Railway Engineering Mathematics

Tutorial Sheet 6

Solutions

Analysing and sketching quadratics

1. Consider the quadratic function $y = x^2 + 3x - 4$
   (a) Where does $y = x^2 + 3x - 4$ cross the y-axis?
   (b) Where does $y = x^2 + 3x - 4$ cross the x-axis?
   (c) Is it $\cup$-shaped, or $\cap$-shaped?
   (d) Sketch the graph of this function.

Solution:

(a) Crosses the y-axis when $x = 0$:

$$\begin{aligned} y &= 0^2 + 3(0) - 4 \\ &= -4 \end{aligned}$$

So at the y-intercept of 4.

(b) The curve crosses the x-axis at $y = 0$:

$$0 = x^2 + 3x - 4$$

Here, $a = 1$, $b = 3$ and $c = -4$, and we can solve this using the quadratic formula:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{3^2 - 4(1)(-4)}}{2 \times 1} \\
&= \frac{-3 \pm \sqrt{25}}{2} \\
&= \frac{-3 \pm 5}{2}
\end{aligned}
$$

Hence we have the roots:

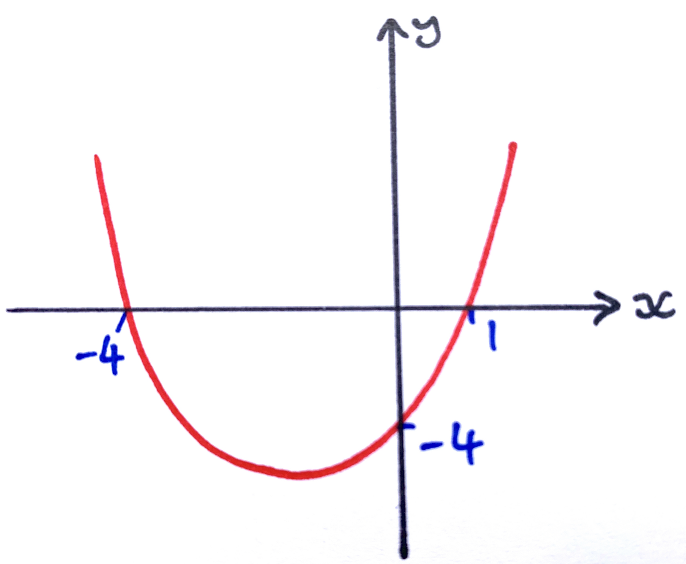
$$x_1 = \frac{-3-5}{2} = \frac{-8}{2} = -4$$

and

$$x_2 = \frac{-3+5}{2} = \frac{2}{2} = 1$$

(c) This graph will be $\cup$-shaped as $a > 0$.

(d) Given the shape of the parabola, and the important points (x and y-intercepts), we can sketch the curve:



2. Repeat the process of Question 1 for the functions:

   (i) $y = 16 - x^2$

   (ii) $y = 2x^2 + 4x - 20$

   Check your results by plotting the curves in EXCEL or other software (such as Desmos or GeoGebra).

   Solution:

   (i) For $y = 16 - x^2$

   (a) Crosses the y-axis when $x = 0$:

   $$\begin{aligned} y &= 16 - 0^2 \\ &= 16 \end{aligned}$$

   (b) Crosses the x-axis at $y = 0$:

   $$0 = 16 - x^2$$

   In this case, we could simply re-write it as:

   $$x^2 = 16$$

   and so taking square roots of both sides, we immediately obtain

   $$x = \pm\sqrt{16} = \pm 4$$

   Or we could still use the quadratic formula. To do this, write the equation as:

   $$-x^2 + 0x + 16 = 0$$

   Here, $a = -1$, $b = 0$ and $c = 16$, and thus:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-0 \pm \sqrt{0^2 - 4(-1)(16)}}{2 \times (-1)} \\
&= \frac{\pm\sqrt{64}}{-2} \\
&= \frac{\pm 8}{-2}
\end{aligned}
$$

Hence we have the roots:

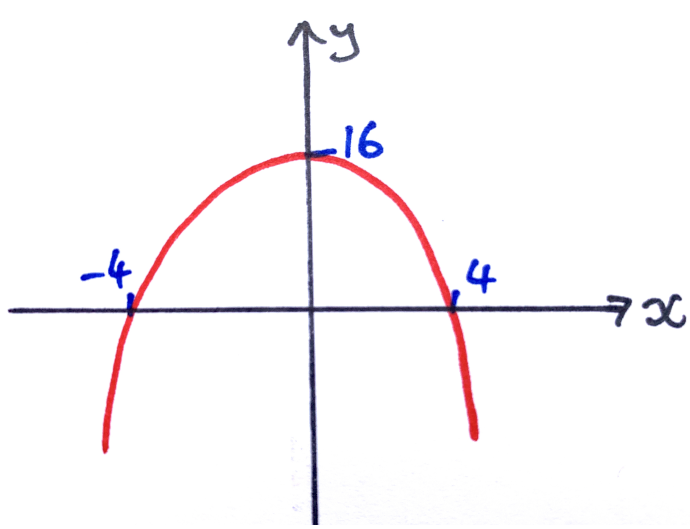
$$x_1 = \frac{-8}{-2} = 4$$

and

$$x_2 = \frac{8}{-2} = -4$$

(c) This graph will be $\cap$-shaped as $a < 0$.

(d) Given the shape of the parabola, and the important points (x and y-intercepts), we can sketch the curve:



(ii) For $y = 2x^2 + 4x - 20$

(a) Crosses the y-axis when $x = 0$:

$$\begin{aligned} y &= 2(0)^2 + 4(0) - 20 \\ &= -20 \end{aligned}$$

(b) Crosses the x-axis at $y = 0$:

$$0 = 2x^2 + 4x - 20$$

Here, $a = 2$, $b = 4$ and $c = -20$, and applying the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-20)}}{2 \times 2} \\ &= \frac{-4 \pm \sqrt{176}}{4} \\ &= \frac{-4 \pm 4\sqrt{11}}{4} \\ &= -1 \pm \sqrt{11} \end{aligned}$$

Hence we have the roots:

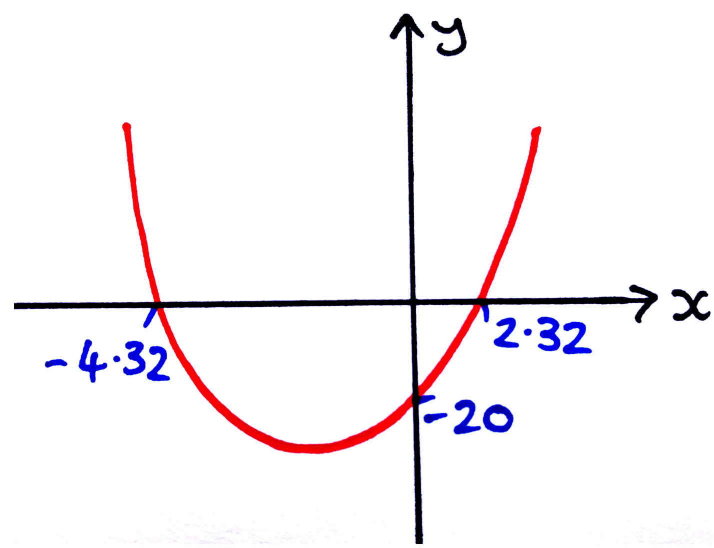
$$x_1 = -1 + \sqrt{11} = 2.32\ldots$$

and

$$x_2 = -1 - \sqrt{11} = -4.32\ldots$$

(c) This graph will be $\cup$-shaped as $a > 0$.

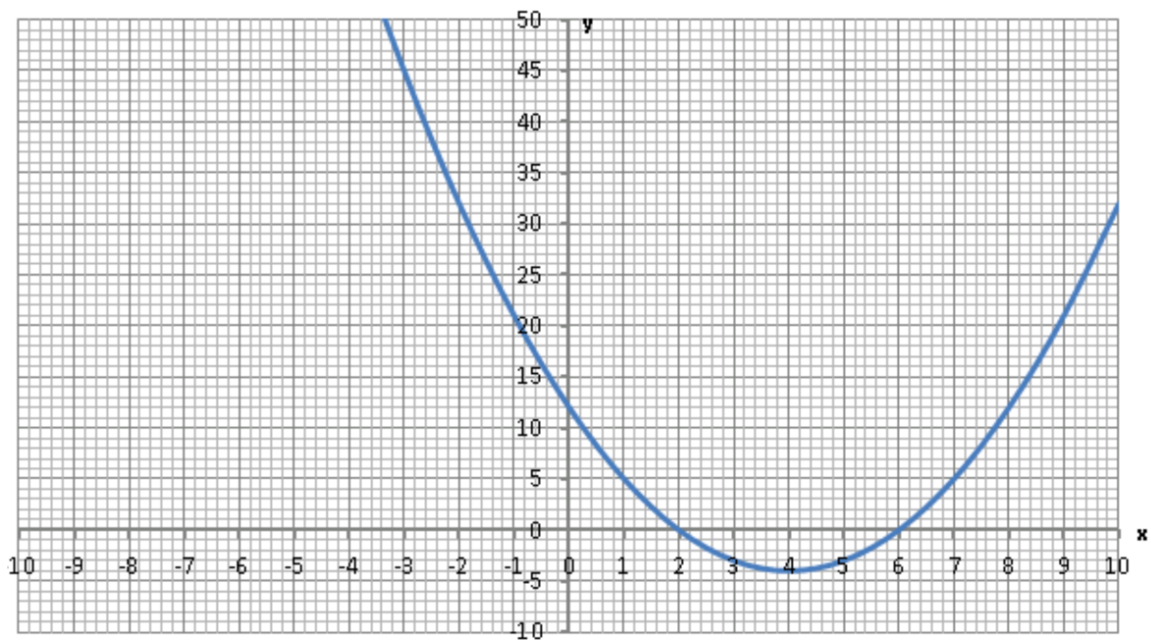
(d) Given the shape of the parabola, and the important points (x and y-intercepts), we can sketch the curve:



Determining the equation of a parabola

3. Determine the quadratic equation of the following graphs, given that they are in the form of either $y = x^2 + bx + c$ or $y = -x^2 + bx + c$ (i.e. note that the x^2 term is already known to have a coefficient equal to either one or minus one).

   (a)



Solution:

Given that the parabola is $\cup$-shaped, we know that the coefficient a of the x^2 term is positive, and since we only have the choice from our prior information of $a = 1$ or $a = -1$, therefore we must have $a =$ and hence:

$$y = x^2 + bx + c$$

Now, from the graph this curve crosses the x-axis at $x_1 = 2$ and $x_2 = 6$. These are the roots, which we can substitute into the factorised formula from earlier. Then expand the brackets and simplify:

$$\begin{aligned}
y &= a(x - x_1)(x - x_2) \\
&= 1(x - 2)(x - 6) \\
&= x^2 - 6x - 2x + 12 \\
&= x^2 - 8x + 12
\end{aligned}$$

Alternatively (or if we did not know the roots, specifically), from the graph we can see that the curve crosses the y-axis at a height of $y = 12$. Thus the y-intercept, and hence the value of parameter c, is equal to 12 and so we have:

$$y = x^2 + bx + 12$$

Now there is only one unknown parameter remaining. If we choose any point (apart from $(0, 12)$ as we have essentially already used that information) that the curve passes through, and substitute those co-ordinates into our equation so far, this will yield an equation where b is the only unknown and so we can solve for it.

For example, choose the point $(1, 5)$ and thus substitute in the values $x = 1$ and $y = 5$:

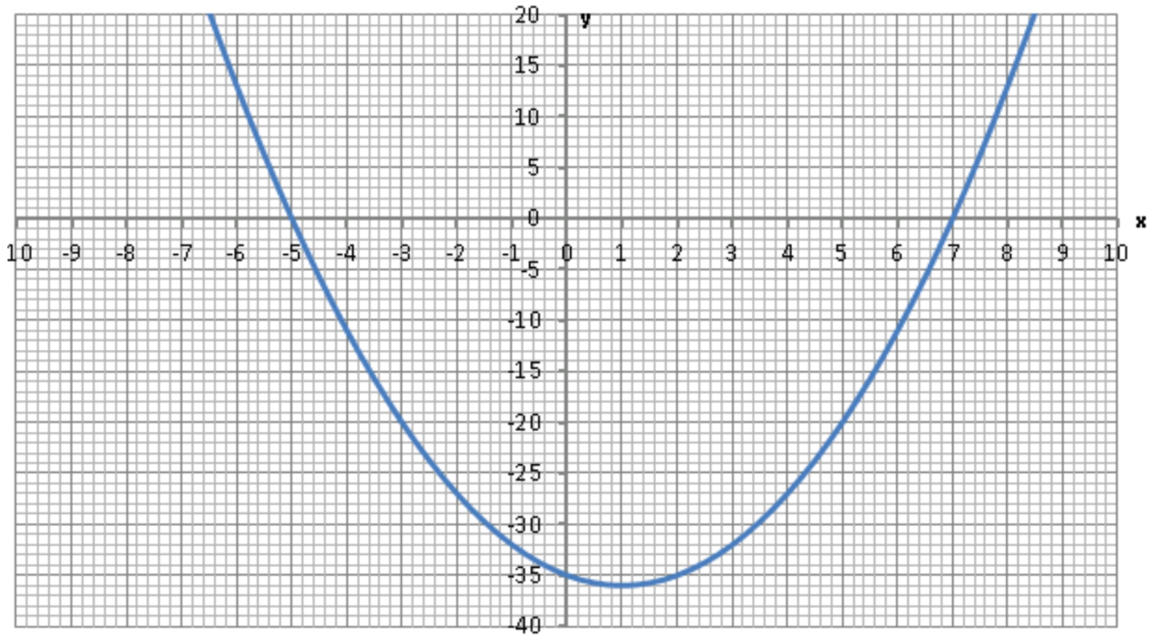
$$\begin{aligned}
& 5 = (1)^2 + (1)b + 12 \\
\therefore \quad & 5 = 1 + b + 12 \\
\therefore \quad & 5 = b + 13 \\
\therefore \quad & b = -8
\end{aligned}$$

and so we have:

$$y = x^2 - 8x + 12$$

which is the same result that we obtained using the other method.

(b)

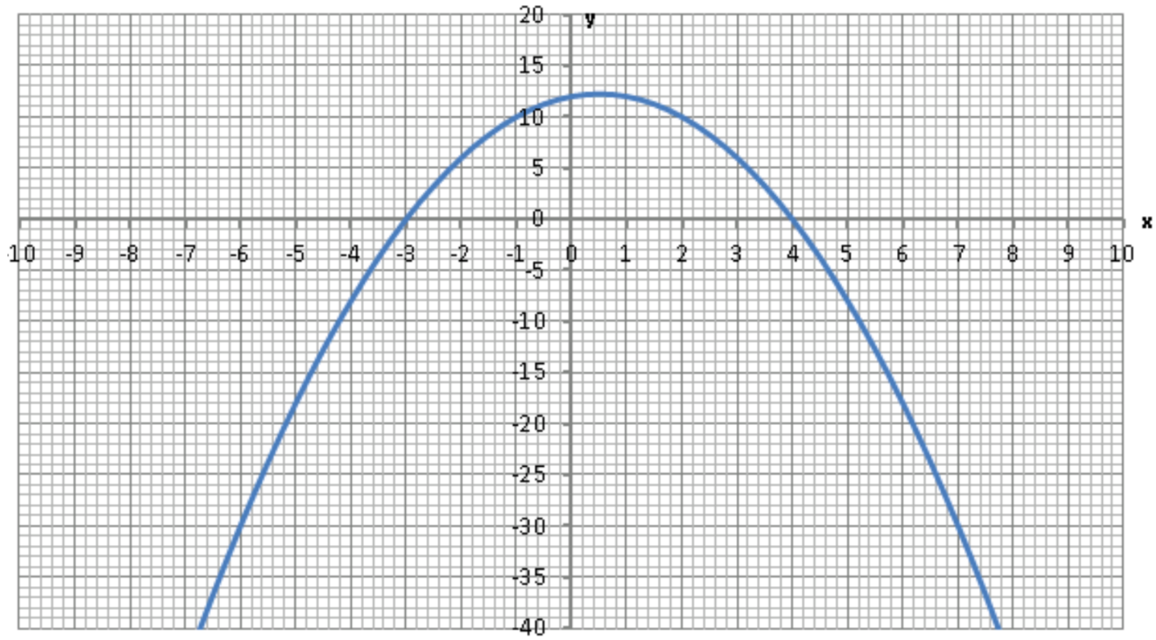


Solution:

This quadratic has roots $x_1 = -5$ and $x_2 = 7$. As it is $\cup$-shaped, we know that $a > 0$ and so it must be $a = 1$. Substituting this into the factorised formula and simplifying:

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= 1(x - (-5))(x - 7) \\ &= x^2 - 7x + 5x - 35 \\ &= x^2 - 2x - 35 \end{aligned}$$

(c)



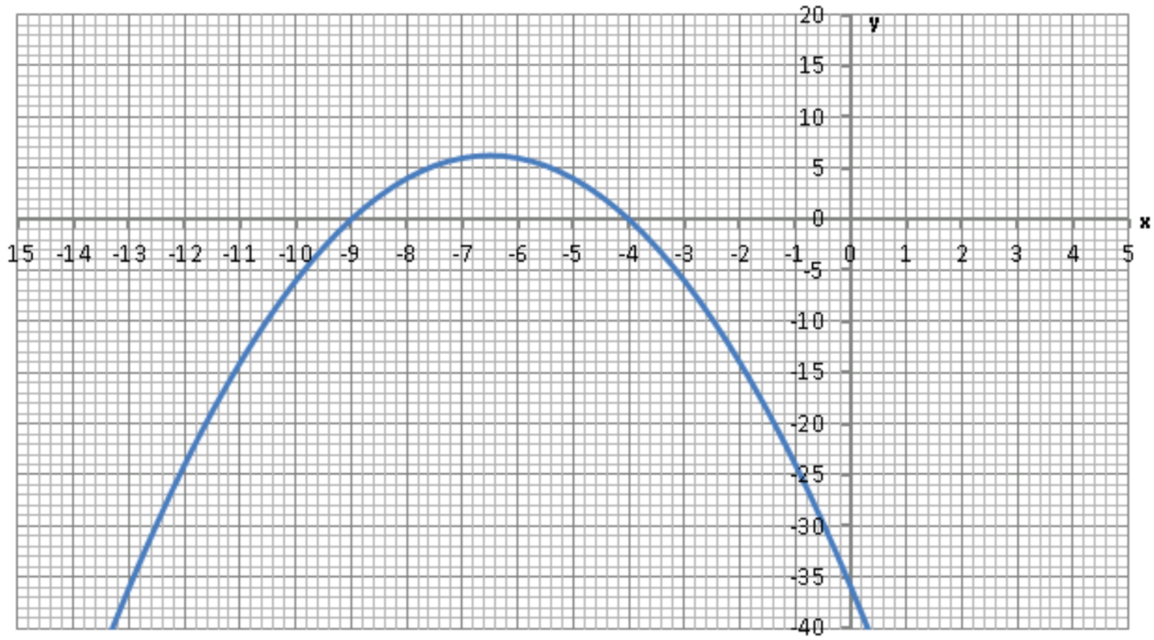
Solution:

Crosses the x-axis at $x_1 = -3$ and $x_2 = 4$. This time the graph is a $\cap$-shaped parabola, so the coefficient a of the x^2 term must be negative and thus in this case we have $a = -1$.

Hence:

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= -1(x - (-3))(x - 4) \\ &= -1(x + 3)(x - 4) \\ &= -(x^2 - 4x + 3x - 12) \\ &= -(x^2 - x - 12) \\ &= -x^2 + x + 12 \end{aligned}$$

(d)



Solution:

The parabola crosses the x-axis at $x_1 = -9$ and $x_2 = -4$, and it is $\cap$-shaped so we must have $a < 0$ and hence from the question $a = -1$:

$$
\begin{aligned}
y &= a(x - x_1)(x - x_2) \\
&= -1(x - (-9))(x - (-4)) \\
&= -1(x + 9)(x + 4) \\
&= -(x^2 + 4x + 9x + 36) \\
&= -(x^2 + 13x + 36) \\
&= -x^2 - 13x - 36
\end{aligned}
$$

Using the quadratic formula

4. Calculate the roots of the following polynomials:

   (a) $y = x^2 + 6x - 27$

   (b) $y = x^2 + 2x - 48$

   (c) $y = 3x^2 + 11x + 8$

   (d) $y = 6x^2 - 23x + 20$

Solution:

(a) $y = x^2 + 6x - 27$

Here, the coefficients are $a = 1$, $b = 6$ and $c = -27$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(-27)}}{2 \times 1} \\ &= \frac{-6 \pm \sqrt{144}}{2} \\ &= \frac{-6 \pm 12}{2} \\ &= -3 \pm 6 \end{aligned}$$

Hence the roots are:

$$x_1 = -3 - 6 = -9$$

and

$$x_2 = -3 + 6 = 3$$

(b) $y = x^2 + 2x - 48$

In this case, the coefficients are $a = 1$, $b = 2$ and $c = -48$. Substituting these into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{6^2 - 4(1)(-48)}}{2 \times 1} \\ &= \frac{-2 \pm \sqrt{196}}{2} \\ &= \frac{-2 \pm 14}{2} \\ &= -1 \pm 7 \end{aligned}$$

Hence the roots are:

$$x_1 = -1 - 7 = -8$$

and

$$x_2 = -1 + 7 = 6$$

(c) $y = 3x^2 + 11x + 8$

Here, the coefficients are $a = 3$, $b = 11$ and $c = 8$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4(3)(8)}}{2 \times 3} \\ &= \frac{-11 \pm \sqrt{25}}{6} \\ &= \frac{-11 \pm 5}{6} \end{aligned}$$

Hence the roots are:

$$x_1 = \frac{-11-5}{6} = \frac{-16}{6} = \frac{-8}{3}$$

and

$$x_2 = \frac{-11+5}{6} = \frac{-6}{6} = -1$$

(d) $y = 6x^2 - 23x + 20$

Here, the coefficients are $a = 6$, $b = -23$ and $c = 20$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-23) \pm \sqrt{(-23)^2 - 4(6)(20)}}{2 \times 6} \\ &= \frac{23 \pm \sqrt{49}}{12} \\ &= \frac{23 \pm 7}{12} \end{aligned}$$

Hence the roots are:

$$x_1 = \frac{23-7}{12} = \frac{16}{12} = \frac{4}{3}$$

and

$$x_2 = \frac{23+7}{12} = \frac{30}{12} = \frac{5}{2}$$

5. If the roots of the quadratic $y = ax^2 + bx + c$ are x_1 and x_2 then it is possible to rewrite the quadratic in its factorised form as follows:

$$y = a(x - x_1)(x - x_2)$$

(a) Use this information to rewrite the four quadratics that appear in Question 4.

(b) Expand the brackets of your new form in each case to demonstrate that it is equivalent to the original form.

Solution:

(a)

For $x^2 + 6x - 27$**:**
Here, we found the roots to be $x_1 = -9$ and $x_2 = 3$.

Given that $a = 1$, we can substitute into the factorised form as follows:

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= 1(x - -9)(x - 3) \end{aligned}$$

hence, the factorised form is:

$$y = (x + 9)(x - 3).$$

For $x^2 + 2x - 48$**:**
Here, we found the roots to be $x_1 = -8$ and $x_2 = 6$.

Given that $a = 1$, we can substitute into the factorised form as follows:

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= 1(x - -8)(x - 6) \end{aligned}$$

hence, the factorised form is:

$$y = (x+8)(x-6).$$

For $3x^2 + 11x + 8$**:**
Here, we found the roots to be $x_1 = -\frac{8}{3}$ and $x_2 = -1$.

Given that $a = 3$, we can substitute into the factorised form as follows:

$$\begin{aligned} y &= a(x-x_1)(x-x_2) \\ &= 3\left(x - -\frac{8}{3}\right)\left(x - -1\right) \\ &= (3x - -8)(x - -1) \end{aligned}$$

hence, the factorised form is:

$$y = (3x+8)(x+1).$$

For $6x^2 - 23x + 20$**:**
Here, we found the roots to be $x_1 = \frac{4}{3}$ and $x_2 = \frac{5}{2}$.

Given that $a = 6$, we can substitute into the factorised form as follows:

$$\begin{aligned} y &= a(x-x_1)(x-x_2) \\ &= 6\left(x - \frac{4}{3}\right)\left(x - \frac{5}{2}\right) \quad \text{[splitting the 6 into factors]} \\ &= 3\left(x - \frac{4}{3}\right) \times 2\left(x - \frac{5}{2}\right) \end{aligned}$$

hence, the factorised form is:

$$y = (3x - 4)(2x - 5).$$

(b)
For parts (a) to (d) multiply out the brackets to confirm the original forms from question 4.

Trickier questions

6. Solve $5x^2 + 2x = 4$, giving the solutions correct to 3 significant figures.

Solution:

Firstly rearrange the equation so that it is equal to zero:

$5x^2 + 2x - 4 = 0.$

Here, the coefficients are $a = 5$, $b = 2$ and $c = -4$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(5)(-4)}}{2 \times 5} \\ &= \frac{-2 \pm \sqrt{84}}{10}. \end{aligned}$$

Hence, the roots are:

$$x_1 = \frac{-2 - \sqrt{84}}{10} = -1.12$$

and

$$x_2 = \frac{-2 + \sqrt{84}}{10} = 0.717$$

7. Solve $x^2 = 4(x-3)^2$

Solution:

Firstly multiply out the brackets:

$$\begin{aligned} x^2 &= 4(x-3)(x-3) \\ x^2 &= 4(x^2 - 6x + 9) \\ x^2 &= 4x^2 - 24x + 36 \end{aligned}$$

Rearranging all terms to the left-hand side gives $-3x^2 + 24x - 36 = 0$.

Here, the coefficients are $a = -3$, $b = 24$ and $c = -36$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-24 \pm \sqrt{24^2 - 4(-3)(-36)}}{2 \times -3} \\ &= \frac{-24 \pm \sqrt{144}}{-6} \\ &= \frac{-24 \pm 12}{-6}. \end{aligned}$$

Hence, the roots are:

$$x_1 = \frac{-24 - 12}{-6} = \frac{-36}{-6} = 6$$

and

$$x_2 = \frac{-24 + 12}{-6} = \frac{-12}{-6} = 2$$

8. Solve $\dfrac{5}{x+2} + \dfrac{9}{x-2} = 2$

Solution: Note that there is more than one way to solve this.

Firstly multiply each term by $x+2$:

$$\begin{aligned} \frac{5}{x+2} + \frac{9}{x-2} &= 2 \\ \frac{5(x+2)}{x+2} + \frac{9(x+2)}{x-2} &= 2(x+2) \\ 5 + \frac{9(x+2)}{x-2} &= 2(x+2). \end{aligned}$$

Now multiply each term by $x-2$:

$$\begin{aligned} 5 + \frac{9(x+2)}{x-2} &= 2(x+2) \\ 5(x-2) + \frac{9(x+2)(x-2)}{x-2} &= 2(x+2)(x-2) \\ 5(x-2) + 9(x+2) &= 2(x+2)(x-2). \end{aligned}$$

Multiplying out and rearranging gives:

$$\begin{aligned} 5(x-2) + 9(x+2) &= 2(x+2)(x-2) \\ 5x - 10 + 9x + 18 &= 2(x^2 - 4) \\ 14x + 8 &= 2x^2 - 8 \\ -2x^2 + 14x + 16 &= 0. \end{aligned}$$

Here, the coefficients are $a = -2$, $b = 14$ and $c = 16$. Substituting these into the quadratic formula gives:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-14 \pm \sqrt{14^2 - 4(-2)(16)}}{2 \times -2} \\
&= \frac{-14 \pm \sqrt{324}}{-4} \\
&= \frac{-14 \pm 18}{-4}.
\end{aligned}$$

Hence, the roots are:

$$x_1 = \frac{-14 - 18}{-4} = \frac{-32}{-4} = 8$$

and

$$x_2 = \frac{-14 + 18}{-4} = \frac{4}{-4} = -1$$

9. The quadratic formula is being used to solve a quadratic equation. After substitution of the coefficients into the formula, we have

$$x = \frac{-7 \pm \sqrt{17}}{4}$$

Work out the original quadratic equation, giving your answer in the form $ax^2 + bx + c = 0$.

Solution:

The quadratic formula is in the form:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore $-b = -7$, hence $b = 7$. From the denominator we can deduce that $2a = 4$. Therefore $a = 2$. Finally we can determine the value of c from the discriminant:

$$\begin{aligned} b^2 - 4ac &= 17 \\ 7^2 - 4 \times 2 \times c &= 17 \\ 49 - 8c &= 17 \\ -8c &= -32 \\ c &= 4 \end{aligned}$$

Therefore the quadratic equation is $2x^2 + 7x + 4 = 0$.

10. Solve the equation $\dfrac{2^{(n^2)}}{2^n \times 2^6} = 1$

Solution:

Before we can use the quadratic equation, the base 2 must be eliminated. Let's use the rules of indices and logs to simplify the equation:

$$\begin{aligned} \frac{2^{(n^2)}}{2^n \times 2^6} &= 1 \quad \text{[add the powers on the denominator]} \\ \frac{2^{(n^2)}}{2^{n+6}} &= 1 \quad \text{[subtract the powers]} \\ 2^{(n^2-n-6)} &= 1 \quad \text{[take } \log_2 \text{ on both sides]} \\ n^2 - n - 6 &= \log_2(1) \\ n^2 - n - 6 &= 0. \end{aligned}$$

This is now a quadratic equation where $a = 1$, $b = -1$ and $c = -6$. Substituting these into the quadratic formula gives:

$$
\begin{aligned}
n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2 \times 1} \\
&= \frac{1 \pm \sqrt{25}}{2} \\
&= \frac{1 \pm 5}{2}.
\end{aligned}
$$

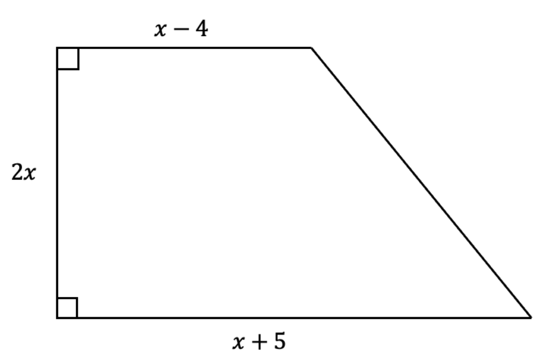
Hence, the solutions are:

$$n_1 = \frac{1-5}{2} = \frac{-4}{2} = -2$$

and

$$n_2 = \frac{1+5}{2} = \frac{6}{2} = 3$$

11. Shown in the diagram below is a trapezium (not drawn to scale) where all measurements are in centimetres.



The area of the trapezium is 351 m^2. Determine the value of x.

Solution:

The area of a trapezium is $A = \frac{h}{2}(a+b)$, where a and b are the lengths and h is the vertical height. Substituting our values into the formula gives:

$$\begin{aligned} A &= \frac{h}{2}(a+b) \\ 351 &= \frac{2x}{2}((x-4)+(x+5)) \\ 351 &= \frac{2x}{2}(2x+1) \quad \text{[cancelling the 2s]} \\ 351 &= x(2x+1) \\ 351 &= 2x^2 + x \\ 0 &= 2x^2 + x - 351. \end{aligned}$$

This is now a quadratic equation where $a = 2$, $b = 1$ and $c = -351$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4(2)(-351)}}{2 \times 2} \\ &= \frac{-1 \pm \sqrt{2809}}{4} \\ &= \frac{-1 \pm 53}{4}. \end{aligned}$$

Hence, the solutions are:

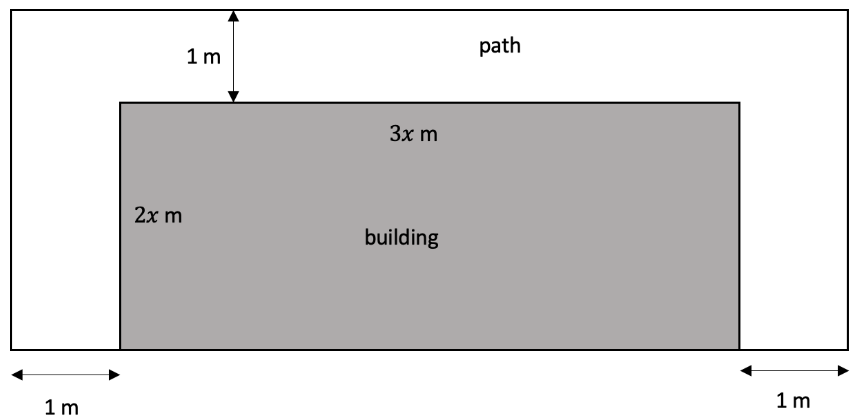
$$x_1 = \frac{-1-53}{4} = \frac{-54}{4} = -13.5$$

and

$$x_2 = \frac{-1+53}{4} = \frac{52}{4} = 13$$

The only sensible answer is $x = 13$ m, as this will result is positive values for the lengths and height of the trapezium.

12. A rectangular building has a length of $3x$ metres and a width of $2x$ metres. The building has a path of width 1 metre on three of its sides, as depicted in the diagram below. Note that the diagram is not drawn to scale.



Given that the total area of the building and path is $100\,\mathrm{m}^2$ calculate the area of the building.

Solution:

The value of x must be determined before we can calculate the area of the building. The area of the building plus the path is known and is rectangular. The area of a rectangle is $A = lw$, where l is the length and w is the width. Substituting our values into the formula gives:

$$\begin{aligned}
A &= lw \\
100 &= (1 + 3x + 1) \times (2x + 1) \\
100 &= (3x + 2) \times (2x + 1) \\
100 &= 6x^2 + 7x + 2 \\
0 &= 6x^2 + 7x - 98
\end{aligned}$$

This is now a quadratic equation where $a = 6$, $b = 7$ and $c = -98$. Substituting these into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-7 \pm \sqrt{7^2 - 4(6)(-98)}}{2 \times 6} \\ &= \frac{-7 \pm \sqrt{2401}}{12} \\ &= \frac{-7 \pm 49}{12}. \end{aligned}$$

Hence, the solutions are:

$$x_1 = \frac{-7 - 49}{12} = \frac{-56}{12} = -\frac{14}{3}$$

and

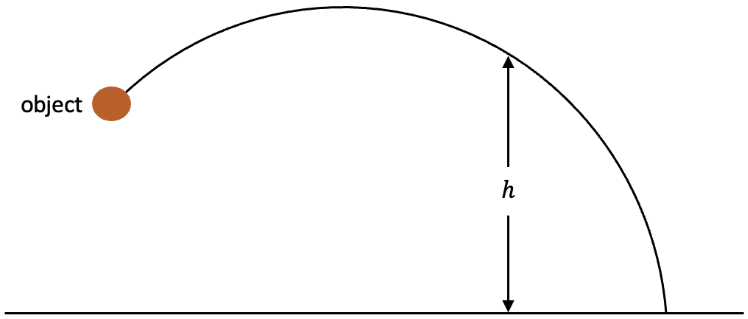
$$x_2 = \frac{-7 + 49}{12} = \frac{42}{12} = 3.5.$$

The only sensible answer is $x = 3.5$ m, as this will result is positive values for the length and width of the building. Now the value of x is known, we are able to calculate the area of the building:

$$\begin{aligned} A &= lw \\ &= 3x \times 2x \\ &= 10.5 \times 7 \\ &= 73.5 \end{aligned}$$

Therefore the area of the building is 73.5 m^2.

13. An object is being launched from an initial height of 2 m above the ground. The object follows a parabola in the form $h = 2 + 6t - 5t^2$, where h is the height in metres above the ground t seconds after it has been launched.



The object hits the ground after T seconds. Calculate the value of T, giving the answer to 2 decimal places.

Solution:

When the object hits the ground the value of h must be zero. Therefore we have $0 = 2 + 6T - 5T^2$. This is a quadratic equation where $a = -5$, $b = 6$ and $c = 2$. Substituting these into the quadratic formula gives:

$$\begin{aligned} T &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(-5)(2)}}{2 \times -5} \\ &= \frac{-6 \pm \sqrt{76}}{-10}. \end{aligned}$$

Hence, the solutions are:

$$T_1 = \frac{-6 - \sqrt{76}}{-10} = 1.47$$

and

$$T_2 = \frac{-6 + \sqrt{76}}{-10} = -0.27.$$

The only sensible answer is $T = 1.47$ s, as this is the only result that is positive. Furthermore, even if both times were positive, the largest value of T would be chosen as the first, whilst a correct result for the solution of the parabola, is not correct for the problem, as the object is thrown from a height of 2 m.