Astro 507; Problem Set 3

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1. ***FIRAS***

TODO: copy over stuff from jupyter notebook

2a. ***Entropy***

Let's derive the entropy for an ideal, non-relativistic Fermi gas (in terms of V, z and T). For this I will follow Lecture 10. The entropy is defined as

$$S = \frac{U + PV - N\mu}{T}, \tag{1}$$

so we just need to use definitions of U, P, N and μ. The definitions of μ is trivial when using the fugacity

$$\mu = k_B T \ln z \tag{2}$$

For the others, I refer to lecture 10 where we can find the definition of n on slide 5

$$N = nV = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V F_{1/2}(z) \tag{3}$$

and the definition of P on slide 7

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \tag{4}$$

For the total energy, we don't have it from the slides but we can derive it in the same way using Fermi-Dirac integrals. It starts in the same way as n but with an extra factor of ϵ in the numerator, then we use the substitution $w = \epsilon / k_B T$

$$U = (2s+1)\frac{V}{h^3} \int \frac{\epsilon}{e^{\frac{\epsilon - \mu}{k_B T}} + 1} \, \mathrm{d}\vec{\mathbf{p}} \tag{5}$$

$$U = \frac{4\pi(2s+1)}{2h^3} (2mk_B T)^{3/2} k_B T \int_0^\infty \mathrm{d}w \, \frac{w^{3/2}}{e^w z^{-1} + 1} \tag{6}$$

$$U = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) \tag{7}$$

Now it's simply a matter of plugging all of these into the entropy expression.

$$S = \frac{U + PV - N\mu}{T} \tag{8}$$
$$= \frac{1}{T}\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) + \frac{4(2s+1)}{3\pi^{1/2}} V k_B T \lambda^{-3} F_{3/2}(z) + \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T \ln z F_{1/2}(z)\right] \tag{9}$$
$$= \frac{2(2s+1)V k_B T}{\pi^{1/2}\lambda^3}\left[F_{3/2}(z) + \frac{2}{3}F_{3/2}(z) + \ln z F_{1/2}(z)\right] \tag{10}$$
$$\boxed{S = \frac{2(2s+1)V k_B T}{\pi^{1/2}\lambda^3}\left[\frac{5}{3}F_{3/2}(z) + F_{1/2}(z)\ln z\right]} \tag{11}$$

2b. ***Expanding pressure***

To start, we can right a simplified expression for P/nk_BT using the expressions from part a.

$$\frac{P}{nk_BT} = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z)\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} k_B T F_{1/2}(z)\right]^{-1} \tag{12}$$
$$\frac{P}{nk_BT} = \frac{2F_{3/2}(z)}{3F_{1/2}(z)} \tag{13}$$

Now we need to expand this in terms of z. Let's start by just expanding the Fermi-Dirac integral to start with. We know this is defined as

$$F_v(z) = \int_0^\infty \mathrm{d}w \, \frac{w^v}{e^w z^{-1} + 1} \tag{14}$$

If we perform a Taylor expansion on this we find

$$F_v(z) \approx \int_0^\infty \left(\sum_{i=1}^\infty e^{-iw} w^v z^i\right) \mathrm{d}w \tag{15}$$

We can switch the order of this sum and integral and also pull out the z terms to find that

$$F_v(z) \approx \sum_{i=1}^\infty \left(z_i \int_0^\infty e^{-iw} w^v \, \mathrm{d}w\right) \tag{16}$$

Let's keep the first two terms of this (since just keeping the first results in just getting 1 as the answer haha). This gives the ratio as

$$\frac{P}{nk_BT} \approx \frac{2}{3}\frac{\frac{3\sqrt{\pi}}{4}z + \frac{3\sqrt{\pi}}{16\sqrt{2}}z^2}{\frac{\sqrt{\pi}}{2}z + \frac{\sqrt{\pi}}{4\sqrt{2}}z^2} \tag{17}$$

Also of those factors of $\sqrt{\pi}$ cancel and we can also taylor expand again to get a nice simple expression as a function of the fugacity

$$\boxed{\frac{P}{nk_BT} \approx 1 + \frac{z}{4\sqrt{2}}} \tag{18}$$

3. Brown Dwarf
3a. ***Fugacity***

The general goal here is to find the number density based on the central temperature and density and then solve for the Fermi-Dirac integral and we can then invert that to find the fugacity. To start, we know that

$$\rho = m_e n_e + m_{\mathrm{H}} n_{\mathrm{H}} + m_{\mathrm{He}} n_{\mathrm{He}} \tag{19}$$

We can rewrite this (and neglect the electron term) as

$$\rho = n_{\mathrm{H}} \left(m_{\mathrm{H}} + m_{\mathrm{He}} \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} \right) \tag{20}$$

Now we can plug in the masses of hydrogen and helium as well as their relative abundance (given as 0.1 in the problem description).

$$n_{\mathrm{H}} = \frac{\rho_c}{1.4 m_p} \tag{21}$$

Now we just relate this to the electron number density simply as

$$n = \left(2 \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} + 1 \right) n_{\mathrm{H}} \tag{22}$$

And combining the two gives an expression for the number density in terms of the central density

$$n = \frac{1.2 \rho_c}{1.4 m_p} \tag{23}$$

The other part that we need is the de-Broglie wavelength which depends on the central temperature as

$$\lambda = \frac{h}{\sqrt{2\pi m_e k_B T_c}} \tag{24}$$

Now let's use this with the equation for the number density from Lecture 10 and solve for the Fermi-Dirac integral (plugging in values in the final equation).

$$n = \frac{2(2s+1)}{\pi^{1/2} \lambda^3} F_{1/2}(z) \tag{25}$$

$$F_{1/2}(z) = \frac{n \pi^{1/2} \lambda^3}{2(2s+1)} \tag{26}$$

$$F_{1/2}(z) = \frac{\pi^{1/2}}{2(2s+1)} \frac{1.2 \rho_c}{1.4 m_p} \left(\frac{h}{\sqrt{2\pi m_e k_B T_c}} \right)^3 \tag{27}$$

$$F_{1/2}(z) = 2.08 \tag{28}$$

I took this value and inverted it using the approximate analytic expressions from the paper (Aymerich-Humet+1981) to find the fugacity.

$$\boxed{z = 1.65} \tag{29}$$

3b. ***Pressure***

In order to calculate the pressure we need to apply the same pressure equation as 2a

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \tag{30}$$

In this specific case, $s = 1/2$, $T = 6 \times 10^6\,\mathrm{K}$, λ is calculated in the same way as part a and we found that $z = 1.65$ in part a. This gives the pressure as

$$\boxed{P = 1.92 \times 10^{16}\,\mathrm{Pa}} \tag{31}$$

3c. ***Relativistic?***

In lecture we found that electrons are relativistic when their fermi momentum is such that

$$p_\mathrm{F} \sim m_e c \tag{32}$$

We showed that in terms of density this was equivalent to

$$\rho_\mathrm{rel} \sim 2 \times 10^6\,\mathrm{g\,cm^{-3}} \cdot \frac{\mu_e}{2m_p} \tag{33}$$

The last term with μ_e is going to be approximately of order unity and so, since the central density is only $325\,\mathrm{g\,cm^{-3}}$ we have that the electrons **are** relativistic

$$\boxed{\rho_c \ll \rho_\mathrm{rel} \implies \text{non-relativistic}} \tag{34}$$

3d. ***Ideal Gas***

For this part we simply need to calculate the pressure assuming that it was an ideal gas. In this case

$$P_\mathrm{ideal} = n k_B T \tag{35}$$

We found n in part a and $T = 6 \times 10^6\,\mathrm{K}$ and thus we can quickly plug these in to find that

$$\boxed{P_\mathrm{ideal} = 1.38 \times 10^{16}\,\mathrm{Pa}} \tag{36}$$

We can therefore see that the degeneracy contributes significantly since the pressure is about 40% larger when including degeneracy effects.

3e. ***Polytropes***

We are given the two following equations

$$\rho_c = 8.44\,\mathrm{g\,cm^{-3}} \frac{M}{R^3} \tag{37}$$

$$RM^{1/3} = \frac{0.13}{(\mu_e F_{1/2}(z))^{2/3}} \tag{38}$$

First, let's quickly find the mass per electron in units of the proton mass, μ_e, knowing that the gas is 90% hydrogen (1 nucleon, 1 electron) and 10% helium (4 nucleons, 2 electrons).

$$\mu_e = 0.9 \cdot 1 + 0.1 \cdot 2 = 1.1 \tag{39}$$

Now we can combine these equations to instead write that

$$M = \frac{\rho_c R^3}{8.44\,\mathrm{g\,cm^{-3}}} \tag{40}$$

$$R = \left[\frac{0.13}{(\mu_e F_{1/2}(z))^{2/3}} \left(\frac{\rho_c}{8.44\,\mathrm{g\,cm^{-3}}}\right)^{-1/3}\right]^{1/2} \tag{41}$$

Plugging in our numbers $\mu_e = 1.1$, $z = 1.65$ and $\rho_c = 325\,\mathrm{g\,cm^{-3}}$ to get R (and then M) gives

$$\boxed{\begin{aligned} R &= 0.15\,\mathrm{R}_\odot & (42) \\ M &= 0.13\,\mathrm{M}_\odot & (43) \end{aligned}}$$