Algebra 2 Honors – Notes 14: Counting, Binomial Theorem & Probability

Name: ____ Period: ______

2024 Dates:

Sun	Mon	Tue	Wed	Thu	Fri	Sat

Contents Included:

Textbook Sections	**Topics**
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11.5	The Binomial Theorem
11.7	Probability

Scoring	
Notes completed fully and accurate on time	_____ / 2
Detailed Summary & Study Guide completed on time	_____ / 2
Total	_____ / 4

11.6 Counting Principles

Counting things is an activity that humans have done for as long as we can tell in recorded history. It doesn't sound hard to do, once you have a numbering scheme. But, counting things can get complicated very fast! The mathematics of counting is also known as *combinatorics* and is typically within a branch of higher mathematics known as *discrete mathematics*. We will explore the basic principles of this branch of mathematics, but I hope that all of you will, in time, experience a fuller course in discrete mathematics, most likely after taking a few years of calculus! ☺ (Something to look forward to!)

In this section we will:

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

"How Many Different Ways…?"

Counting problems often ask how many different ways a particular outcome of a task can occur.

If we can list all the possible outcomes, then we can count how many satisfy the particular outcome we are looking for.

Sometimes, listing them all out is tedious, so we look for patterns that allow us to count using formulas.

Counting sounds easy but can get complicated and non-intuitive. Recognizing the structure of a counting problem can help us apply the appropriate formula.

Remember, you can always start listing out the possibilities to help see a pattern!

Simple Counting Problem - Examples

1. A 6 sided die is rolled and the number is recorded. A die is rolled again and that number is added to the first number. How many ways can you obtain a sum of 8?

2. 6 balls, each numbered 1 thru 6 are placed in a box. Two balls are selected successively, without replacement. The numbers are added together. How many ways could you obtain a sum of 8?

How are the two previous examples the same? How are they different?

The Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways the two events can occur is $m_1 \cdot m_2$.

This extends to more than two events, by continuing to multiply by the number of ways each successive event can occur.

✋ *SKILL*: Applying the Fundamental Counting Principle

1. How many different pairs of letters from the English alphabet are possible?

2. How many different telephone numbers are possible within each area code? (phone numbers do not begin with 0 or 1)

꩜ Counting Principles ꩜

Sometimes, we want to count how many ways we can arrange things when order matters. The Fundamental Counting Principle can be used to develop a formula for finding this count.

***DEFINITION*: Permutation**

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Let's explore. How many ways can I call on everyone in the class to each do one problem?

Number of Permutations of *n* Elements

The number of permutations of n elements is

In other words, there are different ways of ordering n elements.

Remember order matters! B, C, A is a different permutation than A, C, B.

***SKILL*: Counting Permutations**

1. How many ways can you order 5 different textbooks on a shelf?

2. How many ways can you give 18 different teddy bears to 18 children?

Sometimes, we want to count the permutations of a *subset* of the n objects.
Consider this example:

How many 9-person baseball lineups can you make from a team of 11 people?

How can we generalize this counting formula?

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

***CALCULATOR SKILL*: Permutations**

Using nPr on your calculator:

Menu>Probability>Permutations
nPr(n, r)

Try these:

1. ${}_{12}P_3 =$

2. ${}_{6}P_2 =$

3. ${}_{7}P_1 =$

4. ${}_{9}P_9 =$

SKILL: Permutations

1. Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

2. How many different ways can the letters of MEXICO be arranged?

Sometimes, when we want to count the permutations of a *subset* of the n objects, some of the permutations *look exactly alike*. That is, we can get repeats of the same order. Consider this example:

How many different ways can the letters of CANADA be arranged? (How is different than MEXICO?)

Distinguishable Permutations of *n* Elements Taken *r* at a Time

Consider a set of n objects that has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \ldots + n_k$$

The number of distinguishable permutations of the n objects is

Guided Practice: Permutations

1) In how many distinguishable ways can the letters BANANA be written?

2) What about Mississippi?

3) What about Donovan?

4) How many ways can grades of 4 A's, 10 B's, 3 C's, 1 D be given out to a class of 18 students?

5) If a soccer team wins 6 games, loses 2 and ties 3 times, how many ways could that happen?

6) After a softball practice of a team of 14 players, 3 players pick up the bases, 2 players put the equipment away and 5 players rake the field. The remaining players can just go home without doing a job. How many practices would be needed in order to have a unique group do each job exactly once?

Sometimes, order does *not* matter when we want to count.

Consider: How many groups of three students can be chosen to represent a club of eight students?

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

which is equivalent to

***CALCULATOR SKILL*: Combinations**

Using nCr on your calculator:

Menu>Probability>Combinations
nCr(n, r)

Try these:

1. ${}_{12}C_3 =$

2. ${}_{6}C_2 =$

3. ${}_{7}C_1 =$

4. ${}_{9}C_9 =$

✋ *Guided Practice*: Combinations

1) How many 3 topping pizzas can you make out of 8 possible toppings?

2) How many five-card hands can be dealt from a deck of 52 cards?

3) You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Strategy for Approaching Counting Problems

When solving problems involving counting principles, you need to distinguish among the various counting principles to determine which is necessary to solve the problem. To do this, ask yourself the questions below.

- Is the order of the elements important? Permutation
- Is the order of the elements not important? Combination
- Does the problem involve two or more separate events? Fundamental Counting Principle

Some problems are best addressed in stages, with multiple strategies used.

Decide: Permutations or Combinations?

1) Selecting three toppings for ice cream
2) Selecting the batting order for a baseball game.
3) Selecting a team of 5 people for a game of dodgeball.
4) Ranking your two favorite musical groups.
5) Selecting four ingredients for a soup.
6) Selecting three officers for a club.

Guided Practice: Counting Problems

1) How many ways can 7 people line up in cafeteria?

2) How many license plates can be made of 3 letters followed by 3 digits?

3) How many license plates can be made of 6 letters and or digits in any order?

4) How many license plates can be made of 3 letters and 3 digits in any order?

5) A company advertises two job openings, one for a copywriter and one for an artist. If 10 people who are qualified for either position apply, in how many ways can the openings be filled?

6) A company advertises two job openings for computer programmers, both with the same salary and job description. In how many ways can the openings be filled if 10 people apply?

7) For a certain raffle (Q3&4), 845 tickets are sold.
   a. In how many ways can four $50 gift certificates be awarded?

   b. In how many ways can a $100, a $50, a $20, and a $10 gift certificate be awarded?

8) At the Red Lion Diner, an omelet can be ordered plain or with any or all of the following fillings: cheese, onions, peppers. How many different kinds of omelets are possible?

11.5 The Binomial Theorem

Binomial Expansion
Consider the expansion of the following binomial expressions: $(x+y)^0$ $(x+y)^1$ $(x+y)^2$ Recall that we can use a pattern to expand a binomial to any whole power. What is this pattern?

Pascal's Triangle
Row 0 Row 1 Row 2 Row 3 Row 4 Row 5 Row 6 Row 7 Row 8 Row 9 Row 10 . . . We can see that Pascal's triangle gives us the coefficients we need for the terms of the binomial expansion, but can we get these without writing out all the rows of the triangle?

Pascal's Triangle and Combinations

Consider the top row of the triangle as the 0^{th} row, and the leading one in each row to be in position 0.

Then the position in the triangle can be found by nCr(n,r) where n is the row number and r is the position in the row.

What are the coefficients in the expansion of $(x + y)^7$?

How can we find this using combinations?

The Binomial Theorem

In the expansion of $(x + y)^n$

the coefficient of $x^{n-r}y^r$ is

The symbol is often used in place of to denote binomial coefficients.

It is read as " ".

✋*SKILL:* Finding the Coefficient of a Term in Binomial Expansion

Evaluate the binomial coefficient of a term in $(x + y)^n$ and give the term.

1) ${}_5C_3$

2) $\binom{8}{6}$

✋*SKILL:* Expanding Binomials

Expand the following binomials using the binomial theorem:

1) $(2x - y)^3$

2) $(3x - 2y)^4$

3) $(x^2 + y^3)^4$

4) $(2x^3 - 3y^3)^3$

What do you notice about the exponents in the previous two examples?

5) Find the fifth term in the expansion of $(x^4 - 2y)^{10}$

6) Find the coefficient of the term in the expansion of $(x^3 - 5y)^7$ that includes x^3.

The Binomial Theorem – Summation Form

Expanding a binomial produces a sum, with each term according to a pattern.
So, we can express this sum using the sigma notation:

$$(x + y)^n =$$

where n is a non-negative integer.

11.7 Probability

DEFINITIONS: **Experiment, Outcome, Sample Space, Event**
Any happening for which the result is uncertain is an **experiment**. The possible results of the experiment are **outcomes**, the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event**.

✋***SKILL:*** **Listing the Sample Space of an Experiment**
Suppose we roll a six-sided die, toss a coin, and spin a spinner with equal-sized segments of red, yellow and blue.

The Probability of an Event
If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is The number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, so the probability of an event must be a number between 0 and 1, inclusive. That is, When $P(E) = 0$, When $P(E) = 1$,

✋ *GUIDED PRACTICE:* Finding Probabilities

1) Two coins are tossed. Find P(at least one head).

2) One card is drawn from a standard deck of cards. Find P(Ace).

3) Two six sided die are tossed.
What is the probability that the total numbers showing is 7?

4) What is the probability that the total number sum is not 7?

5) In Arizona's The Pick game, a player chooses six different numbers from 1 to 44. If these six numbers match the six numbers drawn (in any order), the player wins (or shares) the top prize. What is the probability of winning the top prize when the player buys one ticket?

***DEFINITION:* Mutually Exclusive Events**
Symbolic notation for "*A* and *B*" is written as   Two events *A* and *B* (from the same sample space) are mutually exclusive when *A* and *B* have no outcomes in common. In the terminology of sets, the intersection of *A* and *B* is the empty set, which implies that

Probability of a Union of Two Events
Symbolic notation for "*A* or *B*" is written as   If *A* and *B* are events in the same sample space, then the probability of *A* or *B* occurring is given by   If *A* and *B* are mutually exclusive, then

***SKILL:* Probability of a Union of Events**
You draw one card at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

✋ ***GUIDED PRACTICE:*** **Finding the Probability of a Union of Events**

Roll two six-sided dice, one red and one green. Find the probabilities:

1. P(Even on red die or odd on the green die)

2. P(Both dice are the same or red die a 6)

Draw a card from a standard 52-card deck. Find the probabilities:

3. P(diamond or black)

4. P(Face card or heart)

5. P(Not a jack)

DEFINITION: **Independent Events**

Two events are independent when the occurrence of one has no effect on the occurrence of the other. For example, rolling a die once and getting a six has no effect on the result of the next roll of the die. To find the probability that two independent events will occur, multiply the probabilities of each.

If A and B are independent events, then the probability that both A and B will occur is

This rule can be extended to any number of independent events.

✋*SKILL:* **Finding the Probability of Independent Events**
1. A random number generator selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?
2. In 2015, approximately 65% of Americans expected much of the workforce to be automated within 50 years. In a survey, researchers selected 10 people at random from the population. What is the probability that all 10 people expected much of the workforce to be automated within 50 years? (Source: Pew Research Center)

DEFINITION: **Complement of an Event**
The **complement of an event** A is the collection of all outcomes in the sample space that are not in A. The complement of event A is denoted by A'. Because A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$.
Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is

✋*SKILL:* **Finding the Probability Using the Complement of an Event**
1. A manufacturer is determined that a certain machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Summary

Topic	Big Ideas; What I Want To Remember; Vocabulary; Notation
11.5: The Binomial Theorem	
11.6: Counting Principles	
11.7 Probability	

Note: If this organizer does not give you enough space, please add pages as needed.