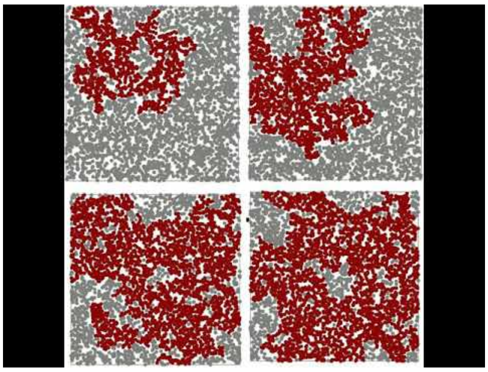
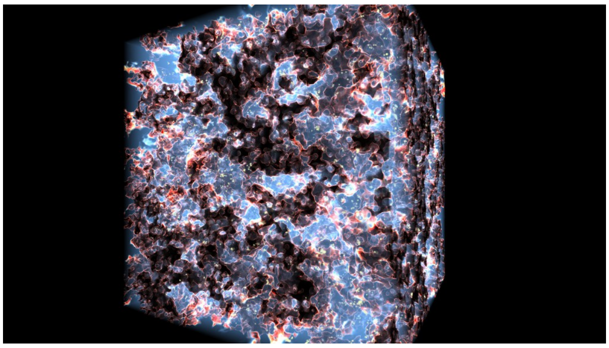
Lecture 8
Percolation



Continuum percolation

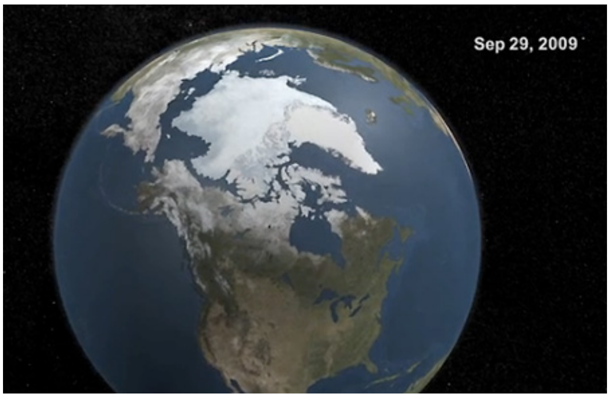
Credit: Youtube



Percolation of ionized regions in the Universe.

Credit Alvarez, Abel & Kaehler

Percolation in the real world



Time lapse of freezing of ocean.

Credit: YouTube & Nasa

Percolation & a model for cluster growth

- Previous lecture: random walks and its connection to diffusion
- Related process: Cluster growth

  a 'cluster' is a set of nearest-neighbour particles - or 'friends of friends' particles

  applications are, for example, growth of a cancer cell, a snowflake, or an iceberg

  'elements' or 'particles' are the individual components of a cluster - e.g. individual cells (cancer), water molecules (snow flake)

  - Start from a seed a cluster with one 'particle'
  - Add particles to current cluster according to some rules
  - Cluster grows larger structure emerges
- We will discuss the Eden and the DLA model for cluster growth

  DLA = diffusion limited aggregation

  Restrict discussion to 2 dimensional clusters, with particles arranged on a regular grid

Cluster growth: Eden model

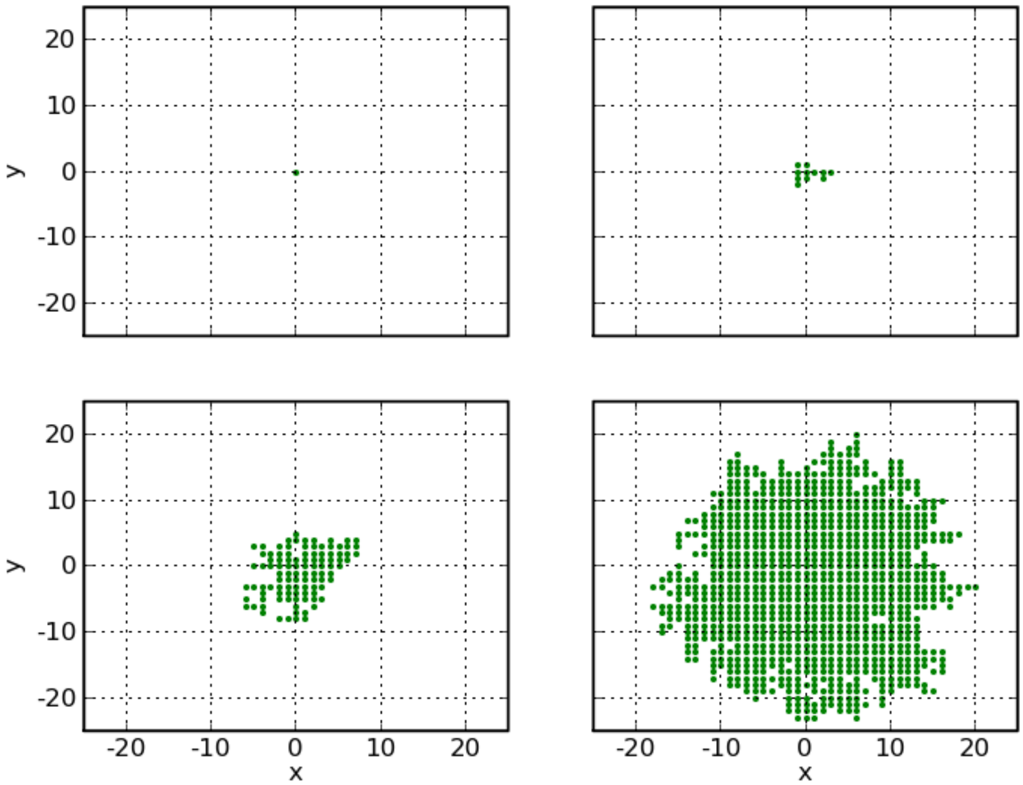
a useful description for the growth of a tumour of cancer cells - hence also known as the 'cancer' model

- Start with a seed for example at location $(x, y) = (0, 0)$
- Growth rule: any unoccupied nearest neighbour is equally eligible for growth

  'nearest neighbour' means differs by $\pm$ one step in x xor y (xor is exclusive or)

  pick any unoccupied nearest neighbour at random, and grow cluster
- Repeat until cluster is finished, according to predefined size, number of sites included, $\cdots$
- Note: unoccupied neighbours can also refer to holes inside the cluster
- After many steps, cluster is approximately circular, with a somewhat "fuzzy" edge and some holes in it.

Cluster growth: Eden model



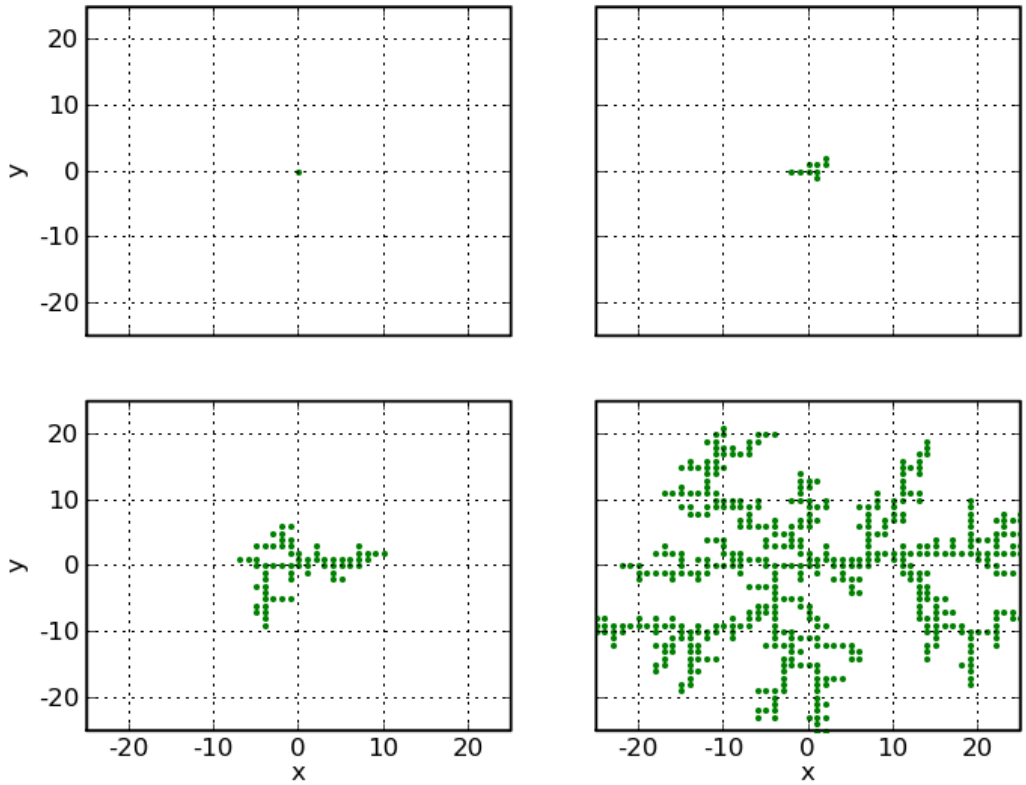
Cluster growth: DLA model

DLA = diffusion limited aggregation - useful model for describing growth of a snowflake

- Start with a seed for example at location $(x, y) = (0, 0)$, as in the Eden model
- Growth rule: Initialise a random walker at a large enough distance from the cluster and let it walk. The random walker is added to cluster when it hits it
- Repeat as in Eden model
- For efficient implementation: discard random walkers moving too far away or "direct" the walk towards the cluster.
- Resulting cluster has significantly different shape from an Eden cluster: a fluffy object of irregular shape, with filaments delineating large empty regions.

  Once a DLA cluster develops a hole, it becomes unlikely or even impossible for the hole to be filled in

Cluster growth: DLA model



Cluster shape: fractal dimension

Eden and DLA clusters have very different shape. One way to quantify a shape is by computing its fractal dimension. We will develop an operational (in contrast to a strictly mathematical) definition to describe this concept.

- What is the dimension of an Eden or a DLA cluster?

  seems like a silly question - they are both 2D structures! So consider following examples.

  - The mass of a disc with radius r is

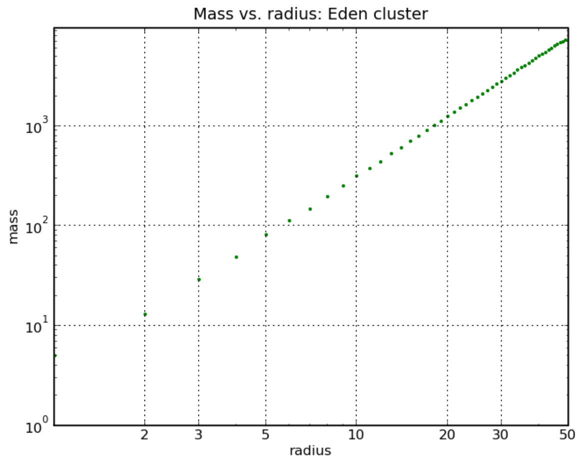
    $$m(r) \propto r^2$$

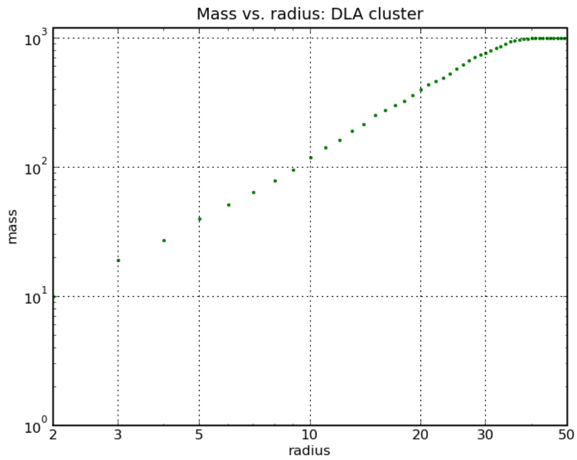
  - The mass of a straight rod with length r is

    $$m(r) \propto r^1 \tag{1}$$

- Therefore $m \propto r^2$ (2D object, e.g disc) and $m \propto r$ (1D object, e.g. rod)
- Suppose an object has $m(r) \propto r^d$
- d is called the fractal dimension of the object

Cluster shape: fractal dimension





Eden cluster: $m \propto r^{1.99} \to d = 1.99$

DLA cluster: $m \propto r^{1.65} \to d = 1.65$

mass increases as a power law in radius, until it saturates (becomes a constant) at large r when whole cluster is inside r and hence mass stops increasing with increasing r

Cluster shape: fractal dimension

- $d \approx 1.99$ for an Eden cluster
- $d \approx 1.65$ for a DLA cluster

fits with our expectations: Eden cluster is almost a disc: its fractal dimension should be close to 2

DLA cluster more filamentary: has smaller dimension than a disc

Percolation

- Percolation as a physical process: for example the *percolation* of ground water through soil, *percolation* of oil oozing through a porous rock, a coffee *percolator*

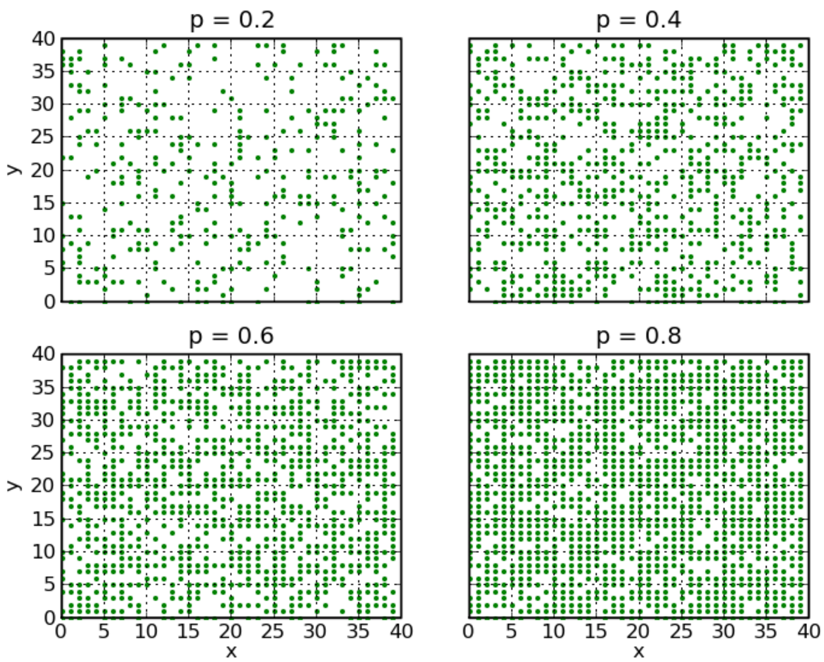
  Fluid follows a path through the substrate (ground, rock, coffee). In on one side, out on the other side.
- Random processes where cells with a finite size within an area or volume are filled or activated closely related to cluster growth
- Large number of applications in science and industry
- Closely related to the physics of phase transitions

  e.g. liquid-solid transition when water freezers, but also ferro-magnetism discussed in next lecture
- We'll simplify: lattice has finite size, the sites are filled randomly, etc. clearly can do better to get more realistic model

Percolation

place green elements randomly on lattice, with some probability p. Clearly closely connected to cluster growth.

- Lattice of dimension 40×40.



for p large enough, large cluster *percolates* - i.e. crosses the whole lattice left to right, or top to bottom, or both

Percolation: phenomenology

- Definition: cluster is structure of connected sites
  'sites' are the particles discussed previously.

  'Connected' means mutual nearest neighbours, $\Delta x = \pm 1$ xor $\Delta y = \pm 1$
- At $p = 0.2$, most clusters contain 1-2 sites
- At $p = 0.4$, most clusters contain 8-10 sites
- At $p = 0.8$, most sites are in a single, large, cluster
- Interesting transition occurs around a value of $p = 0.6$:
  - Typically at this value the first percolating cluster emerges a cluster that connects at least two opposite sides of the lattice. Whether or not there is a percolating cluster depends on actual distribution of occupied sites
  - Presence of such a cluster indicates percolation
  - Often, cluster stops percolating when only a few sites are removed
  - Stated differently: Occupancy of single sites determines average cluster size $\Longrightarrow$ a phase transition between percolation and no percolation. See also next lecture on the Ising model.

Percolation: numerical analysis

- Analyse emergence of percolating cluster as function of p.
- Transition between two regimes no percolating cluster present, or percolating cluster present is sharp and depends on size of lattice d in addition to p
- In the limit of $L \to \infty$ L is the size of the grid the critical probability for appearance of a percolating cluster can be shown to be $p_c \approx 0.593$.

  comfortingly close to the value of $p = 0.6$ we noticed
- How can we verify this numerically? need method to identify clusters
- Simple brute force method: keep increasing number of sites, until a percolating cluster emerges. Record value of p. Repeat process many times.

Percolation: numerical analysis

pseudo-code for cluster identification

Main routine:

1. Begin with an empty lattice, label all sites as empty, '0'
2. Pick a site at random, label it as occupied, '1'

   this is the first site hence the first cluster - label clusters consecutively
3. Repeat step 2 until a percolating cluster emerges
   Pick a site at random. Check for occupied neighbours:
   - No neighbours → new cluster & new integer label for clusters
   - Neighbour(s) → add to existing cluster or join existing clusters
4. If a spanning cluster has emerged keep track of p_c, the fraction of occupied sites.

in workshop: variation of this scheme

Percolation: numerical analysis

Cluster labelling - pseudo-code (cont'd)

Bridging sites:

For every bridging site (BS), examine occupied neighbours:

1. One occupied neighbour: BS inherits label of adjacent cluster.
2. Two or more occupied neighbours:
   - Calculate *minimum* label of adjacent cluster labels, i
   - BS inherits this number
   - All adjacent clusters inherent label i
     $\Longrightarrow$merged cluster has unique label, i

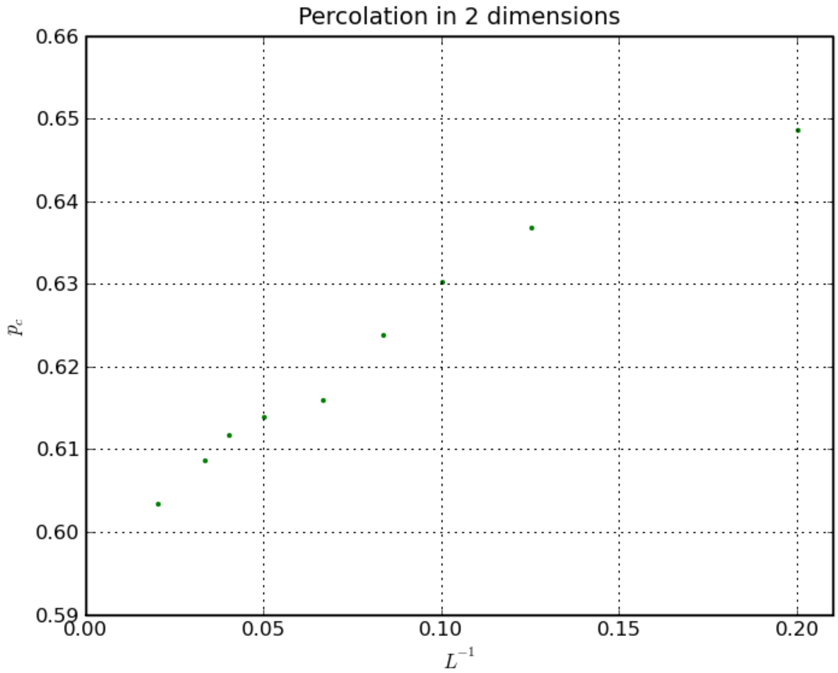
Examine presence of a percolating cluster:

For each cluster keep track of sites at edge of grid - need four booleans (top,bottom,left,right)

Percolation: numerical analysis

value of critical probability, p_c, as a function of $1/L$, the *inverse* of the number of 1D lattice sites

- Lattice with dimension L, sampled over $50L$ runs.
- Statistical fluctuations, linear fit in agreement with 0.593 for $1/L \to 0$



Percolation and phase transitions

- Examine the behaviour near the percolation threshold p_c
- Second order phase transition (first derivative jumps).

  first-order phase transition involves latent heat, for example ice to liquid water. Both states (ice and liquid water) are present at the transition.

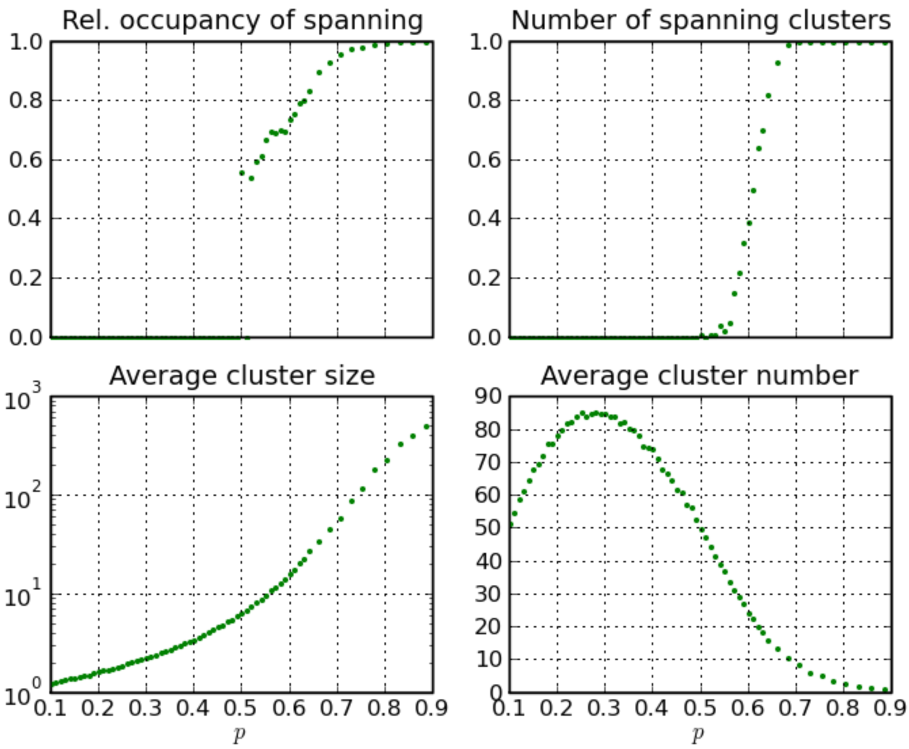
  second-order phase transition: substance is in either one state, or the other
- Here: transition between macroscopically connected and disconnected phases.
- Typical for phase transitions: Singular behaviour of some properties, often described by power laws.
- Property in this case is the fraction of occupied elements that is in the percolating clusters

  therefore percolation is a second-order phase-transition: percolating cluster is either present, or not present!

Percolation and phase transitions

'rel. occupancy of spanning' is the ratio F, where F is the number of sites in the percolating cluster / total number of occupied sites

- Results for $L = 25$ square lattices



Percolation and phase transitions

- Number of percolating clusters and their relative occupancy drops very steeply at around 0.6.
- Write this as $N = N_0(p - p_0)^\gamma$ N is number of percolating clusters and $F = F_0(p - p_0)^\beta$ where p_0, N_0, γ, F_0 and β are fitting parameters
- β, γ known as critical exponents (more in lecture 8)
- Guess: $\mathrm{d}\{F, N\}/\mathrm{d}p \to \infty$ for $p \to p_0$.
- For infinitely large lattices, finite size effects are unimportant, and $p_0 \to p_c$ as $d \to \infty$.
- Also for $d \to \infty$: for **all** two-dimensional lattices

  $$\beta = 5/36.$$
- Also: $F \to 0$ for $d \to \infty$
  $\Longleftrightarrow$ percolating cluster has infinite size but zero volume -
  a fractal!

Summary

- Discussed two closely related processes related to random processes:
  - Cluster growth models (Eden and DLA models)
  - Percolation
- Introduced new way of classifying objects: fractal dimension

  discussed how to identify clusters in a grid of occupied cells - computationally expensive operation
- Described percolation in terms of a phase transition

  more in next lecture
- Interesting outcome: apparently simply models exhibit surprisingly complex behaviour
- Workshop: develop cluster growth models for Eden and DLA cluster. Examine percolation on a 2D lattice