Chapter 5

5.1 Let X be the low points rolled and Y be the high points rolled. These random variables are defined on the sample space consisting of the 36 equiprobable outcomes (i, j) with $1 \leq i, j \leq 6$, where i is the number shown by the first die and j is the number shown by the second die. For $k < l$, the event $\{X = k,\, Y = l\}$ occurs for the outcomes (k, l) and (l, k). This gives $P(X = k,\, Y = l) = \frac{2}{36}$ for $1 \leq k < l \leq 6$. Further, $P(X = k,\, Y = k) = \frac{1}{36}$ for all k.

5.2 Imagine that the 52 cards are numbered as $1, 2, \ldots, 52$. The random variables X and Y are defined on the sample space consisting of all $\binom{52}{13}$ sets of 13 different numbers from $1, 2, \ldots, 52$. Each element of the sample space is equally likely. The number of elements ω for which $X(\omega) = x$ and $Y(\omega) = y$ is equal to $\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}$. Hence the joint probability mass function of X and Y is given by

$$P(X = x,\, Y = y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$$

for all integers x, y with $x, y \geq 0$ and $x + y \leq 13$.

5.3 The joint mass function of X and $Y - X$ is

$$P(X = x,\, Y - X = z) = P(X = x,\, Y = z + x) = \frac{e^{-2}}{x!z!}$$

for $x, z = 0, 1, \ldots$. Since $\sum_{z=0}^{\infty} \frac{e^{-2}}{x!z!} = \frac{e^{-1}}{x!}$ and $\sum_{x=0}^{\infty} \frac{e^{-2}}{x!z!} = \frac{e^{-1}}{z!}$, the marginal distributions of X and $Y - X$ are Poisson distributions with expected value 1. Noting that

$$P(X = x,\, Y - X = z) = P(X = x)P(Y - X = z) \quad \text{for all } x, z,$$

we have by Rule 3.7 that X and $Y - X$ are independent. Thus, using Example 3.14, the random variable Y is Poisson distributed with expected value 2.

5.4 The joint probability mass function of X and Y satisfies

$$P(X = i,\, Y = 4 + i) = \binom{3+i}{i} 0.45^i\, 0.55^4 \quad \text{for } i = 0, 1, 2, 3$$

$$P(X = 4,\, Y = 4 + k) = \binom{3+k}{k} 0.45^4\, 0.55^k \quad \text{for } k = 0, 1, 2, 3.$$

The other $P(X = i,\, Y = j)$ are zero.

5.5 The sample space for X and Y is the set of the $\binom{10}{3} = 120$ combinations of three distinct numbers from 1 to 10. The joint mass function of X and Y is

$$P(X = x,\, Y = y) = \frac{y - x - 1}{120} \quad \text{for } 1 \le x \le 8,\ x + 2 \le y \le 10.$$

The marginal distributions are

$$P(X = x) = \sum_{y=x+2}^{10} \frac{y - x - 1}{120} = \frac{(10 - x)(9 - x)}{240} \quad \text{for } 1 \le x \le 8$$

$$P(Y = y) = \sum_{x=1}^{y-2} \frac{y - x - 1}{120} = \frac{(y - 1)(y - 2)}{240} \quad \text{for } 3 \le y \le 10.$$

Further, for $2 \le k \le 9$,

$$P(Y - X = k) = \sum_{x=1}^{10-k} P(X = x,\, Y = x + k) = \frac{(k - 1)(10 - k)}{120}.$$

5.6 The random variable X and Y are defined on a countably infinite sample space consisting of all pairs (x, y) of positive integers with $x \ne y$. The joint probability mass function of X and Y is given by

$$P(X = x,\, Y = y) = \left(\frac{8}{10}\right)^{x-1} \frac{1}{10} \left(\frac{9}{10}\right)^{y-x-1} \frac{1}{10} \quad \text{for } 1 \le x < y$$

$$P(X = x,\, Y = y) = \left(\frac{8}{10}\right)^{y-1} \frac{1}{10} \left(\frac{9}{10}\right)^{x-y-1} \frac{1}{10} \quad \text{for } 1 \le y < x.$$

Let the random variables V and W be defined by $V = \min(X, Y)$ and $W = \max(X, Y)$. Then,

$$\begin{aligned} P(V = v) &= \sum_{y=v+1}^{\infty} P(X = v,\, Y = y) + \sum_{y=v+1}^{\infty} P(X = x,\, Y = v) \\ &= 2\Big(\frac{8}{10}\Big)^{v-1} \frac{1}{10} \quad \text{for } v = 1, 2, \ldots . \end{aligned}$$

Noting that $\frac{1}{100} \sum_{x=1}^{w-1} (8/10)^{x-1} (9/10)^{w-x-1} = \frac{1}{72} (9/10)^{w} \sum_{x=1}^{w-1} (8/9)^{x}$, we find after some algebra that

$$\begin{aligned} P(W = w) &= \sum_{x=1}^{w-1} P(X = x,\, Y = w) + \sum_{y=1}^{w-1} P(X = w,\, Y = y) \\ &= \frac{2}{9}\Big(\frac{9}{10}\Big)^{w}\Big(1 - \Big(\frac{8}{9}\Big)^{w-1}\Big) \quad \text{for } w = 2, 3, \ldots . \end{aligned}$$

5.7 Using the formula $P(X=x,\, Y=y,\, N=n) = \frac{1}{6}\binom{n}{x}\left(\frac{1}{2}\right)^n \binom{n}{y}\left(\frac{1}{2}\right)^n$, we find that

$$P(X=x,\, Y=y) = \frac{1}{6}\sum_{n=1}^{6}\binom{n}{x}\binom{n}{y}\left(\frac{1}{2}\right)^{2n} \quad \text{for } 0 \le x,\, y \le 6.$$

Since $P(X=Y) = \frac{1}{6}\sum_{n=1}^{6}\left(\frac{1}{2}\right)^{2n}\sum_{x=0}^{n}\binom{n}{x}^2$, it follows that

$$P(X=Y) = \frac{1}{6}\sum_{n=1}^{6}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = 0.3221.$$

5.8 The random variables X, Y and N are defined on a countably infinite state space. The event $\{X=i,\, Y=j,\, N=n\}$ can occur in $\binom{n-1}{i-1}\binom{n-1-(i-1)}{j-1}$ ways. This is the number of ways to choose $i-1$ places for the first $i-1$ heads of coin 1 and to choose $j-1$ non-overlapping places for $j-1$ heads of coin 2 in the first $n-1$ tosses. Thus the joint probability mass function of X, Y and N is given by

$$P(X=i,\, Y=j,\, N=n) = \binom{n-1}{i-1}\binom{n-i}{j-1}\left(\frac{1}{4}\right)^n$$

for $i, j = 1, 2, \ldots$ and $n = i+j-1, i+j, \ldots$. By $P(X=i,\, Y=j) = \sum_{n=i+j-1}^{\infty} P(X=i,\, Y=j,\, N=n)$, it follows that

$$\begin{aligned} P(X=i,\, Y=j) &= \sum_{n=i+j-1}^{\infty}\frac{(n-1)!(n-i)!}{(i-1)!((n-i)!(n-i-j+1)!(j-1)!}\left(\frac{1}{4}\right)^n \\ &= \binom{i+j-2}{i-1}\sum_{n=i+j-1}^{\infty}\binom{n-1}{i+j-2}\left(\frac{1}{4}\right)^n. \end{aligned}$$

Using the identity $\sum_{m=r}^{\infty}\binom{m}{r}a^m = a^m/(1-a)^{m+1}$ for $0 < a < 1$, it follows that

$$\begin{aligned} P(X=i,\, Y=j) &= \binom{i+j-2}{i-1}\sum_{m=i+j-2}^{\infty}\binom{m}{i+j-2}\left(\frac{1}{4}\right)^{m+1} \\ &= \binom{i+j-2}{i-1}\left(\frac{1}{3}\right)^{i+j-1} \end{aligned}$$

By $P(X=i) = \sum_{j=1}^{\infty} P(X=i,\, Y=j)$,

$$P(X=i) = \sum_{j=1}^{\infty}\binom{i+j-2}{i-1}\left(\frac{1}{3}\right)^{i+j-1} = \sum_{n=i-1}^{\infty}\binom{n}{i-1}\left(\frac{1}{3}\right)^{n+1}.$$

Using again the identity $\sum_{m=r}^{\infty}\binom{m}{r}a^m = a^m/(1-a)^{m+1}$ for $0 < a < 1$, it follows that

$$P(X = i) = \left(\frac{1}{2}\right)^i \qquad \text{for } i = 1, 2, \ldots .$$

Further, we have

$$P(X = Y) = \sum_{n=1}^{\infty}\binom{2n-2}{n-1}\left(\frac{1}{3}\right)^{2n-1} = \frac{1}{3}\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{1}{9}\right)^k.$$

Using the identity $\sum_{k=0}^{\infty}\binom{2k}{k}x^k = 1/\sqrt{1-4x}$ for $|x| < \frac{1}{4}$, the numerical value 0.4472 is obtained for $P(X = Y)$.

5.9 The constant c must satisfy

$$1 = c\int_0^{\infty}\int_0^{x} e^{-2x}\,dx\,dy = c\int_0^{x} xe^{-2x}\,dx.$$

Noting that the Erlang probability density $4xe^{-2x}$ integrates to 1 over $(0, \infty)$, we find $c = 4$. By $P((X, Y) \in C) = \iint_C f(x, y)\,dx\,dy$, we have that $Z = X - Y$ satisfies

$$P(Z > z) = \int_0^{\infty} dy \int_{y+z}^{\infty} 4e^{-2x}\,dx = \int_0^{\infty} 2e^{-2(y+z)}\,dy = e^{-2z} \text{ for } z > 0.$$

Thus Z has the exponential density $2e^{-2z}$.

5.10 The constant c is determined by $c\int_0^{\infty}\int_0^{\infty} xe^{-2x(1+y)}\,dx\,dy = 1$. The gamma density satisfies $\int_0^{\infty}\frac{\lambda^k u^{k-1}}{(k-1)!}e^{-\lambda u}\,du = 1$ for any integer $k \geq 1$ and any $\lambda > 0$. Using this identity with $k = 1$ and $\lambda = 2x$, we get

$$c\int_0^{\infty}\int_0^{\infty} xe^{-2x(1+y)}\,dx\,dy = \frac{1}{2}c\int_0^{\infty} e^{-2x}\,dx\int_0^{\infty} 2xe^{-2xy}\,dy$$
$$= \frac{1}{2}c\int_0^{\infty} e^{-2x}\,dx = \frac{1}{4}c = 1$$

and so $c = 4$. Let the random variable Z be defined by $Z = XY$. Then, using the basic formula $P((X, Y) \in C) = \iint_C f(x, y)\,dx\,dy$, we find

$$P(XY \leq z) = 4\int_0^{\infty}\int_0^{z/x} xe^{-2x(1+y)}\,dx\,dy$$
$$= 2\int_0^{\infty} xe^{-2x}\,dx\int_0^{z/x} 2xe^{-2xy}\,dy,$$

and so

$$P(XY \le z) = 2\int_0^\infty e^{-2x}(1 - e^{-2xz/x})\,dx$$
$$= (1 - e^{-2z})\int_0^\infty 2e^{-2x}\,dx = 1 - e^{-2z} \qquad \text{for } z > 0.$$

5.11 Since $c\int_0^1 dx \int_0^1 \sqrt{x+y}\,dy = 1$, the constant $c = (15/4)(4\sqrt{2}-2)^{-1}$. Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, it follows that

$$P(X+Y \le z) = c\int_0^z dx \int_0^{z-x} \sqrt{x+y}\,dy = \frac{2c}{3}\int_0^z (z^{3/2} - x^{3/2})\,dx$$
$$= \frac{2c}{5} z^2\sqrt{z} \qquad \text{for } 0 \le z \le 1$$
$$P(X+Y \le z) = c\int_{z-1}^1 dx \int_{z-x}^1 \sqrt{x+y}\,dy$$
$$= \frac{4c}{15}\left(2^{5/2} - z^{5/2}\right) - \frac{2c}{3}(2-z)\,z^{3/2} \qquad \text{for } 1 \le z \le 2.$$

Differentiation gives that the density function of $X+Y$ is $cz\sqrt{z}$ for $0 < z < 1$ and $c(2-z)\sqrt{z}$ for $1 \le z < 2$.

5.12 The random variable $V = 2\pi\sqrt{X^2+Y^2}$ gives the circumference of the circle. Thus $P(V > \pi) = P(X^2 + Y^2 > \frac{1}{4})$. Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$ with $C = \{(x,y) : x, y \ge 0,\ x^2+y^2 \le \frac{1}{4}\}$, we get

$$P\Big(X^2+Y^2 \le \frac{1}{4}\Big) = \int_0^{0.5} dx \int_0^{\sqrt{0.25-x^2}} (x+y)\,dy$$
$$= \int_0^1 x\sqrt{0.25 - x^2}\,dx + \frac{1}{24}$$
$$= \int_0^{0.25} \frac{1}{2}\sqrt{0.25-u}\,du + \frac{1}{24} = \frac{1}{12}.$$

Therefore $P(V > \pi) = \frac{11}{12}$.

5.13 Let U_1 and U_2 be independent and uniformly distributed on $(0,1)$. Then, for Δx and Δy small,

$$P\big(x < X \le x+\Delta x,\ y < Y \le y + \Delta y\big) = P\big(x < U_1 \le x + \Delta x,$$
$$y < U_2 \le y + \Delta y) + P(x < U_2 \le x + \Delta x,\ y < U_1 \le y + \Delta y\big)$$
$$= 2\,\frac{\Delta x}{a} \times \frac{\Delta y}{a} \qquad \text{for } 0 < x < y < a$$

Therefore the joint density function of X and Y is given by $f(x,y) = \frac{2}{a^2}$ for $0 < x < y < a$ and $f(x,y) = 0$ otherwise. Alternatively, the joint density $f(x,y)$ can be obtained from

$$P(X > x,\, Y \leq y) = \Big(\frac{y-x}{a}\Big)^2 \quad \text{for } 0 \leq x \leq y \leq a.$$

Next, use the identity

$$P(X \leq x,\, Y \leq y) + P(X > x,\, Y \leq y) = P(Y \leq y)$$

and apply $f(x,y) = \frac{\partial}{\partial x}\big(\frac{\partial}{\partial y}P(X \leq x,\, Y \leq y)\big)$.

5.14 The joint density function of (X,Y,Z) is given by $f(x,y,z) = 1$ for $0 < x,y,z < 1$ and $f(x,y,z) = 0$ otherwise. Using the representation $P\big((X,Y,Z) \in D\big) = \iiint_D f(x,y,z)\,dx\,dy\,dz$ with $D = \{(x,y,z) : 0 \leq x,y,z \leq 1, x+y < z\}$, we get

$$P(X+Y < Z) = \int_0^1 dz \int_0^z dy \int_0^{z-y} dx.$$

This integral can be evaluated as

$$\int_0^1 dz \int_0^z (z-y)\,dy = \int_0^1 \frac{1}{2}z^2\,dz = \frac{1}{6}.$$

Since $P\big(\max(X,Y) < Z\big)$ is the probability that Z is the largest of the three components X, Y, and Z, we have by a symmetry argument that $P\big(\max(X,Y) < Z\big) = \frac{1}{3}$. Thus, by $P\big(\max(X,Y) > Z\big) = 1 - P\big(\max(X,Y) < Z\big)$,

$$P\big(\max(X,Y) > Z\big) = 1 - \frac{1}{3} = \frac{2}{3}.$$

Alternatively, $P\big(\max(X,Y) > Z\big) = \int_0^1 dx \int_0^1 dy \int_0^{\max(x,y)} dz$. This integral is $\int_0^1 dx[\int_0^x x\,dy + \int_x^1 y\,dy] = \int_0^1 [x^2 + \frac{1}{2}(1-x)^2]\,dx = \frac{2}{3}$.

5.15 Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, it follows that

$$\begin{aligned} P(X < Y) &= \frac{1}{10}\int_5^{10} dx \int_x^{\infty} e^{-\frac{1}{2}(y+3-x)}\,dy \\ &= \frac{1}{10}\int_5^{10} e^{-\frac{1}{2}(3-x)} 2\,e^{-\frac{1}{2}x} = e^{-\frac{3}{2}}. \end{aligned}$$

5.16 Let $Z = X+Y$. By the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, we get that $P(Z \leq z)$ is given by

$$\frac{1}{2}\int_0^z dx \int_0^{z-x} (x+y)e^{-(x+y)}\,dy = \frac{1}{2}\int_0^z dx \int_x^z ue^{-u}\,du$$
$$= \frac{1}{2}\int_0^z (-ze^{-z} + xe^{-x} + e^{-x} - e^{-z})dx = 1 - e^{-z}(1 + z + \frac{1}{2}z^2)$$

for $z \geq 0$. Hence the density function of $Z = X+Y$ is $f(z) = \frac{1}{2}z^2e^{-z}$ for $z > 0$ and $f(z) = 0$ otherwise. This is the Erlang density with shape parameter 3 and scale parameter 1.

5.17 We have $P(\max(X,Y) > a\min(X,Y)) = P(X > aY) + P(Y > aX)$. Thus, by a symmetry argument,

$$P(\max(X,Y) > a\min(X,Y)) = 2P(X > aY).$$

The joint density of X and Y is $f(x,y) = 1$ for $0 < x, y < 1$ and so

$$P(X > aY) = \int_0^1 dx \int_0^{x/a} dy = \int_0^1 \frac{x}{a}\,dx = \frac{1}{2a}.$$

The sought probability is $\frac{1}{a}$.

5.18 The expected value of the time until the electronic device goes down is given by

$$E(X+Y) = \int_1^\infty \int_1^\infty (x+y)\frac{24}{(x+y)^4}dxdy$$
$$= \int_1^\infty dx \int_1^\infty \frac{24}{(x+y)^3}dy = \int_1^\infty \frac{12}{(x+1)^2}dx = 6.$$

To find the density function of $X+Y$, we calculate $P(X+Y > t)$ and distinguish between $0 \leq t \leq 2$ and $t > 2$. Obviously, $P(X+Y > t) = 1$ for $0 \leq t \leq 2$. For the case of $t > 2$,

$$P(X+Y > t) = \int_1^{t-1} dx \int_{t-x}^\infty \frac{24}{(x+y)^4}dy + \int_{t-1}^\infty dx \int_1^\infty \frac{24}{(x+y)^4}dy$$
$$= \int_1^{t-1} \frac{8}{t^3}dx + \int_{t-1}^\infty \frac{8}{(x+1)^3}dx = \frac{8(t-2)}{t^3} + \frac{4}{t^2}.$$

By differentiation, the density function $g(t)$ of $X+Y$ is $g(t) = \frac{24(t-2)}{t^4}$ for $t > 2$ and $g(t) = 0$ otherwise.

5.19 The time until both components are down is $T = \max(X, Y)$. Noting that $P(T \leq t) = P(X \leq t,\, Y \leq t)$, it follows that

$$P(T \leq t) = \frac{1}{4}\int_0^t dx \int_0^t (2y + 2 - x) dy = 0.125t^3 + 0.5t^2 \text{ for } 0 \leq t \leq 1$$

$$P(T \leq t) = \frac{1}{4}\int_0^t dx \int_0^1 (2y + 2 - x) dy = 0.75t - 0.125t^2 \text{ for } 1 \leq t \leq 2.$$

The density function of T is $0.375t^2 + t$ for $0 < t < 1$ and $0.75 - 0.25t$ for $1 \leq t < 2$.

5.20 Let X and Y be the two random points at which the stick is broken with X being the point that is closest to the left end point of the stick. Assume that the stick has length 1. The joint density function of (X, Y) satisfies $f(x, y) = 2$ for $0 < x < y < 1$ and $f(x, y) = 0$ otherwise. To see this, note that $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$, where U_1 and U_2 are independent and uniformly distributed on $(0, 1)$. For any $0 < x < y < 1$ and $dx > 0$, $dy > 0$ sufficiently small, $P(x \leq X \leq x + dx,\, y \leq Y \leq y + dy)$ is equal to the sum of $P(x \leq U_1 \leq x + dx,\, y \leq U_2 \leq y + dy)$ and $P(x \leq U_2 \leq x + dx,\, y \leq U_1 \leq y + dy)$. By the independence of U_1 and U_2, this gives

$$P(x \leq X \leq x + dx,\, y \leq Y \leq y + dy) = 2dxdy,$$

showing that $f(x, y) = 2$ for $0 < x < y < 1$. All three pieces are no longer than half the length of the stick if and only if $X \leq 0.5$, $Y - X \leq 0.5$ and $1 - Y \leq 0.5$. That is, (X, Y) should satisfy $0 \leq X \leq 0.5$ and $0.5 \leq Y \leq 0.5 + X$. It now follows that

$$P(\text{no piece is longer than half the length of the stick})$$
$$= \int_0^{0.5} dx \int_{0.5}^{0.5+x} 2dy = 2\int_0^{0.5} x\, dx = \frac{1}{4}.$$

5.21 (**a**) Using the basic formula $P((X, Y) \in C) = \iint_C f(x, y)\, dx\, dy$, it follows that the sought probability is

$$P(B^2 \geq 4A) = \int_0^1 \int_0^1 \chi(a, b) f(a, b)\, da\, db,$$

where $\chi(a, b) = 1$ for $b^2 \geq 4a$ and $\chi(a, b) = 0$ otherwise. This leads to

$$P(B^2 \geq 4A) = \int_0^1 db \int_0^{b^2/4} (a + b)\, da = 0.0688.$$

(**b**) Similarly,

$$P(B^2 \geq 4AC) = \int_0^1 \int_0^1 \int_0^1 \chi(a,b,c) f(a,b,c)\, da\, db\, dc,$$

where $\chi(a,b,c) = 1$ for $b^2 \geq 4ac$ and $\chi(a,b,c) = 0$ otherwise. A convenient order of integration for $P(B^2 \geq 4AC)$ is

$$\frac{2}{3}\int_0^1 db \int_0^{b^2/4} da \int_0^1 (a{+}b{+}c)\, dc + \frac{2}{3}\int_0^1 db \int_{b^2/4}^1 da \int_0^{b^2/(4a)} (a{+}b{+}c)\, dc.$$

This leads to $P(B^2 \geq 4AC) = 0.1960$.

5.22 The marginal density of X is given by

$$\begin{aligned} f_X(x) &= \int_0^\infty 4xe^{-2x(1+y)}\, dy = 2e^{-2x} \int_0^\infty 2xe^{-2xy}\, dy \\ &= 2e^{-2x} \quad \text{for } x > 0. \end{aligned}$$

The marginal density of Y is given by

$$\begin{aligned} f_Y(y) &= \int_0^\infty 4xe^{-2x(1+y)}\, dx = \frac{1}{(1+y)^2} \int_0^\infty \Big(2(1+y)\Big)^2 xe^{-2x(1+y)}\, dx \\ &= \frac{1}{(1+y)^2} \quad \text{for } y > 0, \end{aligned}$$

using the fact that the gamma density $\lambda^2 xe^{-\lambda x}$ for $x > 0$ integrates to 1 over $(0, \infty)$.

5.23 The marginal densities of X and Y are $f_X(x) = \int_0^x 4e^{-2x}\, dy = 4xe^{-2x}$ for $x > 0$ and $f_Y(y) = \int_y^\infty 4e^{-2x}\, dx = 2e^{-2y}$ for $y > 0$.

5.24 The marginal density of X is given by

$$\begin{aligned} f_X(x) &= \int_0^{1-x} (3 - 2x - y)\, dy = (3 - 2x)(1 - x) - \frac{1}{2}(1 - x)^2 \\ &= 1.5x^2 - 4x + 2.5 \quad \text{for } 0 < x < 1. \end{aligned}$$

The marginal density of Y is given by

$$\begin{aligned} f_Y(y) &= \int_0^{1-y} (3 - 2x - y)\, dx = 3(1 - y) - (1 - y)^2 - y(1 - y) \\ &= 2 - 2y \quad \text{for } 0 < y < 1. \end{aligned}$$

5.25 The joint density of X and Y is $f(x,y) = 4/\sqrt{3}$ for (x,y) inside the triangle. The marginal density of X is

$$f_X(x) = \begin{cases} \int_0^{x\sqrt{3}} f(x,y)\,dy = 4x & \text{for } 0 < x < 0.5 \\ \int_0^{(1-x)\sqrt{3}} f(x,y)\,dy = 4(1-x) & \text{for } 0.5 < x < 1. \end{cases}$$

The marginal density of Y is

$$f_Y(y) = \int_{y/\sqrt{3}}^{1-y/\sqrt{3}} f(x,y)\,dx = \frac{4}{\sqrt{3}} - \frac{8y}{3} \quad \text{for } 0 < y < \frac{1}{2}\sqrt{3}.$$

5.26 Since $f(x,y) = \frac{1}{x}$ for $0 < x < 1$ and $0 < y < x$ and $f(x,y) = 0$ otherwise, we get $f_X(x) = \int_0^x \frac{1}{x}\,dx = 1$ for $0 < x < 1$ and $f_Y(y) = \int_y^1 \frac{1}{x}\,dx = -\ln(y)$ for $0 < y < 1$.

5.27 Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, we have

$$P(X \le x,\, Y - X \le z) = \int_0^x dv \int_v^{v+z} e^{-w}\,dw = (1 - e^{-x})(1 - e^{-z})$$

for $x, z > 0$. By partial differentiation, we get that the joint density of X and $Z = Y - X$ is $f(x,z) = e^{-x}e^{-z}$ for $x, z > 0$. The marginal densities of X and Z are the exponential densities $f_X(x) = e^{-x}$ and $f_Z(z) = e^{-z}$. The time until the system goes down is Y. The density function of Y is $\int_0^y f(x,y)\,dx = \int_0^y e^{-y}\,dx = ye^{-y}$ for $y > 0$. This is the Erlang density with shape parameter 2 and scale parameter 1.

5.28 The joint density of X and Y is $f(x,y) = 1$ for $0 < x, y < 1$. The area of the rectangle is $Z = XY$. Using the relation $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, it follows that

$$P(Z \le z) = \int_0^z dx \int_0^1 dy + \int_z^1 dx \int_0^{z/x} dy = z - z\ln(z) \text{ for } 0 \le z \le 1.$$

The density function of Z is $f(z) = -\ln(z)$ for $0 < z < 1$. The expected value of Z is $E(Z) = -\int_0^1 z\ln(z)\,dz = \frac{1}{4}$. Note that $E(XY) = E(X)E(Y)$.

5.29 Let X and Y be the packet delays on the two lines. The joint density of X and Y is $f(x,y) = \lambda e^{-\lambda x}\lambda e^{-\lambda y}$ for $x, y > 0$. Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, we obtain

$$P(X - Y > v) = \int_0^\infty \lambda e^{-\lambda y} dy \int_{y+v}^\infty \lambda e^{-\lambda x} dx = \frac{1}{2}e^{-\lambda v}$$

for $v \geq 0$. For any $v \leq 0$, $P(X - Y \leq v) = P(Y - X \geq -v)$. Thus, by symmetry, $P(X - Y \leq v) = \frac{1}{2}e^{\lambda v}$ for $v \leq 0$. Thus the density of $X - Y$ is $\frac{1}{2}\lambda e^{-\lambda|v|}$ for $-\infty < v < \infty$, which is the so-called Laplace density.

5.30 It is easiest to derive the results by using the basic relation $P((X, Y) \in C) = \iint_C f(x, y)\, dx\, dy$. The joint density of X and Y is $f(x, y) = 1$ for $0 < x, y < 1$. Let $V = \frac{1}{2}(X + Y)$. Then $P(V \leq v) = P(X + Y \leq 2v)$. Thus

$$P(V \leq v) = \int_0^{2v} dx \int_0^{2v-x} dy = 2v^2 \quad \text{for } 0 \leq v \leq 0.5$$

$$P(V > v) = \int_{2v-1}^{1} dx \int_{2v-x}^{1} dy = 2 - 4v + 2v^2 \quad \text{for } 0.5 \leq v \leq 1.$$

Thus $f_V(v) = 4v$ for $0 < v \leq \frac{1}{2}$ and $f_V(v) = 4 - 4v$ for $0.5 < v < 1$. This is the triangular density with $a = 0, b = 1, m = 0.5$.

To get the density of $|X - Y|$, note that

$$P(|X - Y| \leq v) = P(X - Y \leq v) - P(X - Y \leq -v) \quad \text{for } 0 \leq v \leq 1.$$

Also, $P(X - Y \leq -v) = P(Y - X \geq v)$ for $0 \leq v \leq 1$. Thus $P(|X - Y| \leq v) = 2P(X - Y \leq v) - 1$ for $0 \leq v \leq 1$. We have

$$P(X - Y \leq v) = P(X \leq Y + v) = \int_0^{1-v} dy \int_0^{y+v} dx + \int_{1-v}^{1} dy \int_0^{1} dx$$

$$= v - \frac{1}{2}v^2 + \frac{1}{2} \qquad \text{for } 0 \leq v \leq 1.$$

Thus $P(|X - Y| \leq v) = 2(v - \frac{1}{2}v^2 + \frac{1}{2}) - 1$ for $0 \leq v \leq 1$. Therefore the density of $|X - Y|$ is $2(1 - v)$ for $0 < v < 1$. This is the triangular density with $a = 0, b = 1, m = 1$.

5.31 We have

$$P(F \leq c) = P(X + Y \leq c) + P(1 \leq X + Y \leq c + 1) \quad \text{for } 0 \leq c \leq 1.$$

Since $P(X + Y \leq c) = \int_0^c dx \int_0^{c-x} dy$ and $P(1 \leq X + Y \leq c + 1) = \int_0^1 dx \int_{1-x}^{\min(c+1-x,\,1)} dy$, we get for any $0 \leq c \leq 1$ that

$$P(X + Y \leq c) = \frac{1}{2}c^2$$

$$P(1 \leq X + Y \leq c) = \int_0^c dx \int_{1-x}^{1} dy + \int_c^1 dx \int_{1-x}^{c+1-x} dy = \frac{1}{2}c^2 + c(1 - c).$$

This gives $P(F \leq c) = c$ for all $0 \leq c \leq 1$, proving the desired result.

5.32 Let X be uniformly distributed on $(0, 24)$ and Y be uniformly distributed on $(0, 36)$, where X and Y are independent. The sought probability is given by $P(X < Y < X + 10) + P(Y < X < Y + 10)$. Since

$$P(X < Y < X + 10) = \int_0^{24} \frac{1}{24} dx \int_x^{x+7} \frac{1}{36} dy = \frac{7}{36}$$
$$P(Y < X < Y + 10) = \int_0^{24} \frac{1}{36} dy \int_y^{\min(24,\, y+7)} \frac{1}{24} dx = \frac{287}{1728},$$

we find that the sought probability is equal to $\frac{7}{36} + \frac{287}{1728} = 0.3605$.

5.33 The joint density function $f(x, y)$ of X and Y satisfies $f(x, y) = f_X(x) f_Y(y)$ and is equal to 1 for all $0 < x, y < 1$ and 0 otherwise. Using the relation $P((X, Y) \in C) = \iint_C f(x, y)\, dx\, dy$ with $C = \{(x, y) : 0 < x < \min(1, yz),\ 0 < y < 1\}$, we get

$$P(Z \leq z) = \int_0^1 dy \int_0^{\min(1,zy)} dx \quad \text{ for } z > 0.$$

Distinguish between the cases $0 \leq z \leq 1$ and $z > 1$. For $0 \leq z \leq 1$.

$$P(Z \leq z) = \int_0^1 dy \int_0^{zy} dx = \int_0^1 zy\, dy = \frac{1}{2} z.$$

For $z > 1$,

$$P(Z \leq z) = \int_0^{1/2} dy \int_0^{zy} dx + \int_{1/z}^1 dy \int_0^1 dx = 1 - \frac{1}{2z}.$$

Hence the density function of Z is $\frac{1}{2}$ for $0 < z \leq 1$ and $\frac{1}{2z^2}$ for $z > 1$. The probability that the first significant digit of Z equals 1 is given by

$$\sum_{n=0}^{\infty} P(10^n \leq Z < 2 \times 10^n) + \sum_{n=1}^{\infty} P(10^{-n} \leq Z < 2 \times 10^{-n})$$
$$= \frac{5}{18} + \frac{1}{18} = \frac{1}{3}.$$

In general, the probability that the first significant digit of Z equals k is

$$\frac{10}{18} \times \frac{1}{k(k+1)} + \frac{1}{18} \quad \text{ for } k = 1, \ldots, 9.$$

5.34 We have

$$P(Z \le z) = \int_0^\infty \lambda e^{-\lambda x}\,dx \int_{x/z}^\infty \lambda e^{-\lambda y}\,dy = \int_0^\infty e^{-\lambda x/z}\lambda e^{-\lambda x}\,dx = \frac{z}{1+z}.$$

Thus the density function of Z is $\frac{1}{(1+z)^2}$.

5.35 We have

$$\begin{aligned} P\big(\max(X,Y) \le t\big) &= P(X \le t,\, Y \le t) = P(X \le t)P(Y \le t) \\ &= \big(1 - e^{-\lambda t}\big)^2 \quad \text{for } t > 0. \end{aligned}$$

Also,

$$P\big(X + \frac{1}{2}Y \le t\big) = \int_0^t \lambda e^{-\lambda x}\,dx \int_0^{2(t-x)} \lambda e^{-\lambda y}\,dy = \big(1 - e^{-\lambda t}\big)^2.$$

5.36 (**a**) The formula is true for $n = 1$. Suppose that the formula has been verified for $n = 1, \ldots, k$. This means that the density function of $X_1 + \cdots + X_k$ satisfies $\frac{s^{k-1}}{(k-1)!}$ for $0 < s < 1$. Then, by the convolution formula, the density function of $X_1 + \cdots + X_k + X_{k+1}$ is given by

$$\int_0^s \frac{(s-y)^{k-1}}{(k-1)!}\,dy = \frac{s^k}{k!} \quad \text{for } 0 < s < 1.$$

This gives

$$P(X_1 + \cdots + X_{k+1} \le s) = \int_0^s \frac{x^k}{k!} = \frac{s^{k+1}}{(k+1)!} \text{ for } 0 \le s \le 1.$$

(**b**) We have $P(N > n) = P(X_1 + \cdots + X_n) = \frac{1}{n!}$, it follows from the formula $E(N) = \sum_{n=0}^\infty P(N > n)$ (see Problem 3.29) that

$$E(N) = \sum_{n=0}^\infty \frac{1}{n!} = e.$$

5.37 Let $X_1, X_2, \ldots$ be a sequence of independent random variables that are uniformly distributed on $(0,1)$, and let $S_n = X_1 + \cdots + X_n$. The sought probability is

$$P(S_1 > a) + \sum_{n=1}^\infty P(S_n \le a,\, a < S_n + X_{n+1} \le 1).$$

Since S_n and X_{n+1} are independent of each other, the joint density $f_n(s,x)$ of S_n and X_{n+1} satisfies $f_n(s,x) = \frac{s^{n-1}}{(n-1)!}$ for $0 < s < 1$ and $0 < x < 1$, using the result (**a**) of Problem 5.36. Therefore,

$$P(S_n \leq a,\, a < S_n + X_{n+1} \leq 1) = \int_0^a ds \int_{a-s}^{1-s} f_n(s,x)\,dx = (1-a)\frac{a^n}{n!}.$$

Thus the sought probability is

$$1 - a + \sum_{n=1}^{\infty}(1-a)\frac{a^n}{n!} = (1-a)e^a.$$

5.38 By the independence of X_1, X_2, and X_3, the joint density function of X_1, X_2, and X_3 is $1 \times 1 \times 1 = 1$ for $0 < x_1, x_2, x_3 < 1$ and 0 otherwise. Let $C = \{(x_1, x_2, x_3) : 0 < x_1, x_2, x_3 < 1,\, 0 < x_2 + x + 3 < x_1\}$. Then

$$P(X_1 > X_2 + X_3) = \iiint_C dx_1 dx_2 dx_3 = \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_1 - x_2} dx_3.$$

This gives

$$\begin{aligned} P(X_1 > X_2 + X_3) &= \int_0^1 dx_1 \int_0^{x_1} (x_1 - x_2)\,dx_2 = \int_0^1 \frac{1}{2}x_1^2\,dx_1 \\ &= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

Since the events $\{X_1 > X_2 + X_3\}$, $\{X_2 > X_1 + X_3\}$ and $\{X_3 > X_1 + X_2\}$ are mutually exclusive, the probability that the largest of the three random variables is greater than the sum of the other two is $3 \times \frac{1}{6} = \frac{1}{2}$.
Note: More generally, let $X_1, X_2, \ldots, X_n$ be independent random numbers chosen from $(0,1)$, then $P(X_1 > X_2 + \cdots + X_n) = \frac{1}{n!}$ for any $n \geq 2$.

5.39 By $P(V > v, W \leq w) = P(v < X \leq w, v < Y \leq w)$ and the independence of X and Y, we have

$$P(V > v,\, W \leq w) = P(v < X \leq w)P(v < Y \leq w) = \left(e^{-\lambda v} - e^{-\lambda w}\right)^2$$

for $0 \leq v \leq w$. Taking partial derivatives, we get that the joint density of V and W is $f(v,w) = 2\lambda^2 e^{-\lambda(v+w)}$ for $0 < v < w$. It follows from

$$P(W - V > z) = \int_0^{\infty} dv \int_{v+z}^{\infty} 2\lambda^2 e^{-\lambda(v+w)}\,dw$$

that $P(W - V > z) = e^{-\lambda z}$ for $z > 0$, in agreement with the memoryless property of the exponential distribution.

5.40 By the substitution rule, the expected value of the area of the rectangle is equal to

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy = \int_0^1 x\Big(\frac{1}{2}x + \frac{1}{3}\Big)\,dx = \frac{1}{3}.$$

5.41 Define the function $g(x,y)$ as $g(x, y = T - \max(x,y)$ if $0 \le x, y \le T$ and $g(x,y) = 0$ otherwise. The joint density function of X and Y is $e^{-(x+y)}$ for $x, y > 0$. Using the memoryless property of the exponential distribution, the expected amount of time the system is down between two inspections is given by

$$E[g(X,Y)] = \int_0^T \int_0^T \big(T - \max(x,y)\big)e^{-(x+y)}\,dx\,dy$$
$$= 2\int_0^T (T-x)(1-e^{-x})e^{-x}\,dx = T - 1.5 + 2e^{-T} - \frac{1}{2}e^{-2T}.$$

5.42 By the substitution rule, the expected value of the time until the system goes down is

$$E[\max(X,Y)] = \frac{1}{4}\int_0^2 dx \int_0^1 \max(x,y)(2y+2-x)\,dy$$
$$= \int_0^1 2x^2\,dx + \frac{1}{4}\int_0^1 \Big[\frac{2}{3}(1-x^3) + \frac{1}{2}(2-x)(1-x^2)\Big]\,dx$$
$$+ \frac{1}{4}\int_1^2 (3x - x^2)\,dx = 0.96875.$$

The expected value of the time between the failures of the two components is $E[\max(X,Y)] - E[\min(X,Y)]$. By the substitution rule,

$$E[\min(X,Y)] = \frac{1}{4}\int_0^2 dx \int_0^1 \min(x,y)(2y+2-x)\,dy = 0.44792$$

and so the expected time between the failures of the two components is 0.52083.

5.43 Using the substitution rule, the expected value of the area of the circle is

$$\int_0^1 \int_0^1 \pi(x^2+y^2)(x+y)\,dx\,dy = \pi\int_0^1 (x^3 + \frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{4})\,dx = \frac{5}{6}\pi.$$

5.44 Using the substitution rule and writing $x+y=2x+y-x$, we get

$$E(X+Y)=\sum_{x=0}^{\infty}\sum_{y=x}^{\infty}(x+y)\frac{e^{-2}}{x!(y-x)!}=\sum_{x=0}^{\infty}2x\frac{e^{-1}}{x!}+\sum_{z=0}^{\infty}z\frac{e^{-1}}{z!}=3.$$

Also, using the substitution rule and writing $xy=x(y-x+x)$

$$E(XY)=\sum_{x=0}^{\infty}\sum_{y=x}^{\infty}xy\frac{e^{-2}}{x!(y-x)!}=\sum_{x=0}^{\infty}x\frac{e^{-1}}{x!}\sum_{z=0}^{\infty}z\frac{e^{-1}}{z!}+\sum_{x=0}^{\infty}x^2\frac{e^{-1}}{x!}=3.$$

5.45 The inverse functions $x=a(v,w)$ and $y=b(v,w)$ are $a(v,w)=vw$ and $b(v,w)=v(1-w)$. The Jacobian $J(v,w)$ is equal to $-v$. The joint density of V and W is

$$f_{V,W}(v,w)=\mu e^{-\mu vw}\mu e^{-\mu v(1-w)}|-v|=\mu^2 ve^{-\mu v}\text{ for } v>0,\ 0<w<1.$$

The marginal densities of V and W are

$$f_V(v)=\int_0^1\mu^2 ve^{-\mu v}\,dw=\mu^2 ve^{-\mu v}\quad\text{for } v>0$$

$$f_W(w)=\int_0^{\infty}\mu^2 ve^{-\mu v}\,dv=1\quad\text{for } 0<w<1.$$

Since $f_{V,W}(v,w)=f_V(v)f_W(w)$ for all v,w, the random variables V and W are independent.

5.46 To find the joint density of V and W, we apply the transformation formula. The inverse functions $x=a(v,w)$ and $y=b(v,w)$ are given by $a(v,w)=vw/(1+w)$ and $b(v,w)=v/(1+w)$. The Jacobian $J(v,w)$ is equal to $-v/(1+w)^2$ and so the joint density of V and W is given by

$$f_{V,W}(v,w)=1\times 1\times|J(v,w)|=\frac{v}{(1+w)^2}\quad\text{for } 0<v<2\text{ and } w>0$$

and $f_{V,W}(v,w)=0$ otherwise. The marginal density of V is

$$f_V(v)=\int_0^{\infty}\frac{v}{(1+w)^2}\,dw=\frac{1}{2}v\quad\text{for } 0<v<2$$

and $f_V(v)=0$ otherwise. The marginal density of W is given by

$$f_W(w)=\int_0^{2}\frac{v}{(1+w)^2}\,dv=\frac{2}{(1+w)^2}\quad\text{for } w>0$$

and $f_W(w)=0$ otherwise. Since $f_{V,W}(v,w)=f_V(v)f_W(w)$ for all v,w, the random variables V and W are independent.

5.47 Since Z^2 has the χ_1^2 density $\frac{1}{\sqrt{2\pi}}u^{-\frac{1}{2}}e^{-\frac{1}{2}u}$ when Z is $N(0,1)$ distributed and the random variables Z_1^2 and Z_2^2 are independent, the joint density of $X = Z_1^2$ and $Y = Z_2^2$ is $\frac{1}{2\pi}(xy)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x+y)}$ for $x, y > 0$. For the transformation $V = X + Y$ and $W = X/Y$, the inverse functions $x = a(v,w)$ and $y = b(v,w)$ are $a(v,w) = \frac{1}{2}(v+w)$ and $b(v,w) = \frac{1}{2}(v-w)$. The Jacobian $J(v,w)$ is equal to $-\frac{1}{2}$. The joint density of V and W is

$$f_{V,W}(v,w) = \frac{1}{4\pi^2}(v^2 - w^2)^{-\frac{1}{2}}\, e^{-\frac{1}{2}v} \text{ for } v > 0,\ -\infty < w < \infty.$$

The random variables V and W are not independent.

5.48 Let $V = Y\sqrt{X}$ and $W = X$. To find the joint density of V and W, we apply the transformation formula. The inverse functions $x = a(v,w)$ and $y = b(v,w)$ are $a(v,w) = w$ and $b(v,w) = v/\sqrt{w}$. The Jacobian $J(v,w)$ is equal to $-1/\sqrt{w}$ and so the joint density of V and W is given by

$$f_{V,W}(v,w) = \frac{1}{\pi}we^{-w(1+v^2/w)}\frac{1}{\sqrt{w}} = \frac{1}{\pi}\sqrt{w}e^{-\frac{1}{2}w}e^{-\frac{1}{2}v^2} \quad \text{for } v, w > 0$$

and $f_{V,W}(v,w) = 0$ otherwise. The densities $f_V(v) = \int_0^\infty f_{V,W}(v,w)dw$ and $f_W(w) = \int_0^\infty f_{V,W}(v,w)dv$ are given by

$$f_V(v) = \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}v^2} \text{ for } v > 0, \quad f_W(w) = \frac{1}{\sqrt{2\pi}}w^{\frac{1}{2}}e^{-\frac{1}{2}w} \text{ for } w > 0.$$

The random variable V is distributed as $|Z|$ with Z having the standard normal distribution and W has a gamma distribution with shape parameter $\frac{3}{2}$ and shape parameter $\frac{1}{2}$. Since $f_{V,W}(v,w) = f_V(v)f_W(w)$ for all v, w, the random variables V and W are independent.

5.49 The inverse functions are $a(v,w) = ve^{-\frac{1}{4}(v^2+w^2)}/\sqrt{v^2+w^2}$ and $b(v,w) = we^{-\frac{1}{4}(v^2+w^2)}/\sqrt{v^2+w^2}$. The Jacobian is $\frac{1}{2}e^{-\frac{1}{2}(v^2+w^2)}$. Since $f_{X,Y}(x,y) = \frac{1}{\pi}$, we get

$$f_{V,W}(v,w) = \frac{1}{\pi} \times \frac{1}{2}e^{-\frac{1}{2}(v^2+w^2)} \quad \text{for} -\infty < v, w < \infty.$$

Noting that $f_{V,W}(v,w) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}w^2}$ for all v, w, it follows that V and W are independent and $N(0,1)$ distributed.

5.50 The joint density function $f(r,\theta)$ of (R,Θ) is given by $1 \times \frac{1}{2\pi}$ for $0 < r < 1$ and $0 < \theta < 2\pi$. The inverse functions $r = a(v,w)$ and $\theta = b(v,w)$ are given by $a(v,w) = \sqrt{v^2+w^2}$ and $b(v,w) = \arctan(\frac{w}{v})$. Using the fact that $\arctan(x)$ has $\frac{1}{1+x^2}$ as derivative, it follows that the Jacobian is given by $\frac{1}{\sqrt{v^2+w^2}}$. Noting that $f(a(v,w),b(v,w))$ is $\frac{1}{2\pi}$ if $v^2+w^2 \leq 1$ and 0 otherwise, it follows from the two-dimensional transformation formula that the joint density $f_{V,W}(v,w)$ of the random vector (V,W) is given by

$$f_{V,W}(v,w) = \frac{1}{2\pi}\frac{1}{\sqrt{v^2+w^2}} \quad \text{for } -1 < v,w < 1,\ v^2+w^2 \leq 1$$

and $f_{V,W}(v,w) = 0$ otherwise. To get the marginal density

$$f_V(v) = \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} f_{V,W}(v,w)dw,$$

we use the following result from calculus:

$$\int_0^x \frac{dt}{\sqrt{1+t^2}} = \ln(x+\sqrt{1+x^2}) \quad \text{for } x > 0.$$

This leads after some algebra to

$$f_V(v) = \frac{1}{\pi}\ln\left(\frac{1}{|v|} + \frac{\sqrt{1-v^2}}{|v|}\right) \quad \text{for } -1 < v < 1.$$

The marginal density of W is of course the same as that of V. The intuitive explanation that (V,W) is not a random point inside the unit circle is as follows. The closer a (small) rectangle within the unit circle is to the center of the circle, the larger the probability of the point (V,W) falling in the rectangle.

5.51 The joint density of X and Y is $\frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}y^{\beta-1}e^{-(x+y)}$. The inverse functions are $a(v,w) = vw$ and $b(v,w) = w(1-v)$. The Jacobian $J(v,w) = w$. Thus the joint density of V and W is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(vw)^{\alpha-1}(w(1-v))^{\beta-1}e^{-v}w \quad \text{for } v,w > 0.$$

This density can be rewritten as

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}v^{\alpha-1}(1-v)^{\beta-1}\frac{w^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}e^{-w} \quad \text{for all } v,w > 0.$$

This shows that V and W are independent, where V has a beta distribution with parameters α and β, and W has a gamma distribution with shape parameter $\alpha + \beta$ and scale parameter 1.

5.52 Since Z and Y are independent, the joint density of Z and Y is

$$f(z, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \times \frac{y^{\frac{1}{2}\nu - 1} e^{-\frac{1}{2}y}}{2^{\frac{1}{2}\nu} \Gamma(\frac{1}{2}\nu)} \quad \text{for } z, y > 0.$$

The inverse functions $z = a(v, w)$ and $y = b(v, w)$ are given by $z = w\sqrt{v/\nu}$ and $y = v$. The Jacobian is $\sqrt{v/\nu}$. Thus, by the two-dimensional transformation formula, the joint density function of V and W is that

$$f_{V,W}(v, w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2 v/\nu} \times \frac{v^{\frac{1}{2}\nu - 1} e^{-\frac{1}{2}v}}{2^{\frac{1}{2}\nu} \Gamma(\frac{1}{2}\nu)} \times \sqrt{\frac{v}{\nu}} \quad \text{for } v, w > 0.$$

Letting $\lambda_w = 1 + \frac{w^2}{\nu}$ for any w, $\alpha = \frac{1}{2}(\nu + 1)$ and using the change of variable $u = \frac{v}{2}$, it follows that $f_W(w) = \int_0^\infty f_{V,W}(v, w) dv$ can be written as

$$f_W(w) = \frac{\lambda_w^{-\alpha} \Gamma(\frac{1}{2}(\nu + 1))}{\sqrt{\pi\nu}\, \Gamma(\frac{1}{2}\nu)} \int_0^\infty \frac{\lambda_w^\alpha u^{\alpha - 1} e^{-\lambda u}}{\Gamma(\alpha)} du = \frac{\lambda_w^{-\alpha} \Gamma(\frac{1}{2}(\nu + 1))}{\sqrt{\pi\nu}\, \Gamma(\frac{1}{2}\nu)},$$

showing the desired result.

Note: This problem shows that the two-dimensional transformation $V = X$ and $W = h(X, Y)$ may be useful when you to want to find the density of a function $h(x, y)$ of a random vector (X, Y) with a given joint density function $f(x, y)$.

5.53 For $\Delta x, \Delta y$ sufficiently small,

$$\begin{aligned} &P\Big(x - \frac{1}{2}\Delta x \leq U_{(1)} \leq x + \frac{1}{2}\Delta x,\ y - \frac{1}{2}\Delta y \leq U_{(n)} \leq y + \frac{1}{2}\Delta y\Big) \\ &= \binom{n}{1}\binom{n-1}{1}(y - x)^{n-2}\Delta x \Delta y \quad \text{for } 0 < x < y < 1. \end{aligned}$$

Therefore the joint density of $U_{(1)}$ and $U_{(n)}$ is

$$f(x, y) = \frac{n!}{(n-2)!}(y - x)^{n-2} \quad \text{for } 0 < x < y < 1.$$

5.54 Let $X = U_{(1)}$ and $Y = U_{(n)}$. The joint density of X and Y is given by $n(n-1)(y-x)^{n-2}$ for $0 < x < y < 1$, see Problem 5.53. For the transformation $V = Y$ and $W = Y - X$, the inverse functions are $a(v,w) = v - w$ and $b(v,w) = v$. The Jacobian $J(v,w) = 1$. Thus the joint density of V and W is given by $n(n-1)w^{n-2}$ for $w < v < 1$ and $0 < w < 1$. The marginal density of the range W is

$$\int_w^1 n(n-1)w^{n-2}\, dv = n(n-1)w^{n-2}(1-w) \quad \text{for } 0 < w < 1.$$

Note: an alternative derivation of the results of the Problems 5.53 and 5.54 can be given. This derivation goes as follows. It follows from $P(X > x, Y \leq y) = P(x < U_i \leq y \text{ for } i = 1, \ldots, n)$ that $P(X > x, Y \leq y) = (y-x)^n$ for $0 \leq x \leq y \leq 1$. Taking partial derivatives, we get that the joint density function of X and Y is given by $f(x,y) = n(n-1)(y-x)^{n-2}$ for $0 < x < y < 1$ and $f(x,y) = 0$ otherwise. Next, we get from $P(Y - X > z) = n(n-1)\int_0^{1-z} dx \int_{z+x}^1 (y-x)^{n-2}\, dy$ that $P(Y - X > z) = n\int_0^{1-z} [(1-x)^{n-1} - z^{n-1}]\, dx$. This gives $P(Y - X > z) = 1 + (n-1)z^n - nz^{n-1}$ for $0 \leq z \leq 1$ and so the density of $Y - X$ is $n(n-1)z^{n-2}(1-z)$ for $0 < z < 1$.

5.55 The marginal distributions of X and Y are the Poisson distributions $p_X(x) = e^{-1}/x!$ for $x \geq 0$ and $p_Y(y) = e^{-2}2^y/y!$ for $y \geq 0$ with $E(X) = \sigma^2(X) = 1$ and $E(Y) = \sigma^2(Y) = 2$. We have

$$E(XY) = \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} xy \frac{e^{-2}}{x!(y-x)!}.$$

Noting that $\sum_{y=x}^{\infty} y \frac{e^{-1}}{(y-x)!} = \sum_{y=x}^{\infty}(y - x + x)\frac{e^{-1}}{(y-x)!} = 1 + x$, we get $E(XY) = 3$. This gives $\rho(X,Y) = 1/\sqrt{2}$.

5.56 Using the marginal densities $f_X(x) = \frac{4}{3}(1 - x^3)$ for $0 < x < 1$ and $f_Y(y) = 4y^3$ for $0 < y < 1$, we obtain

$$E(X) = \frac{2}{5},\ E(Y) = \frac{4}{5},\ \sigma^2(X) = \frac{14}{225}, \ \text{ and } \ \sigma^2(Y) = \frac{2}{75}.$$

By $E(XY) = \int_0^1 dy \int_0^y xy4y^2\, dx = \int_0^1 2y^5\, dy$, we find $E(XY) = \frac{1}{3}$. Hence

$$\rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sigma(X)\sigma(Y)} = 0.3273.$$

5.57 The variance of the portfolio's return is

$$f^2\sigma_A^2 + (1-f)^2\sigma_B^2 + 2f(1-f)\sigma_A\sigma_B\rho_{AB}.$$

Putting the derivative of this function equal to zero, it follows that the optimal fraction f is $\left(\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}\right) / \left(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}\right)$.

5.58 Using the linearity of the expectation operator, it is readily verified from the definition of covariance that

$$\text{cov}(X+Z, Y+Z) = \text{cov}(X,Y) + \text{cov}(X,Z) + \text{cov}(Z,Y) + \text{cov}(Z,Z).$$

Since the random variables X, Y, and Z are independent, we have $\text{cov}(X,Y) = \text{cov}(X,Z) = \text{cov}(Z,Y) = 0$. Further, $\text{cov}(Z,Z) = \sigma^2(Z)$, $\sigma^2(X+Z) = \sigma^2(X)+\sigma^2(Z) = 2$, and $\sigma^2(Y+Z) = \sigma^2(Y)+\sigma^2(Z) = 2$. Therefore $\rho(X+Z, Y+Z) = \frac{1}{2}$.

5.59 (**a**) Let R_A be the rate of return of stock A and R_B be the rate of return of stock B. Since $R_B = -R_A + 14$, the correlation coefficient is -1.
(**b**) Let $X = fR_A + (1-f)R_B$. Since $X = (2f-1)R_A + 14(1-f)$, the variance of X is minimal for $f = \frac{1}{2}$. Invest $\frac{1}{2}$ of your capital in stock A and $\frac{1}{2}$ in stock B. Then the portfolio has a guaranteed rate of return of 7%.

5.60 We have $E(XY) = 6\int_0^1 dx \int_0^x xy(x-y)\,dy = \frac{1}{5}$. The marginal densities of X and Y are $f_X(x) = 3x^2$ for $0 < x < 1$ and $f_Y(y) = 3y^2 - 6y + 3$ for $0 < y < 1$. Then, $E(X) = \frac{3}{4}$, $E(Y) = \frac{1}{4}$, $\sigma(X) = \sigma(Y) = \sqrt{3/80}$. This leads to $\rho(X,Y) = \frac{1}{3}$.

5.61 The joint density of (X,Y) is $f(x,y) = \frac{1}{\pi}$ for (x,y) inside the circle C. Then,

$$E(XY) = \iint_C xy\,\frac{1}{\pi}\,dx\,dy.$$

Since the function xy has opposite signs on the quadrants of the circle, a symmetry argument gives $E(XY) = 0$. Also, by a same argument, $E(X) = E(Y) = 0$. This gives $\rho(X,Y) = 0$, although X and Y are dependent.

5.62 The joint density function of X and Y is $\frac{1}{2}$ on the region D. Since the function xy has opposite signs on the four triangles of the region D, we have $E(XY) = 0$. Also, $E(X) = E(Y) = 0$. Therefore $\rho(X,Y) = 0$.

5.63 The joint density function $f_{V,W}(v,w)$ of V and W is most easily obtained from the relation

$$\begin{aligned}&P(v < V < v+\Delta v,\, w < W < w+\Delta w)\\ &= P(v < X < v+\Delta v,\, w < Y < w+\Delta w)\\ &\quad + P(v < Y < v+\Delta v,\, w < X < w+\Delta w) = 2\Delta v \Delta w\end{aligned}$$

for $0 \leq v < w \leq 1$ when $\Delta v, \Delta w$ are small enough. This shows that $f_{V,W}(v,w) = 2$ for $0 < v < w < 1$. Next it follows that $f_V(v) = 2(1-v)$ for $0 < v < 1$ and $f_W(w) = 2w$ for $0 < w < 1$. This leads to $E(VW) = \frac{1}{4}$, $E(V) = \frac{1}{3}$, $E(W) = \frac{2}{3}$, and $\sigma(V) = \sigma(W) = \frac{1}{3\sqrt{2}}$. Thus $\rho(V,W) = \frac{1}{2}$.

5.64 Let X denote the low points rolled and Y the high points rolled. We have $P(X=i, Y=i) = \frac{1}{36}$ for $1 \leq i \leq 6$ and $P(X=i, Y=j) = \frac{2}{36}$ for $1 \leq i < j \leq 6$, see also Problem 5.1. The marginal distribution of X is given by $P(X=1) = \frac{11}{36}$, $P(X=2) = \frac{9}{36}$, $P(X=3) = \frac{7}{36}$, $P(X=4) = \frac{5}{36}$, $P(X=5) = \frac{3}{36}$, and $P(X=6) = \frac{1}{36}$, while the marginal distribution of Y is $P(Y=1) = \frac{1}{36}$, $P(Y=2) = \frac{3}{36}$, $P(Y=3) = \frac{5}{36}$, $P(Y=4) = \frac{7}{36}$, $P(Y=5) = \frac{9}{36}$, and $P(Y=6) = \frac{11}{36}$. Straightforward calculations yield

$$E(X) = \frac{91}{36},\ E(X^2) = \frac{301}{36},\ E(Y) = \frac{161}{36},\ E(Y^2) = \frac{791}{36}, \sigma(x) = 1.40408$$

$$\sigma(Y) = 1.40408,\ E(XY) = \sum_{x=1}^{6}\sum_{y=x}^{6} xyP(X=x,\, Y=y) = \frac{441}{36}.$$

It now follows that

$$\rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sigma(X)\sigma(Y)} = \frac{441/36 - (91/36)(161/36)}{(1.40408)^2} = 0.479.$$

5.65 The joint probability mass function of X and Y is given by

$$P(X=x,\, Y=y) = \frac{1}{100} \times \frac{1}{x} \quad \text{for } x = 1, 2, \ldots, 100,\ y = 1, \ldots, x.$$

The marginal distributions of X and Y are given by

$$P(X=x) = \frac{1}{100} \quad \text{and} \quad P(Y=y) = \frac{1}{100}\sum_{x=y}^{100}\frac{1}{x}$$

for $1 \le x \le 100$ and $1 \le y \le 100$. Next it follows that

$$E(XY) = \sum_{x=1}^{100} \sum_{y=1}^{x} xy \times \frac{1}{100x} = \frac{1}{100} \sum_{x=1}^{100} \frac{1}{2} x(x+1) = 1717.$$

Further,

$$\begin{aligned}
E(X) &= \frac{1}{100} \sum_{x=1}^{100} x = 50.5, \quad E(X^2) = \frac{1}{100} \sum_{x=1}^{100} x^2 = 3383.5, \\
E(Y) &= \frac{1}{100} \sum_{y=1}^{100} y \sum_{x=y}^{100} \frac{1}{x} = \frac{1}{100} \sum_{x=1}^{100} \frac{1}{x} \sum_{y=1}^{x} y = \frac{1}{200} \sum_{x=1}^{100} (x+1) \\
&= 25.75, \\
E(Y^2) &= \frac{1}{100} \sum_{y=1}^{100} y^2 \sum_{x=y}^{100} \frac{1}{x} = \frac{1}{100} \sum_{x=1}^{100} \frac{1}{x} \sum_{y=1}^{x} y^2 \\
&= \frac{1}{600} \sum_{x=1}^{100} (x+1)(2x+1) = 1153.25.
\end{aligned}$$

Hence the standard deviations of X and Y are $\sigma(X) = \sqrt{3383.5 - 50.5^2} = 28.8661$ and $\sigma(Y) = \sqrt{1153.25 - 25.75^2} = 22.1402$ and so

$$\rho(X,Y) = \frac{1717 - 50.5 \times 25.75}{28.8661 \times 22.1402} = 0.652.$$

5.66 The joint density of X and Y is $f(x,y) = \frac{1}{x}$ for $0 < y < x < 1$ and $f(x,y) = 0$ otherwise. Thus, $E(XY) = \int_0^1 dx \int_0^x xy \frac{1}{x}\, dy = \frac{1}{6}$. The marginal densities of X and Y are $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(y) = \int_y^1 \frac{1}{x} dx = -\ln(y)$ for $0 < y < 1$. This leads to $E(X) = \frac{1}{2}$, $E(Y) = \frac{1}{4}$, $\sigma(X) = \sqrt{\frac{1}{12}}$ and $\sigma(Y) = \sqrt{\frac{7}{144}}$. Therefore $\rho(X,Y) = (\frac{1}{6} - \frac{1}{2} \times \frac{1}{4}) / \sqrt{\frac{1}{12} \times \frac{7}{144}} = 0.655$.

5.67 The joint probability mass function $p(x,y) = P(X = x, Y = y)$ is given by $p(x,y) = r^{x-1} p (r+p)^{y-x-1} q$ for $x < y$ and $p(x,y) = r^{y-1} q (r+q)^{x-y-1} p$ for $x > y$. It is matter of some algebra to get

$$E(XY) = \frac{p}{q} \frac{1}{(1-r)^2} + p \frac{1+r}{(1-r)^3} + \frac{q}{p} \frac{1}{(1-r)^2} + q \frac{1+r}{(1-r)^3}.$$

Also, $E(X) = \frac{1}{p}$ and $E(Y) = \frac{1}{q}$. This leads to $\mathrm{cov}(X,Y) = -1/(1-r)$.

5.68 To obtain the joint density function of X and Y, note that for Δx and Δy small

$$P(x < X \leq x + \Delta x,\, y < Y \leq y + \Delta y) = 6\Delta x(y - x)\Delta y$$

for $0 \leq x < y < 1$, see also Example 5.3. Thus the joint density function of X and Y is given by

$$f(x, y) = 6(y - x) \quad \text{for } 0 < x < y < 1$$

and $f(x, y) = 0$ otherwise. Therefore

$$\begin{aligned} E(XY) &= 6\int_0^1 dx \int_x^1 xy(y - x)\, dy \\ &= 6\int_0^1 x\left[\frac{1}{3}(1 - x^3) - \frac{1}{2}x(1 - x^2)\right] dx = \frac{1}{5}. \end{aligned}$$

The marginal density functions of X and Y are given by

$$f_X(x) = 3(1 - x)^2 \text{ for } 0 < x < 1, \quad f_Y(y) = 3y^2 \text{ for } 0 < y < 1.$$

Simple calculations give $E(X) = \frac{1}{4}$, $E(Y) = \frac{3}{4}$, $\sigma(X) = \sqrt{1/10 - 1/16} = \sqrt{3/80}$, and $\sigma(Y) = \sqrt{3/5 - 9/16} = \sqrt{3/80}$. This leads to

$$\rho(X, Y) = \frac{1/5 - (1/4) \times (3/4)}{\sqrt{(3/80) \times (3/80)}} = \frac{1}{3}.$$

5.69 The "if part" follows from the relations $\text{cov}(X, aX + b) = a\text{cov}(X, X) = a\sigma_1^2$ and $\sigma(aX + b) = |a|\sigma_1$. Suppose now that $|\rho| = 1$. Since

$$\text{var}(V) = \frac{1}{\sigma_2^2}\sigma_2^2 + \frac{\rho^2}{\sigma_1^2}\sigma_1^2 - 2\frac{\rho}{\sigma_1\sigma_2}\text{cov}(X, Y) = 1 - \rho^2,$$

we have $\text{var}(V) = 0$. This result implies that V is equal to a constant and this constant is $E(V) = \frac{E(Y)}{\sigma_2} - \rho\frac{E(X)}{\sigma_1}$. This shows that $Y = aX + b$, where $a = \rho\sigma_2/\sigma_1$ and $b = E(Y) - aE(X)$.

5.70 Using the linearity of the expectation operator, it is readily verified from the definition of covariance that

$$\text{cov}(aX + b, cY + d) = ac\text{cov}(X, Y).$$

Also, $\sigma(aX + b) = |a|\sigma(X)$ and $\sigma(cY + d) = |c|\sigma(Y)$. Therefore $\rho(aX + b, cY + d) = \rho(X, Y)$ if a and c have the same signs.

5.71 (**a**) Suppose that $E(Y^2) > 0$ (if $E(Y^2) = 0$, then $Y = 0$). Let $h(t) = E[(X - tY)^2]$. Then,

$$h(t) = E(X^2) - 2tE(XY) + t^2E(Y^2).$$

The function $h(t)$ is minimal for $t = E(XY)/E(Y^2)$. Substituting this t-value into $h(t)$ and noting that $h(t) \geq 0$, the Cauchy-Schwartz inequality follows.
(**b**) The Cauchy-Schwartz inequality gives $[\mathrm{cov}(X, Y)]^2 \leq \mathrm{var}(X)\mathrm{var}(Y)$ or, equivalently, $\rho^2(X, Y) \leq 1$ and so $-1 \leq \rho(X, Y) \leq 1$.
(**c**) Noting that $E(XY) = E(X)$ and $E(Y^2) = P(X > 0)$, the Cauchy-Schwartz inequality gives $[E(X)]^2 \leq E(X^2)P(X > 0)$. This shows that $P(X > 0) \geq [E(X)]^2/E(X^2)$ and so $P(X = 0) \leq \mathrm{var}(X)/E(X^2)$.

5.72 (**a**) Since $\mathrm{var}(X) = a^2\mathrm{var}(X)$ and $\mathrm{cov}(aX, bY) = ab\mathrm{cov}(X, Y)$, it suffices to verify the assertion for $a_i = 1$ for all i. We use the method of induction to prove that

$$\mathrm{var}\Big(\sum_{j=1}^{k} X_j\Big) = \sum_{j=1}^{k} \mathrm{var}(X_j) + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \mathrm{cov}(X_i, X_j)$$

for all $k \geq 2$. For $k = 2$, the assertion has been proved in Rule 11.5. Suppose the assertion has been proved for $k = 2, \ldots, m$ for some $m \geq 2$. Then, by the induction hypothesis and Rule 11.5 with $X = X_1 + \cdots + X_m$ and $Y = X_{m+1}$, it follows that $\mathrm{var}(\sum_{j=1}^{m+1} X_k)$ is given by

$$\begin{aligned}\mathrm{var}\Big(\sum_{j=1}^{m} X_j\Big) + \mathrm{var}(X_{m+1}) + 2\mathrm{cov}\Big(\sum_{j=1}^{m} X_j, X_{m+1}\Big) = \sum_{j=1}^{m} \mathrm{var}(X_j)\\ + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m} \mathrm{cov}(X_i, X_j) + \mathrm{var}(X_{m+1}) + 2\sum_{i=1}^{m} \mathrm{cov}(X_i, X_{m+1})\\ = \sum_{j=1}^{m+1} \mathrm{var}(X_j) + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m+1} \mathrm{cov}(X_i, X_j).\end{aligned}$$

(**b**) Using the fact that $\sigma^2(aX) = a^2\sigma^2(X)$ for any constant a, it follows that

$$\sigma^2(\overline{X}_n) = \frac{1}{n^2}[n\sigma^2 + 2 \times \frac{1}{2}n(n-1)].$$

(**c**) Since $\mathrm{cov}(aX, bY) = ab\mathrm{cov}(X, Y)$, it suffices to verify the assertion for $a_i = 1$ for all i and $b_j = 1$ for all j. Using the linearity of the

expectation operator, it is immediately verified from the definition of covariance that $\text{cov}(X, Y + Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$. It is readily verified by induction on m that $\text{cov}(X_1, \sum_{j=1}^m Y_j) = \sum_{j=1}^m \text{cov}(X_1, Y_j)$ for all $m \geq 1$. Next, for fixed m, it can be verified by induction on n that $\text{cov}(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m \text{cov}(X_i, Y_j)$.
(**d**) Using the result of (**c**) and the fact that $\text{cov}(X, Y) = 0$ for independent X and Y, we get

$$\begin{aligned}\text{cov}\big(\overline{X}_n, X_i - \overline{X}_n\big) &= \frac{1}{n}\sum_{k=1}^n \text{cov}(X_k, X_i) - \frac{1}{n^2}\sum_{k=1}^n\sum_{j=1}^n \text{cov}(X_k, X_j)\\ &= \frac{1}{n}\sigma^2(X_i) - \frac{1}{n^2}\sum_{k=1}^n \sigma^2(X_k) = 0.\end{aligned}$$

(**e**) Using the result of (**c**), we have

$$\begin{aligned}&\text{cov}(X_1 - X_2, X_1 + X_2) = \sigma^2(X_1) - \text{cov}(X_1, X_2) - \text{cov}(X_2, X_1) - \sigma^2(X_2)\\ &= \sigma^2(X_1) - \sigma^2(X_2) = 0.\end{aligned}$$

5.73 Since $\text{cov}(X_i, X_j) = \text{cov}(X_j, X_i)$, the matrix $\mathbf{C}$ is symmetric. To prove that $\mathbf{C}$ is positive semi-definite, we must verify that $\sum_{i=1}^n \sum_{j=1}^n t_i t_j \sigma_{ij} \geq 0$ for all real numbers $t_1, \ldots, t_n$. This property follows from the formula for $\text{var}(\sum_{i=1}^n t_i X_i)$ in Problem 5.72 and the fact that the variance is always nonnegative.

5.74 Since X and Y are independent $\text{cov}(X, Y) = 0$. Therefore, using the result of Problem 5.72(**c**), $\text{cov}(X, V) = \text{cov(X,X)+cov(X,Y)} = \sigma^2(X) = 1 > 0$, $\text{cov}(V, W) = \text{cov}(X, Y) - a\text{cov}(X, X) + \text{cov}(Y, Y) - a\text{cov}(Y, X) = -a\sigma^2(X) + \sigma^2(Y) = 1 - a > 0$ for $0 < a < 1$, and $\text{cov}(X, W) = \text{cov}(X, Y) - a\text{cov}(X, X) = -a < 0$.

5.75 Let $V = \max(X, Y)$ and $W = \min(X, Y)$. Then $E(V) = 1/\sqrt{\pi}$ and $E(W) = -1/\sqrt{\pi}$, see Problem 4.69. Obviously, $VW = XY$ and so $E(VW) = E(X)E(Y) = 0$, by the independence of X and Y. Thus

$$\text{cov}(V, W) = \frac{1}{\pi}.$$

We have $\min(X, Y) = -\max(-X, -Y)$. Since the independent random variables $-X$ and $-Y$ are distributed as X and Y, it follows that $\min(X, Y)$ has the same distribution as $\min(-X, -Y) = -\max(X, Y)$. Therefore $\sigma^2(V) = \sigma^2(W)$. Also, by $V + W = X + Y$, we have

$\sigma^2(V+W) = \sigma^2(X+Y) = 2$. Using the relation $\sigma^2(V+W) = \sigma^2(V) + \sigma^2(W) + 2\text{cov}(V,W)$, we get $\sigma^2(V) + \sigma^2(W) = 2 - 2/\pi$ and so $\sigma^2(V) = \sigma^2(W) = 1 - 1/\pi$. This leads to

$$\rho(V,W) = \frac{1/\pi}{1 - 1/\pi} = \frac{1}{\pi - 1}.$$

Note: the result is also true when X and Y are $N(\mu, \sigma^2)$ distributed. To see this, use the relations

$$\text{cov}(V,W) = \sigma^2 \text{cov}\Big(\max\Big[\frac{X-\mu}{\sigma}, \frac{Y-\mu}{\sigma}\Big],\ \min\Big[\frac{X-\mu}{\sigma}, \frac{Y-\mu}{\sigma}\Big]\Big) = \frac{\sigma^2}{\pi}$$
$$\text{var}(V) = \text{var}(W) = \sigma^2\Big(1 - \frac{1}{\pi}\Big).$$

5.76 Let $V = \max(X,Y)$ and $W = \min(X,Y)$. The random variable V is exponentially distributed and has $E(V) = \frac{1}{2\lambda}$ and $\sigma^2(V) = \frac{1}{(2\lambda)^2}$. The random variable W satisfies

$$P(W \leq w) = (1 - e^{-\lambda w}) \times (1 - e^{-\lambda w}) \quad \text{for } w \geq 0.$$

It is matter of some algebra to get $E(W) = \frac{3}{2\lambda}$ and $\sigma^2(W) = \frac{5}{4\lambda^2}$. Noting that $E(VW) = E(XY) = E(X)E(Y) = \frac{1}{\lambda^2}$, we find

$$\text{cov}(V,W) = \frac{1}{\lambda^2} - \frac{1}{2\lambda} \times \frac{3}{2\lambda} = \frac{1}{4\lambda^2}.$$

This leads to $\rho(V,W) = \frac{1}{\sqrt{5}}$.

5.77 The linear least square estimate of D_1 given that $D_1 - D_2 = d$ is equal to

$$E(D_1) + \rho(D_1 - D_2,\ D_1)\frac{\sigma(D_1)}{\sigma(D_1 - D_2)}[d - E(D_1 - D_2)].$$

By the independence of D_1 and D_2, $E(D_1 - D_2) = \mu_1 - \mu_2$, $\sigma(D_1 - D_2) = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\text{cov}(D_1 - D_2,\ D_1) = \sigma_1^2$. The linear least square estimate is

$$\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(d - \mu_1 + \mu_2).$$