Limit Definition of the Derivative

Hopefully you took the time to review the various methods of using the limit definition of the derivative. To make sure you've got a good sense of what's going on, let's take a look at some problems.

Problem: Use the limit definition of the derivative to find the first derivative of each of the following functions.

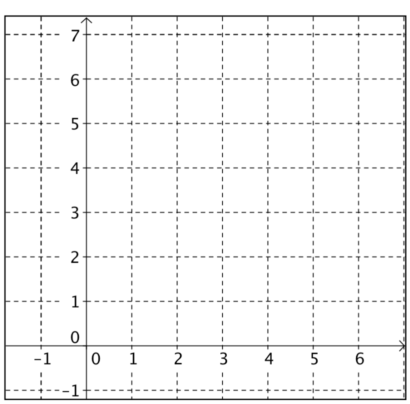
a. $f(x) = x^2 + 3x + 1$

b. $y = \sqrt{x+3}$

c. $f(x) = \dfrac{1}{x+2}$

d. $y = \dfrac{1}{\sqrt{2x+1}}$

Problem: Use one-sided limits to show that $f(x) = |2x-3| + 1$ is not differentiable at $x = 3/2$. Also sketch the graph in the space provided.



You'll be using the limit definition a lot, but mostly you'll be recognizing it. So let's move on for now and circle back to this idea in a bit.

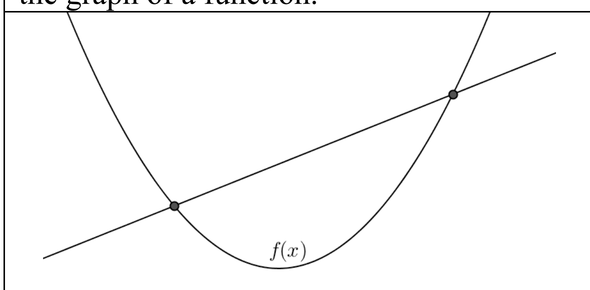
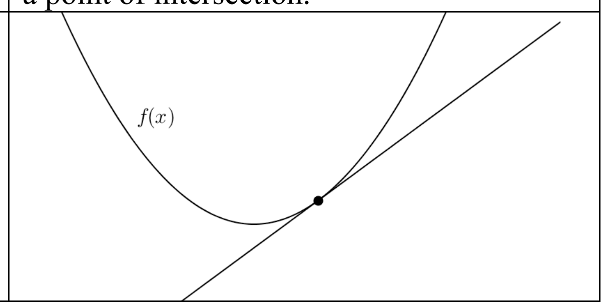
If you're on top of your game coming into this year, you've already reviewed the basic derivative rules that you learned last year. Let's summarize them below.

Summary of Derivatives Rules (So Far…)	
Derivative of a Constant	Power Rule
Constant Multiple Rule	Sum and Difference Rule
Product Rule	Quotient Rule

It's your job to make sure that you are ruthlessly efficient at using all of these rules! There are many problem sets with tons of problems to practice on or you can just make up your own. Check them on your TI-Nspire CAS.

Tangent and Secant Lines

There are two really important types of lines in Calculus. They are called tangent lines and secant lines.

Secant Line	Tangent Line
A line that passes through two points on the graph of a function.	A line that has the same slope as a curve at a point of intersection.
	

Problem: Let $f(x)=\dfrac{x+3}{x-6}$.

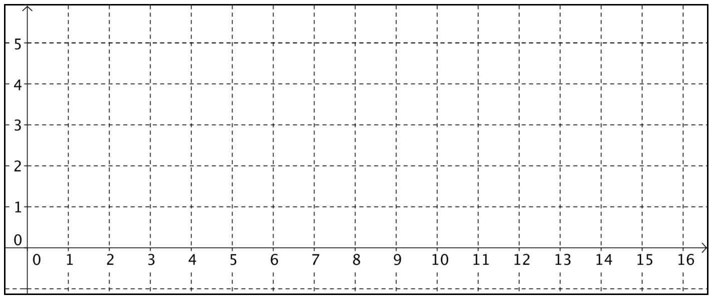
a. Find $f'(x)$ by quotient rule.

b. Write the equation of the secant line through $(5, f(5))$ and $(-3, f(-3))$.

c. Write the equation of the tangent line at $x=3$.

Problem: Given that $f(x)=\sqrt{x}$.

a. Sketch the graph of $f(x)$ below.



b. Find $f'(x)$ using the limit definition of the derivative. Confirm using the Power Rule.

c. Write the equation of the line tangent to $f(x)$ at $x=9$. Add the tangent to your graph.

d. Use the tangent line to estimate $f(9.1)=\sqrt{9.1}$. Based on the shape of the curve, is this an over or underestimate?

e. Write the equation of the line secant to $f(x)$ on the interval $9\le x\le 16$. Add the secant line to your graph.

f. Use the secant line to approximate $f(13)=\sqrt{13}$. Based on the shape of the curve, is this an over or underestimate?

The location of a secant or tangent line will determine what kind of error you get in using it for an approximation. A quick sketch is always useful in making sure you're thinking correctly.

Using your calculator is not optional in this class. In fact, as a personal challenge, I think you should try very hard to do every problem both by hand and by calculator—unless the problem specifically says to use a calculator…some aren't even possible by hand.

Problem: Use your calculator's built-in ability to find numerical derivatives to evaluate each of the following derivatives.

a. $f'(3)$ for $f(x)=\sqrt{x}\cos\left(x^2+1\right)$

b. $g'(-2)$ for $g(x)=\dfrac{x^2+3}{x(x+3)}$

Warning: When entering a function, multiplication between a variable and the opening parentheses is *not* optional. If you don't put it the calculator thinks you're defining a new function instead of doing multiplication.

Problem: Estimate $f(2.8)$ for $f(x)=\sqrt{x}\cos\left(x^2+1\right)$ using the tangent line to $f(x)$ at $x=3$.

You can also use your calculator to graph both the first and second derivative (and any derivative if you have the right calculator).

Problem: Use a calculator to graph and find the zeros of $f'(x)$, the derivative of $f(x)=x^3+3x^2-5x-6$. Make sure to pay attention to how we show our work here.

Problem: Use your calculator to graph the derivative of $f(x)=3x^2+x+\cos(6x)$. Find the *x*-coordinates of all points on the graph of $y=f(x)$ at which the slope is 3.

Derivatives of Sine, Cosine—and the Rest of The Trig Functions

Problem: Use your calculator to graph the derivative of $f(x)=\sin(x)$. What function appears to be the derivative of sine? (It actually *is* the derivative of sine.)

Problem: Use your calculator to graph the derivative of $f(x)=\cos(x)$. What function appears to be the derivative of cosine? (It actually *is* the derivative of cosine.)

You can use your calculator to graph a very accurate version of the derivative of virtually any function. Your calculator is actually using a difference quotient with a very small value for h to calculate the derivative at each value of x on the screen so it can take a long time.

Remember the ratio and reciprocal identities? Sure you do…

Problem: Use the ratio and reciprocal identities and your newfound knowledge of the derivatives of sine and cosine to find the derivatives of $\sec(x)$, $\csc(x)$, $\tan(x)$, and $\cot(x)$.

Derivatives of the Trig Functions	
$\frac{d}{dx}[\sin(x)]=$	$\frac{d}{dx}[\cos(x)]=$
$\frac{d}{dx}[\sec(x)]=$	$\frac{d}{dx}[\csc(x)]=$
$\frac{d}{dx}[\tan(x)]=$	$\frac{d}{dx}[\cot(x)]=$

You need to memorize everything in this table as soon as possible—and preferably in exactly the way I wrote them when I filled this in on the whiteboard.

Derivatives of e^x and $\ln(x)$

Problem: Use your calculator to graph the derivative of $y = e^x$. What function appears to be the derivative of e^x? How weird is that? It's actually the only function for which this is the case.

Problem: Use your calculator to graph the derivative of $y = \ln(x)$. What function appears to be the derivative of $\ln(x)$? What restriction do you need to throw on that?

Derivatives You Now Have Memorized	
$\frac{d}{dx}[\sin(x)] =$	$\frac{d}{dx}[\cos(x)] =$
$\frac{d}{dx}[\sec(x)] =$	$\frac{d}{dx}[\csc(x)] =$
$\frac{d}{dx}[\tan(x)] =$	$\frac{d}{dx}[\cot(x)] =$
$\frac{d}{dx}[e^x] =$	$\frac{d}{dx}[\ln(x)] =$

Now you need to memorize this expanded table!

Problem: Find the first eight derivatives of $f(x) = \sin(x)$. Find the 235^{th} derivative of $\sin(x)$.

Problem: Find the equations of all lines tangent to the graph of $y = \cos(x)$, $0 \le x \le 2\pi$, having slope $\sqrt{3}/2$. Graph on your calculator so that you get faster at using your calculator.

In the space provided, write down the alternate definition of the derivative that results in the slope of the function at a specific point.

Problem: Interpret each of the following limits as a derivative and evaluate the limit accordingly.

a. $\lim_{x \to \pi/6} \frac{\cos(x) - \left(\sqrt{3}/2\right)}{x - (\pi/6)}$

b. $\lim_{x \to 5} \frac{e^{x+3} - e^{8}}{x - 5}$

c. $\lim_{x \to 2} \frac{\sqrt{9x^2} - 6}{x - 2}$

d. $\lim_{x \to \pi/4} \frac{\csc(x) - \sqrt{2}}{x - (\pi/4)}$

It's extremely common to run into limits that you don't immediately (or really ever) know how to do because they're actually just derivatives in disguise. Be on the lookout for that!

Problem: Write each of the following as a limit and evaluate it.

a. The derivative of $\sin(x)$ at $x = 2\pi/3$.

b. The derivative of $\ln(x)$ at $x = 5$.

Problem: Let's find a bunch of derivatives. Keep up!

a. $f(x)=(6x^2+5)(x-4)$

b. $g(x)=\dfrac{x+3}{x+5}$

c. $y=\sqrt[3]{x^5}+5x^{3/2}$

d. $y=(3x-2)^3$

e. $y=\sin(x)\cdot x^3$

f. $f(x)=x^3\ln(x)$

g. $y=\sqrt{12x^7}$

h. $h(t)=-16t^2+42t+8$

Problem: Given that $V=\pi r^2 h$ find $\dfrac{dV}{dr}$ and $\dfrac{dV}{dh}$. What's the difference?

Problem: Given that $V=\dfrac{\pi}{3}r^2 h$ and that both *r* and *h* are functions of *t*, find $\dfrac{dV}{dt}$.

There's a whole set of questions in calculus that you'll encounter and sort of be unsure how to solve. That's okay! It's why you're in the class. Some of those types of problems might look like the ones we're about to do…no time like the present, right?

Problem: Find the equation(s) of all lines tangent to $y = x^3 + 3x$ having a slope of 6. Find any additional points (besides the point of tangency) at which any tangent lines intersect the curve. What important piece of information have you gleaned from this?

Problem: Given $f(x) = x^3$.

a. Find the equation of the line tangent to $y = f(x)$ at the point $(2,8)$.

b. Set up an equation whose solutions give the *x*-coordinates of the points of intersection of the tangent line and the curve.

c. Solve the equation by hand. (Hint: Think about what *x*-value you know for sure is a solution and then use synthetic division.)

Problem: The line $4x + y + 3 = 0$ is tangent to the curve $y = 4x^2 - 2$. Find the point of tangency. This can be done on “two different levels.” See if you can figure out what that means.

Problem: Find the equations of the lines tangent to $f(x) = x^2$ that pass through the point $(4,7)$. Note: That point is *not* on the curve.

Remember All This Stuff

Summary of Derivatives Rules (So Far…)	
Derivative of a Constant	Power Rule
Constant Multiple Rule	Sum and Difference Rule
Product Rule	Quotient Rule

Derivatives You Now Have Memorized

$\frac{d}{dx}\left[\sin(x)\right]=$	$\frac{d}{dx}\left[\cos(x)\right]=$
$\frac{d}{dx}\left[\sec(x)\right]=$	$\frac{d}{dx}\left[\csc(x)\right]=$
$\frac{d}{dx}\left[\tan(x)\right]=$	$\frac{d}{dx}\left[\cot(x)\right]=$
$\frac{d}{dx}\left[e^x\right]=$	$\frac{d}{dx}\left[\ln(x)\right]=$

Two Super Important Lines

Secant Line	Tangent Line

Definition of Derivative

Gives you a function…	Gives you a number…

Note: The definition that gives you a function will often have $x = a$ subbed in and, therefore, will actually give you a number.