Trigonometric Functions and Equations

Railway Engineering Mathematics

Sheffield Hallam University

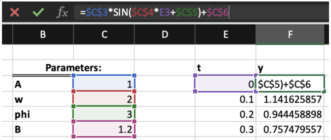
Lecture 10

Learning Outcomes

- Plot general trigonometric functions using EXCEL.
- Identify the formula of a general trig. function from a plot.
- Solve equations involving trigonometric functions.

EXCEL

We can use EXCEL to easily plot a general trig. function with any values of the parameters A, ω, ϕ, B:

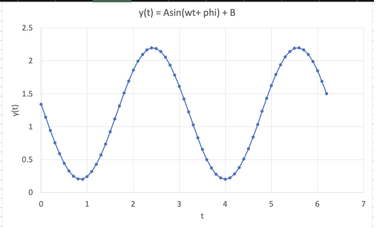


=$C$3*SIN($C$4*E3+$C$5)+$C$6

B	C	D	E	F
Parameters:			t	y
A	1		0	$C$5)+$C$6
w	2		0.1	1.141625857
phi	3		0.2	0.944458898
B	1.2		0.3	0.757479557

(a) Formula

t	y
0	1.34112
0.1	1.1416259
0.2	0.9444589
0.3	0.7574796
0.4	0.5881421
0.5	0.4431975
0.6	0.3284242
0.7	0.2483979
0.8	0.206309
0.9	0.2038354
1	0.2410757
1.1	0.3165453
1.2	0.4272355
1.3	0.5687334
1.4	0.7353978
1.5	0.9205845
1.6	1.1169106
1.7	1.3165492
1.8	1.5115414
1.9	1.6941134
2	1.8569866
2.1	1.9936679
2.2	2.0987081
2.3	2.1679197
2.4	2.1985433
2.5	2.1893582



(b) Plot

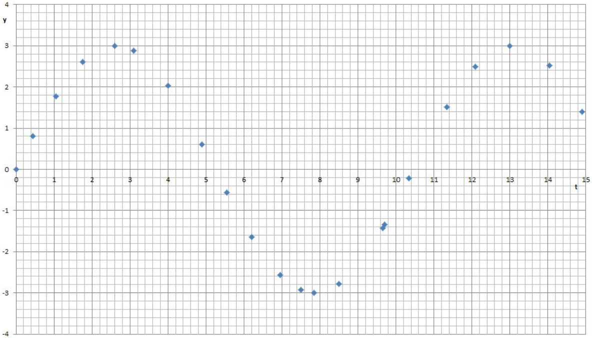
Determining equation from graph

Engineers may encounter sampled data from a periodic signal such as a current or audio signal, and wish to determine the sinusoidal function that matches it. In practice, this means determining the best values of the parameters A, ω, ϕ, B in the general form $y(t) = A\sin(\omega t + \phi) + B$. To do this:

1. Find the maximum and minimum values of the wave. The amplitude A is half the distance between these, and the vertical shift B is their average.
2. Measure the period T, and determine angular frequency from $\omega = 2\pi/T$.
3. Substitute in the values of a point on the plot and solve for ϕ.

Example: Determining equation from graph

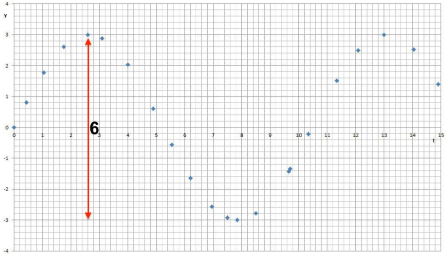
Determine the equation of the curve:



Procedure: Deal with A and B first, then ω, and finally ϕ .

Solution: Determining equation from graph (I/IV)

(i) Amplitude A & vertical shift B



The total vertical displacement between a peak and a trough is:

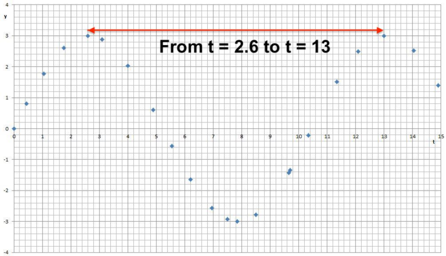
$$3 - (-3) = 6$$

so the amplitude is half this: $A = 3$

As the wave oscillates between -3 and 3, the mean height is zero and so the vertical shift is $B = 0$.

Solution: Determining equation from graph (II/IV)

(ii) Angular frequency ω



The horizontal distance between two neighbouring peaks is:

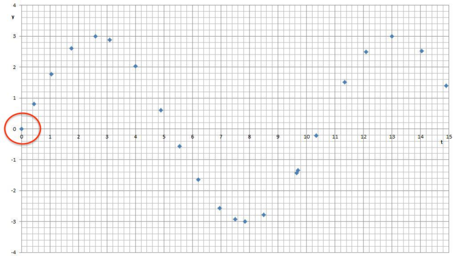
$$13 - 2.6 = 10.4$$

So the period is $T = 10.4$, and the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{10.4} = 0.60 \;\; \text{(2 d.p.)}$$

Solution: Determining equation from graph (III/IV)

(iii) Phase shift ϕ



The graph goes through the point $(0, 0)$, so substituting this in to the equation so far to find ϕ:

$$y = 3\sin(0.60t + \phi)$$

we obtain:

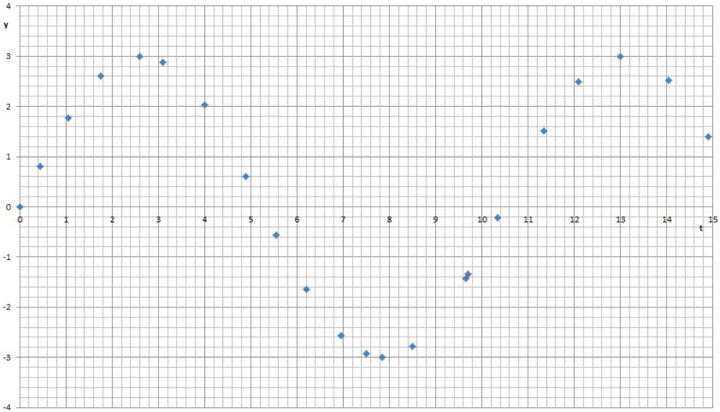
$$0 = 3\sin(0.60 \times 0 + \phi)$$

$$\therefore \quad 0 = \sin(\phi)$$

$$\therefore \quad \phi = \sin^{-1}(0)$$

$$\therefore \quad \phi = 0$$

Solution: Determining equation from graph (IV/IV)



So the equation of this sinusoidal wave is:

$$y = 3\sin(0.60t)$$

Solving equations of the form $A\sin(\omega x + \phi) + B = C$

We have seen how to solve equations involving many kinds of functions (quadratics, logarithms, etc.). How do we solve equations involving trigonometric functions?

Due to the periodicity of sine and cosine, an equation of the type $\sin(x) = c$ will either have no solutions (as sine only takes a certain range of values) or infinitely-many solutions, unless we specify a restricted range of x that we are interested in.

For a general trigonometric equation $A\sin(\omega x + \phi) + B = C$, we follow a procedure to locate all solutions for x within a specified range.

Solving equations of the form $A\sin(\omega x + \phi) + B = C$

1. Define a new variable $u = \omega x + \phi$ to simplify the the trig. function to $\sin(u) = c$, where $c = (C - B)/A$.
2. Calculate the new range in terms of u, by substituting the limits of x into this formula.
3. Determine the set of solutions for u:
   - Use the inverse trigonometric function on your calculator to obtain the **principal value**:

     $$u_0 = \sin^{-1}(c)$$

     This is the first solution, and the value closest to the y-axis.
   - For sine and cosine, use the symmetry of the graph to locate the **other** solution that occurs within the first cycle. This often takes the form $u_1 = \pi - u_0$ for sine, and $u_1 = -u_0$ for cosine.

Solving equations of the form $A\sin(\omega t+\phi)+B=C$

- To find all of the other solutions for u, then:
    - For sine and cosine, add and subtract integer multiples of 2π to **both** u_0 and u_1 until we are outside of the stated range.
    - For tangent, add and subtract multiples of π to u_0.
    - Use high precision when calculating each solution, as errors may compound as we use u_0 to determine u_1 and then use that determine subsequent solutions.

4. Convert solutions for u back to the corresponding solutions for x using:

$$x=\frac{u-\phi}{\omega}$$

5. Verify the final solutions by substituting back into $A\sin(\omega x+\phi)+B$ and evaluating.

Example

Solve

$$\sin(3x + 0.2) = 0.5$$

for $-\pi \leq x \leq \pi$.

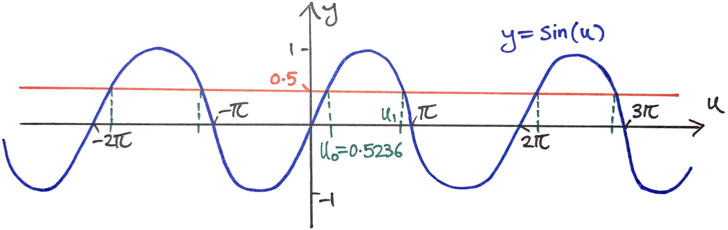
Solution (I/III)

Let $u = 3x + 0.2$, then the range is $-3\pi + 0.2 \leq u \leq 3\pi + 0.2$, or:

$$-9.2248 \leq u \leq 9.6248$$

Now the problem has been converted to:

"Solve $\sin(u) = 0.5$ for $-9.2248 \leq u \leq 9.6248$."



Solution (II/III)

Obtain the principal value:

$$u_0 = \sin^{-1}(0.5) = \frac{\pi}{6} = 0.5236$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.5236 = 2.6180$$

Adding and subtracting multiples of 2π, we find six solutions for u in the acceptable range. Then convert these back to solutions for x using:

$$x = \frac{u - 0.2}{3}$$

Solution (III/III)

u	In Range?	$x = \dfrac{u - 0.2}{3}$
$u_0 = 0.5236$	Yes	$(0.5236 - 0.2)/3 = 0.1079$
$u_0 + 2\pi = 0.5236 + 2\pi = 6.8068$	Yes	2.2022
$u_0 + 4\pi = 0.5236 + 4\pi = 13.090$	No	-
$u_0 - 2\pi = 0.5236 - 2\pi = -5.7596$	Yes	-1.9865
$u_0 - 4\pi = 0.5236 - 4\pi = -12.043$	No	-
$u_1 = 2.6180$	Yes	0.8060
$u_1 + 2\pi = 2.6180 + 2\pi = 8.9012$	Yes	2.9004
$u_1 - 2\pi = 2.6180 - 2\pi = -3.6652$	Yes	-1.2884
$u_1 - 4\pi = 2.6180 - 4\pi = -9.9484$	No	-

So we have six valid solutions:

$x = -1.288, \;\; -1.987, \;\; 0.108, \;\; 0.806, \;\; 2.202, \;\; 2.900$