Railway Engineering Mathematics

Tutorial Sheet 12

Solutions

1. Differentiate the following with respect to the appropriate variable:

(a) $$y = 6\sin(7x-3)$$

Solution:

We take the contents of the sine function as the inner function, and introduce a new variable to replace it. Let $u = 7x - 3$, then we can describe y as the outer function in terms of this new variable:

$$y = 6\sin(u)$$

So u is a function of x, and y is a function of u only. Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}u}\big(6\sin(u)\big) = 6\cos(u) \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(7x-3\big) = 7$$

Then applying the chain rule:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 6\cos(u) \times 7 \\ &= 42\cos(u) \end{aligned}$$

Finally, we must state this answer in terms of the original variable x by substituting back in $u = 7x - 3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 42\cos(7x-3)$$

(b) $$y = 3\sqrt[4]{9x+5}$$

Solution:

First, re-write this in index form:

$$y = 3(9x+5)^{\frac{1}{4}}$$

We will choose the contents of the brackets as the inner function, and replace it with a new variable. Let $u = 9x+5$, then we can write the original function in terms of this new variable:

$$y = 3u^{\frac{1}{4}}$$

Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}u}\left(3u^{\frac{1}{4}}\right) = \frac{3}{4}u^{-\frac{3}{4}} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\big(9x+5\big) = 9$$

Then applying the chain rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \frac{3}{4}u^{-\frac{3}{4}} \times 9 \\ &= \frac{27}{4}u^{-\frac{3}{4}}\end{aligned}$$

Replacing u with $9x+5$ for the final answer, and re-writing the power into a more presentational format using the rules of indices:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{27}{4}(9x+5)^{-\frac{3}{4}} \\ &= \frac{27}{4\sqrt[4]{(9x+5)^3}}\end{aligned}$$

(c) $$y = \left(4x^3 - 3x\right)^6$$

Solution:

Let $u = 4x^3 - 3x$, then rewrite the original function as the outer function in terms only of u:

$$y = u^6$$

Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 6u^5 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 12x^2 - 3$$

Then applying the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (12x^2 - 3) \times 6u^5 \\
&= 6(12x^2 - 3)u^5
\end{aligned}$$

Substituting back in $u = 4x^3 - 3x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6(12x^2 - 3)(4x^3 - 3x)^5$$

(d) $$y = 5\big(2x^2 + 7x - 1\big)^{-4}$$

Solution:

Let $u = 2x^2 + 7x - 1$, then rewrite the original function as:

$$y = 5u^{-4}$$

Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -20u^{-5} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x + 7$$

Then applying the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= -20u^{-5} \times (4x + 7) \\
&= -20(4x + 7)u^{-5}
\end{aligned}$$

Replacing u with $2x^2 + 7x - 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -20(4x + 7)(2x^2 + 7x - 1)^{-5}$$

(e) $$\gamma = \frac{7}{\sqrt{3t^2 + 6t - 9}}$$

Solution:

First, re-writing this function in index form:

$$\gamma = 7(3t^2 + 6t - 9)^{-\frac{1}{2}}$$

Let $u = 3t^2 + 6t - 9$, then rewrite the original function γ in terms of u:

$$\gamma = 7u^{-\frac{1}{2}}$$

Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u} = -\frac{7}{2}u^{-\frac{3}{2}} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 6t + 6$$

Then applying the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{\mathrm{d}\gamma}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t} \\
&= -\frac{7}{2}u^{-\frac{3}{2}} \times (6t + 6) \\
&= -\frac{7}{2}(6t + 6)u^{-\frac{3}{2}} \\
&= -\frac{7}{2}(6t + 6)(3t^2 + 6t - 9)^{-\frac{3}{2}} \\
&= -\frac{7}{2} \times 6(t + 1)(3t^2 + 6t - 9)^{-\frac{3}{2}} \\
&= -21(t + 1)(3t^2 + 6t - 9)^{-\frac{3}{2}} \\
&= \frac{-21(t + 1)}{\sqrt{(3t^2 + 6t - 9)^3}}
\end{aligned}$$

(f) $\psi = -3\,\mathrm{e}^{6x^2-1}$

Solution:

We choose the contents of the index as the inner function. Thus, let $u = 6x^2 - 1$, then rewrite the original function in terms of u as:

$$\psi = -3\,\mathrm{e}^{u}$$

Differentiating the outer and inner functions with respect to their appropriate variables:

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = -3\,\mathrm{e}^{u} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 12x$$

Then applying the chain rule:

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}x} &= \frac{\mathrm{d}\psi}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= -3\,\mathrm{e}^{u} \times 12x \\ &= -36x\,\mathrm{e}^{u} \end{aligned}$$

Finally, substituting back in $u = 6x^2 - 1$ to obtain the final solution in terms of the original variable x:

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = -36x\,\mathrm{e}^{6x^2-1}$$

(g) $$\theta = 7 + 3\sinh(6r^2 - 7r + 9) - 8r + \frac{5}{6r^3}$$

Solution:

Let's begin by re-writing the final term in index form. Also, note that only the sinh term will require the use of the chain rule.

$$\theta = 7 + 3\sinh(6r^2 - 7r + 9) - 8r + \frac{5}{6}r^{-3}$$

Let $u = 6r^2 - 7r + 9$, then rewrite the second term of θ, which we name θ_c, in terms of this new variable u, as:

$$\theta_c = 3\sinh(u)$$

So we are only considering the part of the original function that requires the chain rule for now.

Differentiating both the outer function $\theta_c(u)$ and the inner function $u(r)$ with respect to their appropriate variables:

$$\frac{\mathrm{d}\theta_c}{\mathrm{d}u} = 3\cosh(u) \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}r} = 12r - 7$$

Then substituting both into the chain rule:

$$\begin{aligned} \frac{\mathrm{d}\theta_c}{\mathrm{d}r} &= \frac{\mathrm{d}\theta_c}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}r} \\ &= 3\cosh(u) \times (12r - 7) \\ &= 3(12r - 7)\cosh(u) \end{aligned}$$

Replacing u with $6r^2 - 7r + 9$:

$$\frac{\mathrm{d}\theta_c}{\mathrm{d}r} = 3(12r - 7)\cosh(6r^2 - 7r + 9)$$

Now we can return to the original function θ. Hence, if we differentiate each of its constituent terms and use our chain rule result for the second term:

$$\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(7 + 3\sinh(6r^2 - 7r + 9) - 8r + \frac{5}{6}r^{-3}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}r}(7) + \frac{\mathrm{d}\theta_c}{\mathrm{d}r} - \frac{\mathrm{d}}{\mathrm{d}r}(8r) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{5}{6}r^{-3}\right) \\
&= 0 + 3(12r - 7)\cosh(6r^2 - 7r + 9) - 8 + (-3) \times \frac{5}{6}r^{-3-1} \\
&= 3(12r - 7)\cosh(6r^2 - 7r + 9) - 8 - \frac{5}{2}r^{-4}
\end{aligned}$$

(h) $$\Delta = -4(5t^2-6)^3 + 18\sqrt{t} - \frac{2\sin(2t-8)}{3}$$

Solution:

Note that we will require the chain rule for the first and third terms, which we label Δ_1 and Δ_2 respectively.

First, re-writing the second and final terms to make applying the differentiation rules easier:

$$\Delta = -4(5t^2-6)^3 + 18t^{\frac{1}{2}} - \frac{2}{3}\sin(2t-8)$$

and

$$\Delta_1 = -4(5t^2-6)^3, \qquad \Delta_2 = -\frac{2}{3}\sin(2t-8)$$

For Δ_1, let the inner function be $u = 5t^2 - 6$. Then we can write the outer function in terms of u as:

$$\Delta_1 = -4u^3$$

Differentiating both the inner and outer functions of Δ_1 with respect to their appropriate variables:

$$\frac{\mathrm{d}\Delta_1}{\mathrm{d}u} = -12u^2 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 10t$$

Then applying the chain rule and obtaining the answer solely in terms of t:

$$\begin{aligned}
\frac{\mathrm{d}\Delta_1}{\mathrm{d}t} &= \frac{\mathrm{d}\Delta_1}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t} \\
&= -12u^2 \times 10t \\
&= -120tu^2 \\
&= -120t(5t^2-6)^2
\end{aligned}$$

For Δ_2, let the inner function be $v = 2t - 8$. Then we can write the outer function as:

$$\Delta_2 = -\frac{2}{3}\sin(v)$$

Differentiating both the inner and outer functions of Δ_2 with respect to their appropriate variables:

$$\frac{\mathrm{d}\Delta_2}{\mathrm{d}v} = -\frac{2}{3}\cos(v) \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = 2$$

Then applying the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}\Delta_2}{\mathrm{d}t} &= \frac{\mathrm{d}\Delta_2}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}t} \\
&= -\frac{2}{3}\cos(v) \times 2 \\
&= -\frac{4}{3}\cos(v)
\end{aligned}$$

Substituting $v = 2t - 8$ back in:

$$\frac{\mathrm{d}\Delta_2}{\mathrm{d}t} = -\frac{4}{3}\cos(2t - 8)$$

Combining these terms and the derivative of the middle term, overall we obtain:

$$\begin{aligned}
\frac{\mathrm{d}\Delta}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(- 4(5t^2 - 6)^3 + 18t^{\frac{1}{2}} - \frac{2}{3}\sin(2t - 8)\Big) \\
&= \frac{\mathrm{d}\Delta_1}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\Big(18t^{\frac{1}{2}}\Big) + \frac{\mathrm{d}\Delta_2}{\mathrm{d}t} \\
&= -120t(5t^2 - 6)^2 + 9t^{-\frac{1}{2}} - \frac{4}{3}\cos(2t - 8)
\end{aligned}$$

2. Determine the gradient of:

(a) $$y = 4.5\,\mathrm{e}^{3x+1} \quad \text{at} \quad x = -0.04$$

Solution:

Choose the function in the index as the inner function to be replaced by a new variable. Let $u = 3x + 1$, then we can write the outer function in terms of this new variable:

$$y = 4.5\,\mathrm{e}^{u}$$

Differentiating both functions with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 4.5\,\mathrm{e}^{u} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

Then applying the chain rule and obtaining the solution in terms of the original variable x:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 4.5\,\mathrm{e}^{u} \times 3 \\
&= 13.5\,\mathrm{e}^{u} \\
&= 13.5\,\mathrm{e}^{3x+1}
\end{aligned}$$

Then substituting in $x = -0.04$ and evaluating the derivative to determine the specific value of the gradient at that point:

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-0.04} &= 13.5\,\mathrm{e}^{3(-0.04)+1} \\
&= 32.55 \quad (2\text{ d.p})
\end{aligned}$$

(b) $y = 5\cos^4(9x) \quad \text{at} \quad x = 2.5$

Solution:

First, to make our notation slightly clearer let's re-write this as:

$$y = 5\big(\cos(9x)\big)^4$$

Now, let $u = \cos(9x)$, then we can write the outermost function in terms of this new variable:

$$y = 5u^4$$

Differentiating both with respect to their appropriate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 20u^3 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -9\sin(9x)$$

Then applying the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 20u^3 \times -9\sin(9x) \\
&= -180u^3\sin(9x) \\
&= -180\sin(9x)\big(\cos(9x)\big)^3 \\
&= -180\sin(9x)\cos^3(9x)
\end{aligned}$$

Finally, substituting in $x = 2.5$ to determine the value of the gradient at that point:

$$\begin{aligned}
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2.5} &= -180\sin(9 \times 2.5)\big(\cos(9 \times 2.5)\big)^3 \\
&= -58.41 \quad \text{(2 d.p)}
\end{aligned}$$