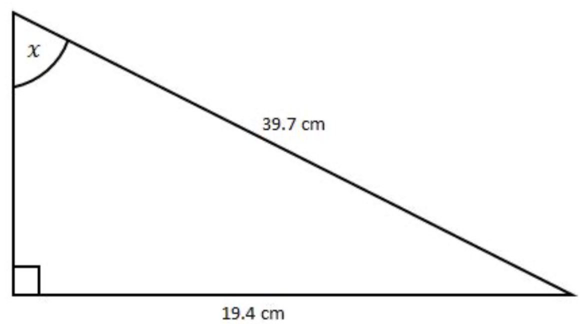
Railway Engineering Mathematics
Tutorial Sheet 9
Solutions

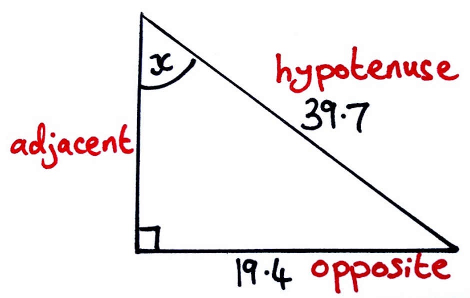
1. Calculate the value of x:

   (a)



Solution:

Label the sides of the triangle:



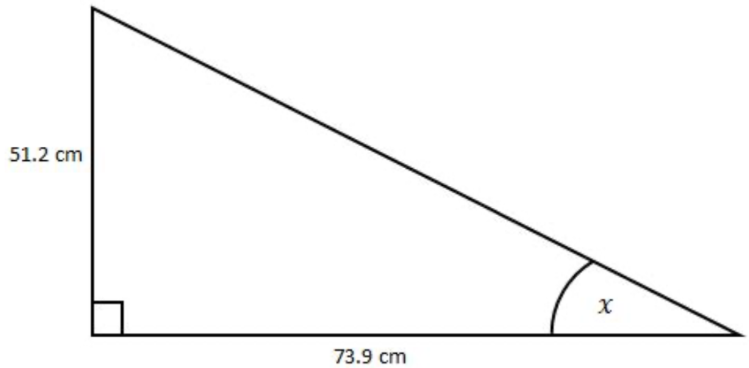
Using $SOH - CAH - TOA$:

$$\sin(x) = \frac{opposite}{hypotenuse} = \frac{19.4}{39.7}$$

Hence,

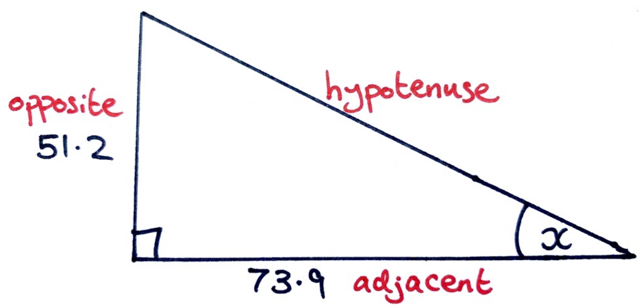
$$x = \sin^{-1}\left(\frac{19.4}{39.7}\right) = 29.25^{\circ}$$

(b)



Solution:

Label the sides of the triangle:



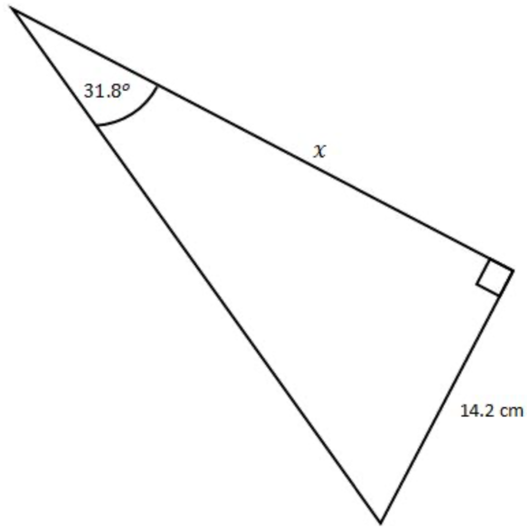
Then the two sides that we know are the opposite and the adjacent, so using $SOH - CAH - TOA$:

$$\tan(x) = \frac{opposite}{adjacent} = \frac{51.2}{73.9}$$

Hence, taking the inverse tan function to obtain x:

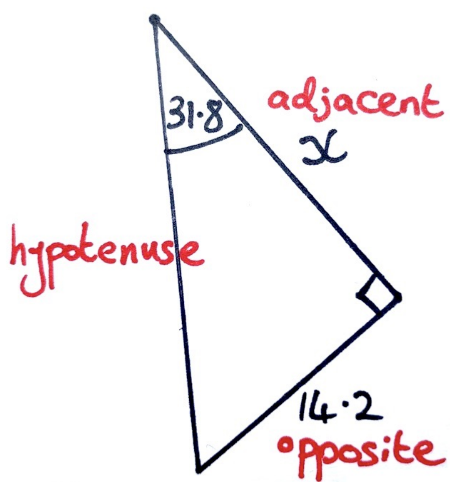
$$x = \tan^{-1}\left(\frac{51.2}{73.9}\right) = 34.72°$$

(c)



Solution:

Label the sides of the triangle:

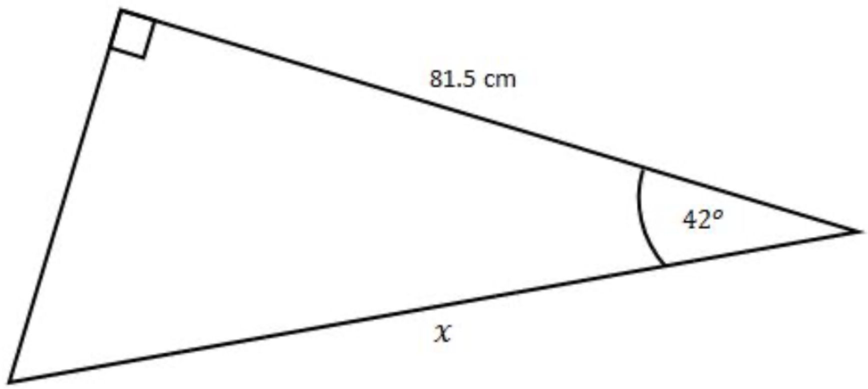


Using $SOH - CAH - TOA$:

$$\tan(31.8) = \frac{opposite}{adjacent} = \frac{14.2}{x}$$

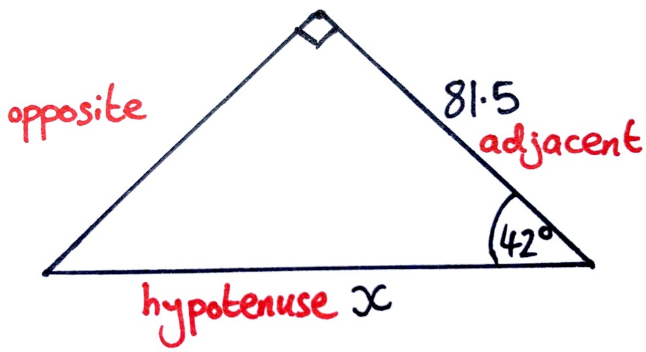
$$\therefore\ x = \frac{14.2}{\tan(31.8)} = 22.90 \text{ cm}$$

(d)



Solution:

Label the sides of the triangle:



So we want to find the hypotenuse, and know the adjacent. Thus, applying $SOH - CAH - TOA$:

$$\cos(42) = \frac{adjacent}{hypotenuse} = \frac{81.5}{x}$$

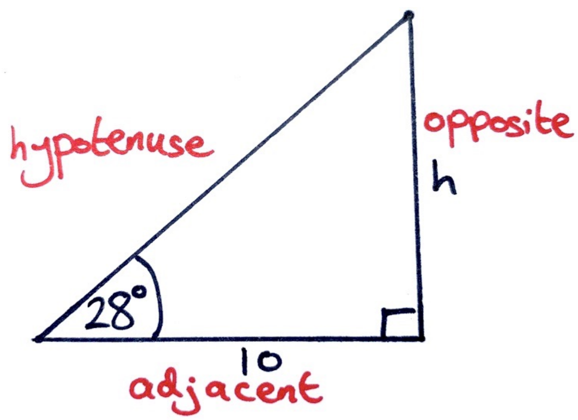
Hence, rearranging to make x the subject:

$$x = \frac{81.5}{\cos(42)} = 109.67 \text{ cm}$$

2. The angle of elevation of the top of a tree from a point on the ground 10m from the base of the tree is 28°. What is the height of the tree, to 1 decimal place?

Solution:

Let the unknown height of the tree be h. Then, begin by sketching the situation and labelling the sides of the resulting right-angled triangle:



Thus, we know the value of the adjacent and need to determine the opposite. Applying $SOH - CAH - TOA$:

$$\tan(28) = \frac{opposite}{adjacent} = \frac{h}{10}$$

Re-arranging to solve for h:

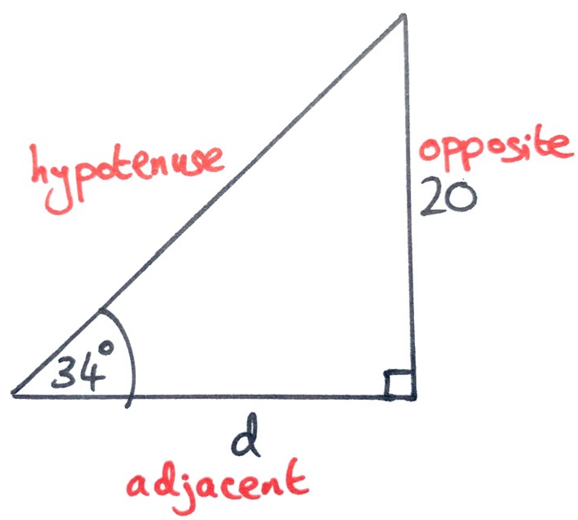
$$h = 10 \times \tan(28) = 5.3\,\mathrm{m}$$

3. The angle of elevation of the top of a 20m high mast from a point at ground level is 34°. How far is the point from the foot of the mast? Give your answer to 2 decimal places.

Solution:

Let the unknown horizontal distance be d.

Sketching the situation and labelling the sides of the resulting right-angled triangle:



Using $SOH - CAH - TOA$:

$$\tan(34) = \frac{opposite}{adjacent} = \frac{20}{d}$$

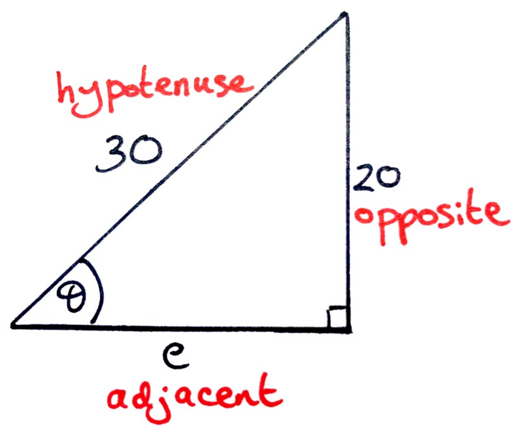
Hence, solving for d:

$$d = \frac{20}{\tan(34)} = 29.65 \text{ m}$$

4. A supporting cable of length 30m is fastened to the top of a 20m high mast. What angle does the cable make with the ground? How far away from the foot of the mast is it anchored to the ground?

Solution:

Let the angle between the cable and the ground be θ, and let the horizontal distance from it's anchor point to the foot of the vertical mast be e.



First, to find the angle θ we use the fact that we know both the opposite and the hypotenuse:

$$\sin(\theta) = \frac{opposite}{hypotenuse} = \frac{20}{30}$$

Taking inverse sine of both sides:

$$\theta = \sin^{-1}\left(\frac{2}{3}\right) = 41.8^{\circ}$$

Then to find the side e, which is the adjacent, use the fact that we know the hypotenuse and now the angle θ as well:

$$\cos(41.8) = \frac{adjacent}{hypotenuse} = \frac{e}{30}$$

$$\therefore\ e = 30 \times \cos(41.8) = 22.4 \text{ m}$$

Note:

We often have several other paths to the solution, for example we could have used Pythagoras's theorem to find e without finding θ first:

$$30^2 = 20^2 + e^2$$

Transposing for e:

$$\begin{aligned} \therefore\ e &= \sqrt{30^2 - 20^2} \\ &= \sqrt{500} \\ &= 22.4\ \mathrm{m} \end{aligned}$$

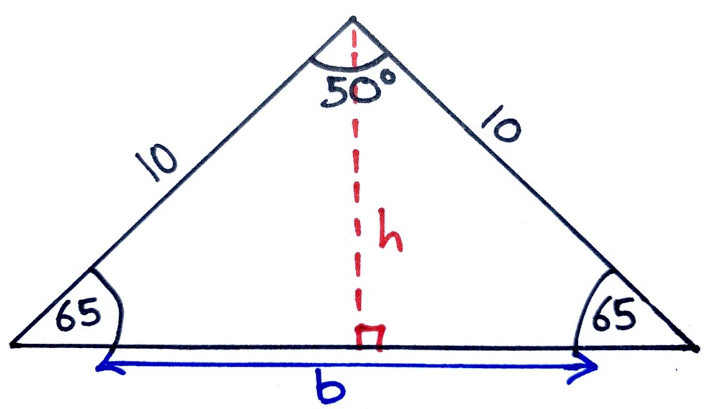
5. What is the height (to 1 decimal place) of an isosceles triangle with base angle 65° and sloping sides with length 10cm? What is the length of the base of this triangle (to 1 decimal place)?

Solution:

Since we know two of the three angles are 65°, the remaining angle is:

$$180 - (65 + 65) = 180 - 130 = 50^\circ$$

Sketching the isosceles triangle with the information given, and labelling the base b and height h:



Next, we find the perpendicular height h. To do this, consider one of the halves, which form right-angled triangles with known hypotenuse of 10 and where h is the opposite side to the known angle of 65°.

Thus, using $SOH - CAH - TOA$:

$$\sin(65) = \frac{opposite}{hypotenuse} = \frac{h}{10}$$

Transposing for h:

$$\begin{aligned} h &= 10 \times \sin(65) \\ &= 9.0630779\ldots \\ &\approx 9.1 \text{ cm} \end{aligned}$$

Then to find the base b, use the fact that half of the base forms the adjacent of each of the right-angled trianges. Again applying $SOH - CAH - TOA$:

$$\cos(65) = \frac{adjacent}{hypotenuse} = \frac{0.5 \times b}{10}$$

Hence:

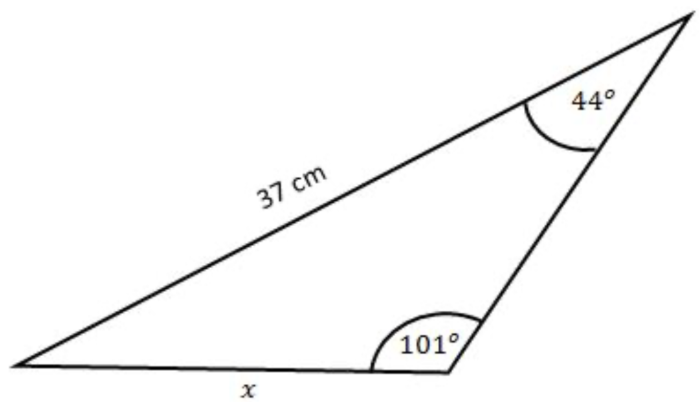
$$\frac{1}{2}b = 10\cos(65) = 4.22618\ldots$$

And so:

$$\begin{aligned} b &= 8.452365\ldots \\ &\approx 8.5 \text{ cm} \end{aligned}$$

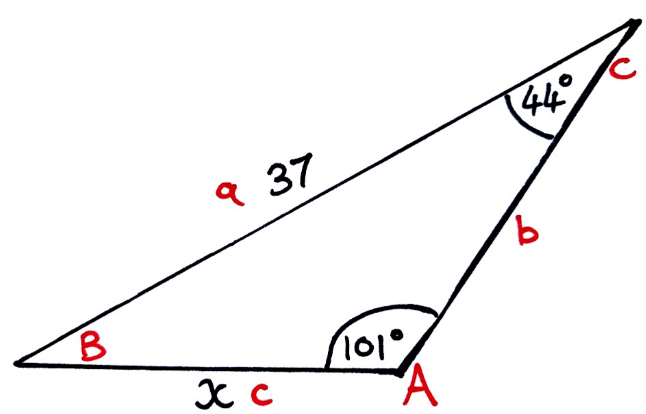
6. Calculate the value of x:

   (a)



Solution:

Labelling the side-angle pairs in the triangle:



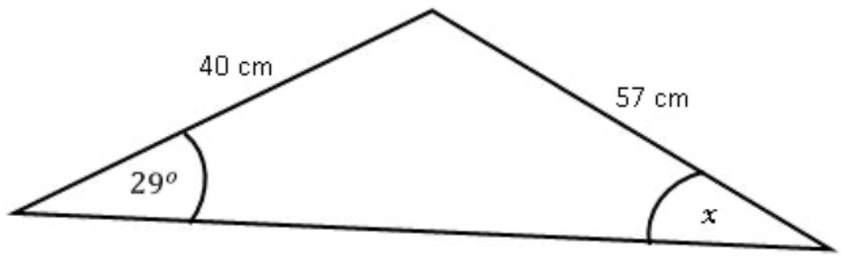
As we know one full side-angle pair (side of length 37 and angle 101°), and we know the angle (44°) opposite the side x that we need to find, we will use the sine rule. Substituting in the values from the triangle:

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \quad \Longrightarrow \quad \frac{37}{\sin(101)} = \frac{x}{\sin(44)}$$

Then solving for x:

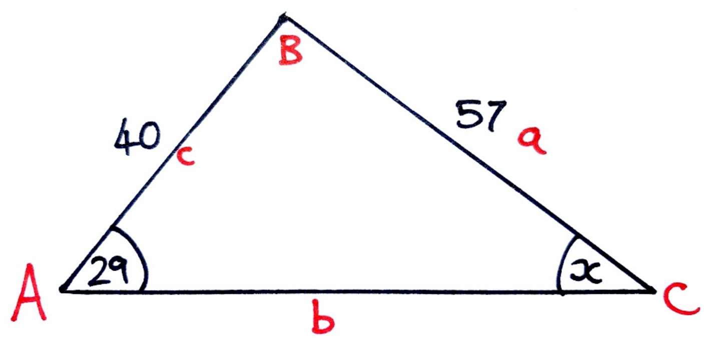
$$x = \sin(44) \times \frac{37}{\sin(101)} = 26.18 \text{ cm}$$

(b)



Solution:

Sketch the triangle and label the side-angle pairs:



We know side (40) facing the angle x that we need to determine, and we know another full side-angle pair. Thus, we can use the sine rule:

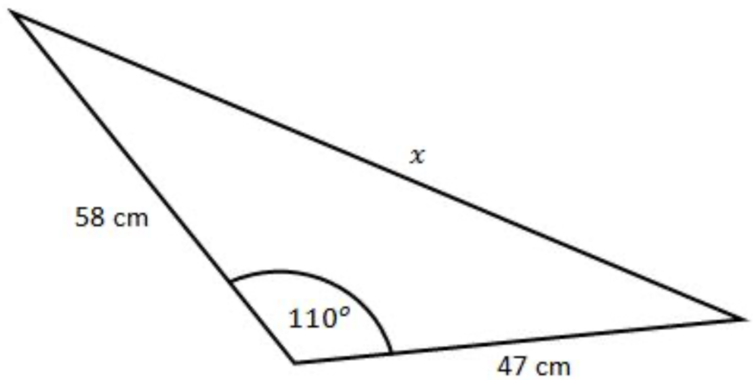
$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} \quad \Longrightarrow \quad \frac{\sin(29)}{57} = \frac{\sin(x)}{40}$$

Re-arranging to make x the subject:

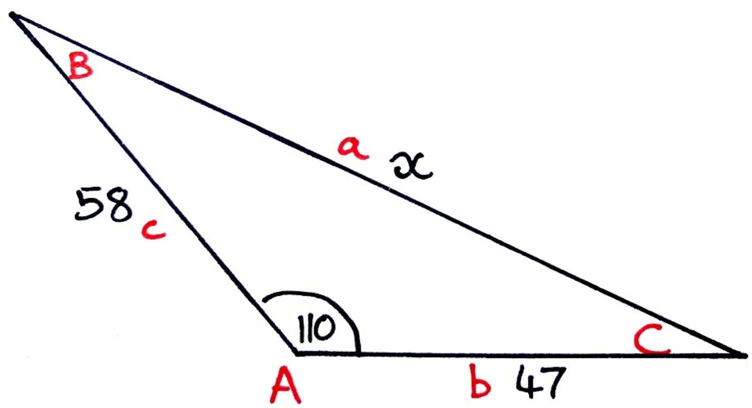
$$\sin(x) = 40 \times \frac{\sin(29)}{57}$$

$$\therefore \quad x = \sin^{-1}\left(\frac{40\sin(29)}{57}\right) = 19.89^\circ$$

(c)



Solution:



In this case, we *don't* know any full side-angle pair so we cannot apply the sine rule. However, we know the other two sides and the angle between them which faces the side we need to find. Thus, using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

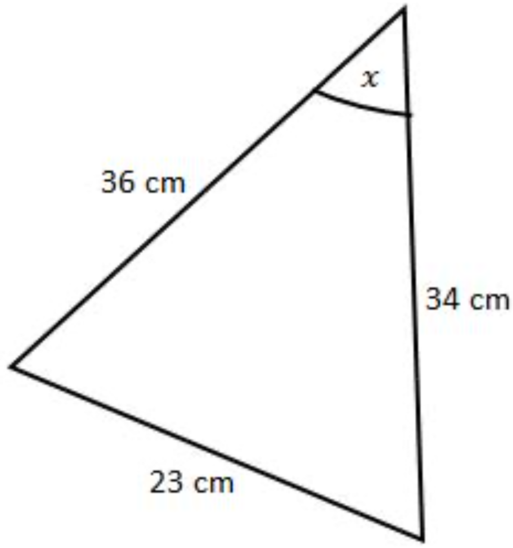
and substituting in our variables:

$$x^2 = 47^2 + 58^2 - 2 \times 47 \times 58 \times \cos(110) = 7437.693821$$

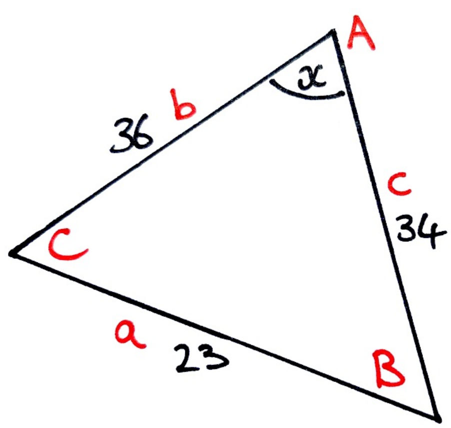
Hence, taking the square root of both sides:

$$x = 86.24 \text{ cm}$$

(d)



Solution:



We know all three sides and want to find an angle, so we can use the cosine rule in this case:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36^2 + 34^2 - 23^2}{2 \times 36 \times 34} = \frac{641}{816}$$

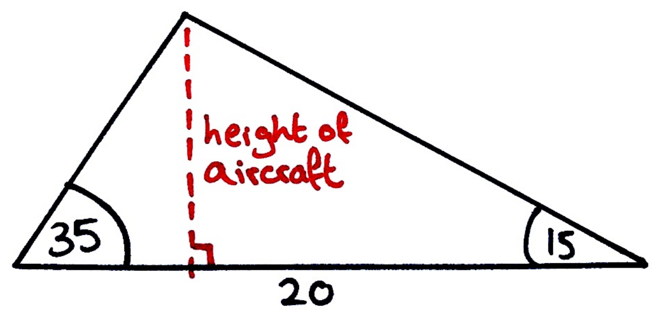
Hence:

$$x = \cos^{-1}\left(\frac{641}{816}\right) = 38.23°$$

7. Two radar stations located 20 miles apart each detect an aircraft situated between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. Determine the altitude of the aircraft.

Solution:

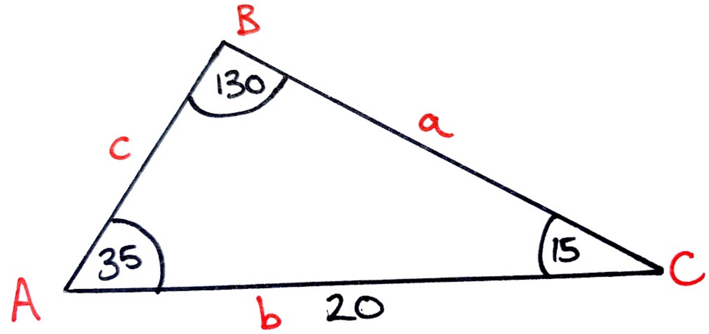
Begin by sketching an overview of the situation:



We are not able to calculate the altitude immediately. If we could find one of the other two sides, then we would know an angle and one side of one of the two right-angled triangles, and we could then use that information and $SOH - CAH - TOA$ to determine the *opposite* side that is the altitude of the plane that we need.

Let's begin by determining the value of the missing angle:

$$180 - 35 - 15 = 130^\circ$$



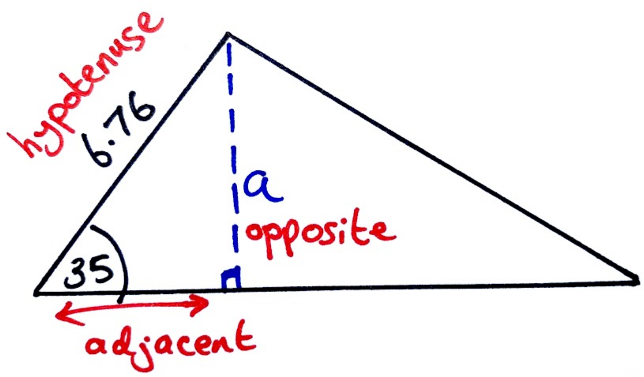
Now we have a complete side-angle pair, so we can apply the sine rule to calculate one of the other sides:

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \implies \frac{20}{\sin(130)} = \frac{c}{\sin(15)}$$

Re-arranging to solve for side c:

$$c = \sin(15) \times \frac{20}{\sin(130)} = 6.76 \text{ miles}$$

Label another sketch to emphasise the right-angled triangle and the key information we now have:



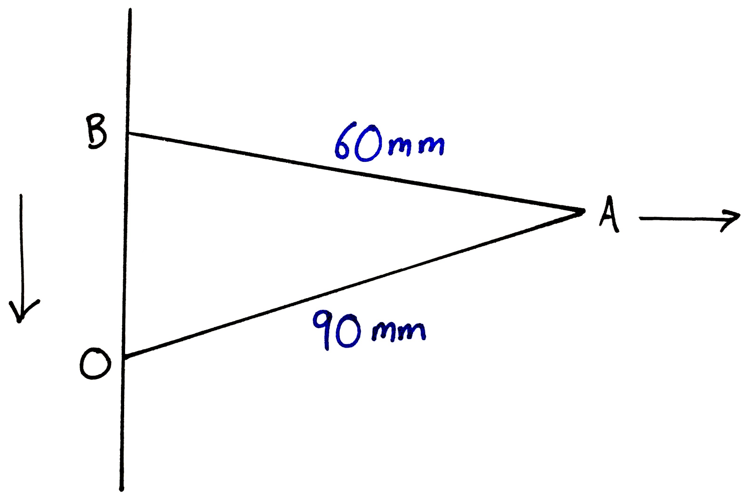
Finally we can calculate the altitude, which is the *opposite*, using $SOH - CAH - TOA$:

$$\sin(35) = \frac{opposite}{hypotenuse} = \frac{a}{6.76}$$

Transposing for altitude a:

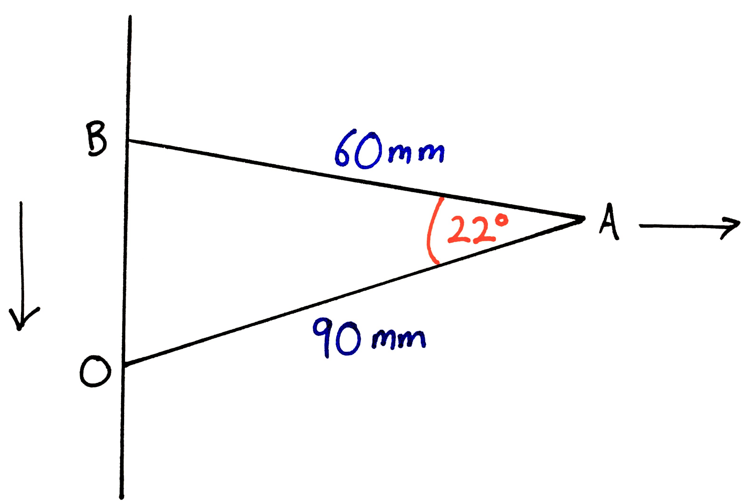
$$a = 6.76 \times \sin(35) = 3.88 \text{ miles}$$

8. The figure shows part of a hinge for a window. As the window is opened, part B moves down to the fixed point O while A moves to the right. When angle A = 22°, determine the length OB.



Solution:

As we want to find a length OB when we know the opposing angle and the other two sides, we can accomplish this using the cosine rule.



$$
\begin{aligned}
(\mathrm{OB})^2 &= (\mathrm{AB})^2 + (\mathrm{OA})^2 - 2(\mathrm{AB})(\mathrm{OA})\cos(22^\circ) \\
&= 60^2 + 90^2 - 2(60)(90)\cos(22^\circ) \\
&= 3600 + 8100 - 10800\cos(22^\circ) \\
&= 11700 - 10800(0.92718385) \\
&= 1686.41437
\end{aligned}
$$

Taking the square root:

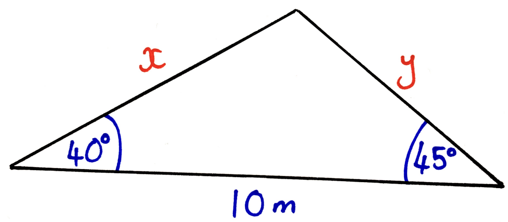
$$\mathrm{OB} = \sqrt{1686.41437} = 41.0660$$

Hence, the length OB is 41.1mm (3 s.f.).

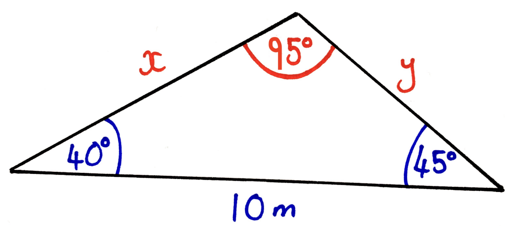
9. A roof of triangular shape has a span (i.e. width) of 10.0m and the two sides make an angle of 40° and 45° at each end. Find the length of these sides.

Solution:

Begin by sketching the scenario, and we label the unknown two sides of the roof as x and y:



Then since we know two angles, we can determine that the remaining angle has size $180 - (45 + 40) = 180 - 85 = 95°$.



To determine the values of x and y, we use the sine rule twice. Beginning with x:

$$\frac{x}{\sin(45°)} = \frac{10}{\sin(95°)}$$

Transposing to solve for x:

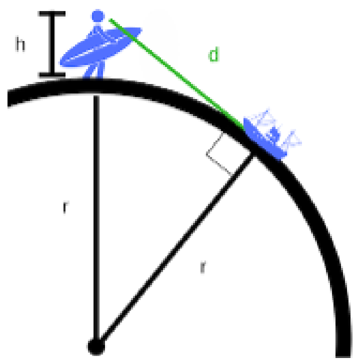
$$x = \frac{10\sin(45°)}{\sin(95°)} = 7.0981$$

and similarly for y (although note that we can alternatively use the cosine rule now, since the opposing angle and both other sides are now known):

$$\frac{y}{\sin(40°)} = \frac{10}{\sin(95°)} \quad \Longrightarrow \quad y = \frac{10\sin(40°)}{\sin(95°)} = 6.4524$$

and so we have $x = 7.10$ m and $y = 6.45$ m (3 s.f.).

10. A surfer is looking out to sea when she observes a ship appearing on the horizon. Given that she has a height of $h = 1.8$m (approximately 0.001 miles) and that the Earth has radius of $r = 3982$ miles, calculate the distance d to the horizon.



Solution:

Using Pythagoras' theorem:

$$(r+h)^2 = d^2 + r^2$$

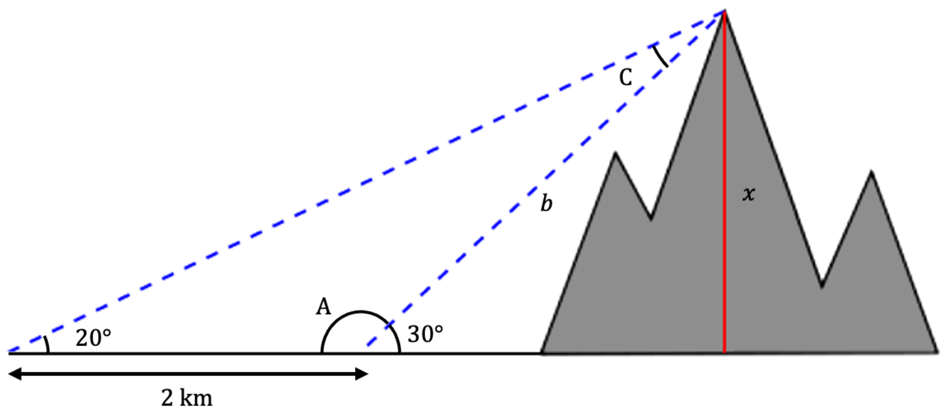
Expanding the brackets and simplifying:

$$\therefore r^2 + h^2 + 2rh = d^2 + r^2$$
$$\therefore h^2 + 2rh = d^2$$

Transposing for d (which must be the positive square root) and evaluating:

$$\begin{aligned}
\therefore d &= +\sqrt{h^2 + 2rh} \\
&= \sqrt{(0.001)^2 + 2(3982)(0.001)} \\
&= \sqrt{10^{-6} + 2 \times 3.982} \\
&= \sqrt{10^{-6} + 7.964} \\
&= \sqrt{7.964001} \\
&= 2.822056\ldots \\
&\approx 2.8 \text{ miles}
\end{aligned}$$

11. A surveyor has been tasked with determining the height x of a mountain. In order to do this, she measures the angular elevation of the summit from two different positions, which are separated on the ground by a distance of 2 km, finding them to be 30° and 20° as indicated on the diagram.



(a) Determine the size of angles A and C indicated in the diagram.

Solution:

Angles on a straight line add to 180°, so we can calculate A as:

$$A = 180 - 30 = 150^\circ$$

Then since the three angles in a triangle also add to 180° and we now have the two other angles, we can calculate C:

$$C = 180 - (20 + 150) = 180 - 170 = 10^\circ$$

(b) By first calculating the length of the side labelled b, determine the height x of the mountain.

Solution:

We know the angle opposite the side b and since we have now obtained C and know the length of its opposing side, we have a side-angle pair and can thus apply the sine rule to calculate b:

$$\frac{b}{\sin(20^\circ)} = \frac{2}{\sin(C)} = \frac{2}{\sin(10^\circ)}$$

Then solving for b:

$$\begin{aligned} b &= \frac{2\sin(20^\circ)}{\sin(10^\circ)} \\ &= 3.93923\ldots \\ &\approx 3.94\,\text{km} \end{aligned}$$

Finally, we use the fact that b is the hypotenuse of a right-angled triangle of height x to calculate the value of x using b and the angular elevation that was measured closest to the mountain:

$$\sin(30^\circ) = \frac{x}{b} = \frac{x}{3.93923}$$

Transposing for x:

$$\begin{aligned} x &= 3.93923\sin(30^\circ) \\ &= 1.9696155\ldots \\ &\approx 1.97\,\text{km} \end{aligned}$$

12. State the amplitude, period and frequency of the following waves:

   (a) $10\sin(2t)$

   (b) $H\sin(\pi t)$

   (c) $220\sin(1000\pi t)$

   (d) $\dfrac{\cos(t/\pi L)}{\pi L}$

   Solution:

   (a) Here the amplitude is $A = 10$.

   The period is calculated using the formula:

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{2} \\ &= \pi \end{aligned}$$

   The frequency is calculated using the formula:

$$\begin{aligned} f &= \frac{1}{T} \\ &= \frac{1}{\pi} \end{aligned}$$

   (b) Here the amplitude is $A = H$.

   The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{\pi} \\
&= 2
\end{aligned}
$$

The frequency is calculated using the formula:

$$
\begin{aligned}
f &= \frac{1}{T} \\
&= \frac{1}{2}
\end{aligned}
$$

(c) Here the amplitude is $A = 220$.

The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{1000\pi} \\
&= \frac{1}{500}
\end{aligned}
$$

The frequency is calculated using the formula:

$$
\begin{aligned}
f &= \frac{1}{0.002} \\
&= 500
\end{aligned}
$$

(d) Here the amplitude is $A = \dfrac{1}{\pi L}$.

The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{1/\pi L} \\
&= 2\pi^2 L
\end{aligned}
$$

The frequency is calculated using the formula:

$$
\begin{aligned}
f &= \frac{1}{T} \\
&= \frac{1}{2\pi^2 L}
\end{aligned}
$$

13. Determine the amplitude, period and phase shift (state lead or lag) of the following:

(a) $4\sin\left(t + \dfrac{\pi}{3}\right)$

(b) $\sin(100\pi t - 0.25)$

(c) $20\sin\left[\left(10t + \dfrac{1}{5}\right)\pi\right]$

(d) $5 + 2\sin\left(2\pi t - \dfrac{\pi}{2}\right)$

Solution:

(a) Here the amplitude is $A = 4$.

The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{1} \\
&= 2\pi
\end{aligned}
$$

The phase shift is $\frac{\pi}{3}$ (lead).

(b) Here the amplitude is $A = 1$.

The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{100\pi} \\
&= 0.02
\end{aligned}
$$

The phase shift is -0.25 (lag).

(c) Here the amplitude is $A = 20$.

The period is calculated using the formula:

$$
\begin{aligned}
T &= \frac{2\pi}{\omega} \\
&= \frac{2\pi}{10\pi} \\
&= \frac{1}{5}
\end{aligned}
$$

The phase shift is $\frac{\pi}{5}$ (lead).

(d) Here the amplitude is $A = 2$.

The period is calculated using the formula:

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{2\pi} \\ &= 1 \end{aligned}$$

The phase shift is $-\dfrac{\pi}{2}$ (lag).