Introduction to Matrices

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 21

Learning Outcomes

- Identify properties of matrices.
- Perform matrix arithmetic:
  - Addition
  - Subtraction
  - Multiplication (three kinds)

Introduction to Matrices

Matrices are used to handle many pieces of information at once, thereby lending themselves to the analysis of systems described by a set of similar equations. Such applications include:

- Circuit theory (sets of linear equations).
- Dynamical systems theory (sets of differential equations).
- Vector graphics and computer imaging.

Definitions and Notation

A matrix is a rectangular array of numbers, for example:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \\ 2 & -8 \end{pmatrix}$$

and are generally represented by a bold capital letter: **A**, **C**, **X** etc., or by underlining the letter, i.e. $\underline{A}$.

The dimensions (or "order") of a matrix are $m \times n$ where m is the number of rows and n is the number of columns. The matrix **A** is a 3×2 matrix.

Order

The **order** of a matrix is a description of its dimensions - the number of rows first, followed by the number of columns.

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$ This is a 2×1 matrix. It is also a *vector*.

$$\begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 4 \end{pmatrix}$$ This is a 2×3 matrix.

$$\begin{pmatrix} 2 & 0 & -1 & 6 \end{pmatrix}$$ This is a 1×4 matrix.

Definitions and Notation

The numbers that make up a matrix are called elements.

An element may be written a_{ij}.

The lowercase a indicates that this is an element of the matrix **A**.

The subscripts i and j refer to the row and column, respectively, in which the element a_{ij} is to be found. Rows are counted top$\rightarrow$bottom and columns left$\rightarrow$right.

Definitions and Notation

For example, if **C** is the 2×3 matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -8 \\ -3 & 2 & 5 \end{pmatrix}$$

then $c_{12} = 0$ and $c_{23} = 5$.

If the 2×2 matrix **D** possessed unspecified elements, we could write it as:

$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

Matrix Addition

It is only possible to add matrices together if they have *exactly* the same order, i.e. the same number of rows and the same number of columns.

We may therefore add a 4×2 matrix to another 4×2, but we cannot add it to a 2×2.

To perform matrix addition we simply add together the corresponding elements of each matrix.

Matrix Addition

For example, if:

$$\underline{E} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix}, \quad \underline{F} = \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix},$$

$$\underline{G} = \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} 0 & -7 & 2 \\ 9 & -1 & 6 \end{pmatrix}$$

then ...

Matrix Addition

$$
\begin{aligned}
\underline{E} + \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 5 + -1 & 3 + 5 \\ -7 + 8 & 2 + -4 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 8 \\ 1 & -2 \end{pmatrix}
\end{aligned}
$$

These could be added as both $\underline{E}$ and $\underline{F}$ are 2×2.

Matrix Addition - Exercises

Determine:

1) $\underline{G} + \underline{H}$

2) $\underline{E} + \underline{H}$

where, as before:

$$\underline{E} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix}, \quad \underline{G} = \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}, \quad \underline{H} = \begin{pmatrix} 0 & -7 & 2 \\ 9 & -1 & 6 \end{pmatrix}$$

Matrix Addition - Solutions

1) Both $\underline{G}$ and $\underline{H}$ have the same order (2×3), so we can proceed:

$$\begin{aligned}
\underline{G} + \underline{H} &= \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -7 & 2 \\ 9 & -1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} -9+0 & -7+(-7) & 2+2 \\ 1+9 & 3+(-1) & -2+6 \end{pmatrix} \\
&= \begin{pmatrix} -9 & -14 & 4 \\ 10 & 2 & 4 \end{pmatrix}
\end{aligned}$$

2) $\underline{E}$ is a 2×2 matrix, while $\underline{H}$ is a 2×3 matrix, so this addition is an invalid operation.

Matrix Subtraction

The same general rules concerning the addition of matrices also applies to subtraction i.e. the two matrices involved must be of the same dimension.

Of course, when subtracting matrices, corresponding elements will undergo a subtraction rather than an addition.

Note: just as in normal arithmetic, addition is **commutative** and subtraction is not, meaning that $\underline{A} + \underline{B} = \underline{B} + \underline{A}$, but it is not necessarily true that $\underline{A} - \underline{B} = \underline{B} - \underline{A}$.

Matrix Subtraction

For example, using the matrices defined earlier (**E**, **F**, **G** and **H**):

$$\begin{aligned} \underline{E} - \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 5 - -1 & 3 - 5 \\ -7 - 8 & 2 - -4 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -2 \\ -15 & 6 \end{pmatrix} \end{aligned}$$

Summary: Matrix Addition and Subtraction

Only matrices with the same order can be added/subtracted.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} \quad \text{INVALID}$$

$$\begin{aligned} \begin{pmatrix} 4 & -1 & -2 \\ -2 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 3 \\ 1 & -4 & -6 \end{pmatrix} &= \begin{pmatrix} 4-2 & -1-0 & -2-3 \\ -2-1 & 3-(-4) & 5-(-6) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & -5 \\ -3 & 7 & 11 \end{pmatrix} \end{aligned}$$

Matrix Multiplication

There are three types of matrix multiplication:

- Multiplication of a matrix by a scalar.
- Multiplication of elements in one matrix by corresponding elements in another matrix.
- Multiplication of a matrix by another matrix.

Scalar Multiplication

To multiply a matrix by a scalar (a real or complex *number*, rather than a vector or matrix) simply by multiplying ("scaling") each element of the matrix by that scalar.

Scalar Multiplication

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

Example: $$-3 \begin{pmatrix} 2 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \times 2 \\ -3 \times 8 \\ -3 \times -5 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \\ 15 \end{pmatrix}$$

Scalar Multiplication

For a further example:

$$
\begin{aligned}
5\underline{G} &= 5\begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 5\times -9 & 5\times -7 & 5\times 2 \\ 5\times 1 & 5\times 3 & 5\times -2 \end{pmatrix} \\
&= \begin{pmatrix} -45 & -35 & 10 \\ 5 & 15 & -10 \end{pmatrix}
\end{aligned}
$$

Element-wise Multiplication

Element-wise multiplication refers to the multiplying of corresponding elements in a pair of different matrices. Note that the matrices must have the *same order*. For example:

$$
\begin{aligned}
\underline{E} \,.\!\times \underline{F} &= \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} .\!\times \begin{pmatrix} -1 & 5 \\ 8 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 5 \times -1 & 3 \times 5 \\ -7 \times 8 & 2 \times -4 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 15 \\ -56 & -8 \end{pmatrix}
\end{aligned}
$$

Matrix Multiplication

Matrix multiplication is a **non-commutative** operation. This means that $\underline{A} \times \underline{B}$ is *not* equivalent to $\underline{B} \times \underline{A}$ and does not necessarily yield the same result.

The direction of matrix multiplication *can not be changed!*

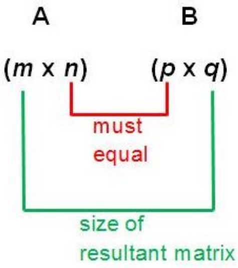
In fact, one direction might not even exist whilst the other does! It is therefore necessary to distinguish between "pre-multiplying" and "post-multiplying", e.g. given the matrices $\underline{X}$ and $\underline{A}$:

- We can pre-multiply $\underline{X}$ by $\underline{A}$ to get $\underline{AX}$
- or post-multiply to get $\underline{XA}$

Matrix Multiplication

As with addition/subtraction, multiplication can only be performed if the two matrices involved have acceptable dimensions.

Criterion: The number of columns in the first matrix must match the number of rows in the second.



Matrix Multiplication

If this is satisfied, the order of the result is given by the remaining dimensions - the same number of rows as the first matrix and columns as the second matrix.

Then to multiply the matrices, imagine setting the rows of the first upon the columns of the second. For example, when computing $\underline{AB}$, we find the element in the i^{th} row and the j^{th} column of $\underline{AB}$ by multiplying the i^{th} row of $\underline{A}$ by the j^{th} column of $\underline{B}$.

$$\underset{2\times 2}{\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}} \quad \underset{2\times 1}{\begin{pmatrix}5\\ 6\end{pmatrix}} = \begin{pmatrix}1\times 5+2\times 6\\ 3\times 5+4\times 6\end{pmatrix} = \underset{2\times 1}{\begin{pmatrix}17\\ 39\end{pmatrix}}$$

Matrix Multiplication - Example

Calculate $\underline{E} \times \underline{G}$:

$$\begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \times \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$2 \times 2 \qquad\qquad 2 \times 3$$

The resultant matrix therefore exists and will be of the form:

$$\underline{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \end{pmatrix}$$

$$2 \times 3$$

Matrix Multiplication - Example

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & & \\ & & \end{pmatrix}$$

Matrix Multiplication - Example

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Matrix Multiplication - Example

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$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & \end{pmatrix}$$

Matrix Multiplication - Example

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$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & (-7 \times 2) + (2 \times -2) \end{pmatrix}$$

Matrix Multiplication - Example

$$\underline{E} \times \underline{G} = \begin{pmatrix} 5 & 3 \\ -7 & 2 \end{pmatrix} \begin{pmatrix} -9 & -7 & 2 \\ 1 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (5 \times -9) + (3 \times 1) & (5 \times -7) + (3 \times 3) & (5 \times 2) + (3 \times -2) \\ (-7 \times -9) + (2 \times 1) & (-7 \times -7) + (2 \times 3) & (-7 \times 2) + (2 \times -2) \end{pmatrix}$$

$$= \begin{pmatrix} -42 & -26 & 4 \\ 65 & 55 & -18 \end{pmatrix}$$

Matrix Multiplication - Exercises

1) Let,

$$\underline{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Calculate $\underline{BC}$ and $\underline{CB}$ if they exist.

2) Let,

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Calculate $\underline{AD}$ if it exists.

Matrix Multiplication: Exercise 1

$$\underline{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

As $\underline{B}$ is a 2×1 and $\underline{C}$ is a 2×2 matrix, $\underline{B} \times \underline{C}$ does not exist as the columns of $\underline{B}$ do not match the number of rows of $\underline{C}$.

However, $\underline{C} \times \underline{B}$ does exist, and the result will be another 2×1 matrix:

$$\underline{CB} = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times 3 + 2 \times -2 \\ 4 \times 3 + 5 \times -2 \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$$

Matrix Multiplication: Exercise 2

$$\underline{A} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \qquad \underline{D} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

The result $\underline{A} \times \underline{D}$ will be another 2×2 matrix:

$$\begin{aligned} \underline{AD} &= \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 0 \times 0 & 2 \times 1 + 0 \times (-2) \\ -1 \times 3 + 1 \times 0 & -1 \times 1 + 1 \times (-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 + 0 & 2 + 0 \\ -3 + 0 & -1 - 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -3 & -3 \end{pmatrix} \end{aligned}$$