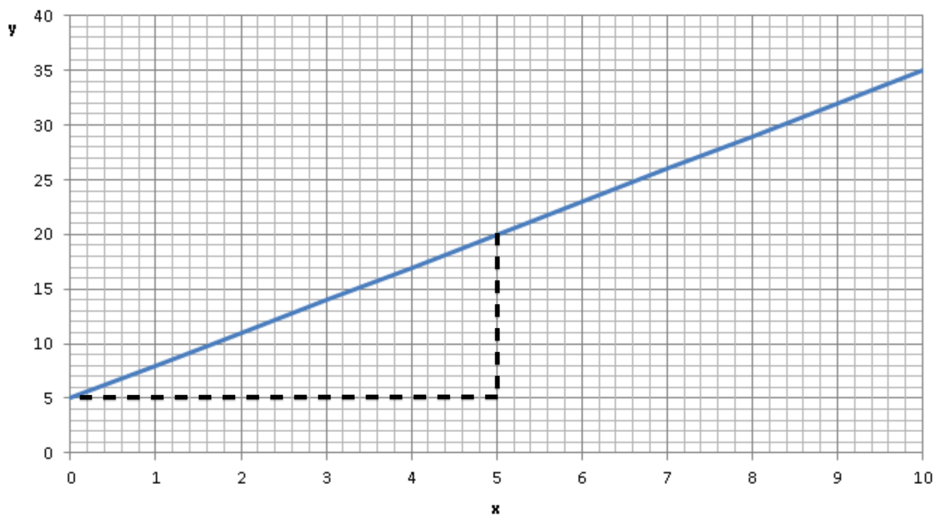
Railway Engineering Mathematics

Tutorial Sheet 5

Solutions

1. Determine the equation of the following graphs:

   (a)



Solution:

In each case, we state the general equation of a straight line $y = mx + c$, then determine the y-intercept (c) from looking at the height of the graph where it crosses the y-axis (at $x = 0$). The gradient (m) is then determined by "rise over run".

For part (a):

The equation of a straight line is $y = mx + c$.

Then since the graph crosses the y-axis at $(0, 5)$, we can determine that $c = 5$.

Choosing this point, with $x_1 = 0$ and $y_1 = 5$, and a second point $(5, 20)$ that the graph passes through so that $x_2 = 5$ and $y_2 = 20$, then the gradient m

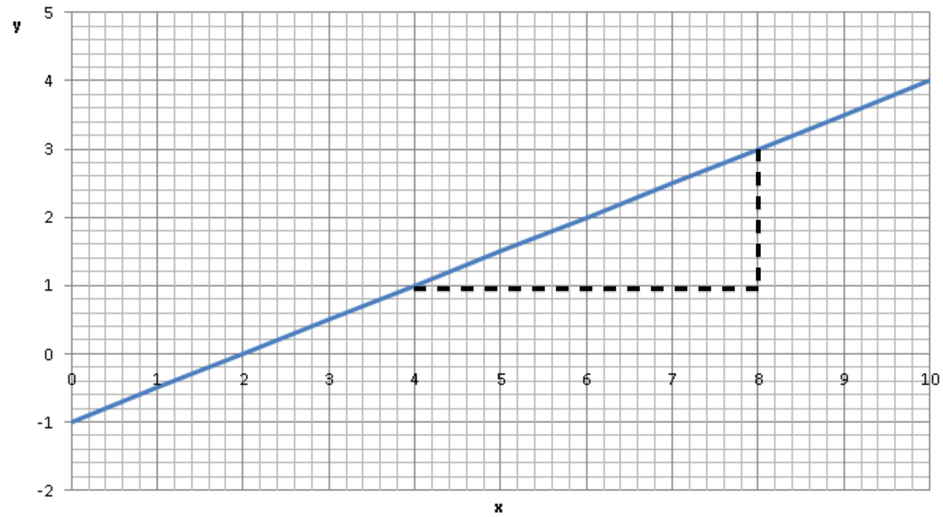
is given by:

$$
\begin{aligned}
m &= \frac{\Delta y}{\Delta x} \\
&= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{20 - 5}{5 - 0} \\
&= \frac{15}{5} \\
&= 3
\end{aligned}
$$

Hence:

$$y = 3x + 5$$

(b)



Solution:

The equation of a straight line is $y = mx + c$.

Then since the graph crosses the y-axis at $(0, -1)$, we can determine that $c = -1$.

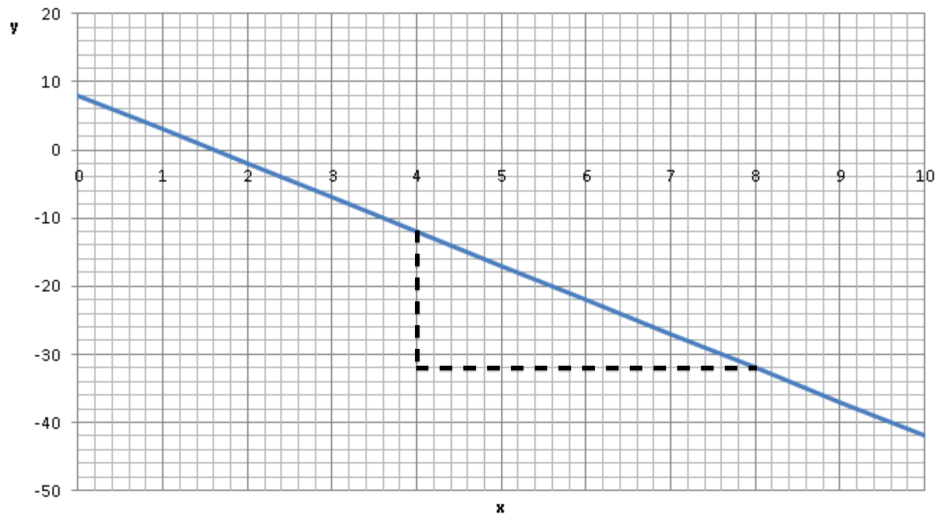
Choosing two points $(4, 1)$ and $(8, 3)$ that the graph passes through so that $x_1 = 4$ and $y_1 = 1$, and $x_2 = 8$ and $y_2 = 3$, then the gradient m is given by:

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - 1}{8 - 4} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

Hence the graph has equation:

$$y = \frac{1}{2}x - 1$$

(c)



Solution:

The equation of a straight line is $y = mx + c$.

Then since the graph crosses the y-axis at $(0, 8)$, the y-intercept is $c = 8$.

Choosing two points $(4, -12)$ and $(8, -32)$ that the graph passes through so that $x_1 = 4$ and $y_1 = -12$, and $x_2 = 8$ and $y_2 = -32$, then the gradient is:

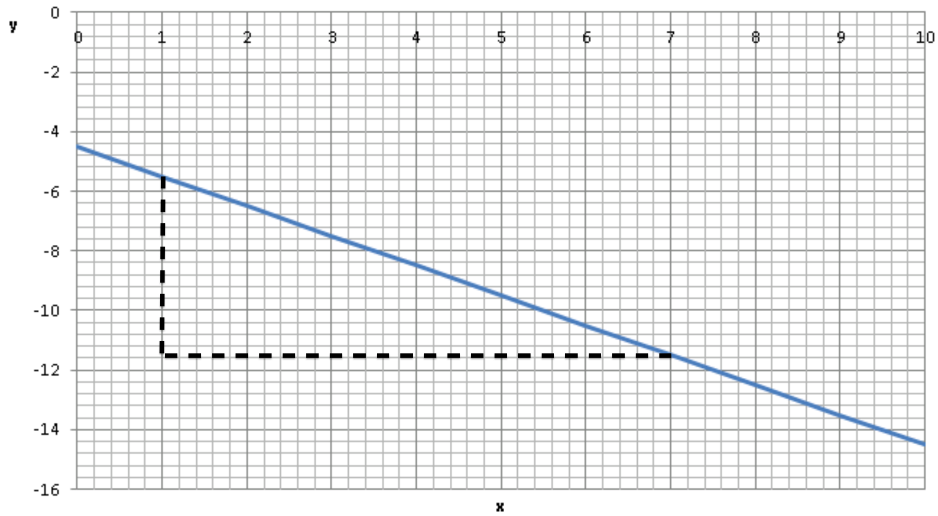
$$
\begin{aligned}
m &= \frac{\Delta y}{\Delta x} \\
&= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-32 - (-12)}{8 - 4} \\
&= \frac{-20}{4} \\
&= -5
\end{aligned}
$$

(Of course, a negative gradient is expected as the graph is sloping downwards.)

Hence:

$$y = -5x + 8$$

(d)



Solution:

The equation of a straight line is $y = mx + c$.

Then since the graph crosses the y-axis at $(0, -4.4)$ (note that each box on the y-axis denotes an increment of 0.4), the y-intercept is $c = -4.4$.

Choosing two points $(1, -5.6)$ and $(7, -11.6)$ that the graph passes through so that $x_1 = 1$ and $y_1 = -5.6$, and $x_2 = 7$ and $y_2 = -11.6$, then the gradient m is:

$$
\begin{aligned}
m &= \frac{\Delta y}{\Delta x} \\
&= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-11.6 - (-5.6)}{7 - 1} \\
&= \frac{-6}{6} \\
&= -1
\end{aligned}
$$

Hence the equation of this graph is:

$$y = -x - 4.4$$

2. Sketch the graph of the the following functions, indicating the points at which the line crosses the x and y axes:

(a) $y = 3x + 2$

(b) $y = 3x - 2$

(c) $y = 3x$

(d) $y = 1 - 2x$

(e) $y = -x + 1$

(f) $y = 1$

(g) $y = 1 + 2x$

Solution:

In each case, the easiest way to do this for a straight line is to determine the y-intercept, and the x-intercept (what is the value of x when we set $y = 0$). Any two points are sufficient to define a straight line, so then we just join them up.

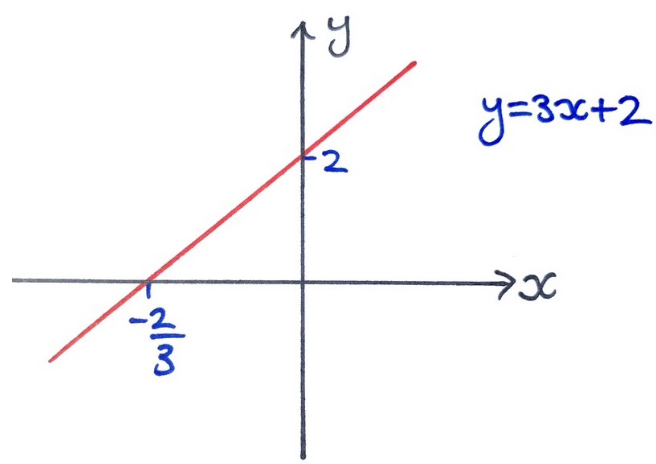
(a) $y = 3x + 2$

y-intercept: $(0, 2)$

Setting $y = 0$ and solving for x:

$3x + 2 = 0$ and thus $x = -\dfrac{2}{3}$

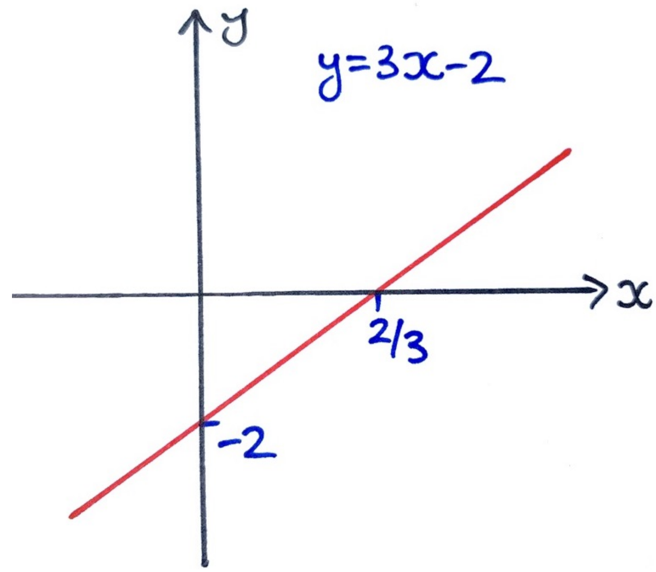
So we find that the x-intercept is at $\left(-\dfrac{2}{3}, 0\right)$.



(b) $y = 3x - 2$

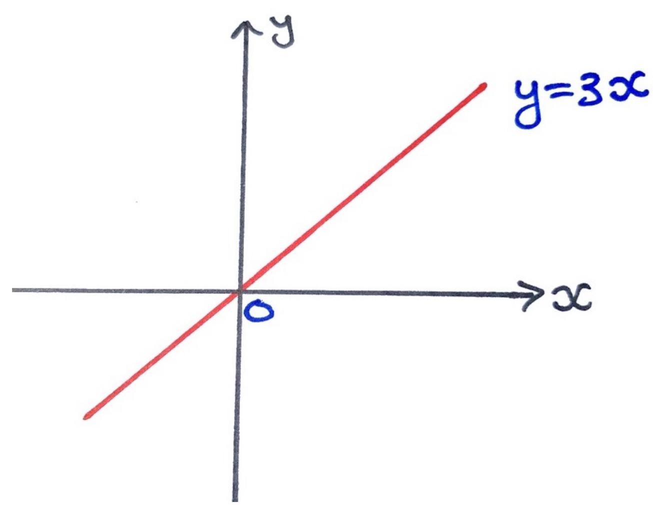
y-intercept: $(0, -2)$

Setting $y = 0$ and solving for x, we find that the x-intercept is at $\left(\frac{2}{3}, 0\right)$.



(c) $y = 3x$

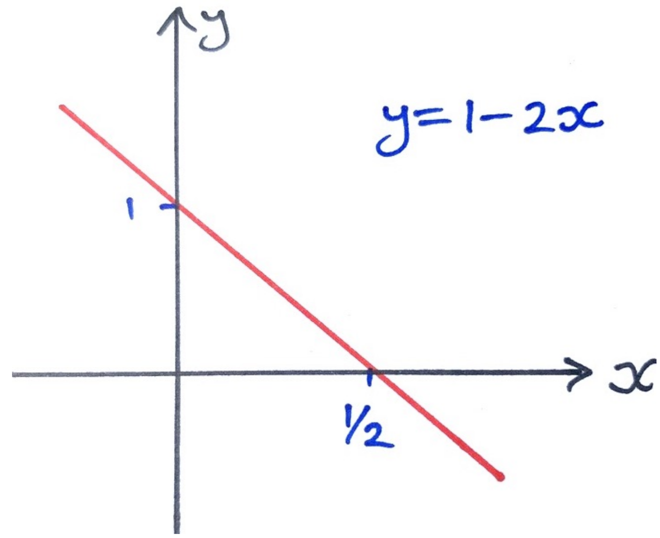
Both the x-intercept and the y-intercept are at the origin $(0, 0)$, so we could choose any other point. For example, let $x = 1$, then $y = 3(1) = 3$, so the graph passes through $(1, 3)$.



(d) $y = 1 - 2x$

y-intercept: $(0, 1)$

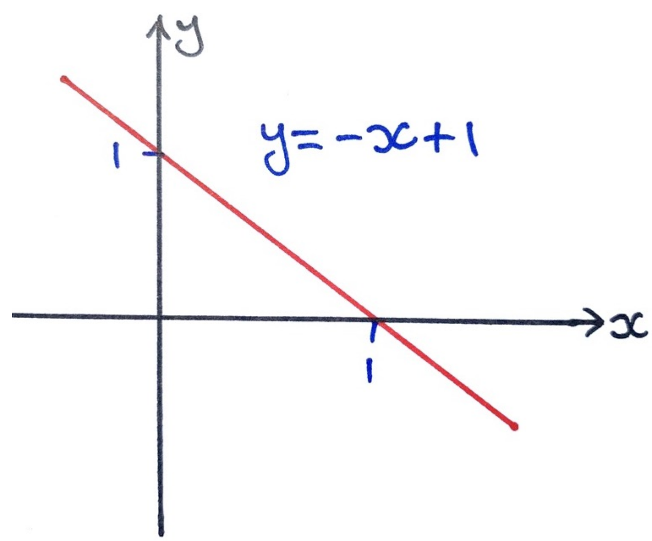
Setting $y = 0$ and solving for x, we find that the x-intercept is at $\left(-\frac{1}{2}, 0\right)$.



(e) $y = -x + 1$

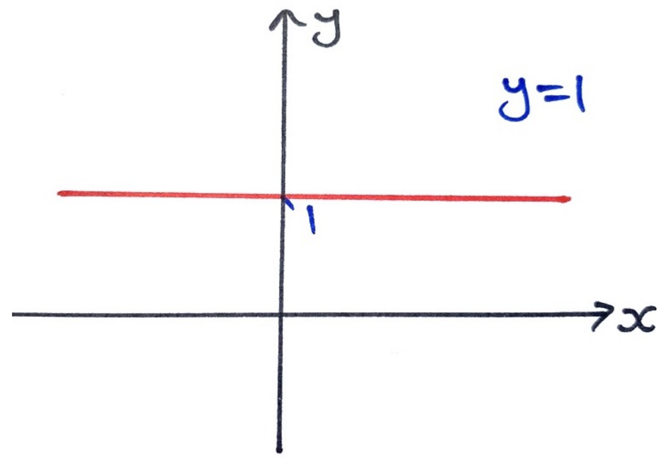
y-intercept: $(0, 1)$

x-intercept: $(1, 0)$



(f) $y = 1$

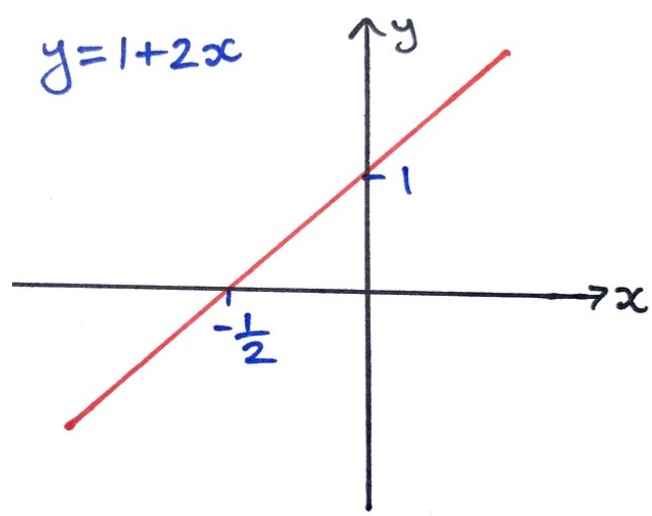
This is a constant function, so it has zero gradient everywhere and thus is a horizontal line with constant height $y = 1$.



(g) $y = 1 + 2x$

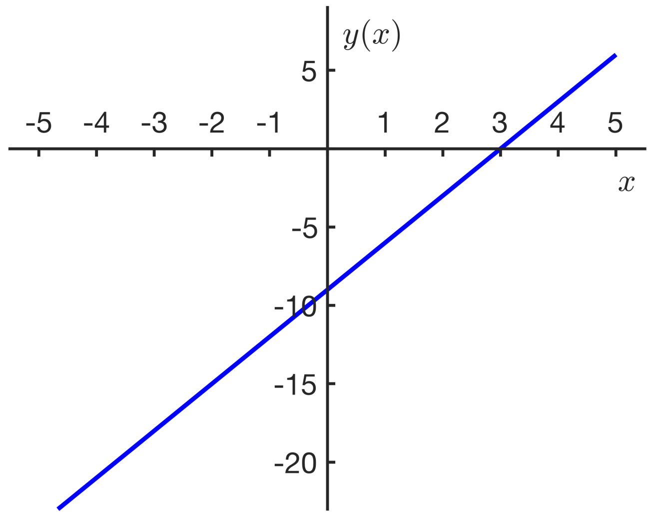
y-intercept: $(0, 1)$

x-intercept: $\left(-\frac{1}{2}, 0\right)$

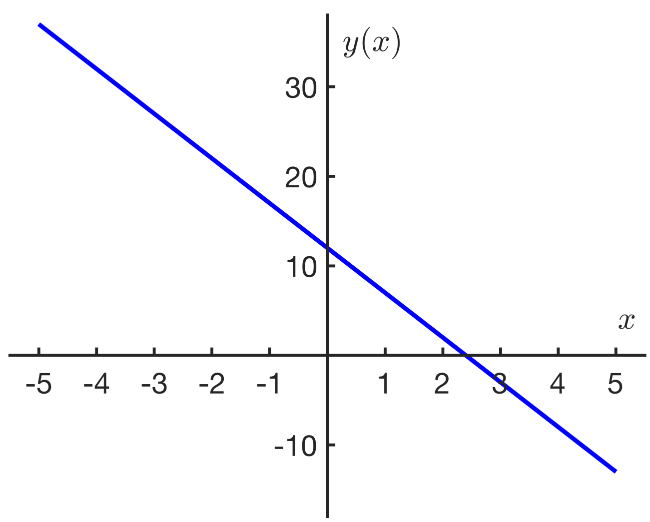


3. Using technology (EXCEL or otherwise), plot the following polynomial functions. You may choose your own range of x, unless otherwise stated.

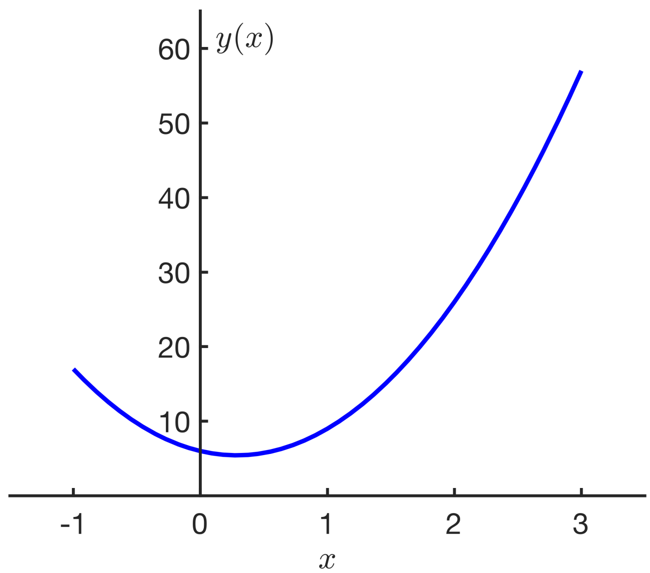
   (a) $y = 3x - 9$



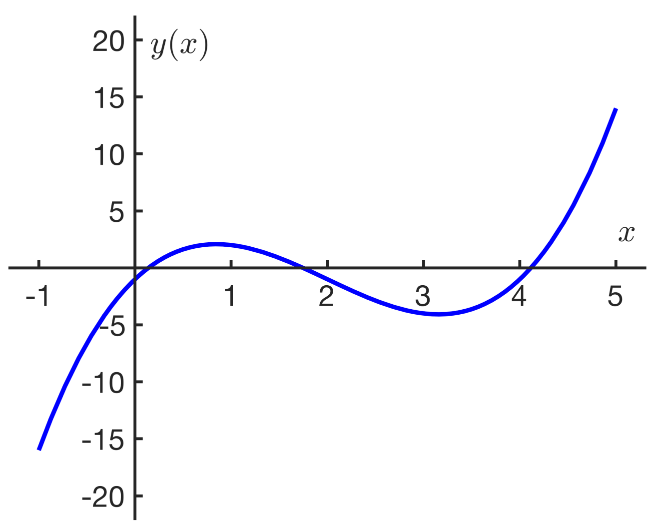
   (b) $y = -5x + 12$



(c) $y = 7x^2 - 4x + 6$ in the range $-1 \le x \le 3$

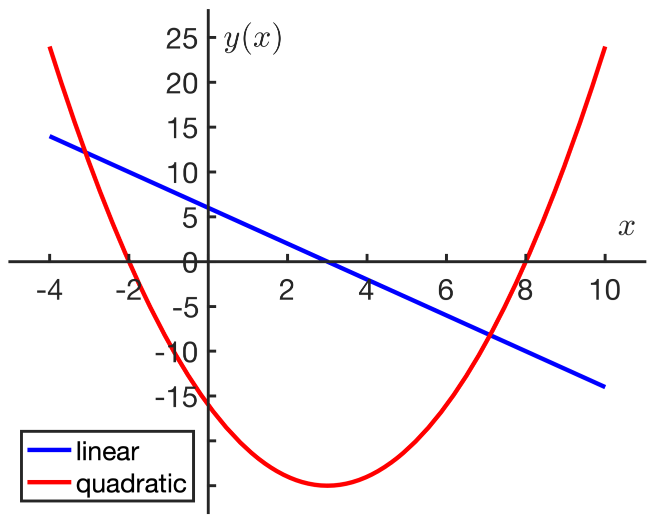


(d) $y = x^3 - 6x^2 + 8x - 1$ in the range $-1 \le x \le 5$



(e) $y = -2x + 6$ and $y = x^2 - 6x - 16$ in the range $-4 \leq x \leq 10$

(Plot both of these functions on the same set of axes.)



4. Determine the equation of the straight line that passes through the points:

(a) $(-2, 9)$ and $(1, -6)$

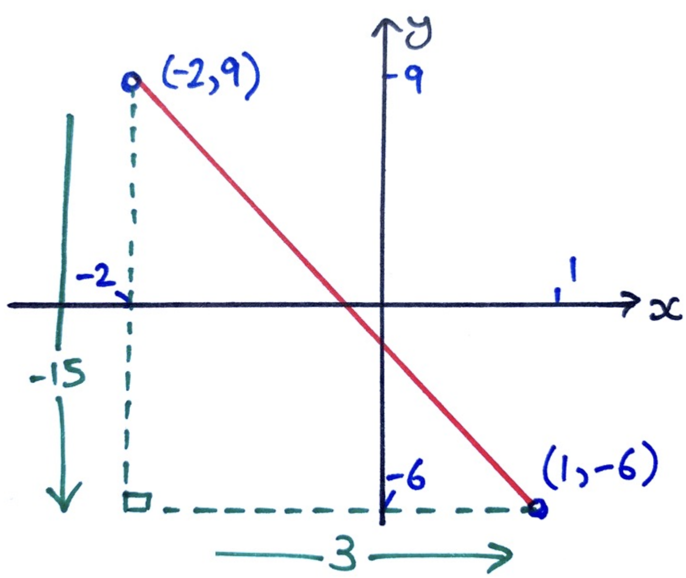
(b) $(-1, -8)$ and $(3, 4)$

(c) $(3, -13)$ and $(5, -17)$

(d) $(1, 5.5)$ and $(6, 3)$

Solution:

(a) $(-2, 9)$ and $(1, -6)$



The equation of a straight line has the form:

$$y = mx + c$$

First, we determine the gradient m using "rise over run":

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{(-6) - 9}{1 - (-2)} \\ &= \frac{-15}{3} \\ &= -5 \end{aligned}$$

So now the equation is:

$$y = -5x + c$$

Substitute the co-ordinates of one of the points into this equation to determine c. We will use $(-2, 9)$:

$$9 = -5(-2) + c$$

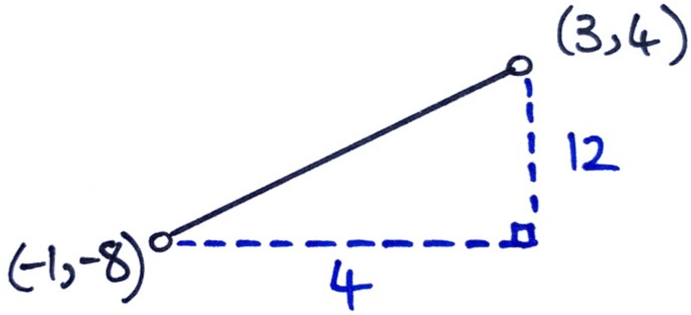
$$\therefore\ 9 = 10 + c$$

$$\therefore\ c = 9 - 10 = -1$$

Hence,

$$y = -5x - 1$$

(b) $(-1, -8)$ and $(3, 4)$



Starting with the equation of the straight line:

$$y = mx + c$$

First, determine the gradient m:

$$m = \frac{\Delta y}{\Delta x} = \frac{4 - (-8)}{3 - (-1)} = \frac{12}{4} = 3$$

So the equation is now:

$$\therefore y = 3x + c$$

Substitute in one of the points that the graph passes through to determine the value of c. We will use $(3, 4)$:

$$4 = 3(3) + c$$

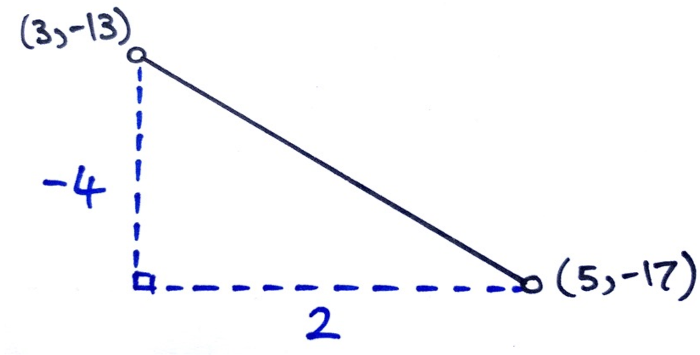
$$\therefore\ 4 = 9 + c$$

$$\therefore\ c = 4 - 9 = -5$$

Hence,

$$y = 3x - 5$$

(c) $(3, -13)$ and $(5, -17)$



Starting with the equation of the straight line:

$$y = mx + c$$

First, determine the gradient m:

$$m = \frac{\Delta y}{\Delta x} = \frac{-17 - (-13)}{5 - 3} = \frac{-4}{2} = -2$$

So the equation becomes:

$$y = -2x + c$$

Substitute in one of the points to determine c. We use $(3, -13)$:

$$-13 = -2(3) + c$$

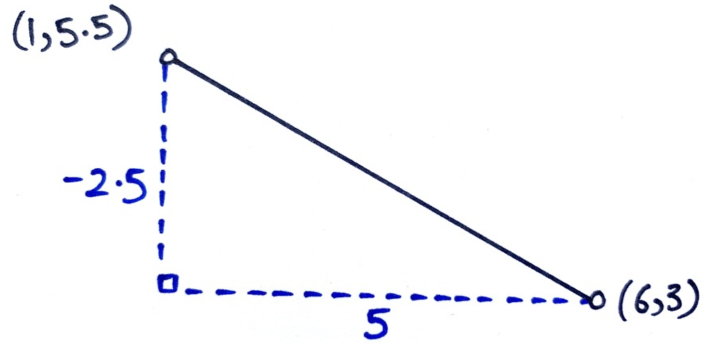
$$\therefore\ -13 = -6 + c$$

$$\therefore\ c = 6 - 13 = -7$$

Hence,

$$y = -2x - 7$$

(d) $(1, 5.5)$ and $(6, 3)$



Starting with the equation of the straight line:

$$y = mx + c$$

First, determine the gradient m:

$$m = \frac{\Delta y}{\Delta x} = \frac{3 - 5.5}{6 - 1} = \frac{-2.5}{5} = -\frac{1}{2}$$

Hence we have:

$$\therefore y = -\frac{1}{2}x + c$$

Substitute in the co-ordinates of point $(6, 3)$ and solve for c:

$$3 = -\frac{1}{2}(6) + c$$

$$\therefore\ 3 = -3 + c$$

$$\therefore\ c = 3 + 3 = 6$$

Hence,

$$y = -\frac{1}{2}x + 6$$

5. In a market, for a time when demand for a product is high relative to supply (that is, before the market becomes saturated with your product or a similar one manufactured by a competitor), we can roughly approximate the relationship between the profit of our company and the number of units manufactured with a linear model.

   You are a financial analyst in the employ of the *Persimmon* company, which has just developed a new product called the *uPhone*. Public reception has been extremely positive, and market research suggests that most people are very keen to puchase a *uPhone* as soon as possible. However, currently the supply of *uPhones* coming out of your factories is relatively low due to the complex engineering processes and rare metals required in the manufacturing process.

   In the first quarter of this year, the company made and sold 1500 units, and reported a profit of £60,000. In the second quarter, we were able to increase production to 6000 units. All of these were sold and a profit of £960,000 was made.

   (a) How many *uPhones* need to be sold each quarter in order to break even during that financial quarter?

   (b) In the third quarter, supply managers are warning that production may slow to 5000 units. Assuming that public demand for the product remains high, how much profit do you forecast?

Solution:

The key point to observe here is in the first paragraph: "we can roughly approximate the relationship between the profit... and the number of units manufactured with a linear model." So we are going to attempt to determine a linear (straight line) relationship between these two quantities of *profit* and the *number of units*, bearing in mind the limited circumstances this relationship will apply to.

For each quarter, let the profit of our company, in pounds, be P. And let the number of units manufactured be N.Then we will seek a relationship of the form:

$$P = mN + c$$

and we will first need to determine the gradient m and the intercept c. The other information we are given that can help with this is in the third paragraph: "In the first quarter of this year, the company made and sold 1500 units, and reported a profit of £60,000. In the second quarter, we were able to increase production to 6000 units. All of these were sold and a profit of £960,000 was made." So, when

$N = 1500$, we have $P = 60000$. And when $N = 6000$, it follows that $P = 960000$. Let's substitute this information in to our linear equation:

$$60000 = 1500m + c$$

$$960000 = 6000m + c$$

We could solve this pair of linear simultaneous equations for m and c in several ways: we could rearrange one of the equations to make c the subject, and then substitute it into the other equation, which would result in an equation only involving m which we could solve. By the end of the module we will learn another method for solving this using matrix algebra. However, since we are dealing with the equation of a straight line we can first find the gradient m using "rise over run":

$$m = \frac{\Delta P}{\Delta N} = \frac{960000 - 60000}{6000 - 1500} = \frac{900000}{4500} = 200$$

So every *uPhone* sold contributes to the overall profit by £200.

Then we have:

$$P = 200N + c$$

And we can substitute either profit-units pair into this and solve for c. For example, when $N = 1500$, we know that $P = 60000$, so:

$$60000 = 200(1500) + c$$

$$\therefore \quad 60000 = 300000 + c$$

$$\therefore \quad c = 60000 - 300000$$

$$\therefore \quad c = -240000$$

And hence we have the linear relationship:

$$P = 200N - 240000$$

(a) How many *uPhones* need to be sold each quarter in order to break even during that financial quarter?

Solution:

To break even, we need move from loss into profit, so we are interested in the point at which we clear the threshold of zero profit. For what number of units N does this occur?

Set $P = 0$ and solve for N:

$$\begin{aligned} & 0 = 200N - 240000 \\ \therefore \quad & 200N = 240000 \\ \therefore \quad & N = 240000/200 \\ \therefore \quad & N = 1200 \end{aligned}$$

So at exactly $N = 1200$ units manufactured (and sold) we have exactly $P = 200(1200) - 240000 = 0$ profit, and the company breaks even.

(b) In the third quarter, supply managers are warning that production may slow to 5000 units. Assuming that public demand for the product remains high, how much profit do you forecast?

Solution:

The model

$$P = 200N - 240000$$

allows us to easily calculate how profit P depends on the number N of units, so simply substitute in $N = 5000$ and evaluate:

$$P = 200(5000) - 240000 = 760000$$

So a profit of £760,000 is anticipated.

Note that this is a reasonable prediction to make because (i) we are told that we are able to assume that the conditions of this model (the high demand) are met, and that (ii) 5000 units per quarter is roughly on the same order as the other quantities that we have been working with. If in the next quarter the company upscaled production to 20 million units we could not be so sure that the model (and hence, the prediction of profit) is reasonable - as many conditions may have changed. Market conditions will certainly change as the *uPhone* becomes much more widely available and retailer start to offer discounts, and it becomes less of a prestige symbol. There will likely also be a lot more overheads as we will have had to build new factories, employ additional workers, or invest in new processes, additional transportation to markets, perhaps a new mass-market advertising campaign, etc. The point here is that we must be aware of assumptions involved in constructing a model and hence the limitations in reliability when using it to make predictions in situations for which it may not be designed. The supply costs involved in producing several thousand of a product may be very different from those involved in supplying it in the millions in the same time interval, so we would have to be very careful about using a model developed for one set of circumstances to make predictions for the other set of circumstances.

It would also be sensible to sketch or plot the model relationship: