Railway Engineering Mathematics

Tutorial Sheet 17

Solutions

Evaluate the following integrals using integration by parts:

1. $$\int -4x\,\mathrm{e}^{7x}\ \mathrm{d}x$$

Solution:

As the integrand consists of the product of two functions of x, neither of which is a composite function, integration by parts is a suitable method. Using “L-A-T-E”, select the “algebraic” (that is, the polynomial) function to be u, and the exponential as v'. Thus, let:

$$u = -4x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{7x}$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -4 \quad \text{and} \quad v = \int \mathrm{e}^{7x}\ \mathrm{d}x = \frac{1}{7}\,\mathrm{e}^{7x}$$

Substituting these four components into the formula for integration by parts:

$$\begin{aligned}
\int -4x\,\mathrm{e}^{7x}\ \mathrm{d}x &= \big(-4x\big)\left(\frac{1}{7}\,\mathrm{e}^{7x}\right) - \int \left(\frac{1}{7}\,\mathrm{e}^{7x}\right)\big(-4\big)\,\mathrm{d}x \\
&= -\frac{4}{7}x\,\mathrm{e}^{7x} + \int \frac{4}{7}\,\mathrm{e}^{7x}\ \mathrm{d}x \\
&= -\frac{4}{7}x\,\mathrm{e}^{7x} + \frac{4}{7}\cdot\frac{1}{7}\,\mathrm{e}^{7x} + c \\
&= -\frac{4}{7}x\,\mathrm{e}^{7x} + \frac{4}{49}\,\mathrm{e}^{7x} + c \\
&= \frac{4}{49}\,\mathrm{e}^{7x}\big(1 - 7x\big) + c
\end{aligned}$$

2. $$\int 12x\cos(9x)\,\mathrm{d}x$$

Solution:

The integrand $12x\cos(9x)$ consists of a product of two simple functions of x, neither of which is a substantial composite, so integration by parts is an appropriated technique to attempt. Using "L-A-T-E", select the "algebraic" (that is, the polynomial) function to be u, and the trigonometric function (i.e. the cosine) as v'.

Thus, let:

$$u = 12x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos(9x)$$

then differentiating u to obtain u' and integrating v' to obtain v:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 12 \quad \text{and} \quad v = \int \cos(9x)\,\mathrm{d}x = \frac{1}{9}\sin(9x)$$

Substituting these into the formula for integration by parts:

$$\begin{aligned}
\int 12x\cos(9x)\,\mathrm{d}x &= \big(12x\big)\left(\frac{1}{9}\sin(9x)\right) - \int \left(\frac{1}{9}\sin(9x)\right)\big(12\big)\,\mathrm{d}x \\
&= \frac{12}{9}x\sin(9x) - \int \frac{12}{9}\sin(9x)\,\mathrm{d}x \\
&= \frac{12}{9}x\sin(9x) - \frac{12}{9}\left(-\frac{1}{9}\right)\cos(9x) + c \\
&= \frac{12}{9}x\sin(9x) + \frac{12}{81}\cos(9x) + c \\
&= \frac{4}{3}x\sin(9x) + \frac{4}{27}\cos(9x) + c
\end{aligned}$$

3. $$\int 5x^2 \ln(2x)\, \mathrm{d}x$$

Solution:

Using "L-A-T-E", select the logarithmic function to be u, and the "algebraic" function (i.e. the polynomial) as v'.

Thus, let:

$$u = \ln(2x) \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 5x^2$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \quad \text{and} \quad v = \int 5x^2\, \mathrm{d}x = \frac{5}{3}x^3$$

Substituting all four components into the formula for integration by parts:

$$\begin{aligned}
\int 5x^2 \ln(2x)\, \mathrm{d}x &= \big(\ln(2x)\big)\left(\frac{5}{3}x^3\right) - \int \left(\frac{5}{3}x^3\right)\left(\frac{1}{x}\right) \mathrm{d}x \\
&= \frac{5}{3}x^3 \ln(2x) - \int \frac{5}{3}x^2\, \mathrm{d}x \\
&= \frac{5}{3}x^3 \ln(2x) - \frac{5}{3} \cdot \frac{1}{3}x^3 + c \\
&= \frac{5}{3}x^3 \ln(2x) - \frac{5}{9}x^3 + c
\end{aligned}$$

4. $$\int_{0.4}^{5.6} -2t\sin(0.3t) + 12t^3 \,\mathrm{d}t$$

Solution:

The second term can be integrated directly, so we shall only need to deal with the first term by parts.

Since the first term consists of a product of a linear function of t (i.e. an "algebraic" part) and a cosine function which is a trigonometric function, using "L-A-T-E" we select $-2t$ to be the function that will be differentiated.

Thus, let:

$$u = -2t \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \sin(0.3t)$$

Then differentiating u with respect to t yields u':

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(-2t\big) = -2$$

And integrating v' to obtain v:

$$v = \int \sin(0.3t)\,\mathrm{d}t = -\frac{1}{0.3}\cos(0.3t) = -\frac{10}{3}\cos(0.3t)$$

For easier notation, let's assign a name to the entire quantity we require:

$$\text{Let} \quad I = \int_{0.4}^{5.6} -2t\sin(0.3t) + 12t^3 \,\mathrm{d}t$$

Substituting u, u', v, v' into the by parts formula and integrating the second term of I directly:

$$
\begin{aligned}
I &= \left[\Big(-2t\Big)\left(-\frac{10}{3}\cos(0.3t)\right)\right]_{0.4}^{5.6} - \int_{0.4}^{5.6}\left(-\frac{10}{3}\cos(0.3t)\right)(-2)\ \mathrm{d}t + \left[\frac{12t^4}{4}\right]_{0.4}^{5.6} \\
&= \left[\frac{20}{3}t\cos(0.3t)\right]_{0.4}^{5.6} - \int_{0.4}^{5.6}\frac{20}{3}t\cos(0.3t)\ \mathrm{d}t + \left[3t^4\right]_{0.4}^{5.6} \\
&= \left[\frac{20}{3}t\cos(0.3t)\right]_{0.4}^{5.6} - \left[\frac{20}{3}\cdot\frac{1}{0.3}\sin(0.3t)\right]_{0.4}^{5.6} + \left[3t^4\right]_{0.4}^{5.6} \\
&= \left[\frac{20}{3}t\cos(0.3t)\right]_{0.4}^{5.6} - \left[\frac{200}{9}\sin(0.3t)\right]_{0.4}^{5.6} + \left[3t^4\right]_{0.4}^{5.6}
\end{aligned}
$$

We could evaluate each of these three at the upper and lower limits for t, but it may be easier to simply combine the three to a single set of brackets:

$$
\begin{aligned}
I &= \left[\frac{20}{3}t\cos(0.3t) - \frac{200}{9}\sin(0.3t) + 3t^4\right]_{0.4}^{5.6} \\
&= \left(\frac{20}{3}(5.6)\cos(0.3\times 5.6) - \frac{200}{9}\sin(0.3\times 5.6) + 3(5.6)^4\right) \\
&\qquad -\left(\frac{20}{3}(0.4)\cos(0.3\times 0.4) - \frac{200}{9}\sin(0.3\times 0.4) + 3(0.4)^4\right) \\
&= 2924.13 \quad \text{(2 d.p.)}
\end{aligned}
$$

5. $$\int_{0.5}^{3.6} 2x\,\mathrm{e}^{-0.7x}\ \mathrm{d}x$$

Solution:

The integrand is the product of a linear function of x and an exponential function. Using "L-A-T-E" to select the linear ("algebraic") part to be differentiated, let:

$$u = 2x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-0.7x}$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \text{and} \quad v = \int \mathrm{e}^{-0.7x}\ \mathrm{d}x = \frac{1}{-0.7}\,\mathrm{e}^{-0.7x} = \frac{-10}{7}\,\mathrm{e}^{-0.7x}$$

Substituting these into the by parts formula:

$$\begin{aligned}
\int_{0.5}^{3.6} 2x\,\mathrm{e}^{-0.7x}\ \mathrm{d}x &= \left[(2x)\left(\frac{-10}{7}\,\mathrm{e}^{-0.7x}\right)\right]_{0.5}^{3.6} - \int_{0.5}^{3.6}\left(\frac{-10}{7}\,\mathrm{e}^{-0.7x}\right)(2)\ \mathrm{d}x \\
&= \left[-\frac{20}{7}x\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} - \int_{0.5}^{3.6} -\frac{20}{7}\,\mathrm{e}^{-0.7x}\ \mathrm{d}x \\
&= \left[-\frac{20}{7}x\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} - \left[-\frac{20}{7}\cdot\frac{1}{-0.7}\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} \\
&= \left[-\frac{20}{7}x\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} - \left[\frac{200}{49}\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} \\
&= \left[-\frac{20}{7}x\,\mathrm{e}^{-0.7x} - \frac{200}{49}\,\mathrm{e}^{-0.7x}\right]_{0.5}^{3.6} \\
&= \left(-\frac{20}{7}(3.6)\,\mathrm{e}^{-0.7\times3.6} - \frac{200}{49}\,\mathrm{e}^{-0.7\times3.6}\right) \\
&\qquad - \left(-\frac{20}{7}(0.5)\,\mathrm{e}^{-0.7\times0.5} - \frac{200}{49}\,\mathrm{e}^{-0.7\times0.5}\right) \\
&= 2.73 \quad (2\text{ d.p.})
\end{aligned}$$

6. $$\int_0^{\frac{\pi}{2}} 2\theta \sin(\theta)\, \mathrm{d}\theta$$

Solution:

The integrand is the product of a simple linear ("algebraic") and sine ("trigonometric") function. Making our choice using "L-A-T-E", let:

$$u = 2\theta \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sin(\theta)$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = 2 \quad \text{and} \quad v = \int \sin(\theta)\, \mathrm{d}\theta = -\cos(\theta)$$

Substituting these four components into the by parts formula and evaluating:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} 2\theta \sin(\theta)\, \mathrm{d}\theta &= \Big[(2\theta)\big(-\cos(\theta)\big)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \big(-\cos(\theta)\big)(2)\, \mathrm{d}\theta \\
&= \big[-2\theta\cos(\theta)\big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2\cos(\theta)\, \mathrm{d}\theta \\
&= \big[-2\theta\cos(\theta)\big]_0^{\frac{\pi}{2}} - \big[-2\sin(\theta)\big]_0^{\frac{\pi}{2}} \\
&= \big[-2\theta\cos(\theta) + 2\sin(\theta)\big]_0^{\frac{\pi}{2}} \\
&= \left(-2 \times \frac{\pi}{2} \times \cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)\right) \\
&\qquad -\Big(-2 \times 0 \times \cos(0) + 2\sin(0)\Big) \\
&= \big(-\pi \times 0 + 2 \times 1\big) - \big(0 + 2 \times 0\big) \\
&= 2
\end{aligned}$$

7. $$\int \frac{4x}{\mathrm{e}^{3x}}\ \mathrm{d}x$$

Solution:

First, let's re-write the integrand as a product rather than a quotient using the rules of indices:

$$\int 4x\,\mathrm{e}^{-3x}\ \mathrm{d}x$$

Then using "L-A-T-E", select the "algebraic" (that is, the polynomial) part to be u, and the exponential as v'.

Thus, let:

$$u = 4x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-3x}$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 4 \quad \text{and} \quad v = \int \mathrm{e}^{-3x}\ \mathrm{d}x = -\frac{1}{3}\,\mathrm{e}^{-3x}$$

Substituting these into the formula for integration by parts:

$$\begin{aligned}
\int \frac{4x}{\mathrm{e}^{3x}}\ \mathrm{d}x &= (4x)\left(-\frac{1}{3}\,\mathrm{e}^{-3x}\right) - \int \left(-\frac{1}{3}\,\mathrm{e}^{-3x}\right)(4)\ \mathrm{d}x \\
&= -\frac{4}{3}x\,\mathrm{e}^{-3x} - \int -\frac{4}{3}\,\mathrm{e}^{-3x}\ \mathrm{d}x \\
&= -\frac{4}{3}x\,\mathrm{e}^{-3x} - \left(-\frac{4}{3}\right)\left(\frac{1}{-3}\right)\mathrm{e}^{-3x} + c \\
&= -\frac{4}{3}x\,\mathrm{e}^{-3x} - \frac{4}{9}\,\mathrm{e}^{-3x} + c
\end{aligned}$$

8. $$\int_{-0.2}^{0.7} -9x\,\mathrm{e}^{-0.45x} + 3.5x^2 \,\mathrm{d}x$$

Solution:

The second term can be integrated directly, so we shall only need to deal with the first term by parts. To do this, let:

$$u = -9x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-0.45x}$$

then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -9 \quad \text{and} \quad v = \int \mathrm{e}^{-0.45x}\,\mathrm{d}x = \frac{1}{-0.45}\,\mathrm{e}^{-0.45x} = -\frac{20}{9}\,\mathrm{e}^{-0.45x}$$

For ease of notation, let the entire integral we wish to evaluate be named I:

$$\text{Let} \quad I = \int_{-0.2}^{0.7} -9x\,\mathrm{e}^{-0.45x} + 3.5x^2 \,\mathrm{d}x$$

Substituting these into the by parts formula and integrating the second term directly:

$$\begin{aligned}
I &= \left[(-9x)\left(-\frac{20}{9}\,\mathrm{e}^{-0.45x}\right)\right]_{-0.2}^{0.7} - \int_{-0.2}^{0.7}\left(-\frac{20}{9}\,\mathrm{e}^{-0.45x}\right)(-9)\,\mathrm{d}x + \left[\frac{7}{6}x^3\right]_{-0.2}^{0.7} \\
&= \left[20x\,\mathrm{e}^{-0.45x}\right]_{-0.2}^{0.7} - \int_{-0.2}^{0.7} 20\,\mathrm{e}^{-0.45x}\,\mathrm{d}x + \left[\frac{7}{6}x^3\right]_{-0.2}^{0.7} \\
&= \left[20x\,\mathrm{e}^{-0.45x}\right]_{-0.2}^{0.7} - \left[20\frac{1}{-0.45}\,\mathrm{e}^{-0.45x}\right]_{-0.2}^{0.7} + \left[\frac{7}{6}x^3\right]_{-0.2}^{0.7} \\
&= \left[20x\,\mathrm{e}^{-0.45x} + \frac{400}{9}\,\mathrm{e}^{-0.45x} + \frac{7}{6}x^3\right]_{-0.2}^{0.7} \\
&= \left(20 \times 0.7 \times \mathrm{e}^{-0.45\times 0.7} + \frac{400}{9}\,\mathrm{e}^{-0.45\times 0.7} + \frac{7}{6}(0.7)^3\right) \\
&\qquad - \left(20 \times (-0.2) \times \mathrm{e}^{-0.45\times -0.2} + \frac{400}{9}\,\mathrm{e}^{-0.45\times -0.2} + \frac{7}{6}(-0.2)^3\right) \\
&= -1.19 \quad (2\text{ d.p.})
\end{aligned}$$