CS412 Solutions to exercise sheet 1

Using basic logic and set notation

1. (a) Show that $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

   Two propositional expressions are equivalent if they have the same truth table. Could also show equivalence by showing that the first implies the second and the second implies the first, but a truth table is simplest for a small case like this.

P	Q	P	$\Rightarrow$	Q	$\neg P$	$\vee$	Q
T	T	T	**T**	T	F	**T**	T
T	F	T	**F**	F	F	**F**	F
F	T	F	**T**	T	T	**T**	T
F	F	F	**T**	F	T	**T**	F

   (b) Show that $(P \wedge Q) \Rightarrow R$ is equivalent to $P \Rightarrow (Q \Rightarrow R)$

P	Q	R	$(P \wedge Q)$	$\Rightarrow$	R	P	$\Rightarrow$	$(Q \Rightarrow R)$
T	T	T	T	**T**			**T**	T
T	T	F	T	**F**			**F**	F
T	F	T	F	**T**			**T**	T
T	F	F	F	**T**			**T**	T
F	T	T	F	**T**			**T**	T
F	T	F	F	**T**			**T**	F
F	F	T	F	**T**			**T**	T
F	F	F	F	**T**			**T**	T

   Once again, these two propositions represent the same truth function and so are equivalent.

   (c) Let $X = P \vee (Q \wedge R)$ and $Y = \neg Q \Rightarrow P$. Show that Y is a logical consequence of X.

   Can either use a truth table to show that, on every line of the table where X is true, Y is also true. Or can use reasoning to show that the X implies Y. Here is how a reasoning approach would work:

$$
\begin{aligned}
P \vee (Q \wedge R) &\equiv (P \vee Q) \wedge (P \vee R) \\
&\Rightarrow P \vee Q \\
&\equiv Q \vee P \\
&\equiv \neg\neg\, Q \vee P \\
&\equiv \neg\, Q \Rightarrow P
\end{aligned}
$$

Here I'm using "informal but informed" reasoning - that is, I'm not quoting the justification at each stage but moving in "obvious" steps.

(d) Show that Q is a logical consequence of $P \wedge \neg P$. That is, anything follows from a contradiction.

Again, there would be a number of ways to do this, but I think a truth table would be the simplest. We just need to show that for every line on which $P \wedge \neg P$ is true, Q is also true. In this case, the result follows trivially since the first statement is never true.

P	Q	P	$\wedge$	$\neg P$	Q
T	T	T	**F**	F	**T**
T	F	T	**F**	F	**F**
F	T	F	**F**	T	**T**
F	F	F	**F**	T	**F**

(e) Suppose Q is a *contradiction* (that is, it's a proposition that's always false such as $R \wedge \neg R$). Show that $P \wedge Q$ is itself a contradiction for any P. Show also that $P \vee Q$ is equivalent to P.

If Q is a *tautology* (that is, it's a proposition that's always true such as $R \vee R$) what can you say about $P \wedge Q$ and $P \vee Q$ for any P?

If Q is a contradiction:

P	Q	$P \wedge Q$	$P \vee Q$
T	F	F	T
F	F	F	F

If Q is a tautology:

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
F	T	F	T

Hence in this case, $P \wedge Q$ is equivalent to P, and $P \vee Q$ is a tautology.

(f) Simplify: $(Q \vee P) \wedge (P \Rightarrow (Q \wedge R)) \wedge \neg P$ (let's call this X)

A good tip here is that if one conjunct of a logical statement implies a second then we can drop the second without affecting the overall meaning. To convince yourself of that you could do a quick truth table to show that $(P \Rightarrow Q) \Rightarrow ((P \wedge Q) \Leftrightarrow P)$ is always true.

For the current exercise, from part (d) above, we know that $(\neg P \wedge P) \Rightarrow (Q \wedge R)$ (anything follows from a contradiction).

Using the equivalence from part (b) we can rewrite this as $\neg P \Rightarrow \underline{(P \Rightarrow (Q \wedge R))}$

Hence the implied conjunct (indicated by the underline) can be dropped from our original proposition, X, leaving the equivalent: $(Q \vee P) \wedge \neg P$.

By distribution, this is equivalent to $(Q \wedge \neg P) \vee (P \wedge \neg P)$

As $(P \wedge \neg P)$ is a contradiction, by part (e), we can simplify further to: $(Q \wedge \neg P)$.

You could confirm this with a truth table.

(g) Give an example to show that $\forall x \bullet (\exists y \bullet P(x, y))$ is not equivalent to $\exists y \bullet (\forall x \bullet P(x, y))$

Eg: in the domain of $\mathbb{N}$, if $P(x, y)$ is interpreted as "y is greater than x". With this, the first statement is true but the second false.

2. If $X = \{1, 2, 3\}$ what are:

(a) $X \times X$
$= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(b) $\mathbb{P} X$
$= \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\{2, 3\}, \{1, 2, 3\}\}$

(c) $\mathbb{P}(X \times \{a\})$
$= \{\{\}, \{(1, a)\}, \{(2, a)\}, \{(3, a)\}, \{(1, a), (2, a)\}, \{(1, a), (3, a)\}, \{(2, a), (3, a)\}, \{(1, a), (2, a), (3, a)\}\}$

3. With $\mathbb{N}$ as the set of natural numbers, use logic and set notation to describe the following.

(a) The set of all numbers between 100 and 200 inclusive.
$\{x \mid x \in \mathbb{N} \wedge x \geq 100 \wedge x \leq 200\}$

(b) The set of all prime numbers.
$\{x \mid x \in \mathbb{N} \wedge \forall y, z \bullet (y \in \mathbb{N} \wedge z \in \mathbb{N} \wedge y * z = x \Rightarrow (y = 1 \vee z = 1)\}$

(c) The set of all finite sets of numbers which contain their own cardinality (size) as a member.
$\{x \mid x \subseteq \mathbb{N} \wedge card(x) \in x\}$

4. Which of the following are true for all sets S, T and U? Justify your answer.

   (a) $(S \cap T) \cup U = (S \cup U) \cap (T \cup U)$
   True. An element x is a member of $(S \cap T) \cup U$ exactly when $(x \in S \wedge x \in T) \vee x \in U)$ which is logically equivalent to $(x \in S \vee x \in U) \wedge (x \in T \vee x \in U)$ which is precisely the condition for x to be a member of $(S \cup U) \cap (T \cup U)$.

   (b) $S \cup T \neq S$
   Not true. If T is a subset of S then $S \cup T = S$.

   (c) $S - (T \cap U) = (S - T) \cap (S - U)$
   Not true. Suppose $S = \{1, 2, 3\}$, $T = \{2\}$, $U = \{3\}$ then:
   LHS $= \{1, 2, 3\} - \{\} = \{1, 2, 3\}$
   RHS $= \{1, 3\} - \{1, 2\} = \{1\}$

5. Suppose a specification uses the set, *PID* of person identifiers and declares the variables:
   $club1, club2 \subseteq PID$
   $committee1, committee2 \subseteq PID$

   (a) Write predicates (statements) to express the following.
      i. For both clubs, the committee must be comprised of members of that club.
         $(committee1 \subseteq club1) \wedge (committee2 \subseteq club2)$
      ii. Members of club2 are not allowed to serve on club1's committee.
         $club2 \cap committee1 = \{\}$
      iii. Some people are not members of either club1 or club2.
         $club1 \cup club2 \neq PID$
   (b) Write expressions to denote:
      i. The number of ordinary (ie: non committee) members of club1.
         $card(club1 - committee1)$
      ii. The set of people who belong to one club but not both.
         $(club1 \cup club2) - (club1 \cap club2)$
      iii. The set of pairs in which the first of each pair is a person and the second is the set of clubs they belong to.
         $\{pp, ss \mid pp \in PID \wedge ss = \{cc \mid cc \in \{club1, club2\} \wedge pp \in cc\}\}$
   (c) A sign in the clubroom for club1 says "If you are a member of club1 then you are entitled to free coffee".
      i. As a natural language statement this would probably be expected to mean that non-members are *not* entitled to free coffee.
      ii. Trying to express this as $\forall\, xx \bullet (xx \in club1 \Rightarrow freecoffee(xx))$ would mean that in the case of $xx \notin club1$ we can deduce *nothing* about xx's eligibility for free coffee. Would need to use $\Leftrightarrow$ if we wanted them to be ineligible.

Note the difference between the first two sections of question 5. The first part involved predicates (statements which are either true or false). The second involved expressions which defined terms ("things") in the specification. There's an important difference between these two "parts of speech".

6. Why can't we use conjunction in the typed univeral quantification? $\forall\, xx \bullet (xx \in T \wedge P)$ could only be true if *all* elements were of type T. For example, $\forall\, xx \bullet (xx \in \mathbb{N} \wedge xx \geq 0)$ is false because not everything is a number.

   Why can't we use implication in the typed existential quantification? $\exists\, xx \bullet (xx \in T \Rightarrow P)$ is trivially satisfied by any xx which is not a member of T. Therefore the overall statement will always be true.