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# On Topological Watersheds

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**Abstract.** In this paper, we investigate topological watersheds (Couprie and Bertrand, 1997). One of our main results is a necessary and sufficient condition for a map $G$ to be a watershed of a map $F$, this condition is based on a notion of extension. A consequence of the theorem is that there exists a (greedy) polynomial time algorithm to decide whether a map $G$ is a watershed of a map $F$ or not. We introduce a notion of "separation between two points" of an image which leads to a second necessary and sufficient condition. We also show that, given an arbitrary total order on the minima of a map, it is possible to define a notion of "degree of separation of a minimum" relative to this order. This leads to a third necessary and sufficient condition for a map $G$ to be a watershed of a map $F$. At last we derive, from our framework, a new definition for the dynamics of a minimum.

**Keywords:** mathematical morphology, discrete topology, graph, watershed, dynamics, separation

## 1. Introduction

The watershed transform [14, 16, 17, 22, 24, 30] of greyscale images is very popular as an important step of image segmentation methods [3, 4, 23]. Nevertheless, most existing approaches have several drawbacks. The watershed algorithms produce a binary result, that is, they lose the greyscale information that is present in the original image. This information may be useful for further processing (e.g., reconnection of corrupted contours). In fact, one may attempt to recover, from a binary watershed, some greyscale values extracted from the original image (e.g., to assign, to each catchment bassin and each watershed arc, the minimal greyscale value of the corresponding region). Unfortunately, it appears that this reconstructed greyscale image does not preserve the contrast of the image (e.g., the altitudes of the passes are not preserved), some counter-examples may be found in [6, 19, 21]. In fact, most popular watersheds algorithms, such as the ones based on the flooding paradigm, produce watersheds which are not necessarily on the most significant contours of the original image. It follows that there is a real difficulty to establish some properties of these watersheds since some basic features of the original image are not always preserved.

In this paper,[1] we investigate a topological approach [6] which allows to precisely define a greyscale watershed transform as an ultimate "$W$-thinning," a $W$-thinning being a kind of thinning which preserves the connected components of the complement of each cross-section of the original image (see also [7]). Here, a greyscale image is considered as a map from the set of vertices of an arbitrary graph to the set of integers. This approach is very general (e.g., it applies to images of arbitrary dimension) and it does keep track of the useful greyscale information. An algorithm was proposed for extracting such a watershed from a map [6]. Nevertheless, at this time, no general property of topological watersheds was proved.

Our main contributions are the following:

(1) We give a first necessary and sufficient condition for a map $G$ to be a $W$-thinning of a map $F$, this condition is based on the notion of extension. A consequence of the theorem is that there exists a (greedy) polynomial time algorithm to decide whether a map $G$ is a watershed of a map $F$ or

not. This is an unexpected result because, in the classical framework of homotopy (and simple points), such an algorithm cannot exist [5].

(2) We introduce a notion of $k$-separation between two points $x$ and $y$ of an image: $x$ and $y$ are $k$-separated if the lowest altitude for joining $x$ and $y$ is precisely $k$ and if this altitude is strictly greater than the altitudes of both $x$ and $y$. We give a second necessary and sufficient condition for a map $G$ to be a $W$-thinning of a map $F$, this condition is composed of the two following sub-conditions:
   - the minima of $G$ must be "extensions" of the minima of $F$; and
   - any couple of points which are $k$-separated for $F$, must be $k$-separated for $G$.

(3) We also show that, given an arbitrary total order on the minima of a map, it is possible to define a notion of "degree of separation of a minimum" relative to this order. This leads to a third necessary and sufficient condition for a map $G$ to be a $W$-thinning of a map $F$. At last we derive, from our framework, a new definition for the dynamics of a minimum [10, 18].

## 2. Basic Definitions

Any discrete function may be represented by its different threshold levels [11, 27, 31]. These levels constitute a "stack". In fact, the datum of a function is equivalent to the datum of a stack. In this section, we introduce new definitions for stacks and related notions, this set of definitions allows to handle both the threshold levels of a discrete function and the complements of these levels.

### 2.1. Discrete Maps and Stacks

Here and subsequently $E$ stands for a non-empty finite set and $K$ stands for an element of $\mathbb{Z}$, with $K > 0$. If $X \subseteq E$, we write $\bar{X} = \{x \in E \mid x \notin X\}$. If $k_1$ and $k_2$ are elements of $\mathbb{Z}$, we define $[k_1, k_2] = \{k \in \mathbb{Z} \mid k_1 \leq k \leq k_2\}$. We set $\mathbb{K} = [-K, +K]$, and $\mathbb{K}^\circ = [-K+1, +K-1]$.

*Definition 1.* Let $F = \{F[k] \subseteq E \mid k \in \mathbb{K}\}$ be a family of subsets of $E$ with index set $\mathbb{K}$, such a family is said to be a $\mathbb{K}$-*family* (*on* $E$). Any subset $F[k], k \in \mathbb{K}$, is a *section of* $F$ (*at level* $k$) or the $k$-*section of* $F$. We set:

$$\bar{F} = \{\bar{F}[k] \mid \bar{F}[k] = \overline{F[k]},\ k \in \mathbb{K}\},$$
$$F^{-1} = \{F^{-1}[k] \mid F^{-1}[k] = F[-k],\ k \in \mathbb{K}\}.$$

The $\mathbb{K}$-families $\bar{F}$ and $F^{-1}$ are, respectively, the *complement of* $F$ and the *symmetric of* $F$.

*Definition 2.* We say that a $\mathbb{K}$-family $F$ is an *upstack on* $E$ if:

$$F[-K] = E,\ F[K] = \emptyset, \quad \text{and} \quad F[j] \subseteq F[i] \text{ whenever } i < j.$$

We say that a $\mathbb{K}$-family $F$ is a *downstack on* $E$ if:

$$F[-K] = \emptyset,\ F[K] = E, \quad \text{and} \quad F[i] \subseteq F[j] \text{ whenever } i < j.$$

A $\mathbb{K}$-family is a *stack* if it is either an upstack or a downstack.

We denote by $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) the family composed of all upstacks on $E$ (resp. downstacks on $E$). We also set $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$, i.e., $\mathcal{S}$ is the family composed of all stacks on $E$ .

Let $F, G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. We say that $G$ is *under* $F$, written $G \subseteq F$ if, for all $k \in \mathbb{K}$, $G[k] \subseteq F[k]$.

If $F$ is an upstack, then $\bar{F}$ and $F^{-1}$ are downstacks. In fact we have:

$$\mathcal{S}^- = \{\bar{F} \mid F \in \mathcal{S}^+\} = \{F^{-1} \mid F \in \mathcal{S}^+\},$$
$$\mathcal{S}^+ = \{\bar{F} \mid F \in \mathcal{S}^-\} = \{F^{-1} \mid F \in \mathcal{S}^-\}.$$

Furthermore, if $F \in \mathcal{S}$, then $[\bar{F}]^{-1} = \overline{[F^{-1}]}$.

*Definition 3.* Let $F \in \mathcal{S}^+$ and let $G \in \mathcal{S}^-$. We define two maps from $E$ on $\mathbb{K}$, also denoted by $F$ and $G$, such that, for any $x \in E$,

$$F(x) = \max\{k \in \mathbb{K} \mid x \in F[k]\},$$
$$G(x) = \min\{k \in \mathbb{K} \mid x \in G[k]\},$$

these maps $F$ and $G$ are, respectively, the *functions induced by the upstack* $F$ *and the downstack* $G$, $F(x)$ and $G(x)$ are, respectively, the *altitudes of* $x$ *for* $F$ *and* $G$.

Let $F \in \mathcal{S}$ and let $x \in E$. We set $S(x, F) = F[k]$, with $k = F(x)$, $S(x, F)$ is the *section of* $x$ *for* $F$.

| $F$ | | | | | | | | | | $F^{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −3 |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | −2 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | −1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| −1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| −2 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 |
| −3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 |
| | a | b | c | d | e | f | g | h | i | |

*Figure 1.* representation of an upstack $F$ (the 1's, $k$ on the left), the downstack $F^{-1}$ (the 1's, $k$ on the right), the downstack $\bar{F}$ (the 0's, $k$ on the left), and the upstack $[\bar{F}]^{-1} = \overline{[F^{-1}]}$ (the 0's, $k$ on the right).

We observe that, if $F \in \mathcal{S}$ and $x \in E$, $S(x, F) = S(x, F^{-1})$. Furthermore, if $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, $x \in E$, and $k \in \mathbb{K}$, we have:

$$F[k] = \{x \in E \mid F(x) \geq k\} \quad \text{and}$$
$$G[k] = \{x \in E \mid G(x) \leq k\},$$
$$-K \leq F(x) < K \quad \text{and}$$
$$-K < G(x) \leq K, \bar{F}(x) = F(x) + 1 \quad \text{and}$$
$$\bar{G}(x) = G(x) - 1, F^{-1}(x) = -F(x) \quad \text{and}$$
$$G^{-1}(x) = -G(x).$$

We also note that, if $F$ and $G$ are both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, then $F \subseteq G$ if and only if, for each $x \in E$, $F(x) \leq G(x)$ (resp. $G(x) \leq F(x)$).

In Fig. 1, a representation of an upstack $F$ is given. For example, we have $F[1] = F^{-1}[-1] = \{b, d, e\}$, $\bar{F}[1] = \{a, c, f, g, h, i\}$, $F(c) = 0$, $\bar{F}(c) = 1$, $S(c, F) = \{b, c, d, e, h\}$, $S(c, \bar{F}) = \{a, c, f, g, h, i\}$.

*Remark 1* (maps). Let $\mathcal{F}$ be the family composed of all maps $f$ from $E$ on $\mathbb{K}$, such that, for all $x$ in $E$, $f(x) \neq K$. Let $f \in \mathcal{F}$. The *upstack induced by* $f$ is the family $F = \{F[k] \mid k \in \mathbb{K}\}$, such that, for each $k \in \mathbb{K}$, $F[k] = \{x \in E \mid f(x) \geq k\}$. We note that $F$ is indeed an upstack on $E$. We also observe that, for each $x \in E$, $f(x) = F(x)$. Thus, we can associate, to each map $f \in \mathcal{F}$, an upstack $F$, the function induced by $F$ being precisely $f$. Conversely, we can associate, to each upstack $F$, an induced function $f \in \mathcal{F}$ (the function induced by the upstack $F$), the upstack induced by $f$ being precisely $F$.

*Remark 2* (duality). Let $F_1, \ldots, F_i$ be elements of $\mathcal{S}$ and $k_1, \ldots, k_j$ be in $\mathbb{K}$. Let $\mathcal{P}(F_1, \ldots, F_i, k_1, \ldots, k_j)$ be a certain proposition which depends on $F_1, \ldots, F_i, k_1, \ldots, k_j$. We will say that $\mathcal{P}(F_1, \ldots, F_i, k_1, \ldots, k_j)$ is *dual* if $\mathcal{P}(F_1, \ldots, F_i, k_1, \ldots, k_j)$ is true whenever $\mathcal{P}(F_1^{-1}, \ldots, F_i^{-1}, -k_1, \ldots, -k_j)$ is true.

For example, if $F \in \mathcal{S}$ and if $k \in \mathbb{K}$, the property "The subset $X$ of $E$ is the $k$-section of $F$" is dual.

Now, if $F \in \mathcal{S}$ and $k \in \mathbb{K}$, let us define the *$k$-cut of* $F$ to be the subset $\{x \in E \mid F(x) \leq k\}$. It may be easily seen that the property "The subset $X$ of $E$ is the $k$-cut of $F$" is not dual.

In this paper, even if not explicitly mentioned, all properties (and all definitions) are dual.

### 2.2. Graphs

Throughout this paper, $\Gamma$ will denote a binary relation on $E$ (thus, $\Gamma \subseteq E \times E$), which is reflexive (for all $x$ in $E$, $(x, x) \in \Gamma$) and symmetric (for all $x$, $y$ in $E$, $(y, x) \in \Gamma$ whenever $(x, y) \in \Gamma$). We say that the pair $(E, \Gamma)$ is a *graph*, each element of $E$ is called a *vertex* or a *point*. We will also denote by $\Gamma$ the map from $E$ to $2^E$ (the set composed of all subsets of $E$), such that, for all $x \in E$, $\Gamma(x) = \{y \in E \mid (x, y) \in \Gamma\}$. If $y \in \Gamma(x)$, we say that $y$ is *adjacent to* $x$. If $X \subseteq E$ and $y \in \Gamma(x)$ for some $x \in X$, we say that *$y$ is adjacent to $X$*.

Let $X \subseteq E$, a *path in* $X$ is a sequence $\pi = \langle x_0, \ldots, x_l \rangle$ such that $x_i \in X$, $i \in [0, l]$, and $x_i \in \Gamma(x_{i-1})$, $i \in [1, l]$. We also say that $\pi$ is a *path from* $x_0$ *to* $x_l$ *in* $X$. Let $x, y \in X$. We say that $x$ and $y$ are *linked for* $X$ if there exists a path from $x$ to $y$ in $X$. We say that $X$ is *connected* if any $x$ and $y$ in $X$ are linked for $X$. We say that $Y \subseteq E$ is a *connected component of* $X \subseteq E$, if $Y \subseteq X$, $Y$ is connected, and $Y$ is maximal for these two properties (i.e., $Y = Z$ whenever $Y \subseteq Z \subseteq X$ and $Z$ is connected).

In the sequel of this paper, we will assume that $E$ is connected. All notions and properties may be easily extended for non-connected graphs.

### 2.3. Stacks and Graphs

We are now in position to give some basic definitions relative to stacks and graphs and to introduce the notion of "connection value" which plays a key role in our framework (see also [25, 26, 29]).

*Definition 4.* Let $F \in \mathcal{S}$ and let $k \in \mathbb{K}$. A connected component of a non-empty $k$-section of $F$ is a *component of* $F$ (*at level* $k$) or a *$k$-component of* $F$. Let $x \in E$

and let $S(x, F)$ be the section of $x$ for $F$. We denote by $C(x, F)$ the connected component of $S(x, F)$ which contains $x$, $C(x, F)$ is the *component of $x$ for $F$*.

We say that $x \in E$ and $y \in E$ are *$k$-linked for $F$* if $x$ and $y$ are linked for $F[k]$, i.e., if $x$ and $y$ belong to the same connected component of $F[k]$.

*Definition 5.* Let $F \in \mathcal{S}$ and let $X \subseteq E$.

The subset $X$ is an *extremum of $F$* if $X$ is a component of $F$ and if $X$ is minimal for this property (i.e., no proper subset of $X$ is a component of $F$). If $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, we also say that an extremum of $F$ is a *maximum of $F$* and that an extremum of $G$ is a *minimum of $G$*.

The subset $X$ is *flat for $F$* if $F(x) = F(y)$ for all $x$, $y$ in $X$. If $X$ is flat for $F$, the *altitude of $X$ for $F$* is the value $F(X)$ such that $F(X) = F(x)$ for every $x \in X$.

Observe that a subset $X \subseteq E$ is an extremum of $F$ if and only if $X$ is a component of $F$ and flat for $F$.

*Definition 6.* Let $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, and let $x$, $y$ be two vertices in $E$. We define:

$$F(x, y) = \max\{k \mid x \text{ and } y \text{ are } k\text{-linked for } F\},$$
$$G(x, y) = \min\{k \mid x \text{ and } y \text{ are } k\text{-linked for } G\},$$

$F(x, y)$ and $G(x, y)$ are the *connection values between $x$ and $y$ for, respectively, $F$ and $G$*. If $X$ and $Y$ are two subsets of $E$, we set

$$F(X, Y) = \max\{F(x, y) \mid x \in X, y \in Y\},$$
$$G(X, Y) = \min\{G(x, y) \mid x \in X, y \in Y\},$$

which are the *connection values between $X$ and $Y$ for, respectively, $F$ and $G$*.

Connection values may be expressed in terms of paths. If $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, and if $\pi = \langle x_0, \ldots, x_l \rangle$ is a path in $E$, let us set:

$$F(\pi) = \min\{F(x_i) \mid i \in [0, l]\},$$
$$G(\pi) = \max\{G(x_i) \mid i \in [0, l]\}.$$

It may be seen that two points $x$ and $y$ are $k$-linked for $F \in \mathcal{S}^+$ (resp. $G \in \mathcal{S}^-$) if and only if there exists a path $\pi$ from $x$ to $y$ such that $F(\pi) \geq k$ (resp. $G(\pi) \leq k$). Thus, if $x$ and $y$ are two points of $E$, and if we denote by $\Pi(x, y)$ the set composed of all paths from $x$ to $y$ in $E$, we have:

$$F(x, y) = \max\{F(\pi) \mid \pi \in \Pi(x, y)\},$$
$$G(x, y) = \min\{G(\pi) \mid \pi \in \Pi(x, y)\}.$$

## 3. Topological Watersheds

We introduce a notion of watershed solely based on topological criteria, more precisely based on the preservation of certain connected components [6]. We first present a definition for subsets, then, in a natural way, we extend this definition to stacks (and, by the way, to functions according to Remark 1).

*Definition 7.* Let $X \subseteq E$ and let $x \in X$.

- $x$ is a *border point of $X$* if $x$ is adjacent to $\bar{X}$;
- $x$ is an *inner point of $X$* if $x$ is not a border point of $X$;
- $x$ is *separating for $X$* if $x$ is adjacent to at least two connected components of $\bar{X}$.
- $x$ is *$W$-simple for $X$* if $x$ is adjacent to exactly one connected component of $\bar{X}$.

Thus, a point $x$ is $W$-simple for $X$ if and only if $x$ is a border point of $X$ and not separating for $X$.

*Definition 8.* Let $X$, $Y$ be subsets of $E$. We say that $Y$ is a *$W$-thinning* of $X$, written $X \searrow^{W} Y$, if:

(i) $Y = X$; or if
(ii) there exists a set $Z$ which is a $W$-thinning of $X$ and there exists a $W$-simple point $x$ for $Z$ such that $Y = Z \setminus \{x\}$.

A $W$-thinning $Y$ of $X$ is a *watershed of $X$* if $Y \searrow^{W} Z$ implies $Z = Y$.

In other words, $Y$ is a $W$-thinning of $X$, if there exists a (possibly empty) sequence $\langle x_0, \ldots, x_l \rangle$ such that $Y = X \setminus \{x_0, \ldots, x_l\}$, and, for any $i \in [0, l]$, $x_i$ is $W$-simple for $X \setminus \{x_j \mid j < i\}$. A subset $Y$ is a watershed of $X$ if $Y$ is a $W$-thinning of $X$ and if there exists no $W$-simple point for $Y$.

*Definition 9.* Let $F \in \mathcal{S}$ and let $x \in E$ such that $F(x) \in \mathbb{K}^{\circ}$. We denote by $S(x, F)$ the section of $x$ for $F$.

- $x$ is a *border point for $F$* if $x$ is a border point of $S(x, F)$;
- $x$ is an *inner point for $F$* if $x$ is an inner point of $S(x, F)$;
- $x$ is *separating for $F$* if $x$ is separating for $S(x, F)$;
- $x$ is *$W$-destructible for $F$* if $x$ is $W$-simple for $S(x, F)$.

We now introduce an operation on stacks which is the extension of the removal of a point from a set.

Let $F \in \mathcal{S}$ and let $x \in E$ such that $F(x) \in \mathbb{K}^{\circ}$. We denote by $F \setminus x$ the element of $\mathcal{S}$ such that $[F\setminus x][k] = F[k] \setminus \{x\}$ if $k = F(x)$, and $[F\setminus x][k] = F[k]$ otherwise. In other words:

(i) if $F \in \mathcal{S}^+$, then $[F\setminus x](x) = F(x) - 1$ and $[F\setminus x](y) = F(y)$ whenever $y \neq x$;
(ii) if $F \in \mathcal{S}^-$, then $[F\setminus x](x) = F(x) + 1$ and $[F\setminus x](y) = F(y)$ whenever $y \neq x$.

*Definition 10.* Let $F, G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. We say that $G$ is a *W-thinning* of $F$, written $F \overset{W}{\searrow} G$, if:
(i) $G = F$; or if
(ii) there exists a $W$-thinning $H$ of $F$ and there exists a $W$-destructible point $x$ for $H$, such that $G = H\setminus x$.

A $W$-thinning $G$ of $F$ is a *watershed of* $F$ if $G \overset{W}{\searrow} H$ implies $H = G$.

In Fig. 2(a), an upstack $F$ on the set $E = \{a, b, \ldots, j\}$ is represented by its induced function (see Remark 1). Thus $F[k] = \{x \in E \mid F(x) \geq k\}$. For example we have $F[6] = \{i, b, h\}$. Let us consider the point $e$. We have $S(e, F) = \{x \in E \mid F(x) \geq 5\}$. Thus $\overline{S(e, F)} = \{c, d, j, g, a\}$. This subset is composed of two connected components but $e$ is adjacent to only one of these connected components, thus $e$ is $W$-destructible. On the other hand, $j$ is not $W$-destructible since it is not a border point. Two topological watersheds of (a) are represented Fig. 2(b) and (c).

Figure 3(a) gives another example of an upstack $F$ on $E$ represented by its induced function, here $E$ is a subset of $\mathbb{Z}^2$ (a rectangle). We choose for $\Gamma$ the well-known "4-adjacency relation" [12], i.e., each point is adjacent to itself and to its North, South, East, West neighbors (whenever they exist). A topological watershed of $F$ (relative to $(E, \Gamma)$) is shown Fig. 3(b).

The following theorem will be of primary importance for this paper. It allows to extend, in a systematic way, results established for the binary case to stacks and functions.

**Theorem 1.** *Let $F, G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. The stack $G$ is a $W$-thinning of $F$ if and only if, for each $k \in \mathbb{K}$, $G[k]$ is a $W$-thinning of $F[k]$.*

**Proof:**

(i) Let $x$ be a $W$-destructible point for $F$. By the very definition of a $W$-destructible point, for each $k \in$

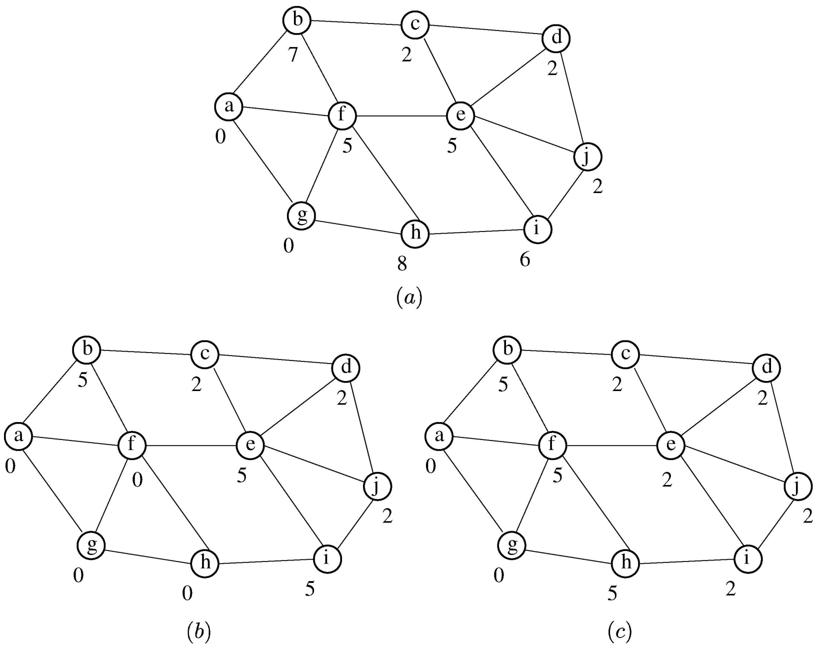


*Figure 2.* (a) An upstack represented by its function, (b) and (c) two topological watersheds of (a).

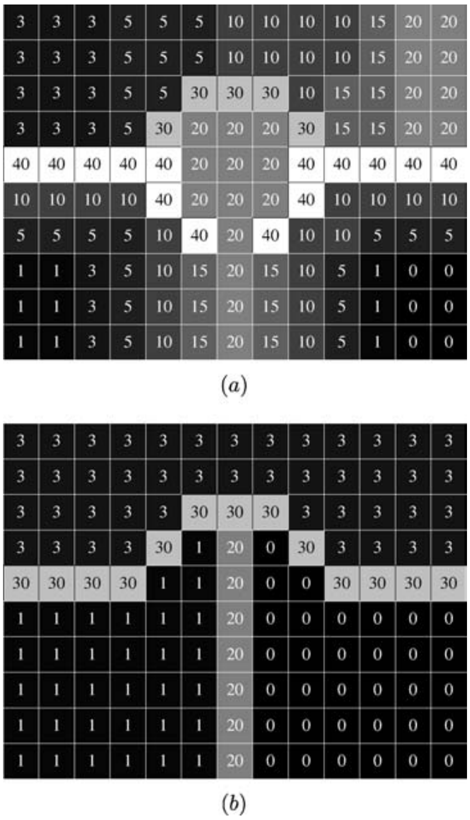


*Figure 3.* (a) Original image, (b) A topological watershed of (a).

$\mathbb{K}$, $[F \backslash x][k]$ is a $W$-thinning of $F[k]$. By induction, if $G$ is a $W$-thinning of $F$, then, for each $k \in \mathbb{K}$, $G[k]$ is a $W$-thinning of $F[k]$.

(ii) Suppose $F$ and $G$ are two upstacks such that, for each $k \in \mathbb{K}$, $G[k]$ is a $W$-thinning of $F[k]$.

Let us consider the $\mathbb{K}$-families $G_i$, $i \in \mathbb{K}$, defined by:

$$\begin{cases} G_i[k] = F[k] & \text{if } k \in [-K, i], \\ G_i[k] = G[k] & \text{if } k \in [i+1, K]. \end{cases}$$

– We observe that these $\mathbb{K}$-families are upstacks. We also note that $G_K = F$ and $G_{-K} = G$. Furthermore, for any $x$ in $E$, if $G_i(x) > i$, then $G_i(x) = G(x)$. Let us consider and index $i$ and a point $x$ in $F[i] \setminus G[i]$. Since $G_i[i] = F[i]$, and since $x$ is an element of $F[i]$, we have $G_i(x) \geq i$. But it is not possible that $G_i(x) > i$, otherwise, by the preceding remark, we would have $G(x) > i$ which contradicts $x \notin G[i]$. Thus, we may affirm that, whenever $x$ is in $F[i] \setminus G[i]$, $G_i(x) = i$.

– Trivially, $G_K$ is a $W$-thinning of $F$. Suppose $G_i$ is a $W$-thinning of $F$, for some $i \in [-K+1, K]$. In order to prove the property by induction, we will prove that $G_{i-1}$ is a $W$-thinning of $F$.

Since $G[i]$ is a $W$-thinning of $F[i]$, there exists a sequence $\langle x_0, \ldots, x_l \rangle$ such that $G[i] = F[i] \setminus \{x_0, \ldots, x_l\}$, and, for any $j \in [0, l]$, $x_j$ is $W$-simple for $F[i] \setminus \{x_0, \ldots, x_{j-1}\}$. By the above remark, we have $G_i(x_j) = i$, for any $j \in [0, l]$. Let us consider the upstacks $G_{i,j}$, $j \in [0, l+1]$, such that $G_{i,0} = G_i$, and $G_{i,j} = G_{i,j-1} \setminus x_{j-1}$ if $j \in [1, l+1]$. It may be seen that $G_{i,j}[i] = G_i[i] \setminus \{x_0, \ldots, x_{j-1}\}$ and $G_{i,j}[k] = G_i[k]$ if $k \neq i$. Thus, since $G_i[i] = F[i]$, $x_j$ is $W$-simple for $G_{i,j}[i]$, for $j \in [0, l]$. Furthermore, $G_{i,j}(x_j) = G_i(x_j) = i$. It follows that $x_j$ is $W$-destructible for $G_{i,j}$ and $G_{i,l+1}$ is a $W$-thinning of $G_{i,0}$. But $G_{i,0} = G_i$, and we observe that $G_{i,l+1}$ is precisely $G_{i-1}$. Since, by hypothesis, $G_i$ is a $W$-thinning of $F$, $G_{i-1}$ is a $W$-thinning of $F$. By induction, we may affirm that $G$ is a $W$-thinning of $F$.

A downstack $G$ is a $W$-thinning of a downstack $F$ if and only if the upstack $G^{-1}$ is a $W$-thinning of the upstack $F^{-1}$. Thus, the property is also true whenever $F$ and $G$ are downstacks. □

Let $F$ and $G$ be two upstacks such that $G$ is a watershed of $F$. We observe that $G[k]$ is not necessarily a watershed of $F[k]$. The reason is that, $G$ being an upstack, we must have $G[k+1] \subseteq G[k]$ : since this constraint, there may exist some $W$-simple point for $G[k]$ which cannot be removed. For example, let us consider the upstack $G$ of Fig. 2(b) which is a watershed of the upstack $F$ of Fig. 2(a). The point b is $W$-simple for $G[2]$, thus $G[2]$ is a $W$-thinning of $F[2]$, but not a watershed of $F[2]$. Thus, Theorem 1 does not hold for watersheds. In order to recover an equivalent of Theorem 1, we introduce the notion of a constrained watershed.

*Definition 11.* Let $X, C$ be subsets of $E$, with $C \subseteq X$. Let $Y$ be a $W$-thinning of $X$, with $C \subseteq Y$. We say that $Y$ is a *watershed of $X$ constrained by $C$* if $Y \overset{W}{\searrow} Z$ and $C \subseteq Z$ both imply $Z = Y$.

In other words $Y$ is a watershed of $X$ constrained by $C$ if $Y$ is a $W$-thinning of $X$ which contains $C$ and if any point in $Y \backslash C$ is not $W$-simple for $Y$.

Let $F$ and $G$ be in $\mathcal{S}^+$ (resp. in $\mathcal{S}^-$). Trivially, from Theorem 1, we may affirm that:

(i) The stack $G$ is a $W$-thinning of $F$ if and only if, for each $k \in \mathbb{K}^\circ$, $G[k]$ is a $W$-thinning of $F[k]$ such that $G[k+1] \subseteq G[k]$ (resp. $G[k-1] \subseteq G[k]$). On the other hand we observe that $G(x) = k$ if and only if $x \in G[k] \backslash G[k+1]$ (resp. $x \in G[k] \backslash G[k-1]$). Thus, by the very definition of a $W$-destructible point, we may affirm that:

(ii) There is no $W$-destructible point for $G$ if and only if, for each $k \in \mathbb{K}^\circ$ and for each $x$ in $G[k] \backslash G[k+1]$ (resp. $G[k] \backslash G[k-1]$), $x$ is not $W$-simple for $G[k]$.

The following Theorem 2 is a direct consequence of properties i) and ii).

**Theorem 2.** *Let $F$ and $G$ be in $\mathcal{S}^+$ (resp. in $\mathcal{S}^-$). The stack $G$ is a watershed of $F$ if and only if, for each $k \in \mathbb{K}^\circ$, $G[k]$ is a watershed of $F[k]$ constrained by $G[k+1]$ (resp. by $G[k-1]$).*

*Remark 3* (emergence). Let $F$ be an upstack. Let $F^K, F^{K-1}, \dots, F^{-K}$ be a sequence of upstacks with $F^K = F$ and such that for any $k = K-1, K-2, \dots -K$:

– $F^k[k]$ is a watershed of $F[k]$ constrained by $F^{k+1}[k+1]$; and
– $F^k[i] = F^{k+1}[i]$, for any $i \neq k$.

Let us consider the upstack $G = F^{-K}$. We see that $G$ has been obtained from $F$ by a certain kind of computation which consists in performing operations on sections of $F$, these sections being processed according to their decreasing levels. Such a kind of computation has been called an *emergence* in [21]. Of course, by the very definition of $G$, we may affirm that $G$ is a $W$-thinning of $F$. But we observe that such an upstack $G$ satisfies the condition of Theorem 2. Two consequences may be derived of this remark:

(i) The upstack $G$ is indeed a watershed of $F$.
(ii) Any watershed of $F$ may be obtained by such an emergence computation.

These properties have been given in [21]. In fact, Theorem 2 allows to derive them in a more direct way.

We give now a characterization of watersheds for a special class of stacks (see also [26] for the definition of a connected function).

Let $F \in \mathcal{S}$. We say that $F$ is *flat* if, for each $k \in \mathbb{K}$, $F[k] = \emptyset$ or $F[k] = E$. In other words, $F$ is flat if $E$ is flat for $F$.

We say that $F \in \mathcal{S}$ is *connected* if, for each $k \in \mathbb{K}$, $F[k]$ is connected. It may be seen that $F$ is connected if and only if $F$ has exactly one extremum.

We leave to the interested reader the proof of the following property.

**Property 3.** *Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, such that $\bar{F}$ is connected and $G \subseteq F$. Let $M$ be the unique extremum of $\bar{F}$. The stack $G$ is a watershed of $F$ if and only if $G$ is flat and $G(E) = F(M)$.*

Thus, we are able to characterize a watershed of a connected stack. The aim of this paper is to investigate such characterizations for arbitrary stacks.

## 4. Extensions

In this section, we will give a first necessary and sufficient condition which allows to characterize $W$-thinnings of any given upstack.

Let $X$ be a subset of $E$. We denote by $\mathcal{C}(X)$ the set composed of all connected components of $X$.

*Definition 12.* Let $X$, $Y$ be non empty subsets of $E$ such that $X \subseteq Y$. We say that $Y$ is an *extension of* $X$ if each connected component of $Y$ contains exactly one connected component of $X$.

We also say that $Y$ is an extension of $X$ if $X$ and $Y$ are both empty. If $Y$ is an extension of $X$, the *extension map relative to* $(X, Y)$ is the bijection $\epsilon$ from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$ such that, for each $C \in \mathcal{C}(X)$, $\epsilon(C)$ is the connected component of $Y$ which contains $C$.

If $Y$ is an extension of $X$ and $Z$ is an extension of $Y$, clearly $Z$ is an extension of $X$. In fact, we have a more remarkable "triangular" property.

**Theorem 4.** *Let $X$, $Y$, $Z$ be subsets of $E$ such that $Z$ is an extension of $X$ and $X \subseteq Y \subseteq Z$.*

*The subset $Y$ is an extension of $X$ if and only if $Z$ is an extension of $Y$.*

**Proof:** The case $X = \emptyset$ is trivial, we suppose that $X \neq \emptyset$.

(i) Suppose $Y$ is an extension of $X$. Let $A$ be a connected component of $Z$, $A$ contains exactly one connected component $B$ of $X$ and $B$ is contained in one connected component $C$ of $Y$. Thus $A \cap C \neq \emptyset$, which, by properties of connected components, implies that $C \subseteq A$. It follows that each connected component of $Z$ contains at least one connected component of $Y$. But a connected component $A$ of $Z$ cannot contain more than one connected component of $Y$, otherwise, since $Y$ is an extension of $X$, $A$ would contain more than one connected component of $X$. Thus, $Z$ is an extension of $Y$.

(ii) Suppose $Z$ is an extension of $Y$. Let $A$ be a connected component of $Y$, $A$ is contained in a connected component $B$ of $Z$, $B$ contains exactly one connected component $C$ of $X$, and $C$ is contained in a connected component $D$ of $Y$. Thus $D \cap B \neq \emptyset$, which, by properties of connected components, implies that $D \subseteq B$. Since $B$ contains one and only one connected component of $Y$, we must have $D = A$. It follows that any connected component of $Y$ contains at least one connected component of $X$. But a connected component $A$ of $Y$ cannot contain more than one connected component of $X$, otherwise the connected component of $Z$ which contains $A$ would contain more than one connected component of $X$. Thus, $Y$ is an extension of $X$.

□

The following property is a direct consequence of Definition 7 and will be used for establishing Theorem 6.

**Property 5.** *Let $X \subseteq E$, let $x \in X$, let $Y = X \setminus \{x\}$, and let $C$ be a connected component of $\bar{Y}$.*

(i) *The subset $C$ contains more than one connected component of $\bar{X}$ if and only if $x$ is a separating point for $X$ such that $x \in C$;*
(ii) *The subset $C$ does not contain any connected component of $\bar{X}$ if and only if $x$ is an inner point for $X$ such that $x \in C$. In this case we must have $C = \{x\}$.*

**Theorem 6.** *Let $X$ and $Y$ be subsets of $E$. The subset $Y$ is a $W$-thinning of $X$ if and only if $\bar{Y}$ is an extension of $\bar{X}$.*

**Proof:** The case $X = E$ is trivial, we suppose that $X \neq E$.

(i) Let $x$ be a $W$-simple point for $X$ and let $X' = X \setminus \{x\}$. By Property 5, each connected component of $\bar{X}'$ contains one and only one connected component of $\bar{X}$: $\bar{X}'$ is an extension of $\bar{X}$. By induction (and by the remark following Definition 12), if $Y$ is a $W$-thinning of $X$, then $\bar{Y}$ is an extension of $\bar{X}$.

(ii) Suppose $\bar{Y}$ is an extension of $\bar{X}$. If $X = Y$, then we are done. Otherwise, we will show that there exists a $W$-simple point $x$ for $X$ such that $Y \subseteq X' = X \setminus \{x\}$. By the first part of the proof, $\overline{X'}$ will be an extension of $\bar{X}$, and, by Theorem 4, $\bar{Y}$ will be an extension of $\bar{X}'$. We see that this will establish the property by induction. Thus, suppose $X \neq Y$. Hence, by definition of an extension, there must exist one connected component $D$ of $\bar{Y}$ which contains exactly one connected component $C$ of $\bar{X}$ and such that $D \neq C$. Let $y \in C$ and $z \in D \setminus C$. Since $D$ is connected, there exists a path $\pi$ from $y$ to $z$ in $D$. We set $\pi = \langle x_0, \ldots, x_l \rangle$, with $x_0 = y$ and $x_l = z$. Let $j$ the highest index such that $x_j \in C$. Thus $x = x_{j+1} \in D \setminus C$. We observe that, since $x$ is adjacent to $x_j \in C$, we have $x \in X$ (otherwise we would have $x \in C$), thus $x$ is a border point for $X$. Furthermore $x$ cannot be a separating point for $X$, otherwise $D$ would contain more than one connected component of $\bar{X}$. It follows that $x$ is $W$-simple for $X$ and we have $Y \subseteq X' = X \setminus \{x\}$.

□

*Definition 13.* Let $F, G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. We say that $G$ is an *extension of $F$* if, for any $k \in \mathbb{K}$, $G[k]$ is an extension of $F[k]$.

As a direct consequence of Theorems 1, 4, and 6, we obtain the following results.

**Theorem 7** (*extension*). *Let $F, G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. The stack $G$ is a $W$-thinning of $F$ if and only if $\bar{G}$ is an extension of $\bar{F}$.*

**Theorem 8** (*confluence*). *Let $F, G, H$ be all in $\mathcal{S}^+$ or all in $\mathcal{S}^-$ such that $H$ is a $W$-thinning of $F$ and $H \subseteq G \subseteq F$.*

*The stack $G$ is a $W$-thinning of $F$ if and only if $H$ is a $W$-thinning of $G$.*

This last "confluence" property shows that $W$-thinnings are related to greedy structures [13].

Let us consider the following recognition problem $\mathcal{P}$: given two upstacks $F$ and $G \subseteq F$ in $\mathcal{S}^+$, decide whether $G$ is a $W$-thinning of $F$ or not. By definition, $G$ is a $W$-thinning of $F$ if $G$ may be obtained from $F$ by iteratively lowering (by one) $W$-destructible points. If we directly use this definition for solving $\mathcal{P}$, we get an exponential method. By Theorem 8, $\mathcal{P}$ may be solved by the following greedy method which is polynomial:

Set $H = F$;

(i) arbitrarily select a point $p$ which is $W$-destructible for $H$ and which satisfies $H(p) > G(p)$;
(ii) do $H = [H \backslash p]$.
Repeat (i) and (ii) until stability; if $H = G$, then $G$ is a $W$-thinning of $F$; otherwise $G$ is not a $W$-thinning of $F$.

The above confluence property does not hold in the framework of homotopic thinnings (by deformation retract [28], or by collapse [9], or by simple points removal [12]). A counter-example is the so-called Bing's house [5]. A 3D-cube $C$ may be thinned till one point $P$, but it may also be thinned till a Bing's house $B$. We may have $P \subseteq B \subseteq C$, but $B$ cannot be thinned till $P$. This very example shows that, in the general case, the above greedy method does not work for the recognition problem in the framework of homotopic thinnings.

Of course, $W$-thinnings preserve less topological characteristics than homotopic thinnings. For example, a $W$-thinning of an object $X \subseteq E$ does not necessarily preserve the connected components of $X$. A $W$-thinning of a simple closed curve in a 3D space may delete this curve, thus, 3D holes (3D tunnels) are not necessarily preserved. Nevertheless, the confluence property ensures that arbitrary $W$-thinnings cannot get "stuck" in some configurations.

## 5. Separation

In this section we introduce the notion of separation which will appear to be closely related to extensions and $W$-thinnings. The proofs of Properties 9, 10, 11 may be easily derived and are left to the reader.

*Definition 14.* Let $F \in \mathcal{S}$ and let $x$ and $y$ be two points in $E$.

The point $x$ *dominates* $y$ *for* $F$ if $y$ belongs to the component of $x$ in $F$. The points $x$ and $y$ are *linked for* $F$ if $x$ dominates $y$ for $F$ or $y$ dominates $x$ for $F$.

We observe that a stack $F \in \mathcal{S}$ is connected if and only if any pair of points in $E$ are linked for $F$ (see also [26]).

**Property 9.** *Let $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, and $x, y \in E$.*

(i) *$x$ dominates $y$ for $F$ (resp. $G$) if and only if $F(x, y) = F(x)$ (resp. $G(x, y) = G(x)$), i.e., if and only if there is a path $\pi$ from $x$ to $y$ such that $F(x) \leq F(z)$ (resp. $G(x) \geq G(z)$), for all $z$ in $\pi$.*
(ii) *$x$ and $y$ are linked for $F$ (resp. $G$) if and only if $F(x, y) = \min\{F(x), F(y)\}$ (resp. $G(x, y) = \max\{G(x), G(y)\}$).*
(iii) *$x$ and $y$ are linked for $F$ (resp. $G$) if and only if, for each $k \in \mathbb{K}$, if $x$ and $y$ are in $F[k]$ (resp. $G[k]$), then $x$ and $y$ are linked for $F[k]$ (resp. $G[k]$).*

**Property 10.** *Let $F \in \mathcal{S}$ and let $\Lambda(F)$ be the relation $\Lambda(F) = \{(x, y) \in E \times E;\ x$ dominates $y$ for $F\}$. If $x \in E$, we set $\Lambda(x, F) = \{y \in E;\ (x, y) \in \Lambda(F)\}$.*

(i) *The relation $\Lambda(F)$ is a preorder, i.e., $\Lambda(F)$ is a reflexive and transitive relation.*
(ii) *If $y \in \Lambda(x, F)$ and $x \in \Lambda(y, F)$, then we have $F(x) = F(y)$. The converse is, in general, not true.*
(iii) *For any $x \in E$, there is at least one extremum $X$ of $F$ such that $X \subseteq \Lambda(x, F)$.*
(iv) *A subset $X$ of $E$ is an extremum of $F$ if and only if, for all points $x$ in $X$, $\Lambda(x, F) = X$.*
(v) *A point $x$ of $E$ belongs to an extremum of $F$ if and only if, for all $y \in \Lambda(x, F)$, $x \in \Lambda(y, F)$.*

*Definition 15.* Let $X \subseteq E$ and let $x, y$ be in $X$. The points $x$ and $y$ are *separated for* $X$ if $x$ and $y$ are not linked for $X$, *i.e.*, if $x$ and $y$ belong to distinct connected components of $X$.

Let $F \in \mathcal{S}$ and let $x$ and $y$ be two points in $E$.

The points $x$ and $y$ are *separated for* $F$ if $x$ and $y$ are not linked for $F$. The points $x$ and $y$ are *$k$-separated for* $F$ if $x$ and $y$ are separated for $F$ and if the connection value for $F$ between $x$ and $y$ is precisely $k$, i.e., if $F(x, y) = k$.

In Fig. 2(a), the point $e$ dominates the point $h$ for $F$, the points $d$ and $a$ are 6-separated for $\bar{F}$.

**Property 11.** *Let $F \in \mathcal{S}^+$, $G \in \mathcal{S}^-$, and $x, y \in E$.*

(i) *$x$ and $y$ are separated for $F$ (resp. $G$) if and only if $F(x, y) < \min\{F(x), F(y)\}$ (resp. $G(x, y) > \max\{G(x), G(y)\}$).*
(ii) *$x$ and $y$ are separated for $F$ (resp. $G$) if and only if there exists some $k \in \mathbb{K}$ such that $x$ and $y$ are separated for $F[k]$ (resp. $G[k]$).*
(iii) *$x$ and $y$ are $k$-separated for $F$ (resp. $G$) if and only if:*
   - *$x$ and $y$ belong to the same connected component of $F[k]$ (resp. $G[k]$), i.e., $x$ and $y$ are linked for $F[k]$ (resp. $G[k]$); and*

– *$x$ and $y$ belong to distinct connected components of $F[k+1]$ (resp. $G[k-1]$), i.e., $x$ and $y$ are separated for $F[k+1]$ (resp. $G[k-1]$).*

(iv) *If $X$ and $Y$ are two distinct extrema of $F$, then, any $x \in X$ and any $y \in Y$ are separated for $F$.*

**Property 12.** *Let $F \in \mathcal{S}$ and let $x$ and $y$ be two points which are $k$-separated for $F$. If $x$ dominates $z$ for $F$, then $z$ and $y$ are $k$-separated for $F$.*

**Proof:** By duality we may suppose that $F \in \mathcal{S}^+$. Let $\pi_1$ be a path from $x$ to $y$ such that $F(\pi_1) = F(x, y) = k$, thus $k < \min\{F(x), F(y)\}$. Let $\pi_2$ be a path from $x$ to $z$ such that $F(\pi_2) = F(x, z) = F(x)$. We denote by $\pi_2^{-1}$ the sequence obtained by reversing $\pi_2$. The path $\pi_3 = \pi_2^{-1} \cdot \pi_1$ is a path from $z$ to $y$ such that $F(\pi_3) = \min\{F(\pi_2^{-1}), F(\pi_1)\} = k$, thus, we must have $F(z, y) \geq k$. Let $\pi_4$ be a path from $z$ to $y$. The path $\pi_5 = \pi_2 \cdot \pi_4$ is a path from $x$ to $y$ such that $F(\pi_5) = \min\{F(x), F(\pi_4)\}$. We must have $F(\pi_4) \leq k$ otherwise we would have $F(\pi_5) > k$ which implies $F(x, y) > k$. Thus, $F(z, y) \leq k$. From the above it follows that $F(z, y) = k$ and, since $F(z) \geq F(x)$, we have $F(z, y) < \min\{F(z), F(y)\}$. Consequently, $z$ and $y$ are $k$-separated. □

*Definition 16.* Let $X$, $Y$ be subsets of $E$ such that $X \subseteq Y$. We say that $Y$ is a *separation of* $X$ if any $x$ and $y$ in $X$ which are separated for $X$, are separated for $Y$, *i.e.*, if each connected component of $Y$ contains at most one connected component of $X$.

Let $F$ and $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $F \subseteq G$. We say that $G$ *is a separation of* $F$ if, for any $k \in \mathbb{K}$, $G[k]$ is a separation of $F[k]$.

The following property may be easily derived.

**Property 13.** *Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $F \subseteq G$. The stack $G$ is a separation of $F$ if and only if any $x$ and $y$ in $E$ which are $k$-separated for $F$, are $k$-separated for $G$.*

The following theorem asserts that it is sufficient to consider extrema of $F$ for deciding whether $G$ is a separation of $F$ or not.

**Theorem 14.** (*restriction to extrema*): *Let $F$ and $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $F \subseteq G$. The stack $G$ is a separation of $F$ if and only if, for all distinct extrema $X$, $Y$ of $F$, we have $F(X, Y) = G(X, Y)$.*

**Proof:** By duality, we may suppose that $F$ and $G$ are in $\mathcal{S}^+$.

(i) Suppose $G$ is a separation of $F$ and let $X$, $Y$ be distinct maxima of $F$. Let $k = F(X, Y)$. For all $x \in X$, $y \in Y$, $x$ and $y$ are $k$-separated for $F$, hence they are $k$-separated for $G$ (Property 13).
Thus, $G(X, Y) = \max\{G(x, y);\ x \in X, y \in Y\} = k = F(X, Y)$.

(ii) Suppose $G$ is not a separation of $F$, *i.e.*, there exist two points $x$ and $y$ which are $k$-separated for $F$ (thus $k < \min\{F(x), F(y)\}$) but not $k$-separated for $G$. If $x$ and $y$ are not $k$-separated for $G$, either $x$ and $y$ are linked for $G$, in which case $G(x, y) = \min\{G(x), G(y)\}$, or $G(x, y) \neq k$. Since $F \subseteq G$, in both cases, we must have $G(x, y) > k$. Let $X$ and $Y$ be two maxima of $F$ such that $X \subseteq \Lambda(x, F)$ and $Y \subseteq \Lambda(y, F)$, thus $F(\{x\}, X) = F(x) > k$ and $F(\{y\}, Y) = F(y) > k$.
By Property 12, we have $F(X, Y) = k$ (which implies $X \neq Y$). Since $G(\{x\}, X) \geq F(\{x\}, X) > k$ and $G(\{y\}, Y) \geq F(\{y\}, Y) > k$, we have $G(X, Y) \geq \min\{G(X, \{x\}), G(x, y), G(\{y\}, Y)\} > k$. Therefore, $G(X, Y) \neq F(X, Y)$. □

## 6. Extrema Extension and Strong Separation

If a stack $G$ is a separation of a stack $F$, it may be seen that $G$ may have more extrema than $F$. Since our purpose is to study $W$-thinnings and since $W$-thinnings cannot generate new extrema, we introduce the following notion.

If $F \in \mathcal{S}$, we denote by $\mathcal{E}(F)$ the set composed of all extrema of $F$.

*Definition 17.* Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $F \subseteq G$.

We say that $G$ is an *extrema extension* or an *e-extension of* $F$ if there is a bijection $\epsilon : \mathcal{E}(F) \to \mathcal{E}(G)$ such that:

(i) for all $X \in \mathcal{E}(F)$, $X \subseteq \epsilon(X)$; and
(ii) for all $X \in \mathcal{E}(F)$, $F(X) = G[\epsilon(X)]$.

We say that $G$ is an *extrema cover* or an *e-cover of* $F$ if any extremum $X$ of $G$ contains at least one extremum $Y$ of $F$ such that $G(X) = F(Y)$.

*Definition 18.* Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $F \subseteq G$. We say that $G$ is a *strong* *separation of F* if $G$ is both a separation of $F$ and an e-extension of $F$.

**Property 15.** *Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$, and such that $G$ is a separation of $F$. If $G$ is an e-cover of $F$, then $G$ is a strong separation of $F$.*

**Proof:** By duality, we may suppose that $F$ and $G$ are in $\mathcal{S}^+$.

Suppose $G$ is both a separation and an e-cover of $F$.

(i) Let $X$ be a maximum of $G$. There exists a maximum $Y$ of $F$ such that $Y \subseteq X$ and $G(X) = F(Y)$. Suppose $x$ and $x'$ are two elements of $X$ such that $x \in Y$, $x' \in Y'$, with $Y' \in \mathcal{E}(F)$. Then we must have $Y = Y'$, otherwise $x$ and $x'$ would be separated for $F$ and linked for $G$. Thus, any maximum $X$ of $G$ contains a unique maximum $Y$ of $F$, furthermore $G(X) = F(Y)$.

(ii) Let $X$ be a maximum of $F$ and let $x \in X$. Let $Y$ be a maximum of $G$ such that $Y \subseteq \Lambda(x, G)$. From (i), there is a unique maximum $X'$ of $F$ such that $X' \subseteq Y$. We must have $X = X'$, otherwise $x$ and any element $x' \in X'$ would be separated for $F$ but not separated for $G$. Thus any maximum of $F$ is contained in a maximum of $G$. Of course, this maximum is unique.

□

We are now in position to prove the equivalence between $W$-thinnings and strong separations. Beforehand, we have to establish the following property relative to extensions (see Definitions 12 and 13).

**Property 16.** *Let $F$, $G$ be in $\mathcal{S}^+$ (resp. in $\mathcal{S}^-$) and such that $G$ is an extension of $F$. For each $k \in \mathbb{K}$, we denote by $\epsilon_k$ the extension map relative to $(F[k], G[k])$. Let $k, l \in \mathbb{K}$, with $k \leq l$ (resp. $k \geq l$), and let $X$, $Y$ be respectively a $k$-component and an $l$-component of $F$. Then $Y \subseteq X$ if and only if $\epsilon_l(Y) \subseteq \epsilon_k(X)$.*

**Proof:** By duality, we may suppose that $F$ and $G$ are in $\mathcal{S}^+$.

(i) If $Y \subseteq X$, we have $\epsilon_l(Y) \cap \epsilon_k(X) \neq \emptyset$ (since $F \subseteq G$) and, $G$ being an upstack, we must have $\epsilon_l(Y) \subseteq \epsilon_k(X)$.

(ii) Suppose $Y \not\subseteq X$. Let $Z$ be the $k$-component of $F$ which contains $Y$, we have $Z \neq X$. From (i), we have $\epsilon_l(Y) \subseteq \epsilon_k(Z)$. It is not possible that $\epsilon_l(Y) \subseteq \epsilon_k(X)$, otherwise, we would have $\epsilon_k(X) \cap \epsilon_k(Z) \neq \emptyset$, which implies $\epsilon_k(X) = \epsilon_k(Z)$. Since $\epsilon_k$ is a bijection, this contradicts $Z \neq X$.

□

**Property 17.** *Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. The stack $G$ is an extension of $F$ if and only if $G$ is a strong separation of $F$.*

**Proof:** By duality, we may suppose that $F$ and $G$ are in $\mathcal{S}^+$.

(i) Suppose $G$ is an extension of $F$. For each $k \in \mathbb{K}$, we denote by $\epsilon_k$ the extension map relative to $(F[k], G[k])$.

- Let $X$ be a maximum of $G$ and let $k = G(X)$. The set $Y = \epsilon_k^{-1}(X)$ is a $k$-component of $F$. It is not possible that there exists a $(k+1)$-component $Z$ of $F$, with $Z \subseteq Y$. Otherwise the component $\epsilon_{k+1}(Z)$ of $G[k+1]$ would be such that $\epsilon_{k+1}(Z) \subseteq X$ (Property 16) which contradicts the fact that $X$ is a maximum for $G$. From this it follows that $Y$ is flat for $F$ and that $F(Y) = k = G(X)$. Thus $Y$ is a maximum for $F$, furthermore $Y \subseteq X$ (by definition of an extension). Hence, $G$ is an e-cover of $F$.
- For each $k \in \mathbb{K}$, any connected component of $G[k]$ contains exactly one connected component of $F[k]$, thus $G$ is a separation of $F$.

  It follows that $G$ is a strong separation of $F$ (Property 15).

(ii) Suppose $G$ is a strong separation of $F$. Let $k \in \mathbb{K}$.

- If $F[k] = \emptyset$, we must have $G[k] = \emptyset$, otherwise it could be seen that $G$ would not be an e-extension of $F$ (a maximum of $G$ which is included in $G[k]$ would not contain a maximum of $F$ at the same altitude).
- Suppose $F[k] \neq \emptyset$. Let $X$ be a connected component of $G[k]$. The (non empty) subset $X$ must contain at least one connected component of $F[k]$, otherwise it could be seen that $G$ would not be an e-extension of $F$ (again, a maximum of $G$ which is included in $X$ would not contain a maximum of $F$ at the same altitude). Since $G[k]$ is a separation of $F[k]$, we may affirm that $X$ contains exactly one connected component of $F[k]$. Thus $G$ is an extension of $F$.

□

Property 17 and Theorem 7 lead to the following characterization of $W$-thinnings.

**Theorem 18** (*strong separation*). *Let $F$, $G$ be both in $\mathcal{S}^+$ or both in $\mathcal{S}^-$. The stack $G$ is a $W$-thinning of $F$ if and only if $\bar{G}$ is a strong separation of $\bar{F}$.*

## 7. Ordered Extrema and the Dynamics

Let us consider the following recognition problem $\mathcal{P}$: given two upstacks $F$ and $G$ such that $F \subseteq G$, decide whether $G$ is a separation of $F$ or not. Recall that $G$ is a separation of $F$ iff any $x$ and $y$ in $E$ which are $k$-separated for $F$ are $k$-separated for $G$ (Property 13). Two points $x$ and $y$ are $k$-separated for $F$ iff $F(x, y) < \min\{F(x), F(y)\}$ and $k = F(x, y)$. This leads to a method for solving $\mathcal{P}$ which involves the computation of the connection values $F(x, y)$, $x \in E$ and $y \in E$. We see that $n(n-1)/2$ connection values relative to $F$, with $n = |E|$ are used to solve $\mathcal{P}$ by using this direct approach (the computation of $F(x, x)$ is unnecessary and $F(x, y) = F(y, x)$). Theorem 14 asserts that, in fact, it is sufficient to consider the connection values between distinct extrema of $F$. Thus $m(m-1)/2$ values relative to $F$ are sufficient, $m$ being the number of extrema of $F$. This shows that the above $n(n-1)/2$ values contain some "redundant information" with respect to $\mathcal{P}$. In this section, we will see that, again, the $m(m-1)/2$ connection values relative to the extrema contain redundant information and that only $(m-1)$ values are necessary to solve $\mathcal{P}$.

*Definition 19.* Let $\mathcal{E}$ be a family composed of non empty subsets of $E$ and let $\prec$ be an *ordering on* $\mathcal{E}$, i.e., $\prec$ is a relation on $\mathcal{E}$ which is transitive and trichotomous (for any $X, Y$ in $\mathcal{E}$, one and only one of $X \prec Y$, $Y \prec X$, $X = Y$ is true). We denote by $X^{\prec}_{\max}$ the element of $\mathcal{E}$ such that, for all $Y \in \mathcal{E}\backslash\{X^{\prec}_{\max}\}$, $Y \prec X^{\prec}_{\max}$.

Let $F$ be in $\mathcal{S}^+$ (resp. in $\mathcal{S}^-$) and let $X \in \mathcal{E}$. The *connection value of $X$ for* $(F, \prec)$ is the number $F(X, \prec)$ such that:

- If $X = X^{\prec}_{\max}$, then $F(X, \prec) = -\infty$ (resp. $F(X, \prec) = \infty$); and
- If $X \neq X^{\prec}_{\max}$, then $F(X, \prec) = \max\{F(X, Y) \mid Y \in \mathcal{E}$ and $X \prec Y\}$ (resp. $F(X, \prec) = \min\{F(X, Y) \mid Y \in \mathcal{E}$ and $X \prec Y\}$).

**Theorem 19** (*ordered extrema*). *Let $F$, $G$ be both in $\mathcal{S}^+$ (or both in $\mathcal{S}^-$) such that $F \subseteq G$. Let $\prec$ be an ordering on the extrema of $F$. The stack $G$ is a separation of $F$ if and only if, for each extremum $X$ of $F$, we have $F(X, \prec) = G(X, \prec)$.*

**Proof:** By duality, we may suppose that $F$ and $G$ are in $\mathcal{S}^+$.

By Theorem 14, if $G$ is a separation of $F$, then, for each maximum $X$ of $F$, we have $F(X, \prec) = G(X, \prec)$.

Suppose $G$ is not a separation of $F$. By Theorem 14, it means that there exist two distinct maxima $X$ and $Y$ of $F$ such that $F(X, Y) \neq G(X, Y)$. We set $k = F(X, Y)$, since $F \subseteq G$, we have $G(X, Y) > k$.

There exist two distinct components $X'$ and $Y'$ of $F[k+1]$ such that $X \subseteq X'$, $Y \subseteq Y'$. Furthermore, there exists a component $C$ of $F[k]$ such that $X' \subseteq C$ and $Y' \subseteq C$.

Let $X''$ (resp. $Y''$) be the maximum of $F$ which is a subset of $X'$ (resp. $Y'$) such that $Z \prec X''$ (resp. $Z \prec Y''$), for all maxima $Z$ of $F$, $Z \subseteq X'$ and $Z \neq X''$ (resp. $Z \subseteq Y'$ and $Z \neq Y''$). We observe that $F(X, X'') > k$, $F(Y, Y'') > k$, and $F(X'', Y'') = k$.

Since $G(X'', Y'') \geq \min\{G(X'', X), G(X, Y), G(Y, Y'')\}$, and since $F \subseteq G$, we have $G(X'', Y'') > k$.

Without loss of generality, suppose $X'' \prec Y''$. We observe that, since all maxima $Z$ for $F$ such that $F(X'', Z) > k$ satisfy $Z \subseteq X'$ and $Z \prec X''$, we must have $F(X'', \prec) \leq k$. Furthermore, since $X'' \prec Y''$, we have $F(X'', \prec) \geq k$. The result is $F(X'', \prec) = k$.

But $G(X'', \prec) > k$, which follows from $G(X'', Y'') > k$ and $X'' \prec Y''$. From this we conclude that $F(X'', \prec) \neq G(X'', \prec)$. □

The above definition of the connection value of an extremum leads to a new notion of dynamics the definition of which is given below. In a forthcoming paper [2], it will be shown that this notion "encodes more topological features" than the original one [10].

Let $F$ be in $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) and let $\prec$ be an ordering on the extrema of $F$. We say that $\prec$ is an *altitude ordering on the extrema of* $F$ if $X \prec Y$ whenever $F(X) < F(Y)$ (resp. $F(X) > F(Y)$).

Let $\prec$ be an altitude ordering of the extrema of $F$ and let $X$ be an extremum for $F$. The *dynamics of $X$ for* $(F, \prec)$ is the value $Dyn(X; F, \prec) = F(X) - F(X, \prec)$ (resp. $Dyn(X; F, \prec) = F(X, \prec) - F(X)$).

In Fig. 4, a topological watershed (b) of the original image (a) is represented. This watershed was computed with the algorithm presented in [6], see also [8,20]. The minima of the watershed (c) illustrate the well-known over-segmentation problem. Using the methodology introduced in mathematical morphology [10] and our notions, we can extract all the minima which have a dynamics (according to an altitude ordering) greater than a given threshold (here 20), and suppress all others

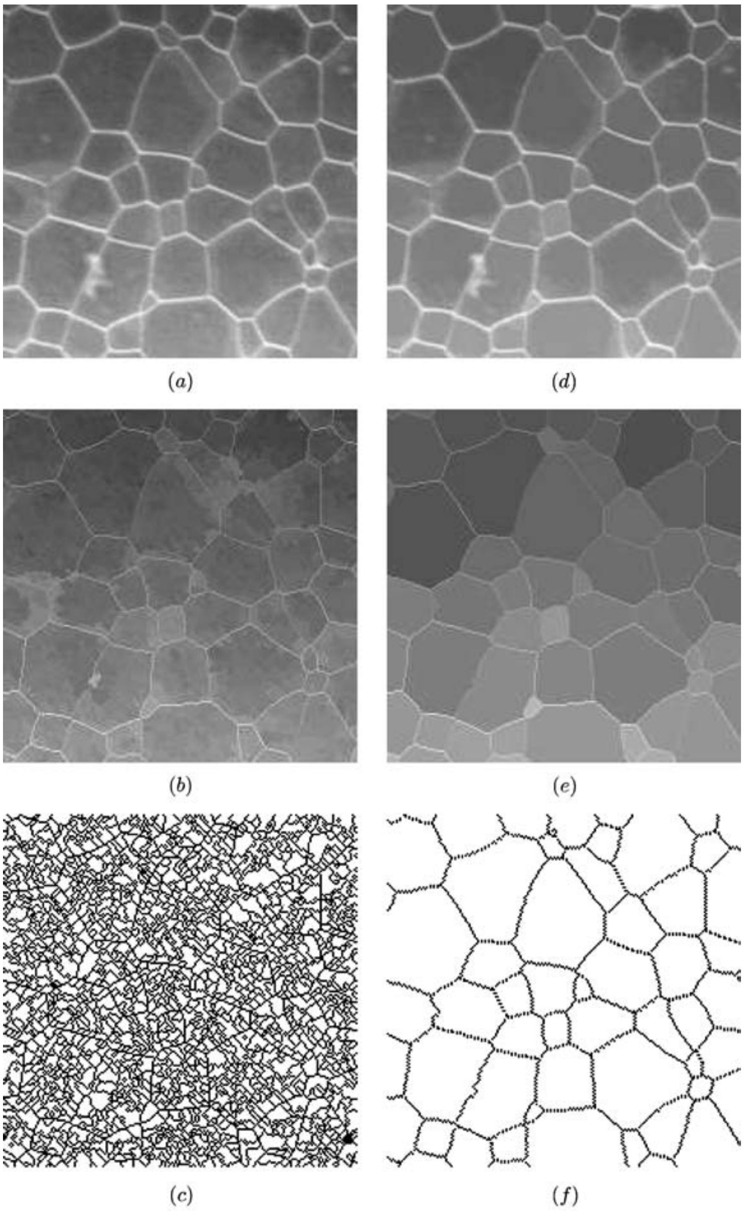


*Figure 4*. (a) original image, (b) a topological watershed of (a), (c) the minima of (b), (d) a filtering of (a) with ordered dynamics, (e) a topological watershed of (d), (f) the minima of (e).

with a geodesic reconstruction. We obtain the image (d), the topological watershed (e), and the minima (f).

## 8. Conclusion

We have seen that a topological watershed of an image satisfies certain basic properties and keeps some features of the original image. Furthermore, it is possible to characterize a $W$-thinning (or a topological watershed) of an upstack (or a map) by several necessary and sufficient conditions.

Forthcoming related papers will include properties of the dynamics, the link between topological watersheds and minimum spanning trees [2], the link between mosaic images and topological watersheds [21], and quasi-linear time algorithms for topological watersheds [8, 20]. See also [8] for comparisons with homotopic thinnings.

## Note

1. This paper is an improved version of a conference paper [1]. We now use stacks instead of functions for deriving our results.

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