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New algorithm for Half-plane Intersection and its Practical Value

— Thesis for Chinese Team Selecting Contest 2006

半平面交的新算法及其实用价值

——中国代表队 2006 年选拔赛论文

By Zeyuan Zhu

声明：因为 Word 兼容问题，本次提交论文分为两个版本，均为中英文。

DOC 版——因为链接错误，而不得不删除部分英文文字链接，算法主保留

PDF 版——我的完整英文版论文配合中文翻译

New algorithm for Half-plane Intersection and its Practical Value

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Abstract

主旨:半平面的交是当今学术界热烈讨论的问题之一，本文将介绍一个全新的 O(nlogn)半平面交算法，强调它在实际运用中的价值，并且在某种程度上将复杂度下降至 O(n)线性。最重要的是，我将介绍的算法非常便于实现.

§1 introduces what half-plane intersection is. §2 prepares a linear algorithm for convex polygon intersection (**abbr. CPI**). Equipped with such knowledge, a common solution for HPI is briefly discussed in §3. Then, my new algorithm emerges in §4 detailedly. Not only as a conclusion of the whole paper, §5 also discuss its further usage practically and compares it with the older algorithm described in §3.

§1 什么是半平面交. §2 凸多边形交预备知识. §3 简要介绍旧 D&C 算法. §4 揭开我的新算法 S&I 神秘面纱. §5 总结和实际运用.

Timestamps: Came up with it in April 2005; implemented partly in June 2005[2]; problem set in July 2005[3]; publicized as a post in USENET, November 6, 2005[4].

[1] E-mail: zhuzeyuan[at]hotmail[dot]com
[2] The sub-problem of HPI appeared in USA Invitational Computing Olympiad contest.
[3] Set an HPI problem in Peking University Online Judge, with brief introduction about the algorithm.
[4] URL: http://groups.google.com/group/sci.math/msg/68e9d9fc634f0d92

1. Introduction

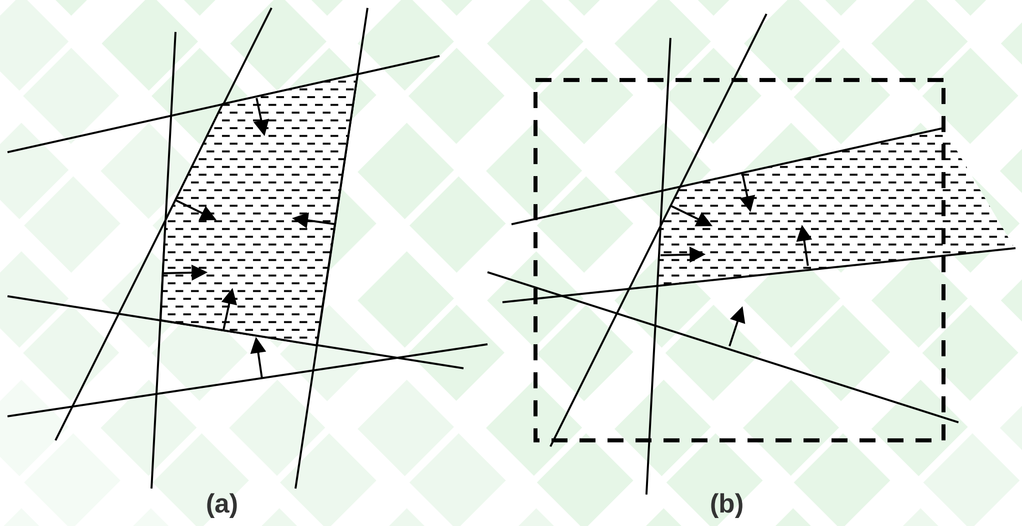
A line in plane is usually represented as *ax+by=c*. Similarly, its inequality form *ax+by≤c* represents a ***half-plane*** (also named ***h-plane*** for short) as one side of this line. Notice that *ax+by≤c* and *-ax-by≤-c* show opposite h-planes unlike *ax+by=c* and *-ax-by=-c*. Half-Plane Intersection (**abbr. HPI**) considers the following problem:

众所周知，直线常用 ax+by=c 表示，类似地半平面以 ax+by≤(≥)c 为定义。

Given *n* half-planes, $a_ix+b_iy\le c_i$ (1≤i≤n), you are to determine the set of all points that satisfying all the *n* inequations. 给定 n 个形如 $a_ix+b_iy\le ci$ 的半平面，找到所有满足它们的点所组成的点集

As **Figure i** Two examples of HPI. (b) gives an example of unbounded intersection area. describes, the feasible region, which is the intersection, forms a shape of convex hull but possibly unbounded. However, we shall add four h-planes forming a rectangle, which is large enough to make sure the area after intersections finite. *In the following sections, we suppose the feasible region is **bounded** with a finite area.*

合并后区域形如凸多边形，可能无界。此时增加 4 个半平面保证面积有限



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Each h-plane builds up at most one side of the convex polygon, hence, the convex region will be bounded by at most *n* edges. Pay attention the intersection sometimes yields a line, a ray, a line-segment, a point or an empty region. 每个半平面最多形成相交区域的一条边，因此相交区域不超过 n 条边。

注意相交后的区域，有可能是一个直线、射线、线段或者点，当然也可能是空集。

2. Convex Polygon Intersection (abbr. CPI)

Intersecting two convex polygons **A** and **B** into a single one, can be properly solved via Line Segment Intersection in $O(n\log n)$ time, when there are $O(n)$ edges. We will sketch out an easier and more efficient way, named ***plane sweep method***.

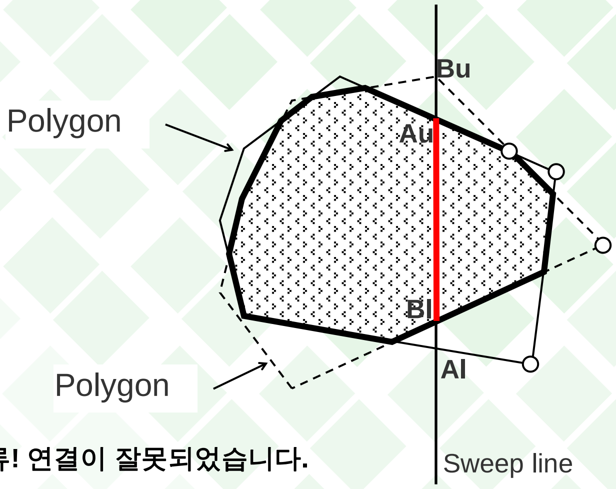
求两个凸多边形 A 和 B 的交（一个新凸多边形）。我们描绘一个平面扫描法。

The main idea is to calculate intersections of edges as cutting points, and break boundaries of A and B, into *outer edges* and *inner edges*. The segments of *inner edges* establish ties to each other, and form the shape of a polygon, which is the expected polygon after intersection. In **Figure ii** How the sweep line works. "○" potentially shows next x-event., *inner edges* are indicated by thick segments, which form a bold outline of the intersection. 主要思想: 以两凸边形边的交点为分界点，将边分为内、外两种。内边互相连接，成为所求多边形。

Suppose there is a vertical sweep line, performing left-to-right sweep. The x-coordinates to be swept are called ***x-events***. At anytime, there are at most four intersections from sweep line to either given polygon[5]:

假设有一个垂直的扫描线，从左向右扫描。我们称被扫描线扫描到的 x 坐标叫做 x 事件。任何时刻，扫描线和两个多边形最多 4 个交点

- to the upper hull of **A** (name that intersection **Au** for short)
- to the lower hull of **A** (name that intersection **Al** for short)
- to the upper hull of **B** (name that intersection **Bu** for short)
- to the lower hull of **B** (name that intersection **Bl** for short)



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[5] Assume there is no edge in polygons parallel to the sweep line. If such situation happens, we could rotate the plane in proper angle, or else, we need good sense to judge a great many special cases.

Look at **Figure ii** How the sweep line works. “○” potentially shows next x-event., the lower one between intersections **Au** and **Bu**, and the upper one between intersections **Al** and **Bl**, form an interval of the current inner region – the red segment in bold. Au、Bu 中靠下的，和 Al、Bl 中靠上的，组成了当前多边形的内部区域。

Obviously, the sweep line may not go through all the x-event with rational coordinates. Call the edges where **Au**, **Al, Bu** and **Bl** are: ***e1, e2, e3*** and ***e4*** respectively. The next x-event should be chosen among four endpoints of ***e1*, *e2*, *e3*** and ***e4***, and four potential intersections: ***e1*∩*e3***, ***e1*∩*e4***, ***e2*∩*e3*** and ***e2*∩*e4***.

当然，我们不能扫描所有有理数！称 Au, Al, Bu, Bl 所在的边叫做 e1,e2,e3,e4，下一个 x 事件将在这四条边的端点，以及两两交点中选出。

The above operation could be implemented with $O(n)$ running time, since there are $O(n)$ x-events, and the maintenance of **Au**, **Al, Bu** and **Bl** takes only $O(1)$.

3. Common solution: Divide-and-Conquer Algorithm (abbr. D&Q)

The basic approach is simple, depends on divide-and-conquer idea.

- DIVIDE: Partition the n h-planes into two sets of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.
- CONQUER: Compute the feasible region (intersection) recursively of both two sub-sets.
- COMBINE: Compute the intersection of two convex polygons, by linear CPI algorithm described in §2.
- 分: 将 n 个半平面分成两个 n/2 的集合.
- 治: 对两子集合递归求解半平面交.
- 合: 将前一步算出来的两个交(凸多边形)利用第 2 章的 CPI 求解.

The total time complexity of the solution can be calculated via recursive equation: 总时间复杂度可以用递归分析法.

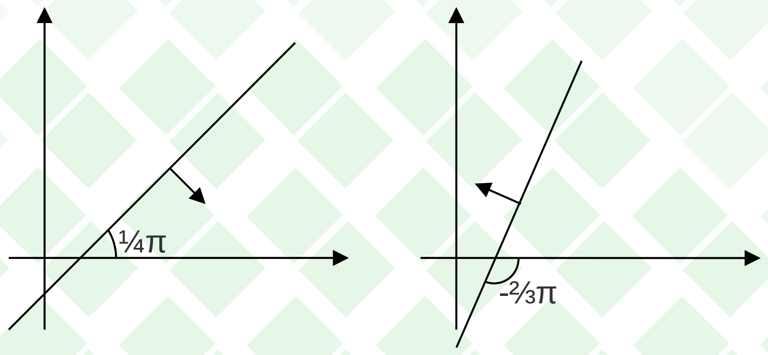
$$T(n) = 2T(\tfrac{n}{2}) + O(n)$$
$$T(n) = O(n \log n)$$

4. My New Solution: Sort-and-Incremental Algorithm (abbr. S&I)

Definition of h-plane’s polar angle:

- for the h-plane like *x-y≥constant,* we define its polar angle to ¼π.
- for the h-plane like *x+y≤constant,* we define its polar angle to ¾π.
- for the h-plane like *x+y≥constant,* we define its polar angle to -¼π.
- for the h-plane like *x-y≤constant,* we define its polar angle to -¾π.

半平面的极角定义: 比如 x-y　常数的半平面，定义它的极角为¼π.



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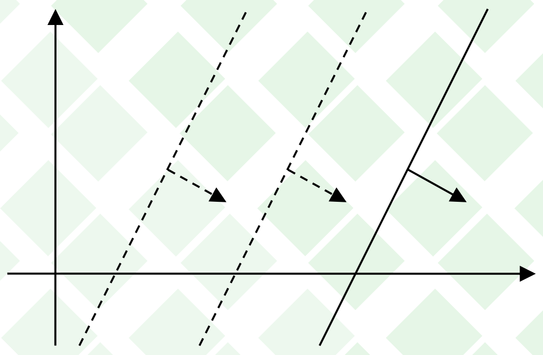
Definition of h-plane’s constant:

- for the h-plane like *ax+by≤c*, we say its constant is *c*.

My new ***Sort-and-Incremental Algorithm*** seems lengthy since I am going to introduce it in details:

Step 1: 将半平面分成两部分，一部分极角范围(-⅟π, ⅟π]，另一部分范围(-π, -⅟π]∪(⅟π, π]



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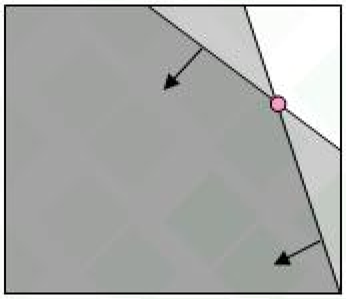
Step 2:考虑(-⅟π, ⅟π]的半平面(另一个集合类似地做 Step3/4)，将他们极角排序。对极角相同的半平面，根据常数项保留其中之一。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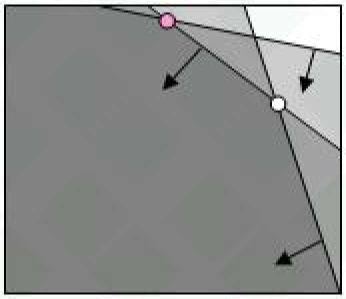
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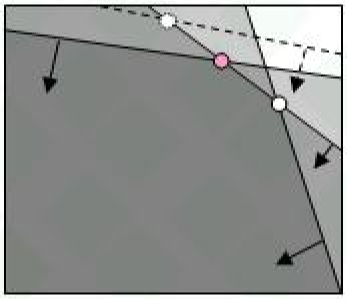
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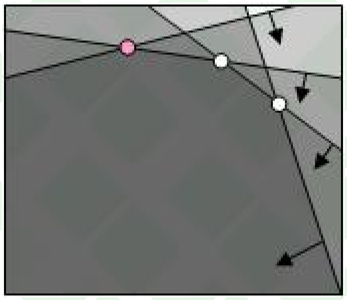
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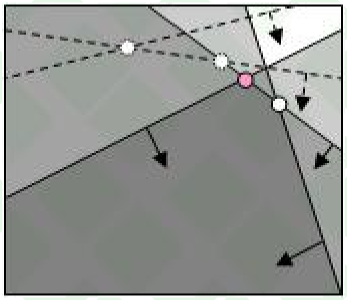
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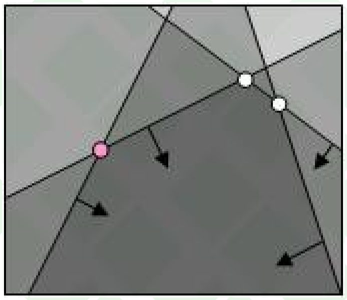
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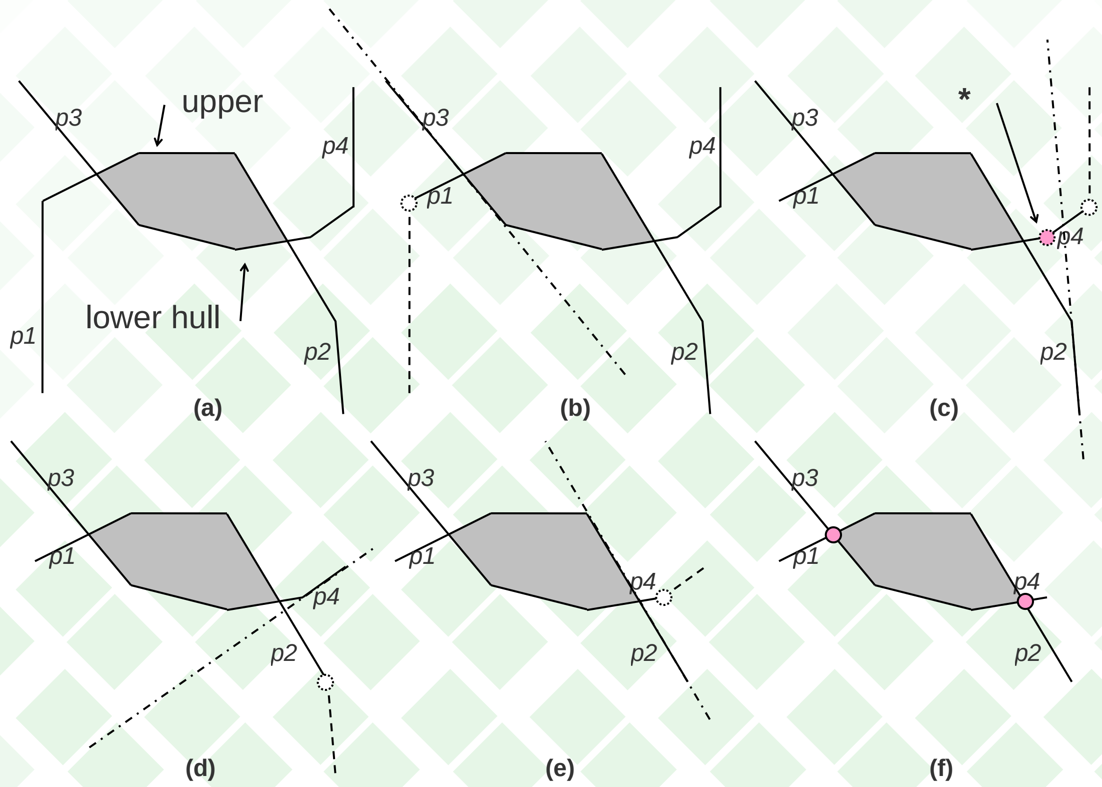
Step 3: 从排序后极角最小两个半平面开始，求出它们的交点并且将他们押入栈。

每次按照极角从小到大顺序增加一个半平面，算出它与栈顶半平面的交点。如果当前的交点在栈顶两个半平面交点的右边，出栈（pop）。前问我们说到出栈，出栈只需要一次么？Nie!我们要继续交点检查，如果还在右边我们要继续出栈，直到当前交点在栈顶交点的左边。

Step 4:相邻半平面的交点组成半个凸多边形。我们有两个点集，(-⅟π, ⅟π]给出上半个，(-π, -⅟π]∪(⅟π, π]给出下半个

初始时候，四个指针 p1, p2, p3 and p4 指向上/下凸壳的最左最右边。p1, p3 向右走，p2, p4 向左走。任意时刻，如果最左边的交点不满足 p1/p3 所在半平面的限制，我们相信这个交点需要删除。p1 或 p3 走向它右边的相邻边。类似地我们处理最右边的交点。重复运作直到不再有更新出现——迭代。



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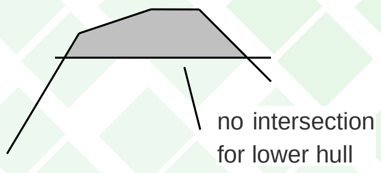

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除了 Step2 中的排序以外，S&I 算法的每一步都是线性的。通常我们用快速排序实现 Step2，总的时间复杂度为 O(nlogn)，隐蔽其中的常数因子很小

5. Practical Value and Linear approach

Great ideas need landing gear as well as wings. S&I 算法似乎和 D&C 算法时间复杂度相同，但是它有着压倒性(overwhelming)的优势。

- 新的 S&I 算法代码容易编写，相对于 D&C 大大简单化，C++程序语言实现 S&I 算法仅需 3KB 不到.

- S&I 算法复杂度中的系数，远小于 D&C，因为我们不再需要 O(nlogn)次交点运算. 通常意义上来讲，S&I 程序比 D&C 快五倍。

  Remark: intersection calculations play the most important role in both algorithms due to its operational speed (so slow, equivalent to hundreds of addition operations). CPI, the preparative algorithm which will be called several times from D&C, requires *O*(*n*) number of calculations, wherefore it rises the total running time up. Besides, S&I calculates *O*(*n*) times in any case.
- 如果给定半平面均在(-⅟ π, ⅟ π]（或任意一个跨度为π的区间），S&I 算法可被显著缩短，C++程序只需要约二十行。USAICO 比赛中就出现了这样一题。

Asymptotical Optimization to linear: The bottleneck of this algorithm is sorting. Pay attention the sorting is NOT a ***comparison sort*** (sorting based on comparison)! The key words for elements to be sorted are polar angles, which can be certainly determined by gradient – a decimal fraction. Since then, we can replace the *O*(*n*log*n*) quick-sort to *O*(*n*) ***radix-sort***. <u>**The asymptotical complexity of algorithm can decrease to *O*(*n*)**</u>. Anyway *O*(*n*) approach usually runs slower than *n*log*n* ones for its additional memory usage!
本算法瓶颈是排序，这里的排序不是比较排序，因此可以将快速排序替换成基数排序，降低程序渐进时间复杂度到线性。

The sentiment of my creation: An invention does not attribute to someone who comes up with ideas. Most people have ideas. The difference between the average person and the inventor is that the inventor for some reason believes only himself, and has the urge to see his ideas through to fruition. *Henri Matisse*, a French painter and sculptor, ever said ‘there is nothing more difficult for a truly creative painter than to paint a rose, because before he can do so, he has first to forget all the roses that were ever painted.’ Equipped with both idealistic and practical spirit of innovation, I am on the way. How about you?

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