Theory Project
Semester 7

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1 The Gravitational Shockwave of a Massless Particle

1.1 Prerequisite calculations

Lets, consider an exact exterior solution of vaccum Einstein's equation.

$$ds^2 = -(1 - \frac{r_g}{r})dt^2 + \frac{1}{(1 - \frac{r_g}{r})}dr^2 + r^2 d\Omega^2$$

Now, lets put $r = (1 + \frac{r_g}{4\rho})^2\rho$
where $\rho^2 = x^2 + y^2 + z^2$

Then the metric becomes

$$ds^2 = -(\frac{1-B}{1+B})^2 dt^2 + (1+B)^4(dx^2 + dy^2 + dz^2)$$

where $B = \frac{r_g}{4\rho}$

Now the metric is

$$ds^2 = -(\frac{1-B}{1+B})^2 dt^2 + (1+B)^4(dx^2 + dy^2 + dz^2)$$

where $B = \frac{m}{2R}$

When $m << 2R$, then $B \sim 0$

Therefore

$$ds^2 \sim -(1-\frac{2m}{R})dt^2 + (1+\frac{2m}{R})(dx^2+dy^2+dz^2)$$

Now lets boost the rest frame with respect to, coordinates (t,x,y,z) via

$$T = t\cosh\beta - z\sinh\beta$$

$$Z = -t\sinh\beta + z\cosh\beta$$

Lets, also set $m = 2pe^{-\beta}$ where p=constant$>$ 0 Introduce null coordintaes

$$u = t - z$$

$$v = t + z$$

Hence, the momentum of the particle is

$$p^a = m[\cosh\beta\delta_t^a + \sinh\beta\delta_z^a]$$

and thus

$$\lim_{\beta\to\infty} p^a = p(\delta_t^a + \delta_z^a)$$

in the limit the particle is massless and moves (at the speed of light) in the v-direction; p is its mommentum and is kept finite.

After doing a bit of algebra we get,

$$Z = -\frac{p}{m}u + \frac{m}{4p}v$$

$$T = \frac{p}{m} + \frac{m}{4p}$$

$$R^2 = x^2 + y^2 + (\frac{p}{m}u - \frac{m}{4p})^2$$

And the metric becomes

$$ds^2 = (1+\frac{2m}{R})(-dudv + dx^2 + dy^2) + \frac{4m}{R}(\frac{p}{m}du + \frac{m}{4p}dv)^2$$

Therefore,

$$\lim_{m\to 0;(u\neq 0,v,x,y)} ds^2 = -du(dv - 4p\frac{du}{|u|}) + dx^2 + dy^2$$

This metric is flat although the coordinate

$$dv^{/} = dv - 4p\frac{du}{|u|}$$

suffers a discontinuity at u=0.

Lets, introduce,

$$\hat{u} = u + \frac{m^2 Z \ln 2R}{pR}$$

$$\hat{v} = v + \frac{4pZ \ln 2R}{R}$$

So, the metric becomes,

$$ds^2 = -d\hat{u}d\hat{v} + dx^2 + dy^2$$

The linearized geodesics of metric(1) are given by,

$$\dot{T} = E(1 + \frac{2m}{R})$$

$$y\dot{Z} - Z\dot{y} = L(1 - \frac{2m}{R})$$

$$\dot{y}^2 + \dot{Z}^2 = -M^2(1 - \frac{2m}{R}) + E^2$$

The dot denotes derivative with respect to affine parameter λ, E is energy, L is angular momentum and M is mass of the test particle. Now, we consider only geodesics(we set M=O(m^2)).

Expanding y, Z , T in power of m and considering only the terms in m, we have,

$$y = y_0 + my_1$$

$$Z = Z_0 + mZ_1$$

$$T = T_0 + mT_1$$

From this we get,

$$\dot{T}_0 = E$$
$$\dot{T}_1 = \frac{2E}{R_0}$$
$$y_0\dot{Z}_0 - Z_0\dot{y}_0 = L$$
$$y_0\dot{Z}_1 - Z_1\dot{y}_0 + y_1\dot{Z}_0 - Z_0\dot{y}_1 = -\frac{2L}{R_0}$$
$$\dot{y}_0{}^2 + \dot{Z}_0{}^2 = E^2$$
$$\dot{y}_0\dot{y}_1 + \dot{Z}_0\dot{Z}_1 = 0$$

where $\mathrm{R}_0^2 = y_0^2 + Z_0^2$

And from here we get

$$u = \frac{mE}{p}\lambda - \frac{m^2}{p}\ln Z_0 + R_0$$
$$v = -4p\ln Z_0 + R_0$$

where we have ignored irrelevent inegration constant and thus,

$$\hat{u} = \frac{mE}{p}\lambda + \frac{m^2}{p}[\frac{Z_0\ln 2R_0}{R_0} - \ln Z_0 + R_0]$$
$$\hat{v} = 4p[\frac{Z_0\ln 2R_0}{R_0}\ln Z_0 + R_0]$$

We now seperate the space into a near region N and a far region F as follows:

$$N = \{|\lambda| < \frac{1}{\sqrt{m}}\}$$
$$F = \{\sqrt{m} < m|\lambda| < \infty\}$$

Now, we have,

$$\lim_{\lambda\to-\infty}\hat{v} = 0$$
$$\lim_{\lambda\to\infty}\hat{v} = -4p\ln y_0^2$$
$$\lim_{\lambda\to+\infty}\hat{u} = \lim_{\lambda\to-\infty}\hat{u} = \frac{mE}{p}\lambda$$

Thus, the total shift in $\hat{v}$ is given by

$$\Delta\hat{v} = -4p\ln y_0^2$$

In the limit m tending to zero, $\lambda is infinite everywhere in F and \hat{u}$ is zero in N, whereas $\hat{u}$ is a good affine parameter in F along the geodesic.
The shift that occurs, for small m, essentially only in N.Thus, in the limit as m goes to zero, the shift occurs at $\hat{u} = 0$ and represents a finite discontinuity in $\hat{v}$ alomg null geodesics. This can also be seen by calculating

$$\lim_{\lambda \to \pm\infty} \dot{\hat{v}} = 0$$

After solving y we get,

$$y = -\frac{L}{E} + m(-\frac{2R_0}{y_0} + AZ_0)$$

where A is a constant.

We are interested in the behaviour of y in far field F for small m. We obtain,

$$\lim_{m \to 0; \hat{u} \neq 0} \frac{\partial y}{\partial \hat{u}} = -\frac{2p}{y_0}\hat{u} - pA$$

Lets, take a metric,

$$d\hat{s}^2 = 2A(u, v)dudv + g(u, v)h_{ij}(x^i)dx^i dx^j$$

This metric is assumed to satisfy Eimstein's vacuum equations. We introduced a shockwave by keeping the metric for $u < 0$ but replacing v by $v + f(x^i)$ for $u > 0$.

$$ds^2 = 2A(u, v + \theta f)du(dv + \theta f_i dx^i) + g(u, v + \theta f)h_{ij}(x^i)dx^i dx^j$$

where $\theta = \theta(u)$ is the usual step function. Changing to coordinates $(\hat{u}, \hat{v}, \hat{x}^i)$ defined by

$$\hat{u} = u$$
$$\hat{v} = v + \theta f$$
$$\hat{x}^i = x^i$$

We obtain

$$ds^2 = 2A(\hat{u}, \hat{v})d\hat{u}(d\hat{v} - \delta(\hat{u})f d\hat{u}) + g(\hat{u}, \hat{v})h_{ij}d(\hat{x})^i d(\hat{x})^j$$

The stress-energy tensor for massless particle located at the origin $\rho = 0$ of the (x^i) 2-surface and at $u = 0$ is

$$T^{ab} = 4p\delta(\rho)\delta(u)\delta_v^a\delta_v^b$$

where p is the momentum of the particle. Thus, the only non-zero component is

$$T_{uu} = 4pA^2\delta(\rho)\delta(u)$$

1.2 Shockwave example

Gravitational field of a massless particle in Minkowski space is described by the metric

$$ds^2 = -d\hat{u}(d\hat{v} + 4p\ln{(\rho^2)}\delta(\hat{u})d\hat{u}) + dx^2 + dy^2 \tag{1}$$

where $\rho^2 = x^2 + y^2$

The particle moves in the $\hat{v}$ direction with momentum p. By calculating geodesics which cross the shock wave which is the located at $u = 0$, we obtain the following two physical effects of such a shock wave: geodesics have s dicontinuity $\Delta\hat{v}$at $u = 0$ and are refracted in the transverse direction. The shift $\Delta\hat{v}$ for photon is given by

$$\Delta\hat{v} = -\frac{4l_{Pl}^2\nu}{c}\ln\frac{\rho_0^2}{l_{Pl}^2} \tag{2}$$

where $E = pc = h\nu and \rho = \rho(u) and \rho(0) = \rho_0$

There is also a refracted effect for photon described by

$$\cot\alpha + \cot\beta = \frac{4l_{Pl}^2\nu}{c\rho_0} \tag{3}$$

Hence,

$$\Delta\frac{\partial y}{\partial\hat{u}} = -\frac{4p}{y_0}$$

1.3 General result

Lets, consider a solution of the vacuum Einstein field equations of the form

$$d\hat{s}^2 = 2A(u,v)dudv + g(u,v)h_{ij}(x^i)dx^i dx^j \tag{4}$$

At $u = 0$ we have

$$A_{,v} = g_{,v} = 0 \;\; ; \quad \frac{A}{g}\Delta f - \frac{g_{,uv}}{g} f = 32\pi p A^2 \delta(p) \tag{5}$$

where $f = f(x^i)$ represents the shift in v, Δf is the laplacian of f with respect to the 2-metric h_{ij} and the resulting metric described by (B.2) and (B.4). Eq(5) represents our main result. We now turn to specific examples.

For a plane wave to a photon in Minkowski space we have

$$ds^2 = -dudv + dx^2 + dy^2$$

And thus

$$A = -\frac{1}{2} \;\; ; \;\; g = 1 \tag{6}$$

Therefore

$$A_{,v} = g_{,v} = 0 \;\; ; \;\; \Delta f = -16\pi p \delta(p) \tag{7}$$

where $\rho^2 = x^2 + y^2$.The solutionn of this equation,unique upto solution of homogeneous equation, is

$$f = -4p \ln \rho^2 \tag{8}$$

which agrees precisely with eq(2) and (A.26).
For a sourceless plane wave in Minkowski space we set p = 0 to obtain

$$\Delta f = 0 \tag{9}$$

For a spherical wave in Minkowski space we write the metric in the form

$$d\hat{s^2} = -dudv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

so that

$$A = -\frac{1}{2} \;\; ; \;\; g = r^2 = -\frac{1}{4}(v-u)^2 \tag{10}$$

But $g_{,v}|_{u=0} \not= 0$. Thus there exists no spherical waves of this form in Minkowski space.

Now, lets take Kruskal-Szekers coordinates

$$d\hat{s}^2 = -\frac{32m^3}{r} \exp -\frac{r}{2m} dudv + r^2(d\theta^2 + \sin^2\theta d\theta^2 d\phi^2)$$

Thus

$$A = -\frac{-16m^3}{r} e^{-\frac{r}{2m}} \;\; ; \;\; g = r^2 \;\; ; \;\; uv = -(\frac{r_{,v}}{2m} - 1)e^{\frac{r}{2m}} \tag{11}$$

So that all derivatives of r are proportional to u. Thus, the conditions on the metric coefficient A and g are satisfied at $u = 0$.
Furthermore, since $g_{u,v} = A$ the condition on f becomes

$$\Delta f - f = 32\pi pgA|_{u=0}\Delta(\theta) = -2\pi\kappa\delta(\theta) \tag{12}$$

where $\kappa = 2^9 m^4 p e^{-\frac{r}{2m}}$ and where we have arranged the coordinates so that the photon is at 0 = 0 = u .

We now solve Eq. (12) by expanding f in terms of spherical harmonics $Y_{lm}(\theta, \phi)$. We see immediately that only spherical harmonics with $m = 0$ contribute; expressing these in terms of the Legendre polynomials $\mathrm{P}_l(x) leadsto$

$$f = \kappa \sum_l \frac{l + \frac{1}{2}}{l(l+1)+1} P_l(\cos\theta) \tag{13}$$

We can obtain an integral expression for f by using the generating function for the Legendre polynomials, namely

$$\sum_{l=0}^{\infty} P_l(x)t^l = (1 - 2xt + t^2)^{\frac{1}{2}}$$

and the fact that

$$\int_{-\infty}^{0} t^l e^{\frac{s}{2}} \cos\frac{1}{2}\sqrt{3}sds = \frac{l+\frac{1}{2}}{l(l+1)+1} \tag{14}$$

where $t = e^s$, to finally obtain

$$f = \kappa \int_0^2 \frac{\sqrt{\frac{1}{2}} \cos\frac{1}{2}\sqrt{3}s}{(\cosh s - \cos\theta)^{\frac{1}{2}}} \tag{15}$$

We have not attempted to perform the integration explicitly. We note that the homogeneous equation (eq. (12) with $p = 0$) has no solution. In the limit of small θ eq.(15) in appropriate coordinates reduces to eq.(8), with well-determined value of the integration constant.

1.4 Discussion

2 On gravitational shock waves in curved space-times

2.1 Prerequisite calculations

<u>**Appendix A:**</u> Corresponding to metric eqn(21) the non vanishing Christoffel symbols are (for notational convenience we omit the hats)

$$
\begin{aligned}
\Gamma^u_{uu} &= -\frac{F_{,v}}{2A} + \frac{A_{,u}}{A}\,, & \Gamma^u_{ij} &= -\frac{g_{,v}}{2A}\, h_{ij}\,, & & \\
\Gamma^v_{uu} &= \frac{F_{,u}}{2A} + \frac{FF_{,v}}{2A^2} - \frac{FA_{,u}}{A^2}\,, & \Gamma^v_{uv} &= \frac{F_{,v}}{2A}\,, & \Gamma^v_{ui} &= \frac{F_{,i}}{2A}\,, \\
\Gamma^v_{vv} &= \frac{A_{,v}}{A}\,, & \Gamma^v_{ij} &= \left(-\frac{g_{,u}}{2A} + \frac{Fg_{,v}}{2A^2}\right) h_{ij}\,, & & \\
\Gamma^i_{uu} &= -\frac{1}{2g}\, h^{ik} F_{,k}\,, & \Gamma^i_{uj} &= \frac{g_{,u}}{2g}\, \delta^i_j\,, & \Gamma^i_{vj} &= \frac{g_{,v}}{2g}\, \delta^i_j\,, \\
\Gamma^i_{jk} &= \frac{1}{2} h^{il} (h_{lk,j} + h_{lj,k} - h_{jk,l})\,. & & & &
\end{aligned}
\tag{A.1}
$$

Using the above expressions we find that the non-vanishing components of the Ricci tensor are (we substitute $F = -2Af\delta$ (from (21)))

$$R_{uu} = \frac{d-2}{2}\left(\frac{g_{,u}A_{,u}}{gA} - \frac{g_{,uu}}{g} + \frac{g_{,u}^2}{2g^2}\right) + \frac{A}{g}\delta\triangle_{h_{ij}} f - \frac{d-2}{2}\frac{g_{,v}}{g}\delta' f$$
$$+\left(2\frac{A_{,uv}}{A} - 2\frac{A_{,u}A_{,v}}{A^2} + \frac{d-2}{2gA}(g_{,u}A_{,v} + g_{,v}A_{,u})\right)\delta f$$
$$+2\left(\frac{A_{,vv}}{A} - \frac{A_{,v}^2}{A^2} + \frac{d-2}{2}\frac{g_{,v}A_{,v}}{gA}\right)\delta^2 f^2 ,$$
$$R_{uv} = \left(\frac{A_{,u}A_{,v}}{A^2} - \frac{A_{,uv}}{A} + \frac{d-2}{4}\frac{g_{,u}g_{,v}}{g^2} - \frac{d-2}{2}\frac{g_{,uv}}{g}\right)$$
$$+\left(\frac{A_{,v}^2}{A^2} - \frac{A_{,vv}}{A} - \frac{d-2}{2}\frac{g_{,v}A_{,v}}{gA}\right)\delta f ,$$

$$R_{ui} = -\left(\frac{d-4}{2}\frac{g_{,v}}{g} + \frac{A_{,v}}{A}\right)\delta f_{,i} ,$$
$$R_{vv} = \frac{d-2}{2}\left(\frac{g_{,v}A_{,v}}{gA} + \frac{g_{,v}^2}{2g^2} - \frac{g_{,vv}}{g}\right) ,$$
$$R_{ij} = R_{ij}^{(d-2)} - \left(\frac{d-4}{2}\frac{g_{,u}g_{,v}}{gA} + \frac{g_{,uv}}{A}\right)h_{ij} - \left(\frac{d-4}{2}\frac{g_{,v}^2}{gA} + \frac{g_{,vv}}{A}\right)h_{ij}\delta f . \quad \text{(A.2)}$$

Appendix B:Backgrounds with a eovariantly constant null Killing vector

Consider the metric with a covariantly constant null Killing vector

$$ds^2 = 2dudv + g_{ij}(u,x)dx^i dx^j$$

where (i,j = 1,2 . . . . . d - 2). The most general matter field energy-momentum tensor consistent with the non-vanishing components of the Ricci tensor corresponding to the metric above has the form

$$T = 2T_{uv}(u,x)dudv + 2T_{ui}(u,x)dudx^i + T_{ij}(u,v,x)dx^i dx^j$$

The analog of metric (21) is

$$ds^2 = 2d\hat{u}d\hat{v} + \hat{F}d\hat{u}^2 + g_{ij}d\hat{x}^i d\hat{x}^j \;\; ; \;\; \hat{F} = F(\hat{u},\hat{x}) = -2\hat{f}\delta$$

.

The tensor (22) is same as the energy momentum tensor that is mentioned in appendix B. And the metric we are working with right now has no v-dependence in the components.Then the condition to be satisfied at u = 0 is the differential equation (again dropping the hats over the symbols)

$$\Delta_{g_{ij}} f = 32\pi p \delta^{(d-2)}(x) \ ; \ \Delta_{g_{ij}} = \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j$$

2.2 General Results

Lets take the metric representing the d-dimensional spacetime

$$ds^2 = 2A(u,v)dudv + g(u,v)h_{ij}(x^i)dx^i dx^j \tag{16}$$

with $(i,j = 1,2,.....,d-2)$. Now lets assume that there are matter fields which is represented by the energy momentum tensor

$$T = 2T_{uv}(u,v,x)dudv + T_{uu}(u,v,x)du^2 + T_{vv}(u,v,x)dv^2 + T_{ij}(u,v,x)dx^i dx^j \tag{17}$$

This kind of energy momentum tensor is consistent with the Ricci tensor of the metric in eq.(16) for $f = 0$.

We consider a massless particle located at u=0 and moving with the speed of light in the v-direction and we want to find out what its effect is on the geometry described by eq.(16). Our ansatz will be that for $u < 0$ the spacetime is described by eq(16) and for $u > 0$ by eq.(16) but with v shifted as $v \to v + f(x)$ where f (x) is a function to be determined. Therefore the resulting spacetime metric and energy-momentum tensor are

$$ds^2 = 2A(u, v+\Theta f)du(dv + \Theta f_{,i}dx^i) + g(u, v+\Theta f)h_{ij}(x^i)dx^i dx^j \tag{18}$$

where $\Theta = \Theta(u)$ is Heaviside step function and

$$T = 2T_{uv}(u, v+\Theta f, x)du(dv + \Theta f_{,i}dx^i) + T_{uu}(u, v+\Theta f, x)du^2$$

$$+\mathrm{T}_{vv}(u, v+\Theta f v, x)(dv + \Theta f_{,i}dx^i)^2 + T_{ij}(u, v+\Theta f, x)dx^i dx^j \tag{19}$$

Lets take new coordinates

$$\hat{u} = u \ ; \ \hat{v} = v + f(x)\Theta(u) \ ; \ \hat{x}^i = x^i \tag{20}$$

in which the metric(18) and the energy momentum tensor take the form

$$ds^2 = 2\hat{A}d\hat{u}\hat{v} + \hat{F}d\hat{u}^2 + \hat{g}\hat{h}_{ij}d\hat{x}^i d\hat{x}^j \ ; \ \hat{F} = F(\hat{u},\hat{v},\hat{x}) = -2\hat{A}\hat{f}\hat{\delta} \tag{21}$$

and

$$T = 2(\hat{T}_{\hat{u}\hat{v}} - \hat{T}_{\hat{v}\hat{v}}\hat{f}\hat{\delta})d\hat{u}d\hat{v} + (\hat{T}_{\hat{u}\hat{u}} + \hat{T}_{\hat{v}\hat{v}}\hat{f}^2\hat{\delta}^2)d\hat{u}^2 + \hat{T}_{\hat{u}\hat{v}}d\hat{v}^2 + \hat{T}_{\hat{i}\hat{j}}d\hat{x}^i d\hat{x}^j \tag{22}$$

The various metric and field components must satisfy Einstein's equations in the presence of matter, which in d-dimensions read (we take $c = \hbar = G = 1$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = -8\pi T_{\mu\nu} R_{\mu\nu} = -8\pi(T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T_{\lambda}{}^{\lambda}) = -8\pi\tilde{T}_{\mu\nu} \tag{23}$$

Obviously, $\tilde{T} = \tilde{T}_{\mu\nu}dx^{\mu}dx^{\nu}$ is also of the form eq.(17). We would like to find solutions of (23)with metric tensor of the form eq.(21) and the energy momentum tensor given by eqn(22) and plus the contribution of the energy momentum tensor for a massless particle located at the origin of the transverse x-space and at $u = 0$ and moving with the speed of light in the v direction

$$T^p = T^p_{uu}du^2 = \hat{T}^p_{\hat{u}\hat{u}}d\hat{u}^2 = -4p\hat{A}^2\hat{\delta}(\hat{u})d\hat{u}^2 \tag{24}$$

where p is the momentum of the particle. All the relevant tensor components appearing in eqn(23) are given in appendix A. To simplify the notation we will also drop the hats over the symbols keeping in mind however the transformation eqn(20). Assuming that the parts of the equations(23) that do not involve the function f are satisfied, one finds by examining the linear in $f\delta$ terms that at u=0 the additional conditions

$$g_{,v} = A_{,v} = T_(vv) = 0\ ,\ \Delta_{h_{ij}}f - \frac{(d-2)}{2}\frac{g_{,uv}}{A}f = 32\pi pgA\delta^{(d-2)}(x) \tag{25}$$

must also be satisfied, where the Laplacian is defined as $\Delta_{h_{ij}} = 1/\sqrt{h}\partial_i\sqrt{h}h^{ij}\partial_j$. In order to cast the differential eqn in (25) into the given form we used the fact that at u=0

$$\frac{A_{,uv}}{A} = -\frac{d-2}{2}\frac{g_{,uv}}{g} + 8\pi\tilde{T}_{uv} \tag{26}$$

This equality follows from the $(\mu,\nu) = (u,v)$ components of (23) computed at $u = 0$ and for$f = 0$. Notice also that the differential equation in (25) does not explicitly depend on the components of the energy-momentum tensor of the matter fields. Its dependence on these fields is only implicit through the functions A, g that are determined from the $f-$independent part of Einstein's equations.

Next we examine the $f^2\delta^2$ type terms. This is important because such terms should also vanish (in a distribution sense), otherwise our considerations are perturbative in powers of f. Using the first line in (2.10) it is easy to see that the coefficient of f282 in the R,u component of the Ricci tensor in (A.2) has terms of order O(u) and O(u 2) (in all of our examples such terms are of order O(u 2) since there is a functional dependence only on the product uv). Remembering that we are really considering all the quantities involving t -functions as distributions to be integrated over smooth functions we find that in this case such an integral vanishes. Moreover, because Tvv = 0 at u = 0 we have that T ,v= O(u) (at least). Therefore we can take the terms in (2.8) involving $f^2\delta^2$ tO zero. A last crucial remark is that because we have assumed that f is a function of the xi's only, the potential v-dependence in (2.10) should drop out for a consistent solution to exist. Mathematically that implies that the coefficient of the order O(u) term in the expansion of g(u, v) in powers of u should be a linear function of v. In fact this is the case in all the examples we explicitly work out. For the 4-dimensional case in the absence of matter fields the condition (2.10) was given in Ref. [6]. It is remarkable that (2.10) has essentially the same form in the vacuum and in the presence of matter except that in the latter case the additional