Integration by Parts

Railway Engineering Mathematics

Sheffield Hallam University

Lecture 17

Learning Outcomes

- Use the “by parts” technique to evaluate complicated integrals that consist of two functions multiplied together.

Why do we need this rule?

Consider the following integrals. They are too complicated to use the standard rules, but it doesn't look like we can use the substitution method either - there is no "inner" function. Instead, these all appear to be the product of two small functions.

$$\int 3x \ln(x) \, \mathrm{d}x$$

$$\int \mathrm{e}^x \ln(5x) \, \mathrm{d}x$$

$$\int x^4 \sin(x) \, \mathrm{d}x$$

$$\int \cos(2x)\left(5x + x^3\right) \, \mathrm{d}x$$

Integration by Parts

These are cases where we are required to integrate products of functions in the form $g(x)f(x)$. In these instances we must use the by parts formula:

The "by parts" rule:

$$\int u\,\frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x}\,\mathrm{d}x \;=\; u\mathrm{v} - \int \mathrm{v}\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Integration by parts is the integral equivalent of the product rule. It is obtained by re-arranging the product rule and integrating it.

Example 1 (I/II)

Evaluate:

$$\int 2x\, e^{3x}\ \mathrm{d}x$$

Let $u = 2x$ and $\frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x} = e^{3x}$ (we will see later how to decide this).

We need to know all **four** terms: $u, \frac{\mathrm{d}u}{\mathrm{d}x}, \mathrm{v}, \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x}$

Therefore, differentiate u:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2x) = 2$$

and integrate the other:

$$\mathrm{v} = \int \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x}\ \mathrm{d}x = \frac{1}{3}e^{3x} \quad \text{(ignore } +C \text{ for now)}$$

Example 1 (II/II)

Substituting these values into the integration by parts rule:

$$\begin{aligned}
\int u \, \frac{\mathrm{dv}}{\mathrm{d}x} \, \mathrm{d}x &= u\mathrm{v} - \int \mathrm{v} \, \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x \\
\int 2xe^{3x} \, \mathrm{d}x &= (2x) \times \left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right) \times (2) \, \mathrm{d}x \\
&= \frac{2}{3}xe^{3x} - \int \frac{2}{3}e^{3x} \, \mathrm{d}x \\
&= \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C
\end{aligned}$$

Integration by parts

How do we decide which function should be u and which should be the derivative of v? It is best to choose u using the following order of priority (**LATE**) from highest to lowest priority:

Logarithmic	$\ln(x)$	
Algebraic	x^n	(e.g. $-13x$, or $4x^2$)
Trigonometric	$\sin(x)$ or $\cos(x)$	
Exponential	e^{nx}	

Example 2 (I/II)

Evaluate:

$$\int 5x\sin(7x)\ \mathrm{d}x$$

Using LATE, choose $u = 5x$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = \sin(7x)$

Therefore, differentiate u:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(5x) = 5$$

and integrate the other to obtain v:

$$\mathrm{v} = \int \frac{\mathrm{dv}}{\mathrm{d}x}\ \mathrm{d}x = \int \sin(7x)\ \mathrm{d}x = -\frac{1}{7}\cos(7x)$$

Example 2 (II/II)

Substituting these values into the integration by parts rule:

$$\begin{aligned}
\int u \, \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}x} \, \mathrm{d}x &= u\mathrm{v} - \int \mathrm{v} \, \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x \\
\int 5x \sin(7x) \, \mathrm{d}x &= (5x)\left(-\frac{1}{7}\cos(7x)\right) - \int \left(-\frac{1}{7}\cos(7x)\right)(5) \, \mathrm{d}x \\
&= -\frac{5}{7}x\cos(7x) + \int \frac{5}{7}\cos(7x) \, \mathrm{d}x \\
&= -\frac{5}{7}x\cos(7x) + \frac{5}{49}\sin(7x) + C
\end{aligned}$$

Definite Integration by Parts

The "by parts" rule for definite integrals:

$$\int_a^b u \, \frac{\mathrm{dv}}{\mathrm{d}x} \, \mathrm{d}x \; = \; \left[u\mathrm{v}\right]_a^b - \int_a^b \mathrm{v} \, \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Both the first term, and the second following the integral, will need to be evaluated at the upper and lower limits.

Example 3 (I/III)

Evaluate:

$$\int_{-1}^{2} -2xe^{4x} \, \mathrm{d}x$$

Using LATE, choose $u = -2x$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = e^{4x}$

We obtain the other terms as normal:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-2x\right) = -2$$

and to obtain v:

$$\mathrm{v} = \int \frac{\mathrm{dv}}{\mathrm{d}x} \, \mathrm{d}x = \int e^{4x} \, \mathrm{d}x = \frac{1}{4}e^{4x}$$

Example 3 (II/III)

Substituting these values into the integration by parts rule:

$$\begin{aligned}
\int_a^b u \, \frac{\mathrm{dv}}{\mathrm{d}x} \, \mathrm{d}x &= \left[u\mathrm{v}\right]_a^b - \int \mathrm{v} \, \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x \\
\int_{-1}^{2} -2xe^{4x} \, \mathrm{d}x &= \left[(-2x)\left(\frac{1}{4}e^{4x}\right)\right]_{-1}^{2} - \int_{-1}^{2} \left(\frac{1}{4}e^{4x}\right)(-2) \, \mathrm{d}x \\
&= \left[-\frac{1}{2}xe^{4x}\right]_{-1}^{2} + \int_{-1}^{2} \frac{1}{2}e^{4x} \, \mathrm{d}x \\
&= \left[-\frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x}\right]_{-1}^{2}
\end{aligned}$$

Example 3 (III/III)

Now we simply have the extra step of evaluating both terms at the upper and lower limits, and determine the difference:

$$\int_{-1}^{2} -2xe^{4x}\ \mathrm{d}x = \left[-\frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x} \right]_{-1}^{2}$$

$$= \left(-\frac{1}{2}(2)e^{4\times 2} + \frac{1}{8}e^{4\times 2} \right) - \left(-\frac{1}{2}(-1)e^{4\times -1} + \frac{1}{8}e^{4\times -1} \right)$$

$$= \left(-\frac{7}{8}e^{8} \right) - \left(\frac{5}{8}e^{-4} \right)$$

$$= -2608.3 \quad \text{(1 d.p.)}$$