Further Matrices and their Applications

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 22

Learning Outcomes

- Perform additional matrix operations such as determining the inverse, transpose and determinant (if they exist).
- Solve systems of linear simultaneous equations using matrices.

Transpose of a Matrix

Taking the transpose of a matrix causes the 1st row to become the first column, the 2nd row to become the second column, etc.

For example, if

$$\underline{M} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \\ 2 & -8 \end{pmatrix}$$

Then

$$\underline{M}^{\mathsf{T}} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & -8 \end{pmatrix}$$

Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**.

The determinant of matrix $\underline{A}$ can be denoted by $det(\underline{A})$ or $|\underline{A}|$.

For a 2×2 matrix $\underline{A}$, the determinant is very simple to calculate by multiplying the diagonal entries:

Determinant of a 2×2 matrix:

$$det(\underline{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: determinant of a 2×2 matrix

Given the square matrix

$$\underline{B} = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$\begin{aligned} det(\underline{B}) &= 3 \times 2 - (-1) \times 4 \\ &= 6 + 4 \\ &= 10 \end{aligned}$$

Identity Matrices

Identity matrices are square matrices in which all elements are zero except for the elements on the leading diagonal; these are all 1, e.g.

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad or \quad \underline{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Pre/post-multiplying by $\underline{\text{I}}$ has no impact, i.e.

$$\underline{A}\,\underline{I} = \underline{A} \quad \text{and} \quad \underline{I}\,\underline{A} = \underline{A}$$

Inverse matrix

For a **square** matrix $\underline{A}$, there may exist an inverse matrix $\underline{A}^{-1}$

Inverse Matrix

$$\underline{A}\underline{A}^{-1} = \underline{I} \quad \text{and} \quad \underline{A}^{-1}\underline{A} = \underline{I}$$

So an inverse matrix is analagous to the reciprocal of a number - it's what you multiply by to get back to 1 (or the identity):

$$5 \times \frac{1}{5} = 1$$

$$\underline{A} \times \underline{A}^{-1} = \underline{I}$$

Calculating the inverse of a 2×2 matrix

For a general 2×2 square matrix $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Inverse of a 2×2 matrix

$$\underline{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{or} \quad \underline{A}^{-1} = \frac{1}{|\underline{A}|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant of a square matrix is equal to zero, then that matrix has no inverse!

Example: Inverse of a 2×2 matrix

To find (if it exists) the inverse of 2×2 square matrix $\underline{A}$:

$$\underline{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

First obtain the determinant:

$$det(\underline{A}) = (1)(2) - (-1)(0) = 2$$

Then as the determinant is non-zero, the inverse exists and is:

$$\underline{A}^{-1} = \frac{1}{det(\underline{A})} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Exercise: Inverse of a 2×2 matrix

For the following square matrices, find the determinant and the inverse matrix if it exists:

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution: Inverse of a 2×2 matrix

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$\underline{B}^{-1} = \frac{1}{(1)(2) - (0)(-3)} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$det(\underline{C}) = (1)(-1) - (1)(-1) = 0$$

Hence $\underline{C}$ has zero determinant $\implies$ its inverse does not exist.

Motivation

Many engineering problems can be modelled as a system of simultaneous equations.

For example, let's say that there are two materials $\underline{A}$ and $\underline{B}$, whose densities are unknown. You have two samples of different composites of these: one is 15% $\underline{A}$ and 85% $\underline{B}$ and has a density of 1 kg m^{-3}, while the other is 40% $\underline{A}$ and 60% $\underline{B}$ but twice as dense. This could be written as:

$$
\begin{aligned}
0.15\underline{A} + 0.85\underline{B} &= 1 \\
0.4\underline{A} + 0.6\underline{B} &= 2
\end{aligned}
$$

We wish to determine the densities of the constituents $\underline{A}$ and $\underline{B}$.

Introduction

This is an example of a pair of simultaneous linear equations.
Another example:

$$3x + 2y = 16$$
$$-x + 4y = 7$$

We will learn to solve them (i.e. find the unique values of x and y for which both equations are true) using a matrix method.

Method (1)

Given a pair of simultaneous equations, ensure they are in this form first:

$$ax + by = p$$
$$cx + dy = q$$

1. Then write the pair of equations as a matrix equation:

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\underline{AX} = \underline{B}$$

Method (2)

1. So the square matrix of coefficients is $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the vector $\underline{X}$ contains x and y which we want to find.
2. Calculate the inverse matrix $\underline{A}^{-1}$
3. Pre-multiply both sides by the inverse matrix to obtain $\underline{X}$:

$$\underline{AX} = \underline{B} \quad \Longrightarrow \quad \underline{A}^{-1}\underline{AX} = \underline{A}^{-1}\underline{B} \quad \Longrightarrow \quad \underline{X} = \underline{A}^{-1}\underline{B}$$

4. From the entries in vector $\underline{X}$, read off the values of x and y.
5. Substitute the values of x and y back into the original equations to verify solutions.

Example 1 (I/II)

Solve for x and y:

$$5x + 2y = 10$$
$$4x - 3y = 14$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $\underline{AX} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Example 1 (II/II)

Then,

$$\underline{A}^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find $x = 58/23$ and $y = -30/23$.

Example 2

Solve for x and y:

$$3x = 7 + 5y$$
$$4y + 2x = 20$$

Example 2 - Solution (I/II)

First, re-write both of these in a consistent format:

$$3x - 5y = 7$$
$$2x + 4y = 20$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

so we have $\underline{AX} = \underline{B}$, where

$$\underline{A} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

Example 2 - Solution (II/II)

Then,

$$\underline{A}^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

and so

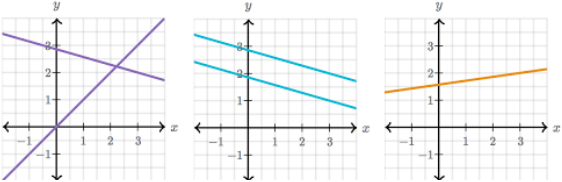
$$\underline{X} = \underline{A}^{-1}\underline{B} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

Thus we find $x = 64/11$ and $y = 23/11$.

Special cases

A linear equation $ax + by = d$ can be re-written in the form $y = mx + c$. In other words, we have been trying to find the co-ordinates of the point where two straight lines intersect.

What if the pair of lines are parallel or actually the same?



In these cases (zero solutions or infinitely many solutions), the matrix of coefficients will be **uninvertible** (its determinant $= 0$).

Special cases

If the matrix of coefficients has determinant $= 0$, examine the two equations and determine if they are the same equation (infinitely-many solutions), or if they are contradictory (zero solutions).

$$x - 3y = 10$$
$$2x - 6y = 20$$

$$-2x + y = 3$$
$$4x - 2y = 17$$

The first pair are the **same**, and the second pair are **contradictory**.