Railway Engineering Mathematics
Tutorial Sheet 20

1. Given the following complex numbers in Cartesian form:

$$z_1 = 7 - j3, \qquad z_2 = -1 - j4, \qquad z_3 = -5 + j, \qquad z_4 = 9 + j6$$

(i) Express in polar form:

(a) z_1

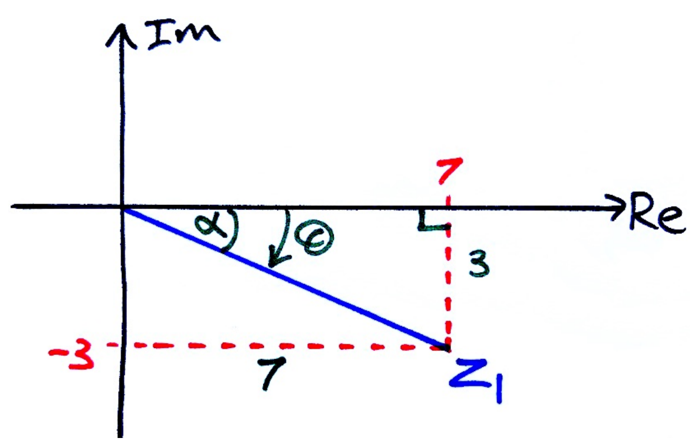
(b) z_2

(c) z_3

(d) z_4

Solution:

(a) First, sketch z_1 to see which quadrant it lies in.



To write in polar form, we need to determine the radius and the argument of z_1.

Starting with radius r, which we obtain using Pythagoras' theorem:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{7^2 + (-3)^2} \\ &= \sqrt{49 + 9} \\ &= 7.62 \quad \text{to 2 d.p.} \end{aligned}$$

The angle marked α can be found using trigonometry:

$$\begin{aligned} \tan(\alpha) &= \frac{y}{x} \\ &= \frac{-3}{7} \end{aligned}$$

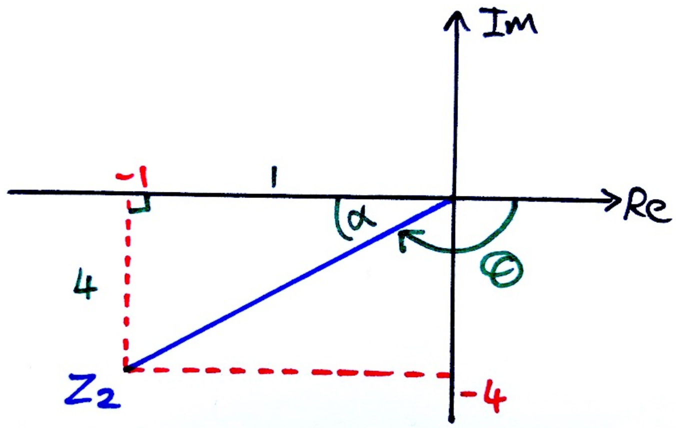
And taking the inverse tan function (with our calculator set to radians):

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{-3}{7}\right) \\ &= -0.40 \end{aligned}$$

Thus, we can write the polar form as:

$$\begin{aligned} z_1 &= 7.62\angle - 0.40 \\ &= 7.62\big(\cos(-0.40) + j\sin(-0.40)\big) \end{aligned}$$

(b) First, let's sketch the Argand diagram of z_2:



To write in polar form, we need to determine the radius r using Pythagoras' theorem:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-1)^2+(-4)^2} \\ &= \sqrt{1+16} \\ &= 4.12 \quad \text{to 2 d.p.} \end{aligned}$$

Calculating the angle marked α by applying $SOH-CAH-TOA$ to the right-angled triangle with sides of length 4 and 1:

$$\begin{aligned} \tan(\alpha) &= \frac{y}{x} \\ &= \frac{-4}{-1} \\ \therefore\ \alpha &= \tan^{-1}(4) \\ &= 1.3258 \end{aligned}$$

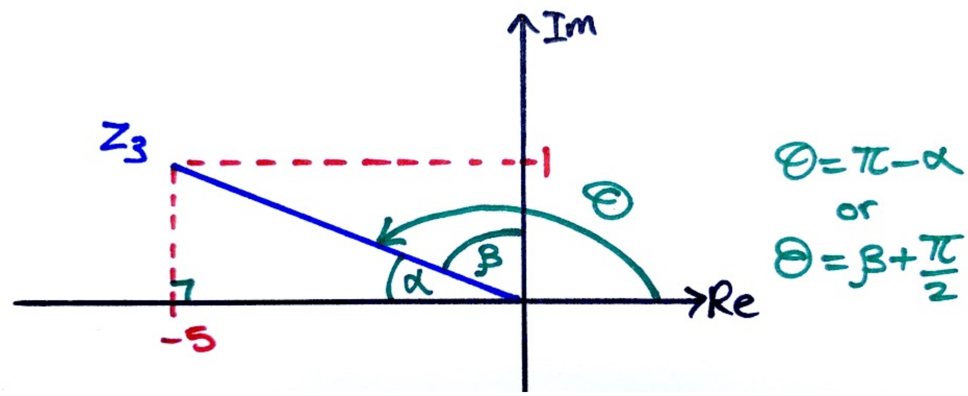
Now, to find the argument, the angle from the positive x-axis to z_2 is going to be the difference $\alpha - \pi$. As this is measured clockwise, rather than anticlockwise, the argument is:

$$\begin{aligned} \theta &= \alpha - \pi \\ &= 1.3258 - \pi \\ &= -1.82 \quad \text{rad} \end{aligned}$$

So in polar form:

$$\begin{aligned} z_2 &= 4.12\angle - 1.82 \\ &= 4.12\big(\cos(-1.82) + j\sin(-1.82)\big) \end{aligned}$$

(c) Sketch the Argand diagram of z_3:



The modulus is:

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
&= \sqrt{(-5)^2+1^2} \\
&= \sqrt{36} \\
&= 5.10
\end{aligned}
$$

To find the argument, we first determine the angle angle α using the sides of the right-angled triangle:

$$
\begin{aligned}
\tan(\alpha) &= \frac{y}{x} \\
&= \frac{1}{5} \\
\therefore\ \alpha &= \tan^{-1}\left(\frac{1}{-5}\right) \\
&= -0.1974
\end{aligned}
$$

Then the argument is obtained by adding π to the value of α (which is the angle corresponding to one whole half-rotation to the negative x-axis):

$$\begin{aligned}\theta &= \alpha + \pi \\ &= -0.1974 + \pi \\ &= 2.94 \text{ rad} \quad \text{to 2 d.p.}\end{aligned}$$

Note: as is often the case, we could have calculated this in another way. If we instead found the angle β (see the Argand diagram), then the argument could be found by adding $\pi/2$ radians, corresponding to the quarter rotation from the positive x-axis to the positive y-axis. Using this method,

$$\theta = \beta + \frac{\pi}{2}$$

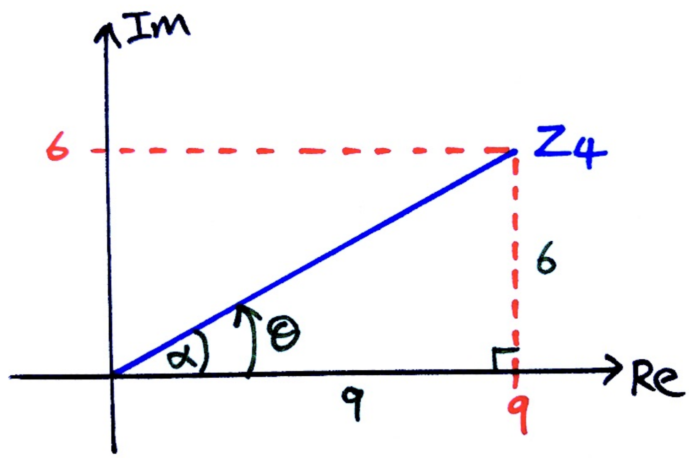
where

$$\beta = \tan^{-1}\left(\frac{5}{1}\right)$$

Regardless of how the argument is obtained, finally we state z_3 in polar form:

$$\begin{aligned}z_3 &= 5.10\angle 2.94 \\ &= 5.10\big(\cos(2.94) + j\sin(2.94)\big)\end{aligned}$$

(d) First, sketch the Argand diagram of z_4:



The modulus r is given by:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{9^2 + 6^2} \\
&= 10.82
\end{aligned}
$$

In this case, z_4 lies in the first quadrant, so we can calculate the argument directly (this is the simplest case): The argument θ is identical to the angle α indicated.

$$
\begin{aligned}
\theta &= \alpha \\
&= \tan^{-1}\left(\frac{y}{x}\right) \\
&= \tan^{-1}\left(\frac{6}{9}\right) \\
&= 0.59 \text{ rad}
\end{aligned}
$$

So in polar form:

$$
\begin{aligned}
z_4 &= 10.82\angle 0.59 \\
&= 10.82\big(\cos(0.59) + j\sin(0.59)\big)
\end{aligned}
$$

(ii) Calculate the following in polar form:

(a) $z_3 z_1$

(b) $z_2 z_4$

(c) $\dfrac{z_3}{z_1}$

(d) $\dfrac{z_4}{z_2}$

(e) $\dfrac{z_1}{z_3}$

(f) $z_3 z_2$

(g) $\dfrac{1}{z_2}$

(h) $\dfrac{1}{z_4}$

Solution:

To multiply complex numbers in polar form, we simply multiply their moduli and add their arguments - this is an advantage over Cartesian form (although it is much easier to add/subtract complex numbers when they are expressed in Cartesian form compared to polar form).

(a)
$$\begin{aligned} z_3 z_1 &= (5.10\angle 2.94) \times (7.62\angle -0.40) \\ &= (5.10 \times 7.62)\angle(2.94 + -0.40) \\ &= 38.86\angle 2.54 \end{aligned}$$

(b)
$$\begin{aligned} z_2 z_4 &= (4.12\angle -1.82) \times (10.82\angle 0.59) \\ &= (4.12 \times 10.82)\angle(-1.82 + 0.59) \\ &= 44.58\angle -1.23 \end{aligned}$$

(c) Dividing complex numbers is also much easier when they are written in polar form: we simply divide their moduli and subtract their arguments.

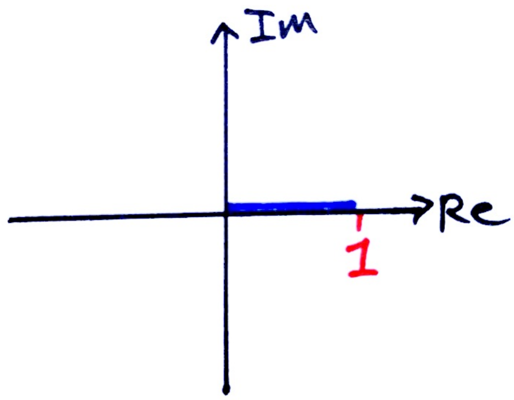
$$\begin{aligned}\frac{z_3}{z_1} &= \frac{5.10\angle 2.94}{7.62\angle -0.40}\\ &= \frac{5.10}{7.62}\angle\big(2.94-(-0.40)\big)\\ &= 0.67\angle 3.34\end{aligned}$$

(d)
$$\begin{aligned}\frac{z_4}{z_2} &= \frac{10.82\angle 0.59}{4.12\angle -1.82}\\ &= \frac{10.82}{4.12}\angle\big(0.59-(-1.82)\big)\\ &= 2.63\angle 2.41\end{aligned}$$

(e)
$$\begin{aligned}\frac{z_1}{z_3} &= \frac{7.62\angle -0.40}{5.10\angle 2.94}\\ &= \frac{7.62}{5.10}\angle\big(-0.40-2.94\big)\\ &= 1.49\angle -3.34\end{aligned}$$

(f)
$$\begin{aligned}z_3z_2 &= \big(5.10\angle 2.94\big)\times\big(4.12\angle -1.82\big)\\ &= 5.10\times 4.12\angle\big(2.94+(-1.82)\big)\\ &= 21.01\angle 1.12\end{aligned}$$

(g) First, note that the real numbers are a subset of the complex numbers, which means that we can consider any real number as being complex but with an imaginary part equal to zero. Thus, by drawing the Argand diagram, we can write 1 as a complex number in polar form, and treat this problem as a question of dividing one complex number by another.



So as 1 lies on the positive real axis, it has argument zero, and the modulus (the straight line distance from the origin) is just equal to 1. Therefore:

$$1 = 1 + j0 = 1\angle 0$$

and so

$$\begin{aligned}\frac{1}{z_2} &= \frac{1\angle 0}{4.12\angle - 1.82} \\ &= \frac{1}{4.12}\angle(0 - (-1.82)) \\ &= 0.24\angle 1.82\end{aligned}$$

(h) Similarly to the previous question:

$$\begin{aligned}\frac{1}{z_4} &= \frac{1\angle 0}{10.82\angle 0.59} \\ &= \frac{1}{10.82}\angle(0 - 0.59) \\ &= 0.09\angle - 0.59\end{aligned}$$

2. Given the following complex numbers in Polar form:

$$z_5 = 2.5\angle -2.9, \qquad z_6 = 4.1\angle -5.1, \qquad z_7 = 0.3\angle 1.7, \qquad z_8 = 7.9\angle 6.1$$

(i) Express in rectangular/Cartesian form:

(a) z_5

(b) z_6

(c) z_7

(d) z_8

Solution:

Given a complex number z expressed in polar form: $z = r\angle\theta$, we can convert it to Cartesian form $z = x + jy$ by calculating the real part x and imaginary part y according to:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta)$$

(a)
$$\begin{aligned} z_5 &= r\cos(\theta) + jr\sin(\theta) \\ &= 2.6\cos(-2.9) + j(2.5)\sin(-2.9) \\ &= -2.43 - j0.60 \end{aligned}$$

(b)
$$\begin{aligned} z_6 &= r\cos(\theta) + jr\sin(\theta) \\ &= 4.1\cos(-5.1) + j(4.1)\sin(-5.1) \\ &= 1.55 + j3.8 \end{aligned}$$

(c)
$$\begin{aligned} z_7 &= r\cos(\theta) + jr\sin(\theta) \\ &= 0.3\cos(1.7) + j(0.3)\sin(1.7) \\ &= -0.04 + j0.3 \end{aligned}$$

(d)
$$\begin{aligned} z_8 &= r\cos(\theta) + jr\sin(\theta) \\ &= 7.9\cos(6.1) + j(7.9)\sin(6.1) \\ &= 7.77 - j1.44 \end{aligned}$$

(ii) Calculate the following in polar form, and then convert the result to Cartesian form:

(a) $z_5 z_6$

(b) $z_7 z_8$

(c) $\dfrac{z_6}{z_8}$

(d) $\dfrac{z_8}{z_5}$

(e) $\dfrac{1}{z_7}$

(f) $\dfrac{1}{z_5}$

Solution:

(a) Undertaking the calculation in polar form, as instructed, the resulting modulus is the product of the constituent moduli and the arguments are added together:

$$\begin{aligned} z_5 z_6 &= (2.5\angle -2.9) \times (4.1\angle -5.1) \\ &= 2.5 \times 4.1\angle\big(-2.9 + (-5.1)\big) \\ &= 10.25\angle -8.0 \end{aligned}$$

Then we convert to Cartesian form by calculating the real part:

$$\begin{aligned} \mathrm{Re}\big(z_5 z_6\big) &= r\cos(\theta) \\ &= 10.25\cos(-8.0) \\ &= -1.49 \end{aligned}$$

And the imaginary part:

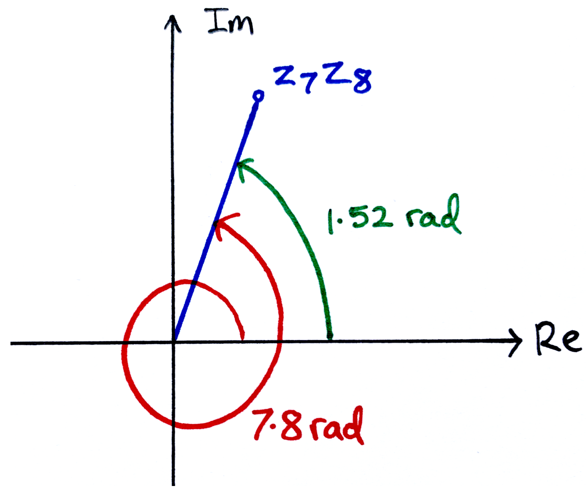
$$\begin{aligned} \mathrm{Im}\big(z_5 z_6\big) &= r\sin(\theta) \\ &= 10.25\sin(-8.0) \\ &= -10.14 \end{aligned}$$

Finally, combining them to express $z_5 z_6$ in Cartesian form:

$$\begin{aligned} z_5 z_6 &= x + jy \\ &= -1.49 - j10.14 \end{aligned}$$

(b)
$$\begin{aligned} z_7 z_8 &= (0.3\angle 1.7) \times (7.9\angle 6.1) \\ &= 0.3 \times 7.9\angle(1.7 + 6.1) \\ &= 2.37\angle 7.8 \end{aligned}$$

A single full rotation consists of $2\pi \approx 6.28$ radians, so an argument of 7.8 is *more* than one whole rotation.



As it is best practice to use a value of the argument in the range $-\pi < \theta \leq \pi$, it would be better to therefore state the argument as:

$$7.8 - 2\pi = 1.52 \text{ rad}$$

Converting to Cartesian form:

$$\begin{aligned} z_7 z_8 &= 2.37\cos(1.52) + j(2.37)\sin(1.52) \\ &= 0.13 + j2.37 \end{aligned}$$

Note: we would obtain the same answer if we had stuck with using the original expression of the argument as 7.8 radians, due to the periodicity of cosine:

$$\cos(1.52) = \cos(1.52 + 2\pi) = \cos(7.80)$$

and similarly for sine.

(c)

$$\begin{aligned}
\frac{z_6}{z_8} &= \frac{4.1\angle - 5.1}{7.9\angle 6.1} \\
&= \frac{4.1}{7.9}\angle(-5.1 - 6.1) \\
&= 0.52\angle - 11.2
\end{aligned}$$

Again, this expression of the argument as -11.2 radians is greater than one whole revolution, so we could add 2π to give the argument as -4.92. Even better, we could add $2 \times 2\pi = 4\pi$ to state the argument as 1.37.

Finally (and regardless of which form of the argument we choose), converting to Cartesian form:

$$\begin{aligned}
\frac{z_6}{z_8} &= 0.52\angle 1.37 \\
&= 0.52\cos(1.37) + j(0.52)\sin(1.37) \\
&= 0.11 + j0.51
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{z_8}{z_5} &= \frac{7.9\angle 6.1}{2.5\angle - 2.9} \\
&= \frac{7.9}{2.5}\angle\big(6.1 - (-2.9)\big) \\
&= 3.16\angle 9.0
\end{aligned}$$

Subtracting one period 2π from the argument to obtain an expression within the principal range $-\pi < \theta < pi$:

$$\begin{aligned}
\frac{z_8}{z_5} &= 3.16\angle 9.0 - 2\pi \\
&= 3.16\angle 2.72
\end{aligned}$$

Converting the final answer to Cartesian form:

$$\begin{aligned}
\frac{z_8}{z_5} &= 3.16\cos(2.72) + j(3.16)\sin(2.72) \\
&= -2.88 + j1.29
\end{aligned}$$

(e) As before, first write 1 as a complex number in polar form $1 \equiv 1\angle 0$:

$$\begin{aligned}
\frac{1}{z_7} &= \frac{1\angle 0}{0.3\angle 1.7} \\
&= \frac{1}{0.3}\angle(0 - 1.7) \\
&= 3.33\angle - 1.7 \\
&= 3.33\cos(-1.7) + j(3.33)\sin(-1.7) \\
&= -0.43 - j3.30
\end{aligned}$$

(f)
$$\begin{aligned}
\frac{1}{z_5} &= \frac{1\angle 0}{2.5\angle - 2.9} \\
&= \frac{1}{2.5}\angle\big(0 - (-2.9)\big) \\
&= 0.4\angle 2.9
\end{aligned}$$

Converting the final answer to Cartesian form:

$$\begin{aligned}
\frac{1}{z_5} &= 0.4\cos(2.9) + j(0.4)\sin(2.9) \\
&= -0.39 + j0.10
\end{aligned}$$

3. When multiple impedances in a electrical circuit are connected in series, the total impedance Z (ohms) is given by the sum of the individual impedances. This is related to the voltage V (volts) and current I (amps) using Ohm's Law, which states that $V = IZ$.

Two impedances $Z_1 = (3 + j6)\ \Omega$ and $Z_2 = (4 - j3)\ \Omega$ are connected in series to a supply voltage of 120 V. What is the magnitude of the current flowing through the circuit?

Solution:

Begin by calculating the total impedance as a single complex number in Cartesian form:

$$
\begin{aligned}
Z &= Z_1 + Z_2 \\
&= (3 + j6) + (4 - j3) \\
&= 7 + j3
\end{aligned}
$$

Then using Ohm's Law, we obtain the current:

$$
\begin{aligned}
I &= \frac{V}{Z} \\
&= \frac{120}{7 + j3}
\end{aligned}
$$

and to simplify this to a complex number in Cartesian form, we must multiply both the numerator and denominator by the conjugate of the denominator:

$$
\begin{aligned}
I &= \frac{120(7-j3)}{(7+j3)(7-j3)} \\
&= \frac{120(7-j3)}{49-j21+j21-j^2 9} \\
&= \frac{120(7-j3)}{49-(-1)9} \quad \text{using } j^2=-1 \\
&= \frac{120(7-j3)}{49+9} \\
&= \frac{120}{58}(7-j3) \\
&= 14.4828 - j6.2069
\end{aligned}
$$

So we have obtained the current as a complex number. Finally, we calculate the magnitude as the question asks:

$$
\begin{aligned}
|I| &= \sqrt{(14.4828)^2+(6.2069)^2} \\
&= \sqrt{209.7515+38.5256} \\
&= \sqrt{248.2721} \\
&= 15.7568\ldots
\end{aligned}
$$

Thus,

$$|I| = 15.76 \text{ A (2 d.p.)}$$