

33 人赞同了该回答

Axiomatically, a parameterized smooth curve is delineated by an arc coordinate , “s” of “t” as follows:

Definition I : $s(t) =: \int_{t_0}^t |\vec{r}'(t)| dt$, in which $\vec{r}(t)$ acts as a ritually defined **position vector** of any specific point on the curve, naturally t_0 is a constant for gauging the arc coordinate.

Then $s'(t) = |\vec{r}'(t)|$

The pertinent propositions about curvature of interest can be viscerally obtained via vector calculus, which I think is the best way to give a curvature at any given point non-singular and of course, not reposing on the boundaris.

Definition II : The curvature $\kappa(t) =: \left| \frac{d\vec{\tau}(t)}{ds} \right|$, in which $\vec{\tau}(t)$ represents the **unit vector tangent** to the curve at the point t.

It happens to come in line with the “ubiquitous” definitions in general textbooks as $\left| \frac{d\theta}{ds} \right|$.

Now just sit and watch some slick gimmicks manipulating Def 1 and 2.

Obviously, $\kappa(t) = \frac{|d\vec{\tau}(t)|}{ds} = \frac{\frac{|d\vec{\tau}(t)|}{dt}}{\frac{ds}{dt}}$, in which $\frac{ds}{dt} = s'(t) = |\vec{r}'(t)|$, and $\vec{r}'(t) = |\vec{r}'(t)| \vec{\tau}(t)$ holds for nothing seductive.

Now we can easily derive something useful.

Lemma I : We have $\vec{\tau}(t) \cdot \vec{\tau}'(t) = 0$.

Proof : We have $\vec{\tau}(t) \cdot \vec{\tau}(t) = |\vec{\tau}(t)|^2 = 1$, as $\vec{\tau}(t)$ is a unit vector.

Then we take the derivative from both sides, there exists $2\vec{\tau}(t) \cdot \vec{\tau}'(t) = 0$.

Lemma 1 predicates $\vec{\tau}(t)$ is perpendicular to $\vec{\tau}'(t)$.

Lemma II &Proof : We can obtain $\vec{r}''(t) = \frac{d}{dt} \left[|\vec{r}'(t)| \vec{\tau}(t) \right] = \frac{d|\vec{r}'(t)|}{dt} \vec{\tau}(t) + |\vec{r}'(t)| \vec{\tau}'(t)$,

now take **the cross product** using the vector $\vec{r}'(t)$ **from both sides**^Q , we have

$$\vec{r}'(t) \times \vec{r}''(t) = |\vec{r}'(t)|^2 \vec{\tau}(t) \times \vec{\tau}'(t) ,$$

now take the magnitude from both sides and note that the unit vector $\vec{\tau}(t)$ is perpendicular to $\vec{\tau}'(t)$, obtaining a compact gadget, $|\vec{r}'(t) \times \vec{r}''(t)| = |\vec{r}'(t)|^2 \left| \frac{d\vec{\tau}(t)}{dt} \right|$. 单位向量长度为1

Now let’s flit back to the original problem. We shall instantaneously give

$$\textbf{Proposition \&Proof} : \kappa(t) = \frac{|d\vec{\tau}(t)|}{ds} = \frac{\frac{|d\vec{\tau}(t)|}{dt}}{\frac{ds}{dt}} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} ,$$

which is **exactly** what we are seeking for.

From **Prop** we can derive all forms of curvature from a hodgepodge of cumbersome derivation methods.

Tentatively, for simplicity I will show how some “domestic” conclusions stand using some results descended from **Prop**, and show you the impeccability and potency of the method.

We single out x for parameterizing the curve $y = f(x)$

Here exists $\vec{r}(x) = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{r}'(x) = \begin{pmatrix} 1 \\ y' \end{pmatrix}$, $\vec{r}''(x) = \begin{pmatrix} 0 \\ y'' \end{pmatrix}$

plug them all in **Prop**, we have

$$\left| \begin{pmatrix} 1 \\ y' \end{pmatrix} \times \begin{pmatrix} 0 \\ y'' \end{pmatrix} \right| = |y''| , \quad |\vec{r}'(x)|^3 = \left(\sqrt{1 + y'^2} \right)^3$$

yielding $\kappa(x) = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}$

$$\vec{r} = \vec{r}(\theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} , \quad \vec{r}'(\theta) = \begin{pmatrix} r' \cos \theta - r \sin \theta \\ r' \sin \theta + r \cos \theta \end{pmatrix} ,$$
$$\vec{r}''(\theta) = \begin{pmatrix} r'' \cos \theta - 2r' \sin \theta - r \cos \theta \\ r'' \sin \theta + 2r' \cos \theta - r \sin \theta \end{pmatrix} .$$

$$\left| \begin{pmatrix} r' \cos \theta - r \sin \theta \\ r' \sin \theta + r \cos \theta \end{pmatrix} \times \begin{pmatrix} r'' \cos \theta - 2r' \sin \theta - r \cos \theta \\ r'' \sin \theta + 2r' \cos \theta - r \sin \theta \end{pmatrix} \right| = |\vec{r}'(\theta) \times \vec{r}''(\theta)| = \left| r^2 + 2r'^2 - rr'' \right|$$

$$|\vec{r}'(\theta)|^3 = \left(\sqrt{(r' \cos \theta - r \sin \theta)^2 + (r' \sin \theta + r \cos \theta)^2} \right)^3 = (r^2 + r'^2)^{\frac{3}{2}} ,$$

yielding $\kappa(\theta) = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{\frac{3}{2}}} .$