Algebra 2 Honors – Notes 15: Quadratic and Polynomial Functions

Name: ____ Period: _________

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Textbook Section	Topics
3.1 Quadratic Functions and Models	1. Analyze graphs of quadratic functions
	2. Write quadratic functions in standard form and use the results to sketch their graphs
	3. Find minimum and maximum values of quadratic functions in real-life applications
4.3 Parabolas	1. Recognize equations of parabolas (vertex at origin)
	2. Graph equations of parabolas (vertex at origin)
	3. Write equations of parabolas (vertex at origin)
3.2 Polynomial Functions of Higher Degree	1. Use transformations to sketch graphs of polynomial functions
	2. Use the Leading Coefficient Test to determine the end behaviors of graphs of polynomial functions
	3. Find real zeros of polynomial functions and use them in sketching
	4. Use the Intermediate Value Theorem to help locate real zeros of polynomial functions
3.4 Zeros of Polynomial Functions	1. Use the Fundamental Theorem of Algebra
	2. Find rational zeros of polynomial functions
	3. Find complex zeros using conjugate pairs
	4. Find zeros of polynomials by factoring
	5. Use Descartes's Rule of Signs and Upper and Lower Bound Rules
	6. Find zeros of polynomials in real-life applications

Scoring	
Notes completed fully and accurate on time	_____ / 2
Detailed Summary & Study Guide completed on time	_____ / 2
Total	_____ / 4

3.1 Quadratic Functions and Models

Polynomial Functions
Recall the definition of a polynomial expression in one variable. A **polynomial function** is a function with a rule that can be written in the form of a polynomial expression. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where n is an integer, $n \geq 0$ and $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real numbers and $a_n \neq 0$.

DEFINITION: **Quadratic Function**
A **quadratic function** is a polynomial function of degree 2. Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is a **quadratic function.**

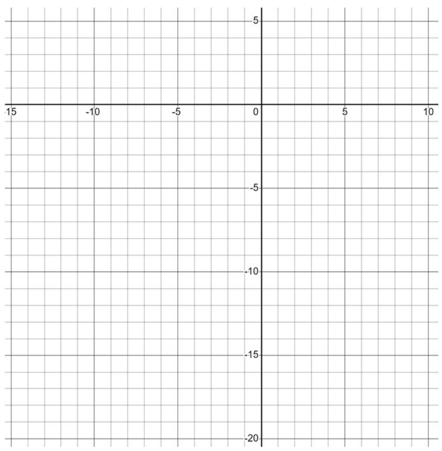
Standard Form of a Quadratic Function
The quadratic function $f(x) = a(x-h)^2 + k,\ a \neq 0$ is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward. When $a < 0$, the parabola opens downward.

Facts About Quadratic Functions
• The graph of a quadratic function is a parabola which is approximately u-shaped.
• The graph of a quadratic function is symmetric with respect to the vertical line through the vertex.
• The domain of a quadratic function is $-\infty < x < \infty$. (All real numbers).
• The y −coordinate of the vertex of a quadratic function is a minimum if $a > 0$ or a maximum if $a < 0$.
• The range of a quadratic function is $y \geq k$ if $a > 0$, or $y \leq k$ if $a < 0$.

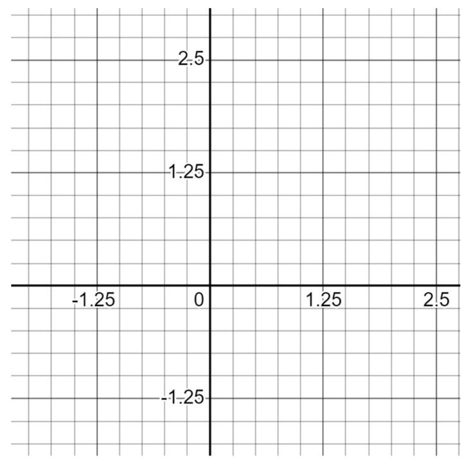
✍ Graphing Quadratic Functions from Standard Form

Graph the following Quadratic Functions.

1. $f(x) = -2(x+2)^2 - 5$



2. $g(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{3}{2}$



✍ Transform Quadratic Functions into Standard Form

Transform quadratic functions into standard form by completing the square. Then sketch a graph.

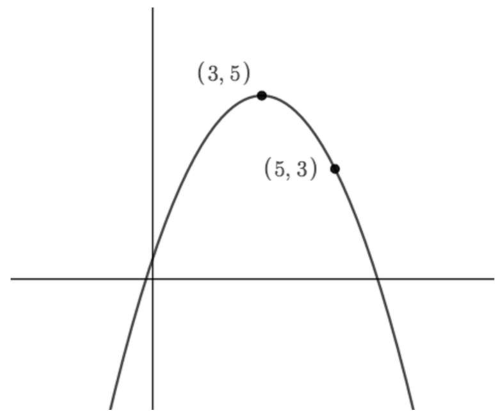
3. $f(x) = 2x^2 + 8x + 7$

4. $f(x) = -x^2 + 6x - 8$

Write a Quadratic Function Given the Vertex and Another Point

Write the standard form of a quadratic function from a graph or when given the vertex and another point on the graph.

5. Write the standard form of the quadratic function shown.



6. Write the standard form of the quadratic function given that the vertex is at $(-3,-2)$ and the y-intercept is $(0,1)$.

Find the *x*-intercepts of a Quadratic Function

The function value is zero at an x-intercept. We find these algebraically by setting the function equal to zero and solving. If there are no real solutions, then there are no x-intercepts on the graph. Note that we *still* call these solutions the zeros of the function.

Find the x-intercepts of each function algebraically.

7. $f(x) = 4x^2 - 2x - 6$

8. $f(x) = -3x^2 - 27$

9. $g(x) = -4(x-2)^2 + 1$

Vertex of a Quadratic Function

For a quadratic function $f(x) = ax^2 + bx + c$

the vertex x-coordinate is given by

and the vertex y-coordinate is given by

Note that if the graph has x-intercepts, the vertex x-coordinate is at the midpoint of the segment between them.
Why is this so?

Applications of Quadratic Functions

There are many ways that quadratic functions can model our world. Often we are looking for the maximum or minimum value of a quadratic function. Keep in mind that **the domain** of a function that is modeling a specific scenario is determined by the input values that are meaningful to the model. Always note the domain and also always include units in your final answers.

10. The path of a punted football is modeled by

$$f(x) = -\frac{16}{2025}x^2 + \frac{9}{5}x + 1.5$$

where $f(x)$ is the height in feet and x is the horizontal distance in feet from the point at which the ball is punted.

a) How high is the ball when it is punted?

b) What is the maximum height of the punt?

c) How long is the punt?

11. A bottled-water manufacturer has a daily production cost given by

$$C(x) = 70000 - 120x + 0.055x^2$$

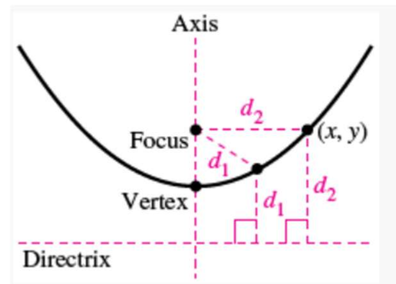
where $C(x)$ is the total cost in dollars and x is the number of units produced. How many units should they produce each day to yield a minimum cost?

4.3 Parabolas

We define a quadratic function in the form $y = f(x)$. But the shape of the function is called a **parabola**, and that shape can open up, open down, or open left or open right.

✍ *DEFINITION:* Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line.
The **vertex** is the midpoint between the focus and the directrix.
The **axis** of the parabola is the line passing through the focus and the vertex.



Note in the figure above that a parabola is symmetric with respect to its axis. The definition of a parabola can be used to derive the **standard form of the equation of a parabola** with vertex at (0,0) and directrix parallel to the x-axis or to the y-axis.

✍ Standard Form of a Parabola with Vertex at (0,0)

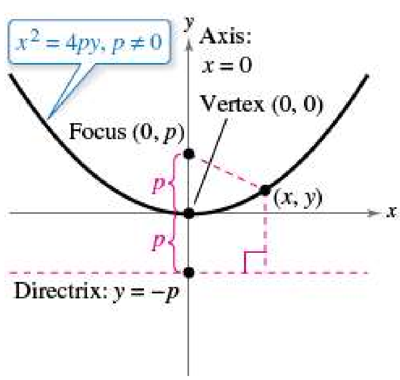
The **standard form of the equation of a parabola** with vertex at (0,0) and **directrix** $y = -p$ is

$$x^2 = 4py,\ p \neq 0 \qquad \textbf{Vertical axis}$$

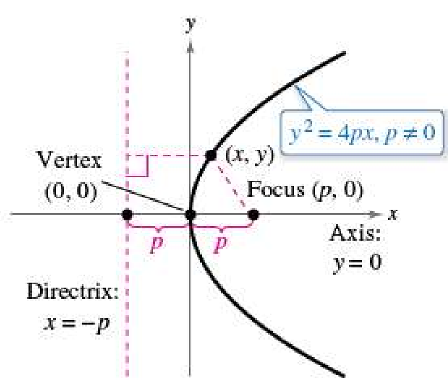
For **directrix** $x = -p$, the equation is

$$y^2 = 4px,\ p \neq 0 \qquad \textbf{Horizontal axis}$$

The **focus** is on the axis p units (directed distance) from the vertex.



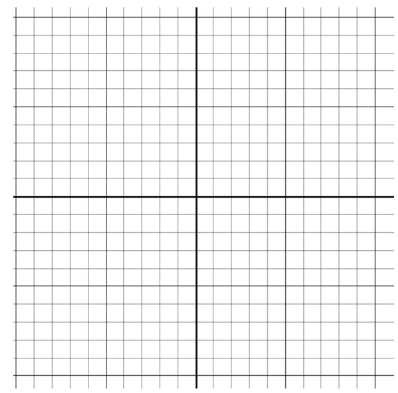
Parabola with vertical axis



Parabola with horizontal axis

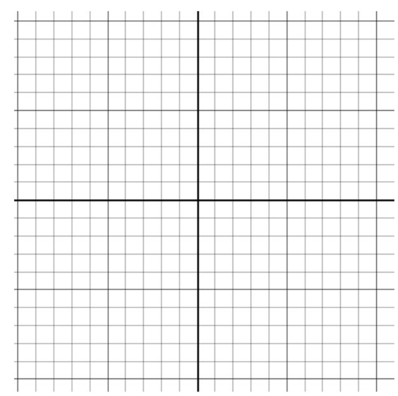
Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex at the origin and focus at (2,0).



Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola $y = -2x^2$. Then sketch the parabola.



3.2 Polynomial Functions of Higher Degree

Characteristics of Polynomial Functions
• All polynomial functions have a domain $-\infty < x < \infty$. (All real numbers).
• Graphs of all polynomial functions are **continuous** and **smooth.** This means they have no **breaks, holes or gaps** and no **sharp turns (cusps**).

End Behavior of Polynomial Functions	
Even though polynomials actually *never* end, we describe the eventual tendency as the graph extends far to the left and far to the right of the coordinate plane as "**end behavior**".	
No matter how high the degree of a polynomial function, *eventually* the end behavior will either tend towards positive infinite (the *y* values get infinitely larger in the positive direction, or *rise*) OR tend towards negative infinite (the *y* values get infinitely larger in the negative direction, or *fall*).	
Note that we typically describe functions *from left to right*, but when we describe end behavior, we say *to the left* or *to the right* depending on which end we are describing.	
Notation for Describing End Behavior of A Function	
We Write...	To Say...
	Graph rises to the right
	Graph falls to the right
	Graph rises to the left
	Graph falls to the left

Leading Coefficient Test	
When the degree of the polynomial is **odd**	
When the degree of the polynomial is **even**	

✍ Describing End Behavior

Describe the left-hand and right-hand end behavior of the graph of each function:

1. $f(x) = -x^3 + 2x$

2. $g(x) = x^4 - x^3 + 5$

3. $\mathrm{h}(x) = x^5 - x^2 - 3$

4. $k(x) = -x^6 - 3x^5 - x^2 + 2$

More Facts About Polynomial Functions

Polynomials are marvelous functions for modeling because they are very predictable. Here are some more facts about polynomials.

For a polynomial function *f* of degree *n*:

① The function *f* has, at most, ____.

② The function *f* has, at most, ____.

✍ The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

This theorem says that, if you know a factor of a polynomial then you can know a zero of the polynomial, and if you know a zero of the polynomial, then you can know a factor of it. In fact, there are several different ways of describing the relationship between a function and a zero of the function.

Real Zeros of Polynomial Functions
When f is a polynomial function and a is a real number, then all the statements below are equivalent. (That is, they are five ways of describing the same relationship between f and a. If any one of them is true, then they are all true.)
❶ $f(a) = 0$
❷ $x = a$ is
❸ $x = a$ is
❹ $(x - a)$ is
❺ $(a, 0)$ is

Find Real Zeros of a Polynomial Function
To find the zeros of a polynomial function, set the function equal to zero and solve.
5. Find the zeros of $f(x) = -2x^4 + 2x^2$

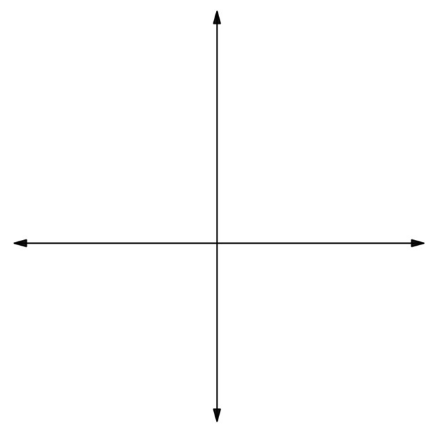
Repeated Zeros
A factor $(x - a)^k$, where k is an integer greater than 1, yields one real zero. However, we can get it from each of the k factors so we say it is a **repeated zero** and we describe the repeat as having **multiplicity *k*.** What does the graph look like when the zeros repeat?
① When k is **odd, the graph crosses the *x*-axis at $x = a$.**
② When k is **even, the graph touches but does not cross the *x*-axis at $x = a$.**
In both cases, $(a, 0)$ is an x-intercept of the graph.

Sketching the Graph of a Polynomial Function

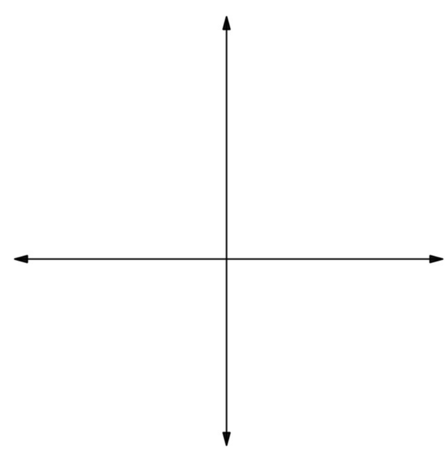
- Determine the right-hand and left-hand end behavior.
- Factor to determine the zeros.
- Use the multiplicity factor to determine how the zeros affect the graph (crosses through or touches *x*-axis).
- Find the *y*-intercept and any additional points as needed.
- Sketch the graph.

Sketch the graph of each function by hand:

6. $f(x) = -3x^4 + 12x^3 - 12x^2$



7. $f(x) = -2x^5 - 2x^4 + 12x^3$

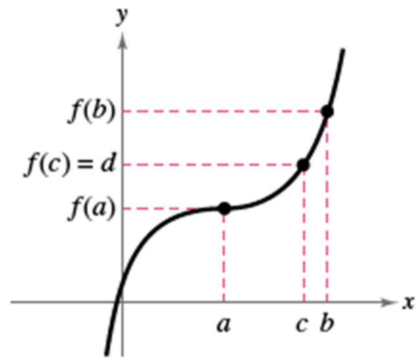


✍ Intermediate Value Theorem

The Intermediate Value Theorem makes sense if you look at a graph.
See the example of a portion of a polynomial function given below:

This theorem only applies to continuous functions.

It says that, if you consider the graph betwe
two points $(a, f(a))$ and $(b, f(b))$
then any *y* value between the two *y* values h
to pair up with at least one *x* value
between the two *x* values.



Make sense? Let's see how the formal theorem is written:

INTERMEDIATE VALUE THEOREM	Let a and b be real numbers such that If f is a polynomial function such that then, in the interval $[a, b]$,

✍ Applying the Intermediate Value Theorem

Use the Intermediate Value Theorem to approximate the real zero of the function

$f(x) = x^3 + x^2 + 1$

3.4 Zeros of Polynomial Functions

The Fundamental Theorem of Algebra
If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Why does this theorem require that the degree of the polynomial is greater than zero?

Linear Factorization Theorem
If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors where

The Rational Zero Test
If $f(x)$ is a polynomial with *integer coefficients* then every rational zero of f has the form Rational Zero = $\frac{p}{q}$ Where

SKILL: Find Possible Rational Zeros

This method is sort of an educated guessing at zeros of a function. It may or may not produce actual zeros, but if there *are* any rational zeros, it will help you find them.

Example. Find (if any) the rational zeros of $f(x) = 2x^3 - x^2 + 2x - 1$

Example. Find (if any) the rational zeros of $f(x) = x^3 - 3x^2 + 6$

Complex Zeros Occur in Conjugate Pairs
Let f be a polynomial function with *real coefficients*. If the complex number $a + bi$, where $b \neq 0$, is a zero of the function, then the complex conjugate $a - bi$ is also a zero of the function.

✋ ***SKILL:*** **Finding a Polynomial Function With Given Zeros**
For these, there are sometimes *many* possible functions that can satisfy the given requirements.
Example. Find a fourth-degree polynomial function f with real coefficients that has zeros of $1, -1$ and $3i$.
For these, there will usually only be *one* possible functions that can satisfy the given requirements.
Example. Find a cubic polynomial function f with real coefficients that has zeros of 2 and $1 - i$, and $f(1) = 3$.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

SKILL: Finding All the Zeros of a Polynomial Function Given Some Zeros

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

SKILL: Write a Polynomial Function As a Product of Linear Factors

Write

$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as a product of linear factors and list all the zeros.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.

2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive nonzero coefficients have opposite signs.

When we count the number of negative or positive real zeros using Descartes's Rule of Signs, **a zero of multiplicity *k* counts as *k* zeros**.

SKILL: Using Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of:

1. $f(x) = 3x^3 - 5x^2 + 6x - 4$

2. $f(x) = x^3 - 3x + 2$

3. $f(x) = 2x^3 + 5x^2 + x + 8$

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Divide $f(x)$ by $(x - c)$ using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of *f*.

2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of *f*.

A real number c is **upper bound** for the real zeros of f when no real zeros are greater than c.

A real number c is **lower bound** for the real zeros of f when no real zeros are less than c.

✋ ***SKILL:*** Using Upper and Lower Bounds

Find all real zeros of:

1. $f(x) = 6x^3 - 4x^2 + 3x - 2$

2. $f(x) = 8x^3 - 4x^2 + 6x - 3$

Topics	Big Ideas; What I Want To Remember; Vocabulary; Notation
Analyze graphs of quadratic functions	
Write quadratic functions in standard form and use the results to sketch their graphs	
Find minimum and maximum values of quadratic functions in real-life applications	
Parabolas: standard form, find directrix and focus; write equations given focus and vertex.	
Use transformations to sketch graphs of polynomial functions	
Use the Leading Coefficient Test to determine the end behaviors of graphs of polynomial functions	
Find real zeros of polynomial functions and use them as sketching aids	

Summary & Study Guide

Use the Intermediate Value Theorem to help locate real zeros of polynomial functions	
Use the Fundamental Theorem of Algebra	
Find rational zeros of polynomial functions	
Find complex zeros using conjugate pairs	
Find zeros of polynomials by factoring	
Use Descartes's Rule of Signs and Upper and Lower Bound Rules	
Find zeros of polynomials in real-life applications	

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