Railway Engineering Mathematics

Tutorial Sheet 10

Solutions

1. There are many inter-relationships between the trigonometric functions. In the following, each of the functions on the left is identical to one of the functions on the right. By observing the graphs, or otherwise, match up the pairs:

$y = \sin(-x)$	$y = -\cos(x)$
$y = \tan(-x)$	$y = \cos(x)$
$y = \cos(-x)$	$y = -\sin(x)$
$y = \sin\left(x + \frac{\pi}{2}\right)$	$y = -\tan(x)$
$y = \sin\left(x - \frac{\pi}{2}\right)$	$y = \sin(x)$

Solution:

$$\sin(-x) = -\sin(x)$$

$$\tan(-x) = -\tan(x)$$

This property is called being "odd", and means that sine and tangent possess a rotational symmetry about the origin.

$$\cos(-x) = \cos(x)$$

This is called being "even", and means cosine has a reflective symmetry across the y-axis.

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

$$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$$

This is because the sine and cosine waves behave identically, but are "out of phase" with each other. That is, they are the same graph but shifted horizontally by $\frac{\pi}{2}$ radians.

2. Download the `Picturebook.xlsx` from the module Blackboard site and navigate to the tab that plots sinusoidal functions of the form:

$$y = A\sin(\omega x + \phi) + B$$

Describe how changing each of the following affects the graph:

(a) A (fix the values of the other parameters at: $\omega = 1, \phi = 0$, and $B = 0$)

(b) ω (fix the values of the other parameters at: $A = 1, \phi = 0$, and $B = 0$)

(c) ϕ (fix the values of the other parameters at: $A = 1, \omega = 1$, and $B = 0$)

(d) B (fix the values of the other parameters at: $A = 1, \omega = 1$, and $\phi = 0$)

Solution:

(a) Varying A stretches the graph in the y-direction by a factor of $\pm A$. If A is negative, then the graph will also be reflected about the x-axis (i.e. it will be flipped upside-down).

(b) Varying ω increases/decreases the number of cycles in any given range (such as between 0 and 2π). So increasing ω compresses the graph in the x-axis, for example doubling the value of ω doubles the frequency of cycles and thus halves the wavelength (period) of one whole cycle.

(c) Changing ϕ shifts the curve by $\mp\phi$ in the x-axis (increasing ϕ moves the graph left, and decreasing it moves the graph to the right).

(d) B shifts the curve by $\pm B$ in the y-axis (increasing the value of B moves the graph vertically up) without changing the shape of the curve.

3. Plot the following functions using EXCEL (choose a sensible range of t):

   (a) $y = 4\sin(t)$

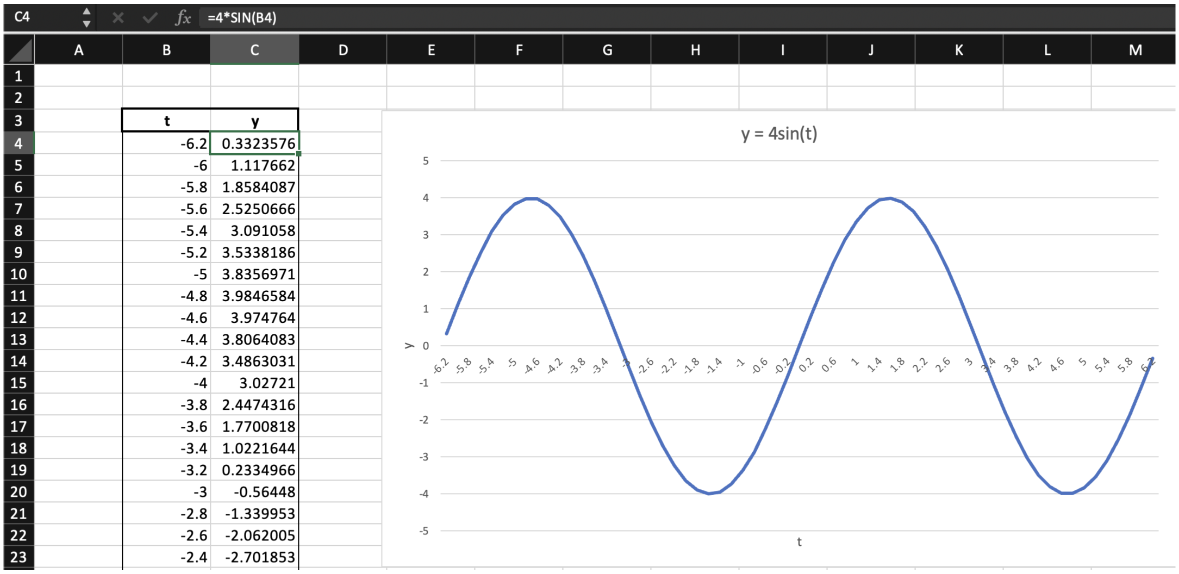
   (b) $y = 3.4\sin(2.4t - 3.2)$

   (c) $y = 2.15\cos(3.6t + 0.5)$

   (d) $y = 7\sin\left(\frac{t}{2} - \frac{3\pi}{4}\right)$

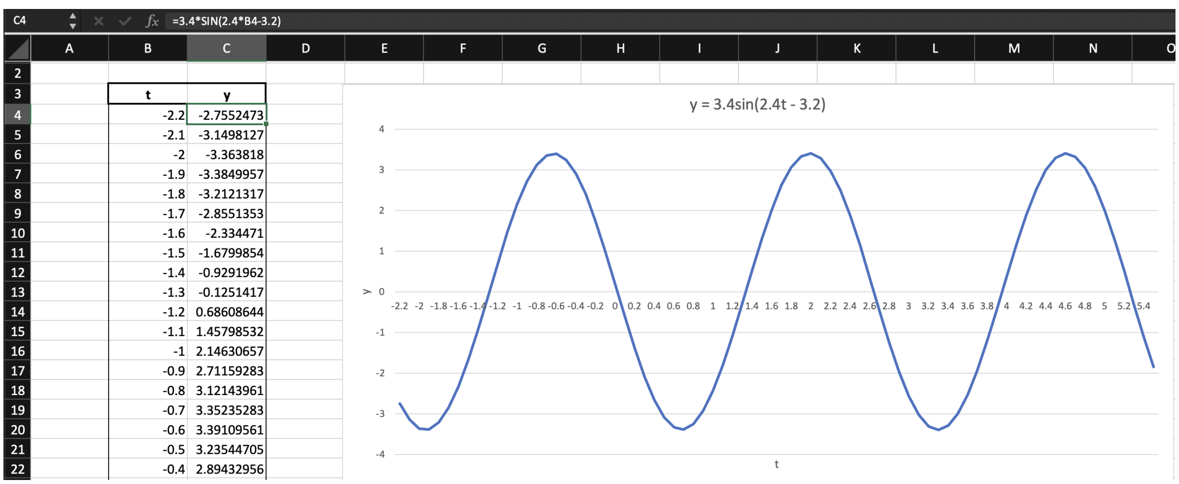
   Solution:

   To view the actual EXCEL spreadsheet used to create the following see `Tutorial_10_Q3_solutions.xlsx`



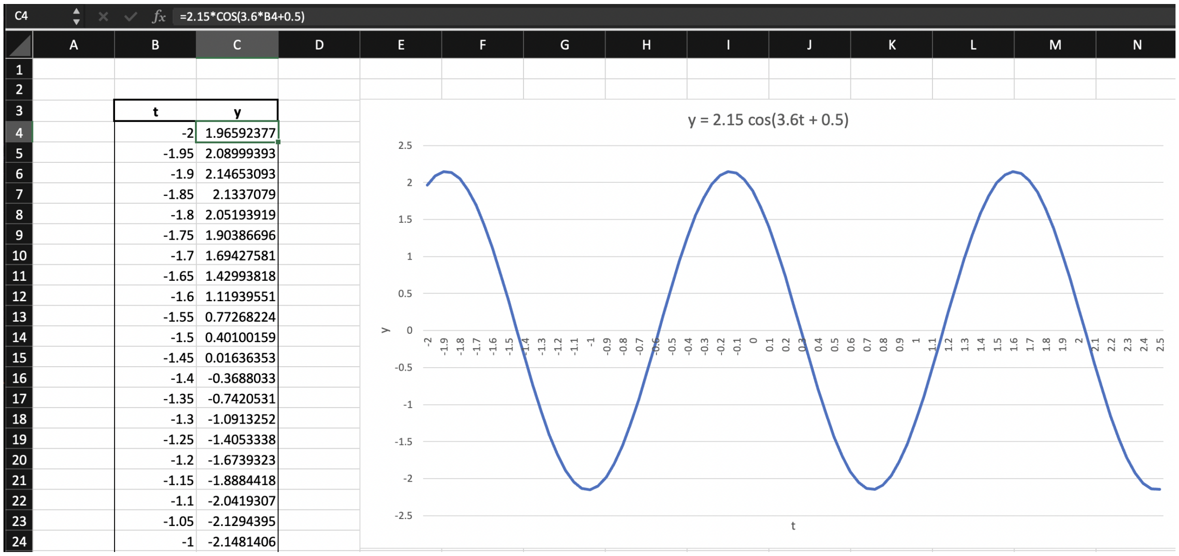
t	y
-6.2	0.3323576
-6	1.117662
-5.8	1.8584087
-5.6	2.5250666
-5.4	3.091058
-5.2	3.5338186
-5	3.8356971
-4.8	3.9846584
-4.6	3.974764
-4.4	3.8064083
-4.2	3.4863031
-4	3.02721
-3.8	2.4474316
-3.6	1.7700818
-3.4	1.0221644
-3.2	0.2334966
-3	-0.56448
-2.8	-1.339953
-2.6	-2.062005
-2.4	-2.701853

(a) $y = 4\sin(t)$



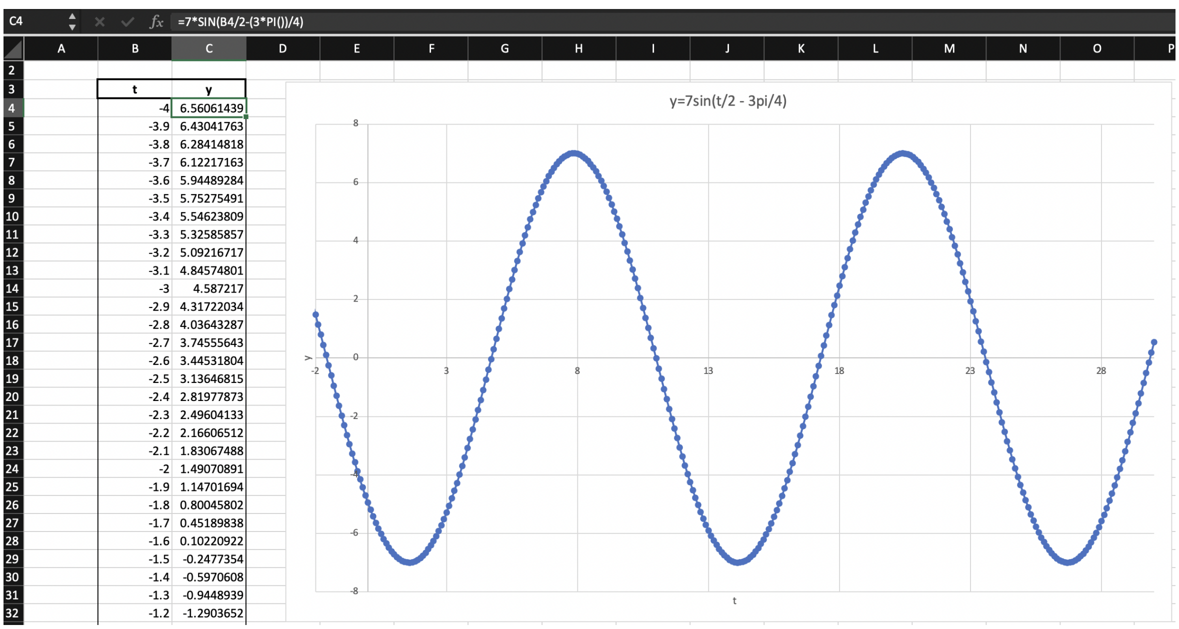
t	y
-2.2	-2.7552473
-2.1	-3.1498127
-2	-3.363818
-1.9	-3.3849957
-1.8	-3.2121317
-1.7	-2.8551353
-1.6	-2.334471
-1.5	-1.6799854
-1.4	-0.9291962
-1.3	-0.1251417
-1.2	0.68608644
-1.1	1.45798532
-1	2.14630657
-0.9	2.71159283
-0.8	3.12143961
-0.7	3.35235283
-0.6	3.39109561
-0.5	3.23544705
-0.4	2.89432956

(b) $y = 3.4\sin(2.4t - 3.2)$



t	y
-2	1.96592377
-1.95	2.08999393
-1.9	2.14653093
-1.85	2.1337079
-1.8	2.05193919
-1.75	1.90386696
-1.7	1.69427581
-1.65	1.42993818
-1.6	1.11939551
-1.55	0.77268224
-1.5	0.40100159
-1.45	0.01636353
-1.4	-0.3688033
-1.35	-0.7420531
-1.3	-1.0913252
-1.25	-1.4053338
-1.2	-1.6739323
-1.15	-1.8884418
-1.1	-2.0419307
-1.05	-2.1294395
-1	-2.1481406

(c) $y = 2.15\cos(3.6t + 0.5)$

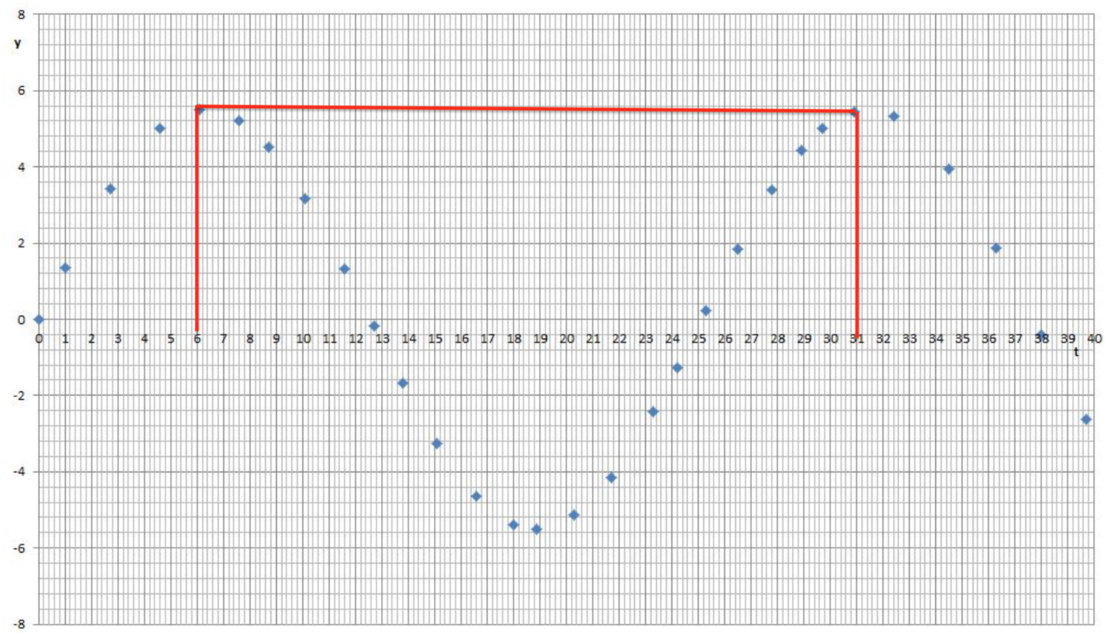


t	y
-4	6.56061439
-3.9	6.43041763
-3.8	6.28414818
-3.7	6.12217163
-3.6	5.94489284
-3.5	5.75275491
-3.4	5.54623809
-3.3	5.32585857
-3.2	5.09216717
-3.1	4.84574801
-3	4.587217
-2.9	4.31722034
-2.8	4.03643287
-2.7	3.74555643
-2.6	3.44531804
-2.5	3.13646815
-2.4	2.81977873
-2.3	2.49604133
-2.2	2.16606512
-2.1	1.83067488
-2	1.49070891
-1.9	1.14701694
-1.8	0.80045802
-1.7	0.45189838
-1.6	0.10220922
-1.5	-0.2477354
-1.4	-0.5970608
-1.3	-0.9448939
-1.2	-1.2903652

(d) $y = 7\sin\left(\frac{t}{2} - \frac{3\pi}{4}\right)$

4. Determine an equation for each of the following sinusoidal curves:

(a)



Solution:

The general form is:

$$y = A\sin(\omega t + \phi) + B$$

(Note that we could instead use the general cosine function, which would simply result in obtaining a different value for the phase shift).

As the graph oscillates between $y = 5.5$ and $y = -5.5$, the amplitude is $A = 5.5$ and there is no vertical shift $B = 0$. To check these with calculations, we could measure half of the distance between a peak and a trough to determine the amplitude:

$$A = \frac{5.5 - (-5.5)}{2} = 5.5$$

And the vertical shift can be found by taking the average value of a peak and a trough:

$$B = \frac{5.5 + (-5.5)}{2} = 0$$

From the graph, we identify the location of two neighbouring peaks at $t = 6$ and $t = 31$. Then a single complete cycle occurs with period given by the horizontal distance between them:

$$T = 31 - 6 = 25$$

Hence the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{25} = 0.25 \quad (2\,\text{d.p.})$$

Therefore, so far we have:

$$y = 5.5\sin(0.25t + \phi)$$

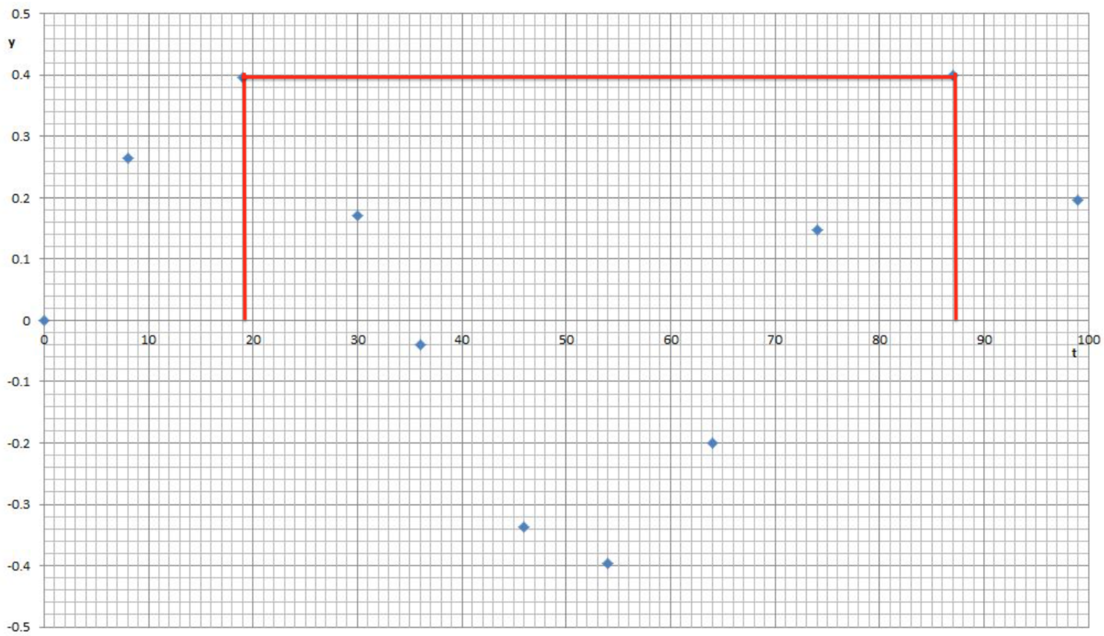
As there is only one unknown variable remaining, we can deduce this by substituting in the co-ordinates of a point that the graph passes through. From the graph, we can observe that when $t = 0$, we have $y = 0$. Thus:

$$\begin{aligned}
&0 = 5.5\sin(0.25 \times 0 + \phi) \\
&\therefore\ 0 = 5.5\sin(\phi) \\
&\therefore\ \phi = \sin^{-1}(0) \\
&\qquad\ = 0
\end{aligned}$$

So the final formula of this graph is:

$$y = 5.5\sin(0.25t)$$

(b)



Solution:

The general form is:

$$y = A\sin(\omega t + \phi) + B$$

As the graph oscillates between $y = 0.4$ and $y = -0.4$, the amplitude is $A = 0.4$ and due to the symmetry it is clear that there is no vertical shift.

From the graph, we measure the location of two neighbouring peaks at $t = 19$ and $t = 87$, then the difference between them means that a single complete cycle occurs with period:

$$T = 87 - 19 = 68$$

Hence the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{68} = 0.09 \quad (2\,\text{d.p.})$$

Therefore, so far we have:

$$y = 0.4\sin(0.09t + \phi)$$

Then to determine the value of ϕ, we can choose a point that the curve passes through such as $(0, 0)$. Thus, substituting $t = 0$ and $y = 0$ into the formula and solving for ϕ:

$$0 = 0.4\sin(0.09 \times 0 + \phi)$$

$$\therefore\ 0 = 0.4\sin(\phi)$$

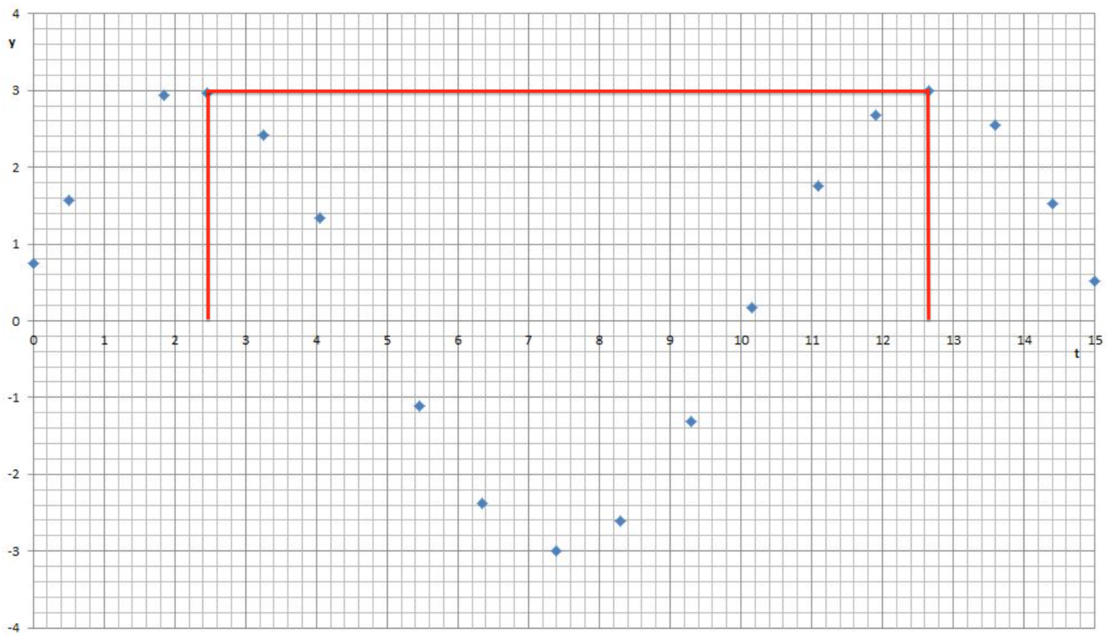
$$\therefore\ \phi = \sin^{-1}(0)$$

$$= 0$$

So finally we have:

$$y = 0.4\sin(0.09t)$$

(c)



Solution:

The general form is:

$$y = A\sin(\omega t + \phi) + B$$

As the graph oscillates between $y = 3$ and $y = -3$, the amplitude is $A = 3$ and there is no vertical shift so $B = 0$.

From the graph, we can identify two neighbouring peaks at $t = 2.45$ and $t = 12.65$, so a single complete cycle occurs with period:

$$T = 12.65 - 2.45 = 10.2$$

Hence the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{10.2} = 0.616$$

Therefore, so far we have found:

$$y = 3\sin(0.616t + \phi)$$

Finally, to determine the value of ϕ, given that the curve passes through the point $(2.45, 3)$, we can substitute into the equation that when $t = 2.45$, we have $y = 3$:

$$3 = 3\sin(0.616 \times 2.45 + \phi)$$
$$\therefore\ 1 = \sin(1.5092 + \phi)$$

Taking inverse sine of both sides:

$$\therefore\ 1.5092 + \phi = \sin^{-1}(1)$$
$$\therefore\ 1.5092 + \phi = \frac{\pi}{2}$$
$$\therefore\ \phi = \frac{\pi}{2} - 1.5092$$
$$= 0.062$$

So finally we have:

$$y = 3\sin(0.616t + 0.062)$$

5. Find all solutions of the following:

(a) $\sin\left(2t + \frac{\pi}{2}\right) = 0.5$ in the range $-\pi \leq t \leq \pi$.

Solution:

Let's make the substitution $u = 2t + \frac{\pi}{2}$. We also need to change the range in which we are solving: when $t = -\pi$, $u = 2 \times -\pi + \frac{\pi}{2} = -\frac{3\pi}{2}$. Similarly when $t = \pi$, $u = 2 \times \pi + \frac{\pi}{2} = \frac{5\pi}{2}$.

Therefore, we are now solving

$$\sin(u) = 0.5 \quad \text{in the range} \quad -\frac{3\pi}{2} \leq u \leq \frac{5\pi}{2} \quad (-4.7124 \leq u \leq 7.854).$$

Next we obtain the principal value:

$$u_0 = \sin^{-1}(0.5) = \frac{\pi}{6} = 0.5236$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.5236 = 2.6180$$

Adding and subtracting multiples of 2π, we find four solutions for u in the acceptable range. Then convert these back to solutions for t using:

$$t = \frac{u - \frac{\pi}{2}}{2}$$

The other solutions can be found as follows:

u	In Range?	$t = \frac{u - \frac{\pi}{2}}{2}$
$u_0 = 0.5236$	Yes	$(0.5236 - \frac{\pi}{2})/2 = -0.5236$
$u_0 + 2\pi = 0.5236 + 2\pi = 6.8068$	Yes	2.6180
$u_0 + 4\pi = 0.5236 + 4\pi = 13.090$	No	-
$u_0 - 2\pi = 0.5236 - 2\pi = -5.7596$	No	-
$u_1 = 2.6180$	Yes	0.5236
$u_1 + 2\pi = 2.6180 + 2\pi = 8.9012$	No	-
$u_1 - 2\pi = 2.6180 - 2\pi = -3.6652$	Yes	-2.6180
$u_1 - 4\pi = 2.6180 - 4\pi = -9.9484$	No	-

So we have four valid solutions: $t = -2.6180, \;\; -0.5236, \;\; 0.5236, \;\; 2.6180$.

(b) $3\sin(5.1t - 1.12) = 0.75$ in the range $0 \le t \le \frac{2\pi}{3}$.

Solution:

Let's make the substitution $u = 5.1t - 1.12$. We also need to change the range in which we are solving: when $t = 0$, $u = 5.1 \times 0 - 1.12 = -1.12$. Similarly when $t = \frac{2\pi}{3}$, $u = 5.1 \times \frac{2\pi}{3} - 1.12 = 9.5614$.

Therefore, we are now solving

$$3\sin(u) = 0.75 \quad \text{in the range} \quad -1.12 \le u \le 9.5614.$$

Next we obtain the principal value:

$$\begin{aligned} 3\sin(u) &= 0.75 \\ \sin(u) &= 0.25 \\ u_0 &= \sin^{-1}(0.25) \\ &= 0.2527 \end{aligned}$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.2527 = 2.8889$$

Adding and subtracting multiples of 2π, we find four solutions for u in the acceptable range. Then convert these back to solutions for t using:

$$t = \frac{u + 1.12}{5.1}$$

The other solutions can be found as follows:

u	In Range?	$t = \frac{u+1.12}{5.1}$
$u_0 = 0.2527$	Yes	$(0.5236 + 1.12)/5.1 = 0.2692$
$u_0 + 2\pi = 0.2527 + 2\pi = 6.5359$	Yes	1.5012
$u_0 + 4\pi = 0.2527 + 4\pi = 12.8191$	No	-
$u_0 - 2\pi = 0.2527 - 2\pi = -6.0305$	No	-
$u_1 = 2.8889$	Yes	0.7861
$u_1 + 2\pi = 2.8889 + 2\pi = 9.1721$	Yes	2.0181
$u_1 - 2\pi = 2.8889 - 2\pi = -3.3943$	No	-

So we have four valid solutions: $t = 0.2692, \;\; 0.7861, \;\; 1.5012, \;\; 2.0181.$

(c) $\dfrac{7}{9}\sin(\pi t + 2.3) = \dfrac{3}{7}$ in the range $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$.

Solution:

Let's make the substitution $u = \pi t + 2.3$. We also need to change the range in which we are solving: when $t = -\dfrac{\pi}{2}$, $u = \pi \times -\dfrac{\pi}{2} + 2.13 = -2.6348$. Similarly when $t = \dfrac{\pi}{2}$, $u = \pi \times \dfrac{\pi}{2} + 2.13 = 7.2348$.

Therefore, we are now solving

$$\frac{7}{9}\sin(u) = \frac{3}{7} \quad \text{in the range} \quad -2.6348 \leq u \leq 7.2348.$$

Next we obtain the principal value:

$$\begin{aligned}
\frac{7}{9}\sin(u) &= \frac{3}{7} \\
\sin(u) &= \frac{21}{49} \\
u_0 &= \sin^{-1}\left(\frac{21}{49}\right) \\
&= 0.5836
\end{aligned}$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.5836 = 2.5580$$

Adding and subtracting multiples of 2π, we find three solutions for u in the acceptable range. Then convert these back to solutions for t using:

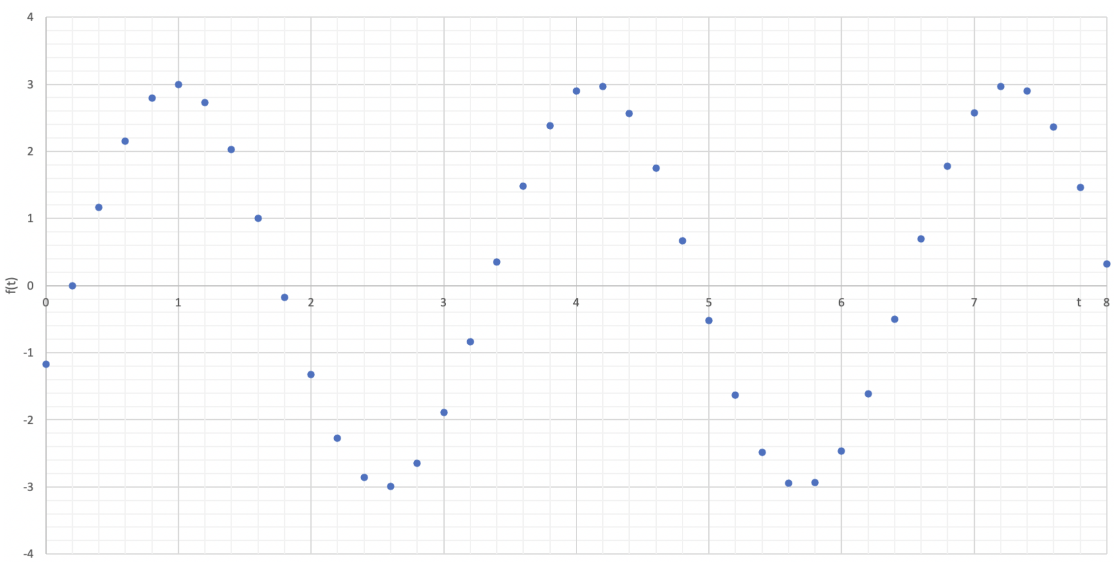
$$t = \frac{u - 2.3}{\pi}$$

The other solutions can be found as follows:

u	In Range?	$t = \dfrac{u - 2.3}{\pi}$
$u_0 = 0.5836$	Yes	$(0.5836 - 2.3)/\pi = -0.5463$
$u_0 + 2\pi = 0.5836 + 2\pi = 6.8668$	Yes	1.4537
$u_0 + 4\pi = 0.5236 + 4\pi = 13.1500$	No	-
$u_0 - 2\pi = 0.5236 - 2\pi = -5.6996$	No	-
$u_1 = 2.5580$	Yes	0.0821
$u_1 + 2\pi = 2.5580 + 2\pi = 8.8412$	No	-
$u_1 - 2\pi = 2.5580 - 2\pi = -3.7252$	No	-

So we have three valid solutions: $t = -0.5463, \quad 0.0821, \quad 1.4537.$

6. Given the signal depicted below:



(a) determine the equation of the function that passes through the points.

Solution:

The general form is:

$$y = A\sin(\omega t + \phi) + B$$

As the graph oscillates between $f(t) = 3$ and $f(t) = -3$, the amplitude is $A = 3$ and there is no vertical shift $B = 0$. To check these with calculations, we could measure half of the distance between a peak and a trough to determine the amplitude:

$$A = \frac{3 - (-3)}{2} = 3$$

And the vertical shift can be found by taking the average value of a peak and a trough:

$$B = \frac{3 + (-3)}{2} = 0$$

From the graph, we identify the location of two neighbouring peaks at $t = 1$ and $t = 4.2$. Then a single complete cycle occurs with period given by the horizontal distance between them:

$$T = 4.2 - 1 = 3.2$$

Hence the angular frequency is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.2} = \frac{5\pi}{8}$$

Therefore, so far we have:

$$f(t) = 3\sin\left(\frac{5\pi}{8}t + \phi\right)$$

As there is only one unknown variable remaining, we can deduce this by substituting in the co-ordinates of a point that the graph passes through. From the graph, we can observe that when $t = 0$, we have $f(t) = -1.2$. Thus:

$$
\begin{aligned}
-1.2 &= 3\sin\left(\frac{5\pi}{8}\times 0+\phi\right)\\
\therefore\ -1.2 &= 3\sin(\phi)\\
\therefore\ -0.4 &= \sin(\phi)\\
\therefore\ \phi &= \sin^{-1}(-0.4)\\
&= -0.41\ \ (2\text{ d.p.})
\end{aligned}
$$

So the final formula of this graph is:

$$f(t) = 3\sin\left(\frac{5\pi}{8}t - 0.41\right)$$

(b) Hence, determine the solutions of t for when $f(t) = 1$ in the range $12 \le t \le 16$.

Solution:

We wish to solve:

$$1 = 3\sin\left(\frac{5\pi}{8}t - 0.41\right) \quad \text{in the range } \ 12 \le t \le 16.$$

Let's make the substitution $u = \dfrac{5\pi}{8}t - 0.41$. We also need to change the range in which we are solving: when $t = 12$, $u = \dfrac{5\pi}{8}\times 12 - 0.41 = 23.1519$. Similarly when $t = 16$, $u = \dfrac{5\pi}{8}\times 16 - 0.41 = 31.0059$.

Therefore, we are now solving

$$3\sin(u) = 1 \quad \text{in the range} \quad 23.1519 \le u \le 31.0059.$$

Next we obtain the principal value:

$$\begin{aligned}
3\sin(u) &= 1 \\
\sin(u) &= \frac{1}{3} \\
u_0 &= \sin^{-1}\left(\frac{1}{3}\right) \\
&= 0.3398
\end{aligned}$$

From the symmetry of the graph, the other solution in the first period is:

$$u_1 = \pi - 0.3398 = 2.8018$$

Adding and subtracting multiples of 2π, we find two solutions for u in the acceptable range. Then convert these back to solutions for t using:

$$t = \frac{8(u + 0.41)}{5\pi}$$

The other solutions can be found as follows:

u	In Range?	$t = \frac{8(u+0.41)}{5\pi}$
$u_0 = 0.3398$	No	-
$u_0 + 2\pi = 0.3398 + 2\pi = 6.6230$	No	-
$u_0 + 4\pi = 0.3398 + 4\pi = 12.9062$	No	-
$u_0 + 6\pi = 0.3398 + 6\pi = 19.1894$	No	-
$u_0 + 8\pi = 0.3398 + 8\pi = 25.4725$	Yes	13.1818
$u_0 + 10\pi = 0.3398 + 10\pi = 31.7557$	No	-
$u_1 = 2.8018$	No	-
$u_1 + 4\pi = 2.8018 + 4\pi = 15.3682$	No	-
$u_1 + 6\pi = 2.8018 + 6\pi = 21.6514$	No	-
$u_1 + 8\pi = 2.8018 + 8\pi = 27.9345$	Yes	14.4357
$u_1 + 10\pi = 2.8018 + 10\pi = 34.2177$	No	-

So we have two valid solutions: $t = 13.1818$ and $t = 14.4357$.

Note on multiple equivalent solutions

Returning to the standard sine equation:

$$y = \sin(x)$$

as sine is periodic with period 2π, it is the case that $y = \sin(x)$ is completely equivalent to:

$$y = \sin(x + 2\pi), \quad y = \sin(x + 4\pi), \quad y = \sin(x - 2\pi), \quad y = \sin(x + 6\pi), \quad \text{etc.}$$

For example, let $x = 0.5$:

$$\sin(0.5) = 0.48$$

$$\sin(0.5 + 2\pi) = 0.48$$

$$\sin(0.5 + 4\pi) = 0.48$$

$$\sin(0.5 + 6\pi) = 0.48$$

So considering a more complicated example, e.g.

$$y = 3\sin(5.6t - 4.2)$$

Then this would be the same as:

$$y = 3\sin(5.6t - 4.2 + 2\pi) = 3\sin(5.6t + 2.08)$$

$$y = 3\sin(5.6t - 4.2 + 4\pi) = 3\sin(5.6t + 8.37)$$

$$y = 3\sin(5.6t - 4.2 - 2\pi) = 3\sin(5.6t - 10.48)$$

Depending on which co-ordinates you chose when calculating the value of the horizontal shift ϕ in Question 5, you might seem to obtain a different value of ϕ to the solutions given here. However, if you check and confirm that it is equal to the same value $\pm$ some integer multiple of 2π then you know that it is an equivalent solution.