Chapter 1

1.1 Imagine a blue and a red die. Take as sample space the set of the ordered pairs (b, r), where b is the number shown on the blue and r is the number shown on the red die. Each of the 36 elements (b, r) is equally likely. There are $2 \times 3 \times 3 = 18$ elements for which exactly one component is odd. Thus the probability that the sum of the two dice is odd equals $\frac{18}{36} = \frac{1}{2}$. There are $3 \times 3 = 9$ elements for which both components are odd. Thus the probability that the product of the two numbers rolled is odd equals $\frac{9}{36} = \frac{1}{4}$. Alternatively, you can obtain the probabilities by using as sample space the set consisting of the four equiprobable elements (odd, odd), $(odd, even)$, $(even, even)$, and $(even, odd)$.

1.2 Label the plumbers as 1 and 2. Take as sample space the set of all possible sequences of ones and twos to the length 3, where a one stands for plumber 1 and a two for plumber 2. The sample space has $2^3 = 8$ equiprobable outcomes. There are 2 outcomes with three ones or three twos. The sought probability is $\frac{2}{8} = \frac{1}{4}$.

1.3 Label the 10 letters of "randomness" as 1 to 10. Take as sample space the set of all permutations of the numbers 1 to 10. All 10! outcomes are equally likely. There are $3 \times 2 \times 8!$ outcomes that begin and end with a vowel and there are $8 \times 3! \times 7!$ outcomes in which the three vowels are adjacent to each other. The probabilities are $3 \times 2 \times 8!/10! = 1/15$ and $8 \times 3! \times 7!/10! = 1/15$.

1.4 Take as sample space the set of all possible sequences of zeros and ones to the length 4, where a zero stands for male gender and an one for female gender. The sample space has $2^4 = 16$ equiprobable outcomes. There are 8 outcomes with exactly three zeros or exactly three ones and 6 outcomes with exactly two zeros. Hence the probability of three puppies of one gender and one of the other is $\frac{8}{16}$. The probability of two puppies of each gender is $\frac{6}{16}$.

1.5 Take as sample space the set of all unordered samples of m different numbers. The sample space has $\binom{n}{m}$ equiprobable elements. There are $\binom{n-1}{m-1}$ samples that contain the largest number. The probability of getting the largest number is $\binom{n-1}{m-1}/\binom{n}{m} = \frac{m}{n}$. Alternatively, you can take as sample space the set of all $n!$ permutations of the integers 1 to n. There are $m \times (n-1)!$ permutations for which the number n

is in one of the first m positions.
Note: More generally, the probability that the largest r numbers are among the m numbers picked is given by both $\binom{n-r}{m-r}/\binom{n}{m}$ and $\binom{m}{r}r!(n-r)!/n!$.

1.6 Take as sample space the set of all possible combinations of two persons who do the dishes. The sample space has $\binom{6}{2}=15$ equally likely outcomes. The number of outcomes consisting of two boys is $\binom{3}{2}=3$. The sought probability is $\frac{3}{15}=\frac{1}{5}$.Alternatively, using an ordered sample space consisting of all 6! possible ordering of the six people and imagining that the first two people in the ordering have to do the dishes, the sought probability can be calculated as $\frac{3\times 2\times 4!}{6!}=\frac{1}{5}$

1.7 Imagine that the balls are labeled as $1,\ldots,n$. It is no restriction to assume that the two winning balls have the labels 1 and 2. Take as sample space the set of all $n!$ permutations of $1,\ldots,n$. For any k, the number of permutations having either 1 or 2 on the kth place is $(n-1)!+(n-1)!$. Thus, the probability that the kth person picks a winning ball is $\frac{(n-1)!+(n-1)!}{n!}=\frac{2}{n}$ for each k.

1.8 Take as sample space the set of all ordered pairs $(i,j): i,j=1,\ldots,6$, where i is the number rolled by player A and j is the number rolled by player B. The sample space has 36 equally likely outcomes. The number of winning outcomes for player B is $9+11=20$. The probability of player A winning is $\frac{16}{36}=\frac{4}{9}$.

1.9 Take as sample space the set of all unordered samples of six different numbers from the numbers 1 to 42. The sample space has $\binom{42}{6}$ equiprobable outcomes. There are $\binom{41}{5}$ outcomes with the number 10. Thus the probability of getting the number 10 is $\binom{41}{5}/\binom{42}{6}=\frac{6}{42}$. The probability that each of the six numbers picked is 20 or more is equal to $\binom{23}{6}/\binom{42}{6}=0.0192$. Alternatively, the probabilities can be calculated by using the sample space consisting of all ordered arrangement of the numbers 1 to 42, where the numbers in the first six positions are the lotto numbers. This leads the calculations $(6\times 41!)/42!=\frac{6}{42}$ and $(\binom{23}{6}\times 6!\times 36!)/42!=0.0192$ for the sought probabilities.

1.10 Take as (unordered) sample space all possible combinations of two candidates to receive a cup of tea from the waiter. The sample space has $\binom{5}{2}=10$ equally likely outcomes. The number of combinations of two people each getting the cup of tea they ordered is 1. The sought probability is $\frac{1}{10}$. Alternatively, using an ordered sample space consisting of all possible orderings of the five people and imagining that the first two people in the ordering get a cup of tea from the waiter, the probability can be calculated as $\frac{2\times 1\times 3!}{5!} = \frac{1}{10}$.

1.11 Label the nine socks as $s_1, \ldots, s_9$. The probability model in which the order of selection of the socks is considered relevant has a sample space with $9 \times 8 = 72$ equiprobable outcomes (s_i, s_j). There are $4 \times 5 = 20$ outcomes for which the first sock chosen is black and the second is white, and there are $5 \times 4 = 20$ outcomes for which the first sock is white and the second is black. The sought probability is $40/72 = 5/9$. The probability model in which the order of selection of the socks is not considered relevant has a sample space with $\binom{9}{2} = 36$ equiprobable outcomes. The number of outcomes for which the socks have different colors is $\binom{5}{1} \times \binom{4}{1} = 20$, yielding the same value $20/36 = 5/9$ for the sought probability.

1.12 This problem can be solved by using either an ordered sample space or an unordered sample space. Label the ten letters of the word Cincinnati as $1, 2, \ldots, 10$. As ordered sample space, take the set of all ordered pairs (i_1, i_2), where i_1 is the label of the first letter dropped and i_2 is the label of the second letter dropped. This sample space has $10 \times 9 = 90$ equally likely outcomes. Let A be the event that the two letters dropped are the same. Noting that in the word Cincinnati the letter c occurs two times and the letters i and n each occur three times, it follows that there are $\binom{2}{2} \times 2! + \binom{3}{2} \times 2! + \binom{3}{2} \times 2! = 14$ outcomes leading to the event A. Hence $P(A) = \frac{14}{90} = \frac{7}{45}$. An unordered sample space can also be used . This sample space consists of all possible sets of two differently labeled letters from the ten letters of Cincinnati. This sample space has $\binom{10}{2} = 45$ equally likely outcomes. The number of outcomes for which the two labeled letters in the set represent the same letter is $\binom{2}{2} + \binom{3}{2} + \binom{3}{2} = 7$. This gives the same value $\frac{7}{45}$ for the probability that the two letters dropped are the same.

1.13 Take as sample space the set of all unordered pairs of two distinct cards. The sample space has $\binom{52}{2}$ equally likely outcomes. There are $\binom{1}{1} \times \binom{51}{1} = 51$ outcomes with the ten of hearts, and $\binom{3}{1} \times \binom{12}{1} = 36$ outcomes with hearts and a ten but not the ten of hearts. The sought probability is $(51 + 36)/\binom{52}{2} = 0.0656$.

1.14 Represent the words chance and choice by chanCe and choiCe. Take as sample space the set of all possible pairs (l_1, l_2), where l_1 is an element from the word chanCe and l_2 is an element from the word choiCe. By distinguishing between c and C, the sample space has $6 \times 6 = 36$ equally likely outcomes. The number of outcomes for which the two chosen letters represent the same letter is $4 + 1 + 1 = 6$. The sought probability is $\frac{1}{6}$.

1.15 Take as sample space the set of all sequences $(i_1, \ldots, i_k)$, where i_k is the number shown on the kth roll of the die. Each element of the sample space is equally likely. The explanation is that there is a one-to-one correspondence between the elements $(i_1, \ldots, i_k)$ with $\sum_{k=1}^{10} i_k = s$ and the elements $(7 - i_1, \ldots, 7 - i_k)$ with $\sum_{k=1}^{10}(7 - i_k) = 70 - s$.

1.16 Take as ordered sample space the set of all sequences $(i_1, \ldots, i_{12})$, where i_k is the number rolled by the kth die. The sample space has 6^{12} equally likely outcomes. The number of outcomes in which each number appears exactly two times is $\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} = 12!/2^6$. The sought probability is $\frac{12!}{2^6 \times 6^{12}} = 0.0034$.

1.17 Take as sample space the set of all possible samples of three residents. This leads to the value $\binom{4}{1}\binom{4}{1}\binom{4}{1}/\binom{12}{3} = \frac{16}{55}$ for the sought probability.

1.18 Take as sample space the set of all ordered arrangements of 10 people, where the people in the first five positions form group 1 and the other five people form group 2. The sample space has 10! equally likely elements. The number of elements for which your two friends and you together are in the same group is $5 \times 4 \times 3 \times 7! + 5 \times 4 \times 3 \times 7!$. The sought probability is $\frac{120 \times 7!}{10!} = \frac{1}{6}$. Alternatively, the probability can be calculated as $\frac{\binom{3}{3}\binom{7}{2} + \binom{3}{0}\binom{7}{5}}{\binom{10}{5}} = \frac{1}{6}$, using as sample space the set of all possible combinations of five people for the first group. A third way to calculate the probability is $\frac{\binom{5}{3} + \binom{5}{3}}{\binom{10}{3}} = \frac{1}{6}$, using as sample space the set of all possible combinations of three positions for the three friends.

1.19 Take as sample space the set of the 9! possible orderings of the nine books. The subjects mathematics, physics and chemistry can be ordered in $3 \times 2 \times 1 = 6$ ways and so the number of favorable orderings is $6 \times 4! \times 3! \times 2!$. The sought probability is $(6 \times 4! \times 3! \times 2!)/9! = 1/210$.

1.20 The sample space is $\Omega = \{(i, j, k) : i, j, k = 0, 1\}$, where the three components corresponds to the outcomes of the three individual tosses of the three friends. Here 0 means heads and 1 means tails. Each element of the sample space gets assigned a probability of $\frac{1}{8}$. Let A denote the event that one of the three friends pays for all the three tickets. The set A is given by $A = \Omega \backslash \{(0,0,0),(1,1,1)\}$ and consists of six elements. The sought probability is $P(A) = \frac{6}{8}$.

1.21 Label the eleven letters of the word Mississippi as $1, 2, \ldots, 11$ and take as sample space the set of the 11^{11} possible ordered sequences of eleven numbers from $1, \ldots, 11$. The four positions for a number representing i, the four positions for a number representing s, the two positions for a number representing p, and the one position for the number representing m can be chosen in $\binom{11}{4} \times \binom{7}{4} \times \binom{3}{2}$ ways. Therefore the number of outcomes in which all letters of the word Mississippi are represented is $\binom{11}{4} \times \binom{7}{4} \times \binom{3}{2} \times 4^4 \times 4^4 \times 2^2$. Dividing this number by 11^{11} gives the value 0.0318 for the sought probability.

1.22 One pair is a hand with the pattern *aabcd*, where a, b, c and d are from distinct kinds of cards. There are 13 kinds and four of each kind in a standard deck of 52 cards. The probability of getting one pair is

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}} = 0.4226.$$

Two pair is a hand with the pattern *aabbc*, where a, b and c are from distinct kinds of cards. The probability of getting two pair is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0475.$$

1.23 Take as sample space the set of all possible combinations of two apartments from the 56 apartments. These two apartment represent the vacant apartments. The sample space has $\binom{56}{2}$ equiprobable elements. The number of elements with no vacant apartment on the top floor is $\binom{48}{2}$. Thus the sought probability is $\left[\binom{56}{2} - \binom{48}{2}\right]/\binom{56}{2} = 0.2675$. Alternatively, using a sample space made up of all permutations of the 56 apartments, the probability can be calculated as $1 - \frac{48 \times 47 \times 54!}{56!} = 0.2675$.

1.24 Imagine that the balls are labeled as 1 to 11, where the white balls get the labels 1 to 7 and the red balls the labels 8 to 11. Take as sample space is the set of all possible permutations of $1, 2, \ldots, 11$. The number of outcomes in which a red ball appears for the first time at the ith drawing is $\binom{7}{i-1} \times (i-1)! \times 4 \times (7-(i-1)+3)!$ for $1 \leq i \leq 8$. The sought probability is

$$\frac{1}{11!}\sum_{k=1}^{4}\binom{7}{2k-1} \times (2k-1)! \times 4 \times (7-(2k-1)+3)! = \frac{13}{33}.$$

1.25 Take as sample space the set of all ordered pairs (i,j), where i is the first number picked and j is the second number picked. There are n^2 equiprobable outcomes. For $r \leq n+1$, the $r-1$ outcomes $(1,r-1),(2,r-2),\ldots,(r-1,1)$ are the only outcomes (i,j) for which $i+j=r$. Thus the probability that the sum of the two numbers picked is r is $\frac{r-1}{n^2}$ for $2 \leq r \leq n+1$. Therefore the probability of getting a sum s when rolling two dice is $\frac{s-1}{36}$ for $2 \leq s \leq 7$. By a symmetry argument, the probability of getting a sum s is the same as the probability of getting a sum $14-s$ for $7 \leq s \leq 12$ (opposite faces of a die always total 7). Thus the probability of rolling a sum s has the value $\frac{14-s-1}{36}$ for $7 \leq s \leq 12$.

1.26 Take as sample space the interval $(0,1)$. The outcome x means that the stick is broken on the point x. The length of the longer piece is at least three times the length of the shorter piece if $x \in (0,\frac{1}{4})$ or $x \in (\frac{3}{4},1)$. The sought probability is $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

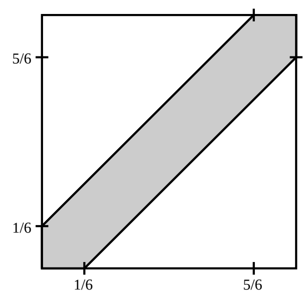
1.27 Take as sample space the square $\{(x,y) : 0 \leq x,y \leq a\}$. The outcome (x,y) refers to the position of the middle point of the coin. The sought probability is given by the probability that a randomly chosen point in the square falls in the subset $\{(x,y) : \frac{d}{2} \leq x,y \leq a-\frac{d}{2}\}$ and is equal to $(a-d)^2/a^2$.

1.28 Take as sample space the interval $(0,1)$. The outcome x means that a randomly chosen point in $(0,1)$ is equal to x. The sought probability is the probability that a randomly chosen point in $(0,1)$ falls into one of the intervals $(0,\frac{1}{12})$ or $(\frac{1}{2},\frac{7}{12})$. The sought probability is $\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$.

1.29 This problem can be solved with the model of picking at random a point inside a rectangle. The rectangle $R=\{(x,y) : 0 \leq x \leq 1,\ \frac{1}{2} \leq y \leq 1\}$ is taken as sample space, where the outcome (x,y) means that you arrive $60x$ minutes past 7 a.m. and your friend arrives $60y$ minutes past 7 a.m. The probability assigned to each subset of the sample space is the area of the subset divided by the area of the rectangle R. The sought probability is $P(A)$, where the set A is the union of the three disjoint subsets $\{(x,y) : \frac{1}{2} < x, y < \frac{1}{2} + \frac{1}{12}\}$, $\{(x,y) : \frac{1}{2} + \frac{1}{12} < x, y < \frac{3}{4}\}$ and $\{(x,y) : \frac{3}{4} < x, y < 1\}$. This gives

$$P(A) = \frac{1}{2}\Big(\frac{1}{12} \times \frac{1}{12} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{4}\Big) = \frac{7}{36}.$$

1.30 Translate the problem into choosing a point at random inside the unit square $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. The probability that the two persons will meet within 10 minutes of each other is given by the probability that a point chosen at random in the unit square will fall inside the shaded region in the figure. The area of the shaded region is calculated as $1 - \frac{5}{6} \times \frac{5}{6} = 0.3056$. This gives the desired probability.



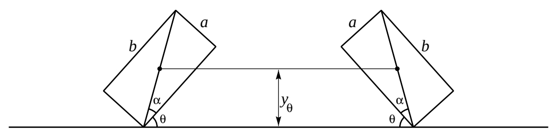
1.31 For $q = 1$, take the square $\{(x,y) : -1 < x, y < 1\}$ as sample space. The sought probability is the probability that a point (x,y) chosen at random in the square satisfies $y \leq \frac{1}{4}x^2$ and is equal to $\frac{1}{4}(2 + \int_{-1}^{1} \frac{1}{4}x^2 dx) = 0.5417$. For the general case, the sought probability is (make a picture!):

$$\frac{1}{4q^2}\Big(2q^2 + \int_{-q}^{q} \frac{1}{4}x^2\,dx\Big) = \frac{1}{2} + \frac{q}{24} \quad \text{for } 0 < q < 4$$

$$\frac{1}{4q^2}\Big(2q^2 + \int_{-2\sqrt{q}}^{2\sqrt{q}} \frac{1}{4}x^2\,dx + 2(q - 2\sqrt{q})q\Big) = 1 - \frac{2}{3\sqrt{q}} \quad \text{for } q \geq 4.$$

1.32 Take as sample space the set $\{(x,y) : 0 \leq x, y \leq 1\}$. The outcome (x,y) means that a randomly chosen point in the unit square is equal to (x,y). The probability that the Manhattan distance from a randomly chosen point to the point $(0,0)$ is no more than a is given by the probability that the randomly chosen point (x,y) satisfies $x + y \leq a$. The area of the region $\{(x,y) : 0 \leq x, y \leq 1 \text{ and } x + y \leq a\}$ is $\frac{1}{2}a^2$. This gives the sought probability. By a symmetry argument, this probability also applies to the case that the point is randomly chosen in the square $\{(x, y) : -1 \leq x, y \leq 1\}$.

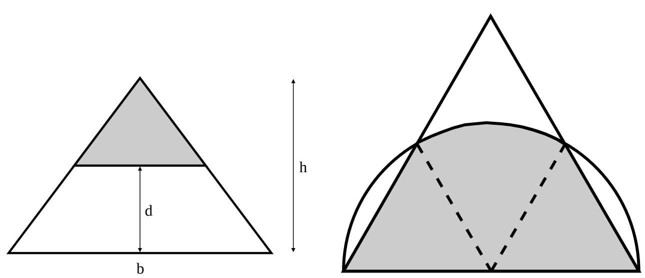
1.33 Take as sample space the set $S = \{(y, \theta) : 0 \leq y \leq \frac{1}{2}D, 0 \leq \theta \leq \frac{\pi}{2}\}$, where y is the distance from the midpoint of the diagonal of the rectangular card to the closest line on the floor and the angle θ is as described in the figure. It is no restriction to assume that $b \geq a$.



Using the figure, it is seen that the card will intersect one of the lines on the floor if and only if the distance y is less than y_θ, where y_θ is determined by $\sin(\alpha + \theta) = y_\theta / (\frac{1}{2}\sqrt{a^2 + b^2})$. Since $\sin(\alpha + \theta) = \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)$ with $\sin(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$, it follows that $y_\theta = \frac{1}{2}\big(a\cos(\theta) + b\sin(\theta)\big)$. The sought probability is the area under the curve $y = \frac{1}{2}\big(a\cos(\theta) + b\sin(\theta)\big)$ divided by the area of the set S and so it is equal to

$$\frac{1}{(1/4)\pi D} \int_0^{\pi/2} \frac{1}{2}\big(a\cos(\theta) + b\sin(\theta)\big)\, d\theta = \frac{2(a+b)}{\pi D}.$$

1.34 The perpendicular distance from the randomly chosen point is larger than d if and only if the point falls inside the shaded region of the triangle in the left figure. Using the fact that the base of the shaded triangle is $\frac{h-d}{h}b$ (the ratio of this base and b equals $(h-d)/h$), it follows that the first probability is given by

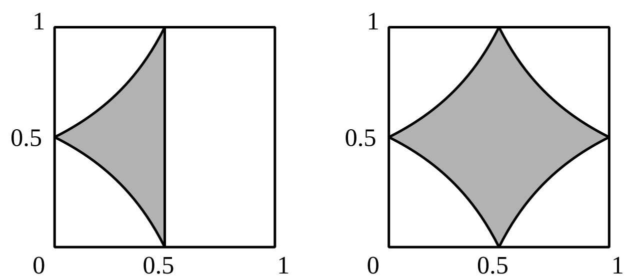


$$\frac{\frac{1}{2}[(h-d) \times (h-d)b/h]}{\frac{1}{2}h \times b} = \frac{(h-d)^2}{h^2}.$$

The randomly chosen point and the base of the triangle form an obtuse triangle if and only if the randomly chosen point falls inside the shaded region in the right figure. The area of the shaded region is the sum of two areas of an equilateral triangle with side lengths $b/2$ plus the area of one sixth of a circle with radius $b/2$. The area of an equilateral triangle with sides of length a is given by $\frac{1}{4}\sqrt{3}a^2$. It now follows that the second probability is given by

$$\frac{(1/4)\sqrt{3}(b/2)^2 + (1/6)\pi(b/2)^2 + (1/4)\sqrt{3}(b/2)^2}{(1/4)\sqrt{3}b^2} = \frac{1}{2} + \frac{\pi}{6\sqrt{3}} = 0.8023.$$

1.35 Take as sample space the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$. The side lengths $v = x$, $w = y \times (1 - v)$ and $1 - v - w$ should satisfy the conditions $v + w > 1 - v - w$, $v + 1 - v - w > w$ and $w + 1 - v - w > v$. These conditions can be translated into $y > \frac{1-2x}{2-2x}$, $y < \frac{1}{2-2x}$ and $x < \frac{1}{2}$. The sought probability is given by the area of the shaded region in the first part of the figure and is equal to $\int_0^{0.5} \frac{1}{2-2x}\, dx - \int_0^{0.5} \frac{1-2x}{2-2x}\, dx = \ln(2) - 0.5$. To find the second probability, let v be the first random



breakpoint chosen on the stick and w be the other breakpoint. The point (v, w) can be represented by $v = x$ and $w = y \times (1 - v)$ if $v < \frac{1}{2}$ and by $v = x$ and $w = y \times v$ if $v > \frac{1}{2}$, where (x, y) is a randomly chosen point in the unit square. The second probability is the area of shaded region in the second part of the figure and is equal to $2\big(\ln(2) - 0.5\big)$.

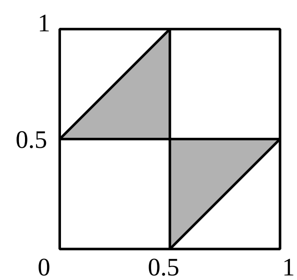
1.36 The problem can be translated into choosing a random point in the unit square. The sample space is $\{(x, y) : 0 \leq x, y \leq 1\}$. For any point (x, y) in the sample space, distinguish between the cases $x > y$ and $x < y$ (the probability that a randomly chosen point (x, y) satisfies $x = y$ is zero). Consider first the case of $x > y$. Then, the three side lengths are y, $x - y$ and $1 - x$. Three side lengths a, b and c form a triangle if and only if

$$a + b > c, \quad a + c > b \quad \text{and} \quad b + c > a.$$

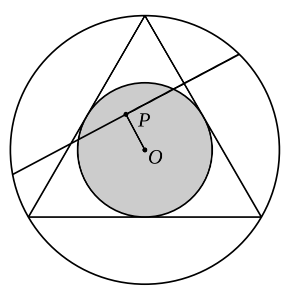
Hence the lengths y, $y - x$ and $1 - y$ must satisfy the three conditions

$$y + x - y > 1 - x, \quad y + 1 - x > x - y \quad \text{and} \quad x - y + 1 - x > y.$$

These three conditions can be rewritten as $x > \frac{1}{2}$, $y > x - \frac{1}{2}$ and $y < \frac{1}{2}$. The set of all points (x, y) satisfying these three conditions is the down-right shaded region in the figure. Next consider the case of $x < y$. Then the three side lengths are x, $y - x$, and $1 - y$. Then (x, y) must satisfy the three conditions as $y > \frac{1}{2}$, $x > y - \frac{1}{2}$ and $x < \frac{1}{2}$. The set of all points (x, y) satisfying these three conditions is the top-left shaded region in the figure. The area of the two shaded regions is $\frac{1}{8} + \frac{1}{8}$. Hence the desired probability is $\frac{1}{4}$.



1.37 The unique chord having the randomly chosen point P as its midpoint is the chord that is perpendicular to the line connecting the point P to the center O of the circle, see the figure. A little geometry tells us



that the chord is longer than the side of the equilateral triangle if and only if the point P falls inside the shaded inner circle in the figure. Thus the sought probability is $\frac{\pi(r/2)^2}{\pi r^2} = \frac{1}{4}$.

1.38 This is a compound experiment that consists of three subexperiments. Take as sample space the set $\Omega = \{(i,j,k) : i,j,k = 0,1\}$, where i=1 (0) if player A beats (is beaten by) player B, the component j=1 (0) if player A beats (is beaten by) player C, and the component k=1 (0) if player B beats (is beaten by) player C. The probabilities $p_{i,j,k}$ assigned to the eight outcomes (i,j,k) are $p_{0,0,0} = 0.5\times 0.3\times 0.6 = 0.09$, $p_{1,0,0} = 0.5\times 0.3\times 0.6 = 0.09$, $p_{0,1,0} = 0.5\times 0.7\times 0.6 = 0.21$, $p_{0,0,1} = 0.5\times 0.3\times 0.4 = 0.06$, $p_{1,1,0} = 0.5\times 0.7\times 0.6 = 0.21$, $p_{1,0,1} = 0.5\times 0.3\times 0.4 = 0.06$, $p_{0,1,1} = 0.5\times 0.7\times 0.4 = 0.14$, and $p_{1,1,1} = 0.5\times 0.7\times 0.4 = 0.14$. Denote by E the event that player A wins at least as many games as any other player, then $E = \{(0,1,0),(1,1,0),(1,0,1),(1,1,1)\}$. Thus, the desired probability $P(E) = 0.21 + 0.21 + 0.06 + 0.14 = 0.62$.

1.39 Take as sample the set of all four-tuples $(\delta_1,\delta_2,\delta_3,\delta_4)$, where $\delta_i = 0$ if component i has failed and $\delta_i = 1$ otherwise. The probability $r_1r_2r_3r_4$ is assigned to $(\delta_1,\delta_2,\delta_3,\delta_4)$, where $r_i = f_i$ if $\delta_i = 0$ and $r_i = 1 - f_i$ if $\delta_i = 1$. Let A_0 be the event that the system fails, A_1 be the event that none of the four components fails and A_i be the event that only component i fails for $i = 2$, 3 and 4. Then, $P(A_1) = (1-f_1)(1-f_2)(1-f_3)(1-f_4)$ and $P(A_2) = (1-f_1)f_2(1-f_3)(1-f_4)$, $P(A_3) = (1-f_1)(1-f_2)f_3(1-f_4)$, and $P(A_4) = (1-f_1)(1-f_2)(1-f_3)f_4$. The events A_k are mutually exclusive and their union is the sample space. Hence the sought probability $P(A_0)$ is $1 - \sum_{i=1}^{4} P(A_i)$.

1.40 Proceeding along the same lines as in Example 1.12, the probability that Bill is the first person to pick a red ball is $\sum_{k=1}^{\infty} \frac{4}{11}\left(\frac{7}{11}\right)^{2k-1} = \frac{7}{18}$.

1.41 Take as sample space the set $\{(1,s),(2,s),\ldots\} \cup \{(1,e),(2,e),\ldots\}$ and assign to the outcomes (i,s) and (i,e) the probabilities $(1-a_7-a_8)^{i-1}a_7$ and $(1-a_7-a_8)^{i-1}a_8$, where $a_7 = \frac{6}{36}$ and $a_8 = \frac{5}{36}$. The probability of getting a total of 8 before a total of 7 is $\sum_{i=1}^{\infty}(1-a_7-a_8)^{i-1}a_8 = a_8/(a_7+a_8) = \frac{5}{11}$.

1.41 Take as sample space the set $\{(1,s),(2,s),\ldots\} \cup \{(1,e),(2,e),\ldots\}$ and assign to the outcomes (i,s) and (i,e) the probabilities $(1-a_7-a_8)^{i-1}a_7$ and $(1-a_7-a_8)^{i-1}a_8$, where $a_7 = \frac{6}{36}$ and $a_8 = \frac{5}{36}$. The probability of getting a total of 8 before a total of 7 is

$$\sum_{i=1}^{\infty}(1-a_7-a_8)^{i-1}a_8 = \frac{a_8}{a_7+a_8} = \frac{5}{11}.$$

1.42 Using the same reasoning as in Example 1.12, the probability that desperado A will be the one to shoot himself dead is

$$\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^{3n}\frac{1}{6}=\frac{36}{91}.$$

The probabilities are $\frac{30}{91}$ for desperado B and $\frac{25}{91}$ for desperado C.

1.43 Take as sample space the set $\{(s_1,s_2) : 2 \leq s_1, s_2 \leq 12\}$, where s_1 and s_2 are the sums rolled by the two persons. The probability $p(s_1,s_2) = p(s_1)\times p(s_2)$ is assigned to the outcome (s_1,s_2), where $p(s)$ is the probability of getting the sum s in a roll of two dice. The probabilities $p(s)$ are given by $p(2)=p(12)=\frac{1}{36}$, $p(3)=p(11)=\frac{2}{36}$, $p(4)=p(10)=\frac{3}{36}$, $p(5)=p(9)=\frac{4}{36}$, $p(6)=p(8)=\frac{5}{36}$, and $p(7)=\frac{6}{36}$. The probability that the sums rolled are different is

$$\sum_{s_1\neq s_2} p(s_1,s_2) = 1-\sum_{s=2}^{12}\big(p(s)\big)^2=\frac{575}{648}.$$

1.44 (**a**) Let the set $C = B\backslash A$ consists of those outcomes that belong to B but do not belong to A. The sets A and C are disjoint and $B = A\cup C$. Then, by Axiom 1.3 (in fact, Rule 1.1 in Section 1.3 should be used), $P(B) = P(A\cup C) = P(A) + P(C)$. By Axiom 1.1, $P(C) \geq 0$ and so we get the desired result $P(B) \geq P(A)$.
(**b**) We can define pairwise disjoint sets $B_1, B_2, \ldots$ such that $\bigcup_{k=1}^{\infty} A_k$ is equal to $\bigcup_{k=1}^{\infty} B_k$. Let $B_1 = A_1$ and let $B_2 = A_2\backslash A_1$. In general, let

$$B_k = A_k\backslash(A_1\cup\cdots\cup A_{k-1}) \quad \text{for } k = 2,3,\ldots.$$

By induction, $B_1\cup\cdots\cup B_k = A_1\cup\cdots\cup A_k$ for any $k\geq 1$ Also, the sets $B_1,\ldots,B_k$ are pairwise disjoint. Hence $\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$ and the sets $B_1, B_2, \ldots$ are pairwise disjoint. Using Axiom 1.3, it now follows that

$$P\Big(\bigcup_{k=1}^{\infty} A_k\Big) = P\Big(\bigcup_{k=1}^{\infty} B_k\Big) = \sum_{k=1}^{\infty} P(B_k).$$

Since $B_k \subseteq A_k$, we have $P(B_k) \leq P(A_k)$ and so the desired result follows.

1.45 The sought probability is at least as large as $1-P(\bigcap_{n=1}^{\infty} B_n)$. We have

$$P(B_n) = \Big(1-\Big(\frac{1}{2}\Big)^r\Big)^n \quad \text{for any } n\geq 1.$$

By the continuity property of probability, $P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n)$ and so $1 - P(\bigcap_{n=1}^{\infty} B_n) = 1$.

1.46 It is intuitively clear that the probability is equal to 0.5. This can be proved as follows. Define A (B) as the event that you see at least 10 consecutive tails (heads) before you see 10 consecutive heads (tails) for the first time if you toss a fair coin indefinitely often. Using the result of Problem 1.45, it follows that $P(A \cup B) = 1$. The events A and B are mutually exclusive and satisfy $P(A) = P(B)$. This proves that $P(A) = P(B) = 0.5$.

1.47 Let A be the event that a second-hand car is bought and B be the event that a Japanese car is bought. Noting that $P(A \cup B) = 1 - 0.55$, it follows from $P(A \cup B) = P(A) + P(B) - P(AB)$ that $P(AB) = 0.25 + 0.30 - 0.45 = 0.10$.

1.48 Let A be the event that a randomly chosen household is subscribed to the morning newspaper and B be the event that a randomly chosen household is subscribed to the afternoon newspaper. The sought probability $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ is $0.5 + 0.7 - 0.8 = 0.4$.

1.49 Let A be the event that the truck is used on a given day and B be the event that the van is used on a given day. Then, $P(A) = 0.75$, $P(AB) = 0.30$ and $P(A^cB^c) = 0.10$. By De Morgan's first law, $P(A \cup B) = 1 - P(A^cB^c)$. By $P(A \cup B) = P(A) + P(B) - P(AB)$, the probability that the van is used on a given day is $P(B) = 0.90 - 0.75 + 0.30 = 0.45$. Since $P(A^cB) + P(AB) = P(B)$, the probability that only the van is used on a given day is $P(A^cB) = 0.45 - 0.30 = 0.15$.

1.50 Since $B \subseteq A \cup B$, it follows that $P(B) \leq P(A \cup B) = \frac{2}{3}$. Using the relation $P(A \cup B) = P(A) + P(B) - P(AB)$, it follows that $P(B) \geq P(A \cup B) - P(A) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$. Hence, $\frac{1}{12} \leq P(B) \leq \frac{2}{3}$.

1.51 The probability that exactly one of the events A and B will occur is given by $P\big((A \cap B^c) \cup (B \cap A^c)\big) = P(A \cap B^c) + P(B \cap A^c)$. Next note that $P(A \cap B^c) = P(A) - P(A \cap B)$ and $P(B \cap A^c) = P(B) - P(A \cap B)$. Thus the probability of exactly one of the events A and B occurring is

$$P(A) + P(B) - 2P(AB).$$

Note: Similarly, we find a formula for the probability of exactly one of the events A, B, and C occurring. This probability is equal to

$P(A \cap B^c \cap C^c) + P(B \cap A^c \cap C^c) + P(C \cap A^c \cap B^c)$. The first term $P(A \cap B^c \cap C^c)$ can be evaluated as $P(A) - [P(A \cap B) + P(A \cap C)] + P(A \cap B \cap C)$. In the same way, the other two terms can be evaluated. Thus the formula for the probability of exactly one of the events A, B, and C occurring is

$$P(A) + P(B) + P(C) - 2P(AB) - 2P(AC) - 2P(BC) + 3P(ABC).$$

A general formula for the probability that exactly r of the events $A_1, \ldots, A_n$ will occur is

$$\sum_{k=0}^{n-r} (-1)^k \binom{r+k}{r} \sum_{j_1 < \cdots < j_{r+k}} P(A_{j_1} \cdots A_{j_{r+k}}).$$

As an illustration, let us determine the probability of getting exactly d different face values when rolling a fair die n times, see also Example 10.6 in the book for another approach. Defining A_i as the event that face value i does *not* appear in n rolls of the die, we get that the desired probability is given by

$$\sum_{k=0}^{\min(n-(6-d),\, d)} (-1)^k \binom{6-d+k}{6-d} \binom{6}{6-d+k} \frac{(d-k)^n}{6^n} \quad \text{for } d = 1, \ldots, 6.$$

If $n = 6$, this probability has the values 1.28×10^{-4}, 0.0199, 0.2315, 0.5015, 0.2315, and 0.0154 for $d = 1, \ldots, 6$.

1.52 In Problem 1.44, the upper bound has already been established. By $P(A \cup B) = P(A) + P(B) - P(AB)$, the lower bound is true for $n = 2$. Suppose the lower bound is verified for $n = 2, \ldots, k$. Then, for $n = k + 1$, let $A = \bigcup_{i=1}^{k} A_i$ and $B = A_{k+1}$. Using the induction hypothesis, we get

$$\begin{aligned}
P\Big(\bigcup_{i=1}^{k+1} A_i\Big) &= P(A) + P(B) - P(AB) \\
&\geq \sum_{i=1}^{k} P(A_i) - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} P(A_i A_j) + P(A_{k+1}) - P\Big(\bigcup_{i=1}^{k} (A_i A_{k+1})\Big) \\
&\sum_{i=1}^{k} P(A_i) - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} P(A_i A_j) + P(A_{k+1}) - \sum_{i=1}^{k} P(A_i A_{k+1}) \\
&= \sum_{i=1}^{k+1} P(A_i) - \sum_{i=1}^{k} \sum_{j=i+1}^{k} P(A_i A_j),
\end{aligned}$$

as was to be verified.

1.53 In this "birthday" problem, the sought probability is

$$1 - \frac{250 \times 249 \times \cdots \times 221}{250^{30}} = 0.8368.$$

1.54 This problem is the birthday problem with m equally likely birthdays and n people. Using the complement rule, the probability that at least one of the outcomes $O_1, \ldots, O_m$ will occur two or more times in n trials is

$$1 - \frac{(m-1)\cdots(m-n+1)}{m^n} = 1 - \left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right).$$

This probability can be approximated by $1 - e^{-1/m} \cdots e^{-(n-1)/m} = 1 - e^{-\frac{1}{2}n(n-1)/m}$ for m large. Solving n from $1 - e^{-\frac{1}{2}n(n-1)/m} = 0.5$ is equivalent to solving n from the quadratic equation $-\frac{1}{2}n(n-1) = m\ln(0.5)$. This yields

$$n \approx 1.177\sqrt{m} + 0.5.$$

Using the complement rule, the probability that the outcome O_1 occurs at least once is $1 - \frac{(m-1)^n}{m^n} \approx 1 - e^{-n/m}$ for m large. Solving n from $1 - e^{-n/m} = 0.5$ gives

$$n \approx 0.6931m.$$

1.55 This problem is a variant of the birthday problem. One can choose two distinct numbers from the numbers $1, 2, \ldots, 25$ in $\binom{25}{2} = 300$ ways. The desired probability is given by

$$1 - \frac{300 \times 299 \times \cdots \times 276}{300^{25}} = 0.6424.$$

1.56 This problem is a birthday problem with $m = \binom{49}{6}$ equally likely birthdays and $n = 3{,}016$ people. Using the solution of Problem 1.54, the probability that in 3,016 drawings some combination of six numbers will appear more than once is about $1 - e^{-\frac{1}{2}n(n-1)/m} = 0.2776$. This approximate value agrees with the exact value in all four decimals.

1.57 The translation step to the birthday problem is to imagine that each of the $n = 500$ Oldsmobile cars gets assigned a "birthday" chosen at random from $m = 2{,}400{,}000$ possible "birthdays". Using the approximate formula in Problem 1.54, the probability that at least one subscriber gets two or more cars can be calculated as $1 - e^{-\frac{1}{2}n(n-1)/m} = 0.051$.

1.58 Imagine that the balls are numbered from 1 to 20. Using the complement rule,we find that the sought probability is

$$1 - \frac{20 \times 10 \times 18 \times 9 \times 16 \times 8}{20 \times 19 \times 18 \times 17 \times 16 \times 15} = 0.8514.$$

1.59 (**a**) The sought probability is

$$1 - \frac{10{,}000 \times 9{,}999 \times \cdots \times 9{,}990}{10{,}000^{11}} = 0.005487.$$

(**b**) The sought probability is

$$1 - (1 - 0.005487)^{300} = 0.8081.$$

1.60 Using the complement rule, this variant of the birthday problem has as solution

$$1 - \frac{450 \times 435 \times 420 \times 405 \times 390 \times 375 \times 360 \times 345 \times 330 \times 315}{450^{10}}$$
$$= 0.8154.$$

1.61 Let A_1 be the event that the card of your first favorite team is not obtained and A_2 be the event that the card of your second favorite team is not obtained. The sought probability is $1 - P(A_1 \cup A_2)$. Since $P(A_1 \cup A_2) = \left(\frac{9}{10}\right)^5 + \left(\frac{9}{10}\right)^5 - \left(\frac{8}{10}\right)^5$, the sought probability is $1 - 0.8533 = 0.1467$.

1.62 (**a**) Let A be the event that you get at least one ace. It is easier to compute the probability of the complementary event A^c that you get no ace in a poker hand of five cards. For the sample space of the chance experiment, we take all ordered five-tuples $(x_1, x_2, x_3, x_4, x_5)$, where x_i corresponds to the suit and value of the ith card you get dealt. The total number of possible outcomes equals $52 \times 51 \times 50 \times 49 \times 48$. The number of outcomes without ace equals $48 \times 47 \times 46 \times 45 \times 44$. Assuming that the cards are randomly dealt, all possible outcomes are equally likely. Then, the event A^c has the probability

$$P(A^c) = \frac{48 \times 47 \times 46 \times 45 \times 44}{52 \times 51 \times 50 \times 49 \times 48} = 0.6588.$$

Hence, the probability of getting at least one ace in a poker hand of five cards is $1 - P(A^c) = 0.3412$. Another possible choice for the sample

space consists of the collection of all unordered sets of five distinct cards, resulting in the probability $1 - \binom{48}{5}/\binom{52}{5} = 0.3412$.
(**b**) It is easiest to take as sample space the collection of all unordered sets of five distinct cards. The sample space has $\binom{52}{5}$ equally likely elements. Let A_i be the event that the five cards of the poker hand are from suit i for $i = 1, \ldots, 4$. Each set A_i has $\binom{13}{5}$ elements. The events $A_1, \ldots, A_4$ are mutually exclusive and so the desired probability $P(A_1 \cup \cdots \cup A_4) = \sum_{i=1}^{4} P(A_i)$. For each i, $P(A_i) = \binom{13}{5}/\binom{52}{5} = 4.95 \times 10^{-4}$. Hence the desired probability is 0.00198.

1.63 It is easiest to compute the complementary probability that more than 5 rolls are needed to obtain at least one five and at least one six. This probability is given by $P(A \cup B)$, where A is the event that no five is obtained in 5 rolls and B is the event that no six is obtained in 5 rolls. We have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (\frac{5}{6})^5 + (\frac{5}{6})^5 - (\frac{4}{6})^5$. Therefore the sought probability is given by $1 - 2(\frac{5}{6})^5 + (\frac{4}{6})^5 = 0.4619$.
Note: The probability that exactly r rolls are needed to obtain at least one five and at least one six is given by $Q_{r-1} - Q_r$, where Q_n is defined as the probability that more than n rolls are needed to obtain at least one five and at least one six. We have $Q_n = (\frac{5}{6})^n + (\frac{5}{6})^n - (\frac{4}{6})^n$.

1.64 The sample space is given by the set $\{(i, j, k) : i, j, k = 1, 2, \ldots, 6\}$. Each element gets assigned a probability of $\frac{1}{216}$. It is easiest to compute the complementary probability $P(A)$, where A is the event that none of the three rolls gives your number chosen. The set A contains $5 \times 5 \times 5 = 125$ elements and so $P(A) = \frac{125}{216}$. Hence the desired probability is $1 - \frac{125}{216} = 0.4213$.

1.65 The sought probability is given by $P(\bigcup_{i=1}^{253} A_i)$. For any i, the probability $P(A_i)$ is equal to $365 \times 1 \times 364 \times 363 \times \cdots \times 344/365^{23} = 0.0014365$. The events A_i are mutually exclusive and so $P(\bigcup_{i=1}^{253} A_i) = \sum_{i=1}^{253} P(A_i)$. Therefore the probability that in a class 23 children exactly two children have a same birthday is equal to $253 \times 0.0014365 = 0.3634$.

1.66 Take as sample space the set of all sequences $(i_1, \ldots, i_6)$, where i_k is the number shown by the kth roll. The sample space has 6^6 equally likely elements. Let A be the event that all six face values appear and B be the event that the largest number rolled is r. The set A has $6!$ elements. Noting that the number of elements for which the largest number rolled does not exceed k equals k^6, we have that the set B has

$r^6 - (r-1)^6$ elements. The sought probabilities are $P(A) = \frac{6!}{6^6} = 0.0154$ and $P(B) = \frac{r^6-(r-1)^6}{6^6}$.

1.67 Take as sample space the set of all ordered sequences of the possible destinations of the five winners. The probability that at least one of the destinations A and B will be chosen is $1 - \frac{1}{3^5}$. Let A_k be the event that none of the winners has chosen the kth destination. The probability $P(A_1 \cup A_2 \cup A_3)$ equals $\frac{2^5}{3^5} + \frac{2^5}{3^5} + \frac{2^5}{3^5} - \frac{1}{3^5} - \frac{1}{3^5} - \frac{1}{3^5} = \frac{31}{81}$.

1.68 Take as sample space the set of all possible combinations of six different numbers from the numbers 1 to 42. The sample space has $\binom{42}{6}$ equally likely elements. Let A be the event that none of the numbers 7, 14, 21, 28, 35, and 42 is drawn and B be the event that exactly one of these numbers is drawn. Then, $P(A) = \binom{36}{6}/\binom{42}{6}$ and $P(B) = \left[\binom{6}{1} \times \binom{36}{5}\right]/\binom{42}{6}$. The events A and B are disjoint. Using the complement rule, the sought probability is $1 - P(A) - P(B) = 0.1975$.

1.69 Let A be the event that there is no ace among the five cards and B be the event that there is neither a king nor a queen among the five cards. The sought probability is

$$1 - P(A \cup B) = 1 - \frac{\binom{48}{5} + \binom{44}{5} - \binom{40}{5}}{\binom{52}{5}} = 0.1765.$$

1.70 Take as sample space the set $\{(i,j) : 1 \leq i, j \leq 6\}$, where i is the number rolled by John and j is the number rolled by Paul. Each element (i,j) of the sample space gets assigned the probability $p(i,j) = \frac{1}{36}$. Let A be the event that John gets a larger number than Paul and B the event that the product of the numbers rolled by John and Paul is odd. The desired probability is $P(A \cup B) = P(A) + P(B) - P(AB)$. This gives that the probability of John winning is $P(A \cup B) = \frac{15}{36} + \frac{6}{36} - \frac{3}{36} = \frac{1}{2}$, using the fact that $P(A) = \sum_{i=2}^{6} \sum_{j=1}^{i-1} p(i,j) = 15/36$, $P(B) = \sum_{i\, odd} \sum_{j\, odd} p(i,j) = \frac{6}{36}$ and $P(AB) = \frac{3}{36}$.

1.71 Take as sample space the set of all 40! possible orderings of the aces and the cards 2 through 10 (the other cards are not relevant). The first probability is $\frac{4 \times 39!}{40!} = \frac{1}{10}$ and the second probability is $\frac{4 \times 3 \times 38!}{40!} = \frac{1}{130}$.

1.72 The sample space consists of the elements $O_1, \overline{O_{12}}\, O_1, \overline{O_{12}}\, \overline{O_{12}}\, O_1, \ldots$, where O_1 occurs if the first trial gives outcome O_1, $\overline{O_{12}}\, O_1$ occurs if the first trial gives neither of the outcomes O_1 and O_2 and the second trial gives the outcome O_1, etc. The probabilities $p_1, (1-p_1-p_2)p_1, (1-p_1-p_2)^2 p_1, \ldots$ are assigned to the elements $O_1, \overline{O_{12}}\, O_1, \overline{O_{12}}\,\overline{O_{12}}\, O_1, \ldots$. The first probability is

$$\sum_{n=1}^{\infty}(1-p_1-p_2)^{n-1}p_1 = \frac{p_1}{p_1+p_2}.$$

To get the second probability, note that the probability of the event $A_{n,k}$ is given by $\binom{n-1}{r-1}\binom{n-r}{k}p_1^{r-1}p_2^k(1-p_1-p_2)^{n-r-k}p_1$. Since the events $A_{n,k}$ are mutually exclusive, it now follows that the second probability is given by

$$\sum_{k=0}^{s-1}\sum_{n=r+k}^{\infty}\binom{n-1}{r-1}\binom{n-r}{k}p_1^r p_2^k(1-p_1-p_2)^{n-r-k}.$$

This sum can be rewritten as

$$\sum_{k=0}^{s-1}\frac{p_1^r\, p_2^k}{(r-1)!\, k!}\sum_{l=0}^{\infty}(l+1)\cdots(l+r+k-1)(1-p_1-p_2)^l.$$

Using the identity $\sum_{l=0}^{\infty}(l+1)\cdots(l+m)\,x^l = m!/(1-x)^{m+1}$, the desired result next follows.

1.73 (**a**) The number of permutations in which the particular number r belongs to a cycle of length k is $\binom{n-1}{k-1}(k-1)!(n-k)!$. The sought probability is

$$\binom{n-1}{k-1}(k-1)!(n-k)!)/n! = \frac{1}{n}.$$

(**b**) For fixed r, s with $r \neq s$, let A_k be the event that r and s belong to a same cycle with length k. The sought probability is

$$P(A_2 \cup \cdots \cup A_n) = \frac{1}{n!}\sum_{k=2}^{n}\binom{n-2}{k-2}(k-1)!(n-k)!\frac{1}{2}.$$

1.74 It is no restriction to assume that the $2n$ prisoners have agreed that the ith prisoner goes to the ith box for $i = 1, 2, \ldots, 2n$. The order in which the names of the prisoners show up in the boxes is a random permutation of the integers $1, 2, \ldots, 2n$. Each prisoner finds his own name after inspecting up to a maximum of n boxes if and only if each cycle of the random permutation has length n or less. In other words, the probability that all prisoners will be released is equal to $1 - P(A_{n+1} \cup \cdots \cup A_{2n})$, where A_k is the event that a random permutation of $1, 2, \ldots, 2n$ contains a cycle of length k. A crucial observation is that, for any $k > n$, any random permutation of the integers $i = 1, 2, \ldots, 2n$ has at most one cycle of length k. Hence $P(A_k) = \binom{2n}{k}(k-1)!(2n-k)!/(2n)! = \frac{1}{k}$. Further, the events $A_{n+1}, \ldots, A_{2n}$ are mutually exclusive and so $P(A_{n+1} \cup \cdots \cup A_{2n}) = \sum_{k=n+1}^{2n} P(A_k)$. Hence the sought probability is

$$1 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}.$$

This probability is about $1 - \ln(2) = 0.3069$ for n large enough. The exact value is 0.3118 for $n = 50$.

1.75 In line with the strategy outlined in Problem 1.74, the person with the task of finding the car first opens door 1. This person next opens door 2 if the car key is behind door 1 and next opens door 3 if the goat is behind door 1. The person with the task of finding the car key first opens door 2. This person next opens door 1 if the car is behind door 2 and next opens door 3 if the goat is behind door 2. Under this strategy the probability of winning the car is $\frac{2}{3}$: the four arrangements (car, key, goat), (car, goat, key), (key, car, goat) and (goat, key, care) are winning, whereas the two arrangements (key, goat, car) and (goat, car, key) are losing.

1.76 Some reflection shows that the game cannot take more than 15 spins (it takes 15 spins if and only if the spins $1, \ldots, 4, 6, \ldots, 9$ and $11, \ldots, 14$ result in "odds," while the spins 5, 10 and 15 result in "heads"). Let A_i be the event that the spinner wins on the ith toss for $i = 3, \ldots, 15$. The events A_i are mutually exclusive. The win probability is given by

$$\sum_{i=3}^{15} P(A_i) = \frac{1}{2^6} + \frac{3}{2^7} + \frac{3}{2^7} + \frac{5}{2^8} + \frac{15}{2^{10}} + \frac{18}{2^{11}} + \frac{19}{2^{12}} + \frac{9}{2^{12}} + \frac{15}{2^{14}} + \frac{5}{2^{14}} + \frac{3}{2^{15}} + \frac{3}{2^{17}} + \frac{1}{2^{18}} = 0.11364.$$

The game is unfavorable to the bettor.

1.77 The probability of correctly identifying five or more wines can be calculated as $\sum_{k=5}^{10} e^{-1}1^k/k! = 0.00366$ when the person is not a connoisseur and just guesses the names of the wines. This small probability is a strong indication that the person is a connoisseur.

1.78 This problem is a variant of the hat-check problem. Let A be the event that exactly one student receives his own paper and B_i be the event that the ith student is the only student who gets back his own paper. Then, by results in Example 1.19, $P(A) = \sum_{k=0}^{14} \frac{(-1)^k}{k!} \approx e^{-1}$. Since $A = \cup_{i=1}^{15} E_i$ and the events E_i are disjoint, $P(A) = \sum_{i=1}^{15} P(E_i)$. For reasons of symmetry $P(E_i) = P(E_1)$ for all i. Thus the sought probability is about $\frac{1}{15}e^{-1}$.

1.79 Let A_i be the event that the ith person gets both the correct coat and the correct umbrella. The sought probability is

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = \sum_{k=1}^{5} (-1)^{k+1} \binom{5}{k} \frac{(5-k)!(5-k)!}{5!5!} = 0.1775.$$

1.80 Label the three Italian wines as 1, 2, and 3. Let A_i be the event that the Italian wine with label i is correctly guessed. The sought probability is $1 - P(A_1 \cup A_2 \cup A_3) = 1 - \left(\binom{3}{1} P(A_1) - \binom{3}{2} P(A_1 A_2) + P(A_1 A_2 A_3)\right)$. We have $P(A_1) = \frac{9!}{10!}$, $P(A_1 A_2) = \frac{8!}{10!}$, and $P(A_1 A_2 A_3) = \frac{7!}{10!}$. The probability that none of the three Italian wines is correctly guessed is

$$1 - \left(3 \times \frac{9!}{10!} - 3 \times \frac{8!}{10!} + \frac{7!}{10!}\right) = 0.7319.$$

1.81 Let A_i be the event that the three choices of five distinct numbers have number i in common. The sought probability is

$$\begin{aligned} P(\bigcup_{k=1}^{25} A_k) &= \sum_{k=1}^{5} (-1)^{k+1} \binom{25}{k} P(A_1 \cdots A_k) \\ &= \sum_{k=1}^{5} (-1)^{k+1} \binom{25}{k} \frac{\binom{25-k}{5-k}^3}{\binom{25}{5}^3} = 0.1891. \end{aligned}$$

1.82 Let A_i be the event that all four cards of kind i are contained in the hand of 13 cards. The desired probability is

$$P(A_1 \cup \cdots \cup A_{13}) = \sum_{k=1}^{3} (-1)^{k+1} \binom{13}{k} P(A_1 \cdots A_k).$$

This probability can be evaluated as

$$\binom{13}{1} \times \frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2} \times \frac{\binom{44}{5}}{\binom{52}{13}} + \binom{13}{3} \times \frac{\binom{40}{1}}{\binom{52}{13}} = 0.0342.$$

1.83 Let A_i be the event that the player's hand does not contain any card of suit i. The desired probability is

$$P(\bigcup_{i=1}^{4} A_i) = \sum_{k=1}^{3}(-1)^{k+1}\binom{4}{k}\frac{\binom{52-13k}{13}}{\binom{52}{13}} = 0.0511$$

for the bridge hand. For the poker hand, the desired probability is

$$\sum_{k=1}^{3}(-1)^{k+1}\binom{4}{k}\frac{\binom{52-13k}{5}}{\binom{52}{5}} = 0.7363.$$

1.84 Take as sample space the set of all possible ordered arrangements of the 12 people. Label the six rooms as $i = 1, \ldots, 6$. The first two people in an arrangement are assigned into room 1, the next two people in the arrangement are assigned into room 2, etc. Let A_i be the event that room i has two people of different nationalities. The sought probability is $1 - P(A_1 \cup \cdots \cup A_5) = 1 - \sum_{k=1}^{4}(-1)^{k+1}\binom{6}{k}P(A_1 \cdots A_k)$. We have

$$P(A_1) = \frac{8 \times 4 \times 2 \times 10!}{12!},\ P(A_1A_2) = \frac{8 \times 4 \times 7 \times 3 \times 2^2 \times 8!}{12!},$$
$$P(A_1A_2A_3) = \frac{8 \times 4 \times 7 \times 3 \times 6 \times 2 \times 2^3 \times 6!}{12!},$$
$$P(A_1A_2A_3A_4) = \frac{8 \times 4 \times 7 \times 3 \times 6 \times 2 \times 5 \times 2^4 \times 4!}{12!}.$$

This leads to the value $\frac{1}{33}$ for the sought probability.

1.85 The possible paths from node n_1 to node n_4 are the four the paths (l_1, l_5), (l_2, l_6), (l_1, l_3, l_6) and (l_2, l_4, l_5). Let A_i be the event that the ith path is functioning. The probability $P(A_1 \cup A_2 \cup A_3 \cup A_4)$ can be evaluated as

$$p_1p_5 + p_2p_6 + p_1p_3p_6 + p_2p_4p_5 - p_1p_2p_5p_6 - p_1p_3p_5p_6 - p_1p_2p_4p_5$$
$$- p_1p_2p_3p_6 - p_2p_4p_5p_6 + p_1p_2p_3p_5p_6 + p_1p_2p_4p_5p_6.$$

This probability reduces to $2p^2(1 + p + p^3) - 5p^4$ when $p_i = p$ for all i.

1.86 Let A_i be the event that number i has not appeared in 30 draws of the Lotto 6/45. The desired probability is given by $P(A_1 \cup \cdots \cup A_{45})$. Noting that $P(A_1 \cdots A_k) = \left[\binom{45-k}{6}/\binom{45}{6}\right]^{30}$, it follows that the desired probability is equal to

$$\sum_{k=1}^{39}(-1)^{k+1}\binom{n}{k}\left[\binom{45-k}{6}/\binom{45}{6}\right]^{30} = 0.4722.$$

1.87 Let A_i be the event that it takes more than r purchases to get the ith coupon. The probability that more than r purchases are needed in order to get a complete set of coupons is

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{(n-k)^r}{n^r}.$$

Using this expression for $n = 38$, 365, and 100, we find that the required number of rolls is $r = 13$, the required number of people is $r = 2{,}287$, and the required number of balls is $r = 497$.

1.88 Let A_i be the event that the ith boy becomes part of a couple. The desired probability is $1 - P(A_1 \cup \cdots \cup A_n)$. For any fixed i, $P(A_i) = \frac{n \times n^{2n-2}}{n^{2n}} = \frac{n}{n^2}$. For any fixed i and j, $P(A_iA_j) = \frac{n\times(n-1)n^{2n-4}}{n^{2n}} = \frac{n(n-1)}{n^4}$ for $i \neq j$. Continuing in this way, we find

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{n(n-1)\cdots(n-k+1)}{n^{2k}}.$$

1.89 Let A_i be the event that the ith person does not share his or her birthday with someone else. The sought probability is given by $1 - P(A_1 \cup \cdots \cup A_n)$ and is equal to

$$1-\frac{1}{365^n}\sum_{k=1}^{\min(n,365)}(-1)^{k+1}\binom{n}{k}(365)\times\cdots\times(365-k+1)\times(365-k)^{n-k}.$$

This probability is 0.5008 for $n = 3{,}064$.

1.90 Let A_i be the event that all of the three random permutations have the same number in the ith position. The desired probability $P(A_1 \cup A_2 \cup \cdots \cup A_{10})$ is given by

$$\sum_{k=1}^{10}(-1)^{k+1}\binom{10}{k}\frac{10\times\cdots\times(10-k+1)\times[(10-k)!]^3}{[10!]^3} = 0.0947.$$

1.91 There are $\binom{10}{2} = 45$ possible combinations of two persons. Let A_k be the event that the two persons from the kth combination have chosen each other's name. Using the fact that $P(A_iA_j) = 0$ for $i \neq j$ when A_i

and A_j have a person in common, we find that the sought probability is given by

$$P(A_1 \cup \cdots \cup A_{45}) = \binom{10}{2}\left(\frac{1}{9}\right)^2 - \frac{1}{2!}\binom{10}{2}\binom{8}{2}\left(\frac{1}{9}\right)^4 + \cdots - \frac{1}{5!}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}\left(\frac{1}{9}\right)^{10} = 0.4654.$$

1.92 Let A_i be the event that the ith person is a survivor. The desired probability is given by $1 - P(A_1 \cup \cdots \cup A_n)$. For any i,

$$P(A_i) = \frac{(n-2)^{n-1}(n-1)}{(n-1)^n}.$$

For any i, j with $i \neq j$,

$$P(A_iA_j) = \frac{(n-3)^{n-2}(n-2)^2}{(n-1)^n}.$$

Continuing in this way, it follows that

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n-2}(-1)^{k+1}\binom{n}{k}\frac{\big(n-(k+1)\big)^{n-k}(n-k)^k}{(n-1)^n}.$$

Note: Suppose that there is a second round with the survivors if there are two or more survivors after the first round, the second round is followed by a third round if there are two or more survivors after the second round, and so on, until there is one survivor or no survivor at all. What is the probability that the game ends with one survivor? This probability is given for several values of n in the table below. These probabilities can be computed by using the powerful method of an absorbing Markov chain, see Chapter 10 of the book. In this method we need the one-step transition probabilities $p_{n,m}$ being the probability of going in one step from state n to state m. In this particular example, the state is the number of survivors and the probability $p_{n,m}$ is the probability of m survivors after a round starting with n survivors. To give these probabilities, we need the generalized inclusion-exclusion formula

$$\begin{aligned}&P(\text{exactly } m \text{ of the events } A_1, \ldots, A_n \text{will occur})\\ &= \sum_{k=0}^{n-m}(-1)^k\binom{m+k}{m}\sum_{i_1<i_2<\cdots<i_{m+k}}P(A_{i_1}A_{i_2}\cdots A_{i_{m+k}}).\end{aligned}$$

n	$P(\text{one survivor})$	n	$P(\text{one survivor})$
2	0.0000	20	0.4693
3	0.7500	30	0.5374
4	0.5926	40	0.4996
5	0.4688	50	0.4720
6	0.4161	60	0.4879
7	0.4389	70	0.5155
8	0.4890	80	0.5309
9	0.5323	90	0.5291
10	0.5547	100	0.5160

1.93 Let A_i be the event that no ball of color i is picked for $i = 1, 2, 3$. The probability of picking at least one ball of each color is

$$1 - P(A_1 \cup A_2 \cup A_3) = 1 - \Big(\sum_{i=1}^{3} P(A_i) - \sum_{i=1}^{2} \sum_{j=i+1}^{3} P(A_i A_j) \Big).$$

If the balls are picked with replacement, then

$$P(A_1) = \frac{12^5}{15^5},\ P(A_2) = \frac{10^5}{15^5},\ P(A_3) = \frac{8^5}{15^5},$$
$$P(A_1A_2) = \frac{7^5}{15^5},\ P(A_1A_3) = \frac{5^5}{15^5},\ \text{ and } P(A_2A_3) = 0.$$

The sought probability is $1 - 0.4763 = 0.5237$. If the balls are picked without replacement, then

$$P(A_1) = \sum_{k=1}^{4} \frac{\binom{5}{k}\binom{7}{5-k}}{\binom{15}{5}},\ P(A_2) = \sum_{k=1}^{3} \frac{\binom{3}{k}\binom{7}{5-k}}{\binom{15}{5}},\ P(A_3) = \sum_{k=1}^{3} \frac{\binom{3}{k}\binom{5}{5-k}}{\binom{15}{5}}$$
$$P(A_1A_2) = \frac{\binom{7}{5}}{\binom{15}{5}},\ P(A_1A_3) = \frac{\binom{5}{5}}{\binom{15}{5}},\ \text{ and } P(A_2A_3) = 0.$$

The probability of picking at least one ball of each color is $1-0.3443 = 0.6557$.

1.94 Take as sample space the set of all possible ordered arrangements of the $2n$ people. The sample space has $(2n)!$ equally likely elements. Imagine that the two people in the positions $2k - 1$ and $2k$ of the ordered arrangement are paired as bridge partners for $k = 1, \ldots, n$. Let A_i the event that couple i is paired as bridge partners. The sought probability is $1 - P(A_1 \cup \cdots \cup A_n)$. The number of elements in the set $A_1 \cap \cdots \cap A_k$ is $n \times (n-1) \times \cdots \times (n-k+1) \times 2^k \times (2n-2k)!$. There are $n(n-1)\cdots(n-k+1)$ possible choices for the couples in the first $2k$ positions of the arrangement and two partners from a couple can be ordered in two ways. The remaining $2n-2k$ people can be ordered in $(2n-2k)!$ ways. Thus we find

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P(A_1 \cdots A_k)$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{n \times (n-1) \times \cdots \times (n-k+1) \times 2^k \times (2n-2k)!}{(2n)!}.$$

Note: the probability that none of the couples will be paired as bridge partners tends to $1 - e^{-\frac{1}{2}} = 0.3935$ as the number of couples gets large, see also Problem 3.85.

1.95 Let A_i be the event that the four cards of rank i are matched. The sought probability is

$$P\Big(\bigcup_{i=1}^{13} A_i\Big) = \sum_{k=1}^{13} (-1)^{k+1} \frac{\binom{13}{k}(4!)^k (52-4k)!}{52!} = 4.80 \times 10^{-5}.$$