Railway Engineering Mathematics
Tutorial Sheet 11
Solutions

1. Differentiate the following with respect to the appropriate variable:

   (a) $$y = 7x^2 - 9x + 8$$

   Solution:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(7x^2 - 9x + 8\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(7x^2\big) - \frac{\mathrm{d}}{\mathrm{d}x}\big(9x\big) + \frac{\mathrm{d}}{\mathrm{d}x}\big(8\big) \\
&= 2 \times 7x^{2-1} - 9 + 0 \\
&= 14x - 9
\end{aligned}$$

   (b) $$y = 4x^5 + 3\sin(6x)$$

   Solution:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(4x^5 + 3\sin(6x)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(4x^5\big) + \frac{\mathrm{d}}{\mathrm{d}x}\big(3\sin(6x)\big) \\
&= 5 \times 4x^{5-1} + 6 \times 3\cos(6x) \\
&= 20x^4 + 18\cos(6x)
\end{aligned}$$

(c) $$y = 8\cos(3x) - \frac{6}{x^4} + \ln(2x) - 8$$

Solution:

First, re-write the second term using the rules of indices. (Note: we do *not* differentiate the other terms at this stage - we are simply rewriting y rather than calculating y'.)

$$y = 8\cos(3x) - 6x^{-4} + \ln(2x) - 8$$

Then differentiating all terms with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(8\cos(3x) - 6x^{-4} + \ln(2x) - 8\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\big(8\cos(3x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\big(6x^{-4}\big) + \frac{\mathrm{d}}{\mathrm{d}x}\big(\ln(2x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\big(8\big) \\
&= -3 \times 8\sin(3x) - 4 \times -6x^{-4-1} + \frac{1}{x} - 0 \\
&= -24\sin(3x) + 24x^{-5} + \frac{1}{x}
\end{aligned}$$

(d) $$y = \sinh(0.3t) - 3\,\mathrm{e}^{2t} - 4t^7$$

Solution:

This time y is a function of t rather than x, so that is the variable it would be meaningful to differentiate it with respect to.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\sinh(0.3t) - 3\,\mathrm{e}^{2t} - 4t^7\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(\sinh(0.3t)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(3\,\mathrm{e}^{2t}\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(4t^7\big) \\
&= 0.3 \times \cosh(0.3t) - 2 \times 3\,\mathrm{e}^{2t} - 7 \times 4t^{7-1} \\
&= 0.3\cosh(0.3t) - 6\,\mathrm{e}^{2t} - 28t^6
\end{aligned}$$

(e) $$y = 9\sqrt{x} + \frac{5x^3}{2}$$

Solution:

First, re-write the square root as an index:

$$y = 9x^{1/2} + \frac{5}{2}x^3$$

Then differentiating both terms in the function with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(9x^{1/2} + \frac{5}{2}x^3\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(9x^{1/2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5}{2}x^3\right) \\
&= \frac{1}{2} \times 9x^{\frac{1}{2}-1} + 3 \times \frac{5}{2}x^{3-1} \\
&= \frac{9}{2}x^{-\frac{1}{2}} + \frac{15}{2}x^2
\end{aligned}$$

(f) $$y = 6\,\mathrm{e}^{-3.5x} - 6.2\cos(x) - \sin\left(\frac{2x}{5}\right)$$

Solution:

To make it slightly more clear how the rules for differentiating functions of the form $a\sin(nx)$ and $a\cos(nx)$ apply, we could write this explicitly as:

$$y = 6\,\mathrm{e}^{-3.5x} - 6.2\cos(1x) - 1\sin\left(\frac{2}{5}x\right)$$

Then differentiating with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(6\,\mathrm{e}^{-3.5x} - 6.2\cos(1x) - 1\sin\left(\frac{2}{5}x\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(6\,\mathrm{e}^{-3.5x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\big(6.2\cos(1x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\left(1\sin\left(\frac{2}{5}x\right)\right) \\
&= 3.5 \times 6\,\mathrm{e}^{-3.5x} - (-1) \times 6.2\sin(x) - \frac{2}{5} \times 1\cos\left(\frac{2}{5}x\right) \\
&= -21\,\mathrm{e}^{-3.5x} + 6.2\sin(x) - \frac{2}{5}\cos\left(\frac{2}{5}x\right)
\end{aligned}$$

Note that we could also have just used a (simpler and less general) rule for the second term:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\cos(x)\big) = -\sin(x)$$

(g) $$x = 4t^3 + 3\ln(4t) + 12$$

Solution:

Differentiating with respect to t:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(4t^3 + 3\ln(4t) + 12\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(4t^3\big) + \frac{\mathrm{d}}{\mathrm{d}t}\big(3\ln(4t)\big) + \frac{\mathrm{d}}{\mathrm{d}t}\big(12\big) \\
&= 3 \times 4t^{3-1} + \frac{3}{t} + 0 \\
&= 12t^2 + \frac{3}{t}
\end{aligned}$$

(h) $$y = 9\psi^2 - \frac{5\cos(2\psi)}{7}$$

Solution:

First, write this as:

$$y = 9\psi^2 - \frac{5}{7}\cos(2\psi)$$

so that we can see more clearly that $5/7$ is the coefficient of the cosine term.

Then differentiating w.r.t ψ:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}\psi} &= \frac{\mathrm{d}}{\mathrm{d}\psi}\left(9\psi^2 - \frac{5}{7}\cos(2\psi)\right) \\
&= 2 \times 9\psi^{2-1} - 2 \times -\frac{5}{7}\sin(2\psi) \\
&= 18\psi + \frac{10}{7}\sin(2\psi)
\end{aligned}$$

2. Determine the gradient of:

(a) $$y = 5x^3 + 4x - 12 \quad \text{at} \quad x = -2$$

Solution:

First, differentiate the function with respect to its variable x to obtain a general formula for the gradient, dependent on the value of x:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(5x^3 + 4x - 12\big)\\ &= 3 \times 5x^{3-1} + 4 - 0\\ &= 15x^2 + 4\end{aligned}$$

So the gradient of a cubic function is described by a quadratic function!

Then to determine the value of the gradient at a specific point, $x = -2$, we substitute that into this formula for the first derivative and evaluate it:

$$\begin{aligned}\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} &= 15(-2)^2 + 4\\ &= 64\end{aligned}$$

(b) $$Q = 9\cos(t) - 2\sin(4t) \quad \text{at} \quad t = \frac{\pi}{2}$$

Solution:

First differentiating Q with respect to t to obtain a formula for the gradient:

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(9\cos(t) - 2\sin(4t)\big) \\
&= -1 \times 9\sin(t) - 4 \times 2\cos(4t) \\
&= -9\sin(t) - 8\cos(4t)
\end{aligned}$$

The substituting in $t = \frac{\pi}{2}$ and evaluating yields the gradient at that point:

$$\begin{aligned}
\left.\frac{\mathrm{d}Q}{\mathrm{d}t}\right|_{t=\frac{\pi}{2}} &= -9\sin\left(\frac{\pi}{2}\right) - 8\cos\left(4 \times \frac{\pi}{2}\right) \\
&= -17
\end{aligned}$$

3. Calculate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, given that:

$$y = \frac{-1}{\sqrt{2x}} - x^{\frac{3}{2}}$$

Solution:

We first re-write the first term. Separating the square root:

$$y = \frac{-1}{\sqrt{2}\sqrt{x}} - x^{\frac{3}{2}}$$

and then using the rule of indices that $\sqrt[n]{x} = x^{\frac{1}{n}}$:

$$\begin{aligned} y &= \frac{-1}{\sqrt{2}x^{\frac{1}{2}}} - x^{\frac{3}{2}} \\ &= \frac{-1}{\sqrt{2}} \cdot \frac{1}{x^{\frac{1}{2}}} - x^{\frac{3}{2}} \end{aligned}$$

then using the rule of indices that $\frac{1}{x^n} = x^{-n}$:

$$y = \frac{-1}{\sqrt{2}} x^{-\frac{1}{2}} - x^{\frac{3}{2}}$$

Both terms are now in the form ax^n and so we can apply our standard rules of differentiation to each:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-1}{\sqrt{2}}x^{-\frac{1}{2}}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(-x^{\frac{3}{2}}\right) \\ &= \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{2}\right)x^{-\frac{1}{2}-1} - \frac{3}{2}x^{\frac{3}{2}-1} \\ &= \frac{1}{2\sqrt{2}}x^{-\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{1}{2\sqrt{2}}x^{-\frac{3}{2}} - \frac{3}{2}\sqrt{x} \end{aligned}$$

To further improve the presentation of this solution, note that again applying rules of indices, we can express the fractional power

$$x^{-\frac{3}{2}} \quad \text{as} \quad \frac{1}{x^{\frac{3}{2}}} \quad \text{and then as} \quad \frac{1}{\sqrt{x^3}}$$

Thus:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{x^3}} - \frac{3}{2}\sqrt{x} \\
&= \frac{1}{2\sqrt{2}\sqrt{x^3}} - \frac{3}{2}\sqrt{x} \\
&= \frac{1}{2\sqrt{2x^3}} - \frac{3}{2}\sqrt{x}
\end{aligned}$$