Astro 507; Problem Set 3

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1. ***FIRAS***

2a. ***Entropy***

Let's derive the entropy for an ideal, non-relativistic Fermi gas (in terms of V, z and T). For this I will follow Lecture 10. The entropy is defined as

$$S = \frac{U + PV - N\mu}{T}, \tag{1}$$

so we just need to use definitions of U, P, N and μ. The definitions of μ is trivial when using the fugacity

$$\mu = k_B T \ln z \tag{2}$$

For the others, I refer to lecture 10 where we can find the definition of n on slide 5

$$N = nV = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V F_{1/2}(z) \tag{3}$$

and the definition of P on slide 7

$$P = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z) \tag{4}$$

For the total energy, we don't have it from the slides but we can derive it in the same way using Fermi-Dirac integrals. It starts in the same way as n but with an extra factor of ϵ in the numerator, then we use the substitution $w = \epsilon / k_B T$

$$U = (2s+1)\frac{V}{h^3} \int \frac{\epsilon}{e^{\frac{\epsilon - \mu}{k_B T}} + 1} \, \mathrm{d}\vec{\mathbf{p}} \tag{5}$$

$$U = \frac{4\pi(2s+1)}{2h^3} (2m k_B T)^{3/2} k_B T \int_0^\infty \mathrm{d}w \, \frac{w^{3/2}}{e^w z^{-1} + 1} \tag{6}$$

$$U = \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) \tag{7}$$

Now it's simply a matter of plugging all of these into the entropy expression.

$$S = \frac{U + PV - N\mu}{T} \tag{8}$$

$$= \frac{1}{T}\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T F_{3/2}(z) + \frac{4(2s+1)}{3\pi^{1/2}} V k_B T \lambda^{-3} F_{3/2}(z) + \frac{2(2s+1)}{\pi^{1/2}\lambda^3} V k_B T \ln z F_{1/2}(z)\right] \quad (9)$$

$$= \frac{2(2s+1)Vk_BT}{\pi^{1/2}\lambda^3}\left[F_{3/2}(z) + \frac{2}{3}F_{3/2}(z) + \ln z F_{1/2}(z)\right] \quad (10)$$

$$\boxed{S = \frac{2(2s+1)Vk_BT}{\pi^{1/2}\lambda^3}\left[\frac{5}{3}F_{3/2}(z) + F_{1/2}(z)\ln z\right]} \quad (11)$$

2b. ***Expanding pressure***

$$\frac{P}{nk_BT} = \frac{4(2s+1)}{3\pi^{1/2}} k_B T \lambda^{-3} F_{3/2}(z)\left[\frac{2(2s+1)}{\pi^{1/2}\lambda^3} k_B T F_{1/2}(z)\right]^{-1} \quad (12)$$

$$= \frac{2F_{3/2}(z)}{3F_{1/2}(z)} \quad (13)$$

TODO: come back to this on Thursday

3. Brown Dwarf
