Railway Engineering Mathematics
Tutorial Sheet 4
Solutions

Solving equations with fractions

1. Solve the following for x:

   (a) $\dfrac{x+11}{2x-5} = 2$

   (b) $\dfrac{5x-24}{x-4} = 3$

   (c) $4y - 9 = \dfrac{2y+3}{3}$

   (d) $\dfrac{x+4}{3} + \dfrac{x+1}{2} = 1$

   (e) $\dfrac{2x-5}{7} - \dfrac{2x-1}{2} = 3$

   (f) $\dfrac{x+1}{2} + \dfrac{2x-1}{4} + \dfrac{x+2}{3} = 1$

   (g) $\dfrac{z-7}{3} + \dfrac{2z-1}{4} = \dfrac{z+3}{6}$

   (h) $\dfrac{x}{2} - \dfrac{2+3x}{5} = 1 + \dfrac{1}{x}$

Solutions:

(a)

$$\begin{aligned}
\frac{x+11}{2x-5} &= 2 && \text{[multiply both sides by } 2x-5\text{]}\\
x+11 &= 2(2x-5) && \text{[multiply out the brackets]}\\
x+11 &= 4x-10 && \text{[subtract } 4x \text{ from both sides]}\\
-3x+11 &= -10 && \text{[subtract 11 from both sides]}\\
-3x &= -21 && \text{[divide both sides by -3]}\\
\therefore \qquad x &= 7
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{5x-24}{x-4} &= 3 && \text{[multiply both sides by } x-4\text{]}\\
5x-24 &= 3(x-4) && \text{[multiply out the brackets]}\\
5x-24 &= 3x-12 && \text{[subtract } 3x \text{ from both sides]}\\
2x-24 &= -12 && \text{[add 24 to both sides]}\\
2x &= 12 && \text{[divide both sides by 2]}\\
\therefore \qquad x &= 6
\end{aligned}$$

(c)

$$
\begin{aligned}
4y-9 &= \frac{2y+3}{3} && \text{[multiply both sides by 3]} \\
3(4y-9) &= 2y+3 && \text{[multiply out the brackets]} \\
12y-27 &= 2y+3 && \text{[subtract } 2y \text{ from both sides]} \\
10y-27 &= 3 && \text{[add 27 to both sides]} \\
10y &= 30 && \text{[divide both sides by 10]} \\
\therefore \qquad y &= 3
\end{aligned}
$$

(d)

$$
\begin{aligned}
\frac{x+4}{3}+\frac{x+1}{2} &= 1 && \text{[multiply both sides by 3]} \\
x+4+\frac{3(x+1)}{2} &= 3 && \text{[multiply both sides by 2]} \\
2(x+4)+3(x+1) &= 6 && \text{[multiply out the brackets]} \\
2x+8+3x+3 &= 6 && \text{[collect like terms]} \\
5x+11 &= 6 && \text{[subtract 11 from both sides]} \\
5x &= -5 && \text{[divide both sides by 5]} \\
\therefore \qquad x &= -1
\end{aligned}
$$

(e)

$$
\begin{aligned}
\frac{2x-5}{7} - \frac{2x-1}{2} &= 3 && \text{[multiply both sides by 7]}\\
2x-5-\frac{7(2x-1)}{2} &= 21 && \text{[multiply both sides by 2]}\\
2(2x-5)-7(2x-1) &= 42 && \text{[multiply out the brackets]}\\
4x-10-14x+7 &= 42 && \text{[collect like terms]}\\
-10x-3 &= 42 && \text{[add 3 to both sides]}\\
-10x &= 45 && \text{[divide both sides by -10]}\\
\therefore \qquad x &= -4.5
\end{aligned}
$$

(f)

$$
\begin{aligned}
\frac{x+1}{2} + \frac{2x-1}{4} + \frac{x+2}{3} &= 1 && \text{[multiply both sides by 2]}\\
x+1+\frac{2(2x-1)}{4} + \frac{2(x+2)}{3} &= 2 && \text{[multiply both sides by 4]}\\
4(x+1)+2(2x-1)+\frac{8(x+2)}{3} &= 8 && \text{[multiply both sides by 3]}\\
12(x+1)+6(2x-1)+8(x+2) &= 24 && \text{[multiply out the brackets]}\\
12x+12+12x-6+8x+16 &= 24 && \text{[collect like terms]}\\
32x+22 &= 24 && \text{[subtract 22 from both sides]}\\
32x &= 2 && \text{[divide both sides by 32]}\\
\therefore \qquad x &= \frac{2}{32} = \frac{1}{16}
\end{aligned}
$$

(g) We could follow the same procedure as above, or multiply through by the lowest common denominator:

$$\begin{aligned}
\frac{z-7}{3}+\frac{2z-1}{4} &= \frac{z+3}{6} && \text{[multiply both sides by 12]}\\
12\left(\frac{z-7}{3}+\frac{2z-1}{4}\right) &= 12\left(\frac{z+3}{6}\right) && \text{[multiply out the brackets]}\\
4(z-7)+3(2z-1) &= 2(z+3) && \text{[multiply out the brackets]}\\
4z-28+6z-3 &= 2z+6 && \text{[collect like terms]}\\
10z-31 &= 2z+6 && \text{[subtract } 2z \text{ from both sides]}\\
8z-31 &= 6 && \text{[add 31 to both sides]}\\
8z &= 37 && \text{[divide both sides by 8]}\\
\therefore \qquad z &= \frac{37}{8}
\end{aligned}$$

(h)
$$\frac{x}{2}-\frac{2+3x}{5}=1+\frac{1}{x}$$

Multiplying both sides by 2 to remove the first fraction on the LHS:

$$\therefore\ x-\frac{2(2+3x)}{5}=2+\frac{2}{x}$$

Multiplying all terms by 5:

$$\therefore\ 5x-2(2+3x)=10+\frac{10}{x}$$

Multiplying all terms by x:

$$\therefore\ 5x^2-2x(2+3x)=10x+10$$

Expand the brackets and gathering like terms: x^2, x and constants:

$$\therefore\ 5x^2 - 4x - 6x^2 = 10x + 10$$

$$\therefore\ -x^2 - 4x = 10x + 10$$

$$\therefore\ x^2 + 14x + 10 = 0$$

This is a quadratic equation that can be solved using the quadratic formula:

$$\begin{aligned} x_{1,2} &= \frac{-14 \pm \sqrt{14^2 - 4(1)(10)}}{2 \times 1} \\ &= \frac{-14 \pm \sqrt{156}}{2} \\ \therefore\ x_1 &= -0.755, \quad \text{and} \quad x_2 = -13.245 \end{aligned}$$

General practice of transposition

2. Transpose the following formulae for the variable stated in the brackets:

   (a) $P = \dfrac{mRT}{V} + \dfrac{mRT_0}{V}$ $\qquad (m)$ and (V)

   (b) $z = 13x - 6 + \alpha x$ $\qquad (x)$

   (c) $\dfrac{2h}{3h - p} = 5p$ $\qquad (h)$

   (d) $5(3m - 2) = 8mk - 9$ $\qquad (m)$

   (e) $x = \dfrac{12}{y}$ $\qquad (y)$

   (f) $a = \dfrac{4}{b} + 2c$ $\qquad (b)$

(g) $y = \dfrac{7}{2x+3}$ $\qquad (x)$

Solutions:

(a) For m:

$$\begin{aligned} P &= \frac{mRT}{V} + \frac{mRT_0}{V} && \text{[multiply both sides by } V\text{]} \\ PV &= mRT + mRT_0 && \text{[factorise]} \\ PV &= m(RT + RT_0) && \text{[divide both sides by } RT + RT_0\text{]} \\ \therefore \quad m &= \frac{PV}{RT + RT_0} = \frac{PV}{R(T+T_0)} \end{aligned}$$

For V:

$$\begin{aligned} P &= \frac{mRT}{V} + \frac{mRT_0}{V} && \text{[factorise]} \\ P &= \frac{mR}{V}(T + T_0) && \text{[multiply both sides by } V\text{]} \\ PV &= mR\,(T + T_0) && \text{[divide both sides by } P\text{]} \\ \therefore \quad V &= \frac{mR}{P}(T + T_0) \end{aligned}$$

(b)

$$\begin{aligned} z &= 13x - 6 + \alpha x && \text{[add 6 to both sides]} \\ z + 6 &= 13x + \alpha x && \text{[factorise]} \\ z + 6 &= x(13 + \alpha) && \text{[divide both sides by } 13 + \alpha\text{]} \\ \therefore \quad x &= \frac{z+6}{13+\alpha} \end{aligned}$$

(c)

$$\frac{2h}{3h-p} = 5p \qquad \text{[multiply both sides by } 3h-p\text{]}$$

$$2h = 5p(3h-p) \qquad \text{[multiply out the brackets]}$$

$$2h = 15hp - 5p^2 \qquad \text{[subtract } 15hp \text{ from both sides]}$$

$$2h - 15hp = -5p^2 \qquad \text{[factorise the LHS]}$$

$$h(2-15p) = -5p^2 \qquad \text{[divide both sides by } 2-15p\text{]}$$

$$\therefore \quad h = \frac{-5p^2}{2-15p} = \frac{5p^2}{15p-2} \text{[it's neater for the numerator to be positive]}$$

(d)

$$5(3m-2) = 8mk - 9 \qquad \text{[multiply out the brackets]}$$

$$15m - 10 = 8mk - 9 \qquad \text{[subtract } 8mk \text{ from both sides]}$$

$$15m - 8mk - 10 = -9 \qquad \text{[add 10 to both sides]}$$

$$15m - 8mk = 1 \qquad \text{[factorise the LHS]}$$

$$m(15-8k) = 1 \qquad \text{[divide both sides by } 15-8k\text{]}$$

$$\therefore \quad m = \frac{1}{15-8k}$$

(e)

$$\begin{aligned} x &= \frac{12}{y} \quad \text{[multiply both sides by } y\text{]} \\ xy &= 12 \quad \text{[divide both sides by } x\text{]} \\ \therefore \quad y &= \frac{12}{x} \end{aligned}$$

(f)

$$\begin{aligned} a &= \frac{4}{b} + 2c \quad \text{[subtract } 2c \text{ from both sides]} \\ a - 2c &= \frac{4}{b} \quad \text{[multiply both sides by } b\text{]} \\ b(a-2c) &= 4 \quad \text{[divide both sides by } a - 2c\text{]} \\ \therefore \quad b &= \frac{4}{a-2c} \end{aligned}$$

(g)

$$\begin{aligned} y &= \frac{7}{2x+3} \quad \text{[multiply both sides by } 2x+3\text{]} \\ y(2x+3) &= 7 \quad \text{[multiply out the brackets]} \\ 2xy + 3y &= 7 \quad \text{[subtract } 3y \text{ from both sides]} \\ 2xy &= 7 - 3y \quad \text{[divide both sides by } 2y\text{]} \\ \therefore \quad x &= \frac{7-3y}{2y} \end{aligned}$$

Powers and roots

3. Transpose the following formulae for the variable stated in the brackets:

(a) $V = \dfrac{4}{3}\pi r^3$ $\quad (r)$

(b) $y = 5 + \sqrt{x-2}$ $\quad (x)$

(c) $9x + \dfrac{3}{P} = \dfrac{1}{2r^2}$ $\quad (r)$

(d) $M = 7t - \sqrt{\dfrac{2}{1-r}}$ $\quad (r)$

Solutions:

(a)

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 && \text{[multiply both sides by 3]} \\ 3V &= 4\pi r^3 && \text{[divide both sides by } 4\pi\text{]} \\ r^3 &= \frac{3V}{4\pi} && \text{[cube root both sides]} \\ \therefore \quad r &= \sqrt[3]{\frac{3V}{4\pi}} \end{aligned}$$

(b)

$$\begin{aligned} y &= 5 + \sqrt{x-2} && \text{[subtract 5 from both sides]} \\ y - 5 &= \sqrt{x-2} && \text{[square both sides]} \\ (y-5)^2 &= x - 2 && \text{[add 2 to both sides]} \\ \therefore \quad x &= (y-5)^2 + 2 \end{aligned}$$

(c)

$$9x + \frac{3}{P} = \frac{1}{2r^2} \qquad \text{[multiply both sides by } r^2\text{]}$$

$$r^2\left(9x + \frac{3}{P}\right) = \frac{1}{2} \qquad \text{[divide both sides by } 9x + \frac{3}{P}\text{]}$$

$$r^2 = \frac{1}{2\left(9x + \frac{3}{P}\right)} \qquad \text{[multiply the RHS by } \frac{P}{P}\text{]}$$

$$r^2 = \frac{P}{2\left(9Px + 3\right)} \qquad \text{[square root both sides]}$$

$$\therefore \quad r = \sqrt{\frac{P}{2\left(9Px + 3\right)}}$$

(d)

$$M = 7t - \sqrt{\frac{2}{1-r}} \qquad \text{[subtract } 7t \text{ from both sides]}$$

$$M - 7t = -\sqrt{\frac{2}{1-r}} \qquad \text{[multiply through by -1]}$$

$$7t - M = \sqrt{\frac{2}{1-r}} \qquad \text{[square both sides both sides]}$$

$$(7t - M)^2 = \frac{2}{1-r} \qquad \text{[multiply both sides by } 1 - r\text{]}$$

$$(1-r)(7t - M)^2 = 2 \qquad \text{[divide both sides by } (7t - M)^2\text{]}$$

$$1 - r = \frac{2}{(7t - M)^2} \qquad \text{[subtract 1 from both sides]}$$

$$-r = \frac{2}{(7t - M)^2} - 1 \qquad \text{[divide both sides by } -1\text{]}$$

$$\therefore \quad r = 1 - \frac{2}{(7t - M)^2}$$

4. Make e the subject of:

$$T = \frac{2v}{g}\left(\frac{1}{1-e}\right)$$

Solution:

$$\begin{aligned}
T &= \frac{2v}{g}\left(\frac{1}{1-e}\right) && \text{[multiply both sides by } 1-e\text{]} \\
(1-e)\,T &= \frac{2v}{g} && \text{[divide both sides by } T\text{]} \\
1-e &= \frac{2v}{gT} && \text{[add } e \text{ to both sides]} \\
1 &= \frac{2v}{gT} + e && \text{[subtract } \tfrac{2v}{gT} \text{ from both sides]} \\
\therefore \quad e &= 1 - \frac{2v}{gT}
\end{aligned}$$

5. Given that:

$$U = V\left(1 - \frac{C}{D\sqrt{N}}\right),$$

find N in terms of U, V, D and C.

Solution:

We will "unwrap" N, first by dividing both sides by V:

$$\therefore \ \frac{U}{V} = 1 - \frac{C}{D\sqrt{N}}$$

Isolate the term containing N:

$$\therefore\ \frac{U}{V} - 1 = -\frac{C}{D\sqrt{N}}$$

Remove N from the denominator of the fraction by multiplication:

$$\therefore\ \sqrt{N}\left(\frac{U}{V} - 1\right) = -\frac{C}{D}$$

Isolate the $\sqrt{N}$ and simplify the RHS:

$$\begin{aligned}
\therefore\ \sqrt{N} &= \frac{-\frac{C}{D}}{\frac{U}{V} - 1} \\
&= \frac{-C}{D\left(\frac{U}{V} - 1\right)} \\
&= \frac{-C}{-D\left(1 - \frac{U}{V}\right)} \\
&= \frac{C}{D\left(1 - \frac{U}{V}\right)} \\
&= \frac{CV}{D(V - U)}
\end{aligned}$$

Finally square both sides to obtain N:

$$\therefore\ N = \left(\frac{CV}{D(V - U)}\right)^2$$

6. Make b the subject of:

$$a(3b-1) = 2b+2$$

Solution:

$$\begin{aligned}
a(3b-1) &= 2b+2 && \text{[multiply out the brackets]}\\
3ab - a &= 2b+2 && \text{[subtract } 2b \text{ from both sides]}\\
3ab - 2b - a &= 2 && \text{[add } a \text{ to both sides]}\\
3ab - 2b &= 2+a && \text{[factorise the LHS]}\\
b(3a-2) &= 2+a && \text{[divide both sides by } 3a-2\text{]}\\
\therefore \quad b &= \frac{2+a}{3a-2}
\end{aligned}$$

7. Make r the subject of:

$$P = \frac{P_0}{1 - r^2}$$

Solution:

$$\begin{aligned}
P &= \frac{P_0}{1 - r^2} && \text{[multiply both sides by } 1 - r^2\text{]} \\
P(1 - r^2) &= P_0 && \text{[multiply out the brackets]} \\
P - Pr^2 &= P_0 && \text{[subtract } P \text{ from both sides]} \\
-Pr^2 &= P_0 - P && \text{[divide both sides by } -P\text{]} \\
r^2 &= \frac{P_0 - P}{-P} && \text{[simplify]} \\
r^2 &= \frac{P - P_0}{P} && \text{[square root both sides]} \\
\therefore \quad r &= \sqrt{\frac{P - P_0}{P}}
\end{aligned}$$

8. Make x the subject of:

$$y = a + \frac{1}{1-x}$$

Solution:

$$\begin{aligned}
y &= a + \frac{1}{1-x} && \text{[subtract } a \text{ from both sides]} \\
y - a &= \frac{1}{1-x} && \text{[multiply both sides by } 1-x\text{]} \\
(1-x)(y-a) &= 1 && \text{[divide both sides by } y-a\text{]} \\
1 - x &= \frac{1}{y-a} && \text{[subtract 1 from both sides]} \\
-x &= \frac{1}{y-a} - 1 && \text{[divide both sides by } -1\text{]} \\
\therefore \quad x &= 1 - \frac{1}{y-a}
\end{aligned}$$

9. Make y the subject of:

$$\frac{y}{y+x} + 5 = x$$

Solution:

$$
\begin{aligned}
\frac{y}{y+x} + 5 &= x && \text{[subtract 5 from both sides]} \\
\frac{y}{y+x} &= x - 5 && \text{[multiply both sides by } y+x\text{]} \\
y &= (x-5)(y+x) && \text{[multiply out the brackets]} \\
y &= xy + x^2 - 5y - 5x && \text{[move the } y \text{ terms to the LHS]} \\
y - xy + 5y &= x^2 - 5x && \text{[factorise both sides simplify]} \\
y(6-x) &= x(x-5) && \text{[divide both sides by } 6-x\text{]} \\
\therefore \quad y &= \frac{x(x-5)}{6-x}
\end{aligned}
$$

10. The equations for a battery with e.m.f. E and an internal resistance r, connected across a resistor R can be expressed as:

$$E = \frac{V(R+r)}{R}$$

where V is the voltage. Find an expression for R.

Solution:

$$\begin{aligned}
E &= \frac{V(R+r)}{R} && \text{[multiply both sides by } R\text{]} \\
ER &= V(R+r) && \text{[multiply out the brackets]} \\
ER &= VR + Vr && \text{[subtract } VR \text{ from both sides]} \\
ER - VR &= Vr && \text{[factorise both sides]} \\
R(E-V) &= Vr && \text{[divide both sides by } E-V\text{]} \\
\therefore \quad R &= \frac{Vr}{E-V}
\end{aligned}$$

11. The impedance, Z, of a circuit containing a resistor of resistance R, a capacitor of capacitance C and an inductor of inductance L is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}, \tag{1}$$

where $X_L = 2\pi f L$ and $X_C = \dfrac{1}{2\pi f C}$.

Determine and expression for C in terms of f, L, R and Z.

Solution:

Firstly, substitute X_L and X_C into equation (1):

$$
\begin{aligned}
Z &= \sqrt{R^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2} && \text{[square both sides]}\\
Z^2 &= R^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2 && \text{[subtract } R^2 \text{ from both sides]}\\
Z^2 - R^2 &= \left(2\pi f L - \frac{1}{2\pi f C}\right)^2 && \text{[square root both sides]}\\
\sqrt{Z^2 - R^2} &= 2\pi f L - \frac{1}{2\pi f C} && \text{[subtract } 2\pi f L \text{ from both sides]}\\
\sqrt{Z^2 - R^2} - 2\pi f L &= -\frac{1}{2\pi f C} && \text{[multiply through by } -1\text{]}\\
\frac{1}{2\pi f C} &= 2\pi f L - \sqrt{Z^2 - R^2} && \text{[multiply both sides by } C\text{]}\\
\frac{1}{2\pi f} &= C\left(2\pi f L - \sqrt{Z^2 - R^2}\right) && \text{[}\div\text{ both sides by } \left(2\pi f L - \sqrt{Z^2 - R^2}\right)\text{]}\\
\therefore \quad C &= \frac{1}{2\pi f\left(2\pi f L - \sqrt{Z^2 - R^2}\right)}
\end{aligned}
$$

12. A system with feedback β and gain A has an output voltage v_{in} given by

$$v_{in} = \left(\frac{1}{A} - \beta\right) v_{out}$$

where v_{out} is the output voltage. Determine the ratio of the output voltage to the input voltage.

Solution:

We are required to determine $\dfrac{v_{out}}{v_{in}}$:

$$v_{in} = \left(\frac{1}{A} - \beta\right) v_{out} \quad \text{[divide both sides by } v_{in} \text{]}$$

$$1 = \left(\frac{1}{A} - \beta\right) \frac{v_{out}}{v_{in}} \quad \text{[divide both sides by } \frac{1}{A} - \beta\text{]}$$

$$\frac{v_{out}}{v_{in}} = \frac{1}{\frac{1}{A} - \beta} \quad \text{[multiply the RHS by } \frac{A}{A}\text{]}$$

$$\frac{v_{out}}{v_{in}} = \frac{A}{A}\left(\frac{1}{\frac{1}{A} - \beta}\right) \quad \text{[multiply out the brackets]}$$

$$\therefore \quad \frac{v_{out}}{v_{in}} = \frac{A}{1 - A\beta}$$

13. As shown in Figure 1, one end of a light inextensible string of length l is attached to a fixed point A and a particle P is attached to the other end. The ends of a second string of the same length are attached to P and to a fixed point B at a distance h ($< 2l$) vertically below A. The particle moves in a horizontal circle with uniform angular speed ω.

    The tension in the second string can be expressed as:

    $$T_2 = \frac{ml}{h}\left(h\omega^2 - 2g\right).$$

    Given that $T_1 > T_2$, both strings will be taut if $T_2 \geq 0$. Determine the least value of ω for which both strings are taut.

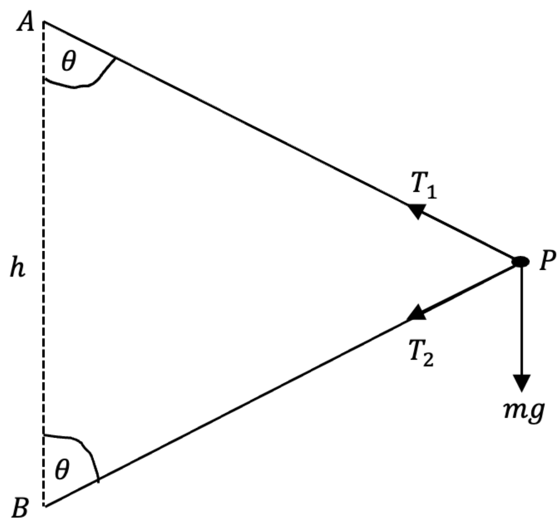

Figure 1

Solution:

Setting $T_2 \geq 0$ we have:

$$T_2 = \frac{ml}{h}\left(h\omega^2 - 2g\right) \geq 0$$

$$\frac{ml}{h}\left(h\omega^2 - 2g\right) \geq 0 \qquad \text{[multiply both sides by } \frac{h}{ml}\text{]}$$

$$h\omega^2 - 2g \geq 0 \qquad \text{[add } 2g \text{ to both sides]}$$

$$h\omega^2 \geq 2g \qquad \text{[divide both sides by } h\text{]}$$

$$\omega^2 \geq \frac{2g}{h} \qquad \text{[square root both sides]}$$

$$\therefore \quad \omega \geq \sqrt{\frac{2g}{h}}$$

Therefore the least value of ω for which both strings are taut is $\sqrt{\frac{2g}{h}}$.

14. In aerodynamics, minimum drag on an aircraft occurs when the lift coefficient L satisfies:

$$kL^2 = Z$$

where Z is the zero lift coefficient and k is a constant. The velocity v of an aircraft satisfies:

$$w = \frac{1}{2}\rho v^2 LA$$

where w is weight, ρ is the density, and A is area. Show that:

$$v^4 = \left(\frac{2w}{\rho A}\right)^2 \cdot \frac{k}{Z}$$

Solution:

Begin with

$$w = \frac{1}{2}\rho v^2 LA$$

and re-write to make v^2 the subject:

$$v^2 = \frac{2w}{\rho LA}$$

Then squaring both sides:

$$\begin{aligned} v^4 &= \left(\frac{2w}{\rho LA}\right)^2 \\ &= \left(\frac{2w}{\rho A}\right)^2 \cdot \frac{1}{L^2} \\ &= \left(\frac{2w}{\rho A}\right)^2 \cdot \frac{1}{Z/k} \qquad \text{using the fact that} \quad kL^2 = Z \\ &= \left(\frac{2w}{\rho A}\right)^2 \cdot \frac{k}{Z} \end{aligned}$$

as required.

15. In thermodynamics, the exit velocity u of a fluid from a nozzle is given by:

$$u = \left\{ \frac{2\gamma P_1 V_1}{\gamma - 1} \left[1 - \frac{P_2 V_2}{P_1 V_1} \right] \right\}^{\frac{1}{2}}$$

where P_1, V_1 represent the entrance pressure and the specific volume respectively, and P_2, V_2 represent the exit pressure and specific volume respectively. γ is the ratio of specific heat capacities. Given that:

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

show that:

$$u^2 = \frac{2\gamma P_1 V_1}{\gamma - 1} \left[1 - \left(\frac{P_2}{P_1} \right)^{1-1/\gamma} \right]$$

then calculate u (correct to 1 d.p.), given the following:

- $\gamma = 1.39$
- $P_1 = 5.2 \times 10^6 \text{ N/m}^2$
- $V_1 = 3.1 \times 10^{-3} \text{ m}^3/\text{kg}$
- $V_2 = 5 \times 10^{-3} \text{ m}^3/\text{kg}$

Solution:

Start by squaring both sides of the formula for u:

$$\begin{aligned} u^2 &= \left(\left\{ \frac{2\gamma P_1 V_1}{\gamma - 1} \left[1 - \frac{P_2 V_2}{P_1 V_1} \right] \right\}^{\frac{1}{2}} \right)^2 \\ &= \frac{2\gamma P_1 V_1}{\gamma - 1} \left[1 - \frac{P_2 V_2}{P_1 V_1} \right] \end{aligned}$$

To obtain the desired expression, we need to eliminate V_2/V_1 from inside the square bracket using a substitution:

$$P_1 V_1^\gamma = P_2 V_2^\gamma \implies \frac{V_2^\gamma}{V_1^\gamma} = \frac{P_1}{P_2} \quad \text{and so} \quad \left(\frac{V_2}{V_1} \right)^\gamma = \frac{P_1}{P_2}$$

Then taking the γ-th root of both sides:

$$\therefore \frac{V_2}{V_1} = \left(\frac{P_1}{P_2}\right)^{1/\gamma} = \left(\frac{P_2}{P_1}\right)^{-1/\gamma}$$

and substituting this into our expression for u^2:

$$\begin{aligned}
u^2 &= \frac{2\gamma P_1 V_1}{\gamma - 1}\left[1 - \frac{P_2}{P_1} \cdot \frac{V_2}{V_1}\right] \\
&= \frac{2\gamma P_1 V_1}{\gamma - 1}\left[1 - \frac{P_2}{P_1} \cdot \left(\frac{P_2}{P_1}\right)^{-1/\gamma}\right] \\
&= \frac{2\gamma P_1 V_1}{\gamma - 1}\left[1 - \left(\frac{P_2}{P_1}\right)^{1-1/\gamma}\right]
\end{aligned}$$

as required.

Now substituting in the given values of γ, P_1, V_1 and V_2 and evaluating u^2. As our formula contains P_2, we will need to evaluate that first:

$$\begin{aligned}
P_2 &= \frac{P_1 V_1^\gamma}{V_2^\gamma} \\
&= \frac{\left(5.2 \times 10^6\right)\left(3.1 \times 10^{-3}\right)^{1.39}}{\left(5 \times 10^{-3}\right)^{1.39}} \\
&= \frac{1694.32258}{0.00063323995} \\
&= 2.6756407 \times 10^6
\end{aligned}$$

Then:

$$
\begin{aligned}
u^2 &= \frac{2(1.39)\big(5.2\times10^6\big)\big(3.1\times10^{-3}\big)}{1.39-1}\left[1-\left(\frac{2.6756407\times10^6}{5.2\times10^6}\right)^{1-\frac{1}{1.39}}\right] \\
&= \frac{10.4\times3.1\times1.39}{0.39}\times1000\left[1-\left(\frac{2.6756407}{5.2}\right)^{1-0.71942446}\right] \\
&= 114906.66667\big[1-0.514546288^{0.2805755}\big] \\
&= 114906.66667\big[1-0.829913388\big] \\
&= 114906.66667\times0.17008661 \\
&= 19544.08599
\end{aligned}
$$

Hence,

$$u=\sqrt{19544.08599}=139.8002\approx139.8\,\mathrm{m/s}$$