Introduction to Complex Numbers

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 19

Learning Outcomes

- Learn about the existence of j, the *imaginary number*.
- Recognise complex numbers.
- Add, subtract, multiply and divide pairs of complex numbers (in "Cartesian form").

Sets of Numbers

Numbers are understood to be organised in nested sets:

Set	Symbol	Examples
Natural numbers	$\mathbb{N}$	0, 1, 2, 3, ...
Integers (i.e. whole numbers)	$\mathbb{Z}$	$\mathbb{N}$ and -1, -2, -3, ...
Rational numbers	$\mathbb{Q}$	$\mathbb{Z}$ and all fractions
Real numbers	$\mathbb{R}$	$\mathbb{Q}$ and all irrationals, e.g. $\sqrt{2}$, π, e, etc

so

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Is $\mathbb{R}$ the biggest set that exists, or are there still more numbers?

Sets of Numbers

Quite often we are required to solve equations, such as

$$x - 7 = 0$$

Here the solution is 7, which is a member of $\mathbb{N}$.

We have also solved equations such as

$$x + 4 = 0,$$

but if do not accept the existence of any numbers other than the members of $\mathbb{N}$, then you cannot solve this equation.

If we define the solution as being $x = -4$, this idea leads to an entirely new set of numbers, the integers: $\mathbb{Z}$.

Sets of Numbers

Once we have accepted the set $\mathbb{Z}$, we can solve a wider variety of equations. For example:

$$x + 12 = 0$$

$$(x + 9)(x + 2) = 0,$$

etc.

Therefore we benefit from accepting the existence of negative numbers such as -1.

The Imaginary Unit

Up until this point, we assumed that there is no solution to the square root of a negative number. So, for example, there is no solution to the equation:

$$x^2 + 1 = 0$$

However, in a similar manner to the previous slides, if we can accept that another number set exists outside of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, then we can solve this equation.

Thus mathematicians define the solution to be the imaginary number j, so that:

$$j^2 + 1 = 0$$

The Imaginary Unit

Let's examine the properties of our newly defined number.

If $j^2 + 1 = 0$ then:

Definition of the imaginary unit:

$$j^2 = -1$$

and

$$j = \sqrt{-1}$$

We call j the imaginary unit, as it is the unit of the imaginary numbers in the same way that 1 is the unit of the real numbers.

Imaginary Numbers

Hence, we can also express the square roots of the other negative numbers in terms of this unit j, for example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{-1 \times 4} \\ &= \sqrt{-1} \times \sqrt{4} \\ &= j2\end{aligned}$$

and

$$\begin{aligned}\sqrt{-25} &= \sqrt{-1 \times 25} \\ &= \sqrt{-1} \times \sqrt{25} \\ &= j5\end{aligned}$$

Complex Numbers

Any multiples of j, such as j, $j6$, $j0.3$, $j\dfrac{7}{3}$ and $j2\sqrt{3}$, are all imaginary numbers.

When we combine these with real numbers through addition/subtraction, we create **complex numbers**, e.g.

$$6 - j5 \quad \text{and} \quad -3 + j$$

Note: Mathematicians (and most software) use i, while engineers use j. Thus, an engineer would write $3 - j5$, while a mathematician would write $3 - 5i$.

Uses of Complex Numbers

Despite their apparent lack of physical meaning, complex numbers are essential for solving some real-world problems.

- Complex numbers can be used to represent certain engineering quantities, particularly in electronics:
  - AC currents and voltages
  - Impedances in AC circuits
- They are also used in:
  - The solution of differential equations
  - Aerodynamics (potential flow in two dimensions)
  - Control theory (stability analysis)
  - Signal processing (spectral analysis)

Complex Numbers

The general (Cartesian/rectangular) form of a complex number is:

Standard Cartesian form of a complex number:

$$z = x + jy$$

where x and y are real numbers ($\mathbb{R}$) and $j = \sqrt{-1}$.

We say that the **real part** of z is x:

$$\text{Re}(z) = x$$

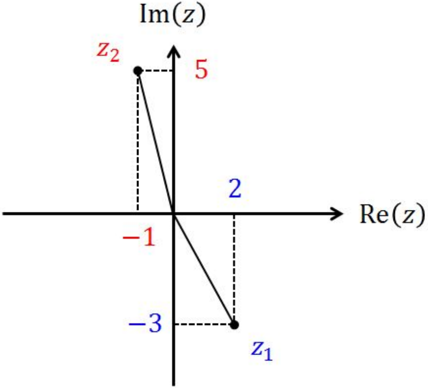
and the **imaginary part** of z is y:

$$\text{Im}(z) = y \qquad (\text{NOT } jy)$$

Complex Numbers - Cartesian form

The Cartesian form tells us how we can represent complex numbers on a two-dimensional plane (called an Argand diagram), by thinking of the real part as the x-coordinate and the imaginary part as the y-coordinate.

Plotting $z_1 = 2 - j3$ and $z_2 = -1 + j5$:



Complex Number Arithmetic: Addition/Subtraction

To add/subtract complex numbers we simply add/subtract the corresponding real and imaginary parts separately.

If $z_1 = 2 - j3$, $\quad z_2 = 6 - j2 \quad$ and $\quad z_3 = -7 + j5$,

Calculate:

1) $\quad z_1 + z_2$
2) $\quad z_2 - z_1$
3) $\quad z_2 - z_3$
4) $\quad z_1 + z_3$

Complex Number Arithmetic: Addition/Subtraction

Solutions:

1)

$$\begin{aligned} z_1 + z_2 &= (2 - j3) + (6 - j2) \\ &= \big(2 + 6\big) + \big(-3 + (-2)\big)j \\ &= 8 - j5 \end{aligned}$$

2)

$$\begin{aligned} z_2 - z_1 &= (6 - j2) - (2 - j3) \\ &= 6 - j2 - 2 + j3 \\ &= (6 - 2) + (-j2 + j3) \\ &= 4 + j \end{aligned}$$

Complex Number Arithmetic: Addition/Subtraction

Solutions:

3)

$$\begin{aligned} z_2 - z_3 &= (6 - j2) - (-7 + j5) \\ &= \big(6 - (-7)\big) + \big(- 2 - 5\big) j \\ &= 13 - j7 \end{aligned}$$

4)

$$\begin{aligned} z_1 + z_3 &= (2 - 3j) + (-7 + j5) \\ &= \big(2 + (-7)\big) + \big((-3) + 6\big) j \\ &= -5 + j2 \end{aligned}$$

Complex Number Arithmetic: Multiplication

To multiply complex numbers we simply expand the brackets; treating j just like any other constant.

Calculate:

1) $z_1 z_2$

2) $z_3 z_2$

Complex Number Arithmetic: Multiplication

Solutions:

1)

$$\begin{aligned} z_1 z_2 &= (2 - j3)(6 - j2) \\ &= 12 - j4 - j18 + j^2 6 \quad \text{(remember } j^2 = -1\text{)} \\ &= 12 - j22 + (-1)(6) \\ &= 6 - j22 \end{aligned}$$

2)

$$\begin{aligned} z_3 z_2 &= (-7 + j5)(6 - j2) \\ &= -42 + j14 + j30 - j^2 10 \\ &= -42 + j44 - (-1)(10) \\ &= -32 + j44 \end{aligned}$$

Complex Number Arithmetic: Division

To divide complex numbers, we first make the denominator *real*.

This can be achieved by multiplying both the numerator and denominator by the **complex conjugate** of the denominator.

Definition of complex conjugates:

For a complex number $z = x + jy$, the complex conjugate of z is:

$$\overline{z} = x - jy$$

Furthermore it can be proven that

$$z\overline{z} = x^2 + y^2$$

a result which is purely real and has no imaginary part.

Complex Number Arithmetic: Division

Calculate:

1) $$\frac{z_3}{z_2}$$

2) $$\frac{z_1}{z_2}$$

Complex Number Arithmetic: Division

Solutions: 1)

$$
\begin{aligned}
\frac{z_3}{z_2} &= \frac{-7+j5}{6-j2} = \frac{(-7+j5)(6+j2)}{(6-j2)(6+j2)} \\
&= \frac{-42-j14+j30+j^2 10}{36+j12-j12-j^2 4} \\
&= \frac{-42+j16-10}{36+4} \\
&= \frac{-52+j16}{40} = \frac{-13+j4}{10} \\
&= \frac{-13}{10} + j\frac{2}{5} = -1.3+j0.4
\end{aligned}
$$

Complex Number Arithmetic: Division

Solutions: 2)

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{2-j3}{6-j2} = \frac{(2-j3)(6+j2)}{(6-j2)(6+j2)} \\
&= \frac{12+j4-j18-j^2 6}{36+j12-j12-j^2 4} \\
&= \frac{12-j14+6}{36+4} \\
&= \frac{18-j14}{40} = \frac{9-j7}{20} \\
&= \frac{9}{20} - j\frac{7}{20} = 0.45 - j0.35
\end{aligned}
$$