Astro 507; Problem Set 3

Tom Wagg

February 23, 2022

1. ***FIRAS***

2a. ***Entropy***

Let's derive the entropy for an ideal, non-relativistic Fermi gas (in terms of V, z and T). For this I will follow Lecture 10. The entropy is defined as

$$S = \frac{U + PV - N\mu}{T}, \tag{1}$$

so we just need to use definitions of U, P, N and μ. The definitions of μ is trivial when using the fugacity

$$\mu = k_B T \ln z \tag{2}$$

For the others, I refer to lecture 10 where we can find the definition of n on slide 5

$$N = nV = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V F_{1/2}(z) \tag{3}$$

and the definition of P on slide 7

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \tag{4}$$

For the total energy, we don't have it from the slides but we can derive it in the same way using Fermi-Dirac integrals. It starts in the same way as n but with an extra factor of ϵ in the numerator, then we use the substitution $w = \epsilon / k_B T$

$$U = (2s+1)\frac{V}{h^3} \int \frac{\epsilon}{e^{\frac{\epsilon - \mu}{k_B T}} + 1} \, \mathrm{d}\vec{\mathbf{p}} \tag{5}$$

$$U = \frac{4\pi(2s+1)}{2h^3}(2mk_B T)^{3/2} k_B T \int_0^\infty \mathrm{d}w \, \frac{w^{3/2}}{e^w z^{-1} + 1} \tag{6}$$

$$U = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) \tag{7}$$

Now it's simply a matter of plugging all of these into the entropy expression.

$$S = \frac{U + PV - N\mu}{T} \tag{8}$$

$$= \frac{1}{T}\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) + \frac{4(2s+1)}{3\pi^{1/2}} V k_B T \lambda^{-3} F_{3/2}(z) + \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T \ln z F_{1/2}(z)\right] \quad (9)$$

$$= \frac{2(2s+1)V k_B T}{\pi^{1/2}\lambda^3}\left[F_{3/2}(z) + \frac{2}{3}F_{3/2}(z) + \ln z F_{1/2}(z)\right] \quad (10)$$

$$\boxed{S = \frac{2(2s+1)V k_B T}{\pi^{1/2}\lambda^3}\left[\frac{5}{3}F_{3/2}(z) + F_{1/2}(z)\ln z\right]} \quad (11)$$

2b. ***Expanding pressure***

$$\frac{P}{n k_B T} = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z)\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} k_B T F_{1/2}(z)\right]^{-1} \quad (12)$$

$$= \frac{2F_{3/2}(z)}{3F_{1/2}(z)} \quad (13)$$

TODO: come back to this on Thursday

3. ***Brown Dwarf***

3a. ***Fugacity***

The general goal here is to find the number density based on the central temperature and density and then solve for the Fermi-Dirac integral and we can then invert that to find the fugacity. To start, we know that

$$\rho = m_e n_e + m_{\mathrm{H}} n_{\mathrm{H}} + m_{\mathrm{He}} n_{\mathrm{He}} \quad (14)$$

We can rewrite this (and neglect the electron term) as

$$\rho = n_{\mathrm{H}}\left(m_{\mathrm{H}} + m_{\mathrm{He}}\frac{n_{\mathrm{He}}}{n_{\mathrm{H}}}\right) \quad (15)$$

Now we can plug in the masses of hydrogen and helium as well as their relative abundance (given as 0.1 in the problem description).

$$n_{\mathrm{H}} = \frac{\rho_c}{1.4 m_p} \quad (16)$$

Now we just relate this to the electron number density simply as

$$n = \left(2\frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} + 1\right) n_{\mathrm{H}} \quad (17)$$

And combining the two gives an expression for the number density in terms of the central density

$$n = \frac{1.2\rho_c}{1.4 m_p} \quad (18)$$

The other part that we need is the de-Broglie wavelength which depends on the central temperature as

$$\lambda = \frac{h}{\sqrt{2\pi m_e k_B T_c}} \quad (19)$$

Now let's use this with the equation for the number density from Lecture 10 and solve for the Fermi-Dirac integral (plugging in values in the final equation).

$$n = \frac{2(2s+1)}{n\pi^{1/2}\lambda^3} F_{1/2}(z) \tag{20}$$

$$F_{1/2}(z) = \frac{n\pi^{1/2}\lambda^3}{2(2s+1)} \tag{21}$$

$$F_{1/2}(z) = \frac{\pi^{1/2}}{2(2s+1)} \frac{1.2\rho_c}{1.4m_p} \left(\frac{h}{\sqrt{2\pi m_e k_B T_c}} \right)^3 \tag{22}$$

$$F_{1/2}(z) = 2.08 \tag{23}$$

I took this value and inverted it using the approximate analytic expressions from the paper (Aymerich-Humet+1981) to find the fugacity.

$$\boxed{z = 1.65} \tag{24}$$

3b. ***Pressure***

In order to calculate the pressure we need to apply the same pressure equation as 2a

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \tag{25}$$

In this specific case, $s = 1/2$, $T = 6 \times 10^6\,\mathrm{K}$, λ is calculated in the same way as part a and we found that $z = 1.65$ in part a. This gives the pressure as

$$\boxed{P = 1.92 \times 10^{16}\,\mathrm{Pa}} \tag{26}$$

3c. ***Relativistic?***

In lecture we found that electrons are relativistic when their fermi momentum is such that

$$p_{\mathrm{F}} \sim m_e c \tag{27}$$

We showed that in terms of density this was equivalent to

$$\rho_{\mathrm{rel}} \sim 2 \times 10^6\,\mathrm{g\,cm^{-3}} \cdot \frac{\mu_e}{2m_p} \tag{28}$$

The last term with μ_e is going to be approximately of order unity and so, since the central density is only $325\,\mathrm{g\,cm^{-3}}$ we have that the electrons **are** relativistic

$$\boxed{\rho_c \ll \rho_{\mathrm{rel}} \implies \text{non-relativistic}} \tag{29}$$