

Astro 507; Problem Set 2

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1. *Heat Capacities*

The definition of heat capacity gives that

$$C \equiv \frac{dQ}{dT} \quad (1)$$

C_P and C_V are C but with the pressure, P , and volume, V , each held constant respectively. Applying the first law of thermodynamics, this gives

$$C_V = \frac{dU}{dT}, \quad C_P = \frac{dU}{dT} + P \frac{dV}{dT} \quad (2)$$

Therefore the difference of the two is found as follows (applying the ideal gas law).

$$C_P - C_V = P \frac{dV}{dT} \quad (3)$$

$$= P \frac{d}{dT} \left(\frac{Nk_B T}{P} \right) \quad (4)$$

$$= P \left(\frac{Nk_B}{P} \right) \quad (5)$$

$$\boxed{C_P - C_V = Nk_B} \quad (6)$$

Now we also need to find the ratio of the two and show that it's equal to the adiabatic index.

TODO: check with Eric that we are doing this for the case of the ideal gas

2. The Fastest Electron

We are given the following information:

$$V = (40 \text{ kpc})^3, \quad n = 0.01 \text{ cm}^{-3}, \quad T = 10^6 \text{ K} \quad (7)$$

This means that the total number of particles in this volume is approximately

$$N = nV = 1.88 \times 10^{67} \quad (8)$$

I think that this means that the fastest particle would have a probability of $1/N$. We of course know that the Maxwell-Boltzmann distribution is

$$f(v) = \left(\frac{m}{2\pi k_B T} \right)^{3/2} 4\pi v^2 \exp \left(-\frac{mv^2}{2k_B T} \right) \quad (9)$$

In order to find the maximum velocity we can set the following integral equal to $1/N$ since we need to find the velocity above which the probability of finding a particle is $1/N$ (such that we expect to find a single particle).

$$\frac{1}{N} = \int_{v_{\max}}^{\infty} f(v) dv \quad (10)$$

$$= \int_{v_{\max}}^{\infty} dv \left(\frac{m}{2\pi k_B T} \right)^{3/2} 4\pi v^2 \exp \left(-\frac{mv^2}{2k_B T} \right) \quad (11)$$

$$= \sqrt{\frac{2}{\pi}} \int_{v_{\max}}^{\infty} dv \left(\frac{m}{2k_B T} \right)^{3/2} v^2 \exp \left(-\frac{mv^2}{2k_B T} \right) \quad (12)$$

$$= \sqrt{\frac{2}{\pi}} \int_{v_{\max}}^{\infty} dv A^{3/2} v^2 \exp(-Av^2) \quad (13)$$

$$\approx \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{A}}{2} v_{\max} \exp(-Av_{\max}^2) \quad (\text{apply hint})$$

$$\frac{1}{N} \approx \sqrt{\frac{A}{2\pi}} v_{\max} \exp(-Av_{\max}^2) \quad (14)$$

At this point I wasn't entirely sure how to solve this so I plugged the whole lot into mathematica (using the values for N and T above plus assuming $m = m_e$). After also asserting that v_{\max} is large, this gives

$$\boxed{v_{\max} = 6.9 \times 10^7 \text{ m s}^{-1}} \quad (15)$$

Given that $0.99c \approx 3 \times 10^8$, v_{\max} is approximately 23% of the speed of cosmic rays. This implies to me that cosmic rays **cannot** be thermal in origin.

3. Atmospheric Escape

3a. Compute number density

We know that the mean free path is defined as

$$\lambda_{\text{esc}} \equiv \frac{1}{n_{\text{esc}} \sigma}, \quad (16)$$

thus we can simply set this equal to the scale height and solve for the number density.

$$\lambda_{\text{esc}} = H \quad (17)$$

$$\frac{1}{n_{\text{esc}} \sigma} = \frac{k_B T}{mg} \quad (18)$$

$$\boxed{n_{\text{esc}} = \frac{mg}{\sigma k_B T}} \quad (19)$$

3b. Particle flux expression

Integration time! Basically just applying the double angle rule and doing some integrals.

$$\phi(v) dv = \frac{f(v)}{4\pi} dv \int_0^{2\pi} d\phi \int_0^{\pi/2} v \cos \theta \sin \theta d\theta \quad (20)$$

$$= \frac{vf(v)}{4\pi} dv \int_0^{2\pi} d\phi \frac{1}{2} \int_0^{\pi/2} \sin(2\theta) d\theta \quad (21)$$

$$= -\frac{vf(v)}{4\pi} dv \int_0^{2\pi} d\phi - \frac{1}{4} [\cos(2\theta)]_0^{\pi/2} \quad (22)$$

$$= -\frac{vf(v)}{4\pi} dv \int_0^{2\pi} d\phi - \frac{1}{4} \cdot (-1 - 1) \quad (23)$$

$$= \frac{vf(v)}{8\pi} dv \int_0^{2\pi} d\phi \quad (24)$$

$$\boxed{\phi(v) dv = \frac{vf(v)}{4} dv} \quad (25)$$

3c. Total particle flux

Right, let's see if I can remember how to do integrals! Let's start by performing a substitution.

$$\phi = \int_{v_{\text{esc}}}^{\infty} \phi(v) dv \quad (26)$$

$$= \frac{1}{4} \int_{v_{\text{esc}}}^{\infty} vf(v) dv \quad (27)$$

$$= \sqrt{\frac{1}{8\pi}} \int_{v_{\text{esc}}}^{\infty} \left(\frac{m}{2k_B T} \right)^{3/2} v^3 \exp\left(-\frac{mv^2}{2k_B T}\right) dv \quad (28)$$

$$= \sqrt{\frac{1}{8\pi}} \int_{v_{\text{esc}}}^{\infty} A^{3/2} v^3 \exp(-Av^2) dv \quad (29)$$

$$u = v^2, \quad \frac{du}{dv} = 2v \quad (30)$$

$$\phi = \sqrt{\frac{1}{8\pi}} \int_{v_{\text{esc}}^2}^{\infty} A^{3/2} v^3 \exp(-Av^2) \frac{du}{2v} \quad (31)$$

$$= \sqrt{\frac{1}{8\pi}} \int_{v_{\text{esc}}^2}^{\infty} \frac{1}{2} A^{3/2} u \exp(-Au) du \quad (32)$$

$$= \sqrt{\frac{A^3}{32\pi}} \int_{v_{\text{esc}}^2}^{\infty} u \exp(-Au) du \quad (33)$$

Okay let's pause for a second. That covers the substitution part, now we need to do integration by parts. Since I'm already using u let's call the two variables in integration by parts a and b such that

$$\int a db = ab - \int b da \quad (34)$$

In our case, we have that

$$a = u, \quad db = \exp(-Au) \quad (35)$$

and thus we can also write that

$$da = 1, \quad b = -\frac{1}{A} \exp(-Au) \quad (36)$$

So now putting that all together, we can write that the integral in full is

$$= \left[u \cdot -\frac{1}{A} \exp(-Au) \right]_{v_{\text{esc}}^2}^{\infty} - \int_{v_{\text{esc}}^2}^{\infty} -\frac{1}{A} \exp(-Au) du \quad (37)$$

$$= \left[-\frac{u}{A} \exp(-Au) \right]_{v_{\text{esc}}^2}^{\infty} + \frac{1}{A} \int_{v_{\text{esc}}^2}^{\infty} \exp(-Au) du \quad (38)$$

$$= \left(0 + \frac{v_{\text{esc}}^2}{A} \exp(-Av_{\text{esc}}^2) \right) - \left[\frac{\exp(-Au)}{A^2} \right]_{v_{\text{esc}}^2}^{\infty} \quad (39)$$

$$= \frac{v_{\text{esc}}^2}{A} \exp(-Av_{\text{esc}}^2) + \frac{\exp(-Av_{\text{esc}}^2)}{A^2} \quad (40)$$

Okay phew, we made it. Now let's go and plug that back in!

$$\phi = \sqrt{\frac{A^3}{32\pi}} \left(\frac{v_{\text{esc}}^2}{A} \exp(-Av_{\text{esc}}^2) + \frac{\exp(-Av_{\text{esc}}^2)}{A^2} \right) \quad (41)$$

$$= \frac{(1 + Av_{\text{esc}}^2) e^{-Av_{\text{esc}}^2}}{\sqrt{32\pi} A} \quad (42)$$

Now let's put this into the format that you asked for. My constant A is related to v_s as $A = v_s^{-2}$ and thus we can write that $Av_{\text{esc}}^2 = \lambda_{\text{esc}}$. Additionally, we want this in terms of the total number of particles and so we multiply by the number density.

$$\boxed{\phi = \frac{n_{\text{esc}} v_s (1 + \lambda_{\text{esc}}) e^{-\lambda_{\text{esc}}}}{\sqrt{32\pi}}} \quad (43)$$

3d. Earth Hydrogen Loss

Let's just quickly write down the constants that we've been given:

$$m \approx 2 \text{ amu} = 3.32 \times 10^{-27} \text{ kg}, \quad T = 1000 \text{ K}, \quad \sigma = \pi \text{ \AA}^2 = \pi \times 10^{-20} \text{ m}^2 \quad (44)$$

where the mass has come from them being hydrogen molecules. Now let's evaluate $\lambda_{\text{esc}}, n_{\text{esc}}, v_s$ so that we can find ϕ .

$$v_{\text{esc}} = 11200 \text{ m s}^{-1} \quad (45)$$

$$n_{\text{esc}} = \frac{m_{\text{H}_2} \cdot g}{\sigma k_B T} = \frac{3.35 \times 10^{-27} \text{ kg} \cdot 10 \text{ m s}^{-2}}{\pi \times 10^{-20} \text{ m}^2 \cdot 1.38 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1} \cdot 1000 \text{ K}} = 7.66 \times 10^{13} \text{ m}^{-3} \quad (46)$$

$$v_s = \sqrt{\frac{2k_B T}{m_{\text{H}_2}}} = \sqrt{\frac{2 \cdot 1.38 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1} \cdot 1000 \text{ K}}{3.35 \times 10^{-27} \text{ kg}}} = 2883 \text{ m s}^{-1} \quad (47)$$

$$\lambda_{\text{esc}} = \left(\frac{11200}{2883} \right)^2 = 15.1 \quad (48)$$

Cool, now it's time to just plug those numbers in to find about ϕ .

$$\phi_{\text{H}_2} = 9.91 \times 10^{10} \text{ m}^{-2} \text{ s}^{-1} \quad (49)$$

Finally, we need to multiply this by the area through which the particles move. Since we are given that $H \ll R$, we can say the surface at this height is essentially the same as the surface area of the Earth.

$$\mathcal{R}_{\text{H}_2} = \phi \cdot 4\pi R_{\oplus}^2 \quad (50)$$

$$= 9.91 \times 10^{10} \text{ m}^{-2} \text{ s}^{-1} \cdot 4\pi (6.4 \times 10^6 \text{ m})^2 \quad (51)$$

$$\boxed{\mathcal{R}_{\text{H}_2} = 5.1 \times 10^{25} \text{ molecules s}^{-1}} \quad (52)$$

Therefore, if we were to evolve forwards for 1 Gyr, the total number of molecules that would be lost are

$$\boxed{N_{\text{H}_2, \text{lost}} = 1.6 \times 10^{42}} \quad (53)$$

and so this is comparable to the current hydrogen content of the Earth's atmosphere, approximately a factor of 6 lower.

3d. Earth Oxygen Loss

This calculation will be exactly the same except we now change the mass from $m \approx 2 \text{ amu}$ to $m \approx 32 \text{ amu}$. This drastically changes things due to the exponential terms and so the rate of oxygen loss is an extremely low

$$\boxed{\mathcal{R}_{\text{O}_2} = 1.6 \times 10^{-71} \text{ molecules s}^{-1}} \quad (54)$$

Meaning that over a billion years we wouldn't expect the loss of *any* oxygen. I therefore conclude that over time the abundance of atmospheric oxygen relative to hydrogen will increase (assuming no other sources or sinks).