Algebra 2 Honors – Notes 13: Sequences and Series

Name: ____ Period: ______

2024 Dates:

Sun	Mon	Tue	Wed	Thu	Fri	Sat

Contents Included:

Textbook Sections	Topics
11.1	Sequences and Series
11.2	Arithmetic Sequences and Partial Sums
11.3	Geometric Sequences and Series

Scoring	
Notes completed fully and accurate on time	_____ / 2
Detailed Summary & Study Guide completed on time	_____ / 2
Total	_____ / 4

11.1 Sequences and Series

Definition of a Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

are the **terms** of the sequence. When the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

When this is the case, the domain includes 0.

Sequence Notation

The **subscripts** of a sequence make up the **domain** of the sequence and serve to **identify the positions of terms within the sequence**.

For example, a_4 is the fourth term of the sequence, and a_n is the nth term of the sequence. Any variable can be a subscript.

The most commonly used variable subscripts in sequence and series notation are i, j, k, and n. We would say "a_i" is the "ith term", "a_j" is the "jth term", "a_k" is the "kth term", and "a_n" is the "nth term".

Sequences are functions, but they are not written with the typical algebraic function notation. This is so that we can distinguish a sequence, which does not take all real numbers as inputs, from real-valued functions, which can.

Consider the sequence of positive integers: 1, 2, 3, 4, 5, . . .

To write this as a sequence, we need a rule that gives each term in the sequence. Since 1, is the first term, and 2 is the second term, and so on, then n is the nth term. Once we know the formula for the nth term of a sequence, that is the rule. So this sequence would be written as:

SKILL: Writing the Terms of a Sequence

a) The first four terms of the sequence given by $a_n = 3n - 2$ are

b) The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

Examples: A Sequence Whose Terms Alternate in Sign

1) Write the first four terms of the sequence given by $a_n = \frac{(-1)^n}{2n+1}$

2) Write the first four terms of the sequence given by $a_n = \frac{(-1)^{n+1}}{2n+1}$

How do these two examples differ?

SKILL: Finding the nth Term of a Sequence

Write an expression for the apparent nth term of each sequence.
(Note that there may be multiple possible formulas. You need only find one if asked, but recognize that a different formula may give the same first few terms but may differ beyond.)

a. $1, 3, 5, 7, \ldots$

b. $2, -5, 10, -17, \ldots$

CALCULATOR SKILL: Graphing a Sequence

Your TI-*n*spire CAS calculator will graph sequences in a Graphs page.
From the entry line, select MENU> Graph Entry/Edit > Sequence > Sequence

The calculator uses typical function notation, using $u1(n)$ for the first sequence formula, $u2(n)$ for a different sequence, etc., the way your functions are labeled $f1(x), f2(x)$, etc.

Type the nth term formula after $u1(n) =$
Leave the initial terms line blank.
Leave the default interval and step for n.

The graph will display with a single dot for each term in the sequence.
Why is this so?

Is it okay to connect the dots? Why or why not?

Definition: Recursive Sequence

Some sequences are defined **recursively**. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms.

Example:
Write the first five terms of the sequence defined recursively as

$a_1 = 3$
$a_k = 2a_{k-1} + 1$ where $k \geq 2$.

Note that in a recursively defined sequence, you cannot find a term without knowing the previous term.

A Very Famous Recursive Sequence – the Fibonacci Sequence

The Fibonacci Sequence is recursively defined as

$a_0 = 1$
$a_1 = 1$
$a_k = a_{k-2} + a_{k-1}$ where $k \geq 2$.

Write the first ten terms of this sequence. (Note that this function starts with the 0^{th} term, so the first ten terms include the 0^{th} term through the 9^{th} term.)

CALCULATOR SKILL: Graphing a Recursive Sequence

Let's graph the Fibonacci Sequence.
From the entry line, select MENU> Graph Entry/Edit > Sequence > Sequence.

Type the nth term formula after $u1(n) =$
List the two initial terms after *Initial Terms:=*
Leave the default interval and step for n. (You can experiment with these on your own ☺)

With a standard viewing window, only a few terms will show. Let's use the table feature to decide how to size our window.
With the graph in view, type CTRL+T to toggle the split screen table view.
Notice that the sequence displays the input value (n) and the sequence term (u1(n)).
How many terms do you want to see in your graph? What viewing window do you need?

To toggle off the table, move your cursor to the graph side of the window and type CTRL+T again.
Change the viewing window using MENU>Window/Zoom > Window Settings …

Definition: Factorial

If n is a positive integer, then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n$$

As a special case, zero factorial is defined as $0! = 1$.

Factorial can also be defined recursively as:

$0! = 1$
$n! = n \cdot (n-1)!$ for $n \geq 1$

SKILL: Simplifying Factorial Expressions

Simplify the following expressions involving factorials. Make sure to follow the order of operations.

1. $4! + 3!$
2. $2(3)!$
3. $(2 \cdot 3)!$
4. $\frac{12!}{3! \cdot 9!}$
5. $\frac{n!}{(n-1)!}$
6. $\frac{(2n+2)!}{(2n+4)!}$

SKILL: Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \frac{2^n}{n!}$, beginning with $n = 0$.

Definition of Summation Notation

A convenient notation for a sum of the terms of a sequence is called **summation notation** or **sigma notation**. It involves the use of the uppercase Greek letter sigma, written as Σ. This is the Greek letter that corresponds to our "S", for "Sum".
(You may be familiar with this symbol if you have used sums in a spreadsheet.)

The sum of the first *n* terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where *i* is the **index of summation**, *n* is the **upper limit of summation**, and 1 is the **lower limit of summation**. (Note that lower limit can be any non-negative integer less than or equal to *n*.)

Note that the index of summation can be any letter. Again, we typically use the letters *i*, *j*, *k* and *n* for these, but it could be anything. The sequence must reference the index of summation.

Summation notation is **an instruction to add the terms of a sequence**.
Note that the upper limit of summation tells you the last term of the sum. So, it is typically a number, not a variable. Summation notation helps you generate the terms of the sequence prior to finding the sum.

Examples:
a)
$$\sum_{i=1}^{5} 3i$$

b)
$$\sum_{k=3}^{6}(1+k^2)$$

c)
$$\sum_{j=0}^{8}\frac{1}{j!}$$

CALCULATOR SKILL: Graphing a Recursive Sequence

Your TI-*n*spire CAS calculator uses sigma notation.
Open a calculator page and press the template button (, next to the 9 key).
Select the template for sigma notation and evaluate the following with your calculator:

1. $\sum_{i=3}^{7}(i^2-3i+2)=$

2. $\sum_{j=6}^{6}j^5=$

3. $\sum_{k=1}^{n}(2k+3)=$

Explain the result of that last sum.

Properties of Sums

These properties follow from the order of operations. However, it is helpful to know them in the sigma notation form.

1. $\sum_{i=1}^{n} c =$

2. $\sum_{i=1}^{n} ca_i =$

3. $\sum_{i=1}^{n} (a_i + b_i) =$

4. $\sum_{i=1}^{n} (a_i - b_i) =$

Definition of Series

A **series** is the sum of terms of a sequence.
Consider the infinite sequence $a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$.

1. The sum of the first *n* terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

2. The sum of *all* the terms of the infinite sequence is called an **infinite series** and is denoted by:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i$$

Note that not all infinite series have a finite sum.

SKILL: Evaluating a Series

Given the series:

$$\sum_{i=1}^{\infty} \frac{3}{10^i}$$

a) find the third partial sum

b) find the sum

APPLICATION: Compound Interest

An investor deposits $5000 in an account that earns 3% interest compounded quarterly. The balance in the account after *n* quarters is given by

$$A_n = 5000\left(1 + \frac{.03}{4}\right)^n, \quad n = 0, 1, 2, \cdots$$

a. Write the first three terms of the sequence.

b. Find the balance in the account after 10 years.

11.2 Arithmetic Sequences and Partial Sums

Definition of Arithmetic Sequence

A sequence whose consecutive terms have a **common difference** is an arithmetic sequence.

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d$$

The number d is the **common difference** of the arithmetic sequence.

Examples of Arithmetic Sequences

a. The sequence whose nth term is $4n + 3$ is arithmetic. The common difference between consecutive terms is ______.

b. The sequence whose nth term is $7 - 5n$ is arithmetic. The common difference between consecutive terms is _______ .

c. The sequence whose nth term is $\frac{1}{4}(n + 3)$ is arithmetic. The common difference between consecutive terms is ________.

What do you notice about the form of an arithmetic sequence?

Deriving the nth Term form of an Arithmetic Sequence
Let's look at successive terms of a sequence with a common difference *d*. $a_1 =$ $a_2 =$ $a_3 =$ $a_4 =$ $a_5 =$ What does the nth term formula appear to be?
The nth term of an arithmetic sequence has the form where *d* is the common difference between consecutive terms of the sequence and a_1 is the first term. You can also write the recursion formula, if you know the nth term and the common difference:

SKILL: **Finding the nth Term of an Arithmetic Sequence**
1) Find a formula for the nth term of the arithmetic sequence whose common difference is 3 and whose first term is 2.
2) Find a formula for the nth term of the arithmetic sequence whose common difference is -7 and whose first term is 20.

Another form of the equation of an arithmetic sequence, if you know two of the terms, say a_n and a_k is:

where d is the common difference between consecutive terms of the sequence.

SKILL: Writing Specified Terms of an Arithmetic Sequence

1) The 4th term of an arithmetic sequence is 20, and the 13th term is 65. Write the nth term formula and write the first 11 terms of this sequence.

2) Find the 10^{th} term of the arithmetic sequence that begins with 7 and 15.

The Sum of a Finite Arithmetic Series

We can express the sum of an arithmetic series with sigma notation.
Every finite arithmetic series has a sum, and it can also be denoted with a capital S subscripted by the number of terms in the sum.

$$S_n = \sum_{i=1}^{n} (a_1 + (i-1)d) =$$

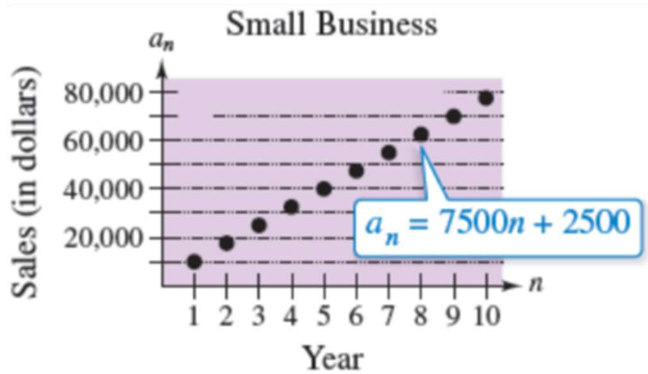
S_n is also called the nth partial sum of the infinite sequence defined by $(a_1 + (i-1)d)$.

✋***SKILL:*** **Sum of a Finite Arithmetic Sequence**
1. Find the sum $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$
2. Find the sum of the first fifty non-negative even numbers.
3. Find the sum of the first N positive integers.

✋***SKILL:*** **Partial Sum of an Arithmetic Sequence**
1. Find the 150th partial sum of the arithmetic sequence 5, 16, 27, 38, 49, . . .
2. Find the 16th partial sum of the arithmetic sequence 100, 95, 90, 85, 80, . . .

✋*APPLICATION:* **Total Sales**

A small business sells \$10,000 worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by \$7,500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.



11.3 Geometric Sequences and Series

Definition of Geometric Sequence
A sequence whose consecutive terms have a **common ratio** is a geometric sequence. A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$ is geometric when there is a number *r* such that $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \quad r \neq 0$ The number *r* is the **common ratio** of the geometric sequence.

Examples of Geometric Sequences
a. The sequence whose nth term is 2^n is geometric. The common ratio of consecutive terms is ______. b. The sequence whose nth term is $4(3^n)$ is geometric. The common ratio of consecutive terms is ______. c. The sequence whose nth term is $\left(\frac{-1}{3}\right)^n$ is geometric. The common ratio of consecutive terms is _____. What do you notice about the form of a geometric sequence?

Deriving the nth Term form of a Geometric Sequence
Let's look at successive terms of a sequence with a common ratio r. $a_1 =$ $a_2 =$ $a_3 =$ $a_4 =$ $a_5 =$ What does the nth term formula appear to be?
The nth term of a geometric sequence has the form where r is the common ratio of consecutive terms of the sequence and a_1 is the first term. So, every geometric sequence can be written in the form: You can also write the recursion formula, if you know the nth term and the common ratio: Another form of the equation of a geometric sequence, if you know two of the terms, say a_n and a_k is: where r is the common ratio of consecutive terms of the sequence.

✋*SKILL:* **Writing Terms of a Geometric Sequence**
1) Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.
2) Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

✋*SKILL:* **Writing the nth Term of a Geometric Sequence**
1) Find a formula for the nth term of the geometric sequence 5, 15, 45, . . . What is the 12th term of the sequence?
2) Find a formula for the nth term of the geometric sequence $2700, -1800, 1200, \ldots$ What is the 8th term of the sequence?

✋*SKILL:* Finding Specified Terms of a Geometric Sequence

1) The 4th term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive. What would happen if we did not assume this?)

2) The 3rd term of a geometric sequence is $-\frac{1}{2}$, and the 6th term is 4. Find the 11th term.

The Sum of a Finite Geometric Series

Every finite geometric series has a sum, and it can also be denoted with a capital S subscripted by the number of terms in the sum.

$$S_n = \sum_{i=1}^{n} \left(a_1 \cdot r^{i-1}\right) =$$

with common ratio $r \neq 1$.

S_n is also called the nth partial sum of the infinite sequence defined by $(a_1 \cdot r^{n-1})$.

✋*SKILL:* Sum of a Finite Geometric Sequence

1. Find the sum:

$$\sum_{i=1}^{12} 4(0.3)^{i-1}$$

2. Find the sum

$$\sum_{i=1}^{12} 4(0.3)^i$$

The Sum of an Infinite Geometric Series

Some infinite geometric series have a sum, and it can also be denoted with a capital *S* with no subscript.

$$S = \sum_{i=1}^{\infty}\left(a_1 \cdot r^{i-1}\right) =$$

when common ratio r satisfies the inequality $|r| < 1$.

When $|r| \geq 1$ the series does not have a sum.

Sometimes the lower limit of summation is zero, not 1. Then we use this formula for the infinite sum. The restriction on *r* is the same.

$$S = \sum_{j=0}^{\infty}\left(a_1 \cdot r^{j}\right) =$$

✋*SKILL:* Finding the Sum of an Infinite Geometric Series

1. Find the sum:

$$\sum_{n=0}^{\infty} 4(0.6)^n$$

2. Find the sum: $5 + 1 + 0.2 + 0.04 + \dots$
3. Find the sum: $\sum_{i=1}^{\infty} 3(0.4)^i$

APPLICATION: Increasing Annuity
An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an increasing annuity.)

Summary

Making Sense of the Big Ideas	
Unit	**Big Ideas and Skills I Learned in This Unit**
11.1: Sequences and Series	
11.2: Arithmetic Sequences and Partial Sums	
11.3 Geometric Sequences and Series	

Note: If this organizer does not give you enough space, please add pages as needed.