Railway Engineering Mathematics
Tutorial Sheet 21
Solutions

Given the following matrices:

$$\underline{A} = \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 6 & 1 & 9 \\ -2 & 0 & -8 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \qquad \underline{E} = \begin{pmatrix} 7 & -6 \\ -9 & 2 \\ 5 & -3 \end{pmatrix}$$

Determine if the following operations are possible, and if so then evaluate them:

1. $\underline{A} + \underline{D}$
2. $\underline{D} - \underline{A}$
3. $2\underline{A} - 5\underline{D}$
4. $\underline{A} + \underline{C}$
5. $\underline{B} + \underline{C}$
6. $3\underline{C} - \underline{B}$
7. $\underline{AB}$
8. $\underline{AD}$
9. $\underline{DA}$
10. $\underline{CD}$
11. $\underline{EA}$
12. $\underline{BE}$

Solution:

1. Matrices $\underline{A}$ and $\underline{D}$ have the same order (both 2×2) so they can be added together, resulting in another 2×2 matrix:

$$\begin{aligned}
\underline{A} + \underline{D} &= \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} + \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 7 + (-5) & 2 + (-4) \\ -3 + 1 & 8 + 6 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 \\ -2 & 14 \end{pmatrix}
\end{aligned}$$

   We should always be able to simplify to the point of a single number for each element of the resulting matrix.

2. Similarly, as they have the exact same order we can obtain their difference:

$$\begin{aligned}
\underline{D} - \underline{A} &= \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} - \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} \\
&= \begin{pmatrix} -5 - 7 & -4 - 2 \\ 1 - (-3) & 6 - 8 \end{pmatrix} \\
&= \begin{pmatrix} -12 & -6 \\ 4 & -2 \end{pmatrix}
\end{aligned}$$

3. Multiplying each by a scalar does not affect the order, so this operation is also permissible as both $2\underline{A}$ and $5\underline{D}$ are still 2×2 matrices:

$$\begin{aligned}
2\underline{A} - 5\underline{D} &= 2\begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} - 5\begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 14 & 4 \\ -6 & 16 \end{pmatrix} - \begin{pmatrix} -25 & -20 \\ 5 & 30 \end{pmatrix} \\
&= \begin{pmatrix} 14-(-25) & 4-(-20) \\ -6-5 & 16-30 \end{pmatrix} \\
&= \begin{pmatrix} 39 & 24 \\ -11 & -14 \end{pmatrix}
\end{aligned}$$

4. $\underline{A}$ is a 2×2 matrix, while $\underline{C}$ is a 2×3 matrix. Therefore they do not have precisely the same order, and so they cannot be added. This is not a valid operation.

5. $\underline{B}$ and $\underline{C}$ have the same order, as both are 2×3 matrices. Therefore they can be added, yielding a new 2×3 matrix:

$$\begin{aligned}
\underline{B} + \underline{C} &= \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix} + \begin{pmatrix} 6 & 1 & 9 \\ -2 & 0 & -8 \end{pmatrix} \\
&= \begin{pmatrix} 4+6 & -1+1 & 7+9 \\ 5+(-2) & -3+0 & -2+(-8) \end{pmatrix} \\
&= \begin{pmatrix} 10 & 0 & 16 \\ 3 & -3 & -10 \end{pmatrix}
\end{aligned}$$

6. $3\underline{C}$ still has order 2×3 and so we can subtract $\underline{B}$ from it:

$$\begin{aligned}
3\underline{C} - \underline{B} &= 3\begin{pmatrix} 6 & 1 & 9 \\ -2 & 0 & -8 \end{pmatrix} - \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 18 & 3 & 27 \\ -6 & 0 & -24 \end{pmatrix} - \begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 18-4 & 3-(-1) & 27-7 \\ -6-5 & 0-(-3) & -24-(-2) \end{pmatrix} \\
&= \begin{pmatrix} 14 & 4 & 20 \\ -11 & 3 & -22 \end{pmatrix}
\end{aligned}$$

7. First, consider the order of each matrix to determine if $\underline{A} \times \underline{B}$ is a valid operation:

   $\underline{A}$ is $2 \times {\color{red}2}$ and $\underline{B}$ is ${\color{red}2} \times 3$

   As the number of columns of the first matrix matches the number of rows of the second (i.e. the two "middle" numbers in red text are equal), this is a valid matrix multiplication *in this ordering* (i.e. it does not necessarily imply that $\underline{B} \times \underline{A}$ is also valid - and in fact, it turns out that that multiplication is invalid!). The outer two numbers determine the order of the result: so we expect to obtain another 2×3 matrix.

   Then, following our procedure of matching the rows of the first matrix with columns of the second:

$$\begin{aligned}
\underline{A} \times \underline{B} &= \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} (7\times 4)+(2\times 5) & (7\times -1)+(2\times -3) & (7\times 7)+(2\times -2) \\ (-3\times 4)+(8\times 5) & (-3\times -1)+(8\times -3) & (-3\times 7)+(8\times -2) \end{pmatrix} \\
&= \begin{pmatrix} 38 & -13 & 45 \\ 28 & -21 & -37 \end{pmatrix}
\end{aligned}$$

8. Both matrix $\underline{A}$ and $\underline{D}$ have order 2×2, so when we write the orders side-by-side:

$$({\color{blue}2} \times {\color{red}2}) \quad \times \quad ({\color{red}2} \times {\color{blue}2})$$

we can easily see that the middle numbers (red) match, and the outer numbers (blue) indicate that the result will be a new 2×2 matrix.

$$\begin{aligned}
\underline{A} \times \underline{D} &= \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} (7 \times -5) + (2 \times 1) & (7 \times -4) + (2 \times 6) \\ (-3 \times -5) + (8 \times 1) & (-3 \times -4) + (8 \times 6) \end{pmatrix} \\
&= \begin{pmatrix} -33 & -16 \\ 23 & 60 \end{pmatrix}
\end{aligned}$$

9. As with the previous question, as both have order 2×2, our condition for a valid multiplication is met, and the result will again be a 2×2 matrix:

$$\begin{aligned}
\underline{D} \times \underline{A} &= \begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix} \\
&= \begin{pmatrix} (-5 \times 7) + (-4 \times -3) & (-5 \times 2) + (-4 \times 8) \\ (1 \times 7) + (6 \times -3) & (1 \times 2) + (6 \times 8) \end{pmatrix} \\
&= \begin{pmatrix} -23 & -42 \\ -11 & 50 \end{pmatrix}
\end{aligned}$$

We can see that this is *not the same* as the result we calculated for multiplying $\underline{A} \times \underline{D}$, even though both ways were valid. This is an example of the non-commutativity of matrix multiplication.

10. Writing the order of each matrix underneath:

$$\underline{C} \times \underline{D} = \underset{2 \times 3}{\begin{pmatrix} 6 & 1 & 9 \\ -2 & 0 & -8 \end{pmatrix}} \times \underset{2 \times 2}{\begin{pmatrix} -5 & -4 \\ 1 & 6 \end{pmatrix}}$$

The middle numbers (3 and 2) do **not** match - that is, the number of columns of the first matrix $\underline{C}$ is not equal to the number of rows of the second matrix $\underline{D}$. Therefore this matrix multiplication is not a valid operation and we can proceed no further.

11. Writing the order of each matrix underneath:

$$\underline{E} \times \underline{A} = \underset{3 \times 2}{\begin{pmatrix} 7 & -6 \\ -9 & 2 \\ 5 & -3 \end{pmatrix}} \times \underset{2 \times 2}{\begin{pmatrix} 7 & 2 \\ -3 & 8 \end{pmatrix}}$$

The number of columns of $\underline{E}$ matches the number of rows of $\underline{A}$, so $\underline{E} \times \underline{A}$ is a valid multiplication. From the outer numbers we deduce that it will yield a 3×2 matrix:

$$\begin{aligned} \underline{E} \times \underline{A} &= \begin{pmatrix} (7 \times 7) + (-6 \times -3) & (7 \times 2) + (-6 \times 8) \\ (-9 \times 7) + (2 \times -3) & (-9 \times 2) + (2 \times 8) \\ (5 \times 7) + (-3 \times -3) & (5 \times 2) + (-3 \times 8) \end{pmatrix} \\ &= \begin{pmatrix} 67 & -34 \\ -69 & -2 \\ 44 & -14 \end{pmatrix} \end{aligned}$$

12. Writing the order of each matrix underneath:

$$\underline{B} \times \underline{E} \quad = \quad \underset{3 \times 2}{\begin{pmatrix} 4 & -1 & 7 \\ 5 & -3 & -2 \end{pmatrix}} \quad \times \quad \underset{2 \times 2}{\begin{pmatrix} 7 & -6 \\ -9 & 2 \\ 5 & -3 \end{pmatrix}}$$

The number of columns of the first matrix and the number of rows of the second (the central numbers above) match, so this is a valid multiplication. The order of the resulting matrix is inherited from the number of rows of the first matrix and the number of columns of the second (i.e. the two outer numbers), hence we know that the result will be a 2×2 matrix:

$$\begin{aligned} \underline{B} \times \underline{E} \quad &= \quad \begin{pmatrix} (4 \times 7) + (-1 \times -9) + (7 \times 5) & (4 \times -6) + (-1 \times 2) + (7 \times -3) \\ (5 \times 7) + (-3 \times -9) + (-2 \times 5) & (5 \times -6) + (-3 \times 2) + (-2 \times -3) \end{pmatrix} \\ &= \quad \begin{pmatrix} 72 & -47 \\ 52 & -30 \end{pmatrix} \end{aligned}$$