \begin{cases} -\frac{k\pi}{2 \cosh(\sqrt{c-\frac{1}{4}}\pi)} I_0(\sqrt{c}(\pi-\theta)), & \text{as } \pi-\theta \ll 1 \\ -\frac{k\pi}{2 \cosh(\sqrt{c-\frac{1}{4}}\pi)}, & \text{as } \pi-\theta \lll 1. \end{cases} \quad (3.16)$$

For the “refraction function” the corresponding results are

$$R(\theta; c) \simeq \begin{cases} k'\sqrt{c} K_1(\sqrt{c}\theta) & \text{as } \theta \ll 1 \\ \frac{k'}{\theta} & \text{as } \theta \lll 1, \end{cases} \quad (3.17)$$

and

$$R(\theta; c) \simeq \begin{cases} \frac{k'\pi\sqrt{c}}{2 \cosh(\sqrt{c-\frac{1}{4}}\pi)} I_1(\sqrt{c}(\pi-\theta)) & \text{as } \pi-\theta \ll 1 \\ \frac{k'\pi c}{4 \cosh(\sqrt{c-\frac{1}{4}}\pi)} (\pi-\theta) & \text{as } \pi-\theta \lll 1, \end{cases} \quad (3.18)$$

⁷ In the following $I_n(x)$ and $K_n(x)$ will denote, as usual, the n th-order modified Bessel functions. The relations: $K'_0 = -K_1$, $I'_0 = I_1$ and the leading order behavior as $x \ll 1$: $I_0 \simeq 1$, $I_1 \simeq x/2$, $K_0 \simeq -\ln x$, $K_1 \simeq 1/x$ will also prove useful.

where $k' = (A/g)_{u=0}k$ is a positive constant. As was expected, as we approach $\theta = 0$ both f and R behave like the corresponding functions in the flat space case [7,6] (see also (B.8)) and both approach infinity. At the southern pole both functions reach their minimum magnitudes and in fact the refraction phenomenon disappears even though a particle trajectory is still discontinuous since $f(\pi; c) \neq 0$.

We end this section with a comment on the case of the extremally charged Reissner–Nordström black hole. It is obvious either from the series representation (D.2) for $f(\theta; c)$, or from the integral one (3.13) that for $c = 0$, or equivalently in the extremal limit $r_- = r_+$, the solution is not well defined. In fact the choice we have made for the constant α in (3.3) and everything that follows are not valid if $r_- = r_+$. It can easily be shown that there is no solution to our problem, or in other words no *single* particle can move with the speed of light in the outer horizon of an extremally charged Reissner–Nordström black hole. Physically this should have been expected because extremality is the situation where the two horizons coincide just before a naked singularity appears. From a more mathematical point of view this happens because the function $F^2(r)$ is not analytic at $r = r_+ = r_-$, as it is in the non-extremal case (see Eq. (3.4)).

4. Constant curvature spaces

In this section we consider the De Sitter and anti-De Sitter spaces where there is a constant curvature $R = 2d/(d-2)\Lambda$ corresponding to a non-vanishing cosmological constant Λ . Even though a cosmological term was not considered in (2.8) its effect can be imitated by an energy–momentum tensor in (2.8) that is proportional to $g_{\mu\nu}$, i.e. $8\pi\tilde{T}_{\mu\nu} = -2/(d-2)\Lambda g_{\mu\nu}$. We also analyze the Schwarzschild–De Sitter black hole case.

4.1. The 4-dimensional cases

The metric for the 4-dimensional De Sitter space is

$$ds_{\text{DS}}^2 = -\left(1 - \frac{r^2}{a^2}\right) dt^2 + \left(1 - \frac{r^2}{a^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.1)$$

whereas for the corresponding anti-De Sitter one

$$ds_{\text{ADS}}^2 = -\left(\frac{r^2}{a^2} - 1\right) dt^2 + \left(\frac{r^2}{a^2} - 1\right)^{-1} dr^2 + r^2 (d\chi^2 + \sinh^2 \chi d\phi^2). \quad (4.2)$$

In both cases the constant a is related to the non-vanishing cosmological constant⁸ as $\Lambda = \pm 3/a^2$. To bring (4.1), (4.2) into the form (2.1) we make the same coordinate transformation as in (3.3) but with $\alpha = a$. Then it turns out that

⁸ Throughout this subsection upper (lower) signs in some quantities correspond to the De Sitter (anti-De Sitter) case.

$$F(r) = \left(\pm \frac{a-r}{r+a} \right)^{1/2}, \quad A(u, v) = \frac{1}{2}(r+a)^2, \quad g(u, v) = r^2. \quad (4.3)$$

The conditions of (2.10) are indeed satisfied at $u = 0$ ($r = a$) and the differential equation f satisfies is of the form (D.1) with

$$c = \mp 2, \quad k = 32pa^4. \quad (4.4)$$

Then according to (D.5) for $l = 1$ the solution for the De Sitter case is

$$f_{\text{DS}}(\theta) = 32pa^4 \left(1 - \frac{1}{2} \cos \theta \ln \left(\cot^2 \frac{\theta}{2} \right) \right) \Theta \left(\frac{\pi}{2} - \theta \right), \quad (4.5)$$

where contributions proportional to $Y_1^m(\theta, \phi)$, $m = -1, 0, 1$ that solve the homogeneous equation have been omitted. As also pointed out in the Appendix D, the use of the Θ -function that essentially restricts the solution to the upper hemisphere is necessary in order to have a solution that does not blow up at the southern pole at $\theta = \pi$, which would be unphysical since no particle is placed there. Comparing with the black hole case considered in the previous section we find a notable difference. The solution goes to minus infinity at $\theta = 0$ and then monotonically increases until it reaches the value $(32pa^4)$ at $\theta = \frac{\pi}{2}$. Therefore there is an angle ($\theta_0 \simeq 33.52^\circ$) where it becomes zero and then it passes from negative to positive values. Thus contrary to intuition the magnitude of the shift has its minimum not at $\theta = \pi$ (as was the case in the previous section) but at an intermediate angle. The corresponding “refraction function” is

$$R_{\text{DS}}(\theta) = 64pa^4 \left(\frac{\cos \theta}{\sin 2\theta} + \frac{1}{2} \sin \theta \ln \left(\cot^2 \frac{\theta}{2} \right) \right) \Theta \left(\frac{\pi}{2} - \theta \right) - 64pa^4 \delta \left(\theta - \frac{\pi}{2} \right). \quad (4.6)$$

The first term is a monotonically decreasing function of θ and it varies from plus infinity to zero as we go from the northern pole to the equator. However, exactly there the second term gives an infinite contribution. This is in effect a consequence of restricting the solution to the upper hemisphere and, in some sense, can be thought of as a source term that displaces the minimum magnitude of $f_{\text{DS}}(\theta)$ from $\theta = \pi/2$ to $\theta = \theta_0$. Notice also that at $\theta = \theta_0$ there is no discontinuity for the geodesic trajectories but there is still a refraction effect.

A natural question to ask is whether or not there is a distribution of massless particles, instead of just a single one, for which the use of a Θ -function in (4.5) is unnecessary. Obviously, there are many such distributions (after all (4.5) is nothing but a Green function). The simplest one is to consider two particles of the *same energy* p , one in each pole of the 2-sphere. Then the solution becomes exactly (4.5) with no Θ -function of course, and in fact it can be found by infinitely boosting a Schwarzschild–De Sitter black hole (see (4.9) below) in the zero mass limit [10].

For the anti-De Sitter case (D.8) for $l = 1$ gives

$$f_{\text{ADS}}(\chi) = 32pa^4 \left(\frac{1}{2} \cosh \chi \ln \left(\coth^2 \frac{\chi}{2} \right) - 1 \right). \quad (4.7)$$

Notice that in the anti-De Sitter case we did not have to “cut” the solution by introducing a Θ -function because (4.7) vanishes by itself for large χ . Of course this is due to the fact that in contrast to the De Sitter case the 2-dimensional space the shift function f takes values on is a non-compact hyperboloid and not the 2-sphere. The corresponding “refraction function” is

$$R_{\text{ADS}}(\chi) = 64pa^4 \left(-\frac{\cosh \chi}{\sinh 2\chi} + \frac{1}{2} \sinh \chi \ln \left(\coth^2 \frac{\chi}{2} \right) \right). \quad (4.8)$$

As a function $f_{\text{ADS}}(\chi)$ ($R_{\text{ADS}}(\chi)$) monotonically decreases (increases) from plus (minus) infinity to zero as we get away from the pole of the hyperboloid at $\chi = 0$. This is similar (but not exactly the same) to the behavior of the corresponding functions in the previous section.

Let us now return to the more general case of the Schwarzschild–De Sitter black hole for which also there is a positive cosmological constant. The metric is

$$ds_{\text{SDS}}^2 = - \left(1 - \frac{2M}{r} - \frac{r^2}{a^2} \right) dt^2 + \left(1 - \frac{2M}{r} - \frac{r^2}{a^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.9)$$

Since it is a straightforward calculation we decided not to give algebraic details concerning the derivation of the various results. The differential equation to be satisfied by the shift function $f(\theta)$ is again of the type (D.1) and it turns out that depending on the ratio a/M there are two branches of solutions for the constants c and k . We will only consider the branch in which the null surface where we will place the massless particle corresponds to a positive value of r (for positive M). In this branch

$$c = 2 \sin \frac{\varphi}{3} \left(\sqrt{3} \cos \frac{\varphi}{3} - \sin \frac{\varphi}{3} \right), \quad \cos \varphi \equiv \sqrt{27} \frac{M}{a} \\ \frac{a}{M} > \sqrt{27} \Rightarrow \varphi \in [-\pi/2, 0) \cup (0, \pi/2], \quad c \in [-2, 0) \cup (0, 1], \quad (4.10)$$

and the constant k is always positive with precise value which is not of any particular interest. The null surface $u = 0$ where the massless particle is placed corresponds to $r = r_3 = 2(a/\sqrt{3}) \cos((\varphi + \pi)/3)$. For positive (negative) values of φ the radial coordinate r lies between $r_3 < r < r_1$ ($r_1 < r < r_3$), where $r_1 = 2(a/\sqrt{3}) \cos((\varphi - \pi)/3)$ (r_1 , r_3 and $r_2 = -2(a/\sqrt{3}) \cos(\varphi/3)$ are the three surfaces where g_{tt} vanishes). The boundary cases $\varphi = -\pi/2, \pi/2$ corresponding to the De Sitter space ($c = -2$) and the Schwarzschild black hole with zero cosmological constant ($c = 1$) have been already discussed. The case $\varphi = 0$ ($a/M = \sqrt{27}$, $c = 0$) is excluded because the situation is similar to the extremal Reissner–Nordström charged black hole. Except for the case $c = -2$ the solution is given by (D.2). For the range $0 < c \leq 1$ the behavior of the solution is exactly the same as in the case of the Reissner–Nordström charged black hole of the previous section. For the range $-2 < c < 0$, similarly to the case of (3.13), we find the following integral representation:

$$f(\theta; c) = -\frac{k}{2c} - k \int_0^\infty ds \cosh(\sqrt{1/4 - c} s) \left(\frac{1/\sqrt{2}}{\sqrt{\cosh s - \cos \theta}} - e^{-s/2} \right). \quad (4.11)$$

Comparing with (3.13) we see that there is a second term in the integrand that essentially regulates the divergent behavior of the first term for large values of s . We note that the solution again blows up at $\theta = 0$ and as in the case of the De Sitter space it is monotonically increasing as we move from the northern to the southern pole of the 2-sphere and changes from negative to positive values at an angle θ_0 that depends on the value of c . For instance for $c = -1$ we have $\theta_0 \simeq 61.59^\circ$. Thus similarly to the De Sitter space case the magnitude of the shift function is reaching its minimum at a point which is not the furthest from the massless particle at $\theta = 0$. The important difference with the De Sitter space case is that the solution is extended to the southern hemisphere as well, i.e. no use of Θ -functions is required. Let us also note that the “refraction function” is a monotonically decreasing function of θ and that the functional dependence in the region around the northern and the southern poles is again given by (3.15), (3.18) (with a different value for k and where the appropriate analytic continuations in the hyperbolic functions should be performed).

Let us also mention that we refrained from presenting here any results concerning analogs of Reissner–Nordström charged black holes (see Ref. [20]) with a cosmological constant because that would have complicated things rather unnecessarily.

4.2. The 3-dimensional black hole

It is interesting to consider the anti-de-Sitter space in $d = 3$. Then the metric is

$$ds^2 = -\left(\frac{r^2}{a^2} - 1\right) dt^2 + \left(\frac{r^2}{a^2} - 1\right)^{-1} dr^2 + r^2 d\chi^2, \quad (4.12)$$

where χ is a non-compact coordinate. Again the constant a is related to the cosmological constant which is negative, i.e. $\Lambda = -1/a^2$. The change of variables that brings (4.12) into the form (2.1) is similar to (4.3) with the lower sign in the expression for $F(r)$. Then the 1-dimensional equation that f satisfies using (2.10) for $d = 3$ turns out to be

$$\frac{\partial^2 f}{\partial \chi^2} - f = 64\pi p a^4 \delta(\chi). \quad (4.13)$$

Its solution that has the correct asymptotic behavior reads (we ignore the obvious solution of the homogeneous equation)

$$f(\chi) \simeq -32\pi p a^4 e^{-|\chi|}. \quad (4.14)$$

A more physical situation arises when the non-compact variable χ is made a compact one by letting $\chi \rightarrow \phi$ and identifying $\phi \equiv \phi + T$. The most physical choice is of course $T = 2\pi$. In this case we have a black hole in $d = 3$ as was shown in Ref. [13]. In fact this solution can also be thought of as a solution to string theory with non-trivial antisymmetric tensor and constant dilaton fields [21]. The corresponding

exact conformal field theory is obtained if one quotients the $SL(2, \mathbb{R})$ WZW model by a discrete subgroup. Because of the above discrete identification the δ -function that appears in (4.13) should be replaced by $\sum_{n=-\infty}^{\infty} \delta(\phi - nT)$. Then instead of (4.14) the appropriate periodic solution satisfying $f(\phi + T) = f(\phi)$ is

$$f(\phi) = -32\pi pa^4 \sum_{n=-\infty}^{\infty} e^{-|\phi - nT|}, \quad |\phi| < \infty. \quad (4.15)$$

After some algebra the infinite sum is computed to give

$$f(\phi) = -64\pi pa^4 \left(\frac{1}{e^T - 1} \cosh \phi + \frac{1}{2} e^{-|\phi|} \right), \quad |\phi| \leq \frac{T}{2}. \quad (4.16)$$

Notice that the range of ϕ is restricted in the fundamental domain and that for $T \rightarrow \infty$ we recover the expression (4.14) as we should. On physical grounds we expect that the absolute value of the shift is always larger than the one predicted in the non-compact case, for any ϕ . This should be the case because the identification $\phi \equiv \phi + T$ “creates” many different sources of infinite curvature (δ -functions) in the entire real line. Using (4.16), (4.13) one easily verifies that statement. In fact $|f(\phi)|$ reaches its maximum (minimum) value at $\phi = 0$ ($|\phi| = T/2$). Finally, the “refraction function” is

$$R(\phi) = -128\pi pa^4 \left(\frac{1}{e^T - 1} \sinh \phi - \frac{1}{2} \text{sign}(\phi) e^{-|\phi|} \right), \quad |\phi| \leq \frac{T}{2}. \quad (4.17)$$

5. String-inspired gravitational solutions

In low energy heterotic string theory we may consider the set of background fields that comprises a metric, an antisymmetric tensor, a dilaton and an electromagnetic field coupled in a 2-dimensional σ -model action. Then the requirement of conformal invariance generates constraints these fields have to obey, the so-called beta-function equations. For our purposes it is convenient to present these equations in the Einstein frame where the metric $g_{\mu\nu}^E$ is related to the σ -model one by

$$g_{\mu\nu}^E = \exp \left(\frac{2}{d-2} \Phi \right) g_{\mu\nu}. \quad (5.1)$$

In the rest of this section we drop the extra index having in mind that all quantities are evaluated in the Einstein frame. To lowest order in the string coupling constant α' , the beta-function equations (see for instance Ref. [22]) are⁹

$$R_{\mu\nu} = \frac{1}{d-2} (D_\mu \Phi D_\nu \Phi - g_{\mu\nu} D^2 \Phi) - \frac{d}{d-2} \left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{d} g_{\mu\nu} F^2 \right) \exp \left(\frac{2}{d-2} \Phi \right)$$

⁹ We will not mention the dilaton beta function since it is not independent from the other ones. Also since in the examples we will shortly give the antisymmetric tensor is locally trivial, i.e. $H_{\mu\nu\rho} = 0$, we restrict to only such cases although it is straightforward to work out the details in the more general case as well.

$$D_\lambda \left(\exp\left(\frac{2\Phi}{d-2}\right) F_\mu{}^\lambda \right) = 0. \quad (5.2)$$

The first of the above equations is of the form of Einstein's equations (2.8) with non-trivial matter energy-momentum tensor that is being created by Φ and $F_{\mu\nu}$ and given by the right-hand side of this equation. In the following we consider metric tensors of the form given by (2.1), and electromagnetic field strength and dilaton field of the form

$$F = 2 F_{uv}(u, v) du \wedge dv + F_{ij}(x) dx^i \wedge dx^j, \\ \Phi = \Phi(u, v). \quad (5.3)$$

Strictly speaking with the above form for the electromagnetic field $F_{\mu\nu}$ the antisymmetric tensor field strength $H_{\mu\nu\rho}$ cannot be set to zero since its source term $dH = -F \wedge F$ does not vanish. However, one may think of having two independent electromagnetic fields F^a , $a = 1, 2$ given by

$$F^1 = 2 F_{uv}^1(u, v) du \wedge dv, \quad F^2 = F_{ij}^2(x) dx^i \wedge dx^j. \quad (5.4)$$

In that case $dH = -F^a \wedge F^a = 0$. The two formulations are equivalent since in both cases the electromagnetic energy-momentum tensors are the same leading to the same equations of motion given by (5.2).

As in Section 1 we “perturb” a given background solution by adding a massless particle moving along the v -direction at $u = 0$. Since the first equation in (5.2) is the same as Einstein's equation (2.8) the conditions on the functions $g(u, v)$, $A(u, v)$ and $f(x)$ are precisely given by (2.10), whereas $T_{vv} = 0$ at $u = 0$ translates into a condition on the dilaton (again at $u = 0$)

$$\Phi_{,v} = 0. \quad (5.5)$$

Examination of the second equation in (5.2) gives the condition $(F_{uv})_{,v} = 0$ at $u = 0$ which however is obeyed thanks to the equations of motion and the aforementioned conditions on $A(u, v)$ and $g(u, v)$.

5.1. An electrically and magnetically charged black hole

As a first application of the above let us consider a solution to low energy heterotic string theory that represents a black hole with both electric and magnetic charges [14,15]

$$ds_E^2 = -\lambda_1(r) dt^2 + \lambda_1^{-1}(r) dr^2 + (r^2 - r_0^2) (d\theta^2 + \sin^2 \theta d\phi^2), \\ \Phi = \ln \left(\frac{r - r_0}{r + r_0} \right), \\ F = \frac{Q_E}{(r - r_0)^2} dt \wedge dr + Q_M \sin \theta d\theta \wedge d\phi, \quad (5.6)$$

where

$$\lambda_1(r) = \frac{(r - r_+)(r - r_-)}{r^2 - r_0^2}, \quad r_0 = \frac{Q_M^2 - Q_E^2}{2M},$$

$$r_{\pm} = M \pm \sqrt{M^2 + r_0^2 - Q_E^2 - Q_M^2}. \quad (5.7)$$

As in the case of (3.1), r_+ and r_- are the outer and inner horizons respectively. The constant r_0 is the so-called dilaton charge that essentially measures the difference between electric and magnetic charges. One easily finds that a transformation of the type (3.3), but with $\alpha = 2(r_+^2 - r_0^2)/(r_+ - r_-)$, takes the metric into the form (2.1) with the relevant functions given by

$$\begin{aligned} F(r) &= e^{r/\alpha} (r - r_-)^{-(r_-^2 - r_0^2)/(2(r_+^2 - r_0^2))} (r - r_+)^{1/2}, \\ A(u, v) &= \frac{\alpha^2}{2} \lambda_1(r) / F(r)^2, \\ g(u, v) &= r^2 - r_0^2. \end{aligned} \quad (5.8)$$

Obviously, after this change of coordinates the expressions for the dilaton and the electromagnetic field strength assume the form (5.3). One easily finds that (2.10), (5.5) are satisfied at $u = 0$ (or $r = r_+$) and that the equation for the shift function f is given by (D.1) with

$$\begin{aligned} c &= r_+ \frac{r_+ - r_-}{r_+^2 - r_0^2}, \\ k &= 32p(r_+^2 - r_0^2)^2 (r_+ - r_-)^{-(r_+^2 - r_-^2)/(r_+^2 - r_0^2)} e^{-r_+(r_+ - r_-)/(r_+^2 - r_0^2)}. \end{aligned} \quad (5.9)$$

Similarly one easily sees that because of the fact that $r_0 \leq r_- < r_+$ we have $0 < c \leq 1$ and therefore the solution to f is given also by the general expression (D.2) or (3.13).

5.2. The coset $SL(2, \mathbb{R})/\mathbb{R} \otimes \mathbb{R}^2$

Finally let us consider the case of the 4-dimensional model that is the tensor product of the 2-dimensional coset $SL(2, \mathbb{R})_{-k}/\mathbb{R}$ [23] with two additional non-compact dimensions. The metric in the Einstein frame (cf. Eq. (5.1)) and the dilaton are

$$\begin{aligned} ds_E^2 &= -2\epsilon du dv + (1 - uv)(dx^2 + dy^2), \\ \Phi &= \ln(1 - uv), \end{aligned} \quad (5.10)$$

with $\epsilon = \text{sign}(k)$, where $(-k)$ is the central extension of the $SL(2, \mathbb{R})_{-k}$ current algebra. For $\epsilon = 1$ the causal structure of the spacetime is that of a black hole [23] with a singularity at future times $t = u + v$, whereas for $\epsilon = -1$ it has the cosmological interpretation of an expanding Universe with no singularity at future times $t = u - v$ (see Ref. [24]). The metric is already of the form (2.1) with

$$A(u, v) = -\epsilon, \quad g(u, v) = 1 - uv. \quad (5.11)$$

It is easy to see that (5.5) is satisfied and that the differential equation for the shift function is

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \epsilon \right) f(\rho) = -16\epsilon p \frac{1}{\rho} \delta(\rho), \quad (5.12)$$

where $\rho^2 = x^2 + y^2$.

Let us first consider the black hole case ($\epsilon = 1$). Then (5.12) is nothing but the modified Bessel equation of order zero with the additional δ -function source term in the right hand side. Its solution with the correct asymptotic behavior is

$$\begin{aligned} f_+(\rho) &= 16p K_0(\rho), \\ R_+(\rho) &= 16p K_1(\rho). \end{aligned} \quad (5.13)$$

In order to obtain a δ -function behavior in the right-hand side of (5.12) one should carefully treat the derivative part of the left-hand side using the fact that $K_0(\rho) \simeq -\ln(\rho)$ for $\rho \ll 1$. Also, if one wishes to compactify one or both of the x, y coordinates one should replace the solution (5.13) by a periodic one that will necessarily involve infinite series as was done in the case of (4.14) and (4.15).

In the cosmological case ($\epsilon = -1$) (5.12) becomes the usual Bessel equation with solution respecting the singular behavior of the source term in the right hand side given by

$$\begin{aligned} f_-(\rho) &= 8\pi p N_0(\rho), \\ R_-(\rho) &= -8\pi p N_1(\rho), \end{aligned} \quad (5.14)$$

where $N_n(\rho)$ denotes Bessel functions of the second kind.¹⁰ Obviously, Eqs. (5.13) and (5.14) are similar to (3.15), (3.17) since in both cases the metric in the transverse x -plane is the flat one (notice: $N_0(\rho) \simeq (2/\pi) \ln(\rho)$ and $N_1(\rho) \simeq -2/(\pi\rho)$ for $\rho \ll 1$). However, whereas in the black hole case the shift and the “refraction function” never change sign, in the cosmological case they do so an infinite number of times until they die off completely at $\rho = \infty$. This type of behavior most likely will show up in other cosmological models as well. The difference in behavior is more dramatic far away from the particle’s position at $\rho = 0$ since for $\rho \gg 1$ $K_0(\rho) \simeq \sqrt{\pi/(2\rho)} e^{-\rho}$ whereas $N_0(\rho) \simeq \sqrt{\pi/(2\rho)} \sin(\rho - (\pi/4))$.

6. Discussion and concluding remarks

In this paper we have found the necessary and sufficient conditions for being able to introduce a gravitational shock wave in a quite general class of background solutions in Einstein’s general relativity coupled to non-trivial matter (with or without a cosmological constant) and string theory. We have also applied the general formalism to many cases that are of particular interest such as charged black holes and constant curvature spacetimes. There are also a few points we would like to further discuss.

(1). It is quite interesting to notice that the addition of the massless particle in a background geometry creates in effect a perturbation that can be described, in the context of string theory in curved spacetime, in terms of a massless vertex operator.

¹⁰ A more systematic way to obtain the solution of (5.12) is to expand as $f(\rho) = \int_0^\infty dk k A(k) J_0(k\rho)$ and use the fact that $\delta(\rho)/\rho = \int_0^\infty dk k J_0(k\rho)$. Then easily we obtain $A(k) = 16p\epsilon/(k^2 + \epsilon)$. Performing the integral we get for the shift function the expression in (5.13) or (5.14) depending on the value of ϵ .

To be precise consider Einstein's equations in the vacuum or in a bosonic string theory language a 2-dimensional σ -model with zero antisymmetric tensor and dilaton fields. Let us take

$$V = F_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu, \quad (6.1)$$

as a candidate for a massless vertex operator.¹¹ If we want to perturb the σ -model by this operator we should require that in every order in perturbation theory in the string coupling constant α' , its anomalous dimension is 2 so that the perturbation stays marginal. That gives a set of consistency conditions $F_{\mu\nu}$ is constrained to satisfy. To leading order in the string coupling they read [25]

$$\begin{aligned} -D^2 F_{\mu\nu} - D_\mu D_\nu F_\lambda{}^\lambda + R_\mu{}^{\rho\sigma}{}_\nu F_{\rho\sigma} + D_{(\mu} D^\lambda F_{\lambda\nu)} &= 0 \\ D^2 F_\lambda{}^\lambda - D^\rho D^\sigma F_{\rho\sigma} + R^{\rho\sigma} F_{\rho\sigma} &= 0. \end{aligned} \quad (6.2)$$

It is a straightforward (though a bit lengthy) calculation to verify that

$$F_{uu} = -2A(u, v) f(x) \delta(u), \quad (6.3)$$

as is read from (2.6), is indeed a massless vertex operator corresponding to the background metric (2.1) provided that, as in (2.10), at $u = 0$ we have $g_{,v} = 0$ and $A_{,v} = 0$ and that $f(x)$ satisfies the *homogeneous* differential equation in (2.10). The vanishing of the right-hand side occurs because we have not included the energy-momentum tensor corresponding to this vertex (2.9). This is not usually done if the background is flat because then the resulting homogeneous equation has a solution (with a different ansatz for a vertex operator (see Ref. [26])), which is not however always true, for instance, as we have seen in the cases of black holes.

(2). What is the meaning of the diverging solution (see Eq. (3.13)) as we go to the extremal limit of a Reissner–Nordström charged black hole and moreover the no-solution result we have found working exactly at the extremal limit? In general the existence of a solution means that the original background responds to the “perturbation” created by the massless particle. A particle geodesic, in a charged close to the extreme limit black hole, gets shifted and refracted by large amounts due to the large gravitational field created by the massless particle. Exactly at the extremal limit there is no solution with a *single* particle, but there is one if in addition to the *positive* energy massless particle at the northern pole we place another one with *negative* energy of the same magnitude at the southern pole (both at $r = r_+ = r_-$). Then the solution with the correct singular behavior at $\theta = 0, \pi$ is $f(\theta) = -kQ_0(x) = -\frac{1}{2}k \ln(\cot^2(\theta/2))$. In other words, we might think that as $r_+ \rightarrow r_-$ and the gravitational field becomes infinite it is necessary to take the *antiparticle* into consideration as well. The fact that there is a solution in the case of an extremal Reissner–Nordström black hole only if we use a *particle* and its *antiparticle* is somewhat analogous to the “semiclassical” explanation of Kleins’s paradox using the concept of a “sea” filled with negative energy particles.

¹¹ In general [25] one may add the term $\alpha' R^{(2)} F(X)$, where $R^{(2)}$ is the curvature of the 2-dimensional worldsheet and $F(X)$ is a tachyon-like operator. However, this is not necessary here.

As a final comment on the Reissner–Nordström charged black hole let us mention what the result would be if we had placed the massless particle in the inner horizon instead of the outer one. The differential equation for the shift function is again of the form (D.1) with

$$c = 1 - \frac{r_+}{r_-} \quad (-\infty < c < 0), \quad k = -32pr_-^4 (r_+ - r_-)^{(r_+^2 - r_-^2)/r_-^2} e^{(r_+ - r_-)/r_-}. \quad (6.4)$$

Thus c, k are basically given by (3.10) with r_+ and r_- interchanged. The solution for the shift function is given by (D.2) which one can show is equivalent to

$$f_n(\theta) = -k \sum_{l=0}^{n-1} \frac{l + \frac{1}{2}}{l(l+1) + c} P_l(\cos \theta) - k \int_0^\infty ds \cosh(\sqrt{1/4 - c} s) \left(\frac{1/\sqrt{2}}{\sqrt{\cosh s - \cos \theta}} - \sum_{l=0}^{n-1} e^{-(l+\frac{1}{2})s} P_l(\cos \theta) \right), \quad (6.5)$$

$$-n(n+1) < c < -n(n-1), \quad n = 1, 2, \dots$$

Using (6.5) one finds that the shift function starts from plus infinity at $\theta = 0$ becomes zero and changes sign as many times as the value of n and eventually assumes the finite constant value predicted by (3.16). Notice also that (4.11) corresponds to the case $n = 1$ of the above formula. Of course for the cases where $c = -n(n+1)$, $n = 1, 2, \dots$ (D.5) or (D.6) gives the solution. The case $c = -2$ ($n = 1$) is exactly the same as in the De Sitter space and the solution is unique. For higher values of n however, there are $(n-1)$ undetermined coefficients which can probably be fixed with some additional physical requirements for the shift or the “refraction function”.

(3). Can we find the shock-wave type of disturbances created by two or more massless particles moving along parallel surfaces (different values of u) just by a coordinate shift? Obviously this requires $A_{,v} = g_{,v} = 0$ at these values of u . This is not possible for black hole type solutions. It can however be done for backgrounds with a covariantly constant null Killing vector (see Appendix B) that include plane waves and the flat space as special cases. Then at each null surface we have a shock wave with a shift function obtained by solving (B.6).

Finally, it would be obviously very interesting to explicitly compute, using techniques similar to those in Refs. [2,4,5], particle scattering amplitudes in the shock wave curved geometries we have derived. In particular one would like to know how these amplitudes depend on the different behaviors of the shift and the “refraction function” we have demonstrated in the present paper.

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Note added

Towards finishing typing this paper we became aware that the authors of Ref. [27] dealt with some of the issues of this paper. However, we disagree with many of their findings. The main points of disagreement are (equation labels starting with “LS.” refer to Ref. [27]):

(1) For the general d -dimensional case we do not agree with the expressions for the Ricci tensor (LS.5-9). Our (A.2) contains some additional terms with explicit d -dependence.

(2) We do not quite agree with the conditions (LS.17-18) (we find instead (2.10)) mainly because of the explicit appearance of the cosmological constant in (LS.18) and the absence of a condition on T_{vv} in (LS.17).

(3) For the case of a Reissner–Nordström charged black hole (LS.35) agrees with (D.2) (with $a = c$). However, we find the different integral representation (3.13) instead of (LS.36).

Appendix A. Components of useful tensors

The corresponding to the metric (2.6) non-vanishing Christoffel symbols are (for notational convenience we omit the hats)

$$\begin{aligned}
 \Gamma_{uu}^u &= -\frac{F_{,v}}{2A} + \frac{A_{,u}}{A}, & \Gamma_{ij}^u &= -\frac{g_{,v}}{2A} h_{ij}, \\
 \Gamma_{uu}^v &= \frac{F_{,u}}{2A} + \frac{FF_{,v}}{2A^2} - \frac{FA_{,u}}{A^2}, & \Gamma_{uv}^v &= \frac{F_{,v}}{2A}, & \Gamma_{ui}^v &= \frac{F_{,i}}{2A}, \\
 \Gamma_{vv}^v &= \frac{A_{,v}}{A}, & \Gamma_{ij}^v &= \left(-\frac{g_{,u}}{2A} + \frac{Fg_{,v}}{2A^2}\right) h_{ij}, \\
 \Gamma_{uu}^i &= -\frac{1}{2g} h^{ik} F_{,k}, & \Gamma_{uj}^i &= \frac{g_{,u}}{2g} \delta_j^i, & \Gamma_{vj}^i &= \frac{g_{,v}}{2g} \delta_j^i, \\
 \Gamma_{jk}^i &= \frac{1}{2} h^{il} (h_{lk,j} + h_{lj,k} - h_{jk,l}).
 \end{aligned} \tag{A.1}$$

Using the above expressions we find that the non-vanishing components of the Ricci tensor are (we substitute $F = -2A f \delta$ (see (2.6)))

$$\begin{aligned}
 R_{uu} &= \frac{d-2}{2} \left(\frac{g_{,u} A_{,u}}{gA} - \frac{g_{,uu}}{g} + \frac{g_{,u}^2}{2g^2} \right) + \frac{A}{g} \delta \triangle_{h_{ij}} f - \frac{d-2}{2} \frac{g_{,v}}{g} \delta' f \\
 &\quad + \left(2 \frac{A_{,uv}}{A} - 2 \frac{A_{,u} A_{,v}}{A^2} + \frac{d-2}{2gA} (g_{,u} A_{,v} + g_{,v} A_{,u}) \right) \delta f \\
 &\quad + 2 \left(\frac{A_{,vv}}{A} - \frac{A_{,v}^2}{A^2} + \frac{d-2}{2} \frac{g_{,v} A_{,v}}{gA} \right) \delta^2 f^2, \\
 R_{uv} &= \left(\frac{A_{,u} A_{,v}}{A^2} - \frac{A_{,uv}}{A} + \frac{d-2}{4} \frac{g_{,u} g_{,v}}{g^2} - \frac{d-2}{2} \frac{g_{,uv}}{g} \right) \\
 &\quad + \left(\frac{A_{,v}^2}{A^2} - \frac{A_{,vv}}{A} - \frac{d-2}{2} \frac{g_{,v} A_{,v}}{gA} \right) \delta f,
 \end{aligned}$$

$$\begin{aligned}
R_{ui} &= - \left(\frac{d-4}{2} \frac{g_{,v}}{g} + \frac{A_{,v}}{A} \right) \delta f_{,i}, \\
R_{vv} &= \frac{d-2}{2} \left(\frac{g_{,v} A_{,v}}{gA} + \frac{g_{,v}^2}{2g^2} - \frac{g_{,vv}}{g} \right), \\
R_{ij} &= R_{ij}^{(d-2)} - \left(\frac{d-4}{2} \frac{g_{,u} g_{,v}}{gA} + \frac{g_{,uv}}{A} \right) h_{ij} - \left(\frac{d-4}{2} \frac{g_{,v}^2}{gA} + \frac{g_{,vv}}{A} \right) h_{ij} \delta f. \quad (\text{A.2})
\end{aligned}$$

In the 4-dimensional case (A.2) reduce to the Ricci tensor given in Ref. [6]. In the case we are dealing with string theory as in Section 5 it might be useful to compute the tensor $D_\mu D_\nu \Phi$ with $\Phi = \Phi(u, v)$. Its non-vanishing components read

$$\begin{aligned}
D_u D_u \Phi &= \left(\Phi_{,uu} - \frac{A_{,u} \Phi_{,u}}{A} \right) - \frac{1}{A} (A_{,u} \Phi_{,v} + A_{,v} \Phi_{,u}) \delta f + \Phi_{,v} \delta' f - 2 \frac{A_{,v} \Phi_{,v}}{A} \delta^2 f^2 \\
D_u D_v \Phi &= \Phi_{,uv} + \frac{A_{,v} \Phi_{,v}}{A} \delta f, \quad D_u D_i \Phi = \Phi_{,v} f_{,i} \delta \\
D_v D_v \Phi &= \Phi_{,vv} - \frac{A_{,v} \Phi_{,v}}{A} \\
D_i D_j \Phi &= \frac{1}{2A} (g_{,u} \Phi_{,v} + g_{,v} \Phi_{,u}) h_{ij} + \frac{g_{,v} \Phi_{,v}}{A} h_{ij} \delta f. \quad (\text{A.3})
\end{aligned}$$

Appendix B. Backgrounds with a covariantly constant null Killing vector

Consider the metric with a covariantly constant null Killing vector

$$ds^2 = 2 du dv + g_{ij}(u, x) dx^i dx^j, \quad (\text{B.1})$$

where $(i, j = 1, 2, \dots, d-2)$. The most general matter field energy-momentum tensor consistent with the non-vanishing components of the Ricci tensor corresponding to (B.1) (see (B.5) below) has the form

$$T = T_{uu}(u, x) du^2 + 2 T_{ui}(u, x) du dx^i + T_{ij}(u, x) dx^i dx^j. \quad (\text{B.2})$$

The analog of (2.6) is

$$\begin{aligned}
ds^2 &= 2 \hat{u} d\hat{u} d\hat{v} + \hat{F} d\hat{u}^2 + \hat{g}_{ij} d\hat{x}^i d\hat{x}^j, \\
\hat{F} &= F(\hat{u}, \hat{x}) = -2 \hat{f} \hat{\delta}, \quad (\text{B.3})
\end{aligned}$$

whereas that of (2.7) is exactly the same tensor (B.2) since no v -component of the energy-momentum tensor appears in (B.2).

The non-zero components of the Christoffel symbols and the Ricci tensor are (once again we drop the hats and also we denote derivatives with respect to u with a dot)

$$\begin{aligned}
\Gamma_{uu}^v &= \frac{1}{2} \dot{F}, & \Gamma_{ui}^v &= \frac{1}{2} F_{,i}, & \Gamma_{ij}^v &= -\frac{1}{2} \dot{g}_{ij}, \\
\Gamma_{uu}^i &= -\frac{1}{2} g^{ik} F_{,k}, & \Gamma_{uj}^i &= \frac{1}{2} g^{ik} \dot{g}_{jk}, & \Gamma_{jk}^i &= \frac{1}{2} g^{il} (g_{lk,j} + g_{lj,k} - g_{jk,l}), \quad (\text{B.4})
\end{aligned}$$

and

$$\begin{aligned}
R_{uu} &= \left(-\frac{1}{2} \ddot{g}_{ij} g^{ij} + \frac{1}{4} g^{ik} g^{jl} \dot{g}_{ij} \dot{g}_{kl} \right) + \delta \Delta_{g_{ij}} f, \\
R_{ui} &= -D_{[i} \dot{g}_{k]j} g^{jk}, \\
R_{ij} &= R_{ij}^{(d-2)}.
\end{aligned} \tag{B.5}$$

Then the condition to be satisfied at $u = 0$ is the differential equation (again we drop the hats over the symbols)

$$\Delta_{g_{ij}} f = 32\pi p \delta^{(d-2)}(x), \quad \Delta_{g_{ij}} = \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j. \tag{B.6}$$

Let us next consider string backgrounds with a covariantly constant null Killing vector, i.e. with metric in the Einstein frame as in (B.1), and in addition with non-trivial antisymmetric tensor field strength components $H_{ijk} = H_{ijk}(u, x)$, $H_{iju} = H_{iju}(u, x)$ as well as a dilaton field $\Phi = \Phi(u, x)$. It can easily be shown that the “effective” matter energy–momentum tensor due to these background fields is again of the form (B.2). Thus, the equation for the shift function is again given by (B.6), where the metric is the Einstein frame one. For completeness, the non-vanishing components of $D_\mu D_\nu \Phi$ are

$$\begin{aligned}
D_u D_u \Phi &= \ddot{\Phi} - \delta g^{kl} \Phi_{,k} f_{,l}, \\
D_u D_i \Phi &= \dot{\Phi}_{,i} - \frac{1}{2} g^{kl} \dot{g}_{li} \Phi_{,k}, \\
D_i D_j \Phi &= \Phi_{,ij} - \Gamma_{ij}^k \Phi_{,k}.
\end{aligned} \tag{B.7}$$

Let us also note that for the particular case of plane waves where all scalars and tensors depend only on the light cone coordinate u and not on the x^i 's, Eq. (B.6) is the same as the one corresponding to the flat space case and therefore its general solution is [7,3]

$$f(\rho) = \begin{cases} 16p \ln \rho, & \text{if } d = 4 \\ -\frac{32\pi p}{(d-4)\Omega_{d-2}} \frac{1}{\rho^{d-4}}, & \text{if } d > 4, \end{cases} \tag{B.8}$$

where in general $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the “area” of the unit sphere in D dimensions and $\rho^2 \equiv g_{ij}(0)x^i x^j$. Since we can freely shift u by a constant the choice of $u = 0$ should be made in such a way that the metric is non-degenerate at that point, i.e. $u = 0$ is not a focusing point of null rays.

Appendix C. Geodesic equations

For the metric (2.6) the geodesic equations obtained by varying v and x^i are (in this Appendix the dots over the various symbols denote derivatives with respect to the affine parameter τ)

$$\begin{aligned}
\ddot{u} + \frac{A_{,u}}{A} \dot{u}^2 - \frac{g_{,v}}{2A} h_{ij} \dot{x}^i \dot{x}^j + f \frac{A_{,v}}{A} \delta \dot{u}^2 &= 0, \\
\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + \frac{g_{,u}}{g} \dot{u} \dot{x}^i + \frac{g_{,v}}{g} \dot{v} \dot{x}^i + \frac{A}{g} \delta f_{,j} h^{ji} \dot{u}^2 &= 0,
\end{aligned} \tag{C.1}$$

whereas the one obtained from the variation of u is

$$\begin{aligned} \ddot{v} + \frac{A_{,v}}{A} \dot{v}^2 - \frac{g_{,u}}{2A} h_{ij} \dot{x}^i \dot{x}^j \\ + \left(f \frac{A_{,u}}{A} \dot{u}^2 - 2f \frac{A_{,v}}{A} \dot{u} \dot{v} - 2f_{,i} \dot{u} \dot{x}^i - \frac{g_{,v}}{A} f h_{ij} \dot{x}^i \dot{x}^j \right) \delta \\ - f \delta' \dot{u}^2 + 2f^2 \delta^2 \frac{A_{,v}}{A} \dot{u}^2 = 0. \end{aligned} \quad (\text{C.2})$$

From (2.6) the energy corresponding to the geodesic is given by

$$2A\dot{u}\dot{v} + gh_{ij}\dot{x}^i\dot{x}^j - 2f\delta A\dot{u}^2 = \alpha, \quad (\text{C.3})$$

where $\alpha = -1, 0, 1$ depending on whether the geodesic is timelike, null or spacelike respectively. It is clear from (C.3) that the \dot{u}^2 -term can be compensated by a discontinuity in v at $u = 0$. Taking $v = v_0 + \Theta(u)\Delta v$, where v_0 is a solution of the geodesic equations for $f = 0$, and integrating (C.3) over a small interval around $u = 0$ we obtain immediately the relation (2.13). Performing the same integration in the first of (C.1) gives no new information because $A_{,v} = 0$ at $u = 0$. However, the same procedure in the second equation in (C.1) gives

$$\dot{x}^i \Big|_{u=0^-} - \dot{x}^i \Big|_{u=0^+} = \frac{A}{g} \Big|_{u=0} f_{,j} h^{ji} \dot{u}. \quad (\text{C.4})$$

Our next step is to express the variation with respect to the affine parameter τ at $u = 0$ as a variation with respect to u itself. This can be done by noticing that the first equation in (C.1) evaluated at $u = 0$ is

$$\ddot{u} + \frac{A_{,u}}{A} \dot{u}^2 \Big|_{u=0} = 0, \quad (\text{C.5})$$

and has as its solution

$$\dot{u} = \frac{1}{A} \Big|_{u=0}. \quad (\text{C.6})$$

Therefore (C.4) takes the form of (2.14). It is important to emphasize that (C.6) is really a solution of (C.5) only at $u = 0$ where $A_{,v} = 0$. Finally, integrating (C.2) over a small interval around $u = 0$ gives no additional conditions as it eventually reduces to (C.5).

Appendix D. Solutions of the equation for the shift function f

Let us consider the differential equation

$$\begin{aligned} \Delta_{(2)} f - c f = 2\pi k \delta(x-1) \delta(\phi), \\ \Delta_{(2)} = \partial_x (1-x^2) \partial_x + \frac{\partial_\phi^2}{1-x^2}, \quad x = \cos \theta, \end{aligned} \quad (\text{D.1})$$

on the unit 2-sphere with metric $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$, where k, c are real constants. Except for the δ -function in the right-hand side this is nothing but the usual Legendre equation of order ν , where ν is a solution of $\nu(\nu + 1) + c = 0$. Therefore its solutions depend very much on the value of the constant c .

We first consider the case of $c \neq -n(n + 1)$, $n = 0, 1, \dots$. Since the eigenvalues of the Laplacian operator are $E_n = -n(n + 1)$, $n = 0, 1, \dots$ the modified Laplacian $(\Delta_{(2)} - c)$ has no zero modes and therefore it is invertible. The general solution of (D.1) can be given in terms of Legendre polynomials as

$$f(\theta; c) = -k \sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l+1) + c} P_l(\cos \theta), \quad c \in \mathbb{R} - \{-n(n+1), n = 0, 1, \dots\}. \quad (\text{D.2})$$

Since in this case c does not coincide with an eigenvalue of the Laplacian there is no solution to the homogeneous equation in (D.1).

If $c = -l(l + 1)$, $l = 0, 1, \dots$, then we should make use of the eigenfunctions of the Laplacian that are not regular at the poles at $x = \pm 1$, namely $Q_l(x)$ that are defined as

$$Q_l(x) = \frac{1}{2} P_l(x) \ln \left(\frac{1+x}{1-x} \right) - \sum_{m=1}^l \frac{1}{m} P_{m-1}(x) P_{l-m}(x). \quad (\text{D.3})$$

If the solution for f is proportional to such a function then a careful treatment of the derivative part in (D.1) will produce δ -functions at the poles of the sphere at $x = \pm 1$. However, according to the right-hand side of (D.1) there should not be any singularity at $x = -1$. Therefore we are forced to consider a solution of the form $f \sim Q_l(x) \Theta(x - x_0)$, where x_0 is to be determined by requiring that f satisfies (D.1). After some algebraic manipulations and using the fact that $(d/dx)\Theta(x) = \delta(x)$ we obtain the condition

$$(1 - x^2) \frac{dQ_l(x)}{dx} \delta(x - x_0) = 0. \quad (\text{D.4})$$

Therefore x_0 must be one of the points where the slope of $Q_l(x)$ vanishes. In fact there are l such points $\{x_1, x_2, \dots, x_l\}$, symmetrically distributed around $x = 0$ which always satisfies (D.4) if l is odd. Thus the general solution is given by

$$f(\theta) = -k Q_l(x) \left(A_0 \Theta(x) + \sum_{m=1}^{(l-1)/2} [A_m \Theta(x - x_m) + B_m \Theta(x + x_m)] \right), \quad x = \cos \theta$$

$$A_0 + \sum_{m=1}^{\frac{l-1}{2}} (A_m + B_m) = 1, \quad l = 1, 3, \dots \quad (\text{D.5})$$

if l is odd and by

$$f(\theta) = -k Q_l(x) \sum_{m=1}^{l/2} (A_m \Theta(x - x_m) + B_m \Theta(x + x_m)), \quad x = \cos \theta$$

$$\sum_{m=1}^{l/2} (A_m + B_m) = 1, \quad l = 2, 4, \dots \quad (\text{D.6})$$

if l is even. The A_m and the B_m are arbitrary constants. The constraints on their sums to be unity is necessary because at $x \simeq 1$ the solution should behave like $f = -kQ_l(x) \simeq (k/2) \ln(1-x)$ in order to have the correct normalization. Notice that the solutions (D.5), (D.6) are discontinuous at the points x_m and that for $l = 0$ there is no solution (unless $k = 0$) because there exists no point where the slope of $Q_0(x)$ is zero. It should be noted that we can add in (D.5), (D.6) any solution of the homogeneous equation in (D.1), namely a linear combination of the spherical harmonics $Y_m^l(\theta, \phi)$, $m = -l, -l+1, \dots, l$.

Let us next consider the differential equation (D.1) but with Laplacian

$$\Delta_{(2)} = \partial_x(x^2 - 1)\partial_x + \frac{\partial_\phi^2}{x^2 - 1}, \quad x = \cosh \chi, \quad (\text{D.7})$$

on the hyperboloid with the Lobachevsky metric $ds^2 = d\chi^2 + \sinh^2 \chi d\phi^2$. We consider the case when $c = l(l+1)$, $l = 0, 1, \dots$. Then the solution is given by

$$f(\chi) = kQ_l(x), \quad x = \cosh \chi, \quad l = 0, 1, \dots \quad (\text{D.8})$$

Notice that in this case there is a solution for $l = 0$. Also we did not need to use any Θ -functions because $f(\chi)$ as given by (D.8) becomes asymptotically zero for large χ as it should be.

Finally let us present the solution of the equation for the shift function (3.8) which is valid in any number of dimensions $d \geq 4$. This can be written down using the generalized spherical functions [28], which are representations functions of $SO(d-1)$. If the number of spacetime dimensions is even they are defined in terms of usual Legendre polynomials as

$$P_l^{(2r)}(\cos \theta) = \frac{l+r-\frac{1}{2}}{(2\pi)^r} \left(\frac{d}{d \cos \theta} \right)^{r-1} P_{l+r-1}(\cos \theta), \quad d = 2r + 2, \quad r = 1, 2, \dots, \quad (\text{D.9})$$

whereas if it is odd in terms of Chebyshev polynomials as

$$P_l^{(2r-1)}(\cos \theta) = 2 \frac{l+r-1}{(2\pi)^r} \left(\frac{d}{d \cos \theta} \right)^{r-2} \frac{\sin(l+r-1)\theta}{\sin \theta}, \quad d = 2r + 1, \quad r = 2, 3, \dots \quad (\text{D.10})$$

The only properties of the generalized spherical harmonics we need here are

$$\begin{aligned} \Delta_{(d-2)} P_l^{(d-2)} &= -l(l+d-3) P_l^{(d-2)}, \\ \delta^{(d-2)}(x) &= \sum_{l=0}^{\infty} P_l^{(d-2)}(\cos \theta). \end{aligned} \quad (\text{D.11})$$

Using the above formulae we find that the solution for the shift function is

$$f_d(\theta; a) = -2\pi b \sum_{l=0}^{\infty} \frac{P_l^{(d-2)}(\cos \theta)}{l(l+d-3) + a}. \quad (\text{D.12})$$

Obviously the above solution is not valid if a coincides with one of the eigenvalues of the Laplacian, i.e. if $a = -n(n+d-3)$, $n = 0, 1, \dots$. Then one has to use the analogs of the Q_l 's (see Ref. [28]) but we will not elaborate on that any more here.

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