Indices

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 2

Learning Outcomes

- Use rules of indices.

Indices

Indices are also known as exponents, powers and orders. The index of a number is the power to which you are raising it, e.g. the index of 3^4 is 4 (the "base" is 3).

If no index is explicitly stated, then it must be 1.

Example:

$$5 = 5^1 \quad \text{and} \quad x = x^1$$

We will need to know how to interpret and simplify various expressions involving indices.

Rules of Indices

When multiplying identical bases, we can use the rule:

Multiplying indices

$$a^m \times a^n = a^{m+n}$$

Example:

$$4^3 \times 4^5 = 4 \times 4 \times 4 \quad \times \quad 4 \times 4 \times 4 \times 4 \times 4 = 4^8$$

Or more simply using this rule:

$$4^3 \times 4^5 = 4^{3+5} = 4^8$$

Rules of Indices

When dividing identical bases, we can use the rule:

Dividing indices

$$a^m \div a^n \quad \text{or} \quad \frac{a^m}{a^n} \;=\; a^{m-n}$$

Example:

$$\frac{6^5}{6^2} = \frac{6 \times 6 \times 6 \times 6 \times 6}{6 \times 6} = 6 \times 6 \times 6 = 6^3$$

Or more simply using this rule:

$$\frac{6^5}{6^2} = 6^{5-2} = 6^3$$

Rules of Indices

Now consider the expression $(a^6)^2$

$$
\begin{aligned}
(a^6)^2 &= (a \times a \times a \times a \times a \times a)^2 \\
&= (a \times a \times a \times a \times a \times a) \times (a \times a \times a \times a \times a \times a) \\
&= a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \\
&= a^{12}
\end{aligned}
$$

So it turns out that:

$$(a^6)^2 = a^{12} = a^{6 \times 2}$$

Rules of Indices

From this we obtain a general rule for powers of powers:

Powers of indices

$$(a^m)^n = a^{m\times n} = (a^n)^m$$

Example:

$$(5.03^{0.75})^{1.8} = 5.03^{0.75\times 1.8} = 5.03^{1.35} = 8.85 \quad \text{(2 d.p.)}$$

Confirm this on your calculator by checking both $(5.03^{0.75})^{1.8}$ and $5.03^{1.35}$

Rules of Indices

Now consider the expression $\dfrac{a^2}{a^6}$

We already know that:

$$\frac{a^2}{a^6} = a^{2-6} = a^{-4}$$

However:

$$\frac{a^2}{a^6} = \frac{aa}{aaaaaa} = \frac{1}{aaaa} = \frac{1}{a^4}$$

Therefore, we can see that these expressions are equivalent:

$$a^{-4} = \frac{1}{a^4}$$

Rules of Indices

Thus we discern a further rule for negative indices:

Negative indices

$$a^{-m} = \frac{1}{a^m}$$

Negative indices denote reciprocals.

Example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Rules of Indices

Now consider the expression $\frac{a^2}{a^2}$

We know that:

$$\frac{a^2}{a^2} = a^{2-2} = a^0$$

But it is also true that:

$$\frac{a^2}{a^2} = \frac{aa}{aa} = 1$$

Therefore, we can conclude:

$$a^0 = 1$$

Rules of Indices

This is again a general rule, that *any* number raised to the power of zero is exactly one:

Zero index

$$a^0 = 1$$

Example:

$$17^0 = 1 \qquad \pi^0 = 1 \qquad (47.01\pi + 13)^0 = 1 \qquad (zy - d)^0 = 1$$

Rules of Indices

Finally, consider the expression $a^{\frac{1}{2}}\,a^{\frac{1}{2}}$. We know that:

$$a^{\frac{1}{2}}\,a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

But that is also the definition of the square root of a:

$$\sqrt[2]{a}\ \sqrt[2]{a} = a$$

Therefore, we can conclude that these must be the same thing:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Rules of Indices

This is true not just for square roots, but more generally for n^{th}-roots:

Fractional indices

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example:

$$64^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

Rules of Indices

We also know that:

$$(\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{1}{n}\times m} = a^{\frac{m}{n}}$$

Therefore, we can conclude that:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

So fractional indices indicate roots and powers.

Example:

$$25^{\frac{3}{2}} = (\sqrt{25})^3 = 5^3 = 125$$

Summary

For any numbers a, m, n:

Rules of indices

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{m \times n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$a^0 = 1 \quad \text{and} \quad a^1 = a$$

We will use these rules constantly throughout the module. You must know them instinctively.