Railway Engineering Mathematics
Tutorial Sheet 16
Solutions

Evaluate the following integrals using integration by substitution:

1. $$\int \sqrt{1+2x}\,\mathrm{d}x$$

Solution:

First, re-write the integrand in index form:

$$\int (1+2x)^{\frac{1}{2}}\,\mathrm{d}x$$

We choose to replace the inner function with a new variable, so let $u = 1+2x$. Thus:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{2}\mathrm{d}u$$

Substitute both into the integral, so that we eliminate x and obtain a simpler integral solely in terms of our new variable u, which we can evaluate:

$$\begin{aligned}
\int (1+2x)^{\frac{1}{2}}\,\mathrm{d}x &= \int \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u \\
&= \frac{1}{2}\int u^{\frac{1}{2}}\,\mathrm{d}u \\
&= \frac{1}{2}\cdot\frac{1}{3/2}u^{\frac{3}{2}} + c
\end{aligned}$$

However, we must give our answer in terms of the original variable x:

$$\begin{aligned}
\int (1+2x)^{\frac{1}{2}}\,\mathrm{d}x &= \frac{1}{2}\cdot\frac{2}{3}(1+2x)^{\frac{3}{2}} + c \\
&= \frac{1}{3}\sqrt{(1+2x)^3} + c
\end{aligned}$$

2. $$\int \frac{4}{4x-1}\,\mathrm{d}x$$

Solution:

Typically with complicated fractions in the integrand, we substitute the denominator for a new variable. Thus, let $u = 4x - 1$.

Then:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 4 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{4}\mathrm{d}u$$

Substitute both of these into the integral:

$$\begin{aligned} \int \frac{4}{4x-1}\,\mathrm{d}x &= \int \frac{4}{u} \cdot \frac{1}{4}\,\mathrm{d}u \\ &= \int \frac{1}{u}\,\mathrm{d}u \end{aligned}$$

So we have obtained an integral in terms only of u, which we can evaluate:

$$\begin{aligned} \int \frac{4}{4x-1}\,\mathrm{d}x &= \ln(u) + c \\ &= \ln(4x-1) + c \end{aligned}$$

As before, we *must* ensure that we substitute the original variable back in, rather than giving our answer in terms of the variable u which we introduced in the course of the solution.

3. $$\int_0^3 \frac{10x-3}{5x^2-3x+2}\,\mathrm{d}x$$

Solution:

Replace the denominator with a new variable, so we let $u = 5x^2 - 3x + 2$.

Then differentiating u with respect to x and transposing for $\mathrm{d}x$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x - 3 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{10x-3}\mathrm{d}u$$

Substitute both into the integral:

$$\int_{x=0}^{x=3} \frac{10x-3}{u}\frac{1}{10x-3}\,\mathrm{d}u = \int_{x=0}^{x=3} \frac{1}{u}\,\mathrm{d}u$$

This is a definite integral, and the limits are in terms of x, so these must also be converted to obtain an integral solely in terms of new variable u:

$$\begin{aligned} &\text{Lower limit:} \quad x = 0 \implies u = 5(0)^2 - 3(0) + 2 = 2 \\ &\text{Upper limit:} \quad x = 3 \implies u = 5(3)^2 - 3(3) + 2 = 38 \end{aligned}$$

So replacing these and evaluating the resulting integral:

$$\begin{aligned} \int_{x=0}^{x=3} \frac{10x-3}{5x^2-3x+2}\,\mathrm{d}x &= \int_{u=2}^{u=38} \frac{1}{u}\,\mathrm{d}u \\ &= \big[\ln(u)\big]_2^{38} \\ &= \ln(38) - \ln(2) \\ &= 2.94 \quad \text{to 2 d.p.} \end{aligned}$$

4. $$\int_{-\pi}^{\pi/3} 2\sin\big(5t+7\big)\ \mathrm{d}t$$

Solution:

This is a composite function, and we replace the inner function (inside the sine) with a new variable. Let $u = 5t + 7$.

Thus:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 5 \quad \text{and so} \quad \mathrm{d}t = \frac{1}{5}\mathrm{d}u$$

As this is a definite integral, again we must *also* convert the limits to be in terms of u, so the new limits are:

$$t = -\pi \implies u = 5(-\pi) + 7 = -8.71$$
$$t = 2\pi \implies u = 5(2\pi) + 7 = 12.24$$

Substituting these and evaluating the integral:

$$\begin{aligned}
\int_{t=-\pi}^{t=\pi/3} 2\sin\big(5t+7\big)\ \mathrm{d}t &= \int_{u=-8.71}^{u=12.24} 2\sin(u)\frac{1}{5}\ \mathrm{d}u \\
&= \int_{-8.71}^{12.24} \frac{2}{5}\sin(u)\ \mathrm{d}u \\
&= \Big[-\frac{2}{5}\cos(u)\Big]_{-8.71}^{12.24} \\
&= \Big(-\frac{2}{5}\cos(12.24)\Big) - \Big(-\frac{2}{5}\cos(-8.71)\Big) \\
&= -0.68 \quad \text{(2 d.p.)}
\end{aligned}$$

We use radians by default when evaluating trigonometric functions, unless explicitly discussing angles in shapes.

5. $$\int (16x-7)(8x^2-7x+5)^3 \, \mathrm{d}x$$

Solution:

The integrand consists of the product of two functions, but the larger set of brackets raised to a power is a composite function with the contents of the brackets as the inner function. This is what we will choose to substitute.

Hence, let $u = 8x^2 - 7x + 5$.

Then differentiating u w.r.t. x and transposing for $\mathrm{d}x$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 16x - 7 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{16x-7}\mathrm{d}u$$

Substituting these to obtain an integral in u:

$$\begin{aligned}
\int (16x-7)(8x^2-7x+5)^3 \, \mathrm{d}x &= \int (16x-7)u^3 \frac{1}{16x-7} \, \mathrm{d}u \\
&= \int u^3 \, \mathrm{d}u \\
&= \frac{1}{4}u^4 + c \\
&= \frac{1}{4}(8x^2-7x+5)^4 + c
\end{aligned}$$

Note that this was possible because the smaller bracket was cancelled out by our substitution with $\mathrm{d}x$, resulting in an integral entirely in terms of u. If that had not been the case, we would have needed to try a different substitution or a different method altogether. The key reason why substitution works in this case, is because the smaller bracket is exactly a multiple of the derivative of the inner part of the larger brackets. In other words, this integrand is the result of differentiation by the chain rule, and so we can perfectly reverse that process!

6. $$\int (6t^2+8)(t^3+4t-7)^5 \, \mathrm{d}t$$

Solution:

Let $u = t^3 + 4t - 7$, as this is the inner part of the composite function of t.

Thus:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 3t^2 + 4 \quad \text{and so} \quad \mathrm{d}t = \frac{1}{3t^2+4}\mathrm{d}u$$

Substituting these to obtain an integral in terms only of u and evaluating it:

$$\begin{aligned}
\int (6t^2+8)(t^3+4t-7)^5 \, \mathrm{d}t &= \int (6t^2+8)u^5 \frac{1}{3t^2+4} \, \mathrm{d}u \\
&= \int 2(3t^2+4)u^5 \frac{1}{3t^2+4} \, \mathrm{d}u \\
&= 2\int u^5 \, \mathrm{d}u \\
&= 2 \cdot \frac{1}{6}u^6 + c \\
&= \frac{1}{3}(t^3+4t-7)^6 + c
\end{aligned}$$

7. $$\int_{-2}^{4} 6(7x-9)^3 \, \mathrm{d}x$$

Solution:

Choosing to substitute the inner function, we let $u = 7x - 9$.

Hence:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 7 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{7}\mathrm{d}u$$

As this is a definite integral, we also determine the limits in terms of u:

$$x = -2 \implies u = 7(-2) - 9 = -23$$
$$x = 4 \implies u = 7(4) - 9 = 19$$

Substitute all of these into the integral:

$$\begin{aligned}
\int_{x=-2}^{x=4} 6(7x-9)^3 \, \mathrm{d}x &= \int_{u=-23}^{u=19} 6u^3 \frac{1}{7} \, \mathrm{d}u \\
&= \frac{6}{7} \int_{-23}^{19} u^3 \, \mathrm{d}u \\
&= \frac{6}{7} \left[\frac{1}{4} u^4 \right]_{-23}^{19} \\
&= \frac{6}{7} \left\{ \left(\frac{1}{4}(19)^4 \right) - \left(\frac{1}{4}(-23)^4 \right) \right\} \\
&= -32040
\end{aligned}$$

8. $$\int_{1.7}^{3.9} t\,\mathrm{e}^{-0.6t^2}\ \mathrm{d}t$$

Solution:

Choose the index function as the inner function to substitute, so let $u = -0.6t^2$.

Differentiating with respect to t:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -1.2t \quad \text{and so} \quad \mathrm{d}t = \frac{1}{-1.2t}\mathrm{d}u = -\frac{5}{6t}\mathrm{d}u$$

And the new limits are:

$$t = 1.7 \implies u = -0.6 \times (1.7)^2 = -1.734$$
$$t = 3.9 \implies u = -0.6 \times (3.9)^2 = -9.126$$

Substituting these and evaluating the integral:

$$\begin{aligned}
\int_{t=1.7}^{t=3.9} t\,\mathrm{e}^{-0.6t^2}\ \mathrm{d}t &= \int_{u=-1.734}^{u=-9.126} t\,\mathrm{e}^{u}\left(-\frac{5}{6t}\right)\mathrm{d}u \\
&= -\frac{5}{6}\int_{-1.734}^{-9.126} \mathrm{e}^{u}\ \mathrm{d}u \\
&= -\frac{5}{6}\Big[\,\mathrm{e}^{u}\,\Big]_{-1.734}^{-9.126} \\
&= -\frac{5}{6}\Big\{\big(\,\mathrm{e}^{-9.126}\,\big) - \big(\,\mathrm{e}^{-1.734}\,\big)\Big\} \\
&= 0.147 \quad \text{(3 d.p.)}
\end{aligned}$$

9. $$\int (2x-5)^7 \, \mathrm{d}x$$

Solution:

Let $u = 2x - 5$.

Then differentiating u with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{2}\mathrm{d}u$$

Substitute both into the integral to obtain a simple integral in terms of u, and evaluating it:

$$\begin{aligned}
\int (2x-5)^7 \, \mathrm{d}x &= \int u^7 \cdot \frac{1}{2} \, \mathrm{d}u \\
&= \frac{1}{2} \int u^7 \, \mathrm{d}u \\
&= \frac{1}{2} \cdot \frac{1}{8} u^8 + c
\end{aligned}$$

Replacing u with $2x - 5$ again, to give our final answer in terms only of the original variable x:

$$\int (2x-5)^7 \, \mathrm{d}x = \frac{1}{16}(2x-5)^8 + c$$

10. $$\int \frac{4}{5x-3}\,\mathrm{d}x$$

Solution:

Choosing to make a substitution for the denominator of the fraction, we let $u = 5x - 3$.

Thus:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \quad \text{and so} \quad \mathrm{d}x = \frac{1}{5}\mathrm{d}u$$

Substituting these to obtain an integral solely in terms of u:

$$\begin{aligned}
\int \frac{4}{5x-3}\,\mathrm{d}x &= \int \frac{4}{u} \cdot \frac{1}{5}\,\mathrm{d}u \\
&= \frac{4}{5}\int \frac{1}{u}\,\mathrm{d}u \\
&= \frac{4}{5}\ln(u) + c \\
&= \frac{4}{5}\ln(5x-3) + c
\end{aligned}$$

11. $$\int_0^{\pi/6} 24\sin^5(\theta)\cos(\theta)\,\mathrm{d}\theta$$

Solution:

First, let's re-write this notation to be clearer:

$$\int_0^{\pi/6} 24\big(\sin(\theta)\big)^5\cos(\theta)\,\mathrm{d}\theta$$

So $\sin(\theta)$ is the inner function, and thus we let $u = \sin(\theta)$. Differentiating:

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \cos(\theta) \quad \text{and so} \quad \mathrm{d}\theta = \frac{1}{\cos(\theta)}\mathrm{d}u$$

The limits in terms of u are:

$$\text{Lower limit:} \quad \theta = 0 \implies u = \sin(0) = 0$$

$$\text{Upper limit:} \quad \theta = \frac{\pi}{6} \implies u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Substituting these into the integral and evaluating:

$$\begin{aligned}
\int_{\theta=0}^{\theta=\pi/6} 24\big(\sin(\theta)\big)^5\cos(\theta)\,\mathrm{d}\theta &= \int_{u=0}^{u=1/2} 24u^5\cos(\theta)\frac{1}{\cos(\theta)}\,\mathrm{d}u \\
&= 24\int_0^{1/2} u^5\,\mathrm{d}u \\
&= 24\left[\frac{1}{6}u^6\right]_0^{1/2} \\
&= 4\big[u^6\big]_0^{1/2} \\
&= 4\left\{\left(\left(\frac{1}{2}\right)^6\right) - \left(0^6\right)\right\} \\
&= 4\left(\frac{1}{2}\right)^6 \\
&= 0.0625
\end{aligned}$$

12. $$\int_0^1 3x\,\mathrm{e}^{2x^2-1}\ \mathrm{d}x$$

Solution:

Let $u = 2x^2 - 1$. Thus:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 4x \quad \text{and so} \quad \mathrm{d}x = \frac{1}{4x}\mathrm{d}u$$

Determining the limits of terms of this new variable u:

$$\text{Lower limit:} \quad x = 0 \implies u = 2(0)^2 - 1 = -1$$
$$\text{Upper limit:} \quad x = 1 \implies u = 2(1)^2 - 1 = 1$$

Substitute these into the integral and evaluate:

$$\begin{aligned}
\int_{x=0}^{x=1} 3x\,\mathrm{e}^{2x^2-1}\ \mathrm{d}x &= \int_{u=-1}^{u=1} 3x\,\mathrm{e}^{u}\,\frac{1}{4x}\ \mathrm{d}u \\
&= \frac{3}{4}\int_{-1}^{1} \mathrm{e}^{u}\ \mathrm{d}u \\
&= \frac{3}{4}\Big[\,\mathrm{e}^{u}\,\Big]_{-1}^{1} \\
&= \frac{3}{4}\Big\{\,\mathrm{e}^{1} - \mathrm{e}^{-1}\,\Big\} \\
&= 1.76 \quad (2\,\text{d.p.})
\end{aligned}$$