Chapter 4

4.1 Since $c\int_0^{10}(10-x)\,dx$ must be equal to 1, we get $c=\frac{1}{50}$. The probabilities are $P(X\leq 5)=\int_0^5 \frac{1}{50}(10-x)\,dx=\frac{3}{4}$ and $P(X>2)=\int_2^{10}\frac{1}{50}(10-x)\,dx=\frac{16}{25}$.

4.2 The constant c follows from the requirement $c\int_0^1(3x^2-8x-5)=1$. This gives $c=-\frac{1}{8}$. Since $f(x)=(5+8x-3x^2)/8$ is positive for $0<x<1$, we have that $f(x)$ represents indeed a probability density function. The cumulative probability distribution function $F(x)=P(X\leq x)$ is given by

$$F(x)=\frac{1}{8}\int_0^x(5+8y-3y^2)\,dy=\frac{1}{8}(5x+4x^2-x^3)\quad \text{for } 0\leq x\leq 1.$$

Further $F(x)=0$ for $x<0$ and $F(x)=1$ for $x\geq 1$.

4.3 Noting that $\int_0^a 2cxe^{-cx^2}dx=\int_0^{ca^2}e^{-u}du=1-e^{-ca^2}$, we get

$$P(X\leq 15)=\int_0^{15}2cxe^{-cx^2}dx=0.3023,$$
$$P(X>30)=\int_{30}^{\infty}2cxe^{-cx^2}dx=0.2369,$$
$$P(20<X\leq 25)=\int_{20}^{25}2cxe^{-cx^2}dx=0.1604.$$

4.4 Let the random variable X be the length of any particular phone call made by the travel agent. Then,

$$P(X>7)=\int_7^{\infty}0.25e^{-0.25x}\,dx=-e^{-0.25x}\big|_7^{\infty}=e^{-1.75}=0.1738.$$

4.5 The proportion of pumping engines that will not fail before 10,000 hours use is $P(X>10)=\int_{10}^{\infty}0.02xe^{-0.01x^2}\,dx$. Since

$$\int_a^{\infty}0.02xe^{-0.01x^2}\,dx=\int_{a^2/100}^{\infty}e^{-y}\,dy=e^{-a^2/100},$$

we get $P(X>10)=e^{-1}$. Also $P(X>5)=e^{-0.25}$. Therefore the probability that the engine will survive for another 5,000 hours given that it has functioned properly during the past 5,000 hours is

$$P(X>10\mid X>5)=\frac{P(X>10)}{P(X>5)}=\frac{e^{-1}}{e^{-0.25}}=0.4724.$$

4.6 The cumulative distribution function $P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$ is given by $F(x) = \frac{1}{50}(x-115)^2$ for $115 \leq x \leq 120$ and $F(x) = 1-\frac{1}{50}(125-x)^2$ for $120 \leq x \leq 125$. Since $P(117 < X < 123) = F(123) - F(117) = \frac{21}{25}$, the proportion of non-acceptable strain gauges is $\frac{4}{25}$.

4.7 The cumulative distribution function $F(x) = P(X \leq x)$ of the random variable X is $F(x) = 105 \int_0^x y^4(1-y)^2\, dy = x^5(15x^2 - 35x + 21)$ for $0 \leq x \leq 1$. The solution of the equation $1 - F(x) = 0.05$ is $x = 0.8712$. Thus the capacity of the storage tank in thousands of gallons should be 0.8712.

4.8 A stockout occurs if and only if the demand X is larger than Q. Thus

$$P(\text{stockout}) = \int_Q^{\infty} f(x)\, dx = 1 - \int_0^Q f(x)\, dx.$$

4.9 Let the random variable Y be the area of the circle. Then $Y = \pi X^2$. Since $P(X \leq x) = x$ for $0 \leq x \leq 1$ and $P(Y \leq y) = P(X \leq \sqrt{y/\pi})$, we get $P(Y \leq y) = \sqrt{y/\pi}$ for $0 \leq y \leq \pi$. Differentiation of $P(Y \leq y)$ gives that the density function of Y is $1/\big(2\sqrt{\pi y}\big)$ for $0 < y < \pi$ and 0 otherwise.

4.10 To find the density function of $Y = \frac{1}{X}$, we determine $P(Y \leq y)$. Obviously, $P(Y \leq y) = 0$ for $y \leq 1$. For $y > 1$,

$$P(Y \leq y) = P\left(X \geq \frac{1}{y}\right) = 1 - P\left(X \leq \frac{1}{y}\right) = 1 - F\left(\frac{1}{y}\right),$$

where $F(x)$ is the probability distribution function of X. By differentiation, it follows that the density function $g(y)$ of Y is given by

$$g(y) = f\left(\frac{1}{y}\right) \times \frac{1}{y^2} = \frac{6}{7}\left(\frac{1}{y^3} + \frac{1}{y^2\sqrt{y}}\right) \qquad \text{for } y > 1$$

and $g(y) = 0$ otherwise.

4.11 The cumulative distribution function of $Y = X^2$ is

$$P(Y \leq y) = P(X \leq \sqrt{y}) = F(\sqrt{y}) \qquad \text{for } y \geq 0,$$

where $F(x) = P(X \leq x)$. Differentiation gives that the density function of Y is $\frac{1}{2} f(\sqrt{y})/\sqrt{y}$ for $y > 0$ and 0 otherwise. The cumulative distribution function of $W = V^2$ is

$$P(W \leq w) = P(-\sqrt{w} \leq V \leq \sqrt{w}) = \frac{2\sqrt{w}}{2a} \qquad \text{for } 0 \leq w \leq a^2.$$

The density function of W is $1/\big(2a\sqrt{w}\big)$ for $0 < w < a^2$ and 0 otherwise.

4.12 (**a**) Let the random variable V be the sum of the coordinates of the point Q. For $0 \leq v \leq 1$, the random variable V takes on a value smaller than or equal to v if and only if the point Q falls in a right triangle with legs of length v (draw a picture). The area of this triangle is $\frac{1}{2}v^2$. Hence

$$P(V \leq v) = \frac{1}{2}v^2 \quad \text{for } 0 \leq v \leq 1.$$

For $1 \leq v \leq 2$, the random variable V takes on a value larger than v if and only if the point Q falls in a right triangle with legs of length $1-(1-v) = 2-v$. The area of this triangle is $\frac{1}{2}(2-v)^2$ and so $P(V > v) = \frac{1}{2}(2-v)^2$ for $1 \leq v \leq 2$. This gives

$$P(V \leq v) = 1 - \frac{1}{2}(2-v)^2 \quad \text{for } 1 \leq v \leq 2.$$

By differentiation, it now follows that the density function $f_V(v)$ of V satisfies $f_V(v) = v$ for $0 < v \leq 1$, $f(v) = 2 - v$ for $1 < v \leq 2$ and $f_V(v) = 0$ otherwise.

(**b**) Let the random variable W be the product of the coordinates of the randomly chosen point Q. A point (x, y) in the unit square satisfies $xy \leq w$ for any given $0 \leq w \leq 1$ if and only if either the point belongs to the set $\{(x, y) : 0 \leq x \leq w, 0 \leq y \leq 1\}$ or the point satisfies $w \leq x \leq 1$ and is below the graph $y = \frac{w}{x}$ (draw a figure). This gives

$$P(W \leq w) = w + \int_w^1 \frac{w}{x}\,dx = w - w\ln(w).$$

The density function of W is $f_W(w) = -\ln(w)$ for $0 < w < 1$ and $f_W(w) = 0$ otherwise.

4.13 The random variable $V = X/(1-X)$ satisfies

$$P(V \leq v) = P\Big(X \leq \frac{v}{1+v}\Big) = \frac{v}{1+v} \quad \text{for } v \geq 0.$$

Thus the density function of V is $\frac{1}{(1+v)^2}$ for $v > 0$ and 0 otherwise. To get the density of $W = X(1-X)$, note that the function $x(1-x)$ has $\frac{1}{4}$ as its maximal value on $(0, 1)$ and that the equation $x(1-x) = w$

has the solutions $x_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1-4w}$ and $x_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-4w}$ for $0 \leq w \leq \frac{1}{4}$. Thus

$$P(W > w) = P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} 1\, dx = \sqrt{1-4w} \quad \text{for } 0 \leq w \leq \frac{1}{4}.$$

Thus the density function of W is $2/\sqrt{1-4w}$ for $0 < w < \frac{1}{4}$ and 0 otherwise.

4.14 Let the random variable U be a number chosen at random from the interval (0,1). Using the fact that $P(U \leq u) = u$ for $0 \leq u \leq 1$, it follows that

$$P(X \leq x) = P(0 \leq U \leq x) + P(1-x \leq U \leq 1) = 2x \quad \text{for } 0 \leq x \leq 0.5.$$

Hence X has the density function $f(x) = 2$ for $0 < x < 0.5$ and $f(x) = 0$ otherwise. Let the random variable $Y = X/(1-X)$. Then,

$$P(Y \leq y) = P\left(X \leq \frac{y}{1+y}\right) = \frac{2y}{1+y} \quad \text{for } 0 \leq y \leq 1.$$

The density function of $Y = X/(1-X)$ is $f_Y(y) = \frac{2}{(1+y)^2}$ for $0 < y < 1$ and $f_Y(y) = 0$ otherwise.

4.15 The sample space of the experiment is $\{(x, y) : 0 \leq x, y \leq 1\}$. Noting that $\max(x, y) \leq v$ if and only if $x \leq v$ and $y \leq v$, it follows that the random variable V takes on a value smaller than or equal to v if and only if the randomly chosen point falls in the set $A = \{(x, y) : 0 \leq x, y \leq v\}$. Hence the probability $P(V \leq v)$ is equal to the area of the set A and so

$$P(V \leq v) = v^2 \quad \text{for } 0 \leq v \leq 1.$$

Hence the density function of V is $f_V(v) = 2v$ for $0 < v < 1$ and $f_V(v) = 0$ otherwise. Noting that $\min(x, y) > w$ if and only if $x > w$ and $y > w$, the probability $P(W > w)$ can be calculated as the area of the set $B = \{(x, y) : 1 > x, y > w\}$. This gives

$$P(W \leq w) = 1 - (1-w)^2 \quad \text{for } 0 \leq w \leq 1.$$

Hence the density function of W is given by $f_W(w) = 2(1-w)$ for $0 < w < 1$ and $f_W(w) = 0$ otherwise.

4.16 Drawing a figure and using the symmetry in the model, we can conclude that the height X above the ground can be modeled as $X = 15 + 15\cos(\Theta)$, where Θ is a randomly chosen angle between 0 and π. Then,

$$\begin{aligned} P(X \leq x) &= P(15 + 15\cos(\Theta) \leq x) = P\left(\Theta \geq \arccos\left(\frac{x}{15} - 1\right)\right) \\ &= -\frac{1}{\pi}\arccos\left(\frac{x}{15} - 1\right) \quad \text{for } 0 \leq x \leq 30, \end{aligned}$$

where the last equality uses the fact that the randomly chosen angle Θ has the density $\frac{1}{\pi}$ on $(0, \pi)$. In particular, $P(X \leq 22.5) = 2/3$ and $P(X \leq 7.5) = 1/3$. The derivative of $\arccos(z)$ is $-\frac{1}{\sqrt{1-z^2}}$. Hence the density function of X is given by

$$f(x) = \begin{cases} \frac{1}{15\pi\sqrt{1-(x/15-1)^2}} & \text{for } 0 < x < 30 \\ 0 & \text{otherwise.} \end{cases}$$

4.17 We have $E(X) = \frac{1}{625}\int_{50}^{75} x(x-50)\,dx + \frac{1}{625}\int_{75}^{100} x(100-x)\,dx = 75$ hundred hours.

4.18 By partial integration,

$$E(X) = \int_0^{0.25} x\pi\sqrt{2}\cos(\pi x)\,dx = \sqrt{2}x\sin(\pi x)\,\Big|_0^{0.25} + \frac{\sqrt{2}}{\pi}\cos(\pi x)\,\Big|_0^{0.25}.$$

This gives $E(X) = 0.1182$ seconds.

4.19 The density function of the distance X thrown by Big John is $f(x) = \frac{x-50}{600}$ for $50 < x < 80$, $f(x) = \frac{90-x}{200}$ for $80 < x < 90$, and $f(x) = 0$ otherwise. This gives $E(X) = \int_{50}^{90} xf(x)\,dx = 73\frac{1}{3}$ meters.

4.20 The expected values of the random variables in the Problems 4.2, 4.4, and 4.6 are $\frac{53}{96}$, 4, and 120.

4.21 Let the random variable X be the distance from the randomly chosen point to the base of the triangle. Using a little geometry, it follows that $P(X > x)$ is equal to the ratio of $\frac{1}{2}[(h-x) \times (h-x)b/h]$ and $\frac{1}{2}h \times b$. Differentiation shows that the density function of X is $2(h-x)/h^2$ for $0 < x < h$ and 0 otherwise. Then

$$E(X) = \int_0^h x\frac{2(h-x)}{h^2}\,dx = \frac{1}{3}h.$$

4.22 The expected value of the price paid for the collector's item is $E(X) = \int_0^1 x\, 90(x^8 - x^9)\, dx = \frac{9}{11}$.

4.23 Let X be the distance from the point to the origin. Then

$$P(X \leq a) = \frac{1}{4}\pi a^2 \quad \text{for } 0 \leq a \leq 1$$
$$P(X \leq a) = \frac{1}{4}\pi a^2 - 2\int_1^a \sqrt{a^2 - x^2}\, dx = \frac{1}{4}\pi a^2 - a^2 \arccos(\frac{1}{a}) + \sqrt{a^2 - 1}$$

for $1 < a \leq \sqrt{2}$. The density function $f(x)$ of X satisfies

$$f(x) = \begin{cases} \frac{1}{2}\pi x & \text{for } 0 < x < 1, \\ \frac{1}{2}\pi x - 2x \arccos(\frac{1}{x}) & \text{for } 1 < x < \sqrt{2}. \end{cases}$$

Numerical integration leads to $E(X) = \int_0^{\sqrt{2}} x f(x)\, dx = 0.765$.

4.24 The range of the random variable X is the interval $(0, 0.5)$. Let A be the subset of points from the unit square for which the distance to the closest side of the rectangle is larger than x, where $0 < x < 0.5$. Then A is a square whose sides have the length $1 - 2x$ and so the area of A is $(1 - 2x)^2$. It now follows that

$$P(X \leq x) = 1 - (1 - 2x)^2 \quad \text{for } 0 \leq x \leq 0.5.$$

The probability density $f(x)$ of X is given by $f(x) = 4(1 - 2x)$ for $0 < x < 0.5$ and $f(x) = 0$ otherwise. The expected value of X is $\int_0^{0.5} x\, 4(1 - 2x)\, dx = 0.1667$.

4.25 (**a**) By $P(A \mid B) = P(AB)/P(B)$, we get

$$P(X \leq x \mid X > a) = \frac{\int_a^x f(v)\, dv}{P(X > a)}.$$

Thus the conditional density of X given that $X > a$ is $f(x)/P(X > a)$ for $x > a$ and 0 otherwise. In the same way, the conditional density of X given that $X \leq a$ is $f(x)/P(X \leq a)$ for $x < a$ and 0 otherwise.
(**b**) By $E(X) = \frac{1}{1-a}\int_a^1 x\, dx$, we get $E(X) = \frac{1}{2}(1 + a)$.
(**c**) $E(X \mid X > a) = a + \frac{1}{\lambda}$ and $E(X \mid X \leq a) = \frac{1 - e^{-\lambda a} - \lambda a e^{-\lambda a}}{\lambda(1 - e^{-\lambda a})}$ for any $a > 0$.

4.26 (**a**) By $P(X > x) = \int_x^\infty f(y)\,dy$, it follows that

$$\int_0^\infty P(X > x)\,dx = \int_{x=0}^\infty dx \int_{y=x}^\infty f(y)\,dy.$$

By interchanging the order of integration, the last integral becomes

$$\int_{y=0}^\infty f(y)\,dy \left(\int_{x=0}^y dx \right) = \int_0^\infty y f(y)\,dy = E(X),$$

proving the desired result.
(**b**) Let the random variable X be the smallest of n independent random numbers from $(0,1)$. Then $P(X > x) = (1-x)^n$ for $0 \leq x \leq 1$ and $P(X > x) = 0$ for $x > 1$. This gives $E(X) = \int_0^1 (1-x)^n\,dx = \frac{1}{n+1}$.

4.27 (**a**) The function $g(x) = \frac{1}{x}$ is convex for $x > 0$. Therefore, by Jensen's inequality, $E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)}$. Since $E(X) = \frac{3}{5}$, we get $E\left(\frac{1}{X}\right) \geq \frac{5}{3}$.
(**b**) $E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x}\,12x^2(1-x)\,dx = 2$ and $E[\left(\frac{1}{X}\right)^2] = \int_0^1 \frac{1}{x^2}\,12x^2(1-x)\,dx = 6$, and so $\sigma(\frac{1}{X}) = \sqrt{2}$.

4.28 The random variable U has density function $f(u) = 1$ for $0 < u < 1$ and $f(u) = 0$ otherwise. By the substitution rule, we find for $V = \sqrt{U}$ and $W + U^2$ thatthat

$$E(V) = \int_0^1 \sqrt{u}\,du = \frac{2}{3} \quad \text{and} \quad E(V^2) = \int_0^1 u\,du = \frac{1}{2}$$
$$E(W) = \int_0^1 u^2\,du = \frac{1}{3} \quad \text{and} \quad E(W^2) = \int_0^1 u^4\,du = \frac{1}{5}.$$

Hence, the expected value and standard deviation of V are given by $\frac{2}{3}$ and $\sqrt{\frac{1}{2} - \frac{4}{9}} = 0.2357$. The expected value and standard deviation of W are given by $\frac{1}{3}$ and $\sqrt{\frac{1}{5} - \frac{1}{9}} = 0.2981$.

4.29 Let Y be the amount paid by the supplement policy. Then $Y = g(X)$, where $g(x)$ is $\min(500, x - 450)$ for $x > 450$ and 0 otherwise. By the substitution rule,

$$E(Y) = \int_{450}^{950} (x - 450)\frac{1}{1250}\,dx + 500 \int_{950}^{1250} \frac{1}{1250}\,dx = 220 \text{ dollars}.$$

4.30 Let random variable X be the amount of waste (in thousands of gallons) produced during a week and Y be the total costs incurred during a week. Then the random variable Y can be represented as $Y = g(X)$, where the function $g(x)$ is given by

$$g(x) = \begin{cases} 1.25 + 0.5x & \text{for } 0 < x < 0.9, \\ 1.25 + 0.5 \times 0.9 + 5 + 10(x - 0.9) & \text{for } 0.9 < x < 1. \end{cases}$$

and $g(x) = 0$ otherwise. By the substitution rule, the expected value of the weekly costs is given by

$$E(Y) = 105 \int_0^1 g(x) x^4 (1-x)^2 \, dx = 1.6975.$$

To find the standard deviation of the weekly costs, we first calculate

$$E(Y^2) = \int_0^1 g^2(x) x^4 (1-x)^2 \, dx = 3.6204.$$

Thus the standard deviation of the weekly costs is $\sqrt{E(Y^2) - E^2(Y)} = 0.8597$.

4.31 The net profit $Y = g(X)$, where $g(x) = 2x$ for $0 \leq x \leq 250$ and $g(x) = 2 \times 250 - 0.5(x - 250)$ for $x > 250$. By the substitution rule,

$$E(Y) = \int_0^{250} 2x f(x) \, dx + \int_{250}^{\infty} [500 - 0.5(x - 250)] f(x) \, dx = 194.10$$

dollars. The probability of a stockout is $P(X > 250) = 1 - \int_0^{250} f(x) \, dx = 0.0404$.

4.32 The insurance payment (in thousands of dollars) is a so-called mixed random variable S, where

$$S = \begin{cases} 20 - 1 & \text{with probability } 0.01 \\ \max(0, X - 1) & \text{with probability } 0.02 \\ 0 & \text{with probability } 0.97, \end{cases}$$

where X represents the cost of a repairable damage. The random variable X has the density function $f(x) = \frac{1}{200}(20 - x)$ for $0 < x < 20$. Thus,

$$\begin{aligned} E(S) &= 0.01 \times 19 + 0.02 \Big[\int_0^1 0 \times f(x) dx + \int_1^{20} (x-1) f(x) dx \Big] + 0.97 \times 0 \\ &= 0.19 + 0.02 \int_1^{20} (x-1) \frac{20 - x}{200} \, dx = 0.19 + 0.11432 = 0.30432. \end{aligned}$$

The expected value of the insurance payment is 304.32 dollars.

4.33 Let U be the random point in $(0,1)$ and define $g(u) = 1-u$ if $u < s$ and $g(u) = u$ if $u \geq s$. Then $L = g(U)$ is the length of the subinterval covering the point s. By the substitution rule,

$$E(L) = \int_0^s (1-u)\,du + \int_s^1 u\,du = s - s^2 + \frac{1}{2}.$$

4.34 In the Problems 4.2, 4.4, and 4.6, $E(X)$ has the values $\frac{53}{96}$, 4, and 120 and the second moment $E(X^2)$ has the values 0.3833, 32, and 14404.2 Therefore the standard deviation $\sigma(X)$ has the values 0.2802, 4, and 2.0412.

4.35 The area of the circle is $Y = \pi X^2$, where X has the density function $f(x) = 1$ for $0 < x < 1$. By the substitution rule,

$$E(Y) = \int_0^1 \pi x^2\,dx = \frac{\pi}{3} \text{ and } E(Y^2) = \int_0^1 \pi^2 x^4\,dx = \frac{\pi^2}{5}.$$

The expected value and the standard deviation of Y are $\frac{\pi}{3}$ and $\frac{2\pi}{3\sqrt{5}}$.

4.36 Let the random variable X be the distance from the center of the sphere to the point Q. Using the fact that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$, we get

$$P(X \leq x) = \frac{x^3}{r^3} \quad \text{for } 0 \leq x \leq r.$$

Hence X has the density function $f(x) = \frac{3x^2}{r^3}$ for $0 < x < r$ and $f(x) = 0$ otherwise. The expected value and the standard deviation of the random variable X are $\frac{3}{4}r$ and $\sqrt{3/80}\,r$.

4.37 (**a**) $E\left[(X-c)^2\right] = E(X^2) - 2cE(X) + c^2$ and is minimal for $c = E(X)$, as follows by differentiation. The minimal value is the variance of X. (**b**) $E(|X-c|) = \int_{-\infty}^{c}(c-x)f(x)dx + \int_c^{\infty}(x-c)f(x)dx$. The derivative of $E(|X-c|)$ is $2P(X \leq c) - 1$. The minimizing value of c satisfies $P(X \leq c) = \frac{1}{2}$ and is the median of X.

4.38 The height above the ground is given by the random variable $X = 15 + 15\cos(\Theta)$, where Θ is uniformly distributed on $(0, \pi)$. Using the substitution rule and the relation $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$, we get

$$E(X) = \frac{1}{\pi}\int_0^{\pi}[15 + 15\cos(x)]\,dx = 15$$

$$E(X^2) = \frac{1}{\pi}\int_0^{\pi} 225[1 + 2\cos(x) + \cos^2(x)]\,dx$$

$$= 337.5 + \frac{225}{2\pi}\int_0^{\pi}\cos(2x)\,dx = 337.5.$$

The standard deviation of X is $\sigma(X) = \sqrt{337.5 - 15^2} = 10.61$ meters. *Note*: An alternative method to calculate $E(X^2)$ is to use the density function $h(x)$ of X, see Problem 4.16. However, it seems that numerical integration must be used to obtain the value of $\int_0^{30} x^2 h(x)\,dx$ (as a sanity check, the numerical computation of the integral gives also the answer 337.5). It is much simpler to use the substitution rule to get this answer.

4.39 Let Y be the amount of demand that cannot be satisfied from stock on hand and define $g(x) = x - s$ for $x > s$ and $g(x) = 0$ otherwise. By the substitution rule,

$$E(Y) = \int_s^{\infty}(x - s)\lambda e^{-\lambda x}\,dx \text{ and } E(Y^2) = \int_s^{\infty}(x - s)^2\lambda e^{-\lambda x}\,dx.$$

These integrals can be evaluated as $\frac{1}{\lambda}e^{-\lambda s}$ and $\frac{2}{\lambda^2}e^{-\lambda s}$. This leads to

$$E(Y) = \frac{1}{\lambda}e^{-\lambda s} \text{ and } \sigma(Y) = \frac{1}{\lambda}[e^{-\lambda s}(2 - e^{-\lambda s})]^{1/2}.$$

4.40 (**a**) The expected value of X is given by

$$\begin{aligned} E(X) &= \int_\beta^{\infty} x(\alpha/\beta)(\beta/x)^{\alpha+1}\,dx = \int_\beta^{\infty}\alpha\beta^{\alpha}x^{-\alpha}\,dx \\ &= \frac{\alpha\beta^{\alpha}}{1-\alpha}x^{-\alpha+1}\Big|_\beta^{\infty} = \frac{\alpha\beta}{\alpha - 1}, \end{aligned}$$

provided that $\alpha > 1$; otherwise $E(X) = \infty$. For $\alpha > 2$,

$$\begin{aligned} E(X^2) &= \int_\beta^{\infty} x^2(\alpha/\beta)(\beta/x)^{\alpha+1}\,dx = \int_\beta^{\infty}\alpha\beta^{\alpha}x^{-\alpha+1}\,dx \\ &= \frac{\alpha\beta^{\alpha}}{2-\alpha}x^{-\alpha+2}\Big|_\beta^{\infty} = \frac{\alpha\beta^2}{\alpha - 2}. \end{aligned}$$

For $0 < \alpha \leq 2$, $E(X^2) = \infty$. Thus, for $\alpha > 2$,

$$\mathrm{var}(X) = \frac{\alpha\beta^2}{\alpha - 2} - \left(\frac{\alpha\beta}{\alpha - 1}\right)^2 = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}.$$

For any $\alpha > 0$,

$$P(X \leq x) = \int_\beta^x (\alpha/\beta)(\beta/y)^{\alpha+1}\, dy = 1 - (\beta/x)^\alpha \quad \text{for } x > \beta.$$

Putting $P(X \leq x) = 0.5$ gives the for the median m the value

$$m = 2^{1/\alpha}\beta.$$

(**b**) The mean of the income is 4,500 dollars and the median is 3,402 dollars. The percentage of the population with an income between 25 and 40 thousand dollars is 0.37%, as follows from

$$\begin{aligned} P(25 < X \leq 40) &= P(X \leq 40) - P(X \leq 25) \\ &= (2.5/25)^{2.25} - (2.5/40)^{2.25} = 0.0037. \end{aligned}$$

(**c**) The Pareto distribution shows rather well the way that a larger portion of the wealth in a country is owned by a smaller percentage of the people in that country. The explanation is that the Pareto density $f(x)$ decreases from $x = \beta$ onwards and has a long tail. Thus, most realizations of a Pareto distributed random variable tend to be small but occasionally the realizations will be very large. This is quite typical for income distributions. Also, the Pareto distribution has the property that the mean is always larger than the median.

4.41 The age of the bulb upon replacement is $Y = g(X)$, where $g(x) = x$ for $x \leq 10$ and $g(x) = 10$ for $x > 10$. Then $E(Y) = \int_2^{10} x\,\frac{1}{10}\,dx + \int_{10}^{12} 10\,\frac{1}{10}\,dx$ and $E(Y^2) = \int_2^{10} x^2\,\frac{1}{10}\,dx + \int_{10}^{12} 10^2\,\frac{1}{10}\,dx$. This leads to $E(Y) = 6.8$ and $\sigma(Y) = 2.613$.

4.42 Let the random variable X be the thickness of a sheet of steel and Y be the thickness of a non-scrapped sheet of steel. Then

$$P(Y > y) = P(X > y \mid X > 125) \quad \text{for } 125 \leq y \leq 150.$$

The random variable X is uniformly distributed on $(120, 150)$ and so $P(X > x) = \frac{150-x}{30}$ for $120 \leq x \leq 150$. This implies that

$$P(Y > y) = \frac{150 - y}{25} \quad \text{for } 125 \leq y \leq 150.$$

In other words, the random variable Y is uniformly distributed on $(125, 150)$. Hence the expected value and the standard deviation of a non-scrapped sheet of steel are given by $\frac{125+150}{2} = 137.5$ millimeters and $\frac{25}{\sqrt{12}} = 7.217$ millimeters.

4.43 Since $P\big(\min(X,Y) > t\big) = P(X > t,\, Y > t) = P(X > t)P(Y > t)$, we get

$$P\big(\min(X,Y) > t\big) = e^{-\alpha t}e^{-\beta t} = e^{-(\alpha+\beta)t} \qquad \text{for all } t > 0,$$

and so $\min(X,Y)$ is exponentially distributed. Using this result and the memoryless property of the exponential distribution, we have that the time to failure of the reliability system is distributed as $T_1+T_2+T_3$, where T_1, T_2 and T_3 are independent and exponentially distributed with respective parameters 5λ, 4λ and 3λ. Thus

$$E(T_1 + T_2 + T_3) = \frac{1}{5\lambda} + \frac{1}{4\lambda} + \frac{1}{3\lambda} = \frac{47}{60\lambda}$$

$$\sigma(T_1 + T_2 + T_3) = \Big(\frac{1}{25\lambda^2} + \frac{1}{16\lambda^2} + \frac{1}{9\lambda^2}\Big)^{0.5} = \frac{\sqrt{769}}{60\lambda}.$$

4.44 By the memoryless property of the exponential distribution, the time from three o'clock in the afternoon until the next departure of a limousine has an exponential distribution with an expected value of 20 minutes. Using the fact that the standard deviation of an exponential density is the same as the expected value of the density, the expected value and the standard deviation of your waiting time are both equal to 20 minutes.

4.45 Since the sojourn time of each bus is exactly half an hour, the number of buses on the parking lot at 4 p.m is the number of buses arriving between 3:30 p.m and 4 p.m. Taking the hour as unit of time, the buses arrive according to a Poisson process with rate $\lambda = \frac{4}{3}$. Using the memoryless property of the Poisson process, the number of buses arriving between 3:30 p.m and 4 p.m is Poisson distributed with expected value $\lambda \times \frac{1}{2} = \frac{2}{3}$.

4.46 Take the hour as unit of time. The average number of arrivals per hour between 6 p.m and 10 p.m is 1.2. The random variable X measuring the time from 6 p.m until the first arrival after 6 p.m is exponentially distributed with parameter $\lambda = 1.2$. Hence the expected value of X is $\frac{1}{1.2} = \frac{10}{12}$ hours or 50 minutes. The median of X follows by solving

$1 - e^{-1.2x} = 0.5$ and is equal to $-\ln(0.5)/1.2 = 0.5776$ hours or 34.66 minutes. The probability that the first call occurs between 6:20 p.m and 6:45 p.m is given by

$$P\left(\frac{1}{3} \leq X \leq \frac{3}{4}\right) = e^{-1.2\times 1/3} - e^{-1.2\times 3/4} = 0.2638.$$

Let the random variable Y be the time measured from 7 p.m until the first arrival after 7 p.m. The probability of no arrival between 7 p.m and 7:20 p.m and at least one arrival between 7:20 p.m and 7:45 p.m is $P(\frac{1}{3} < Y \leq \frac{3}{4})$. By the memoryless property of the exponential distribution, the random variable Y has the same exponential distribution as X. Hence the probability $P(\frac{1}{3} < Y \leq \frac{3}{4})$ is also equal to $e^{-1.2\times 1/3} - e^{-1.2\times 3/4} = 0.2638$.

4.47 The probability that the time between the passings of two consecutive cars is more than c seconds is given by $p = \int_c^\infty \lambda e^{-\lambda t}\, dt = e^{-\lambda c}$. By the lack of memory of the exponential distribution, $p = e^{-\lambda c}$ gives also the probability that no car comes around the corner during the c seconds measured from the moment you arrive at the road. The number of passing cars before you can cross the road has the shifted geometric distribution $\{(1-p)^k p,\, k = 0, 1, \ldots\}$.

4.48 By the lack of memory of the exponential distribution, the remaining washing time of the car being washed in the station has the same exponential density as a newly started washing time. Hence the probability that the car in the washing station will need no more than five other minutes is equal to

$$\int_0^5 \frac{1}{15} e^{-\frac{1}{15}t}\, dt = 1 - e^{-5/15} = 0.2835.$$

The probability that you have to wait more than 20 minutes before your car can be washed is equal to $P(X_1 + X_2 > 20)$, where X_1 is the remaining service time of the car in service when you arrive and X_2 is the service time of the other car. The random variables X_1 and X_2 are independent. By the memoryless property of the exponential distribution, X_1 has the same exponential distribution as X_2. The random variable $X_1 + X_2$ has an Erlang-2 distribution and the sought probability is given by

$$P(X_1 + X_2 > 20) = e^{-20/15} + \frac{20}{15} e^{-20/15} = 0.6151.$$

Alternatively, this answer can be seen from Rule 4.3 by noting that $P(X_1 + X_2 > 20)$ is the probability of at most one service completion in the 20 minutes.

4.49 The probability of having a replacement because of a system failure is given by

$$\sum_{n=0}^{\infty} P\big(nT < X \leq (n+1)T - a\big) = \sum_{n=0}^{\infty} \big(e^{-\mu nT} - e^{-\mu[(n+1)T-a]}\big).$$

This probability is equal to $(1 - e^{-\mu(T-a)})/(1 - e^{-\mu T})$. The expected time between two replacements is

$$\sum_{n=1}^{\infty} nTP\big((n-1)T < X \leq nT)\big) = \frac{T}{1 - e^{-\mu T}}.$$

4.50 The probability that the closest integer to the random observation is odd is equal to

$$\sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2}) = \sum_{k=0}^{\infty} \int_{2k+\frac{1}{2}}^{2k+1+\frac{1}{2}} e^{-x}\, dx$$

$$= \sum_{k=0}^{\infty} \big[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})}\big] = e^{-\frac{1}{2}} \left(\frac{1 - e^{-1}}{1 - e^{-2}}\right) = \frac{e^{-\frac{1}{2}}}{1 + e^{-1}}.$$

The conditional probability that the closest integer to the random observation is odd given that it is larger than the even integer r is equal to

$$\sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2} \mid X > r)$$

$$= \frac{1}{P(X > r)} \sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2},\ X > r)$$

$$= \frac{1}{e^{-r}} \sum_{k=r/2}^{\infty} \int_{2k+\frac{1}{2}}^{2k+1+\frac{1}{2}} e^{-x}\, dx = \frac{1}{e^{-r}} \sum_{k=r/2}^{\infty} \big[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})}\big]$$

Since $\sum_{k=r/2}^{\infty} \big[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})}\big] = e^{-r} \sum_{l=0}^{\infty} \big[e^{-(2l+\frac{1}{2})} - e^{-(2l+1+\frac{1}{2})}\big]$, the conditional probability that the closest integer to the random observation is odd given that it is larger than r is equal to

$$\sum_{l=0}^{\infty} \big[e^{-(2l+\frac{1}{2})} - e^{-(2l+1+\frac{1}{2})}\big] = e^{-\frac{1}{2}} \left(\frac{1}{1 - e^{-2}} - \frac{e^{-1}}{1 - e^{-2}}\right) = \frac{e^{-\frac{1}{2}}}{1 + e^{-1}}.$$

The conditional probability is the same as the unconditional probability that the closest integer to the random observation from the exponential density is odd. This result can also be explained from the memoryless property of the exponential distribution.

4.51 Your win probability is the probability of having exactly one signal in (s, T). This probability is $e^{-\lambda(T-s)}\lambda(T-s)$, by the memoryless property of the Poisson process. Putting the derivative of this expression equal to zero, we get that the optimal value of s is $T - \frac{1}{\lambda}$. The maximal win probability is e^{-1}.

4.52 Let $N(t)$ be the number of events to occur in $(0, t)$. Then,

$$P(N(a) = k \mid N(a+b) = n) = \frac{P(N(a) = k,\, N(a+b) - N(a) = n-k)}{P(N(a+b) = n)}$$

for any $0 \leq k \leq n$. We have $P(N(a) = k,\, N(a+b) - N(a) = n-k) = P(N(a) = k)P(N(a+b) - N(a) = n-k)$, by the independence of $N(a)$ and $N(b) - N(a)$. Thus, for $k = 0, 1, \ldots, n$,

$$\begin{aligned} P(N(a) = k \mid N(a+b) = n) &= \frac{e^{-\lambda a}(\lambda a)^k/k! \times e^{-\lambda b}(\lambda b)^{n-k}/(n-k)!}{e^{-\lambda(a+b)}\big(\lambda(a+b)\big)^n/n!} \\ &= \binom{n}{k}\Big(\frac{a}{a+b}\Big)^k\Big(\frac{b}{a+b}\Big)^{n-k}. \end{aligned}$$

In view of the characteristic properties of the Poisson process, it is not surprising that the conditional distribution of $N(a)$ is the binomial distribution with parameters n and $\frac{a}{a+b}$.

4.53 Take the minute as time unit. Let $\lambda = \frac{8}{60}$ and $T = 10$. The probability that the ferry will leave with two cars is $1 - e^{-\lambda T} = 0.7364$. Let the generic variable X be exponentially distributed with an expected value of $\frac{1}{\lambda} = 7.5$ minutes. The expected value of the time until the ferry leaves is

$$\frac{1}{\lambda} + E\big(\min(X, T)\big) = \frac{1}{\lambda} + \int_0^T t\lambda e^{-\lambda t}\, dt + T\int_T^{\infty} \lambda e^{-\lambda t}\, dt$$

minutes. This leads to an expected value of $\frac{1}{\lambda} + \frac{1}{\lambda}(1 - e^{-\lambda T}) = 13.02$ minutes.

4.54 Noting that major cracks on the highway occur according to a Poisson process with rate $\frac{1}{10}$, it follows that the probability that there are no major cracks on a specific 15-mile stretch of the highway is $e^{-15/10} = 0.2231$ and the probability of two or more major cracks on that part of the highway is $1 - e^{-15/10} - (15/10)e^{-15/10} = 0.4422$.

4.55 In view of Rule 4.3, we can think of failures occurring according to a Poisson process with a rate of 4 per 1,000 hours. The probability of no more than five failures during 1,000 hours is given by the Poisson probability

$$\sum_{k=0}^{5} e^{-4}\,\frac{4^k}{k!} = 0.7851.$$

The smallest value of n such that $\sum_{k=0}^{n} e^{-4}\,\frac{4^k}{k!} \geq 0.95$ is $n = 8$.

4.56 The probability of no bus arriving during a wait of t minutes at the bus stop is $e^{-t/10}$. Putting $e^{-t/10} = 0.05$ gives $t = 29.96$. You must leave home no later than 7:10 a.m.

4.57 (**a**) Since the number of goals in the match is Poisson distributed with an expected value of $90 \times \frac{1}{30} = 3$, the answer is $1 - \sum_{k=0}^{2} e^{-3}\,\frac{3^k}{k!} = 0.5768$.
(**b**) The numbers of goals in disjoint time intervals are independent of each other and so the answer is $e^{-1.5}\,\frac{1.5^2}{2!} \times e^{-1.5}\,1.5 = 0.0840$.
(**c**) Let, for $k = 0, 1, \ldots$,

$$a_k = e^{-3\times(12/25)}\,\frac{(3\times(12/25))^k}{k!} \quad \text{and} \quad b_k = e^{-3\times(13/25)}\,\frac{(3\times(13/25))^k}{k!}.$$

Then, by the results of Rule 3.12, we get that the probability of a draw is equal to $\sum_{k=0}^{\infty} a_k \times b_k = 0.2425$ and the probability of a win for team A is equal to $\sum_{k=1}^{\infty} a_k \times \sum_{n=0}^{k-1} b_n = 0.3524$.

4.58 The probability of having no other emergence unit within a distance r of the incident is given by the probability of no emergence unit in a circle with radius r around the point of the incident. The probability of no Poisson event in a region with area πr^2 is $e^{-\alpha\pi r^2}$ and so the desired probability is $1 - e^{-\alpha\pi r^2}$.

4.59 The answer is $\big(1 - \Phi(\frac{20}{16})\big) \times 100\% = 10.56\%$.

4.60 The solution of $1-\Phi(x) = 0.05$ is given by the percentile $z_{0.95} = 1.6449$. Thus the cholesterol level of $5.2 + 1.6449 \times 0.65 = 6.27$ mmol/L is exceeded by 5% of the population.

4.61 An estimate for the standard deviation σ of the demand follows from the formula $50+\sigma z_{0.95}=75$, where $z_{0.95}=1.6449$ is the 95% percentile of the standard normal distribution. This gives the estimate $\sigma=1.2$.

4.62 The proportion of euro coins that are not accepted by the vending machine is

$$\Phi\Big(\frac{22.90-23.25}{0.10}\Big)+1-\Phi\Big(\frac{23.60-23.25}{0.10}\Big)=2[1-\Phi(3.5)]=0.0046.$$

4.63 By $P(X<20)=P(X\leq 20)=P\big(\frac{X-25}{2.5}\leq\frac{20-25}{2.5}\big)=\Phi(-2)$, we have $P(X<20)=0.0228$. Finding the standard deviation σ of the thickness of the coating so that $P(X<20)=0.01$ translates into solving σ from $\Phi\big(\frac{20-25}{\sigma}\big)=0.01$. The 0.01th percentile of the $N(0,1)$ distribution is -2.3263, and so $-5/\sigma=-2.3263$, or $\sigma=2.149$.

4.64 The proportion of the mills output that can be used by the customer is equal to

$$\Phi\Big(\frac{10.15-10}{0.07}\Big)-\Phi\Big(\frac{9.85-10}{0.07}\Big)=0.9839.$$

4.65 We have $P(|X-\mu|>k\sigma)=P(|Z|>k)=P(Z\leq -k)+P(Z\geq k)$, where Z is $N(0,1)$ distributed. Since $P(Z\geq k)=P(Z\leq -k)$ and $P(Z\geq k)=1-\Phi(k)$, the sought result follows.

4.66 Let the random variable $Y=aX+b$. To evaluate $P(Y\leq y)$, distinguish between the two cases $a\geq 0$ and $a<0$. For the case that $a\geq 0$,

$$\begin{aligned}P(Y\leq y) &= P\left(X\leq\frac{y-b}{a}\right)=P\left(\frac{X-\mu}{\sigma}\leq\frac{y-b-a\mu}{a\sigma}\right)\\ &= \Phi\left(\frac{y-b-a\mu}{a\sigma}\right),\end{aligned}$$

showing that Y is $N(a\mu+b,a^2\sigma^2)$ distributed. For the case that $a<0$,

$$\begin{aligned}P(Y\leq y) &= P\left(X\geq\frac{y-b}{a}\right)=P\left(\frac{X-\mu}{\sigma}\geq\frac{y-b-a\mu}{a\sigma}\right)\\ &= 1-\Phi\left(\frac{y-b-a\mu}{a\sigma}\right)=1-\Phi\left(\frac{-y+b+a\mu}{|a|\sigma}\right).\end{aligned}$$

Using the fact that $\Phi(-x)=1-\Phi(x)$ for any $x>0$, it next follows that $P(Y\leq y)=\Phi\left(\frac{y-b-a\mu}{|a|\sigma}\right)$. In other words, Y is $N(a\mu+b,a^2\sigma^2)$ distributed.

4.67 The number of heads in 10,000 tosses of a fair coin is approximately normally distributed with expected value 5,000 and standard deviation 50. The outcome of 5,250 heads lies five standard deviations above the expected value. Without doing any further calculations we can conclude that the claim is highly implausible ($1-\Phi(5) = 2.87\times 10^{-7}$).

4.68 (**a**) For any $z \geq 0$, we have $P(|Z| \leq z) = P(-z \leq Z \leq z) = \Phi(z)-\Phi(-z)$. Differentiation gives that $|Z|$ has the probability density function

$$\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad \text{for } z > 0.$$

Using the change of variable $v = z^2$, we get

$$E(|Z|) = \int_0^\infty z\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}v}\,dv = \frac{\sqrt{2}}{\sqrt{\pi}}.$$

Also, noting that $E(|Z|^2) = E(Z^2)$ and using the fact that $E(Z^2) = 1$ for the $N(0,1)$ distributed random variable Z, we get $E(|Z|^2) = 1$. This gives $\sigma^2(|Z|) = 1 - \frac{2}{\pi}$ and so $\sigma(|Z|) = \sqrt{1-\frac{2}{\pi}}$.
(**b**) Let $V = \max(Z-c, 0)$. Since $E(V) = \int_0^\infty P(V > v)\,dv$ (see Problem 4.26), we have

$$E(V) = \int_0^\infty [1-\Phi(v+c)]\,dv = \int_c^\infty [1-\Phi(x)]\,dx.$$

By partial integration, we next get

$$E(V) = x[1-\Phi(x)]\Big|_c^\infty + \frac{1}{\sqrt{2\pi}}\int_c^\infty xe^{-\frac{1}{2}x^2}\,dx.$$

The integral $\int_c^\infty xe^{-\frac{1}{2}x^2}\,dx$ is $\int_{\frac{1}{2}c^2}^\infty e^{-y}\,dy = e^{-\frac{1}{2}c^2}$. Thus we get

$$E(V) = -c[1-\Phi(c)] + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}c^2}.$$

4.69 Since $X-Y$ is $N(0, 2\sigma^2)$ distributed, $(X-Y)/(\sigma\sqrt{2})$ is $N(0,1)$ distributed. Thus, using Problem 4.68, $E(|X-Y|) = \sigma\sqrt{2}\sqrt{2/\pi}$. Also, $E(X+Y) = 2\mu$. The formulas for $E(|X-Y|)$ and $E(X+Y)$ give two equations in $E[\max(X,Y)]$ and $E[\min(X,Y)]$, yielding the sought result.
Note: $\max(X,Y)$ and $\min(X,Y)$ have each standard deviation $\sigma\sqrt{1-\frac{1}{\pi}}$.

4.70 The random variable D_n can be represented as

$$D_n = |X_1 + \cdots + X_n|,$$

where the random variable X_i is equal to 1 if the ith step of the drunkard goes to the right and is otherwise equal to -1. The random variables $X_1, \ldots, X_n$ are independent and have the same distribution with expected value $\mu = 0$ and standard deviation $\sigma = 1$. The central limit theorem now tells us that $X_1+\cdots+X_n$ is approximately normally distributed with expected value 0 and standard deviation $\sqrt{n}$ for n large. Thus,

$$P(D_n \leq x) \approx \Phi\left(\frac{x}{\sqrt{n}}\right) - \Phi\left(\frac{-x}{\sqrt{n}}\right) \qquad \text{for } x > 0.$$

In Problem 4.68, the expected value and the standard deviation of $V = |X|$ are given for a standard normally distributed random variable X. Using this result and the fact that $(X_1+\cdots+X_n)/\sqrt{n}$ is approximately $N(0,1)$ distributed, the approximations for the expected value and the standard deviation of D_n follow.

4.71 Let X_1 and X_2 be the two measurement errors. Since X_1 and X_2 are independent, $\frac{1}{2}(X_1 + X_2)$ is normally distributed with expected value 0 and standard deviation $\frac{1}{2}\sqrt{0.006^2 l^2 + 0.004^2 l^2} = l\sqrt{52}/2{,}000$. The sought probability is

$$\begin{aligned} P\Big(\frac{1}{2}|X_1 + X_2| \leq 0.005l\Big) &= P(-0.01l \leq X_1 + X_2 \leq 0.01l) \\ &= \Phi\Big(\frac{20}{\sqrt{52}}\Big) - \Phi\Big(-\frac{20}{\sqrt{52}}\Big) = 0.9945. \end{aligned}$$

4.72 The desired probability is $P(|X_1 - X_2| \leq a)$. Since the random variables X_1 and X_2 are independent, the random variable $X_1 - X_2$ is normal distributed with expected value $\mu = \mu_1 - \mu_2$ and standard deviation $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$. It now follows that

$$\begin{aligned} P(|X_1 - X_2| \leq a) &= P(-a \leq X_1 - X_2 \leq a) \\ &= P\left(\frac{-a-\mu}{\sigma} \leq \frac{X_1 - X_2 - \mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{a - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) - \Phi\left(\frac{-a - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right). \end{aligned}$$

4.73 (**a**) The profit of Joe and his brother after 52 weeks is $\sum_{i=1}^{52} X_i$, where the X_i are independent with $P(X_i = 10) = \frac{1}{2}$ and $P(X_i = -5) = \frac{1}{2}$. The expected value and the standard deviation of the X_i are 2.5 and $\sqrt{62.5 - 2.5^2} = 7.5$ dollars. The sought probability is

$$P\Big(\sum_{i=1}^{52} X_i \geq 100\Big) \approx 1 - \Phi\Big(\frac{100 - 52 \times 2.5}{7.5\sqrt{52}}\Big) = 0.7105.$$

(**b**) Let X_i be the score of the ith roll. Then

$$P\Big(\sum_{i=1}^{80} X_i \leq 300\Big) \approx \Phi\Big(\frac{300 - 80 \times 3.5}{1.7078\sqrt{80}}\Big) = 0.9048.$$

4.74 The random variable $Y_n = \frac{1}{n}(X_1 + \cdots + X_n) - \mu$ has expected value 0 and variance $\frac{\sigma^2}{n}$. Using the central limit theorem, Y_n is approximately $N(0, \frac{\sigma^2}{n})$ distributed for n large. Since $P(|Y_n| > c) = P(Y_n < -c) + P(Y_n > c)$, we have

$$P(|Y_n| > c) \approx \Phi\Big(\frac{-c\sqrt{n}}{\sigma}\Big) + 1 - \Phi\Big(\frac{c\sqrt{n}}{\sigma}\Big) = 2\Big[1 - \Phi\Big(\frac{c\sqrt{n}}{\sigma}\Big)\Big].$$

4.75 (**a**) The number of sixes in one throw of $6r$ dice is distributed as the binomial random variable $S_{6r} = \sum_{k=1}^{6r} X_k$, where the X_k are independent 0−1 variables with expected value $\mu = \frac{1}{6}$ and standard deviation $\sigma = \frac{1}{6}\sqrt{5}$. We have

$$P(S_{6r} \geq r) = P\Big(\frac{S_{6r} - 6r\mu}{\sigma\sqrt{6r}} \geq 0\Big).$$

By the central limit theorem, this probability tends to $1 - \Phi(0) = \frac{1}{2}$ as $r \to \infty$.

(**b**) Let $X_1, \ldots X_n$ be independent and Poisson distributed random variables with expected value 1. The sum $X_1 + \cdots + X_n$ is Poisson distributed with expected value n. Therefore,

$$P(X_1 + \cdots + X_n \leq n) = e^{-n}\Big(1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\Big)$$

Next repeat the arguments in (**a**).

4.76 The probability is about $1 - \Phi(2.828) = 0.0023$.

4.77 The number of even numbers in any given drawing of the lotto 6/45 has a hypergeometric distribution with expected value $\mu = \frac{132}{45}$ and standard deviation $\sigma = \frac{1}{15}\sqrt{299}$. By the central limit theorem, the total number of even numbers that will be obtained in 52 drawings of the lotto 6/45 is approximately normally distributed with expected value $52\mu = 152.533$ and standard deviation $\sigma\sqrt{52} = 8.313$. The outcome 162 lies $(162 - 152.533)/8.313 = 1.14$ standard deviations above the expected value. This outcome does not cast doubts on the unbiased nature of the lotto drawings.

4.78 The probability distribution of the total rainfall (in millimeters) next year in Amsterdam can be modeled by a normal distribution with expected value 799.5 and standard deviation 121.39. The sought probability is equal to

$$1 - \Phi\big(\frac{1000 - 799.5}{121.39}\big) = 0.0493.$$

4.79 The payoff per game has an expected value of $\mu = 12$ dollars and a standard deviation of $\sigma = \sqrt{6{,}142 - 144} = 77.447$ dollars. By the central limit theorem, the probability of the casino losing money in a given week is approximately

$$\Phi\Big(-\frac{5,000 \times 3}{77.447\sqrt{5,000}}\Big) = 1 - \Phi(2.739) = 3.1 \times 10^{-3}.$$

4.80 The total number of bets lost by the casino is $X_1 + \cdots + X_n$, where the random variable X_i is equal to 1 if the casino loses the ith bet and X_i is otherwise equal to 0. We have $E(X_i) = p$ and $\sigma(X_i) = \sqrt{p(1-p)}$. By the central limit theorem, $X_1 + \cdots + X_n$ has approximately a normal distribution with expected value np and standard deviation $[p(1-p)]^{\frac{1}{2}}\sqrt{n}$ for large n. The casino loses money to the player if and only if the casino loses $\frac{1}{2}n + 1$ or more bets (assume that n is even). The probability of this is approximately equal to $1 - \Phi\left(\beta_n\right)$, where

$$\beta_n = \frac{\frac{1}{2}n + 1 - np}{\big(p(1-p)\big)^{1/2}\sqrt{n}}.$$

The loss probability is about 0.1876, 0.0033, and 6.1×10^{-18} for $n =$ 1,000, 10,000 and 100,000. Assuming one dollar is staked on each bet, then for n plays the profit of the casino over the gamblers equals $W_n = n - 2\,(X_1 + \cdots + X_n)$. We have

$$E(W_n) = n(1-2p) \quad \text{and} \quad \sigma(W_n) = 2[p(1-p)]^{\frac{1}{2}}\sqrt{n}.$$

The random variable W_n is approximately normally distributed for large n The standard normal density has 99% of its probability mass to the right of point -2.326. This means that, with a probability of approximately 99%, the profit of the casino over the player is greater than $n(1-2p) - 2.326 \times 2[p(1-p)]^{\frac{1}{2}}\sqrt{n}$.

4.81 The premium c should be chosen such that $P\big(rc - (X_1 + \cdots + X_n) \geq \frac{1}{10}rc\big)$ is at least 0.99, where X_i is the amount claimed by the ith policy holder. This probability can be approximated by $\Phi\big((\frac{9}{10}rc - r\mu)/(\sigma\sqrt{r})\big)$. Thus c should be chosen such that $(\frac{9}{10}rc - r\mu)/(\sigma\sqrt{r})$ equals the 0.99th percentile 2.326 of the standard normal distribution. Therefore,

$$c \approx \frac{10}{9}\Big(\mu + \frac{2.326\sigma}{\sqrt{r}}\Big).$$

4.82 The probability mass function of the number of copies of the appliance to be used when an infinite supply would be available is a Poisson distribution with expected value of $\frac{150}{2} = 75$. Suppose that Q copies of the appliance are stored in the space ship. Let the exponentially distributed random variable X_i be the lifetime (in days) of the ith copy used. Then the probability of a shortage during the space mission is $P(X_1 + \cdots + X_Q \leq 150)$. The random variables $X_1, \ldots, X_Q$ are independent and have an expected value of $\frac{1}{\lambda}$ days and a standard deviation of $\frac{1}{\lambda}$ days, where $\lambda = \frac{1}{2}$. By the central limit theorem,

$$\begin{aligned} P(X_1 + \cdots + X_Q \leq 150) &= P\left(\frac{X_1 + \cdots + X_Q - 2Q}{2\sqrt{Q}} \leq \frac{150 - 2Q}{2\sqrt{Q}}\right) \\ &\approx \Phi\left(\frac{150 - 2Q}{2\sqrt{Q}}\right). \end{aligned}$$

The 0.001th percentile of the standard normal distribution is -3.0902. Solving the equation $\frac{150-2Q}{2\sqrt{Q}} = -3.0902$ gives $Q = 106.96$ and so the normal approximation suggests to store 107 units.
Note: The exact value of the required stock follows by finding the smallest value of Q for which the Poisson probability$\sum_{k>Q} e^{-75}\frac{75^k}{k!}$ is smaller than or equal to 10^{-3}. This gives $Q = 103$.

4.83 Let X_i be the amount of dollars the casino owner loses on the ith bet. Then the X_i are independent random variables with $P(X_i = 10) = \frac{18}{37}$ and $P(X_i = -5) = \frac{19}{37}$. Then, $E(X_i) = \frac{85}{37}$ and $\sigma(X_i) = \frac{45}{37}\sqrt{38}$. The amount of dollars lost by the casino owner is $\sum_{i=1}^{2{,}500} X_i$ and is approximately $N(\mu, \sigma^2)$ distributed with $\mu = 2{,}500 \times \frac{85}{37}$ and $\sigma = 50 \times \frac{45}{37}\sqrt{38}$. The casino owner will lose more than 6,500 dollars with a probability of about

$$1 - \Phi\Big(\frac{6{,}500 - \mu}{\sigma}\Big) = 0.0218.$$

4.84 By a change to polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with $dxdy = rdrd\theta$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy &= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{1}{2}r^2} r\,dr\,d\theta \\ &= 2\pi\int_0^{\infty} re^{-\frac{1}{2}r^2}\,dr = \pi\int_0^{\infty} e^{-\frac{1}{2}r^2}\,dr^2 = 2\pi,\end{aligned}$$

using the fact that $\int_0^{\infty} e^{-\frac{1}{2}y}dy = 2$. This proves the result $I = \sqrt{2\pi}$. *Note*: This result implies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The change of variable $t = \frac{1}{2}x^2$ in $I = 2\int_0^{\infty} e^{-\frac{1}{2}x^2}dx$ leads to $\sqrt{2\pi} = \frac{2}{\sqrt{2}}\int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt$, showing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4.85 Let V_n be the bankroll (in dollars) of the gambler after the nth bet. Then $V_n = (1-\alpha)V_{n-1} + \alpha V_{n-1}R_n$, where $\alpha = 0.05$, $V_0 = 1{,}000$ and the R_i are independent random variables with $P(R_i = \frac{1}{4}) = \frac{19}{37}$ and $P(R_i = 2) = \frac{18}{37}$. Iterating this equality gives $V_n = (1-\alpha+\alpha R_1) \times \cdots \times (1-\alpha+\alpha R_n)V_0$. This leads to

$$\ln(V_n/V_0) = \sum_{i=1}^{n} \ln(1-\alpha+\alpha R_i).$$

The random variables $X_i = \ln(1-\alpha+\alpha R_i)$ are independent. The expected value and the variance of these random variables are

$$\begin{aligned}\mu &= \frac{19}{37}\ln(0.9625) + \frac{18}{37}\ln(1.05) \\ \sigma^2 &= \frac{19}{37}\ln^2(0.9625) + \frac{18}{37}\ln^2(1.05) - \mu^2.\end{aligned}$$

By the central limit theorem, the random variable $\ln(V_{100}/V_0)$ is approximately $N(100\mu, 100\sigma^2)$ distributed (the gambler's bankroll after 100 bets is approximately lognormally distributed). The probability that the gambler will take home more than d dollars is $P(V_n > V_0 + d) = P\big(\ln(V_n/V_0) > \ln(1 + d/V_0)\big)$. This probability is approximately equal to

$$1 - \Phi\Big(\frac{\ln(1 + d/V_0) - 100\mu}{10\sigma}\Big)$$

and has the values 0.8276, 0.5494, 0.2581, and 0.0264 for $d = 0$, 500, 1,000, and 2,500.

4.86 Denoting by the random variable F_n the factor at which the size of the population changes in the nth generation, the size S_n of the population after n generations is distributed as $(F_1 \times \cdots \times F_n)s_0$. By the central limit theorem,

$$\ln(S_n) = \sum_{i=1}^{n} \ln(F_i) + \ln(s_0)$$

has approximately a normal distribution with expected value $n\mu_1 + \ln(s_0)$ and standard deviation $\sigma_1\sqrt{n}$ for n large, where μ_1 and σ_1 are the expected value and the standard deviation of the $\ln(F_i)$. The numerical values of μ_1 and σ_1 are given by

$$\begin{aligned}\mu_1 &= 0.5\ln(1.25) + 0.5\ln(0.8) = 0\\ \sigma_1 &= \sqrt{0.5[\ln(1.25)]^2 + 0.5[\ln(0.8)]^2} = 0.22314.\end{aligned}$$

Since $\ln(S_n)$ has approximately a normal distribution with expected value $\ln(s_0)$ and standard deviation $0.22314\sqrt{n}$, the probability distribution of S_n can be approximated by a lognormal distribution with parameters $\mu = \ln(s_0)$ and $\sigma = 0.22314\sqrt{n}$.

4.87 The distance can be modeled as $\sqrt{X^2 + Y^2}$, where X and Y are independent $N(0, 1)$ random variables. The random variable $X^2 + Y^2$ has the chi-square density $f(v) = \frac{1}{2}e^{-\frac{1}{2}v}$. We have

$$P(\sqrt{X^2 + Y^2} \le r) = P(X^2 + Y^2 \le r^2) = \int_0^{r^2} f(x)\,dx \quad \text{for } r > 0.$$

Hence the probability density of the distance from the center of the target to the point of impact is $2rf(r^2) = re^{-\frac{1}{2}r^2}$ for $r > 0$. The expected value of the distance is $\int_0^\infty r^2 e^{-\frac{1}{2}r^2}\,dr = \frac{1}{2}\sqrt{2\pi}$. The density

$re^{-\frac{1}{2}r^2}$ assumes its maximum value $1/\sqrt{e}$ at $r = 1$. Thus the mode of the distance is 1. The median of the distance is $\sqrt{2\ln(2)}$, as follows by solving the equation $\int_0^x re^{-\frac{1}{2}r^2}\,dr = 0.5$.

4.88 If the random variable X is positive, the result follows directly from Rule 4.6 by noting that $\frac{1}{x}$ is strictly decreasing for $x > 0$ and has $-\frac{1}{x^2}$ as its derivative. If X can take on both positive and negative values, we use first principles. Then, by $P(Y \le y) = P\big(\frac{1}{X} \le y\big)$, we have

$$P(Y \le y) = \begin{cases} P(X \le 0) + P\big(0 < X \le \frac{1}{y}\big) & \text{for } y > 0 \\ P\big(\frac{1}{y} \le X \le 0\big) & \text{for } y \le 0. \end{cases}$$

Differentiation gives that Y has the density function $\frac{1}{y^2} f\big(\frac{1}{y}\big)$. The desired result next follows by noting that $\frac{1}{y^2}\frac{1}{\pi(1+1/y^2)} = \frac{1}{\pi(1+y^2)}$.

4.89 The inverse of the function $y = \frac{1}{2}ms^2$ is $s = \sqrt{\frac{2y}{m}}$. We have $\frac{ds}{dy} = \sqrt{\frac{1}{2ym}}$. An application of Rule 4.6 gives that the probability density of the kinetic energy E is

$$\frac{2}{c^3}\sqrt{\frac{y}{\pi m}}\, e^{-my/c^2} \qquad \text{for } y > 0.$$

4.90 The conditions of Rule 4.6 are not satisfied for the random variable $\ln(|X|^a)$. Noting that $0 < |X|^a < 1$ and so $\ln(|X|^a) < 0$, we get by first principles that

$$P(\ln(|X|^a) \le x) = P(\ln(|X|) \le \frac{x}{a}) = P(|X| \le e^{x/a}) \ \text{ for } x \le 0.$$

Therefore, using the fact that X is uniformly distributed on $(-1, 1)$,

$$P(\ln(|X|^a) \le x) = P(-e^{x/a} \le X \le e^{x/a}) = \frac{2e^{x/a}}{2} \ \text{ for } x \le 0.$$

This shows that the probability density of $\ln(|X|^a)$ is $\frac{1}{a}e^{x/a}$ for $x < 0$ and is 0 otherwise.
Note: For $a < 0$, the probability density of $\ln(|X|^a)$ is $-\frac{1}{a}e^{x/a}$ for $x > 0$ and 0 otherwise. This result readily follows by noting that $\ln(|X|^a) = -\ln(|X|^{-a})$ for $a < 0$.

4.91 It follows from $P(Y \leq y) = P\big(X \leq \frac{\ln(y)}{\ln(10)}\big)$ that

$$P(Y \leq y) = \frac{\ln(y)}{\ln(10)} \quad \text{for } 1 \leq y \leq 10.$$

Next, differentiation shows that Y has the density function $\frac{1}{\ln(10)y}$ for $1 < y < 10$.

4.92 (**a**) The Weibull distributed random variable has the cumulative probability distribution function $F(x) = 1 - e^{-(\lambda x)^\alpha}$ for $x \geq 0$. Letting u be random number between 0 and 1, the solution of the equation $F(x) = u$ gives the random observation $x = \frac{1}{\lambda}[-\ln(1-u)]^{1/\alpha}$ from the Weibull distribution. Since $1 - u$ is also a random number between 0 and 1, one can also take

$$x = \frac{1}{\lambda}[-\ln(u)]^{1/\alpha}$$

as a random observation from the Weibull distribution. The Weibull distribution with $\alpha = 1$ is the exponential distribution.
(**b**) Let u be random number between 0 and 1, then solving the equation $F(x) = u$ for $F(x) = e^x/(1 + e^x)$ gives $e^x = \frac{u}{1-u}$. Thus $x = \ln\big(\frac{u}{1-u}\big)$ is a random observation from the logistics distribution.

4.93 Generate n random numbers $u_1, \ldots, u_n$ from the interval $(0, 1)$. Then the number $-\frac{1}{\lambda}\big[\ln(u_1) + \cdots + \ln(u_n)\big] = -\frac{1}{\lambda}\ln(u_1 \times \cdots \times u_n)$ is a random observation from the gamma distributed random variable.

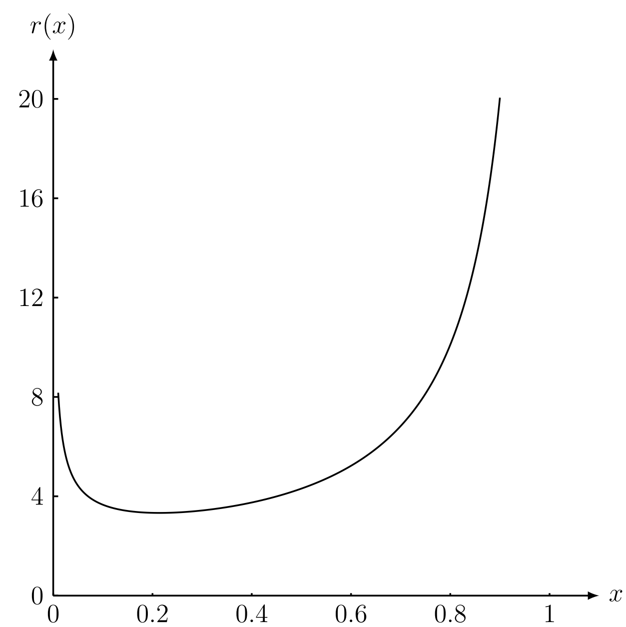
4.94 It is easiest to consider first the case of a random variable V having a triangular density on $(0, 1)$ with m_0 as its most likely value. The density function $f(v)$ of V is given by $f(v) = 2v/m_0$ for $0 < v \leq m_0$ and $f(v) = 2(1 - v)/(1 - m_0)$ for $m_0 < v < 1$. The probability distribution function of V is

$$P(V \leq v) = \begin{cases} \frac{v^2}{m_0} & \text{for } 0 \leq v \leq m_0 \\ \frac{2v - v^2 - m_0}{1 - m_0} & \text{for } m_0 \leq v \leq 1. \end{cases}$$

To obtain random observation from V, generate a random number u between 0 and 1 and solve v from the equation $P(V \leq v) = u$. The solution is $v = \sqrt{m_0 u}$ if $0 < u \leq m_0$ and $v = 1 - \sqrt{(1 - m_0)(1 - u)}$ for $m_0 < u < 1$. If X has a triangular density on (a, b) with m as its most likely value, then the random variable $V = (X - a)/(b - a)$ has a triangular density on $(0,1)$ with $m_0 = (m-a)(b-a)$. Hence a random observation from X is given by $x = a + (b-a)\sqrt{m_0 u}$ if $0 < u \leq m_0$ and $x = a + (b-a)[1 - \sqrt{(1-m_0)(1-u)}]$ if $m_0 < u < 1$.

4.95 (**a**) The random variable X is with probability p distributed as an exponential random variable with parameter λ_1 and with probability $1-p$ as an exponential random variable with parameter λ_2. Hence generate two random numbers u_1 and u_2 from $(0,1)$. The random observation is $-(1/\lambda_1)\ln(u_1)$ if $u_1 \leq p$ and is $-(1/\lambda_2)\ln(u_2)$ otherwise.
(**b**) Let V be exponentially distributed with parameter λ. Then the random variable X is with probability p distributed as V and with probability $1-p$ as $-V$. Hence generate two random numbers u_1 and u_2 from $(0,1)$. The random observation is $-\frac{1}{\lambda}\ln(u_2)$ if $u_1 \leq p$ and $\frac{1}{\lambda}\ln(u_2)$ otherwise. This simulation method is called the composition method.

4.96 Apply the definition $r(x) = \frac{f(x)}{1-F(x)}$ to get the expression for $r(x)$. It is a matter of elementary but tedious algebra to show that $r(x)$ is first decreasing and then increasing with a bathtub shape. As an illustration of the bathtub shape of $r(x)$, we give in the figure the graph of $r(x)$ for $\nu = 0.5$.



4.97 Since $P(V > x) = P(X_1 > x, \ldots X_n > x)$, it follows from the independence of the X_k and the failure rate representation of the reliability function that

$$\begin{aligned} P(V > x) &= P(X_1 > x) \cdots P(X_n > x) \\ &= e^{-\int_0^x r_1(y)\,dy} \ldots e^{-\int_0^x r_n(y)\,dy} \quad \text{for } x \geq 0. \end{aligned}$$

This gives $P(V > x) = e^{-\int_0^x [\sum_{k=1}^n r_k(y)]dy}$, proving the desired result.

4.98 Noting that $1 - F(x) = e^{-\int_0^x r(t)dt}$, we get

$$1 - F(x) = e^{-\ln(1+x)} = \frac{1}{1+x} \quad \text{for } x \geq 0.$$

4.99 Let $F(x) = P(X \leq x)$. Since

$$\int_0^x r(t)\,dt = \lambda \int_0^x d\ln(1+t^\alpha) = \lambda \ln(1+x^\alpha),$$

it follows that $F(x) = 1 - e^{-\lambda \ln(1+x^\alpha)}$. Thus the reliability function is given by

$$1 - F(x) = (1+x^\alpha)^{-\lambda} \quad \text{for } x \geq 0.$$

The derivative $r'(x) = (\alpha - 1 - x^\alpha)/(1+x^\alpha)^2$. For the case that $\alpha > 1$, the derivative is positive for $x < (\alpha-1)^{1/\alpha}$ and negative for $x > (\alpha-1)^{1/\alpha}$, showing that $r(x)$ first increases and then decreases.

4.100 Let $F_i(x) = P(X_i \leq x)$. Then

$$P(X_i > s+t \mid X_i > s) = \frac{1 - F(s+t}{1 - F(s)} = \frac{e^{-\int_0^{s+t} r_i(x)dx}}{e^{-\int_0^s r_i(x)dx}}.$$

Since $r_1(x) = \frac{1}{2} r_2(x)$, we get

$$P(X_1 > s+t \mid X_1 > s) = \frac{e^{-\frac{1}{2}\int_0^{s+t} r_2(x)dx}}{e^{-\frac{1}{2}\int_0^s r_2(x)dx}} = \Big(\frac{e^{-\int_0^{s+t} r_2(x)dx}}{e^{-\int_0^s r_2(x)dx}}\Big)^{\frac{1}{2}}.$$

showing that $P(X_1 > s+t \mid X_1 > s) = \sqrt{P(X_2 > s+t \mid X_2 > s)}$.

4.101 The integral $\int_{1{,}000}^\infty e^{-(y/1{,}250)^{2.1}}\,dy / e^{-(1{,}000/1{,}250)^{2.1}}$ gives the mean residual life time $m(1{,}000)$. By numerical integration, $m(1{,}000) = 516.70$ hours.

4.102 In order to minimize $-\sum_{i=1}^{\infty} p_i \log p_i$ subject to $p_i \geq 0$ for all i, $\sum_{i=1}^{\infty} p_i = 1$ and $\sum_{i=1}^{\infty} ip_i = \mu$, form the Lagrange function

$$F(p_1, p_2, \ldots, \lambda_1, \lambda_2) = -\sum_{i=1}^{\infty} p_i \log p_i + \lambda_1 \Big(\sum_{i=1}^{\infty} p_i - 1 \Big) + \lambda_2 \Big(\sum_{i=1}^{\infty} ip_i - \mu \Big),$$

where λ_1 and λ_2 are the Lagrange multipliers. Putting $\partial F / \partial p_i = 0$ gives the equations $-1 - \log p_i + \lambda_1 + \lambda_2 i = 0$ for $i \geq 1$ and so

$$p_i = e^{\lambda_1 - 1 + \lambda_2 i} \quad \text{for } i \geq 1.$$

The condition $\sum_{i=1}^{\infty} p_i = 1$ implies that $\lambda_2 < 0$ and $e^{\lambda_1 - 1} = (1 - e^{\lambda_2})\, e^{-\lambda_2}$. Hence we have $p_i = (1 - e^{\lambda_2})\, e^{\lambda_2 (i-1)}$ for $i \geq 1$. Letting $r = 1 - e^{\lambda_2}$, we get the geometric distribution $p_i = (1-r)^{i-1} r$ for $i \geq 1$. The condition $\sum_{i=1}^{\infty} ip_i = \mu$ implies that $\frac{1}{r} = \mu$ and so $r = \frac{1}{\mu}$.

4.103 Form the Lagrange function

$$F(p_1, \ldots, p_n, \lambda_1, \lambda_2) = -\sum_{i \geq 1} p_i \log p_i + \lambda_1 \Big(\sum_{i=1}^{n} p_i - 1 \Big) + \lambda_2 \Big(\sum_{i=1}^{n} p_i E_i - E \Big),$$

where λ_1 and λ_2 are Lagrange multipliers. Putting $\partial F / \partial p_i = 0$ results in $-1 - \log p_i + \lambda_1 + \lambda_2 E_i = 0$ and so $p_i = e^{\lambda_1 - 1 + \lambda_2 E_i}$ for all i. Substituting p_i into the constraint $\sum_{i=1}^{n} p_i = 1$ gives $e^{\lambda_1 - 1} = 1 / \sum_{i=1}^{n} e^{\lambda_2 E_i}$. Thus

$$p_i = \frac{e^{\lambda_2 E_i}}{\sum_{k=1}^{n} e^{\lambda_2 E_k}} \quad \text{for all } i.$$

Substituting this into $\sum_{i=1}^{n} p_i E_i = E$ gives the equation

$$\sum_{i=1}^{n} E_i e^{\lambda_2 E_i} - E \sum_{k=1}^{n} e^{\lambda_2 E_k} = 0$$

for λ_2. Replacing λ_2 by $-\beta$, we get the desired expression for the p_i^* and the equation for the unknown β.

4.104 Apply Rule 10.8 with $E_1 = 4.50$, $E_2 = 6.25$, $E_3 = 7.50$, and $E = 5.75$. Using a numerical root-finding method, we obtain $\beta = 0.218406$. The maximum entropy probabilities of a customer ordering a regular cheeseburger, a double cheeseburger and a big cheeseburger are 0.4542, 0.30099 and 0.2359.