Integration by Substitution

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 16

Learning Outcomes

- Solve more challenging integrals using the technique **integration by substitution**.

Integration Using Substitution

Integrations of the form:

$$\int (5-x)^3 \, \mathrm{d}x$$

$$\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x$$

$$\int \frac{8x}{4x^2-3} \, \mathrm{d}x$$

require a substitution to be made before we can evaluate them.

Integration Using Substitution

This is somewhat equivalent to the *chain rule* in differentiation.

We choose the interior component and call that a new variable (usually u unless that has already been used in the problem). We then convert *everything in the integral* to be in terms of this new variable *only*.

If this was the right technique to employ, the new version of the integral will be something easier that we can deal with using standard techniques.

$$\text{Hard integral about } x \xrightarrow{\text{substitution}} \text{Easy integral about } u$$

Integration Using Substitution

Given a complicated integral in terms of variable x...

General steps:

1. Define the substitution: what is u as a function of x?
2. Calculate the derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ and re-arrange to make $\mathrm{d}x$ the subject.
3. [If definite integration:] convert the limits from x to u.
4. Make all the substitutions!
5. Do we have a simple integral that is solely in terms of the new variable u? If so, evaluate it. Otherwise, try a different substitution for u, or a different technique.

Example 1 (I/III)

Determine $\int 7(2-6x)^5 \, \mathrm{d}x$ using substitution.

First, let $u = 2 - 6x$ (usually the object inside the brackets).

Next, we rewrite as: $\int 7u^5 \, \mathrm{d}x$.

The problem here is that we are trying to integrate the expression $7u^5$ with respect to x, so we must next deal with the $\mathrm{d}x$ part.

As $u = 2 - 6x$, we can differentiate this to give $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -6$.
Rearranging this gives $\dfrac{\mathrm{d}u}{-6} = \mathrm{d}x$

Example 1 (II/III)

To summarise, we are trying to determine $\int 7(2-6x)^5 \, \mathrm{d}x$, and have $u = 2-6x$ and $\frac{\mathrm{d}u}{-6} = \mathrm{d}x$.

Substituting both in:

$$\int 7u^5 \, \frac{\mathrm{d}u}{-6} \qquad \text{or rather,} \qquad \int -\frac{7}{6}u^5 \, \mathrm{d}u$$

This is now a simple integral *entirely in terms of* u (no x left over!) and we can evaluate it.

Example 1 (III/III)

$$\int -\frac{7}{6}u^5 \, \mathrm{d}u = -\frac{7}{6}\frac{u^6}{6} + C$$

$$= -\frac{7u^6}{36} + C$$

To finish the integral, substitute the u back to obtain the final answer in terms of the original variable x:

$$\int 7(2-6x)^5 \, \mathrm{d}x = -\frac{7(2-6x)^6}{36} + C$$

Example 2

Evaluate the following definite integral using substitution:

$$\int_0^1 (5-x)^3 \, \mathrm{d}x$$

With definite integrals, we will also have to substitute the *limits*!

Example 2 - Solution (I/II)

First, we make the following substitution:

$$u = 5 - x$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -1 \quad \text{so, rearranging gives:} \quad \mathrm{d}x = -\mathrm{d}u$$

The limits are in terms of x, so they *also* need to be converted to corresponding limits for u:

$$x = 0 \quad \Longrightarrow \quad u = 5 - (0) = 5$$

$$x = 1 \quad \Longrightarrow \quad u = 5 - (1) = 4$$

Example 2 - Solution (II/II)

Now substitute all of these into the integral:

$$\int_{x=0}^{x=1} (5-x)^3 \; \mathrm{d}x = \int_{u=5}^{u=4} u^3(-1) \; \mathrm{d}u = -\int_{u=5}^{u=4} u^3 \; \mathrm{d}u$$

So now this is a simple integral entirely in terms of u:

$$\begin{aligned} -\int_{u=5}^{u=4} u^3 \; \mathrm{d}u &= -\left[\frac{1}{4}u^4\right]_5^4 \\ &= -\left\{\left(\frac{1}{4}(4)^4\right) - \left(\frac{1}{4}(5)^4\right)\right\} \\ &= 92.25 \end{aligned}$$

Integration Using Substitution - Extra rules

We can make use of the following general results:

$$\int \frac{f'(x)}{f(x)}\ \mathrm{d}x = \ln|f(x)| + C$$

and

$$\int f'(x)\,[f(x)]^n\ \mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

and

$$\int f'(x)\ g\left(f(x)\right)\ \mathrm{d}x = \int g(u)\,\mathrm{d}u$$

Example 3 (I/II)

For example, one could evaluate the integral:

$$\int \frac{14x + 3}{7x^2 + 3x - 2} \, \mathrm{d}x$$

by recognising that it is of the form:

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln |f(x)| + C$$

So we can instantly see that the result is just the log of the denominator:

$$\therefore \int \frac{14x + 3}{7x^2 + 3x - 2} \, \mathrm{d}x = \ln |7x^2 + 3x - 2| + C$$

Example 3 (II/II)

We could alternatively use the full substitution method. In this case, choose the more complicated function (the denominator) to substitute:

$$u = 7x^2 + 3x - 2$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 14x + 3 \quad \text{so, rearranging:} \quad \mathrm{d}x = \frac{\mathrm{d}u}{14x + 3}$$

Substituting in, the $14x + 3$ on the numerator is cancelled out:

$$\therefore \quad \int \frac{14x + 3}{u} \cdot \frac{\mathrm{d}u}{14x + 3} = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln|7x^2 + 3x - 2| + C$$

Exercises

Integrate the following:

1) $$\int \frac{\tau}{3} \sin\left(\tau^2\right)\, \mathrm{d}\tau$$

2) $$\int (6x+7)(3x^2+7x-8)^5\, \mathrm{d}x$$

3) $$\int \frac{8x}{4x^2-3}\, \mathrm{d}x$$

Exercises - Solutions (I/VI)

1) $\displaystyle\int \frac{\tau}{3} \sin\left(\tau^2\right)\, \mathrm{d}\tau$

Let:

$$u = \tau^2$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 2\tau \quad \text{so, rearranging:} \quad \mathrm{d}\tau = \frac{\mathrm{d}u}{2\tau}$$

Substituting both in, notice that the extra τ's cancel. If this didn't happen, we could not proceed!

$$\int \frac{\tau}{3} \sin\left(\tau^2\right)\, \mathrm{d}\tau \;=\; \int \frac{\tau}{3} \sin(u)\, \frac{\mathrm{d}u}{2\tau} \;=\; \frac{1}{6} \int \sin(u)\, \mathrm{d}u$$

Exercises - Solutions (II/VI)

Due to the cancellation, this is now a simple integral in terms of u only:

$$\begin{aligned}\frac{1}{6}\int \sin(u)\ \mathrm{d}u &= \frac{1}{6}\big(-\cos(u)\big)+C\\ &= -\frac{1}{6}\cos(u)+C\\ &= -\frac{1}{6}\cos\left(\tau^2\right)+C\end{aligned}$$

Being sure to give the final answer in terms of τ and not u.

Exercises - Solutions (III/VI)

2) $$\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x$$

It's not always obvious, but let's try the "biggest" internal function:

$$u = 3x^2 + 7x - 8$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x + 7 \quad \text{so, rearranging:} \quad \mathrm{d}x = \frac{\mathrm{d}u}{6x+7}$$

Notice that, again, when we make the substitutions the remaining x term is perfectly cancelled away! If this didn't happen, we would have to reconsider our method - either the wrong choice of u, or a different approach may be required altogether.

Exercises - Solutions (IV/VI)

$$\begin{aligned}
\int (6x+7)(3x^2+7x-8)^5 \, \mathrm{d}x &= \int (6x+7)u^5 \, \frac{\mathrm{d}u}{6x+7} \\
&= \int u^5 \, \mathrm{d}u \\
&= \frac{1}{6}u^6 + C \\
&= \frac{1}{6}\left(3x^2+7x-8\right)^6 + C
\end{aligned}$$

Exercises - Solutions (V/VI)

3) $\int \frac{8x}{4x^2 - 3} \, \mathrm{d}x$

Usually if there are fractions, try substituting the denominator:

$$\text{Let} \quad u = 4x^2 - 3$$

Then differentiate it:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 8x \quad \text{so, rearranging:} \quad \mathrm{d}x = \frac{\mathrm{d}u}{8x}$$

Yet again, this will cancel the x on the numerator, giving an integral only involving u - so this is definitely the right method.

Exercises - Solutions (VI/VI)

$$\begin{aligned}
\int \frac{8x}{4x^2-3}\,\mathrm{d}x &= \int \frac{8x}{u}\cdot\frac{1}{8x}\,\mathrm{d}u \\
&= \int \frac{1}{u}\,\mathrm{d}u \\
&= \ln|u| + C \\
&= \ln|4x^2-3| + C
\end{aligned}$$

Or we could have solved this instantly (but still explaining our working) by recognising that, with $f(x) = 4x^2 - 3$, this has form:

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$