

# Astro 507; Problem Set 3

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February 25, 2022

## 1. *FIRAS*

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TODO: copy over stuff from jupyter notebook

### 2a. *Entropy*

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Let's derive the entropy for an ideal, non-relativistic Fermi gas (in terms of  $V$ ,  $z$  and  $T$ ). For this I will follow Lecture 10. The entropy is defined as

$$S = \frac{U + PV - N\mu}{T}, \quad (1)$$

so we just need to use definitions of  $U$ ,  $P$ ,  $N$  and  $\mu$ . The definitions of  $\mu$  is trivial when using the fugacity

$$\mu = k_B T \ln z \quad (2)$$

For the others, I refer to lecture 10 where we can find the definition of  $n$  on slide 5

$$N = nV = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V F_{1/2}(z) \quad (3)$$

and the definition of  $P$  on slide 7

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \quad (4)$$

For the total energy, we don't have it from the slides but we can derive it in the same way using Fermi-Dirac integrals. It starts in the same way as  $n$  but with an extra factor of  $\epsilon$  in the numerator, then we use the substitution  $w = \epsilon/k_B T$

$$U = (2s+1) \frac{V}{h^3} \int \frac{\epsilon}{e^{\frac{\epsilon-\mu}{k_B T}} + 1} d\vec{p} \quad (5)$$

$$U = \frac{4\pi(2s+1)}{2h^3} (2mk_B T)^{3/2} k_B T \int_0^\infty dw \frac{w^{3/2}}{e^w z^{-1} + 1} \quad (6)$$

$$U = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) \quad (7)$$

Now it's simply a matter of plugging all of these into the entropy expression.

$$S = \frac{U + PV - N\mu}{T} \quad (8)$$

$$= \frac{1}{T} \left[ \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) + \frac{4(2s+1)}{3\pi^{1/2}} V k_B T \lambda^{-3} F_{3/2}(z) + \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T \ln z F_{1/2}(z) \right] \quad (9)$$

$$= \frac{2(2s+1)V k_B}{\pi^{1/2}\lambda^3} \left[ F_{3/2}(z) + \frac{2}{3} F_{3/2}(z) + \ln z F_{1/2}(z) \right] \quad (10)$$

$$\boxed{S = \frac{2(2s+1)V k_B}{\pi^{1/2}\lambda^3} \left[ \frac{5}{3} F_{3/2}(z) + F_{1/2}(z) \ln z \right]} \quad (11)$$

## 2b. *Expanding pressure*

To start, we can right a simplified expression for  $P/nk_B T$  using the expressions from part a.

$$\frac{P}{nk_B T} = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \left[ \frac{2(2s+1)}{\pi^{1/2}\lambda^3} k_B T F_{1/2}(z) \right]^{-1} \quad (12)$$

$$\frac{P}{nk_B T} = \frac{2F_{3/2}(z)}{3F_{1/2}(z)} \quad (13)$$

Now we need to expand this in terms of  $z$ . Let's start by just expanding the Fermi-Dirac integral to start with. We know this is defined as

$$F_v(z) = \int_0^\infty dw \frac{w^v}{e^{wz^{-1}} + 1} \quad (14)$$

If we perform a Taylor expansion on this we find

$$F_v(z) \approx \int_0^\infty \left( \sum_{i=1}^\infty (-1)^{i-1} e^{-iw} w^v z^i \right) dw \quad (15)$$

We can switch the order of this sum and integral and also pull out the  $z$  terms to find that

$$F_v(z) \approx \sum_{i=1}^\infty \left( (-1)^{i-1} z_i \int_0^\infty e^{-iw} w^v dw \right) \quad (16)$$

Let's keep the first two terms of this (since just keeping the first results in just getting 1 as the answer haha). This gives the ratio as

$$\frac{P}{nk_B T} \approx \frac{2}{3} \frac{\frac{3\sqrt{\pi}}{4} z - \frac{3\sqrt{\pi}}{16\sqrt{2}} z^2}{\frac{\sqrt{\pi}}{2} z - \frac{\sqrt{\pi}}{4\sqrt{2}} z^2} \quad (17)$$

Also of those factors of  $\sqrt{\pi}$  cancel and we can also taylor expand again to get a nice simple expression as a function of the fugacity

$$\boxed{\frac{P}{nk_B T} \approx 1 + \frac{z}{4\sqrt{2}}} \quad (18)$$

### 3. Brown Dwarf

#### 3a. Fugacity

The general goal here is to find the number density based on the central temperature and density and then solve for the Fermi-Dirac integral and we can then invert that to find the fugacity. To start, we know that

$$\rho = m_e n_e + m_H n_H + m_{He} n_{He} \quad (19)$$

We can rewrite this (and neglect the electron term) as

$$\rho = n_H \left( m_H + m_{He} \frac{n_{He}}{n_H} \right) \quad (20)$$

Now we can plug in the masses of hydrogen and helium as well as their relative abundance (given as 0.1 in the problem description).

$$n_H = \frac{\rho_c}{1.4 m_p} \quad (21)$$

Now we just relate this to the electron number density simply as

$$n = \left( 2 \frac{n_{He}}{n_H} + 1 \right) n_H \quad (22)$$

And combining the two gives an expression for the number density in terms of the central density

$$n = \frac{1.2 \rho_c}{1.4 m_p} \quad (23)$$

The other part that we need is the de-Broglie wavelength which depends on the central temperature as

$$\lambda = \frac{h}{\sqrt{2\pi m_e k_B T_c}} \quad (24)$$

Now let's use this with the equation for the number density from Lecture 10 and solve for the Fermi-Dirac integral (plugging in values in the final equation).

$$n = \frac{2(2s+1)}{\pi^{1/2} \lambda^3} F_{1/2}(z) \quad (25)$$

$$F_{1/2}(z) = \frac{n \pi^{1/2} \lambda^3}{2(2s+1)} \quad (26)$$

$$F_{1/2}(z) = \frac{\pi^{1/2}}{2(2s+1)} \frac{1.2 \rho_c}{1.4 m_p} \left( \frac{h}{\sqrt{2\pi m_e k_B T_c}} \right)^3 \quad (27)$$

$$F_{1/2}(z) = 2.08 \quad (28)$$

I took this value and inverted it using the approximate analytic expressions from the paper (Aymerich-Humet+1981) to find the fugacity.

$$\boxed{z = 1.65} \quad (29)$$

### 3b. *Pressure*

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In order to calculate the pressure we need to apply the same pressure equation as 2a

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \quad (30)$$

In this specific case,  $s = 1/2$ ,  $T = 6 \times 10^6$  K,  $\lambda$  is calculated in the same way as part a and we found that  $z = 1.65$  in part a. This gives the pressure as

$$\boxed{P = 1.92 \times 10^{16} \text{ Pa}} \quad (31)$$

### 3c. *Relativistic?*

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In lecture we found that electrons are relativistic when their fermi momentum is such that

$$p_F \sim m_e c \quad (32)$$

We showed that in terms of density this was equivalent to

$$\rho_{\text{rel}} \sim 2 \times 10^6 \text{ g cm}^{-3} \cdot \frac{\mu_e}{2m_p} \quad (33)$$

The last term with  $\mu_e$  is going to be approximately of order unity and so, since the central density is only  $325 \text{ g cm}^{-3}$  we have that the electrons **are** relativistic

$$\boxed{\rho_c \ll \rho_{\text{rel}} \implies \text{non-relativistic}} \quad (34)$$

### 3d. *Ideal Gas*

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For this part we simply need to calculate the pressure assuming that it was an ideal gas. In this case

$$P_{\text{ideal}} = nk_B T \quad (35)$$

We found  $n$  in part a and  $T = 6 \times 10^6$  K and thus we can quickly plug these in to find that

$$\boxed{P_{\text{ideal}} = 1.38 \times 10^{16} \text{ Pa}} \quad (36)$$

We can therefore see that the degeneracy contributes significantly since the pressure is about 40% larger when including degeneracy effects.

### 3e. *Polytropes*

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We are given the two following equations

$$\rho_c = 8.44 \text{ g cm}^{-3} \frac{M}{R^3} \quad (37)$$

$$RM^{1/3} = \frac{0.13}{(\mu_e F_{1/2}(z))^{2/3}} \quad (38)$$

First, let's quickly find the mass per electron in units of the proton mass,  $\mu_e$ , knowing that the gas is 90% hydrogen (1 nucleon, 1 electron) and 10% helium (4 nucleons, 2 electrons).

$$\mu_e = 0.9 \cdot 1 + 0.1 \cdot 2 = 1.1 \quad (39)$$

Now we can combine these equations to instead write that

$$M = \frac{\rho_c R^3}{8.44 \text{ g cm}^{-3}} \quad (40)$$

$$R = \left[ \frac{0.13}{(\mu_e F_{1/2}(z))^{2/3}} \left( \frac{\rho_c}{8.44 \text{ g cm}^{-3}} \right)^{-1/3} \right]^{1/2} \quad (41)$$

Plugging in our numbers  $\mu_e = 1.1$ ,  $z = 1.65$  and  $\rho_c = 325 \text{ g cm}^{-3}$  to get R (and then M) gives

$$\boxed{R = 0.15 \text{ R}_\odot} \quad (42)$$

$$\boxed{M = 0.13 \text{ M}_\odot} \quad (43)$$