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Axiomatically, a parameterized smooth curve is delineated by an arc coordinate ，“s”of “t” as follows:

Definition I : $s(t) =: \int_{t_0}^{t} \left|\vec{r}'(t)\right| dt$, in which $\vec{r}(t)$ acts as a ritually defined position vector of any specific point on the curve, naturally t_0 is a constant for gauging the arc coordinate.

Then $s'(t) = \left|\vec{r}'(t)\right|$

The pertinent propositions about curvature of interest can be viscerally obtained via vector calculus, which I think is the best way to give a curvature at any given point non-singular and of course, not reposing on the boundaris.

Definition II : The curvature $\kappa(t) =: \left|\frac{d\vec{\tau}(t)}{ds}\right|$, in which $\vec{\tau}(t)$ represents the unit vector tangent to the curve at the point t.

It happens to come in line with the “ubiquitous” definitions in general textbooks as $\left|\frac{d\theta}{ds}\right|$.

Now just sit and watch some slick gimmicks manipulating Def 1 and 2.

Obviously, $\kappa(t) = \frac{|d\vec{\tau}(t)|}{ds} = \frac{\frac{|d\vec{\tau}(t)|}{dt}}{\frac{ds}{dt}}$, in which $\frac{ds}{dt} = s'(t) = \left|\vec{r}'(t)\right|$, and $\vec{r}'(t) = \left|\vec{r}'(t)\right|\vec{\tau}(t)$ holds for nothing seductive.

Now we can easily derive something useful.

Lemma I : We have $\vec{\tau}(t) \cdot \vec{\tau}'(t) = 0$.

Proof : We have $\vec{\tau}(t) \cdot \vec{\tau}(t) = |\vec{\tau}(t)|^2 = 1$, as $\vec{\tau}(t)$ is a unit vector.

Then we take the derivative from both sides, there exists $2\vec{\tau}(t) \cdot \vec{\tau}'(t) = 0$.

Lemma 1 predicates $\vec{\tau}(t)$ is perpendicular to $\vec{\tau}'(t)$.

Lemma II &Proof : We can obtain $\vec{r}''(t) = \frac{d}{dt}\left[\left|\vec{r}'(t)\right|\vec{\tau}(t)\right] = \frac{d\left|\vec{r}'(t)\right|}{dt}\vec{\tau}(t) + \left|\vec{r}'(t)\right|\vec{\tau}'(t)$, now take the cross product using the vector $\vec{r}'(t)$ from both sides , we have $\vec{r}'(t) \times \vec{r}''(t) = \left|\vec{r}'(t)\right|^2 \vec{\tau}(t) \times \vec{\tau}'(t)$,

now take the magnitude from both sides and note that the unit vector $\vec{\tau}(t)$ is perpendicular to $\vec{\tau}'(t)$, obtaining a compact gadget, $\left|\vec{r}'(t) \times \vec{r}''(t)\right| = \left|\vec{r}'(t)\right|^2 \left|\frac{d\vec{\tau}(t)}{dt}\right|$.

Now let's flit back to the original problem. We shall instantaneously give

Proposition &Proof : $\kappa(t) = \frac{|d\vec{\tau}(t)|}{ds} = \frac{\frac{|d\vec{\tau}(t)|}{dt}}{\frac{ds}{dt}} = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$,

which is **exactly** what we are seeking for.

From ***Prop*** we can derive all forms of curvature from a hodgepodge of cumbersome derivation methods.

Tentatively, for simplicity I will show how some “domestic” conclusions stand using some results descended from ***Prop***, and show you the impeccability and potency of the method.

We single out x for parameterizing the curve $y = f(x)$

Here exists $\vec{r}(x) = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{r}'(x) = \begin{pmatrix} 1 \\ y' \end{pmatrix}$, $\vec{r}''(x) = \begin{pmatrix} 0 \\ y'' \end{pmatrix}$

plug them all in ***Prop***, we have

$$\left|\begin{pmatrix} 1 \\ y' \end{pmatrix} \times \begin{pmatrix} 0 \\ y'' \end{pmatrix}\right| = |y''| \text{ , } \left|\vec{r}'(x)\right|^3 = \left(\sqrt{1+y'^2}\right)^3$$

yielding $\kappa(x) = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$

$$\vec{r} = \vec{r}(\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \text{ , } \vec{r}'(\theta) = \begin{pmatrix} r'\cos\theta - r\sin\theta \\ r'\sin\theta + r\cos\theta \end{pmatrix} \text{ , }$$
$$\vec{r}''(\theta) = \begin{pmatrix} r''\cos\theta - 2r'\sin\theta - r\cos\theta \\ r''\sin\theta + 2r'\cos\theta - r\sin\theta \end{pmatrix} \text{ .}$$

$$\left|\begin{pmatrix} r'\cos\theta - r\sin\theta \\ r'\sin\theta + r\cos\theta \end{pmatrix} \times \begin{pmatrix} r''\cos\theta - 2r'\sin\theta - r\cos\theta \\ r''\sin\theta + 2r'\cos\theta - r\sin\theta \end{pmatrix}\right| = \left|\vec{r}'(\theta) \times \vec{r}''(\theta)\right| = \left|r^2 + 2r'^2 - rr''\right|$$

$$\left|\vec{r}'(\theta)\right|^3 = \left(\sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\right)^3 = (r^2 + r'^2)^{\frac{3}{2}} \text{ ,}$$

yielding $\kappa(\theta) = \frac{\left|r^2 + 2r'^2 - rr''\right|}{(r^2 + r'^2)^{\frac{3}{2}}}$.

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