

Introduction to Integration

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 15

Learning Outcomes

- Understand what integration is, in terms of (i) the inverse of differentiation, and (ii) finding the area under a curve.
- Learn the standard rules for integration, using the tables in the formula booklet.
- Understand the difference between **definite** and **indefinite** integration.

Introduction

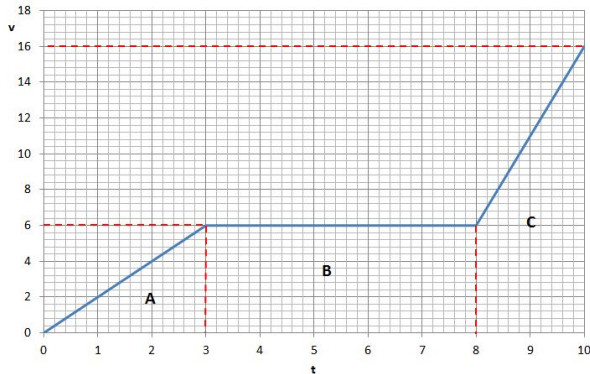
Integration is used to solve many problems, such as determining how much of a quantity has accumulated over time:

- Integrate power over time to determine total energy.
- Integrate force over distance to determine energy spent (valuable in potential energy problems).
- Integrate flow rate over time to determine accumulation of the flowing quantity e.g. mass, volume, charge etc.

Integration concerns calculating the **area underneath a curve**.

Example

Consider again this piece-wise graph which illustrates the velocity v (m/s) of an object over time t (s).



Example

Considering region B we can determine that the object is travelling at 6 m/s between $t = 3$ s and $t = 8$ s.

So in this region the object has travelled 6 m after 1 second, 12 m after 2 seconds and 30 m after 5 seconds.

But we could instead calculate the total distance travelled in region B by calculating the **area underneath the curve**:

$$\begin{aligned}\text{Area} &= \text{Width} \times \text{Height} \\ &= 5 \text{ (s)} \times 6 \text{ (ms}^{-1}\text{)} \\ &= 30 \text{ m}\end{aligned}$$

Integration: Opposite of Differentiation

We could similarly calculate the displacement in regions A and C , using the formulae for the area of a trapezium.

However, calculating the area underneath a more general **curve** requires approximations by trapezia, and is a non-trivial process.

So, just as we calculated gradients by referring to our table of derivatives, we will now look at integrating functions by referring to a table of integrals.

Integration is the **inverse of differentiation**, just as division is the inverse of multiplication and subtraction is the inverse of addition. This means that many of the rules will be familiar, but reversed!

Integration Table - find this in the formula booklet!

y	$\int y \, dx$
a (any constant)	$ax + C$
ax^n	$\frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$
ae^{nx}	$\frac{ae^{nx}}{n} + C$
$\frac{a}{x}$	$a \ln x + C$
$\frac{a}{nx+b}$	$\frac{a \ln(nx+b)}{n} + C$
$a \sin nx$	$\frac{-a \cos nx}{n} + C$
$a \cos nx$	$\frac{a \sin nx}{n} + C$

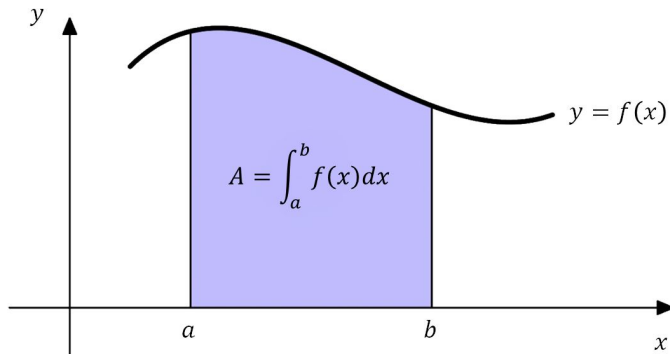
Definite Integration

Integration can be either **definite** or **indefinite**.

Definite integration allows us to calculate the exact area enclosed between:

- a curve,
- two vertical lines and
- the x -axis.

Notation for Definite Integration



Area A is given by the definite integral of function $f(x)$ between the limits $x = a$ and $x = b$.

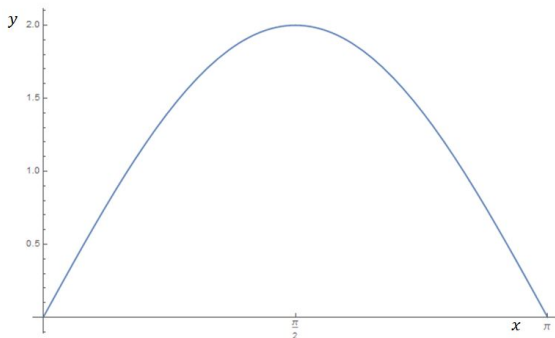
Definite Integration Example 1

Using formal integration, let's confirm the answer from the earlier example. In region B , between $x = 3$ and $x = 8$, the function was a constant: $y = 6$

$$\begin{aligned}\text{disp} &= \int_3^8 6 \, dx \\ &= [6x]_3^8 \quad \text{Now sub. in upper and lower limits:} \\ &= (6 \times 8) - (6 \times 3) \quad (\text{upper}) - (\text{lower}) \\ &= 48 - 18 \\ &= 30\text{m} \quad \text{as expected.}\end{aligned}$$

Definite Integration Example 2

Determine the area under the curve $y = 2 \sin x$ between the limits of $x = 0$ and $x = \pi$. Note that we will always use radians for trigonometric functions in this context unless stated otherwise.



Definite Integration Example 2 - Solution

$$\begin{aligned}
 \text{Area} &= \int_0^{\pi} 2 \sin x \, dx \\
 &= \left[-2 \cos x \right]_0^{\pi} \\
 &= (-2 \cos(\pi)) - (-2 \cos(0)) \quad (\text{use radians!}) \\
 &= 2 + 2 \\
 &= 4 \text{ units squared}
 \end{aligned}$$

Physical Example 3

Determine the displacement, S m, of an object between the times $t = 2$ s and $t = 5$ s given that the expression for its velocity is:

$$v = 3t^2 - 6t + 7$$

Physical Example 3 - Solution

$$\begin{aligned} S &= \int_2^5 v \, dt \\ &= \int_2^5 3t^2 - 6t + 7 \, dt \\ &= \left[\frac{3t^3}{3} - \frac{6t^2}{2} + 7t \right]_2^5 \\ &= [t^3 - 3t^2 + 7t]_2^5 \\ &= (5^3 - 3 \times 5^2 + 7 \times 5) - (2^3 - 3 \times 2^2 + 7 \times 2) \\ &= 85 - 10 = 75 \text{ m} \end{aligned}$$

Indefinite Integration

The other kind is **indefinite integration**, where we do not have specific upper and lower limits. The solutions will always contain the constant of integration, C , to account for this uncertainty.

Example 4: Integrate the function $5x^2 + 7x - 2$ with respect to x :

$$\int 5x^2 + 7x - 2 \, dx = \frac{5x^3}{3} + \frac{7x^2}{2} - 2x + C$$

Note:

- only one $+C$ is required at the end of the answer, as if there were multiple unknown constants they could be merged.
- $+C$ could appear in definite integration, but it would disappear when subtracting the lower limit from the upper.

Indefinite Integration Example 5

Integrate:

$$y = \int 4x^3 - 7e^{3x} \, dx$$

Indefinite Integration Example 5 - Solution

$$\begin{aligned}y &= \int 4x^3 - 7e^{3x} \, dx \\&= \frac{4x^4}{4} - \frac{7e^{3x}}{3} + C \\&= x^4 - \frac{7}{3}e^{3x} + C\end{aligned}$$

We can only determine the constant of integration if we are given additional information about the curve or the physical problem, such as a coordinate, or an initial condition.

Indefinite Integration Example 5

For example, if we were also told that $y = 2$ when $x = 0$, then substitute this in and solve for C :

$$y = x^4 - \frac{7e^{3x}}{3} + C$$

$$\text{At } (0, 2) : \quad 2 = 0^4 - \frac{7e^{3 \times 0}}{3} + C$$

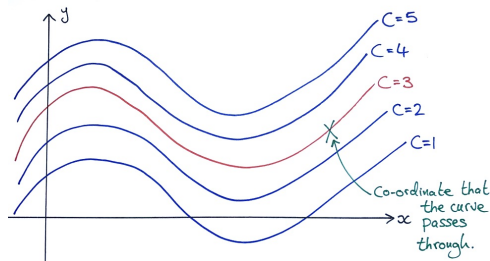
$$2 = -\frac{7}{3} + C$$

$$\therefore \quad C = 2 + \frac{7}{3} = \frac{13}{3}$$

Therefore, the particular solution is: $y = x^4 - \frac{7e^{3x}}{3} + \frac{13}{3}$

Indefinite integration

How does a condition help us find C ? Remember that integration is the inverse of differentiation, so indefinite integrate shows us the set of curves whose gradient all obey the same function. Thus, they are all parallel and vertically-shifted depending on the value of C .



A co-ordinate assigns a value to C , selecting a particular curve.

Exercises

Integrate the following:

1) $\int 6 \sin(3x) + 5x^{-2} \, dx$

2) $\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x} \, dx$

3) $\int_1^3 \frac{2}{x^3} + 2x + 8 \, dx$

4) Determine the particular solution of $y = \int 3 - 7x + 12x^2 \, dx$,
given that when $x = -2$, $y = 0.5$.

Exercises - Solutions (I/IV)

1) Indefinite integration:

$$\begin{aligned}\int 6 \sin(3x) + 5x^{-2} \, dx &= -\frac{6}{3} \cos(3x) + \frac{5}{-1} x^{-1} + C \\ &= -2 \cos(3x) - \frac{5}{x} + C\end{aligned}$$

2) Indefinite integration:

$$\begin{aligned}\int \frac{7x^2}{3} + e^{-2x} + \frac{5}{x} \, dx &= \frac{7}{9} x^3 + \frac{1}{-2} e^{-2x} + 5 \ln(x) + C \\ &= \frac{7}{9} x^3 - \frac{1}{2} e^{-2x} + 5 \ln(x) + C\end{aligned}$$

Exercises - Solutions (II/IV)

3) Definite integration:

$$\begin{aligned}\int_1^3 \frac{2}{x^3} + 2x + 8 \, dx &= \int_1^3 2x^{-3} + 2x + 8 \, dx \\&= \left[\frac{2}{-2}x^{-2} + \frac{2}{2}x^2 + 8x \right]_1^3 \\&= \left[-x^{-2} + x^2 + 8x \right]_1^3 \\&= \left(\frac{-1}{9} + 9 + 24 \right) - \left(\frac{-1}{1} + 1 + 8 \right) \\&= 32 \frac{8}{9} - 8 = 24 \frac{8}{9}\end{aligned}$$

Exercises - Solutions (III/IV)

4) First, the indefinite integral:

$$\begin{aligned}y &= \int 3 - 7x + 2x^2 \, dx \\&= 3x - \frac{7}{2}x^2 + \frac{2}{3}x^3 + C\end{aligned}$$

Then substitute in the condition $x = -2$ and $y = 0.5$:

$$0.5 = 3(-2) - \frac{7}{2}(-2)^2 + \frac{2}{3}(-2)^3 + C$$

Exercises - Solutions (IV/IV)

Simplifying, and solving for C :

$$0.5 = -6 - \frac{7}{2} \times 4 + \frac{2}{3} \times -8 + C$$

$$0.5 = -6 - 14 - \frac{16}{3} + C$$

$$C = \frac{1}{2} + 20 + \frac{16}{3} = \frac{155}{6}$$

So the particular solution is:

$$y = 3x - \frac{7}{2}x^2 + \frac{2}{3}x^3 + \frac{155}{6}$$