Triangle Geometry and Introduction to Trigonometry

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 9

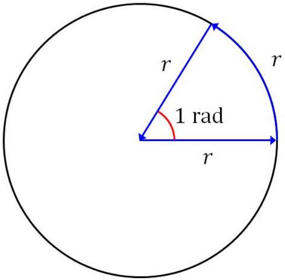
Learning Outcomes

- Introducing trigonometric functions and radian measure.
- Calculate lengths and angles using right-angled triangle rules.
- Rules for calculating lengths and angles in more general triangles.
- General form of trigonometric functions.

Degrees and Radians

Degrees are a measure of angle, where one full rotation is equivalent to an angle of 360^o.

Radians are more commonly used throughout mathematics except when discussing angles of shapes. One radian is the angle between two radii that create a circular arc with length equal to the radius:



Degrees and Radians

Since the circumference of a circle has a length of 2π radii, there must be 2π radians in a full rotation.

Therefore:

Radians - degrees exchange rate

$$2\pi \text{ rad} = 360^o$$

and also

$$1 \text{ rad} = 57.3^o \text{ to 1 d.p.}$$

Radians	Degrees
0	0
$\frac{\pi}{2}$	90
π	180
$\frac{3\pi}{2}$	270
2π	360
1	57.3

The trigonometric functions

Trigonometric functions take an angle (usually measured in radians) as their input.

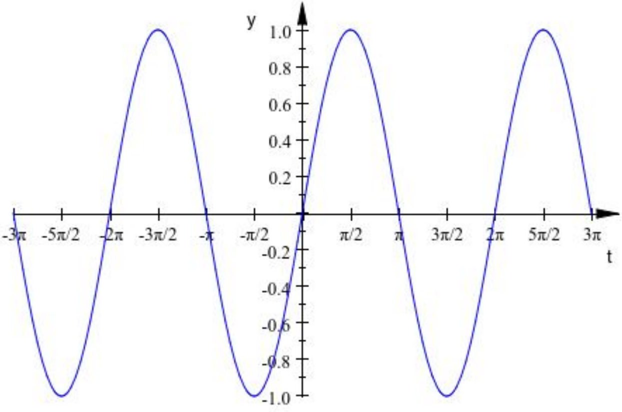
They are **periodic**, meaning that they repeat a pattern indefinitely.

There are three main functions of this kind, and we need to be familiar with their appearance:

- $y = \sin(t)$
- $y = \cos(t)$
- $y = \tan(t)$

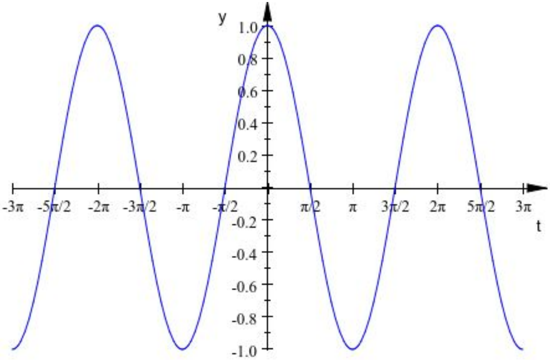
Sketching $\sin(t)$

$\sin(t) = 0$ when $t = 0$. It varies between -1 and 1 and takes 2π rad (or 360^o) to complete one full cycle.



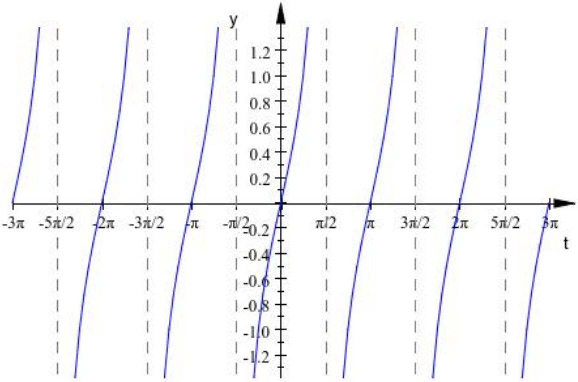
Sketching $\cos(t)$

$\cos(t) = 1$ when $t = 0$. It varies between -1 and 1 and takes 2π rad (or 360^o) to complete one full cycle. It is identical to sine, but shifted left by $\pi/2$ radians.

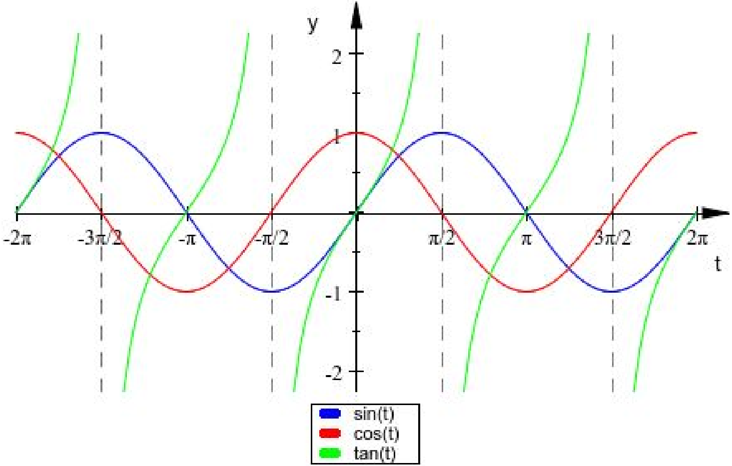


Sketching $\tan(t)$

$\tan(t) = 0$ when $t = 0$. It possesses asymptotes at $t = \pm\frac{\pi}{2}$ rad (or $\pm 90^o$) and takes π rad (or 180^o) to complete one full cycle.

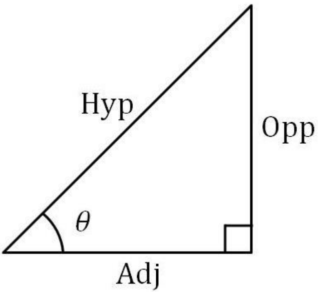


Sketching Sinusoids - summary



Trigonometric Ratios: Right-angled Triangles

You may have encountered the trigonometric rules of a right-angled triangle:



Right-angled triangle trigonometric rules: SOH-CAH-TOA

$$\sin(\theta) = \frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$$

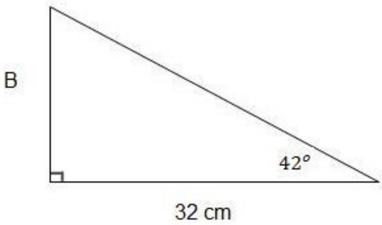
$$\cos(\theta) = \frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$$

$$\tan(\theta) = \frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$$

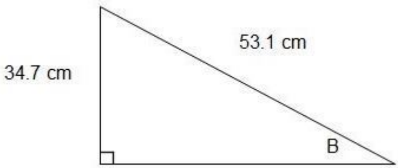
We can use these ratios to find unknown angles or side lengths.

Example 1

Calculate B in each of the following right-angled triangles:

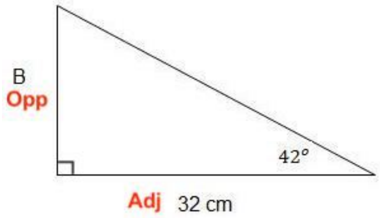


(a)



(b)

Example 1 - Solution (a)



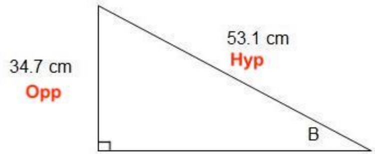
As the two sides of interest are the **o**pposite and the **a**djacent, we can use the **t**angent ratio (because of **TOA**):

$$\tan(42^\circ) = \frac{\text{Opp}}{\text{Adj}} = \frac{B}{32}$$

Transposing:

$$B = 32\tan(42^\circ) = 28.81\text{cm} \quad \text{(2 d.p.)}$$

Example 1 - Solution (b)



As the two sides of interest are the **o**pposite and the **h**ypotenuse, we can use the **s**ine ratio (because of **SOH**):

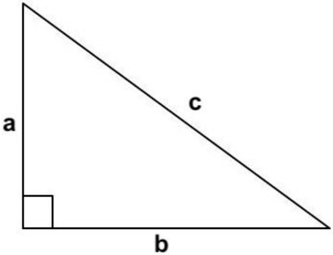
$$\sin(B^\circ) = \frac{\text{Opp}}{\text{Hyp}} = \frac{34.7}{53.1} = 0.65348\ldots$$

Using the inverse sine (or *arcsin*):

$$B = \sin^{-1}(0.65348) = 40.80^\circ \quad \text{(2 d.p.)}$$

Pythagoras' Theorem

In a right-angled triangle we can also calculate the length of a side, if the other two sides are known, using the **Pythagorean theorem**:



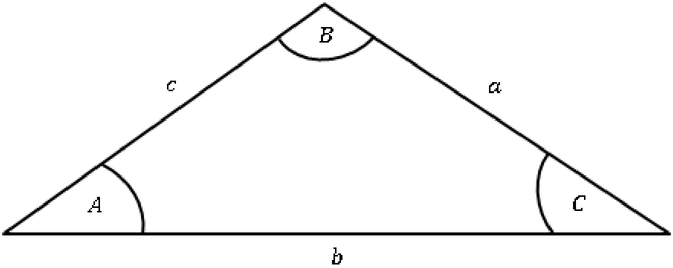
Pythagoras' Theorem

$$a^2 + b^2 = c^2$$

where c is the length of the hypotenuse.

Trigonometric Ratios: Non-right-angled Triangles

More general rules relate the sides and angles of non-right angled triangles:



There are two rules that can be used depending on what information we have and what requires calculation.

The Sine Rule

Sine rule:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{1}$$

Or, equivalently

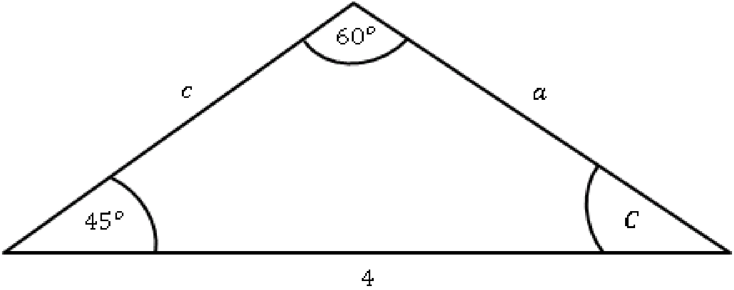
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} \tag{2}$$

These are equivalent, but it is easier to use equation (1) if a side is the unknown and equation (2) if an angle is the unknown.

Note: in order to use the Sine Rule, a complete pair must be known, i.e. a and A (a side and the angle facing it)

Example 2 (I/IV)

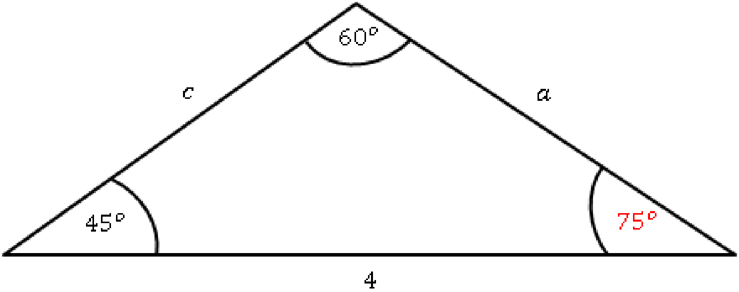
Find a, c and C.



First, calculate C by the fact that angles in a triangle sum to 180^o

$\therefore C = 180 - 60 - 45 = 75^o$

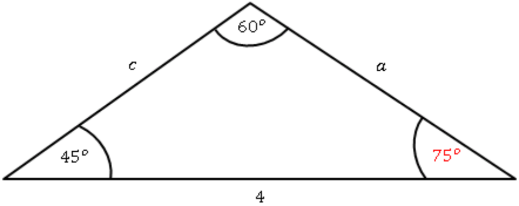
Example 2 (II/IV)



We have a complete pair (4 and 60^o), so we can use the Sine rule.

Let's use it to calculate a, as we also have the angle opposing it.

Example 2 (III/IV)



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

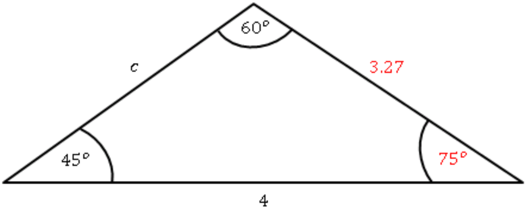
$$\frac{a}{\sin(45)} = \frac{4}{\sin(60)}$$

$$\therefore \quad a = \sin(45) \times \frac{4}{\sin(60)}$$

$$\therefore \quad a = 3.27 \text{ to 2 d.p.}$$

Example 2 (IV/IV)

Now for c:



$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

$$\frac{4}{\sin(60)} = \frac{c}{\sin(75)}$$

$$\therefore \quad c = \sin(75) \times \frac{4}{\sin(60)}$$

$$\therefore \quad c = 4.46 \text{ to 2 d.p.}$$

The Cosine Rule

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\,\cos(A) \tag{3}$$

Or

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} \tag{4}$$

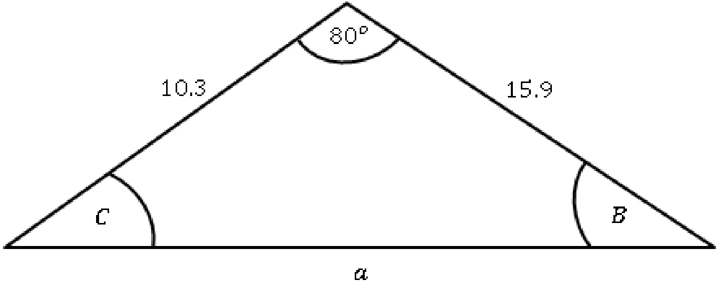
Equation (3) would be used to calculate a side and equation (4) would be used to calculate an angle.

Note: we use the Cosine Rule when we know either:

- **all three sides** a, b, c or
- **two sides** b, c **and the angle** A **inbetween them.**

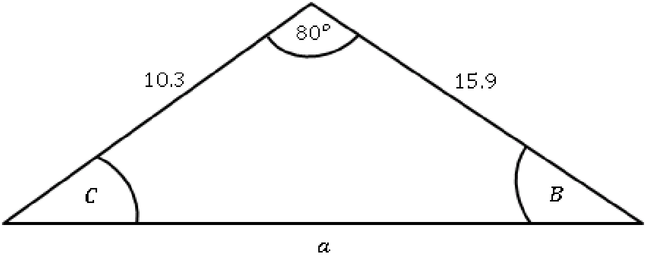
Example 3

Determine a:



Here, we don't have a complete pair so we cannot use the Sine rule. However, we *do* know two sides and the angle (80°) inbetween, so we can use the Cosine Rule.

Example 3 - Solution



$$a^2 = b^2 + c^2 - 2bc\,\cos(A)$$

$$\therefore \qquad a^2 = 10.3^2 + 15.9^2 - 2 \times 10.3 \times 15.9 \times \cos(80)$$

$$\therefore \qquad a^2 = 302.0233$$

$$\therefore \qquad a = 17.38 \text{ to 2 d.p.}$$

The General Form

It is common in engineering to encounter quantities that vary in a sinusoidal fashion over time t (e.g. alternating currents). If y was such a quantity, we could say that:

General wave equation/sinusoidal function:

$$y = A\sin(\omega t + \phi) + B$$

where A is the **amplitude**, B is the **vertical shift**, ω is the **angular frequency** and ϕ is the **phase shift**.

Amplitude A

Amplitude is the maximum vertical displacement of the wave from its equilibrium (central position).

Changing A has the visible effect of stretching the wave in the vertical axis.

- The standard sine wave $y = \sin(x)$ varies between -1 and $+1$, so the amplitude is exactly 1.
- $y = 2\sin(x)$ varies between -2 and $+2$, so the amplitude is 2.
- As $y = \sin(x) + 3$ varies between 2 and 4, its amplitude is still just 1.

Vertical shift B

The **vertical shift** (also known as the mean value) describes how much the wave is shifted up by, compared to the standard wave. We can find it by locating the vertical mean value of the graph.

- The standard sine wave $y = \sin(x)$ varies between -1 and +1, so the vertical shift is 0.
- $y = 2\sin(x)$ varies between -2 and +2, so the vertical shift is still $\dfrac{-2+2}{2} = 0$.
- $y = \sin(x) + 3$ varies between 2 and 4, so in this case the vertical shift is $\dfrac{2+4}{2} = 3$.

Phase shift ϕ

The **phase shift** describes how much the graph is shifted to the left by, compared to the standard sine wave.

For example, $y = \sin(x + 0.1)$ crosses the x-axis at $x = -0.1$, so the phase shift is 0.1 and the entire curve is shifted 0.1 to the left.

However, its effect is influenced by the angular frequency, as we shall see...

Angular frequency ω

The **angular frequency** is the number of times the wave repeats in a distance of 2π along the horizontal axis. If the period (wavelength) is T the angular frequency is given by:

$$\omega = \frac{2\pi}{T}$$

The frequency is the number of complete wavelengths per unit along the horizontal axis: $f = T^{-1}$

If the horizontal axis is time (seconds), then the frequency f units are Hertz (cycles/second) and the angular frequency is measured in radians/second.

Angular frequency ω

Frequency and period formulae:

$$T = \frac{2\pi}{\omega} \qquad T = \frac{1}{f} \qquad \omega = 2\pi f$$

The angular frequency has the visual effect of compressing the wave along the horizontal axis, so that the new period is given by $T = \dfrac{2\pi}{\omega}$ rather than 2π.

We said earlier that the phase shift ϕ moves the graph horizontally. Combined with the effect of the angular frequency, together the graph moves right by an amount $-\phi/\omega$