The Gravitational Shockwave of a Massless Particle

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Outline

Aim

- To summarize the properties of the shock wave due to a massless particle in Minkowski space by doing the explicit calculation of null geodesics across the shockwave
- To determine the necessary and sufficient conditions upon the spacetime metric and the form of the coordinate shift for the existence of shockwave

An exact exterior solution of vaccum Einstein's equation

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$$ds^2 = -(1 - \frac{r_g}{r})dt^2 + \frac{1}{(1 - \frac{r_g}{r})}dr^2 + r^2 d\Omega^2$$

Put $r = (1 + \frac{r_g}{4\rho})^2 \rho$, where $\rho^2 = x^2 + y^2 + z^2$

The metric becomes

$$ds^2 = -(\frac{1 - B}{1 + B})^2 dt^2 + (1 + B)^4 (dx^2 + dy^2 + dz^2)$$

where $B = \frac{r_g}{4\rho} = \frac{m}{2R}$

Small mass approximation

$$m << 2R \implies B \sim 0$$

$$ds^2 \sim -(1 - \frac{2m}{R})dt^2 + (1 + \frac{2m}{R})(dx^2 + dy^2 + dz^2)$$

Boosting the rest frame

Boost the rest frame with respect to, coordinates (t,x,y,z) via

$$T = t\cosh\beta - z\sinh\beta$$

$$Z = -t\sinh\beta + z\cosh\beta$$

Set $m = 2pe^{-\beta}$ where p=constant$>$ 0Introduce null coordintaes

$$u = t - z$$

$$v = t + z$$

The momentum of the particle is

$$p^a = m[\cosh(\beta)\delta^a_t + \sinh(\beta)\delta^a_z]$$

$$\lim_{\beta\to\infty} p^a = p(\delta^a_t + \delta^a_z)$$

Further algebra

After doing a bit of algebra we get,

$$Z = -\frac{p}{m}u + \frac{m}{4p}v$$

$$T = \frac{p}{m}u + \frac{m}{4p}v$$

$$R^2 = x^2 + y^2 + (\frac{p}{m}u - \frac{m}{4p})^2$$

And the metric becomes

$$ds^2 = (1 + \frac{2m}{R})(-dudv + dx^2 + dy^2) + \frac{4m}{R}(\frac{p}{m}du + \frac{m}{4p}dv)^2$$

Therefore,

$$\lim_{m \to 0; (u \neq 0, v, x, y)} ds^2 = -du(dv - 4p\frac{du}{|u|}) + dx^2 + dy^2$$

New coordinates $\hat{u}$ and $\hat{v}$

Lets, introduce,

$$\hat{u} = u + \frac{m^2 Z \ln 2R}{pR}$$

$$\hat{v} = v + \frac{4pZ \ln 2R}{R}$$

So, the metric becomes,

$$ds^2 = -d\hat{u}d\hat{v} + dx^2 + dy^2$$

Linearized geodesic of metric

The linearized geodesics of the metric are given by,

$$\dot{T} = E(1 + \frac{2m}{R})$$

$$y\dot{Z} - Z\dot{y} = L(1 - \frac{2m}{R})$$

$$\dot{y}^2 + \dot{Z}^2 = -M^2(1 - \frac{2m}{R}) + E^2$$

The dot denotes derivative with respect to affine parameter λ, E is energy, L is angular momentum and M is mass of the test particle. Now, we consider only geodesics(we set M=O(m^2)).

First order perturbative analysis of the geodesics

Expanding y, Z , T in power of m and considering only the terms in m, we have,

$$y = y_0 + my_1 \;\; ; \;\; Z = Z_0 + mZ_1 \;\; ; \;\; T = T_0 + mT_1$$

From this we get,

$$\dot{T}_0 = E$$

$$\dot{T}_1 = \frac{2E}{R_0}$$

$$y_0\dot{Z}_0 - Z_0\dot{y}_0 = L$$

$$y_0\dot{Z}_1 - Z_1\dot{y}_0 + y_1\dot{Z}_0 - Z_0\dot{y}_1 = -\frac{2L}{R_0}$$

$$\dot{y_0}^2 + \dot{Z_0}^2 = E^2$$

$$\dot{y}_0\dot{y}_1 + \dot{Z}_0\dot{Z}_1 = 0$$

where $R_0^2 = y_0^2 + Z_0^2$

Expressing coordintes in terms of Z_0 and R_0

$$u = \frac{mE}{p}\lambda - \frac{m^2}{p}\ln(Z_0 + R_0)$$

$$v = -4p\ln(Z_0 + R_0)$$

where we have ignored irrelevent inegration constant and thus,

$$\hat{u} = \frac{mE}{p}\lambda + \frac{m^2}{p}[\frac{Z_0\ln(2R_0)}{R_0} - \ln(Z_0 + R_0)]$$

$$\hat{v} = 4p[\frac{Z_0\ln(2R_0)}{R_0} - \ln(Z_0 + R_0)]$$

Seperating the space into near region N and far region F

We now seperate the space into a near region N and a far region F as follows:

$$N = \{|\lambda| < \frac{1}{\sqrt{m}}\}$$

$$F = \{\sqrt{m} < m|\lambda| < \infty\}$$

Boundary values of coordinates

$$\lim_{\lambda \to -\infty} \hat{v} = 0$$

$$\lim_{\lambda \to \infty} \hat{v} = -4p \ln y_0^2$$

$$\lim_{\lambda \to \pm\infty} \hat{u} = \frac{mE}{p} \lambda$$

Thus, the total shift in $\hat{v}$ is given by

$$\Delta \hat{v} = -4p \ln y_0^2$$

In the limit m tending to zero, λ is infinite everywhere in F and $\hat{u}$ is zero in N, whereas $\hat{u}$ is a good affine parameter in F along the geodesic.

The shift that occurs, for small m, essentially only in N.Thus, in the limit as m goes to zero, the shift occurs at $\hat{u} = 0$ and represents a finite discontinuity in $\hat{v}$ along null geodesics. This can also be seen by calculating

$$\lim_{\lambda \to \pm\infty} \dot{\hat{v}} = 0$$

Boundary values of coordinates continued

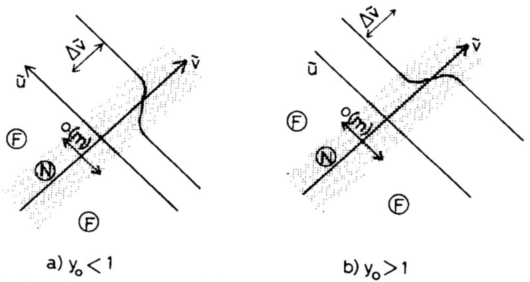


Fig. 2. The path of a null geodesic in the $(\hat{u}, \hat{v})$ plane as described by eq. (2) for $m \ll 1$, $\rho_0 \gg m$, and (a) $\rho_0 < 1$, (b) $\rho_0 > 1$. The near region N and the far region F, as well as the shift $\Delta\hat{v}$, are indicated.

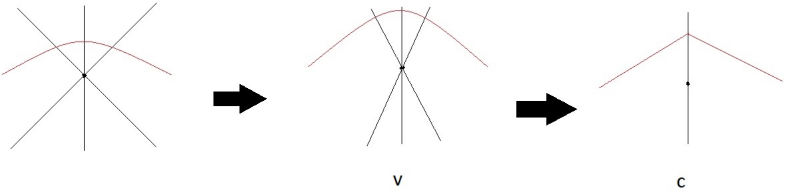
Behaviour of y

$$y = -\frac{L}{E} + m(-\frac{2R_0}{y_0} + AZ_0)$$

where A is a constant.

In far field F for small m,

$$\lim_{m \to 0; \hat{u} \neq 0} \frac{\partial y}{\partial \hat{u}} = -\frac{2p}{y_0}\hat{u} - pA$$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Introducing shockwave into a vacuum Einstein's solution

Lets, take a metric,

$$d\hat{s}^2 = 2A(u,v)dudv + g(u,v)h_{ij}(x^i)dx^i dx^j$$

This metric is assumed to satisfy Eimstein's vacuum equations. We introduced a shockwave by keeping the metric for $u < 0$ but replacing v by $v + f(x^i)$ for $u > 0$.

$$ds^2 = 2A(u, v + \theta f)du(dv + \theta f_i dx^i) + g(u, v + \theta f)h_{ij}(x^i)dx^i dx^j$$

where $\theta = \theta(u)$ is the usual step function.

Introducing new coordinates

Changing to coordinates $(\hat{u}, \hat{v}, \hat{x}^i)$ defined by

$$\hat{u} = u$$

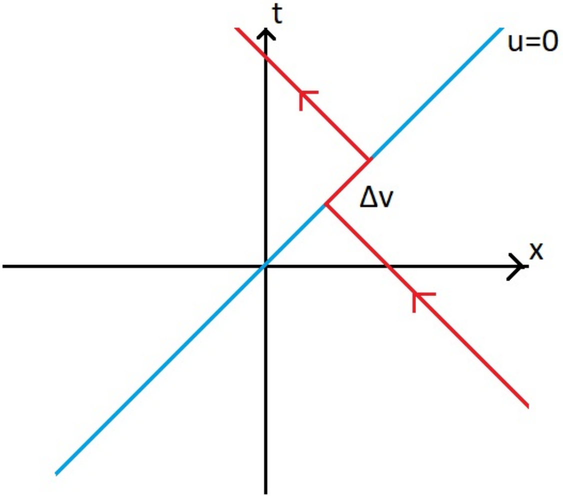
$$\hat{v} = v + \theta f$$

$$\hat{x}^i = x^i$$

We obtain

$$ds^2 = 2A(\hat{u}, \hat{v})d\hat{u}(d\hat{v} - \delta(\hat{u})fd\hat{u}) + g(\hat{u}, \hat{v})h_{ij}d(\hat{x})^i d(\hat{x})^j$$

Introducing New Coordinates continued



Energy-momentum tensor of the particle

The stress-energy tensor for massless particle located at the origin $\rho = 0$ of the (x^i) 2-surface and at $u = 0$ is

$$T^{ab} = 4p\delta(\rho)\delta(u)\delta^a_v\delta^b_v$$

where p is the momentum of the particle. Thus, the only non-zero component is

$$T_{uu} = 4pA^2\delta(\rho)\delta(u)$$

Condition on A and g for shockwave to exist

Now, if we put T_{uu} and R_{uu} in Einstein's equation of general relativity and partially integrating δ' and putting $A_{,v} = g_{,v} = 0$ we get necessary conditions for there to exist shockwave in the given spacetime.

Examples of Shockwave

Gravitational field of a massless particle in Minkowski space is described by the metric

$$ds^2 = -d\hat{u}(d\hat{v} + 4p\ln(\rho^2)\delta(\hat{u})d\hat{u}) + dx^2 + dy^2 \quad (1)$$

where $\rho^2 = x^2 + y^2$

$$\Delta\hat{v} = -4p\ln\rho_0^2 \quad (2)$$

Refracted effect for photon described by

$$\cot\alpha + \cot\beta = \frac{4p}{y_0} \quad (3)$$

$$\Delta\frac{\partial y}{\partial\hat{u}} = -\frac{4p}{y_0}$$

Examples of Shockwave continued

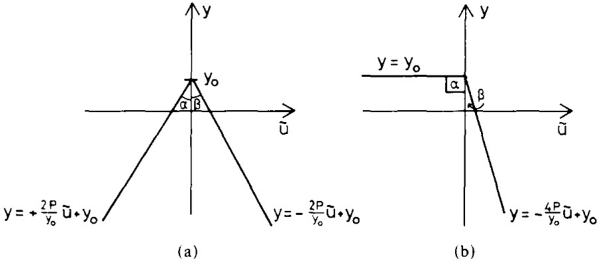


Fig. 3. The "spatial refraction" of null geodesics as described by eq. (3) for the two special cases (a) $\alpha = \beta$, and (b) $\alpha = \frac{1}{2}\pi$.

General Results

Lets, consider a solution of the vacuum Einstein field equations of the form

$$d\hat{s}^2 = 2A(u,v)dudv + g(u,v)h_{ij}(x^i)dx^i dx^j \tag{4}$$

At $u = 0$ we have

$$A_{,v} = g_{,v} = 0 \;\; ; \qquad \frac{A}{g}\Delta f - \frac{g_{,uv}}{g} f = 32\pi p A^2 \delta(p) \tag{5}$$

where $f = f(x^i)$ represents the shift in v, Δf is the laplacian of f with respect to the 2-metric h_{ij}.

Shockwave in Minkowski Spacetime

For a plane wave to a photon in Minkowski space we have

$$ds^2 = -dudv + dx^2 + dy^2$$

And thus

$$A = -\frac{1}{2} \;\; ; \;\; g = 1 \tag{6}$$

Therefore

$$A_{,v} = g_{,v} = 0 \;\; ; \;\; \Delta f = -16\pi p \delta(p) \tag{7}$$

where $\rho^2 = x^2 + y^2$.The solutionn of this equation,unique upto solution of homogeneous equation, is

$$f = -4p \ln \rho^2 \tag{8}$$

Spherical wave in Minkowski Space?

For a spherical wave in Minkowski space we write the metric in the form

$$d\hat{s^2} = -dudv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

so that

$$A = -\frac{1}{2} \;\; ; \;\; g = r^2 = -\frac{1}{4}(v-u)^2 \tag{9}$$

But $g_{,v}|_{u=0} \neq 0$.

No spherical waves in Minkowski space!

Shockwave in Kruskal-Szekers Coordinates

Now, lets take Kruskal-Szekers coordinates

$$d\hat{s}^2 = -\frac{32m^3}{r}\exp -\frac{r}{2m}dudv + r^2(d\theta^2 + \sin^2\theta d\theta^2 d\phi^2)$$

Thus

$$A = -\frac{-16m^3}{r}e^{-\frac{r}{2m}} \;\; ; \;\; g = r^2 \;\; ; \;\; uv = -(\frac{r_{,v}}{2m} - 1)e^{\frac{r}{2m}} \tag{10}$$

All derivatives of r are proportional to u. Thus, the conditions on the metric coefficient A and g are satisfied at $u = 0$.
As, $g_{u,v} = A$ the condition on f becomes

$$\Delta f - f = 32\pi pgA|_{u=0}\Delta(\theta) = -2\pi\kappa\delta(\theta) \tag{11}$$

where $\kappa = 2^9 m^4 pe^{-\frac{r}{2m}}$ and where we have arranged the coordinates so that the photon is at $\theta = 0 = u$.

Reference

- The Gravitational Shockwave of Massless Particle;
  Tevian Dray and Gerard 't Hooft

The End