Railway Engineering Mathematics

Tutorial Sheet 18

Solutions

1. Find the area between the curve

$$y = 4x - x^2$$

and the x-axis.

Solution:

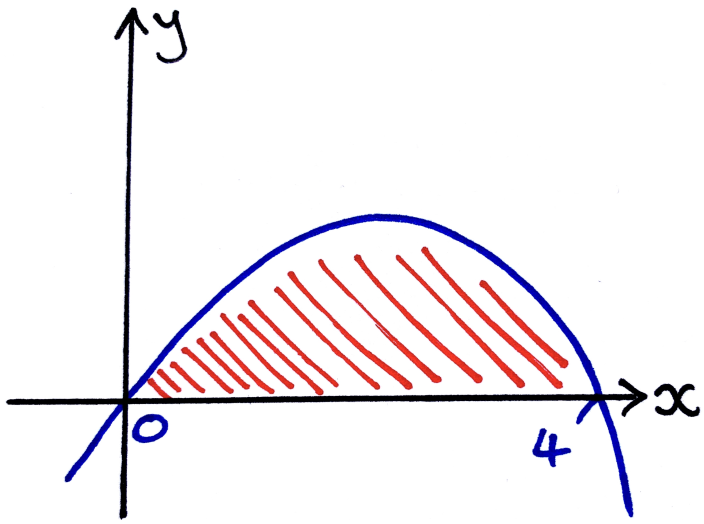
To obtain the limits of our integral, we will need to know the points of intersection between the x-axis and the curve. So we must set $y = 0$ and solve for x:

$$0 = 4x - x^2 \qquad \Longrightarrow \qquad x^2 - 4x = 0$$

This is a quadratic equation, which we can solve either by factorising to $x(x-4) = 0$ or using the quadratic formula with $a = 1$, $b = -4$ and $c = 0$:

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(0)}}{2 \times 1} \\ &= \frac{4 \pm \sqrt{16}}{2} \\ &= \frac{4 \pm 4}{2} \\ &= 2 \pm 2 \\ &= 0 \quad \text{or} \quad 4 \end{aligned}$$

Sketching the curve and the area of interest:



Hence, the area that we would describe as "enclosed between the curve and the x-axis" is the area (in red) under the curve in the range $0 < x < 4$. Since there are no other intersections with the x-axis, we will only need to evaluate one integral and the area is given by:

$$
\begin{aligned}
\text{Area} &= \int_0^4 4x - x^2 \, \mathrm{d}x \\
&= \left[\frac{4}{2}x^2 - \frac{1}{3}x^3\right]_0^4 \\
&= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 \\
&= \left(2(4)^2 - \frac{1}{3}(4)^3\right) - \left(2(0)^2 - \frac{1}{3}(0)^3\right) \\
&= \frac{32}{3} \\
&= 10.67 \text{ units}^2 \quad (2 \text{ d.p.})
\end{aligned}
$$

2. The velocity v of a vehicle (in m/s) t seconds after a certain instant is given by:

$$v = 3t^2 + 4$$

Determine how far it moves in the interval from $t = 1$ s to $t = 5$ s.

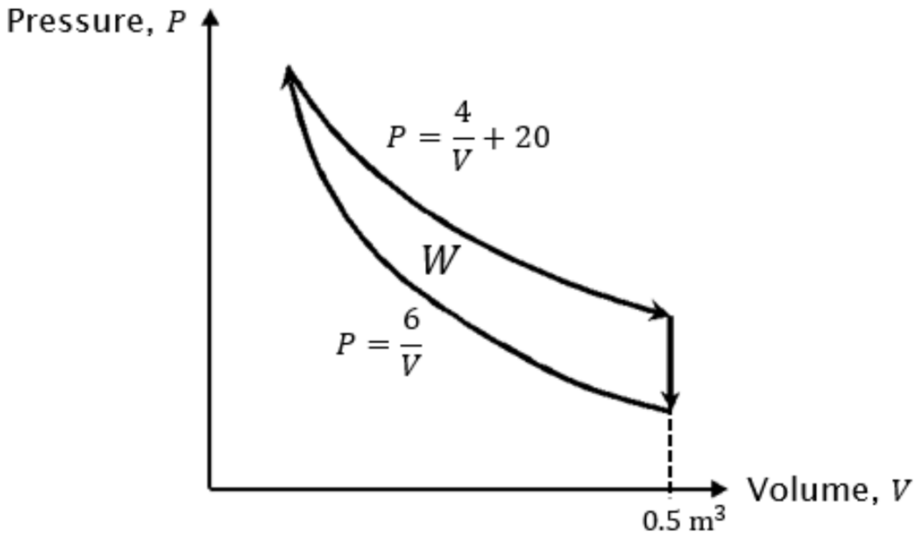
Solution:

Displacement s is given by the integral of velocity with respect to time, thus the net change in displacement over the time interval $1 < t < 5$ is given by the definite integral of velocity between these limits:

$$\begin{aligned}
s &= \int v \; \mathrm{d}t \\
&= \int_1^5 3t^2 + 4 \; \mathrm{d}t \\
&= \left[\frac{3}{3}t^3 + 4t\right]_1^5 \\
&= \left[t^3 + 4t\right]_1^5 \\
&= (5^3 + 4 \times 5) - (1^3 + 4 \times 1) \\
&= 140 \;\; \mathrm{m}
\end{aligned}$$

3. The work done W by a thermodynamic process in one cycle is equal to the area enclosed by the curves shown.

   Calculate W.



Solution:

The upper limit of the integral is already given as $v = 0.5$. To determine the lower limit, we first need to calculate the point(s) of intersection between the two curves. To do this, set the formulae for the two curves (i.e. the heights of the curves) equal to each other, and solve for v to find where they cross:

$$\frac{4}{v} + 20 = \frac{6}{v}$$

$$\therefore \quad 4 + 20v = 6$$

$$\therefore \quad 20v = 2$$

$$\therefore \quad v = \frac{2}{20} = 0.1$$

So this is the value of v (on the horizontal axis) at the point of intersection, which will be the lower limit of our integral(s).

Now, to calculate W we need the area "enclosed between" the curves. This means the area under the "upper" curve subtract the area under the "lower" curve anywhere that makes a closed-off region given the intersection and the upper limit of $v = 0.5$. In the entire enclosed range, we can see that this is consistent and the function $P = \frac{4}{v} + 20$ is always above $P = \frac{6}{v}$. If there was an additional intersection (resembling a twist) within the region such that there was an *enclosed* area where $P = \frac{6}{v}$ was the upper function, we would have to consider that separately as a different integral.

Thus, we can obtain the enclosed area W from a single definite integral of the difference between the upper and lower curves, across the range from the lower to upper limit:

$$
\begin{aligned}
W &= \int_{0.1}^{0.5} \left(\frac{4}{v} + 20\right) - \left(\frac{6}{v}\right) \mathrm{d}v \\
&= \big[4\ln(v) + 20v - 6\ln(v)\big]_{0.1}^{0.5} \\
&= \big[20v - 2\ln(v)\big]_{0.1}^{0.5} \\
&= \big(20 \times 0.5 - 2\ln(0.5)\big) - \big(20 \times 0.1 - 2\ln(0.1)\big) \\
&= 4.78 \ \text{J} \quad (2\ \text{d.p.})
\end{aligned}
$$

Be sure to include suitable units to physical problems, where possible and appropriate.

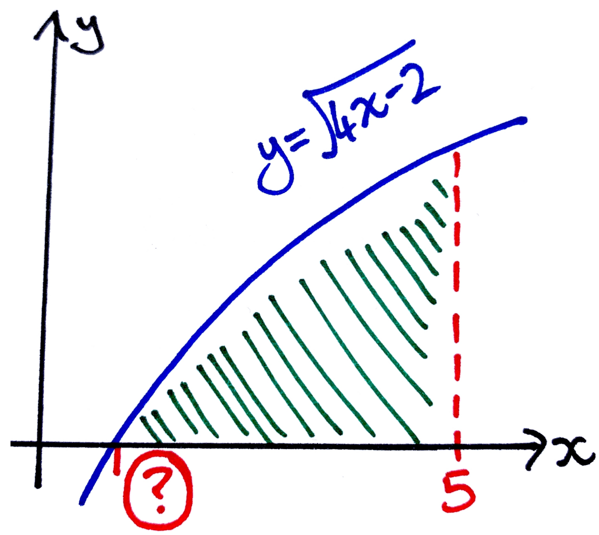
4. Find the area enclosed by the x-axis, the curve

$$y^2 = 4x - 2$$

and the line $x = 5$.

Solution:

Sketching the curve:



So $x = 5$ is the upper bound, and we need to determine the lower bound - which in this case is the value of x where the curve satisfies $y = 0$:

$$0 = \sqrt{4x - 2}$$

$$\therefore\ 0 = 4x - 2$$

$$\therefore\ 4x = 2$$

$$\therefore\ x = 0.5$$

Hence the area we are interested in is given by the integral:

$$\text{Area} = \int_{0.5}^{5} (4x - 2)^{\frac{1}{2}}\ \mathrm{d}x$$

In this case, we need to use the substitution technique to evaluate this integral. Choose the inner part of the composite function to replace with a new variable:

$$\text{Let} \quad u = 4x - 2 \quad \text{thus} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 4 \quad \text{and} \quad \mathrm{d}x = \frac{1}{4}\mathrm{d}u$$

Obtaining limits in terms of the new variable:

$$\text{Lower limit:} \quad x = 0.5 \implies u = 4(0.5) - 2 = 0$$

$$\text{Upper limit:} \quad x = 5 \implies u = 4(5) - 2 = 18$$

Then applying all of this to convert the integral to be in terms of the new variable u only and evaluating it:

$$\begin{aligned}
\text{Area} &= \int_0^{18} u^{\frac{1}{2}} \frac{1}{4} \, \mathrm{d}u \\
&= \frac{1}{4} \int_0^{18} u^{\frac{1}{2}} \, \mathrm{d}u \\
&= \frac{1}{4} \left[\frac{1}{3/2} u^{\frac{3}{2}} \right]_0^{18} \\
&= \frac{1}{6} \left[u^{\frac{3}{2}} \right]_0^{18} \\
&= \frac{1}{6} \left\{ \left(18^{\frac{3}{2}}\right) - \left(0^{\frac{3}{2}}\right) \right\} \\
&= 12.73 \text{ units}^2 \quad \text{(2 d.p.)}
\end{aligned}$$

5. The acceleration of a particle is given by $a = 4t^3$ and has an initial velocity of 3 m/s.

   Find the velocity after 1.5 s.

   Solution:

   Velocity v is given by the integral of acceleration. Thus:

$$
\begin{aligned}
v &= \int a \, \mathrm{d}t \\
&= \int 4t^3 \, \mathrm{d}t \\
&= \frac{4}{4}t^4 + c \\
&= t^4 + c
\end{aligned}
$$

   Then to determine the value of c, we can substitute in the initial condition $v = 3$ when $t = 0$:

$$3 = 0^4 + c \quad \Longrightarrow \quad c = 3$$

   So we have obtained a full description of the particle's velocity as a function of time:

$$v(t) = t^4 + 3$$

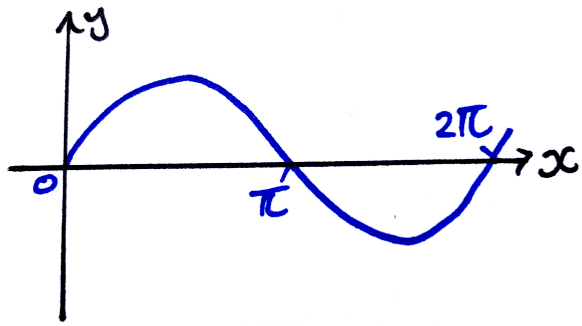
   Now, we can evaluate this to find v when $t = 1.5$:

$$
\begin{aligned}
v(t = 1.5) &= (1.5)^4 + 3 \\
&= 8.1 \text{ m/s}
\end{aligned}
$$

6. Find the area enclosed between the curve $y = \sin(x)$ and the x-axis between 0 and 2π.

Solution:

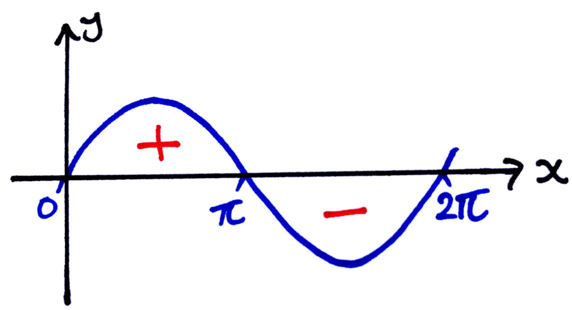
Start with a basic sketch of the sine wave over the specified interval:



Remember that when looking for area "enclosed by" curves, if the curve crosses the x-axis within the range we are interested in, then we must consider each part separately where the curve is exclusively above or below the x-axis. If we did *not* do this, and simply performed the single integral:

$$\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0$$

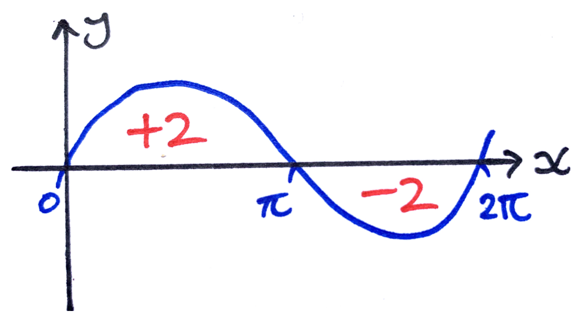
then the result obtained would to be *zero*, as the (informally) "positive area" contributed from $[0, \pi]$ above the curve would be exactly cancelled out by the "negative area" below the curve in the range $[\pi, 2\pi]$ due to the symmetry of the sine function.



This is the "net" area enclosed by the curve over the interval - the amount by which the area above the x-axis is greater than any below the x-axis.

Hence, to find the true total of the area (which is always a positive value) enclosed between the curve and the axis, we must calculate the absolute value (magnitude) of the integrals over each range where the curve is purely above or below the x-axis:

$$
\begin{aligned}
\text{Area} &= \left|\int_0^{\pi} \sin(x)\ \mathrm{d}x\right| + \left|\int_{\pi}^{2\pi} \sin(x)\ \mathrm{d}x\right| \\
&= \left|\Big[-\cos(x)\Big]_0^{\pi}\right| + \left|\Big[-\cos(x)\Big]_{\pi}^{2\pi}\right| \\
&= \Big|(-\cos(\pi)) - (-\cos(0))\Big| + \Big|(-\cos(2\pi)) - (-\cos(\pi))\Big| \\
&= |1-(-1)| + |-1-1| \\
&= |2| + |-2| \\
&= 2+2 \\
&= 4
\end{aligned}
$$



Note that, due again to the symmetry of the sine curve, we could instead have determined this particular area simply by integrating over the range $0 < x < \pi$ and then doubling the result, as the area (in positive terms) under the curve from $\pi < x < 2\pi$ is exactly the same in magnitude:

$$2\int_0^{\pi} \sin(x)\ \mathrm{d}x = 4$$

7. Consider a thin rod orientated on the x-axis over the interval $\left[\frac{\pi}{2}, \pi\right]$.
The density of the rod is given by $\rho(x) = \sin(x)$.

Determine the mass of the rod given that:

$$m = \int_a^b \rho(x)\,\mathrm{d}x$$

where m is the mass, a the lower bound and b the upper bound of the interval.

Solution:

Simply substitute the values of a and b and the function for $\rho(x)$ provided, and evaluate the resulting definite integral:

$$\begin{aligned}
m &= \int_a^b \rho(x)\,\mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \sin(x)\,\mathrm{d}x \\
&= \big[-\cos(x)\big]_{\pi/2}^{\pi} \\
&= \big(-\cos(\pi)\big) - \big(-\cos(\pi/2)\big) \\
&= 1 - 0 \\
&= 1\ \text{kg}
\end{aligned}$$

8. Find the area enclosed between the curve

$$y = x^2 + 8x + 15$$

and the x-axis.

Solution:

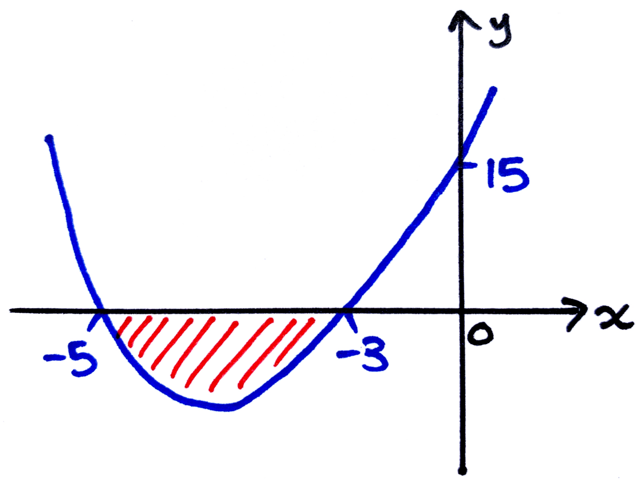
To find the limits of the integral(s), we must locate the points where the y-curve intercepts the x-axis. Setting $y = 0$:

$$x^2 + 8x + 15 = 0$$

This is a quadratic equation, which could be factorised, or solving it using the quadratic formula with $a = 1, b = 8$ and $c = 15$:

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 15}}{2 \times 1} \\ &= \frac{-8 \pm \sqrt{4}}{2} \\ &= \frac{-8 \pm 2}{2} \\ &= -4 \pm 1 \\ &= -5 \quad \text{or} \quad -3 \end{aligned}$$

Then sketching the curve (given that the y-intercept is $c = 15$, and $a = 1 > 0$ means a $\cup$-shaped parabola):



Thus, the area enclosed between the curve and the x-axis occurs in the range $-5 < x < -3$ and we can see that the curve is always below the x-axis throughout this region. This means that we require one integral, which will yield a negative value, the magnitude of which describes the area of this region:

$$
\begin{aligned}
\text{Area} &= \left| \int_{-5}^{-3} x^2 + 8x + 15 \, \mathrm{d}x \right| \\
&= \left| \left[\frac{1}{3}x^3 + \frac{8}{2}x^2 + 15x \right]_{-5}^{-3} \right| \\
&= \left| \left(\frac{1}{3}(-3)^3 + \frac{8}{2}(-3)^2 + 15 \times (-3) \right) - \left(\frac{1}{3}(-5)^3 + \frac{8}{2}(-5)^2 + 15 \times (-5) \right) \right| \\
&= \left| -\frac{4}{3} \right| \\
&= \left| -1.33 \right| \text{ units}^2 \quad (2 \text{ d.p.})
\end{aligned}
$$

So the area of 1.33 units^2 lies beneath the x-axis.

9. Evaluate the following expression for x:

$$x = \int 3t^2 - 7t + 6 \, \mathrm{d}t$$

given that when $t = 0$, $x = 2$.

Solutions:

Evaluating the integral:

$$\begin{aligned} x &= \int 3t^2 - 7t + 6 \, \mathrm{d}t \\ &= \frac{3}{3}t^3 - \frac{7}{2}t^2 + 6t + c \\ &= t^3 - \frac{7}{2}t^2 + 6t + c \end{aligned}$$

Then substitute in the initial condition $x(0) = 2$ to determine the value of c:

$$2 = 0^3 - \frac{7}{2}(0)^2 + 6 \times 0 + c$$

$$\therefore \; 2 = 0 + c$$

$$\therefore \; c = 2$$

Hence, the particular solution is:

$$x = t^3 - \frac{7}{2}t^2 + 6t + 2$$

10. Evaluate this expression for T:

$$T = \int 5\,\mathrm{e}^{-2x} - 3x \,\mathrm{d}x$$

given that $T(x=0) = \frac{5}{3}$.

Solution:

$$\begin{aligned} T &= \int 5\,\mathrm{e}^{-2x} - 3x \,\mathrm{d}x \\ &= \frac{5}{-2}\,\mathrm{e}^{-2x} - \frac{3}{2}x^2 + c \\ &= -\frac{5}{2}\,\mathrm{e}^{-2x} - \frac{3}{2}x^2 + c \end{aligned}$$

To find the value of c, substitute in the condition that when $x = 0$, we have $T = \frac{5}{3}$:

$$\frac{5}{3} = -\frac{5}{2}\,\mathrm{e}^{-2\times 0} - \frac{3}{2}(0)^2 + c$$

$$\therefore \quad \frac{5}{3} = -\frac{5}{2} \cdot 1 - 0 + c$$

$$\therefore \quad c = \frac{5}{3} + \frac{5}{2}$$

$$\therefore \quad c = \frac{25}{6}$$

Thus, the final expression is:

$$T = -\frac{5}{2}\,\mathrm{e}^{-2x} - \frac{3}{2}x^2 + \frac{25}{6}$$

11. Alex attempts to steal a diamond from a bank vault.

   (a) She needs to get out the door of the bank vault, located 4m from the jewel's pedestal, within 3 seconds of snatching the diamond or she will be trapped in the vault by the automatically-closing doors. Following simulations in a replica vault, it is determined that whilst dodging the vault's laser security system, her velocity (in the direction of the vault door) is given by:

   $$v(t) = (6t^2 - 8t)\sin(t^3 - 2t^2)$$

   Will she make it out? (Remember to use radians, not degrees.)

   Solution:

   The criteria for Alex's escape is whether she can cover the distance (i.e. displacement) at time $t = 4$, so we need to obtain a formula for displacement that we can evaluate at that time. Displacement is the integral of velocity, so displacement s at time τ is given by:

   $$s(\tau) = \int_0^{\tau} v(t)\,\mathrm{d}t$$

   So displacement after three seconds:

   $$\begin{aligned} s(3) &= \int_0^3 v(t)\,\mathrm{d}t \\ &= \int_0^3 (6t^2 - 8t)\sin(t^3 - 2t^2)\,\mathrm{d}t \end{aligned}$$

   We can solve this with a substitution:

   Let $u = t^3 - 2t^2$, then $\frac{\mathrm{d}u}{\mathrm{d}t} = 3t^2 - 4t$ and so $\mathrm{d}t = \frac{1}{3t^2-4t}\mathrm{d}u$.

   Calculating the new limits: when $t = 0, u = 0$ and when $t = 3, u = 27-18 = 9$.

Thus:

$$\begin{aligned}
s(3) &= \int_{t=0}^{t=3} (6t^2 - 8t)\sin(t^3 - 2t^2)\,\mathrm{d}t \\
&= \int_{u=0}^{u=15} \frac{2(3t^2 - 4t)}{3t^2 - 4t}\sin(u)\,\mathrm{d}u \\
&= \int_{u=0}^{u=15} 2\sin(u)\,\mathrm{d}u \\
&= \big[-2\cos(u)\big]_{u=0}^{u=9} \\
&= \big(-2\cos(9)\big) - \big(-2\cos(0)\big) \\
&= 2(1 - \cos(9)) \\
&= 3.8223m < 4m
\end{aligned}$$

So she doesn't make it out in time.

Note: it is not essential to use definite integration.Alternatively, we could evaluate the indefinite integral and then assume an initial displacement of zero metres ($s(0) = 0$) and use this initial condition to determine the value of the constant of integration C. Finally, evaluate either $s(u = 9)$ or $s(t = 3)$, to obtain the same solution as above.

(b) At the sentencing, the judge decides that the sentences (in years) should be given as the area between the curve $y = -x^2 + x + 12$ and the x-axis between $x = -5$ and $x = 6$.

How long is she going to prison for?

Your solution should include a sketch of the curve.

Solution:

First, we need to determine the roots of the quadratic, in case they lie within the range $-5 < x < 6$. Either by factorising or using the quadratic formula is acceptable.

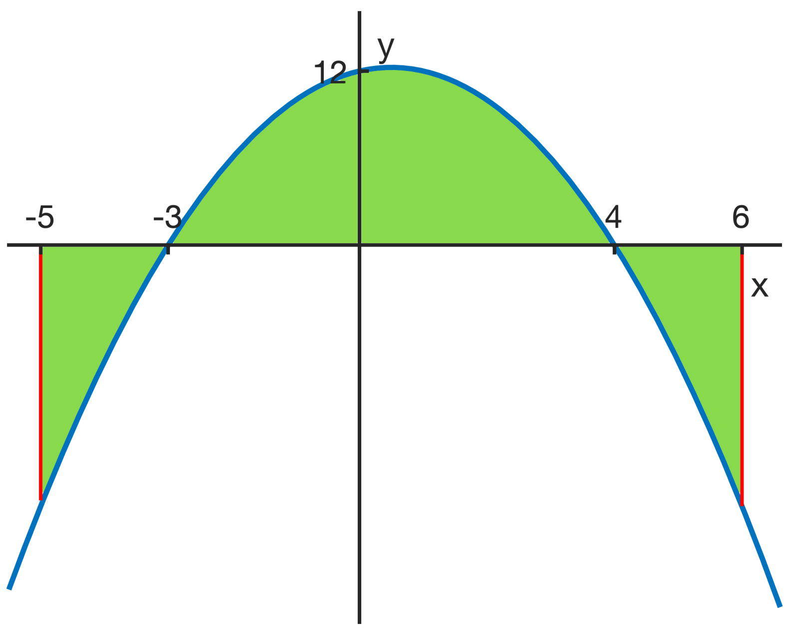
$$y = 0$$

$$-x^2 + x + 12 = 0$$

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

$$x = -3 \quad \text{or} \quad x = 4$$



Then integrate over these three sub-ranges and take the absolute value:

$$
\begin{aligned}
A &= \left|\int_{-5}^{-3} -x^2 + x + 12 \,\mathrm{d}x\right| + \left|\int_{-3}^{4} -x^2 + x + 12 \,\mathrm{d}x\right| + \left|\int_{4}^{6} -x^2 + x + 12 \,\mathrm{d}x\right| \\
&= \left|\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 12x\right]_{-5}^{-3}\right| + \left|\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 12x\right]_{-3}^{4}\right| + \left|\left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 12x\right]_{4}^{6}\right| \\
&= \left|\left(-\frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 + 12(-3)\right) - \left(-\frac{1}{3}(-5)^3 + \frac{1}{2}(-5)^2 + 12(-5)\right)\right| \\
&\quad + \left|\left(-\frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 + 12(4)\right) - \left(-\frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 + 12(-3)\right)\right| \\
&\quad + \left|\left(-\frac{1}{3}(6)^3 + \frac{1}{2}(6)^2 + 12(6)\right) - \left(-\frac{1}{3}(4)^3 + \frac{1}{2}(4)^2 + 12(4)\right)\right| \\
&= \left|\left(9 + \frac{9}{2} - 36\right) - \left(\frac{125}{3} + \frac{25}{2} - 60\right)\right| + \left|\left(-\frac{64}{3} + 8 + 48\right) - \left(9 + \frac{9}{2} - 36\right)\right| \\
&\quad + \left|\left(-\frac{216}{3} + 18 + 72\right) - \left(-\frac{64}{3} + 8 + 48\right)\right| \\
&= \left|33 - 8 - \frac{125}{3}\right| + \left|83 - \frac{64}{3} - \frac{9}{2}\right| + \left|34 + \frac{64}{3} - 72\right| \\
&= \left|\frac{75}{3} - \frac{125}{3}\right| + \left|\frac{498}{6} - \frac{128}{6} - \frac{27}{6}\right| + \left|\frac{64}{3} - \frac{114}{3}\right| \\
&= \left|-\frac{50}{3}\right| + \left|\frac{343}{6}\right| + \left|-\frac{50}{3}\right| \\
&= \frac{50}{3} + \frac{343}{6} + \frac{50}{3} \\
&= \frac{543}{6} \\
&= \frac{181}{2} \\
&= 90.5 \text{ units}^2
\end{aligned}
$$

So a prison sentence of 90 years and 6 months.

12. Starting from a stationary position, a car undergoes a period of constant acceleration at 2 ms^{-2} for 5 seconds. It then maintains speed for a further 12 seconds. What is the final displacement of the car from its initial location?

Solution:

As there are fundamentally two different behaviours, we must calculate the displacement separately for each period of time. For the first 5 seconds $0 < t < 5$, there is constant acceleration of $a(t) = 2$. Integrating with respect to time will yield a linear function for velocity during this period:

$$v_1(t) = \int a(t)\,\mathrm{d}t = \int 2\,\mathrm{d}t = 2t + C$$

And since the car begins from stationary, initial velocity is zero. That is, $v_1(0) = 0$. Hence:

$$0 = 2(0) + C \implies C = 0 \implies v_1(t) = 2t$$

Now integrate again to obtain a formula for the displacement s_1 during this time period:

$$s_1(t) = \int v_1(t)\,\mathrm{d}t = \int 2t\,\mathrm{d}t = t^2 + D$$

And since we are measuring the displacement from the starting position at $t = 0$, by definition the initial displacement is $s_1 = 0$. Hence:

$$0 = (0)^2 + D \implies D = 0 \implies s_1(t) = t^2$$

So after five seconds the displacement from the start is:

$$s_1(5) = (5)^2 = 25\,\text{m}$$

Evaluate $v_1(5)$ to determine the final velocity that is then maintained for the subsequent 12 seconds:

$$v_1(5) = 2(5) = 10\,\text{ms}^{-1}$$

So for the period $5 < t < 17$, the velocity is the constant value $v_2(t) = 10$. To determine the displacement during this time, evaluate the definite integral:

$$
\begin{aligned}
s_2(t) &= \int_5^{17} v_2(t)\,\mathrm{d}t \\
&= \int_5^{17} 10\,\mathrm{d}t \\
&= [10t]_5^{17} \\
&= (10 \times 17) - (10 \times 5) \\
&= 120\,\mathrm{m}
\end{aligned}
$$

But note that **because velocity was constant** during this time, it is also true in this period that "velocity equals change in displacement divided by change in time". That is:

$$v_2 = \frac{\Delta s_2}{\Delta t} \implies \Delta s_2 = v_2 \times \Delta t = 10 \times 12 = 120$$

But you must be clear that this is **only true in cases where the velocity is a constant!!!**

In total, the displacement of the car after 17 seconds is then given by the sum of the displacements over the constituent time periods:

$$25 + 120 = 145\,\mathrm{m}$$

13. Evaluate the following integral:

$$y = \int_0^{10} x(t)\,\mathrm{d}t$$

where $x(t) = 6t^2$ if $0 < t < 3$, and $x(t) = \dfrac{108}{t}$ if $3 < t < 10$.

Solution:

In this case, $x(t)$ is what is called a "piecewise" function, where it exhibits at least two fundamentally different behaviours depending on the input. To integrate across the whole range of t, we must separate the integral into two - splitting up the regions where each behaviour of x occurs. That is, we must separately integrate over the range $0 < t < 3$ and over $3 < t < 10$, and then combine the results at the end.

$$\begin{aligned}
y &= \int_0^3 x(t)\,\mathrm{d}t + \int_3^{10} x(t)\,\mathrm{d}t \\
&= \int_0^3 6t^2\,\mathrm{d}t + \int_3^{10} \frac{108}{t}\,\mathrm{d}t \\
&= \left[\frac{6t^3}{3}\right]_0^3 + \left[108\ln(t)\right]_3^{10} \\
&= \left[2t^3\right]_0^3 + \left[108\ln(t)\right]_3^{10} \\
&= \big(2(27) - 2(0)\big) + \big(108\ln(10) - 108\ln(3)\big) \\
&= 54 + 108\ln\left(\frac{10}{3}\right) \\
&= 184.03 \quad (2\,\text{d.p.})
\end{aligned}$$