Railway Engineering Mathematics

Tutorial Sheet 22

Solutions

Given the following matrices:

$$\underline{A} = \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 6 & 1 & 9 \\ -2 & 0 & -8 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \qquad \underline{F} = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$$

1. Determine if the following operations are possible, and if so then evaluate them:

   (a) $|\underline{A}|$

   (b) $|\underline{C}|$

   (c) $\underline{D}^{-1}$

   (d) $\underline{B}^{-1}$

   (e) $\underline{F}^{-1}$

   (f) $\underline{A}^T$

   (g) $\underline{B}^T$

Solution:

(a) $\underline{A}$ is a square matrix, so it does have a determinant, which we can calculate. Using the formula for the determinant of a 2×2 matrix, whereby we calculate the difference between the product of the diagonal entries:

$$\begin{aligned} |\underline{A}| &= \begin{vmatrix} 7 & 2 \\ -3 & 8 \end{vmatrix} \\ &= (7 \times 8) - (2 \times -3) \\ &= 56 - (-6) \\ &= 62 \end{aligned}$$

Note that this may also be denoted as $\det(\underline{A})$.

(b) The determinant is *only* defined for square matrices. As $\underline{C}$ has order 2×3, it is not square and so $|\underline{C}|$ can not be calculated.

(c) A matrix has an inverse if and only if it is a square matrix with non-zero determinant. This is satisfied by $\underline{D}$, as it has order 2×2 and the determinant is:

$$
\begin{aligned}
|\underline{D}| &= \begin{vmatrix} -5 & -4 \\ 1 & 6 \end{vmatrix} \\
&= (-5 \times 6) - (-4 \times 1) \\
&= -26 \neq 0
\end{aligned}
$$

Thus, the determinant is non-zero and we can proceed with applying the formula for the inverse of $\underline{D}$:

$$
\begin{aligned}
\underline{D}^{-1} &= \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix}^{-1} \\
&= \frac{1}{|\underline{D}|} \begin{pmatrix} 6 & -(-4) \\ -(1) & -5 \end{pmatrix} \\
&= \frac{1}{(-5 \times 6) - (-4 \times 1)} \begin{pmatrix} 6 & 4 \\ -1 & -5 \end{pmatrix} \\
&= -\frac{1}{26} \begin{pmatrix} 6 & 4 \\ -1 & -5 \end{pmatrix}
\end{aligned}
$$

And we can finally simplify this, by evaluating the scalar multiplication, to:

$$
\underline{D}^{-1} = \begin{pmatrix} -3/13 & -2/13 \\ 1/26 & 5/26 \end{pmatrix}
$$

We can verify our solution by multiplying the inverse by the original matrix. By definition, $\underline{D}\underline{D}^{-1} = \underline{I}$ and $\underline{D}^{-1}\underline{D} = \underline{I}$, where $\underline{I}$ is the identity matrix. For example:

$$\begin{aligned}
\underline{D}\underline{D}^{-1} &= \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -3/13 & -2/13 \\ 1/26 & 5/26 \end{pmatrix} \\
&= \begin{pmatrix} (-5)(-3/13) + (-4)(1/26) & (-5)(-2/13) + (-4)(5/26) \\ (1)(-3/13) + (6)(1/26) & (1)(-2/13) + (6)(5/26) \end{pmatrix} \\
&= \begin{pmatrix} 15/13 - 4/26 & 10/13 - 20/26 \\ -3/13 + 6/26 & -2/13 + 30/26 \end{pmatrix} \\
&= \begin{pmatrix} 30/26 - 4/26 & 20/26 - 20/26 \\ -6/26 + 6/26 & -4/26 + 30/26 \end{pmatrix} \\
&= \begin{pmatrix} 26/26 & 0 \\ 0 & -26/26 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \underline{I}
\end{aligned}$$

As required! And we would obtain the same result if we similarly checked the value of $\underline{D}^{-1}\underline{D}$.

(d) As $\underline{B}$ has order 2×3, it is not square and so does not meet the prerequisites of invertibility. Thus, $\underline{B}^{-1}$ is not defined.

(e) $\underline{F}$ is a square (2×2) matrix. However:

$$\begin{aligned}
|\underline{F}| &= \begin{vmatrix} 1 & -3 \\ -3 & 9 \end{vmatrix} \\
&= (1 \times 9) - (-3 \times -3) \\
&= 9 - 9 \\
&= 0
\end{aligned}$$

So it has determinant zero. This means that $\underline{F}$ is a "singular" matrix, and it has no inverse. Thus, $\underline{F}^{-1}$ is not defined.

(f) To obtain the transpose, swap the rows and columns (or equivalently, reflect all elements of the matrix across the lead diagonal):

$$\begin{aligned}
\underline{A}^T &= \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix}^T \\
&= \begin{pmatrix} 7 & -3 \\ 2 & 8 \end{pmatrix}
\end{aligned}$$

Note that elements that lie on the lead diagonal itself (7 and 8 in this case), do not change position.

(g) In this case, as $\underline{B}$ has order 2×3, its transpose will have order 3×2:

$$\begin{aligned}
\underline{B}^T &= \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix}^T \\
&= \begin{pmatrix} 4 & 5 \\ -1 & -3 \\ 7 & -2 \end{pmatrix}
\end{aligned}$$

2. Solve the following systems of linear simultaneous equations for x and y using a matrix method:

(a)
$$5x + 3y = -23$$
$$-4x - 9y = 58$$

Solution:

Re-writing this as a matrix equation:

$$\begin{pmatrix} 5 & 3 \\ -4 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -23 \\ 58 \end{pmatrix}$$

so we have $\underline{A}\underline{X} = \underline{B}$, where:

$$\underline{A} = \begin{pmatrix} 5 & 3 \\ -4 & -9 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} -23 \\ 58 \end{pmatrix}$$

Then the matrix A of coefficients is a 2×2 square matrix, so we can determine its inverse (assuming that the determinant does not turn out to be zero):

$$\begin{aligned} \underline{A}^{-1} &= \begin{pmatrix} 5 & 3 \\ -4 & -9 \end{pmatrix}^{-1} \\ &= \frac{1}{(5)(-9) - (3)(-4)} \begin{pmatrix} -9 & -3 \\ 4 & 5 \end{pmatrix} \\ &= \frac{-1}{33} \begin{pmatrix} -9 & -3 \\ 4 & 5 \end{pmatrix} \end{aligned}$$

The determinant of the matrix is $-33 \neq 0$ so the inverse of $\underline{A}$ is well-defined (i.e. meaningful) and there exists a unique solution to the system of linear equations.

We could evaluate the scalar multiplication, but it will actually be easier to keep the factor of $-1/33$ outside of the matrix for now, and only "bring it in" at the very end of the following matrix multiplication.

Then:

$$\begin{aligned}
\underline{X} &= \underline{A}^{-1}\underline{B} \\
&= \frac{-1}{33}\begin{pmatrix} -9 & -3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} -23 \\ 58 \end{pmatrix} \\
&= \frac{-1}{33}\begin{pmatrix} (-9 \times -23) + (-3 \times 58) \\ (4 \times -23) + (5 \times 58) \end{pmatrix} \\
&= \frac{-1}{33}\begin{pmatrix} 33 \\ 198 \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{1}{33} \times 33 \\ -\dfrac{1}{33} \times 198 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ -6 \end{pmatrix}
\end{aligned}$$

Thus we find:

$$x = -1 \quad \text{and} \quad y = -6$$

Finally, we can verify our solutions by substituting them back into the original equations:

$$5(-1) + 3(-6) = -5 - 18 = -23$$

and

$$-4(-1) - 9(-6) = 4 + 54 = 58$$

as required.

(b) $$x_2 = 7.1 + 7x_1$$

$$2x_1 + 3x_2 = 28.2$$

Solution:

First, we note that the two equations are not in the exact same format of [amount of first variable] + [amount of second variable] = [constant] so we will need to begin by re-writing them in this consistent format, and we choose to put x_1 first in each case:

$$-7x_1 + x_2 = 7.1$$

$$2x_1 + 3x_2 = 28.2$$

Then interpreting this system as a single matrix equation:

$$\begin{pmatrix} -7 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7.1 \\ 28.2 \end{pmatrix}$$

so we have $\underline{A}\underline{X} = \underline{B}$, where:

$$\underline{A} = \begin{pmatrix} -7 & 1 \\ 2 & 3 \end{pmatrix}, \quad \underline{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} 7.1 \\ 28.2 \end{pmatrix}$$

Then calculating the inverse of the matrix $\underline{A}$ of coefficients (and checking that it has non-zero determinant, or else we would need to stop):

$$\begin{aligned} \underline{A}^{-1} &= \frac{1}{(-7)(3) - (1)(2)} \begin{pmatrix} 3 & -1 \\ -2 & -7 \end{pmatrix} \\ &= \frac{-1}{23} \begin{pmatrix} 3 & -1 \\ -2 & -7 \end{pmatrix} \end{aligned}$$

The determinant of matrix $\underline{A}$ is -23 which is non-zero, so the matrix of coefficients is invertible and this system will have a unique solution for x_1 and x_2 that we can proceed to find.

So we can calculate the vector $\underline{X}$ containing the variables we wish to solve for:

$$
\begin{aligned}
\underline{X} &= \underline{A}^{-1}\underline{B} \\
&= \frac{-1}{23}\begin{pmatrix} 3 & -1 \\ -2 & -7 \end{pmatrix}\begin{pmatrix} 7.1 \\ 28.2 \end{pmatrix} \\
&= \frac{-1}{23}\begin{pmatrix} (3 \times 7.1) + (-1 \times 28.2) \\ (-2 \times 7.1) + (-7 \times 28.2) \end{pmatrix} \\
&= \frac{-1}{23}\begin{pmatrix} -6.9 \\ -211.6 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{23} \times -6.9 \\ -\frac{1}{23} \times -211.6 \end{pmatrix} \\
&= \begin{pmatrix} 0.3 \\ 9.2 \end{pmatrix}
\end{aligned}
$$

Thus we obtain:

$$x_1 = 0.3 \quad \text{and} \quad x_2 = 9.2$$

Finally, we can check our solutions by substituting them back into the original equations:

$$-7(0.3) + (9.2) = -2.1 + 9.2 = 7.1$$

and

$$2(0.3) + 3(9.2) = 0.6 + 27.6 = 28.2$$

as required.

3. The *UltraEmissions* car manufacturer has two models currently in production: the 1-litre *Despondent* front-engine front-wheel drive 6-seater people carrier[1], and the ever-popular *Merciless* rear-wheel drive V8 roadster. Two shipping containers containing cars from this brand are delivered to the port, where you have been employed by HM Revenue and Customs to assist the port authorities in issuing import duty.According to the manifest, one of the containers holds five of the *Despondent* model and eight of the *Merciless*, while the second container is storing ten *Despondent* and six of the *Merciless* model. The first container weighs 35 tonnes, and the second weighs 52 tonnes. The shipping company informs you that an empty shipping container weighs exactly three tonnes.

   The table below lists the import duty per vehicle, dependent on the weight class of that vehicle. How much is the total duty to be paid by the *UltraEmissions* company for this particular delivery?

Weight (tonnes)	Duty (£/tonne)
0-1	200
1.01-2	220
2.01-3	250
3.01-4	300
4.01-5	340
5.01-6	400

   Table 1: Import duty on vehicle weight classes

   Solution:

   Because the tax rate on each car is dependent on the weight of the individual car, we can't just issue tax based on the total weight of the two containers. We will indeed need to determine the weight of each model of car, and then calculate the tax for each one and sum them up.

   Let's begin by assigning variable names to the unknown quantities in this problem, which are the masses of the individual models of car. Let D be the mass of a single *Despondent* car, and let M be the mass of a single *Merciless* model.

   Then we can form an equation for the mass of each of the two containers. The first container has five *Despondent*, eight*Merciless*, and together with the three tonne

[1]Only available in beige.

mass of the container the combined weight should be 35 tonnes. Hence:

$$5D + 8M + 3 = 35$$

Similarly, consider the constituents of the mass of the second container, which also weights three tonnes on its own and contains ten *Despondent* and six *Merciless*:

$$10D + 6M + 3 = 52$$

Then if we simplify each of these equations:

$$5D + 8M = 32$$

$$10D + 6M = 49$$

Now we can see that this problem is represented as a pair of simultaneous linear equations, and we can solve them using the matrix method. Let:

$$\underline{A} = \begin{pmatrix} 5 & 8 \\ 10 & 6 \end{pmatrix}, \quad \text{and} \quad \underline{X} = \begin{pmatrix} D \\ M \end{pmatrix}, \quad \text{and} \quad \underline{B} = \begin{pmatrix} 32 \\ 49 \end{pmatrix}$$

Then representing this system as a matrix equation: $\underline{AX} = \underline{B}$.

A is a 2×2 square matrix, and we calculate its determinant:

$$\det(A) = \begin{vmatrix} 5 & 8 \\ 10 & 6 \end{vmatrix} = (5)(6) - (8)(10) = 30 - 80 = -50$$

This is not equal to zero, so A is invertible, and we find:

$$\begin{aligned} \underline{A}^{-1} &= \begin{pmatrix} 5 & 8 \\ 10 & 6 \end{pmatrix}^{-1} \\ &= \frac{1}{-50} \begin{pmatrix} 6 & -8 \\ -10 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -3/25 & 4/25 \\ 1/5 & -1/10 \end{pmatrix} \end{aligned}$$

Then we calculate the vector of solutions:

$$\begin{aligned}
\underline{X} &= \underline{A}^{-1}\underline{B} \\
&= \begin{pmatrix} -3/25 & 4/25 \\ 1/5 & -1/10 \end{pmatrix} \begin{pmatrix} 32 \\ 49 \end{pmatrix} \\
&= \begin{pmatrix} (-3/25)(32) + (4/25)(49) \\ (1/5)(32) + (-1/10)(49) \end{pmatrix} \\
&= \begin{pmatrix} -96/25 + 196/25 \\ 32/5 - 49/10 \end{pmatrix} \\
&= \begin{pmatrix} 100/25 \\ 64/10 - 49/10 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 15/10 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 1.5 \end{pmatrix}
\end{aligned}$$

And so $D = 4$ and $M = 1.5$. That is, one *Despondent* weighs 4 tonnes. According to the duty table, it is therefore taxed at a rate of £300/tonne, and so there is a total of £1200 of import tax to be paid on each *Despondent*. On the other hand, each *Merciless* weighs 1.5 tonnes, and is therefore in the second tax bracket of £220/tonne, with a total tax of £330 to be paid on each one.

In total, 15 *Despondent* and 14 *Merciless* were present in this shipment, so we calculate a total duty of:

$$15(1200) + 14(330) = 22620$$

So £22,620 is to be paid.