Railway Engineering Mathematics

Tutorial Sheet 8

Solutions

1. A rectangular football pitch has its length equal to twice its width and a perimeter of 360 m. Find its length and width.

Solution:

Here, we have two unknown values: the length and width. However, we are informed that the perimeter is 360 m.

The perimeter, P of a rectangle is:

$$P = 2l + 2w,$$

where l and w represent the length and width. We know that the perimeter is 360 m, therefore

$$\begin{aligned} P &= 2l + 2w \\ 360 &= 2l + 2w. \end{aligned}$$

We are unable to determine the two knowns without further information, but we know that the length is equal to twice the width, therefore we can write

$$\begin{aligned} 360 &= 2l + 2w \\ 360 &= 2l + l \\ 360 &= 3l \\ l &= \frac{360}{3} \\ l &= 120 \text{ m} \end{aligned}$$

Given that the length of the football pitch is 120 m, the width is half of that value, $\therefore$ 60 m.

2. (a) Given that $y = \sqrt{x}$, show that the equation

$$\sqrt{x} + \frac{10}{\sqrt{x}} = 7 \qquad (1)$$

maybe written as

$$y^2 - 7y + 10 = 0$$

(b) Hence solve equation (1).

Solution:

(a) Making the substitution $y = \sqrt{x}$:

$$\begin{aligned} \sqrt{x} + \frac{10}{\sqrt{x}} &= 7 \\ y + \frac{10}{y} &= 7 \end{aligned}$$

Multiplying through by y,

$$\begin{aligned} y + \frac{10}{y} &= 7 \\ y^2 + 10 &= 7y \\ y^2 - 7y + 10 &= 0 \end{aligned}$$

(b) This is a quadratic equation and therefore can be solved using the quadratic formula:

$$
\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
y &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 1 \times 10}}{2 \times 1} \\
y &= \frac{7 \pm \sqrt{9}}{2} \\
y &= \frac{7 \pm 3}{2}
\end{aligned}
$$

Therefore,

$$y_1 = \frac{4}{2} = 2 \quad \text{and} \quad y_2 = \frac{10}{2} = 5.$$

Our original task was to solve equation (1). Using the substitution we declared earlier: $y = \sqrt{x}$:

$$\sqrt{x_1} = 2 \quad \text{and} \quad \sqrt{x_2} = 5.$$

Squaring both sides of the equations will give the final solutions:

$$x_1 = 4 \quad \text{and} \quad x_2 = 25.$$

3. A person drove from the Dead Sea up to Amman, and their altitude increased at a constant rate. When they began driving, their altitude was 400 metres below sea level. When they arrived in Amman 2 hours later, their altitude was 1000 metres above sea level.

   After how many hours does the person take to reach an altitude of 1750 metres?

   Solution:

We must first establish the relationship between the altitude (dependant axis) and the number of hours driving (independent axis). Let a represent the altitude in metres (relative to sea level) and t the time in hours.

We have two coordinates connecting the altitude and time: (0,-400) and (2,1000). Since the altitude is increasing at a constant rate, the relationship is linear, which is represented by the the equation $a = mt + c$. From the coordinate, the values of the gradient and a-intercept can be determined:

the gradient m:

$$m = \frac{\Delta a}{\Delta t} = \frac{1000 - -400}{2 - 0} = \frac{1400}{2} = 700$$

Hence we have:

$$a = 700t + c$$

Substitute in the co-ordinates of point $(0, -400)$ and solve for c:

$$\begin{aligned} -400 &= 700(0) + c \\ -400 &= c \end{aligned}$$

Hence, the relationship between altitude and the time is

$$a = 700t - 400.$$

Now that the relationship is establish, we are able to calculate the time it takes to reach an altitude of 1750 m:

$$\begin{aligned} a &= 700t - 400 \\ 1750 &= 700t - 400 \\ 2150 &= 700t \\ t &= \frac{2150}{700} \approx 3.07 \text{ hours.} \end{aligned}$$

4. The manager of a factory finds that it costs £2200 to produce 100 parts in one day and £4800 to produce 300 parts in one day.

   (a) Express the cost as a function of the number of parts produced, assuming that it is linear.

   (b) Plot a graph of the function from part (a), using any software.

   (c) What does the gradient of the graph represent?

   (d) What does the y-intercept represent?

Solution:

(a) Firstly, let's define the variables: let C represent the cost to produce the parts and let p represent the number of parts produced.

Given that we can assume the relationship is linear, we can use the formula:

$$C = mp + d,$$

where m and d represent the gradient and y-intercept, respectively.

First, determine the gradient m:

$$m = \frac{\Delta C}{\Delta p} = \frac{4800 - 2200}{300 - 100} = \frac{2600}{200} = 13$$

Hence we have:

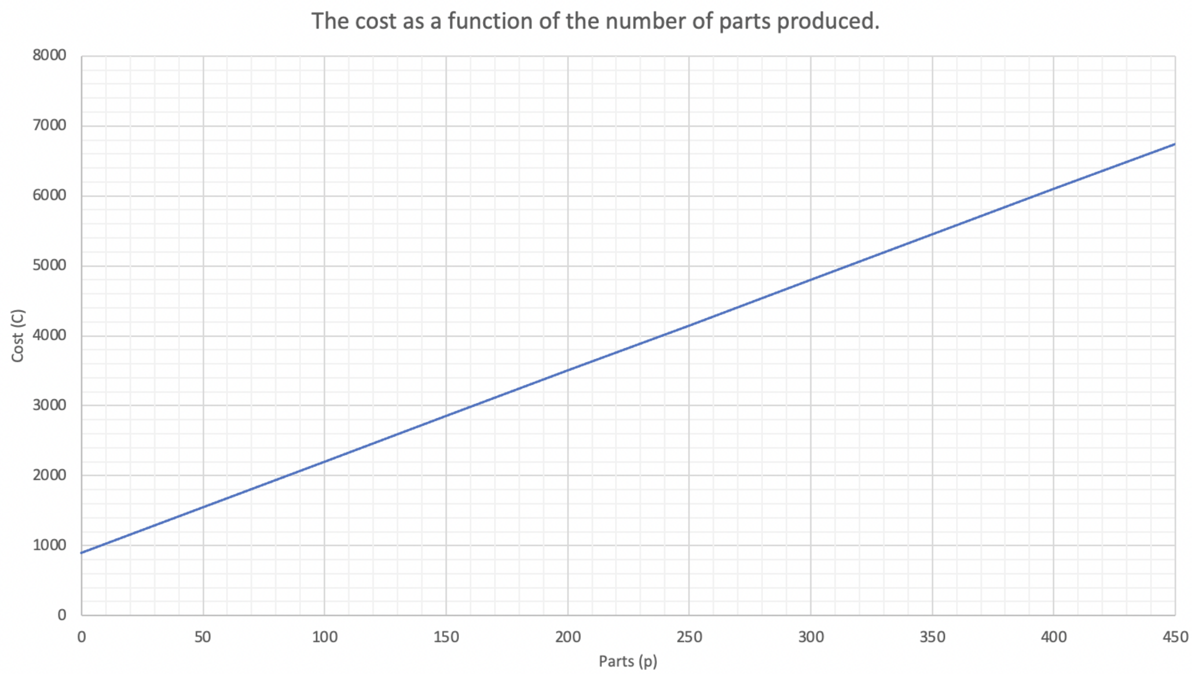
$$C = 13p + d$$

Substitute in the co-ordinates of point $(100, 2200)$ and solve for d:

$$\begin{aligned} 2200 &= 13(100) + d \\ 2200 &= 1300 + d \\ d &= 2200 - 1300 = 900 \end{aligned}$$

Hence,

$$C = 13p + 900$$



(b)

(c) The gradient represents how much is costs to produce one part.

(d) The y-intercept represents the cost of starting production before one part is produced.

5. A bacterial culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

(d) Graph the population function and estimate the time for the population to reach 100,000.

Solution:

(a) Since 3 hours is equal to 6 half hours, the culture will have doubled 6 times. Therefore, there will be

$$500 \cdot 2^6 = 32,000 \text{ bacteria.}$$

(b) Since t hours is equal to $2t$ half hours, the culture will have doubled $2t$ times. Therefore, there will be

$$500 \cdot 2^{2t} \text{ bacteria.}$$

(c) There are two possible answers depending on how you interpret the set-up to the problem. If each bacterium in the culture doubles once every half hour on the half hour, then each one will double after exactly 30 minutes, and then not again until 60 minutes have passed. In that case, there will be

$$500 \cdot 2 = 1000 \text{ bacteria after 40 minutes.}$$

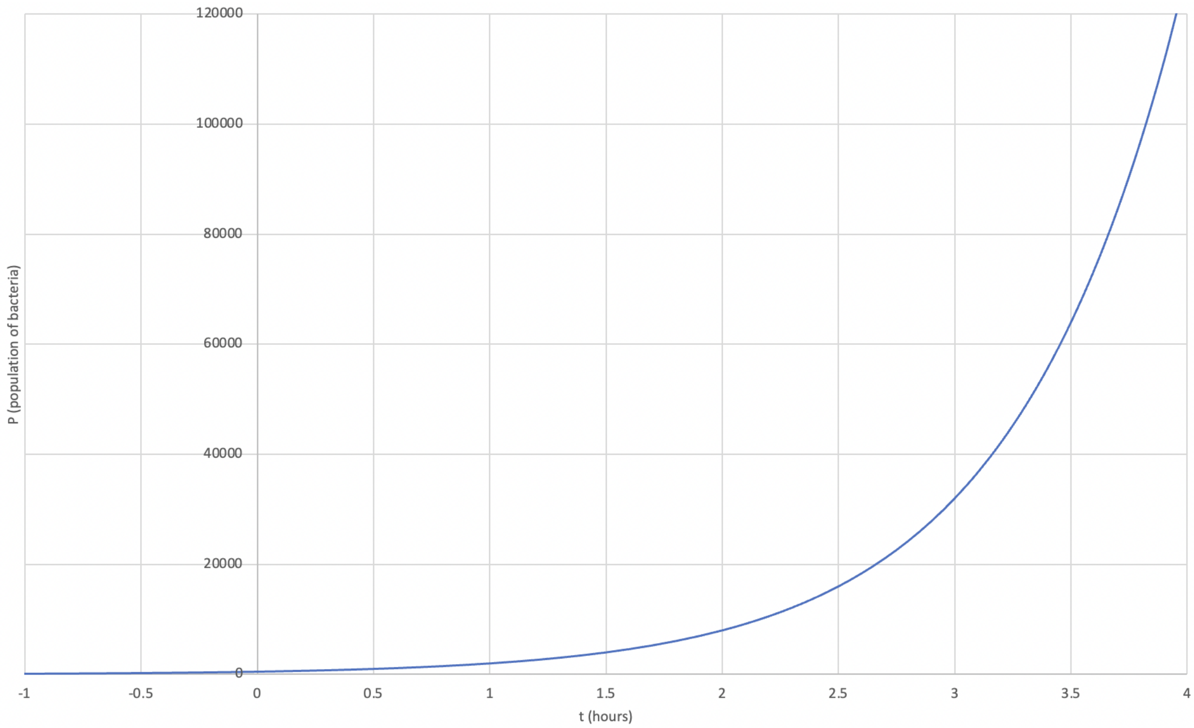
On the other hand, rather than each bacterium doubling exactly once per half hour, it increases at some random time within that half hour, then it makes sense to think of the population function $P(t) = 500 \cdot 2^{2t}$ as continuous (where P is the population of the bacteria and t is the time in hours). In that case, since 40 minutes is

$$\frac{40}{60} = \frac{2}{3}$$

of an hour, the population will be

$$500 \cdot 2^{2\times\frac{2}{3}} = 500 \cdot 2^{\frac{4}{3}} \approx 1259 \text{ bacteria after 40 minutes.}$$

(d) Graphing the function $P(t) = 500 \cdot 2^{2t}$ in EXCEL:

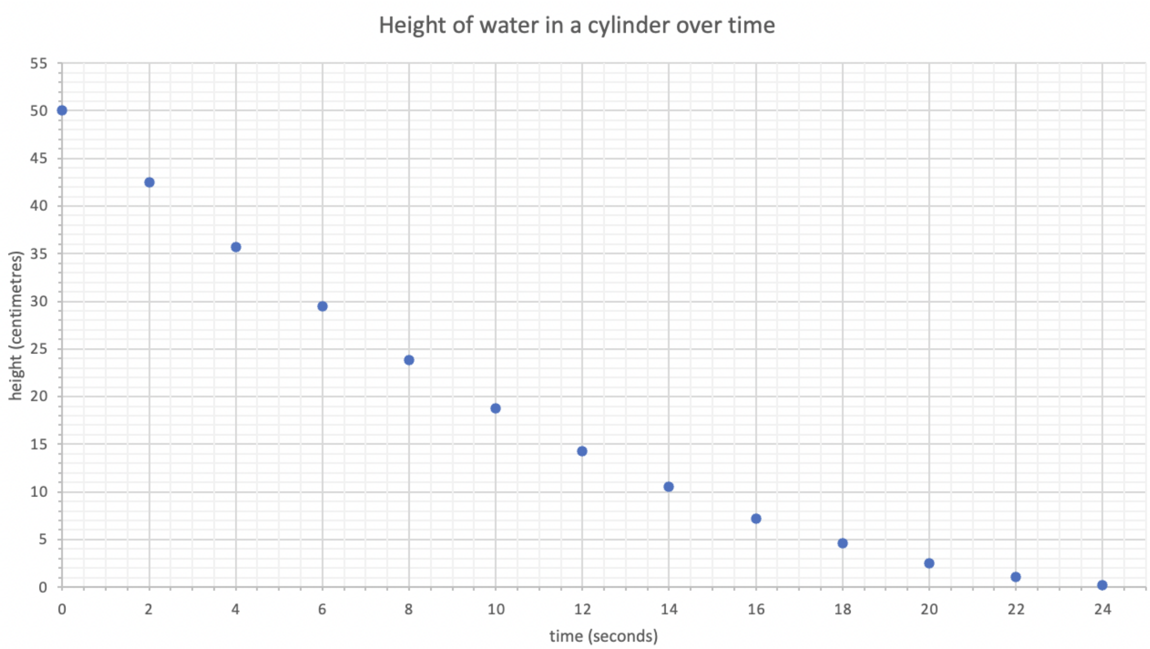


Here it can be seen that the function $P(t) = 500 \cdot 2^{2t}$ passes 100,000 at approximately $t = 3.8$.

6. A cylinder is filled with water to a height of 50 cm. The cylinder has a hole at the bottom which is covered with a stopper. The stopper is released at time $t = 0$ seconds and is allowed to empty. The following data shows the height of the water in the cylinder at different times:

Time (seconds)	Height (cm)
0	50
2	42.5
4	35.7
6	29.5
8	23.8
10	18.8
12	14.3
14	10.5
16	7.2
18	4.6
20	2.5
22	1.1
24	0.2

Depicted in the graph below is the same data:



Assume that the relationship between the height of the water and the time is linear.

(a) With a ruler, draw a linear line of best fit on the graph and determine the equation of the line.

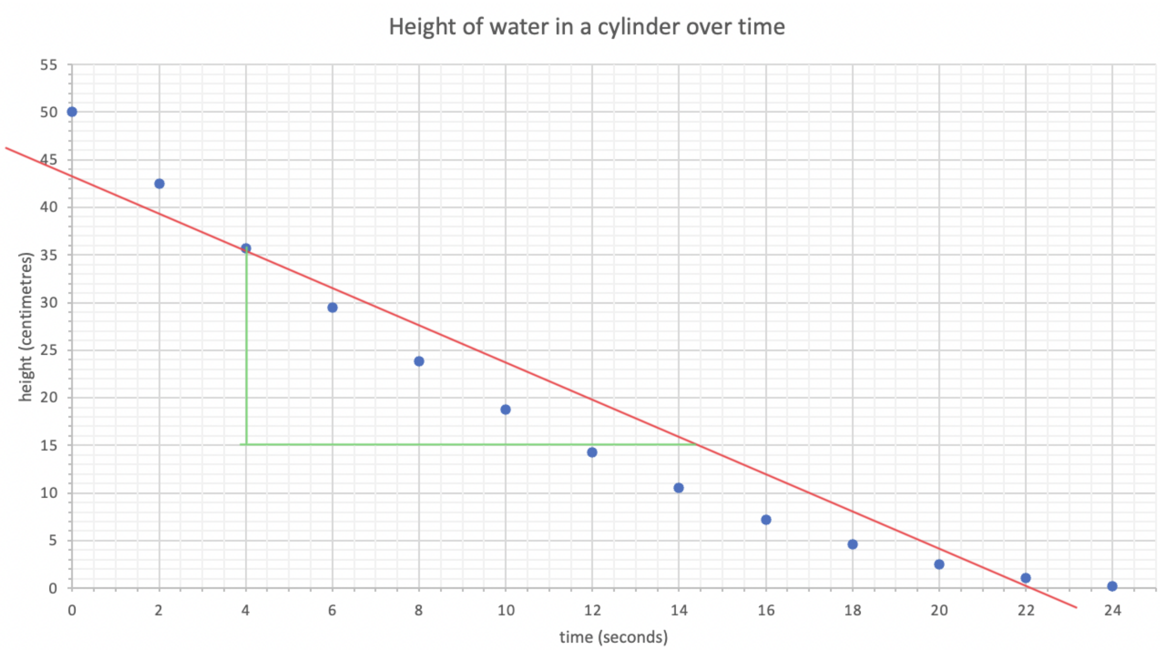
(b) Using the equation obtained in part (a) calculate the height of the water at $t = 1$ s, $t = 12$ s and $t = 24$ s.

(c) What can you conclude about the linear fit in comparison to the original data?

(d) Suggest a better model for the data, determine the equation and test its viability plot plotting the equation and original data on the same axes.

Solution:

(a) Note that there are an infinite number of ways that the line of best fit could be drawn and therefore an infinite number of equations could be calculated.



From this line of best fit, we are able to determine the equation of the straight line $h = mt + c$, where h is the height (in cm) and t is the time (in seconds).

The h intercept can be taken directly from the graph: $c \approx 43$. The gradient can be calculated as follows:

$$
\begin{aligned}
m &= \frac{\Delta h}{\Delta t} \\
&= \frac{15 - 36}{14.5 - 4} \\
&= \frac{-21}{10.5} \\
m &= -2
\end{aligned}
$$

Therefore the equation for the line of best fit is $h = -2t + 43$.

(b) The height of the water at $t = 1$ is $h = -2 \times 1 + 43 = 41$ cm.

The height of the water at $t = 12$ is $h = -2 \times 12 + 43 = 19$ cm.

he height of the water at $t = 24$ is $h = -2 \times 24 + 43 = -5$ cm.

(c) Fitting a linear line to the data is not ideal, as the data forms a curve. From the graph alone, it can be seen that for $t \leq 4$ the model heights are lower than the actual heights recorded. for $6 \leq t \leq 20$ the model provides an over-estimate of the height of the water. For $t > 22$ the model provides a negative value for the height of the water, which is unrealistic.

(d) The original data points are curved which suggests that an exponential model or a quadratic model would be a better model. To check which of these models would be most accurate we can look at the patterns between the original data points.

A linear model: Here equal increments in x yield equal increments in y (arithmetic sequence). We have a constant increment of 2 in the t data the difference between the values is inconsistent. This is why a linear model is not a good fit.

An exponential model: Here there is a common ratio between each y value (given equal increments in x) (geometric sequence). From the table, the first ratio is $\frac{42.5}{50} = 0.85$, the second is $\frac{35.7}{42.5} = 0.84$, the third is $\frac{29.5}{35.7} = 0.83$, etc.

A quadratic model: Data from quadratic functions have a different pattern from arithmetic and geometric sequences. If the data has.a common second difference, then it can be modelling by a quadratic function.

Time (seconds)	Height (cm)	1st difference	2nd difference	common ratio
0	50			
2	42.5	-7.5		0.85
4	35.7	-6.8	0.7	0.84
6	29.5	-6.2	0.6	0.83
8	23.8	-5.7	0.5	0.81
10	18.8	-5	0.7	0.79
12	14.3	-4.5	0.5	0.76
14	10.5	-3.8	0.7	0.73
16	7.2	-3.3	0.5	0.69
18	4.6	-2.6	0.7	0.64
20	2.5	-2.1	0.5	0.54
22	1.1	-1.4	0.7	0.44
24	0.2	-0.9	0.5	0.18

There is a fairly consistent 3rd difference, which would indicate that a cubic model would be a good fit. This, however, is beyond the scope of this module.

The differences and ratios do not indicate a clear model, but the quadratic model may yield the better results.

Quadratic model - A quadratic is in the form $h = at^2 + bt + c$. From the graph we know that $c = 50$. Therefore the equation becomes $h = at^2 + bt + 50$. We now have one equation and two unknowns, so we must use two coordinates from the original data.

Substituting $(2, 42.5)$ and $(22, 1.1)$ gives the two equations:

$$42.5 = 4a + 2b + 50 \; (1) \quad \text{and} \quad 1.1 = 484a + 22b + 50 \; (2)$$

subtracting 50 from both sides gives:

$$-7.5 = 4a + 2b \; (1) \quad \text{and} \quad -48.9 = 484a + 22b \; (2)$$

Multiplying 1 by 11 gives

$$-82.5 = 44a + 22b \quad (3) \quad \text{and} \quad -48.9 = 484a + 22b \quad (2)$$

Subtracting equation 2 from equation 3 gives

$$-33.6 = -440a$$

$$a = \frac{-33.6}{-440}$$

$$a = 0.076$$

Substituting this value of a into equation 1 gives:

$$-7.5 = 4 \times 0.076 + 2b$$

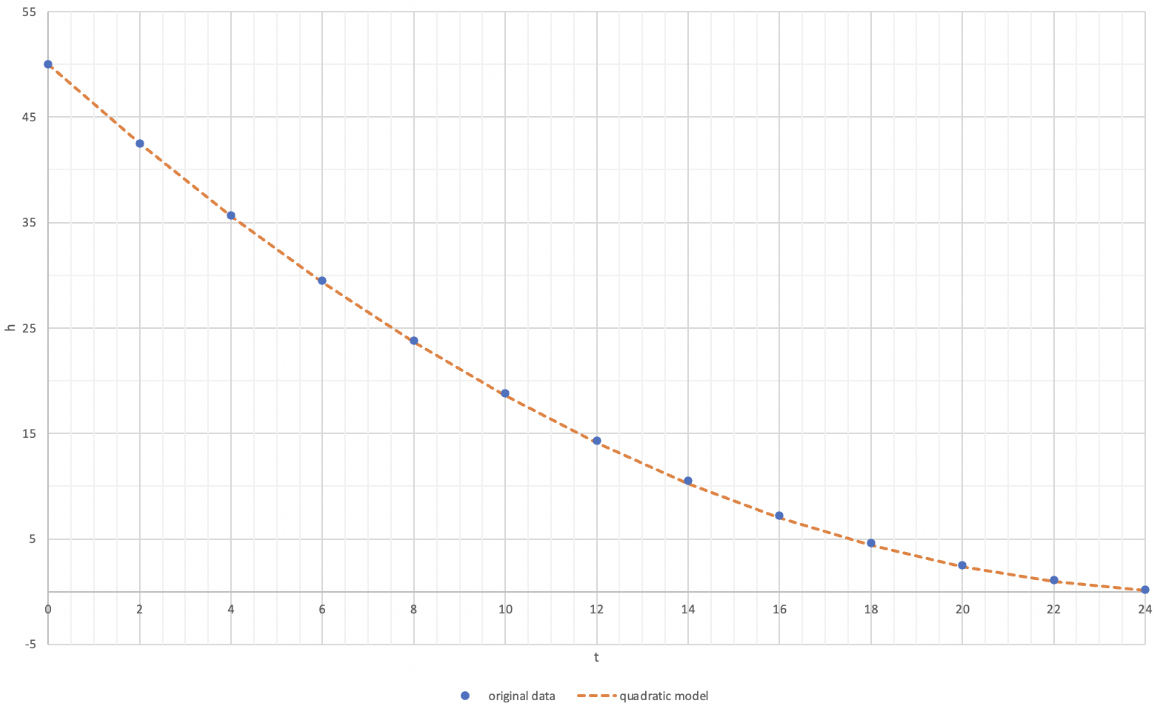
$$-7.5 = 0.304 + 2b$$

$$-7.804 = 2b$$

$$b = \frac{-7.804}{2}$$

$$b = -3.902$$

Therefore the quadratic equation representing the curve in the track is $h = 0.076t^2 - 3.902t + 50$. Shown in the graph below is the original data and the quadratic model curve. We can see that the model fits the data well.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