Railway Engineering Mathematics
Tutorial Sheet 14
Solutions

1. An alternating current, i amperes, is given by:

   $$i = 10\sin(2\pi f t)$$

   where f is the frequency in hertz and t the time in seconds.

   Determine the rate of change of the current when $t = 20$ ms, given that $f = 150$ Hz.

   Solution:

   Substituting in the parameter value $f = 150$ to obtain the particular function for current:

   $$\begin{aligned} i(t) &= 10\sin(2\pi \times 150t) \\ &= 10\sin(300\pi t) \end{aligned}$$

   To calculate a general function for the rate of change of current, differentiate with respect to time t to obtain:

   $$\begin{aligned} \frac{\mathrm{d}i}{\mathrm{d}t} &= 300\pi \times 10\cos(300\pi t) \\ &= 3000\pi\cos(300\pi t) \end{aligned}$$

   Finally, substitute in $t = 20$ms, that is $t = 20 \times 10^{-3} = 0.02$s:

   $$\begin{aligned} \left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{t=0.02} &= 3000\pi\cos(300\pi \times 0.02) \\ &= 3000\pi \quad \text{or} \quad 9424.8 \text{ amp/s} \end{aligned}$$

2. The distance x metres moved by a car in a time t seconds is given by:

$$x = 3t^3 - 2t^2 + 4t - 1$$

Determine the velocity and acceleration when:

(a) $t = 0$

(b) $t = 1.5\,\mathrm{s}$

Solution:

Differentiate our formula for displacement x w.r.t time to determine a formula for velocity:

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(3t^3 - 2t^2 + 4t - 1\big) = 9t^2 - 4t + 4$$

then differentiate velocity to obtain a formula for acceleration:

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(9t^2 - 4t + 4\big) = 18t - 4$$

Note that acceleration is equivalent to the second derivative of displacement $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

Finally, evaluating $v(t)$ and $a(t)$ at the specific times of interest:

(a) At $t = 0$,

$$v(t = 0) = 9(0)^2 - 4(0) + 4 = 4\ m/s$$

$$a(t = 0) = 18(0) - 4 = -4\ m/s^2$$

(b) At $t = 1.5$,

$$v(t = 1.5) = 9(1.5)^2 - 4(1.5) + 4 = 18.25\ m/s$$

$$a(t = 1.5) = 18(1.5) - 4 = 23\ m/s^2$$

3. Find the number of revolutions per minute n which gives the minimum frictional couple F on a bearing where:

$$F = \frac{180000 - 1200n - 45n^2 + n^3}{80000}$$

Check your answer by plotting an appropriate graph.

Solution:

First, separate out the terms:

$$\begin{aligned} F &= \frac{180000}{80000} - \frac{1200n}{80000} - \frac{45n^2}{80000} + \frac{n^3}{80000} \\ &= \frac{9}{4} - \frac{3}{200}n - \frac{9}{16000}n^2 + \frac{1}{80000}n^3 \end{aligned}$$

To locate the stationary points, differentiate with respect to n to obtain a formula for the gradient:

$$\frac{\mathrm{d}F}{\mathrm{d}n} = -\frac{3}{200} - \frac{9}{8000}n + \frac{3}{80000}n^2$$

Alternatively, rather than expanding the fraction, we could have used the linearity of differentiation to remove the *constant factor* of $80,000^{-1}$ from inside the derivative:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}n} &= \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{180000 - 1200n - 45n^2 + n^3}{80000}\right) \\ &= \frac{1}{80000} \cdot \frac{\mathrm{d}}{\mathrm{d}n}\big(180000 - 1200n - 45n^2 + n^3\big) \\ &= \frac{1}{80000}\big(-1200 - 90n + 3n^2\big) \end{aligned}$$

Convince yourself that this is the same as the formulation for $\frac{\mathrm{d}F}{\mathrm{d}n}$ above!

Regardless of which method you use above, once we have obtained the first derivative we can set $\frac{\mathrm{d}F}{\mathrm{d}n} = 0$ and solve for n to locate the stationary points:

$$-\frac{3}{200} - \frac{9}{8000}n + \frac{3}{80000}n^2 = 0$$

Multiply through by 80000:

$$3n^2 - 90n - 1200 = 0$$

Then divide by 3:

$$n^2 - 30n - 400 = 0$$

This is the simplest form of this quadratic equation. Solving this either by factorising to $(n-40)(n+10) = 0$, or using the quadratic formula (with $a = 1$, $b = -30$ and $c = -400$):

$$\begin{aligned} n &= \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(-400)}}{2(1)} \\ &= \frac{30 \pm \sqrt{900 + 1600}}{2} \\ &= \frac{30 \pm 50}{2} \end{aligned}$$

Thus there are two stationary points, at:

$$n_1 = -10 \quad \text{and} \quad n_2 = 40$$

To classify the nature of these two stationary points (and so determine which, if either, is a minimum) we require the second derivative:

$$\begin{aligned} \frac{\mathrm{d}^2F}{\mathrm{d}n^2} &= \frac{\mathrm{d}}{\mathrm{d}n}\left(-\frac{3}{200} - \frac{9}{8000}n + \frac{3}{80000}n^2\right) \\ &= \frac{-9}{8000} + \frac{3}{40000}n \end{aligned}$$

Evaluating the second derivative when $n = -10$:

$$\left.\frac{\mathrm{d}^2F}{\mathrm{d}n^2}\right|_{n=-10} = \frac{-9}{8000} + \frac{3}{40000}(-10) = -0.0019 < 0$$

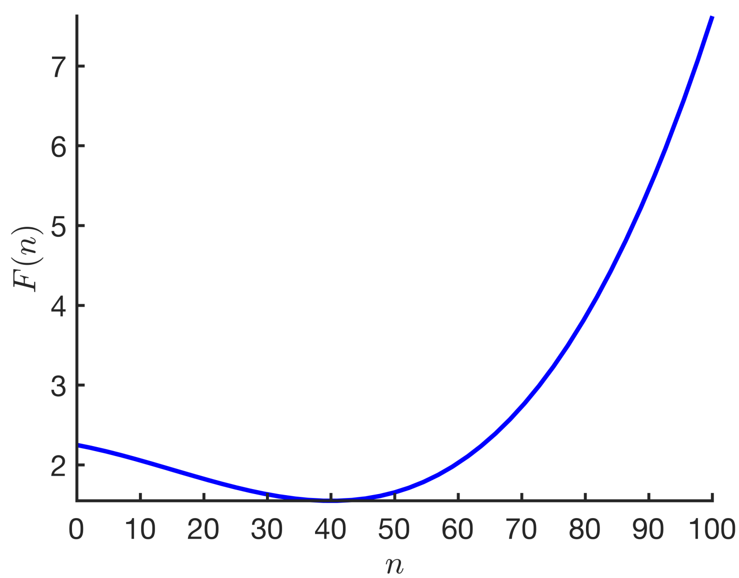
Given that the second derivative is negative at this point, the curve has a maximum here according to the second derivative test.

When $n = 40$:

$$\left.\frac{\mathrm{d}^2F}{\mathrm{d}n^2}\right|_{n=40} = \frac{-9}{8000} + \frac{3}{40000}(40) = 0.0019 > 0$$

As the second derivative is positive here, this is a minimum stationary point at $n = 40$, and at this point the value of F is:

$$\begin{aligned} F(n = 40) &= \frac{180000 - 1200 \times 40 - 45(40)^2 + (40)^3}{80000} \\ &= \frac{124000}{80000} \\ &= 1.55 \end{aligned}$$



From a plot of $F(n)$, we can see that the minimum at $(40, 1.55)$ is accurate.

4. A wooden packing case is made in the form of a rectangular block without a lid and standing on a square base. Its volume is 0.2 metres cubed. If x (in metres) is the length of a side of the base, find the value of x for which the case's surface area is a minimum.

Solution:

If the block has height h, then the surface area is given by the sum of the areas of the base and the four sides:

$$A = x^2 + 4xh$$

As the volume V, which for a cuboid is the product of length, breadth and height, is $0.2m^3$:

$$V = hx^2 = 0.2$$

Rearranging this to make one of the variables the subject:

$$h = \frac{0.2}{x^2}$$

and substituting into the formula for area:

$$\begin{aligned} A &= x^2 + 4x\left(\frac{0.2}{x^2}\right) \\ &= x^2 + 0.8x^{-1} \end{aligned}$$

So now we have obtained a formula for surface area (which is the quantity that we wish to minimise) in terms of only one variable x, and thus we can differentiate it with respect to x:

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 0.8x^{-1}\right) \\ &= 2x - 0.8x^{-2} \end{aligned}$$

To locate the extreme values, set $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$ and solve for x:

$$2x - 0.8x^{-2} = 0$$

$$\therefore\ 2x = 0.8x^{-2}$$

$$\therefore\ 2x^3 = 0.8$$

$$\therefore\ x^3 = 0.4$$

Taking the cubic root:

$$\begin{aligned} x &= \sqrt[3]{0.4} \\ &= 0.737\, m \end{aligned}$$

As this is the only solution, we could reasonably assume it to be the value of x that gives the minimum surface area, but we should check that it is indeed a minimum and not a maximum, using the second derivative test:

$$\begin{aligned} \frac{\mathrm{d}^2A}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x - 0.8x^{-2}\right) \\ &= 2 + 1.6x^{-3} \end{aligned}$$

Thus, evaluating the second derivative at $x = 0.737$:

$$\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right|_{x=0.737} = 2 + 1.6(0.737)^{-3} = 5.9968 > 0$$

As this is positive, this value of x gives a minimum surface area. This minimum is:

$$\begin{aligned} A(x = 0.737) &= (0.737)^2 + 0.8(0.737)^{-1} \\ &= 1.6287\, m^2 \end{aligned}$$

5. The formula:

$$x = \frac{20t^3}{3} - \frac{23t^2}{2} + 6t + 5$$

represents the distance x, in metres, moved by a body in t seconds.

Determine:

(a) The velocity and acceleration at the start

(b) the velocity and acceleration when $t = 3$ s

(c) the values of t when the body is at rest.

Solution:

Differentiating to obtain formulae for velocity and acceleration:

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = 20t^2 - 23t + 6$$

and

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = 40t - 23$$

Evaluating these at the specific times required:

(a) Initially (at $t = 0$):

$$v(t = 0) = 20(0)^2 - 23(0) + 6 = 6\ m/s$$

$$a(t = 0) = 40(0) - 23 = -23\ m/s^2$$

(b) When $t = 3$ s:

$$v(t = 3) = 20(3)^2 - 23(3) + 6 = 117\ m/s$$

$$a(t = 3) = 40(3) - 23 = 97\ m/s^2$$

(c) When the body is at rest, it has zero velocity. Thus, set $v = 0$ and solve for t:

$$20t^2 - 23t + 6 = 0$$

Solving this quadratic equation where $a = 20$, $b = -23$ and $c = 6$ using the quadratic formula:

$$\begin{aligned} t &= \frac{-(-23) \pm \sqrt{(-23)^2 - 4(20)(6)}}{2(20)} \\ &= \frac{23 \pm \sqrt{529 - 480}}{40} \\ &= \frac{23 \pm \sqrt{529 - 480}}{40} \\ &= \frac{23 \pm 7}{40} \\ &= 0.75\text{s} \quad \text{and} \quad 0.4\text{s} \end{aligned}$$

6. A manufacturer develops a formula to determine the demand for its product depending on the price in dollars. The formula is:

$$D = 2000 + 120P - 6P^2$$

where P is the price per unit, and D is the number of units in demand.

What is the maximum demand possible?

Solution:

Begin by differentiating D with respect to P:

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}P} &= \frac{\mathrm{d}}{\mathrm{d}P}\left(2000 + 120P - 6P^2\right) \\ &= 120 - 12P\end{aligned}$$

Then set this equal to zero and solve for P in order to locate the price that gives a stationary point of demand:

$$\frac{\mathrm{d}D}{\mathrm{d}P} = 0$$

$$\therefore\ 120 - 12P = 0$$

$$\therefore\ P = \frac{120}{12} = 10$$

So a price of \$10 gives a stationary point with demand of:

$$\begin{aligned}D(P = 10) &= 2000 + 120(10) - 6(10)^2 \\ &= 2000 + 1200 - 600 \\ &= 2600 \text{ units}\end{aligned}$$

Finally, we must apply the second derivative test to confirm that this stationary point is a maximum in particular:

$$\begin{aligned}\frac{\mathrm{d}^2 D}{\mathrm{d}P^2} &= \frac{\mathrm{d}}{\mathrm{d}P}\big(120 - 12P\big)\\ &= -12\end{aligned}$$

So the second derivative has a constant value of -12 regardless of the value of P.

Thus, this is also true at the stationary point in particular:

$$\left.\frac{\mathrm{d}^2 D}{\mathrm{d}P^2}\right|_{P=10} = -12 < 0$$

So the second derivative test confirms that we obtain maximum demand of $D = 2600$ units at a price of $P = \$10$.

7. The downward deflection y of a cantilever of length L with a load W is given by the equation:

$$y = \frac{W}{6EI}(3L^2x - x^3)$$

where the constants E and I are related to the physical properties of the cantilever.

Find the maximum deflection as x varies.

Solution:

Expanding the brackets for y:

$$y = \frac{WL^2}{2EI}x - \frac{W}{6EI}x^3$$

Then differentiate w.r.t. x to obtain the first derivative. The key here is that x is the only variable - L, W, E and I are all constants:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{WL^2}{2EI}x - \frac{W}{6EI}x^3\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{WL^2}{2EI}x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W}{6EI}x^3\right) \\
&= \frac{WL^2}{2EI} - \frac{W}{2EI}x^2
\end{aligned}$$

Alternatively, rather than expanding the brackets at all, we could have used the linearity of differentiation to remove the *constant factor* from inside the derivative:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W}{6EI}(3L^2x - x^3)\right) \\
&= \frac{W}{6EI} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(3L^2x - x^3) \\
&= \frac{W}{6EI}(3L^2 - 3x^2)
\end{aligned}$$

But remember that this is only possible because that fraction is a constant factor!

To locate the stationary points, set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and solve for x:

$$\frac{WL^2}{2EI} - \frac{W}{2EI}x^2 = 0$$

$$\therefore \frac{W}{2EI}x^2 = \frac{WL^2}{2EI}$$

$$\therefore x^2 = L^2$$

Taking the square root of both sides:

$$\begin{aligned} x &= \pm\sqrt{L^2} \\ &= \pm L \end{aligned}$$

So there are two values of x where the deflection has a stationary point. Then calculating the value of y at each of these:

$$y(x = +L) = \frac{W}{6EI}(3L^2L - L^3) = \frac{WL^3}{3EI}$$

and

$$y(x = -L) = \frac{W}{6EI}(3L^2(-L) - (-L)^3) = \frac{-WL^3}{3EI}$$

Assuming that W, L, E, I are *positive* constants, it seems clear that the maximum value of deflection y will occur at $x = +L$.

However, to rigorously confirm the nature of the two stationary points, we need to calculate the second derivative:

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W}{6EI}\big(3L^2 - 3x^2\big)\right) \\
&= \frac{W}{6EI} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\big(3L^2 - 3x^2\big) \\
&= \frac{W}{6EI} \cdot \big(0 - 6x\big) \\
&= -\frac{W}{EI}x
\end{aligned}$$

Evaluating the second derivative at $x = +L$:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=+L} = -\frac{WL}{EI} < 0 \qquad \therefore \text{ Maximum}$$

and at $x = -L$:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=-L} = \frac{WL}{EI} > 0 \qquad \therefore \text{ Minimum}$$

So maximum deflection of $y = \dfrac{WL^3}{3EI}$ occurs when $x = +L$.

8. While on annual leave, Gavin holds up an armoured car, and attempts to flee the scene using a jetpack of his own creation. In the initial movement, the vertical height s (metres) of the jetpack at time t (seconds) is given by the function:

$$s(t) = 2t^3 + t^2$$

It is estimated that Gavin needs to achieve an acceleration of at least 40m/s^2 after 3 seconds, in order to escape the large nets employed by the security guards.

Does he succeed?

Solution:

The criteria for his escape concerns his acceleration at a certain time. Acceleration is the second derivative of displacement:

$$\begin{aligned} a(t) &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(6t^2 + 2t\big) \\ &= 12t + 2 \end{aligned}$$

So evaluating after three seconds:

$$\begin{aligned} a(t=3) &= 12(3) + 2 \\ &= 38\, m/s^2 \\ &< 40\, m/s^2 \end{aligned}$$

So he does *not* escape.

9. A mechanical component vibrates such that its vertical displacement y (m) at time t (s) obeys:

$$y(t) = t^3 - 9t^2 + 24t + 5$$

valid for the range $0 < t < 10$.

(i) Determine the value(s) of displacement that occur at stationary points(s) within this range.

(ii) Are the values identified in part (i) local maxima or minima?

(iii) Are the values identified in part (i) the most extreme that occur in the range? Plot the function $y(t)$ in EXCEL to find out.

Solution:

(i) Obtain the first derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 18t + 24$$

Then setting this equal to zero and solving for t to locate stationary points:

$$\begin{aligned} y' = 0 \quad &\Longrightarrow \quad 3t^2 - 18t + 24 = 0 \\ \therefore \quad & \quad t^2 - 6t + 8 = 0 \\ \therefore \quad & \quad (t-2)(t-4) = 0 \\ \therefore \quad & \quad t_1 = 2 \quad t_2 = 4 \end{aligned}$$

Both of these are within the range specified $0 < t < 10$. Evaluating the displacement at these:

$$\begin{aligned} y(t_1) &= (2)^3 - 9(2)^2 + 24(2) + 5 \\ &= 8 - 36 + 48 + 5 \\ &= 25\ \mathrm{m} \end{aligned}$$

and

$$
\begin{aligned}
y(t_2) &= (4)^3 - 9(4)^2 + 24(4) + 5 \\
&= 64 - 9(16) + 96 + 5 \\
&= 69 + 96 - 144 \\
&= 21 \text{ m}
\end{aligned}
$$

(ii) The second derivative is:

$$
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(3t^2 - 18t + 24\right) = 6t - 18
$$

Evaluating at the location of each stationary point:

$$
\left.\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right|_{t=2} = 6(2) - 18 = -6 < 0
$$

So there is a local maximum of 25m at $t_1 = 2$.

$$
\left.\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right|_{t=4} = 6(4) - 18 = 6 > 0
$$

So there is a local minimum of 21m at $t_2 = 4$.

(iii) One way to check this is to evaluate the function at the endpoints:

$$
y(0) = 5 < 21
$$

and

$$
\begin{aligned}
y(10) &= (10)^3 - 9(10)^2 + 24(10) + 5 \\
&= 1000 - 900 + 240 + 5 \\
&= 345 > 25
\end{aligned}
$$

Hence, we see that at the endpoints, the values of the function are more extreme (both larger and smaller) than those at the stationary points. So the displacement at the stationary points are local extrema, but not global extrema. See the EXCEL plot to confirm this visually.

10. The displacement s cm of the end of a stiff spring at time t seconds is given by:

$$s = \mu\,\mathrm{e}^{-kt}\sin(2\pi f t)$$

Determine the velocity of the end of the spring after one second if $\mu = 2$, $k = 0.9$ and $f = 5$.

Solution:

Velocity is the time-derivative of displacement: $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$. As the displacement function is the product of an exponential function and a sinusoidal function, neither of which are constant, we will use the product rule to differentiate it. Note: As we will be using v for velocity, use u and w as the additional variables.

$$\text{Let } u = \mu\,\mathrm{e}^{-kt} \quad \text{and} \quad w = \sin(2\pi f t)$$

then $s = u \cdot v$, and differentiating each component with respect to time:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -k\mu\,\mathrm{e}^{-kt} \quad \text{and} \quad \frac{\mathrm{d}w}{\mathrm{d}t} = 2\pi f\cos(2\pi f t)$$

and combining them according to the product rule:

$$\begin{aligned}
v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} &= u \cdot \frac{\mathrm{d}w}{\mathrm{d}t} + w \cdot \frac{\mathrm{d}u}{\mathrm{d}t} \\
&= \big(\mu\,\mathrm{e}^{-kt}\big)\big(2\pi f\cos(2\pi f t)\big) + \big(\sin(2\pi f t)\big)\big(-k\mu\,\mathrm{e}^{-kt}\big) \\
&= \mathrm{e}^{-kt}\big(2\pi\mu f\cos(2\pi f t) - k\mu\sin(2\pi f t)\big)
\end{aligned}$$

Then evaluating velocity at time $t = 1$, and substituting in the values of the constants:

$$
\begin{aligned}
v(1) &= \mathrm{e}^{-(0.9)(1)}\left(2\pi(2)(5)\cos(2\pi(5)(1)) - (0.9)(2)\sin(2\pi(5)(1))\right) \\
&= \mathrm{e}^{-0.9}\left(20\pi\cos(10\pi) - 1.8\sin(10\pi)\right) \\
&= \mathrm{e}^{-0.9}\left(20\pi \times 1 - 1.8 \times 0\right) \\
&= 20\pi\,\mathrm{e}^{-0.9} \\
&= 25.55\,\mathrm{cm/s}
\end{aligned}
$$

Remember to use radians by default, so $\sin(10\pi) = 0$ and $\cos(10\pi) = 1$.

11. A missile fired from ground level rises x metres vertically upwards in t seconds and

$$x = 100t - \frac{25}{2}t^2$$

Find:

(a) The initial velocity of the missile;

(b) The time when the height of the missile is a maximum;

(c) The maximum height reached;

(d) The velocity with which the missile strikes the ground.

Solution:

(a) Velocity is the derivative with respect to time, of displacement:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(100t - \frac{25}{2}t^2\right) = 100 - 25t$$

Then evaluating at $t = 0$ yields the initial velocity:

$$v(t = 0) = 100 - 25(0) = 100\ \mathrm{m/s}$$

(b) The maximum height is achieved at a stationary point of height x. Physically, this is when the rocket stops gaining height, and stops momentarily, before falling back to the ground. Thus, set $v = 0$ and solve for t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 100 - 25t = 0 \quad \Longrightarrow \quad t = \frac{100}{25} = 4\ \mathrm{s}$$

And we use the second derivative test to confirm that this is a local maximum:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -25 \ \text{ at any value of time.}$$

Hence, at $t = 4$, we have $\ddot{x} < 0$ and thus it is a local maximum of height.

(c) Evaluate this maximum height at the stationary point (where the missile has zero velocity):

$$\begin{aligned} x(4) &= 100(4) - \frac{25}{2}(4)^2 \\ &= 400 - 25 \times 8 \\ &= 200\,\mathrm{m} \end{aligned}$$

We confirm that this is a positive value, and check that $x(0) = 0$, indicating that this really is the global maximum in the range under consideration.

(d) First we need to know the time at which the missile strikes the ground. When this happens, it has zero height, so set $x = 0$ and solve for t:

$$\begin{aligned} & 100t - \frac{25}{2}t^2 = 0 \\ \therefore \quad & 8t - t^2 = 0 \\ \therefore \quad & t(8 - t) = 0 \end{aligned}$$

So there are two solutions: the missile has zero height at the moment of launch ($t = 0$) and later at $t = 8$, which must be the moment it returns to the ground. Substitute this into the formula for velocity to determine the value at the moment of impact:

$$v(8) = 100 - 25(8) = -100\,\mathrm{m/s}$$

So it is travelling vertically downwards with speed of 100 m/s.

12. At any time t seconds, the displacement x metres of a particle moving in a straight line from a fixed point is given by:

$$x = 4t + \ln(1-t)$$

Determine the acceleration of the particle after 1.5 seconds.

Solution:

Acceleration is the second derivative w.r.t. time of displacement:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(4t + \ln(1-t)\big) = 4 - \frac{1}{1-t} = 4 - (1-t)^{-1}$$

Then to undertake the second derivative, we make a substitution of the inner function $u = 1 - t$. Hence $\dot{x} = 4 - u^{-1}$ and the derivatives are:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = u^{-2} \text{ and } \frac{\mathrm{d}u}{\mathrm{d}t} = -1$$

Then applying the chain rule:

$$\ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) \cdot \frac{\mathrm{d}u}{\mathrm{d}t} = -u^{-2} = -(1-t)^{-2}$$

So the formula for acceleration of the particle at time t seconds is:

$$a(t) = \frac{-1}{(1-t)^2}$$

Evaluating at 1.5 seconds:

$$a(1.5) = \frac{-1}{(1-1.5)^2} = \frac{-1}{(-1/2)^2} = \frac{-1}{1/4} = -4\,\mathrm{ms}^{-2}$$

So it accelerating in the reverse direction at that time.