ASTR 507, Winter 2022, Homework #4
Due: March 16, 2022
1. [7 pt] Impossible astronomy

Suppose that an astronomer finds one of the following objects. Explain why the astronomer must be misinterpreting the data as these objects should not exist in nature (or they have made a discovery of profound importance!). Use equations and explain the physical principle(s) that support your arguments, where possible.

(a) [2 pt] A planet of three times the mass of Jupiter and the size of the Earth. **Quantify** (Hint: for iron, the densest common element in the Universe, $Z = 26$ and $A = 56$).

(b) [1 pt] A *very cold* planet of three times the mass of Jupiter and half the size of the Sun (about 5 times the radius of Jupiter).

(c) [1 pt] A neutron star of four times the mass of the Sun.

(d) [1 pt] A white dwarf of twice the mass of the Sun.

(e) [1 pt] A white dwarf which is 60% of the mass of the Sun and has a surface temperature of 1000 K.

(f) [1 pt] A stellar remnant black hole less than the mass of the Sun.

2. [7 pt] Mass-radius relation of water/ice worlds.

The approximate equation of state for water/ice given in Seager et al. (2007) is:

$$\rho(P) = \rho_0 + KP^n \tag{1}$$

with $\rho_0 = 1.46$ g cm^{-3}, $K = 3.11 \times 10^{-6}$ g cm^{-3} Pa^{-n}, where Pa = Pascal = 10 g cm^{-1} sec^{-2}[1], $n = 0.513$. Note: for this problem, c does not mean the speed of light, K does not mean Kelvin, and n does not mean the polytropic index.

a) [1 pt] Sketch the density versus pressure for this water EOS. In the high pressure limit, this equation of state becomes a polytrope. What is the adiabatic index, Γ, in this limit? Is water softer or stiffer compared to an adiabatic ideal gas at high pressure?

b) [2 pt] Estimate the transition pressure between the constant density at low pressure and rising density at high pressure, and mark this on your diagram. Why is the density constant at low pressure? Why does it rise at high pressure?

c) [1 pt] There is no temperature dependence to this equation of state. Explain why this is a good approximation for the interiors of planets when

[1] Note: this means you need to express pressure in this equation in units of Pascals.

the temperature is less than 10^3 K.

d) [3 pt] Using the one-zone approximation, up to what planet mass is a constant density a good approximation given the pressure transition from part (b)? How does this compare to the mass of Earth, 6×10^{27} g? How do you expect the power- law slope of radius versus mass to differ above this mass (qualitatively)?

3. [7 pt] Radiation-dominated star (drop all constants of order unity in the first two parts of this problem)

a) [1 pt] Using the one-zone model, express the internal temperature of a radiation-dominated star in terms of G, M, R, and a, where $P_{rad} \approx a_{rad}T^4$.

b) [2 pt] Using the fact that in a radiation-dominated star, $P_{rad} > P_{gas}$, show that the lower-limit on the mass of a radiation-dominated star is: $M_{rad} = m_p \left(\frac{Gm_p^2}{ch}\right)^{-3/2}$. You can use $a_{rad} \approx k_B^4/(c^3h^3)$ (dropping factors of order unity). The ratio $Gm_p^2/(ch)$ is sometimes called the gravitational fine-structure constant.

c) [1 pt] Evaluate the mass limit, M_{rad}, in terms of $M_\odot$. Despite the missing factors of order unity, does this estimate make sense given what you know about stars?

d) [2 pt] If the equation of state in the interior of a radiation-dominated star is expressed as $P = K\rho^\Gamma = K\rho^{1+1/n}$, what value is Γ close to, and what is the polytropic index n? What assumption is made in this equation such that the pressure is just a function of density and not temperature?

e) [1 pt] Explain how you would use the solution to the Lane-Emden equation (assuming that most of the interior of the star were radiation-dominated, and that the polytropic equation of state from part d applies) to derive the internal structure for stars above the radiation-dominated mass limit.