Astro 507; Problem Set 2

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1. Heat Capacities

The definition of heat capacity gives that

$$C \equiv \frac{\mathrm{d}Q}{\mathrm{d}T} \tag{1}$$

C_P and C_V are the same as C but with the pressure, P, and volume, V, each held constant respectively. Applying the first law of thermodynamics, this gives

$$C_V = \left(\frac{\mathrm{d}U}{\mathrm{d}T}\right)_{V,N} \qquad C_P = \left(\frac{\mathrm{d}U}{\mathrm{d}T}\right)_P + P\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)_P \tag{2}$$

Now let's apply the equipartition theorem and therefore write that

$$U = \frac{1}{2} N f k_B T \tag{3}$$

where f is the number of degrees of freedom. Now let's do some derivatives and apply the ideal gas law ($V = N k_B T/P$)

$$C_V = \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{1}{2} N f k_B T\right) \tag{4}$$

$$= N k_B \cdot \frac{f}{2} \tag{5}$$

$$C_P = \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{1}{2} N f k_B T\right) + P \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{N k_B T}{P}\right) \tag{6}$$

$$= \frac{1}{2} N f k_B + N k_B \tag{7}$$

$$= N k_B \left(1 + \frac{f}{2}\right) \tag{8}$$

$$\tag{9}$$

Therefore, the difference between the two is

$$\boxed{C_P - C_V = N k_B} \tag{10}$$

And dividing the two gives the adiabatic index

$$\boxed{\frac{C_P}{C_V} = 1 + \frac{2}{f} = \gamma} \tag{11}$$

2. The Fastest Electron

We are given the following information:

$$V = (40\,\mathrm{kpc})^3, \qquad n = 0.01\,\mathrm{cm}^{-3}, \qquad T = 10^6\,\mathrm{K} \tag{12}$$

This means that the total number of particles in this volume is approximately

$$N = nV = 1.88 \times 10^{67} \tag{13}$$

I think that this means that the fastest particle would have a probability of $1/N$. We of course know that the Maxwell-Boltzmann distribution is

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{14}$$

In order to find the maximum velocity we can set the following integral equal to $1/N$ since we need to find the velocity above which the probability of finding a particle is $1/N$ (such that we expect to find a single particle).

$$\begin{aligned}
\frac{1}{N} &= \int_{v_{\max}}^{\infty} f(v)\,\mathrm{d}v && (15)\\
&= \int_{v_{\max}}^{\infty} \mathrm{d}v \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) && (16)\\
&= \sqrt{\frac{2}{\pi}} \int_{v_{\max}}^{\infty} \mathrm{d}v \left(\frac{m}{2k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) && (17)\\
&= \sqrt{\frac{2}{\pi}} \int_{v_{\max}}^{\infty} \mathrm{d}v\, A^{3/2} v^2 \exp\left(-Av^2\right) && (18)\\
&\approx \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{A}}{2} v_{\max} \exp\left(-Av_{\max}^2\right) && \text{(apply hint)}\\
\frac{1}{N} &\approx \sqrt{\frac{A}{2\pi}} v_{\max} \exp\left(-Av_{\max}^2\right) && (19)
\end{aligned}$$

At this point I wasn't entirely sure how to solve this so I plugged the whole lot into mathematica (using the values for N and T above plus assuming $m = m_e$). After also asserting that $v_{\max}$ is large, this gives

$$\boxed{v_{\max} = 6.9 \times 10^7\,\mathrm{m\,s^{-1}}} \tag{20}$$

Given that $0.99c \approx 3 \times 10^8$, $v_{\max}$ is approximately 23% of the speed of cosmic rays. This implies to me that cosmic rays **cannot** be thermal in origin.

3. Atmospheric Escape

3a. Compute number density

We know that the mean free path is defined as

$$\lambda_{\mathrm{esc}} \equiv \frac{1}{n_{\mathrm{esc}}\sigma}, \tag{21}$$

thus we can simply set this equal to the scale height and solve for the number density.

$$\lambda_{\mathrm{esc}} = H \tag{22}$$

$$\frac{1}{n_{\mathrm{esc}}\sigma} = \frac{k_B T}{mg} \tag{23}$$

$$\boxed{n_{\mathrm{esc}} = \frac{mg}{\sigma k_B T}} \tag{24}$$

3b. Particle flux expression

Integration time! Basically just applying the double angle rule and doing some integrals.

$$\Phi(v)\,\mathrm{d}v = \frac{f(v)}{4\pi}\,\mathrm{d}v \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} v\cos\theta\sin\theta\,\mathrm{d}\theta \tag{25}$$

$$= \frac{vf(v)}{4\pi}\,\mathrm{d}v \int_0^{2\pi} \mathrm{d}\phi\, \frac{1}{2} \int_0^{\pi/2} \sin(2\theta)\,\mathrm{d}\theta \tag{26}$$

$$= -\frac{vf(v)}{4\pi}\,\mathrm{d}v \int_0^{2\pi} \mathrm{d}\phi - \frac{1}{4}[\cos(2\theta)]_0^{\pi/2} \tag{27}$$

$$= -\frac{vf(v)}{4\pi}\,\mathrm{d}v \int_0^{2\pi} \mathrm{d}\phi - \frac{1}{4}\cdot(-1-1) \tag{28}$$

$$= \frac{vf(v)}{8\pi}\,\mathrm{d}v \int_0^{2\pi} \mathrm{d}\phi \tag{29}$$

$$\boxed{\Phi(v)\,\mathrm{d}v = \frac{vf(v)}{4}\,\mathrm{d}v} \tag{30}$$

3c. Total particle flux

Right, let's see if I can remember how to do integrals! Let's start by performing a substitution.

$$\Phi = \int_{\mathrm{v_{esc}}}^{\infty} \Phi(v)\,\mathrm{d}v \tag{31}$$

$$= \frac{1}{4}\int_{\mathrm{v_{esc}}}^{\infty} vf(v)\,\mathrm{d}v \tag{32}$$

$$= \sqrt{\frac{1}{8\pi}} \int_{\mathrm{v_{esc}}}^{\infty} \left(\frac{m}{2k_B T}\right)^{3/2} v^3 \exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v \tag{33}$$

$$= \sqrt{\frac{1}{8\pi}} \int_{\mathrm{v_{esc}}}^{\infty} A^{3/2} v^3 \exp\left(-Av^2\right) \mathrm{d}v \tag{34}$$

$$u = v^2, \qquad \frac{\mathrm{d}u}{\mathrm{d}v} = 2v \tag{35}$$

$$\Phi = \sqrt{\frac{1}{8\pi}} \int_{\mathrm{v_{esc}^2}}^{\infty} A^{3/2} v^3 \exp\left(-Av^2\right) \frac{\mathrm{d}u}{2v} \tag{36}$$

$$= \sqrt{\frac{1}{8\pi}} \int_{\mathrm{v_{esc}^2}}^{\infty} \frac{1}{2} A^{3/2} u \exp\left(-Au\right) \mathrm{d}u \tag{37}$$

$$= \sqrt{\frac{A^3}{32\pi}} \int_{\mathrm{v_{esc}^2}}^{\infty} u \exp\left(-Au\right) \mathrm{d}u \tag{38}$$

Okay let's pause for a second. That covers the substitution part, now we need to do integration by parts. Since I'm already using u let's call the two variables in integration by parts a and b such that

$$\int a \, \mathrm{d}b = ab - \int b \, \mathrm{d}a \tag{39}$$

In our case, we have that

$$a = u, \qquad \mathrm{d}b = \exp(-Au) \tag{40}$$

and thus we can also write that

$$\mathrm{d}a = 1, \qquad b = -\frac{1}{A} \exp(-Au) \tag{41}$$

So now putting that all together, we can write that the integral in full is

$$= \left[u \cdot -\frac{1}{A} \exp(-Au) \right]_{\mathrm{v_{esc}^2}}^{\infty} - \int_{\mathrm{v_{esc}^2}}^{\infty} -\frac{1}{A} \exp(-Au) \, \mathrm{d}u \tag{42}$$

$$= \left[-\frac{u}{A} \exp(-Au) \right]_{\mathrm{v_{esc}^2}}^{\infty} + \frac{1}{A} \int_{\mathrm{v_{esc}^2}}^{\infty} \exp(-Au) \, \mathrm{d}u \tag{43}$$

$$= \left(0 + \frac{v_{\mathrm{esc}}^2}{A} \exp\left(-Av_{\mathrm{esc}}^2\right) \right) - \left[\frac{\exp(-Au)}{A^2} \right]_{\mathrm{v_{esc}^2}}^{\infty} \tag{44}$$

$$= \frac{v_{\mathrm{esc}}^2}{A} \exp\left(-Av_{\mathrm{esc}}^2\right) + \frac{\exp(-Av_{\mathrm{esc}}^2)}{A^2} \tag{45}$$

Okay phew, we made it. Now let's go and plug that back in!

$$\Phi = \sqrt{\frac{A^3}{32\pi}} \left(\frac{v_{\mathrm{esc}}^2}{A} \exp\left(-Av_{\mathrm{esc}}^2\right) + \frac{\exp(-Av_{\mathrm{esc}}^2)}{A^2} \right) \tag{46}$$

$$= \frac{(1 + Av_{\mathrm{esc}}^2) e^{-Av_{\mathrm{esc}}^2}}{\sqrt{32\pi A}} \tag{47}$$

Now let's put this into the format that you asked for. My constant A is related to v_s as $A = v_s^{-2}$ and thus we can write that $Av_{\mathrm{esc}}^2 = \lambda_{\mathrm{esc}}$. Additionally, we want this in terms of the total number of particles and so we multiply by the number density.

$$\boxed{\Phi = \frac{n_{\mathrm{esc}} v_s (1 + \lambda_{\mathrm{esc}}) e^{-\lambda_{\mathrm{esc}}}}{\sqrt{32\pi}}} \tag{48}$$

3d. Earth Hydrogen Loss

Let's just quickly write down the constants that we've been given:

$$m \approx 2\,\mathrm{amu} = 3.32 \times 10^{-27}\,\mathrm{kg}, \quad T = 1000K, \quad \sigma = \pi \mathring{\mathrm{A}}^2 = \pi \times 10^{-20}\,\mathrm{m}^2 \tag{49}$$

where the mass has come from them being hydrogen molecules. Now let's evaluate $\lambda_{\mathrm{esc}}, n_{\mathrm{esc}}, v_s$ so that we can find ϕ.

$$v_{\mathrm{esc}} = 11200\,\mathrm{m\,s^{-1}} \tag{50}$$

$$n_{\mathrm{esc}} = \frac{m_{\mathrm{H2}} \cdot g}{\sigma k_B T} = \frac{3.35 \times 10^{-27}\,\mathrm{kg} \cdot 10\,\mathrm{m\,s^{-2}}}{\pi \times 10^{-20}\,\mathrm{m}^2 \cdot 1.38 \times 10^{-23}\,\mathrm{m^2\,kg\,s^{-2}\,K^{-1}} \cdot 1000\,\mathrm{K}} = 7.66 \times 10^{13}\,\mathrm{m}^{-3} \tag{51}$$

$$v_s = \sqrt{\frac{2k_B T}{m_{\mathrm{H2}}}} = \sqrt{\frac{2 \cdot 1.38 \times 10^{-23}\,\mathrm{m^2\,kg\,s^{-2}\,K^{-1}} \cdot 1000\,\mathrm{K}}{3.35 \times 10^{-27}\,\mathrm{kg}}} = 2883\,\mathrm{m\,s^{-1}} \tag{52}$$

$$\lambda_{\mathrm{esc}} = \left(\frac{11200}{2883}\right)^2 = 15.1 \tag{53}$$

Cool, now it's time to just plug those numbers in to find about ϕ.

$$\phi_{\mathrm{H2}} = 9.91 \times 10^{10}\,\mathrm{m^{-2}s^{-1}} \tag{54}$$

Finally, we need to multiply this by the area through which the particles move. Since we are given that $H \ll R$, we can say the surface at this height is essentially the same as the surface area of the Earth.

$$\mathcal{R}_{\mathrm{H2}} = \phi \cdot 4\pi R_\oplus^2 \tag{55}$$

$$= 9.91 \times 10^{10}\,\mathrm{m^{-2}s^{-1}} \cdot 4\pi (6.4 \times 10^6\,\mathrm{m})^2 \tag{56}$$

$$\boxed{\mathcal{R}_{\mathrm{H2}} = 5.1 \times 10^{25}\,\mathrm{molecules\,s^{-1}}} \tag{57}$$

Therefore, if we were to evolve forwards for 1 Gyr, the total number of molecules that would be lost are

$$\boxed{N_{\mathrm{H2,lost}} = 1.6 \times 10^{42}} \tag{58}$$

and so this is comparable to the current hydrogen content of the Earth's atmosphere, approximately a factor of 6 lower.

3d. Earth Oxygen Loss

This calculation will be exactly the same except we now change the mass from $m \approx 2\,\mathrm{amu}$ to $m \approx 32\,\mathrm{amu}$. This drastically changes things due to the exponential terms and so the rate of oxygen loss is an extremely low

$$\boxed{\mathcal{R}_{\mathrm{O2}} = 1.6 \times 10^{-71}\,\mathrm{molecules\,s^{-1}}} \tag{59}$$

Meaning that over a billion years we wouldn't expect the loss of *any* oxygen. I therefore conclude that over time the abundance of atmospheric oxygen relative to hydrogen will increase (assuming no other sources or sinks).