Railway Engineering Mathematics

Tutorial Sheet 7

Solutions

1. Download the `Picturebook.xlsx` from the module Blackboard site. Navigate to the "logarithmic" tab, featuring the general logarithmic equation:

   $$y = A\ln(x) + B$$

   Determine how the two parameters affect the shape of the graph:

   (a) A

   (b) B

   What happens when you try to calculate $\ln(0)$ or log of a negative number?

   Solution:

   (a) Increasing A stretches the curve in the vertical direction.

   (b) B shifts the curve up/down.

   Logs are not defined for non-positive inputs. i.e. log of zero or any negative number does not exist.

2. Use the laws of logarithms to match each expression on the left with an equivalent expression on the right:

$\ln(4)+\ln(7)$	$-\ln(3)$
$\ln(e^5)$	$\ln(49)$
$\ln(8)-\ln(2)$	$\ln(28)$
$\ln\left(\dfrac{9}{27}\right)$	0
$2\ln(7)$	5
$\ln(1)$	$\ln(4)$

Solution:

- $\ln(4)+\ln(7)=\ln(4\times 7)=\ln(28)$
- $\ln(\mathrm{e}^5)=5\ln(e)=5\times 1=5$
- $\ln(8)-\ln(2)=\ln\left(\dfrac{8}{2}\right)=\ln(4)$
- $\ln\left(\dfrac{9}{27}\right)=\ln\left(\dfrac{1}{3}\right)=\ln(1)-\ln(3)=0-\ln(3)=-\ln(3)$
- $2\ln(7)=\ln(7^2)=\ln(49)$
- $\ln(1)=0$

3. Download the `Picturebook.xlsx` from the module Blackboard site. Navigate to the "exponential" tab, featuring the general exponential function:

$$y = A\,\mathrm{e}^{Bx} + C$$

Determine how the two parameters affect the shape of the graph:

(a) A

(b) B

(c) C

What conditions must be satisfied by B in order to obtain exponential growth, or to obtain exponential decay?

Solution:

(a) Increasing the magnitude of A stretches the curve in the vertical axis.

(b) B controls the rate of growth/decay. If $B > 0$ then the curve exhibits exponential growth, while if $B < 0$ then the curve exhibits exponential decay.

(c) C shifts the curve up/down without changing the shape.

4. Solve the following exponential equations for x:

(a) $e^x = 9.5$

(b) $5\,e^x = 50.6$

(c) $27.3\,e^x - 12 = 112.7$

(d) $-14.6\,e^x + 7.5 = 4.2$

(e) $29 - 6.3\,e^x = 15.4$

(f) $34.7\,e^{4.2x} + 4.3 = 25.1$

(g) $117.1\,e^{-0.4x} - 15.7 = -3.8$

Solution:

(a) $e^x = 9.5$

Taking the natural log of both sides:

$$\begin{aligned} \ln\left(e^x\right) &= \ln(9.5) \\ \therefore\ x &= \ln(9.5) \\ &= 2.25 \end{aligned}$$

(b) $5\,\mathrm{e}^x = 50.6$

First divide both sides by 5 so that the entire LHS is purely an exponential function:

$$\begin{aligned} \mathrm{e}^x &= \frac{50.6}{5} \\ \therefore\ \mathrm{e}^x &= \frac{253}{25} \end{aligned}$$

Then taking the natural log of both sides:

$$\begin{aligned} x &= \ln\left(\frac{253}{25}\right) \\ &= 2.31 \end{aligned}$$

(c) $27.3\,\mathrm{e}^x - 12 = 112.7$

Begin by isolating the exponential function on the LHS:

$$\begin{aligned} 27.3\,\mathrm{e}^x &= 112.7 + 12 \\ \therefore\ 27.3\,\mathrm{e}^x &= \frac{1247}{10} \\ \therefore\ \mathrm{e}^x &= \frac{1247}{273} \end{aligned}$$

Then taking ln of both sides:

$$\begin{aligned} x &= \ln\left(\frac{1247}{273}\right) \\ &= 1.52 \end{aligned}$$

(d) $-14.6\,\mathrm{e}^x + 7.5 = 4.2$

Isolating the exponential function on the LHS:

$$\begin{aligned} -14.6\,\mathrm{e}^x &= 4.2 - 7.5 \\ \therefore\ -14.6\,\mathrm{e}^x &= -\frac{33}{10} \\ \therefore\ \mathrm{e}^x &= \frac{33}{146} \end{aligned}$$

Then taking the natural log of both sides:

$$\begin{aligned} x &= \ln\left(\frac{33}{146}\right) \\ &= -1.49 \end{aligned}$$

(e) $29 - 6.3\,\mathrm{e}^x = 15.4$

Isolating the exponential function:

$$\begin{aligned} -6.3\,\mathrm{e}^x &= 15.4 - 29 \\ \therefore\ -6.3\,\mathrm{e}^x &= -\frac{68}{5} \\ \therefore\ \mathrm{e}^x &= \frac{136}{63} \end{aligned}$$

Then taking ln of both sides:

$$\begin{aligned} x &= \ln\left(\frac{136}{63}\right) \\ &= 0.77 \end{aligned}$$

(f) $34.7\,\mathrm{e}^{4.2x} + 4.3 = 25.1$

Isolating the exponential function:

$$\begin{aligned} 34.7\,\mathrm{e}^{4.2x} &= 25.1 - 4.3 \\ \therefore\ 34.7\,\mathrm{e}^{4.2x} &= \frac{104}{5} \\ \therefore\ \mathrm{e}^{4.2x} &= \frac{208}{347} \end{aligned}$$

Then taking logs of both sides:

$$\begin{aligned} 4.2x &= \ln\left(\frac{208}{347}\right) \\ \therefore\ x &= \frac{1}{4.2}\ln\left(\frac{208}{347}\right) \\ &= -0.12 \end{aligned}$$

(g) $117.1\,\mathrm{e}^{-0.4x} - 15.7 = -3.8$

First, isolating the exponential function on the LHS:

$$\begin{aligned} 117.1\,\mathrm{e}^{-0.4x} &= -3.8 + 15.7 \\ \therefore\ 117.1\,\mathrm{e}^{-0.4x} &= \frac{119}{10} \\ \therefore\ \mathrm{e}^{-0.4x} &= \frac{119}{1171} \end{aligned}$$

Then taking the natural log of both sides:

$$\begin{aligned} -0.4x &= \ln\left(\frac{119}{1171}\right) \\ \therefore\ x &= \frac{1}{-0.4}\ln\left(\frac{119}{1171}\right) \\ &= 5.72 \end{aligned}$$

5. Solve the following logarithmic equations for x:

(a) $\ln(x) = 0.6$

(b) $15 - \ln(2x) = 10.3$

(c) $12.8 = \log_{10}(0.3x + 6)$

Solution:

(a) $\ln(x) = 0.6$

As the desired variable x is contained within a natural log function, we can extract it by taking an exponential function of both sides:

$$
\begin{aligned}
\mathrm{e}^{\ln(x)} &= \mathrm{e}^{0.6} \\
\therefore\ x &= \mathrm{e}^{0.6} \\
&= 1.82 \quad (2\text{ d.p.})
\end{aligned}
$$

(b) $15 - \ln(2x) = 10.3$

First, isolate the natural log term:

$$
\begin{aligned}
\ln(2x) &= 15 - 10.3 \\
\therefore\ \ln(2x) &= 4.7
\end{aligned}
$$

Then taking an exponential function of both sides:

$$
\begin{aligned}
\mathrm{e}^{\ln(2x)} &= \mathrm{e}^{4.7} \\
\therefore\ 2x &= \mathrm{e}^{4.7} \\
\therefore\ x &= \frac{1}{2}\mathrm{e}^{4.7} \\
&= 54.97 \quad (2\text{ d.p.})
\end{aligned}
$$

(c) $2.8 = \log_{10}(0.3x + 6)$

In this example, one side consists of a log with base 10, rather than the natural log which has base e. To invert this and extract the contents, we take both sides as powers of 10 rather than e:

$$10^{2.8} = 10^{\log_{10}(0.3x+6)}$$

$$\therefore\ 10^{2.8} = 0.3x + 6$$

Then transposing for x as usual:

$$0.3x = 10^{12.8} - 6$$

$$\therefore\ x = \frac{1}{0.3}\left(10^{2.8} - 6\right)$$

$$= 2083.19 \quad (2\text{ d.p.})$$

6. The pressure p pascals at height h metres above ground level is given by:

$$p = p_0 \, \mathrm{e}^{-\frac{h}{C}}$$

where p_0 is the pressure at ground level and C is a constant.

Find the pressure p when $p_0 = 1.012 \times 10^5$ Pa, height is $h = 1420$m and $C = 71500$.

Solution:

As we already have a formula for pressure, simply substitute in the constants and the specific height, and evaluate:

$$\begin{aligned} p &= p_0 \, \mathrm{e}^{-\frac{h}{C}} \\ &= 1.012 \times 10^5 \times \mathrm{e}^{-1420/71500} \\ &= 99209.98 \text{ Pa} \end{aligned}$$

For physical problems such as these, be sure to include appropriate units, and rounding if necessary.

7. The current i flowing in a capacitor at time t is given by:

$$i = 12.5\left(1 - e^{-\frac{t}{CR}}\right)$$

where $R = 30$ kilohms, and the capacitance C is 20 micro-farads. Determine:

(a) the current flowing after 0.5 seconds

(b) the time for the current to reach 10 amperes

Solution:

(a) Substitute in the parameters and $t = 0.5$, and evaluate the formula we have for current i:

$$\begin{aligned}
i &= 12.5\left(1 - e^{-\frac{t}{CR}}\right) \\
&= 12.5\left(1 - e^{\frac{-0.5}{20\times10^{-6}\times30\times10^{3}}}\right) \\
&= 12.5\left(1 - e^{\frac{-0.5}{6\times10^{-1}}}\right) \\
&= 12.5\left(1 - e^{-\frac{5}{6}}\right) \\
&= 7.07 \text{ amperes}
\end{aligned}$$

(b) Substitute in the parameters and $i = 10$, and simplify first:

$$10 = 12.5\left(1 - \mathrm{e}^{-\frac{t}{20\times10^{-6}\times30\times10^{3}}}\right)$$

$$\therefore\ 10 = 12.5\left(1 - \mathrm{e}^{-\frac{t}{0.6}}\right)$$

Now re-arrange the formula to make t the subject:

$$\therefore\ \frac{10}{12.5} = 1 - \mathrm{e}^{-\frac{t}{0.6}}$$

$$\therefore\ \frac{4}{5} - 1 = -\,\mathrm{e}^{-\frac{t}{0.6}}$$

$$\therefore\ -\frac{1}{5} = -\,\mathrm{e}^{-\frac{t}{0.6}}$$

$$\therefore\ \mathrm{e}^{-\frac{t}{0.6}} = \frac{1}{5}$$

Then taking the natural log of both sides:

$$\therefore\ -\frac{t}{0.6} = \ln\left(\frac{1}{5}\right)$$

$$\therefore\ t = -0.6\ln\left(\frac{1}{5}\right)$$

$$= 0.97 \text{ seconds (2 d.p.)}$$

8. The resistance R of an electrical conductor at temperature $\theta°C$ is given by $R = R_0\,\mathrm{e}^{\alpha\theta}$, where α is a constant and $R_0 = 5 \times 10^3$ ohms.

Determine the value of α, correct to 4 significant figures, when $R = 6 \times 10^3$ ohms and $\theta = 1500°C$.

Also, find the temperature, correct to the nearest degree, when the resistance R is 5.4×10^3 ohms.

Solution:

First, determining the value of α when $R = 6 \times 10^3$. Substitute in the values of R, R_0 and θ and transpose for α using logs:

$$R = R_0\,\mathrm{e}^{\alpha\theta}$$

$$\therefore\ 6 \times 10^3 = 5 \times 10^3 \times \mathrm{e}^{1500\alpha}$$

$$\therefore\ \frac{6}{5} = \mathrm{e}^{1500\alpha}$$

$$\therefore\ 1500\alpha = \ln\left(\frac{6}{5}\right)$$

$$\therefore\ \alpha = \frac{1}{1500}\ln\left(\frac{6}{5}\right)$$

$$= 1.215 \times 10^{-4} \quad (4\,\text{s.f.})$$

Now to find θ when $R = 5.4 \times 10^3$:

$$5.4 \times 10^3 = 5 \times 10^3 \times \mathrm{e}^{1.215 \times 10^{-4}\theta}$$

$$\therefore\ \frac{27}{25} = \mathrm{e}^{1.215 \times 10^{-4}\theta}$$

$$\therefore\ \ln\left(\frac{27}{25}\right) = 1.215 \times 10^{-4}\theta$$

$$\therefore\ \theta = \frac{1}{1.215 \times 10^{-4}}\ln\left(\frac{27}{25}\right)$$

$$= 633°\mathrm{C} \quad (\text{nearest degree})$$

9. A research group models the population P (in millions) of a particular country by following formula:

$$P = A\,\mathrm{e}^{kt}$$

where t is the time (in years) since 1980, the value of k is 0.0241, and A is a constant that is yet to be determined.

In the year 2000, the population of the country was recorded as 11 million.

(a) What is the population projected to be in 2020?

(b) When is the population forecast to exceed 25 million?

(c) Plot the projected population in EXCEL between the years 1980 and 2050. From the graph, describe the behaviour of this exponential function.

(d) From the graph, determine when the population is equal to 27 million. Indicate clearly on the graph how you obtain your solution.

(e) Your colleague uses this formula to predict the population of the country in the year 2500. Discuss the reliability of this prediction.

Solution:

First, substitute in $P(t = 20) = 11$ and solve to determine the value of A.

$$\begin{aligned}
11 &= A\,\mathrm{e}^{20\times 0.0241} \\
11 &= A\,\mathrm{e}^{0.4820} \\
A &= \frac{11}{\mathrm{e}^{0.482}} = 6.7930
\end{aligned}$$

(a) In 2020, $t = 40$. Then:

$$P(t = 40) = 6.7930\,\mathrm{e}^{40\times 0.0241} = 17.8124$$

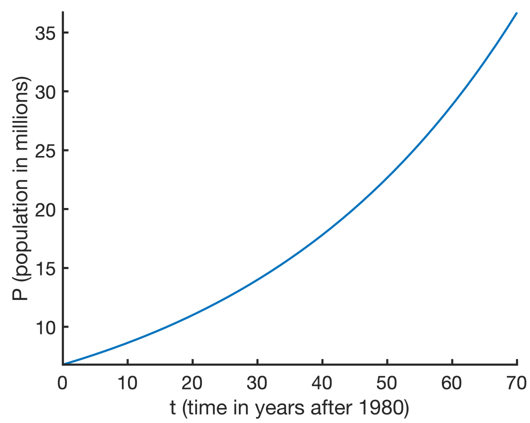
So the population is projected to be 17.8 million in 2020.

(b) Solve for t when $P = 25$:

$$\begin{aligned}
25 &= 6.7930\,\mathrm{e}^{0.0241t} \\
\mathrm{e}^{0.0241t} &= \frac{25}{6.7930} = 3.6803 \\
0.0241t &= \ln(3.6803) = 1.3030 \\
t &= \frac{1.3030}{0.0241} = 54.0657
\end{aligned}$$

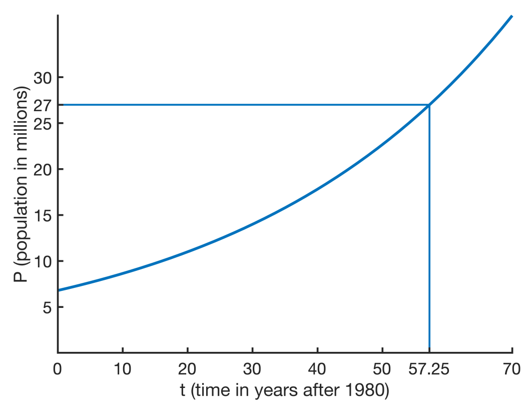
So 54 years after 1980, hence the year is 2034.

(c) Graph:



Grows with increasing rapidity and without bound.

(d) Graph:



Thus, 57 years after 1980, so in 2037.

(e) This is a much greater timeframe than the model is calibrated for. Models are not a guaranteed prediction, they are based on the assumption that current circumstances and conditions will never change. Of course, it is very likely that if this pattern were to continue, eventually limitations such as population density, resource, climate and disease that are not accounted for in the current model could become significant.

10. The temperatures θ_1 and θ_2 of a pipe with inner radius r_1 and outer radius r_2 are given by:

$$\theta_1 = -\frac{Q}{2\pi kL}\ln(r_1) \quad \text{and} \quad \theta_2 = -\frac{Q}{2\pi kL}\ln(r_2)$$

where Q is the heat transfer rate, L is the length of the pipe and k is the thermal conductivity. Show that:

$$Q = \frac{2\pi kL(\theta_1 - \theta_2)}{\ln\left(\dfrac{r_2}{r_1}\right)}$$

Solution:

Combining our expressions for θ_1 and θ_2:

$$\begin{aligned}
\theta_1 - \theta_2 &= -\frac{Q}{2\pi kL}\ln(r_1) - \left(-\frac{Q}{2\pi kL}\ln(r_2)\right) \\
&= -\frac{Q}{2\pi kL}\big(\ln(r_1) - \ln(r_2)\big) \\
&= -\frac{Q}{2\pi kL}\big(-\big(\ln(r_2) - \ln(r_1)\big)\big) \\
&= -\frac{Q}{2\pi kL}\left(-\ln\left(\frac{r_2}{r_1}\right)\right) \\
&= \frac{Q}{2\pi kL}\ln\left(\frac{r_2}{r_1}\right)
\end{aligned}$$

where we have re-arranged to ensure that $\ln\left(\dfrac{r_2}{r_1}\right)$ occurs in the expression.

Now transpose to make Q the subject:

$$2\pi kL(\theta_1 - \theta_2) = Q\ln\left(\frac{r_2}{r_1}\right) \quad \Longrightarrow \quad Q = \frac{2\pi kL(\theta_1 - \theta_2)}{\ln\left(\dfrac{r_2}{r_1}\right)}$$

as required.

11. The voltage v and current i of an inductor is given by:

$$i = 5\left(\mathrm{e}^{-200t} - \mathrm{e}^{-800t}\right) \quad \text{and} \quad v = \mathrm{e}^{-200t} + 400\,\mathrm{e}^{-800t}$$

Find an expression for the power $p = vi$ of the inductor.

Solution:

Expanding the brackets and combining the exponents using the laws of indices:

$$\begin{aligned}
p &= vi \\
&= 5\left(\mathrm{e}^{-200t} - \mathrm{e}^{-800t}\right)\left(\mathrm{e}^{-200t} + 400\,\mathrm{e}^{-800t}\right) \\
&= 5\left(\mathrm{e}^{-200t-200t} - \mathrm{e}^{-200t-800t} + 400\,\mathrm{e}^{-800t-200t} - 400\,\mathrm{e}^{-800t-800t}\right) \\
&= 5\left(\mathrm{e}^{-400t} + 399\,\mathrm{e}^{-1000t} - 400\,\mathrm{e}^{-1600t}\right) \\
&= 5\,\mathrm{e}^{-1600t}\left(\mathrm{e}^{1200t} + 399\,\mathrm{e}^{600t} - 400\right)
\end{aligned}$$