Railway Engineering Mathematics

Tutorial Sheet 19

Solutions

1. Given the following complex numbers in Cartesian form:

$$z_1 = 2 + j4, \qquad z_2 = 3 - j, \qquad z_3 = -7 + j5, \qquad z_4 = 9 - j6$$

Calculate:

(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $z_2 - z_1$

(d) $z_3 + z_4$

(e) $z_3 z_1$

(f) $z_2 z_4$

(g) $\bar{z}_3$

(h) $\dfrac{z_4}{z_2}$

(i) $\dfrac{z_1}{z_3}$

(j) $z_4 \bar{z}_4$

(k) $\mathrm{Re}\big(z_1 z_3\big)$

(l) $\mathrm{Im}\big(z_1 \bar{z}_1\big)$

Solution:

When adding or subtracting two complex numbers in Cartesian form, consider the "real parts" and the "imaginary parts" as two separate calculations. We are always able to simplify our final answer to the most simple Cartesian form of one real term plus (or minus) one imaginary term.

(a)

$$\begin{aligned} z_1 + z_2 &= (2 + j4) + (3 - j) \\ &= 2 + j4 + 3 - j \\ &= (2 + 3) + (4 - 1)j \\ &= 5 + j3 \end{aligned}$$

(b)

$$\begin{aligned} z_1 - z_2 &= (2 + j4) - (3 - j) \\ &= 2 + j4 - 3 + j \\ &= (2 - 3) + j(4 + 1) \\ &= -1 + j5 \end{aligned}$$

(c)

$$\begin{aligned} z_2 - z_1 &= (3 - j) - (2 + j4) \\ &= 3 - j - 2 - j4 \\ &= (3 - 2) + j(-1 - 4) \\ &= 1 - j5 \end{aligned}$$

(d)

$$\begin{aligned} z_3 + z_4 &= (-7 + j5) + (9 - j6) \\ &= -7 + j5 + 9 - j6 \\ &= (-7 + 9) + j(5 - 6) \\ &= 2 - j \end{aligned}$$

(e) Multiplying a pair of complex numbers in Cartesian form involves expanding the brackets - we should initially obtain four terms from pairing up all the possible combinations. Then we can simplify the result to at most one real and one imaginary number.

$$
\begin{aligned}
z_3 z_1 &= (-7 + j5)(2 + j4) \\
&= -14 - j28 + j10 + j^2 20 \\
&= -14 - j18 + (-1) \times 20 \\
&= -14 - j18 - 20 \\
&= -34 - j18
\end{aligned}
$$

Remember:

- By definition, $j^2 = -1$
- We should always be able to simplify down to just two terms - one real part and one imaginary part. If you have more, you haven't fully simplified yet!

(f)
$$
\begin{aligned}
z_2 z_4 &= (3 - j)(9 - j6) \\
&= 27 - j18 - j9 + j^2 6 \\
&= 27 - j27 + (-1) \cdot 6 \\
&= 27 - j27 - 6 \\
&= 21 - j27
\end{aligned}
$$

(g) By definition, to obtain the complex conjugate we just change the sign of the imaginary term:

$$\bar{z}_3 = \overline{(-7 + j5)} = -7 - j5$$

(h) When dividing a pair of complex numbers in Cartesian form, we will need to simplify by multiplying both the numerator and denominator by the **complex conjugate** of the denominator, so that we obtain a purely real denominator overall:

$$
\begin{aligned}
\frac{z_4}{z_2} &= \frac{9-j6}{3-j} \\
&= \frac{9-j6}{3-j} \times \frac{\overline{(3-j)}}{\overline{(3-j)}} \\
&= \frac{9-j6}{3-j} \times \frac{3+j}{3+j} \\
&= \frac{(9-j6)(3+j)}{(3-j)(3+j)} \\
&= \frac{27+j9-j18-j^2 6}{3^2-3j+3j-j^2} \\
&= \frac{27-j9+6}{9+1} \\
&= \frac{33-j9}{10} \\
&= \frac{33}{10} - j\frac{9}{10} \quad \text{or} \quad 3.3-j0.9
\end{aligned}
$$

By obtaining a real denominator,we have again been able to simplify the result to a combination of a single real and a single imaginary term.

(i)
$$
\begin{aligned}
\frac{z_1}{z_3} &= \frac{2+j4}{-7+j5} \\
&= \frac{2+j4}{-7+j5} \times \frac{-7-j5}{-7-j5} \\
&= \frac{-14-j10-j28-j^2 20}{7^2+5^2} \\
&= \frac{-14-j38+20}{49+25} \\
&= \frac{6-j38}{74} \\
&= \frac{6}{74} - j\frac{38}{74} \quad \text{or} \quad 0.081 - j0.514 \quad \text{(3 d.p.)}
\end{aligned}
$$

(j) Multiplying z_4 by its own complex conjugate:

$$
\begin{aligned}
z_4\bar{z}_4 &= (9-j6)(9+j6) \\
&= 81 + j54 - j54 - j^2 36 \\
&= 81 + 36 \\
&= 117
\end{aligned}
$$

There is a more general truth here: taking the product of a complex number and its own conjugate gives a result that is always purely real, and in particular gives the sum of the squares of the real and imaginary coefficients.

That is, for a general complex number $a + jb$ where $a, b \in \mathbb{R}$:

$$
\begin{aligned}
(a+jb)(a-jb) &= a^2 - jab + jab - j^2 b \\
&= a^2 + b^2
\end{aligned}
$$

which is the sum of two real numbers, and thus real, regardless of the actual values of a and b.

(k) This example requires us to first take the product of z_1 and z_3, then state the real part of what we obtain as a result:

$$\begin{aligned}
\mathrm{Re}\big(z_1 z_3\big) &= \mathrm{Re}\big((2+j4)(-7+j5)\big) \\
&= \mathrm{Re}\big(-14+j10-j28+j^2 20\big) \\
&= \mathrm{Re}\big(-14-j18-20\big) \\
&= \mathrm{Re}\big(-34-j18\big) \\
&= -34
\end{aligned}$$

(l)
$$\begin{aligned}
\mathrm{Im}\big(z_1 \bar{z}_1\big) &= \mathrm{Im}\big((2+j4)(2-j4)\big) \\
&= \mathrm{Im}\big(4-j8+j8-j^2 16\big) \\
&= \mathrm{Im}\big(4+16\big) \\
&= \mathrm{Im}\big(20\big) \\
&= 0
\end{aligned}$$

as 20 is purely real.

Note: we could have predicted this, as we discussed earlier that multiplying *any* complex number by its own conjugate always yields a purely real result - so the imaginary part of such a product will always be zero.

Hence, for *any* complex number $z \in \mathbb{C}$:

$$\mathrm{Im}\big(z \cdot \bar{z}\big) = 0$$

2. Calculate the roots of the following polynomial functions:

   (a) $y = x^2 + 14x + 58$

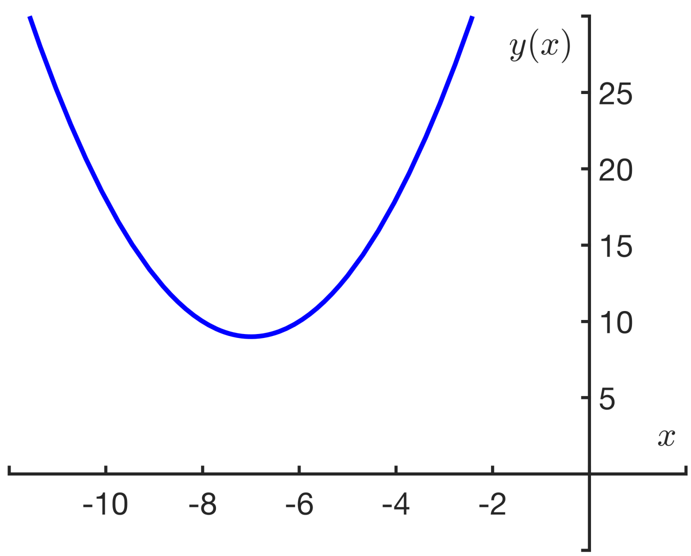
   Solution:

   Remember that the roots are the values of x where $y = 0$, and we can obtain these using the quadratic formula (in the particular cases we deal with here, factorisation is not possible).

   In this case, we can substitute in the coefficients $a = 1$, $b = 14$ and $c = 58$:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 58}}{2 \times 1} \\
&= \frac{-14 \pm \sqrt{-36}}{2}
\end{aligned}
$$

   Before we encountered complex numbers, we would have to give up at this stage: there are no solutions to $\sqrt{-36}$. More precisely, there are no *real* solutions! So this corresponds to a parabola (the $\cup$ or $\cap$-shaped curve described by a quadratic function) that does not cross the x-axis:



But now that we know about complex numbers, we can *always* obtain the square root of any number - whether it is positive or negative.

In particular, we can write the square root of any negative number as some multiple of the imaginary unit j.

Thus:

$$\begin{aligned}
x &= \frac{-14 \pm \sqrt{-36}}{2} \\
&= \frac{-14 \pm \sqrt{-1 \times 36}}{2} \\
&= \frac{-14 \pm \sqrt{-1}\sqrt{36}}{2} \\
&= \frac{-14 \pm j6}{2} \\
&= -7 \pm j3
\end{aligned}$$

So the two solutions are a complex conjugate pair:

$$x_1 = -7 + j3 \quad \text{and} \quad x_2 = -7 - j3$$

So now, by using the quadratic formula, we can *always* find solutions to any quadratic equation. Of course, if the contents of the square root (the discriminant) are positive then we would obtain a pair of real solutions just like before:

Discriminant $\Delta = b^2 - 4ac$	**Roots**
Positive	Two real, distinct roots.
Zero	One real, repeated root.
Negative	Complex conjugate roots.

(b) $y = 4x^2 - 12x + 10$

Solution:

In this case, obtaining the roots means setting $y = 0$ and thus solving:

$$4x^2 - 12x + 10 = 0$$

for x. This is again a quadratic expression, this time with $a = 4$, $b = -12$ and $c = 10$. So, using the quadratic formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 4 \times 10}}{2 \times 4} \\
&= \frac{12 \pm \sqrt{144 - 160}}{8} \\
&= \frac{12 \pm \sqrt{-16}}{8} \\
&= \frac{12 \pm \sqrt{-1 \times 16}}{8} \\
&= \frac{12 \pm \sqrt{-1} \times \sqrt{16}}{8} \\
&= \frac{12 \pm j4}{8} \\
&= \frac{3 \pm j}{2} \\
&= \frac{3}{2} \pm j\frac{1}{2}
\end{aligned}$$

Hence, the pair of solutions are:

$$x_1 = 1.5 + j0.5 \quad \text{and} \quad x_2 = 1.5 - j0.5$$

(c) $y = x^2 + 3x + 12$

Solution:

To find the roots, we wish to set $y = 0$ and thus solve the following equation for x:

$$x^2 + 3x + 12 = 0$$

This is a quadratic expression, with $a = 1$, $b = 3$ and $c = 12$:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 12}}{2 \times 1} \\
&= \frac{-3 \pm \sqrt{-39}}{2} \\
&= \frac{-3 \pm \sqrt{-1}\sqrt{39}}{2} \\
&= \frac{-3 \pm j\sqrt{39}}{2} \\
&= \frac{-3}{2} \pm j\frac{\sqrt{39}}{2}
\end{aligned}$$

So the pair of complex conjugate solutions are:

$$x_1 = -1.5 + j3.12 \quad \text{and} \quad x_2 = -1.5 - j3.12$$

3. In the hydrogen atom, the angular momentum p of the de Broglie wave is given by:

$$p\Psi = -\left(\frac{jh}{2\pi}\right)\big(\pm jm\Psi\big)$$

Determine a simplified expression for p.

Solution:

Divide both sides by Ψ to obtain a formula for p alone:

$$p = -\left(\frac{jh}{2\pi}\right)\big(\pm jm\Psi\big) \cdot \frac{1}{\Psi} = -\left(\frac{jh}{2\pi}\right)\big(\pm jm\big)$$

We do not have any information about h and m, but j is the imaginary number, and as it occurs twice we can simplify using the fact that $j^2 = -1$:

$$\begin{aligned} p &= -\frac{h}{2\pi}\big(\pm m\big)j^2 \\ &= -\frac{h}{2\pi}\big(\pm m\big)(-1) \\ &= \frac{h}{2\pi}\big(\pm m\big) \\ &= \pm\frac{hm}{2\pi} \end{aligned}$$

Without any additional information, we cannot simplify further.