Applications of Differentiation

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 14

Learning Outcomes

- Use the principles of differential calculus to solve engineering problems.
- Determine higher order derivatives.
- Locate stationary points and classify their nature.

Introduction - Rates of change

We have seen that differentiation allows us to calculate the gradient of a curve at any point, indicating how quickly the variable on the y-axis is changing as a result of change in the variable on the x-axis. Therefore, as mentioned previously, gradients represent **rates of change**.

Rates of Change: Physical examples

x-axis	y-axis	Gradient
Time: t	Displacement: S	Velocity: $\nu = \dfrac{\mathrm{d}S}{\mathrm{d}t}$
	Velocity: ν	Acceleration: $a = \dfrac{\mathrm{d}\nu}{\mathrm{d}t}$
	Energy: E	Power: $P = \dfrac{\mathrm{d}E}{\mathrm{d}t}$
	Charge: q	Current: $I = \dfrac{\mathrm{d}q}{\mathrm{d}t}$
	Momentum: p	Force: $F = \dfrac{\mathrm{d}p}{\mathrm{d}t}$
	Angular displacement: θ	Angular velocity: $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

Example 1

A projectile is thrown directly upwards such that its vertical displacement S m changes over time t s in accordance with the formula:

$$S = 2.4t - 4.9t^2$$

Determine a formula for its velocity and, hence, the velocity of the projectile after 4 seconds.

Example 1 - Solution

First, we differentiate to obtain a formula for the velocity ν at any time t (*do* **not** *substitute in* $t = 4$ *until after this step is done!*)

$$\begin{aligned} \nu(t) &= \frac{\mathrm{d}S}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(2.4t - 4.9t^2\right) \\ &= 2.4 - 9.8t \end{aligned}$$

Then evaluate this at $t = 4$ to determine the velocity at that time:

$$\nu(t = 4) = 2.4 - 9.8 \times 4 = -36.8 \text{ m/s}$$

Example 2

When charging up, the charge q (C) held by a capacitor varies with time t (s) such that:

$$q = 10^{-7}\left(1 - e^{-50t}\right)$$

Determine the current flow in the circuit at $t = 8$ s.

Example 2 - Solution

Current is the rate of change of charge, so differentiate w.r.t. time:

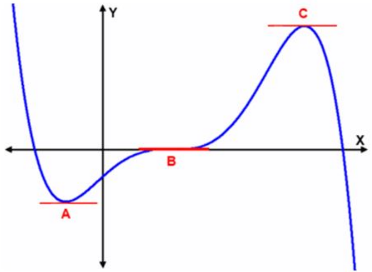
$$\begin{aligned} I(t) &= \frac{\mathrm{d}q}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(10^{-7}\left(1 - e^{-50t}\right)\right) \\ &= 10^{-7} \times 50e^{-50t} \\ &= 5 \times 10^{-6}e^{-50t} \end{aligned}$$

Evaluate at $t = 8$:

$$I(t=8) = 5 \times 10^{-6}e^{-50\times 8} = 9.58 \times 10^{-180} \approx 0$$

Stationary Points

Consider the curve:



Points A, B and C are all points on the curve where the gradient is zero:

Stationary points:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

- A is a minimum
- B is a point of inflection
- C is a maximum

Stationary Points and optimisation

- So **extreme values** (maxima and minima) occur at **stationary points** (or at the edge of the range under consideration).
- Finding values of x that provide a maximum or a minimum value of y may be relevant to an optimisation problems, e.g. what number of check-out staff will maximise profits?
- We can find these by differentiating the function and solving for "**where is the derivative equal to zero?**"

Example 3

The approximate annual cost C (in £100's) of carrying out maintenance on a machine part at a frequency of f (per year) is given by:

$$C = 5e^{-0.5f} + 0.6f$$

Determine the maintenance frequency f that will incur the lowest overall cost, i.e. the optimal maintenance frequency.

Example 3 - Solution

As the cost C is the quantity to optimise, we must obtain a formula for its derivative w.r.t. f:

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}f} &= \frac{\mathrm{d}}{\mathrm{d}f}\left(5e^{-0.5f} + 0.6f\right) \\ &= -2.5e^{-0.5f} + 0.6 \end{aligned}$$

Now set this equal to zero, and solve for f:

$$\frac{\mathrm{d}C}{\mathrm{d}f} = 0$$

Example 3 - Solution

$$\therefore\ -2.5e^{-0.5f} + 0.6 = 0$$

$$\therefore\ e^{-0.5f} = \frac{0.6}{2.5} = \frac{6}{25} = 0.24$$

$$\therefore\ -0.5f = \ln(0.24)$$

$$\therefore\ f = \frac{1}{-0.5}\ln(0.24) = -2\ln(0.24) = 2.85 \quad \text{to (2 d.p.)}$$

So 2.85 times per year, which incurs a cost of

$$C = 5e^{-0.5\times 2.85} + 0.6 \times 2.85 = 2.91 \implies £291$$

This is the only value of f that gives an extreme value of C, but how can we be sure that it is a *minimum* specifically?

Higher Order Derivatives

We can differentiate $y = 2x^3$ once to get the **first derivative**:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$$

We can differentiate again to obtain the **second derivative**:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(6x^2) = 12x$$

We could also differentiate for a third and fourth time, etc.:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 12 \quad \text{and} \quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 0$$

Higher Order Derivatives

For an engineering application, acceleration $a(t)$ is the 2^{nd} order derivative of displacement $S(t)$, since:

$$\nu = \frac{\mathrm{d}S}{\mathrm{d}t} \quad \text{and} \quad a = \frac{\mathrm{d}\nu}{\mathrm{d}t}$$

hence,

$$a = \frac{\mathrm{d}}{\mathrm{d}t}(\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}S}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 S}{\mathrm{d}t^2}$$

Higher Order Derivatives

2^{nd} order derivatives are used to **classify stationary points.**

Second derivative test

If $y = f(x)$ has a stationary point at $x = a$ then:

- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ at $x = a$ then it is a **maximum** point at a.
- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ at $x = a$ then it is a **minimum** point at a.
- if $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = a$ then the nature is unknown and needs further investigation.

Example 4

For a limited speed range, the torque-speed relationship for an AC induction motor is approximated by the formula:

$$\tau = -0.0016\omega^3 + 0.17\omega^2 - 3.4\omega + 250$$

where τ is the torque generated as a percentage of full-load torque and ω is (angular) speed as a percentage of synchronous (angular) speed.

Find the *maximum* and *minimum* torque points and check the result with a plot of τ against ω.

Example 4 - Solution I/VI

To obtain extreme values of τ, we first differentiate it w.r.t. ω:

$$\begin{aligned}\frac{\mathrm{d}\tau}{\mathrm{d}\omega} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\big(-0.0016\omega^3 + 0.17\omega^2 - 3.4\omega + 250\big)\\ &= -0.0048\omega^2 + 0.34\omega - 3.4\end{aligned}$$

Set $\frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0$ and solve for ω:

$$\begin{aligned}-0.0048\omega^2 + 0.34\omega - 3.4 &= 0\\ -48\omega^2 + 3400\omega - 34000 &= 0 \quad \text{simplifying}\ldots\\ 6\omega^2 - 425\omega + 4250 &= 0\end{aligned}$$

Example 4 - Solution II/VI

Now use the quadratic formula with $a = 6$, $b = -425$, $c = 4250$:

$$\begin{aligned}
\omega &= \frac{-(-425) \pm \sqrt{(-425)^2 - 4 \times 6 \times 4250}}{2 \times 6} \\
&= \frac{425 \pm \sqrt{78625}}{12} \\
&= \frac{425 \pm 280.4015}{12} \\
&= 12.0499 \quad \text{or} \quad 58.7835
\end{aligned}$$

So these values of ω are the "locations" of the extreme values of τ.

Example 4 - Solution III/VI

Substitute these values into the original function to determine the extreme values of τ that occur at these points:

$$\begin{aligned} \tau(\omega = 12.0499) &= -0.0016(12.0499)^3 + 0.17(12.0499)^2 \\ &\qquad -3.4(12.0499) + 250 \\ &= 230.9149\ldots \end{aligned}$$

and

$$\begin{aligned} \tau(\omega = 58.7835) &= -0.0016(58.7835)^3 + 0.17(58.7835)^2 \\ &\qquad -3.4(58.7835) + 250 \\ &= 312.5869\ldots \end{aligned}$$

Example 4 - Solution IV/VI

To confirm the classifications, find the second derivative:

$$
\begin{aligned}
\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2} &= \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\omega}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega}\big(-0.0048\omega^2 + 0.34\omega - 3.4\big) \\
&= -0.0096\omega + 0.34
\end{aligned}
$$

And evaluate this at each stationary point.

Example 4 - Solution V/VI

$$\left.\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2}\right|_{\omega=12.0499} = -0.0096 \times 12.0499 + 0.34 = +0.2243 > 0$$

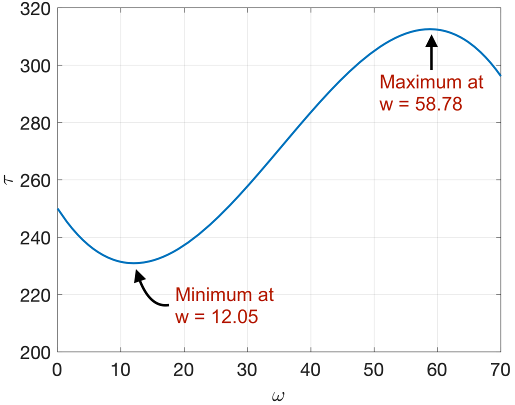
So at $\omega = 12.05$ there is a **minimum** of $\tau = 230.91$.

$$\left.\frac{\mathrm{d}^2\tau}{\mathrm{d}\omega^2}\right|_{\omega=58.7835} = -0.0096 \times 58.7835 + 0.34 = -0.2243 < 0$$

Confirming that at $\omega = 58.78$ there is a **maximum** of $\tau = 312.59$.

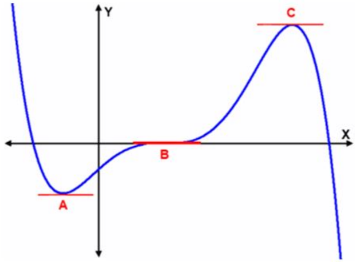
Example 4 - Solution VI/VI

Plotting this function in Excel or MATLAB to check our results:



Global vs. Local extrema

Recall our original curve:



Stationary points A and C are either the maximum or minimum in a "neighbourhood" around that point.

But stationary points may not be the largest or smallest overall. Hence, they are called "local" extrema.

In this case, the endpoints give the max and min overall values of the function over the range shown. These are the "global" extrema.

Stationary points **may or may not** be global extrema, depending on the function and range.