Railway Engineering Mathematics
Tutorial Sheet 13
Solutions

1. Differentiate the following with respect to the appropriate variable, using either the product rule or the quotient rule as required:

   (a) $$y = 8x^5 \sin(2x)$$

   Solution:

   This consists of two simple non-constant functions of x multiplied together, and so differentiating it will require the product rule.

   $$\text{Let} \quad u = 8x^5 \quad \text{and} \quad v = \sin(2x), \quad \text{so that} \quad y = u \cdot v$$

   Then differentiating each term with respect to x:

   $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(8x^5\big) = 40x^4 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(2x)\big) = 2\cos(2x)$$

   Then substituting all four components (u, u', v, v') into the product rule:

   $$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (8x^5)\big(2\cos(2x)\big) + \big(\sin(2x)\big)(40x^4) \\
&= 16x^5\cos(2x) + 40x^4\sin(2x) \\
&= 8x^4\big(2x\cos(2x) + 5\sin(2x)\big)
\end{aligned}$$

(b) $$y = \mathrm{e}^{-2t}\cosh(7t)$$

Solution:

This will require the product rule, as y consists of a product of two simple functions of t.

Let $u = \mathrm{e}^{-2t}$ and $v = \cosh(7t)$

Differentiating each term with respect to t:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\,\mathrm{e}^{-2t}\,\big) = -2\,\mathrm{e}^{-2t} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(\cosh(7t)\big) = 7\sinh(7t)$$

Now we have obtained the four components (u and v and their derivatives) required, and substituting these into the product rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= u\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}u}{\mathrm{d}t} \\
&= \big(\,\mathrm{e}^{-2t}\,\big)\big(7\sinh(7t)\big) + \big(\cosh(7t)\big)\big(-2\,\mathrm{e}^{-2t}\,\big) \\
&= 7\,\mathrm{e}^{-2t}\sinh(7t) - 2\,\mathrm{e}^{-2t}\cosh(7t) \\
&= \mathrm{e}^{-2t}\big(7\sinh(7t) - 2\cosh(7t)\big)
\end{aligned}$$

(c) $$y = \frac{9x^{-3} + 27}{3\sin(6x)}$$

Solution:

The function $y(x)$ consists of a fraction (quotient) of two non-constant functions of x, so differentiating it will require the quotient rule.

$$\text{Let} \quad u = 9x^{-3} + 27 \quad \text{and} \quad v = 3\sin(6x), \quad \text{so that} \quad y = \frac{u}{v}$$

Differentiating each term with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -27x^{-4} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 18\cos(6x)$$

Then substituting these into the quotient rule and simplifying as much as possible:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \\
&= \frac{\big(3\sin(6x)\big)(-27x^{-4}) - (9x^{-3} + 27)\big(18\cos(6x)\big)}{\big(2\sin(6x)\big)^2} \\
&= \frac{-81x^{-4}\sin(6x) - 18(9x^{-3} + 27)\cos(6x)}{9\sin^2(6x)} \\
&= \frac{-9\big(9x^{-4}\sin(6x) + 2(9x^{-3} + 27)\cos(6x)\big)}{9\sin^2(6x)} \\
&= \frac{-9x^{-4}\sin(6x) - 18(x^{-3} + 3)\cos(6x)}{\sin^2(6x)}
\end{aligned}$$

(d) $$x = \frac{7+\cos(t)}{6t^3} - 5t^2 + 7t - 9$$

Solution:

Differentiating the first term, which we label x_1, will require using the quotient rule:

$$\text{Let} \quad x_1 = \frac{7+\cos(t)}{6t^3}$$

Then, let:

$$u = 7 + \cos(t) \quad \text{and} \quad v = 6t^3$$

such that

$$x_1 = \frac{u}{v}$$

Differentiating both u and v with respect to t:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\sin(t) \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = 18t^2$$

Substituting these into the quotient rule to obtain the derivative of the term x_1:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}t} - u\dfrac{\mathrm{d}v}{\mathrm{d}t}}{v^2} \\
&= \frac{(6t^3)\big(-\sin(t)\big) - \big(7+\cos(t)\big)(18t^2)}{(6t^3)^2} \\
&= \frac{-6t^3\sin(t) - 18t^2(7+\cos(t))}{36t^6} \\
&= \frac{-6t^2\big(t\sin(t) + 3(7+\cos(t))\big)}{6t^2(6t^4)} \\
&= \frac{-\big(t\sin(t) + 3(7+\cos(t))\big)}{6t^4}
\end{aligned}$$

Thus the derivative of the full original function x is:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{7+\cos(t)}{6t^3} - 5t^2 + 7t - 9\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(x_1\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(5t^2\big) + \frac{\mathrm{d}}{\mathrm{d}t}\big(7t\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(9\big) \\
&= \frac{-\big(t\sin(t) + 3(7+\cos(t))\big)}{6t^4} - 2 \times 5t^{2-1} + 7 - 0 \\
&= \frac{-\big(t\sin(t) + 3(7+\cos(t))\big)}{6t^4} - 10t + 7
\end{aligned}$$

Or:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{6}\big(21 + 3\cos(t) + t\sin(t)\big)t^{-4} - 10t + 7$$

(e) $\Delta = (12t^3 - 5t + 3)(3\cos(4t) + 8t)$

Solution:

Δ consists of two functions of t multiplied together, and thus will require the product rule.

Let $u = 12t^3 - 5t + 3$ and $v = 3\cos(4t) + 8t$

Differentiating each term with respect to t:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 36t^2 - 5 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -12\sin(4t) + 8$$

Then substituting u, v and their derivatives into the product rule:

$$\begin{aligned}
\frac{\mathrm{d}\Delta}{\mathrm{d}t} &= u\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}u}{\mathrm{d}t} \\
&= (12t^3 - 5t + 3)(-12\sin(4t) + 8) + (3\cos(4t) + 8t)(36t^2 - 5) \\
&= -12\sin(4t)\big(12t^3 - 5t + 3\big) + 3\cos(4t)\big(36t^2 - 5\big) + 8\big(48t^3 - 10t + 3\big)
\end{aligned}$$

(f) $$Q = 4\sqrt{T} + \frac{6\,\mathrm{e}^{-3T}}{8T+9}$$

Solution:

Differentiating the second term, which we label Q_1, will require using the quotient rule:

$$\text{Let} \quad Q_1 = \frac{6\,\mathrm{e}^{-3T}}{8T+9}$$

Then, let:

$$u = 6\,\mathrm{e}^{-3T} \qquad \text{and} \qquad v = 8T+9$$

Such that:

$$Q_1 = \frac{u}{v}$$

Differentiating both u and v with respect to T:

$$\frac{\mathrm{d}u}{\mathrm{d}T} = -18\,\mathrm{e}^{-3T} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}T} = 8$$

Then substituting these into the quotient rule to determine the derivative of the second term Q_1:

$$\begin{aligned}
\frac{\mathrm{d}Q_1}{\mathrm{d}T} &= \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}T} - u\dfrac{\mathrm{d}v}{\mathrm{d}T}}{v^2} \\
&= \frac{(8T+9)(-18\,\mathrm{e}^{-3T}) - (6\,\mathrm{e}^{-3T})(8)}{(8T+9)^2}
\end{aligned}$$

Simplifying as much as possible:

$$
\begin{aligned}
\frac{\mathrm{d}Q_1}{\mathrm{d}T} &= \frac{-18(8T+9)\,\mathrm{e}^{-3T} - 48\,\mathrm{e}^{-3T}}{(8T+9)^2} \\
&= \frac{-144T\,\mathrm{e}^{-3T} - 162\,\mathrm{e}^{-3T} - 48\,\mathrm{e}^{-3T}}{(8T+9)^2} \\
&= \frac{-144T\,\mathrm{e}^{-3T} - 210\,\mathrm{e}^{-3T}}{(8T+9)^2} \\
&= \frac{-6\,\mathrm{e}^{-3T}\left(24T+35\right)}{(8T+9)^2}
\end{aligned}
$$

Thus the derivative of the full original function is:

$$
\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}T} &= \frac{\mathrm{d}}{\mathrm{d}T}\left(4\sqrt{T} + \frac{6\,\mathrm{e}^{-3T}}{8T+9}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}T}(4\sqrt{T}) + \frac{\mathrm{d}}{\mathrm{d}T}(Q_1) \\
&= 2T^{-\frac{1}{2}} - \frac{6\,\mathrm{e}^{-3T}\left(24T+35\right)}{(8T+9)^2} \\
&= \frac{2}{\sqrt{T}} - 6\,\mathrm{e}^{-3T}\,\frac{\left(24T+35\right)}{(8T+9)^2}
\end{aligned}
$$

(g) $$Z = (2x^3 - 5)\,\mathrm{e}^{-7x}$$

Solution:

This will require the product rule.

$$\text{Let} \quad u = 2x^3 - 5 \quad \text{and} \quad v = \mathrm{e}^{-7x}$$

Differentiating each with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x^2 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -7\,\mathrm{e}^{-7x}$$

Then substituting these four components into the product rule and simplifying:

$$\begin{aligned}
\frac{\mathrm{d}Z}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (2x^3 - 5)(-7\,\mathrm{e}^{-7x}) + (\mathrm{e}^{-7x})(6x^2) \\
&= -7(2x^3 - 5)\,\mathrm{e}^{-7x} + 6x^2\,\mathrm{e}^{-7x} \\
&= \mathrm{e}^{-7x}\left(-14x^3 + 6x^2 + 35\right)
\end{aligned}$$

(h) $$y = \frac{5 - 6\,\mathrm{e}^{-7x}}{10x - \cos\left(\dfrac{8x}{3}\right)}$$

Solution:

This will require the quotient rule.

Let $u = 5 - 6\,\mathrm{e}^{-7x}$ and $v = 10x - \cos\left(\dfrac{8x}{3}\right)$

Differentiating both u and v with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 42\,\mathrm{e}^{-7x} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 10 + \frac{8}{3}\sin\left(\frac{8x}{3}\right)$$

Then substituting these into the quotient rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \\
&= \frac{\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)\left(42\,\mathrm{e}^{-7x}\right) - \left(5 - 6\,\mathrm{e}^{-7x}\right)\left(10 + \dfrac{8}{3}\sin\left(\dfrac{8x}{3}\right)\right)}{\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)^2} \\
&= \frac{42\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)\mathrm{e}^{-7x} - \left(5 - 6\,\mathrm{e}^{-7x}\right)\left(10 + \dfrac{8}{3}\sin\left(\dfrac{8x}{3}\right)\right)}{\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)^2} \\
&= \frac{126\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)\mathrm{e}^{-7x} - 2\left(5 - 6\,\mathrm{e}^{-7x}\right)\left(15 + 4\sin\left(\dfrac{8x}{3}\right)\right)}{3\left(10x - \cos\left(\dfrac{8x}{3}\right)\right)^2}
\end{aligned}$$

2. Determine the gradient of:

(a) $$y = \frac{2x^3 - 5x + 7}{5\,\mathrm{e}^{8x}} \quad \text{at} \quad x = 3.5$$

Solution:

This will require the quotient rule.

$$\text{Let} \quad u = 2x^3 - 5x + 7 \quad \text{and} \quad v = 5\,\mathrm{e}^{8x}$$

Differentiating each term:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x^2 - 5 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 40\,\mathrm{e}^{8x}$$

Then substituting these into the quotient rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \\
&= \frac{(5\,\mathrm{e}^{8x})(6x^2 - 5) - (2x^3 - 5x + 7)(40\,\mathrm{e}^{8x})}{(5\,\mathrm{e}^{8x})^2} \\
&= \frac{5(6x^2 - 5)\,\mathrm{e}^{8x} - 40(2x^3 - 5x + 7)\,\mathrm{e}^{8x}}{25\,\mathrm{e}^{16x}} \\
&= \frac{5\,\mathrm{e}^{8x}}{5\,\mathrm{e}^{8x}}\left[\frac{6x^2 - 5 - 8(2x^3 - 5x + 7)}{5\,\mathrm{e}^{8x}}\right] \\
&= \frac{-16x^3 + 6x^2 + 40x - 61}{5\,\mathrm{e}^{8x}}
\end{aligned}$$

Then substituting in $x = 3.5$ to evaluate the gradient at that point:

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3.5} &= \frac{-16(3.5)^3 + 6(3.5)^2 + 40(3.5) - 61}{5\,\mathrm{e}^{8\times 3.5}} \\
&= -7.38 \times 10^{-11} \quad \text{(3 s.f.)}
\end{aligned}$$

(b) $$y = (8x^2 - 5x + 7)\ln(4x) + 9x \quad \text{at} \quad x = 7$$

Solution:

Differentiating the first term will require the product rule.

$$\text{Let} \quad y_1 = (8x^2 - 5x + 7)\ln(4x)$$

Then, let:

$$u = 8x^2 - 5x + 7 \quad \text{and} \quad v = \ln(4x), \quad \text{such that} \quad y_1 = u \cdot v$$

Differentiating both u and v with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 16x - 5 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

Then substituting these four components into the product rule to determine the derivative of the first term y_1:

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (8x^2 - 5x + 7)\frac{1}{x} + (\ln(4x))(16x - 5) \\
&= \frac{8x^2 - 5x + 7}{x} + (16x - 5)\ln(4x)
\end{aligned}$$

Thus the derivative of the full function y is given by:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(y_1\big) + \frac{\mathrm{d}}{\mathrm{d}x}\big(9x\big) \\
&= \frac{8x^2 - 5x + 7}{x} + (16x - 5)\ln(4x) + 9
\end{aligned}$$

Finally, substitute in $x = 7$ to determine the value of the gradient at that point:

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=7} &= \frac{8(7)^2 - 5 \times 7 + 7}{7} + (16 \times 7 - 5)\ln(4 \times 7) + 9 \\
&= 417.5 \quad \text{(1 d.p.)}
\end{aligned}$$