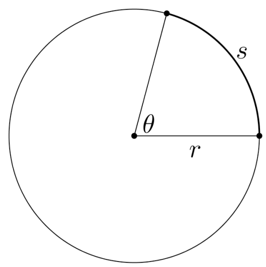
Radians: The Cosmic Angle Measure

Why are there $360°$ in a circle? It's fairly arbitrary, right? You could make a case that if life exists elsewhere in the universe they most likely don't measure a complete revolution to be $360°$. What might they use? They're very likely to use an angle measure called a *radian*. Let's figure out what a radian is, how to convert back and forth from radians to degrees, and what new sorts of things you need to start memorizing immediately.



The circumference of a circle is found using the formula $2\pi r$, where r is the radius of the circle. In geometry you learned how to find the length of an arc along the circumference of a circle.

To calculate the length of an arc you need to know the measure of the central angle. A central angle is an angle whose vertex is the center of the circle and whose rays intersect the circumference of the circle.

Question: In the following table r is the radius of a circle, ϕ is the measure of a central angle of the circle measured in degrees, and s is the length of the arc subtended by the central angle ϕ. Use your calculator in a clever way to find the exact value of s/r. Note: After some quick sketching you should be using your calculator to find the exact values.

Radius	ϕ	Arc length (s)	$s/r =$
4	$30°$		
r	$30°$		
12	$45°$		
r	$45°$		
5	$60°$		
r	$60°$		

The values in the *s/r* column are the *radian* measures of the central angles. Notice that a $30°$ angle is equivalent to $\pi / 6$ radians regardless of the radius of the circle.

Geometrically radians are calculated by finding the ratio of the intercepted arc to the radius of the circle.

Given an arc length, s, and the radius of its circle, r, the radian measure of the central angle is calculated by

$$\theta = \frac{s}{r}.$$

Definition of a radian:

> The *radian* is a unit of angular measure such that 1 radian is the measure of a central angle that intercepts an arc equal in length to the radius of its circle.

Because the radian is a unit of measure based on geometry, and not on some arbitrary notion, it is the preferred angle measure in mathematics. Learn to love it!

Question: We've established that $\theta = s/r$ will calculate angles in radians. Write θ in radians in terms of ϕ, in degrees, by using the formula for arc length, s.

Question: Summarize the formulas for converting between degrees and radians.

> To find θ in radians, multiply ϕ in degrees by _______________.
>
> To find ϕ in degrees, multiply θ in radians by _______________.

Note that in this course (and pretty much the rest of math from now on) if you see an angle that doesn't have a degree sign, then that angle is in radians.

Problem: Convert each of the following. Get used to showing work!

a. $35°$

b. $\dfrac{5\pi}{6}$

c. $40°$

d. $\dfrac{\pi}{12}$

e. $110°$

f. $\dfrac{7\pi}{36}$

g. $75°$

h. π

i. $135°$

j. 3

Question:

One radian is about how many degrees?

One degree is about how many radians?

Question: How many radians are in a circle (really in "one rotation")?

Here are a few things you should get really comfortable with:

> Any angle that is a multiple of
> - $30°$ can be written in radians with a denominator of 6.
> - $45°$ can be written in radians with a denominator of 4.
> - $60°$ can be written in radians with a denominator of 3.

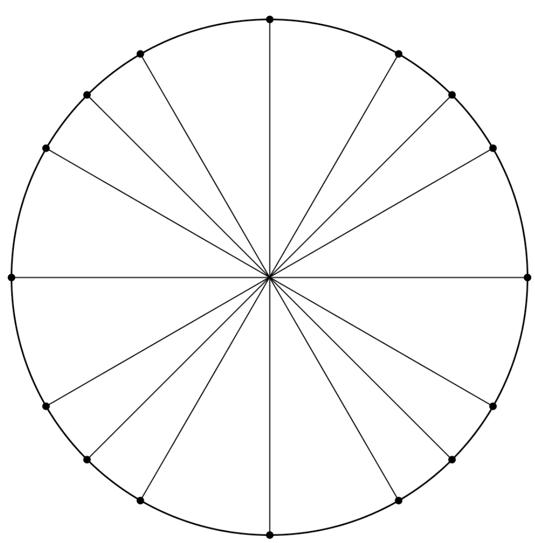
There are seventeen famous angles that you need to memorize the radian measures of so we might as well get started now, right?

Problem: Convert each of the following to radians and then memorize them.

$0°$	$30°$	$45°$	$60°$
$90°$	$120°$	$135°$	$150°$
$180°$	$210°$	$225°$	$240°$
$270°$	$300°$	$315°$	$330°$
$360°$			

Problem: Circle, box, and triangle the $\pi/6$ angles, $\pi/4$, and $\pi/3$ angles, respectively.

These angles are famous for a variety of reasons but importantly they are arrayed around the circle below.
Problem: Identify the angles below in radians (and only radians).



Lifetime assignment: Memorize the angles around this circle!

Let's talk a little more about arc length while we're still sort of thinking about the geometry of the radian.

Problem: Write a formula for finding arc length if the central angle is measured in degrees and another if it is measured in radians.

a. Arc length given degrees

b. Arc length given radians

When the angle is measured in radians the formula works out nicely because radians are a much more ideal angle measure—being based on geometry, not based on…whatever.

Using the basic formula for arc length, $s = r \cdot \theta$, is important. Keep in mind that θ must be measured in radians for the formula to be valid. Since there are three variables, if you know any two of them you can always find the third.

Problems:

a. Find the measure of a central angle of a circle with radius 8 intersecting an arc of length 72 units.

b. Find the length of an arc intersected by a central angle of measure $\theta = \dfrac{3\pi}{8}$ in a circle having radius 24.

c. Find the radius of a circle if a central angle with measure $36°$ intercepts an arc of length 64π. Get used to radians by converting first, then doing the problem.

Basics of Trigonometry in the Plane

Trigonometry is

- traditionally the study of the measures of sides and angles of a triangle
- expanded to be the study of functions, sometimes called circular functions, whose domains are the real numbers

It will probably be helpful to you if you start thinking of angles as a rotation of a single ray, rather than as a sort of physical thing like the measure of the angle in the corner of a room.

Every angle has

- a vertex
- an initial position
- a terminal position

Draw an angle here. Show the direction of rotation.

An angle in *standard position* has

- vertex at the origin;
- initial position on the positive x-axis;
- terminal position anywhere it wants.

Pretty much every angle we look at will be in standard position. In fact, the circle that we filled in before had all of its angles measured from standard position.

Draw an angle in standard position here…

Measuring angles is important and we need to know the established conventions:

- Positive angle measures are made counter-clockwise from the positive x-axis
- Negative angle measures are made clockwise from the positive x-axis
- Angles can have a measure of any real number, which differs from geometry

Draw a positive angle in standard position.

Draw a negative angle in standard position.

Draw an angle larger than $360°$.

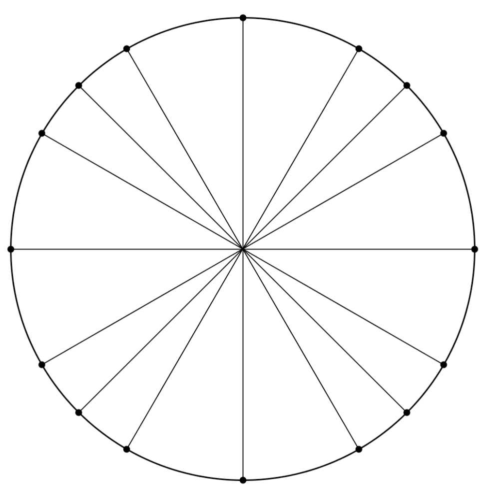
The most commonly used variables for angles fall into three categories:

- Greek letters: $\alpha,\ \beta,\ \gamma,\ \theta,\ \phi$
- Capital letters: A, B, C
- Subscripts: $\theta_1, \theta_2, \theta_3$, etc.

Question: Draw each of the indicated angles in standard position in the space provided.

$\alpha = 120°$	$\beta = 225°$	$\theta = 330°$
$\alpha_2 = 420°$	$A = -30°$	$\phi = -720°$
$\theta_3 = \dfrac{7\pi}{2}$	$\alpha' = \dfrac{11\pi}{4}$	$\beta' = \dfrac{15\pi}{4}$

You almost never really need to draw an angle that is greater than one full rotation, either positive or negative but you want to know how anyway. Let's fill in these angles from memory again.



The reason you never really have to draw an angle greater than 2π is because every angle greater than 2π actually just terminates in the same place as an angle between 0 and 2π a—think of the hands of a clock. These angles are one full rotation apart (or some multiple of a full rotation). When their terminal sides fall in the same exact place, we call the angles coterminal.

Coterminal angles are angles that differ by some integer multiple of $360°$ or 2π , in degrees and radians, respectively.

Coterminal angles are a huge deal and you need to get very good at working with them.

Problem: Show that each set of angles is coterminal.

a. 1090° and 10°

b. $\frac{15\pi}{7}$ and $\frac{43\pi}{7}$

If an angle is not in the interval $0° \le \theta < 360°$ or $0 \le \theta < 2\pi$ it's often useful to find the angle in that interval coterminal to it. An angle coterminal to a given angle, but falling in the first complete rotation is called the principal coterminal angle.

Problem: Given the angle $48°$, find the coterminal angle reached by rotating the indicated number of times in the given direction. In this case $48°$ is the principal coterminal angle.

a. 5 times counterclockwise

b. 10 times counterclockwise

c. 43 times clockwise

d. 3 times clockwise

Problem: Describe a general method of finding all angles coterminal to $48°$.

Problem: Find the minimum value of x such that x is coterminal to $48°$ and $x > 25{,}000°$.

Coterminal angles can and often are used to figure out the quadrant in which an angle terminates. This is important when angles are huge—or even just big. But for fun we'll deal with a huge angle first.

Problem: Find the quadrant in which $234056°$ terminates.

Problem: Each angle in the table either is of the form $a + n \cdot 2\pi$, $0 \le a < 2\pi$, or $a + n \cdot 360°$, $0 \le a < 360°$, $n \in \mathbb{Z}$. Find a, n, and the quadrant in which the angle terminates. If the angle is in degrees work in degrees; if radians work in radians. In this table a is the principal coterminal angle.

Angle	Work	a	n	Quad.
$750°$				
$6350°$				
$4605°$				
$-630°$				
$-7430°$				
$\frac{11\pi}{3}$				
$\frac{27\pi}{4}$				
$\frac{26\pi}{5}$				
$-\frac{17\pi}{6}$				

Question: How do you find n for each angle in the table. (Your method should work for all values in the table.) What function do you know that appears to be related to n?

When you're using your calculator, you need to use := to define a function instead of just using =. It even has its own button—kind of—if you press ctrl and the templates button.

Problem: Write a function that finds the value of *n* from the table on the previous page and another that finds the value of *a* directly. (Why do you need different functions for degrees and radians?) Store the functions on your calculator.

Problem: Use your stored functions to write each of the angles below in terms of its principal coterminal angle.

a. $77562°$

b. $\dfrac{2584\pi}{15}$

You already know this but…two angles whose sum is $90°$ (or $\pi/2$) are *complementary*. Two angles whose sum is $180°$ (or π) are *supplementary*. Nothing new here but they can look weird when they're radians not in terms of π. Also, sometimes an angle will have a negative complement or supplement—we live in strange times. We need that fact often, though. It does differ a bit from what you learned in geometry.

Problem: Find the complement and supplement (if they exist) for the angles.

Angle	$32°$	$142°$	$3\pi/7$	$6\pi/11$	1.2	2.7
Complement						
Supplement						

Problem: A wheel moves with constant speed and makes a complete revolution every 20 seconds. At $t = 3$ seconds, a point on the wheel is at the exact top. Write functions that give the exact times at which the point will be at each of the top, bottom, left most, and right most points. Write another function that generates each of these times, but doesn't distinguish.

Place Value for Degrees: Degrees, Minutes, Seconds

Degrees can be divided up into minutes and seconds. This is a bit of an albatross hung about our necks by a bunch of ancient mariners.

- There are 360 degrees in a circle.
- There are 60 minutes in a degree.
- There are 60 seconds in a minute.

If you follow the logic there are 3600 seconds in a degree. Degrees, minutes, seconds has largely been replaced with decimal degrees (in which we talk about tenths, hundredths, etc. of a degree) but degrees, minutes, and seconds (abbreviated DMS) are still in use in some applications (especially GPS measurements in our everyday lives).

You will be asked to use DMS on a number of questions. You'll often use your calculator for these problems…but not for the next bunch.

Problem: Convert $3.415°$ from DD to DMS by hand.

Problem: Convert $37°25'18"$ from DMS to DD by hand.

Question: Convert each of the following by hand. Compare with a partner or two.

DD	Work	*DMS*
$36.254°$		
$49.985°$		
		$52°14'32"$
		$15°27'43"$

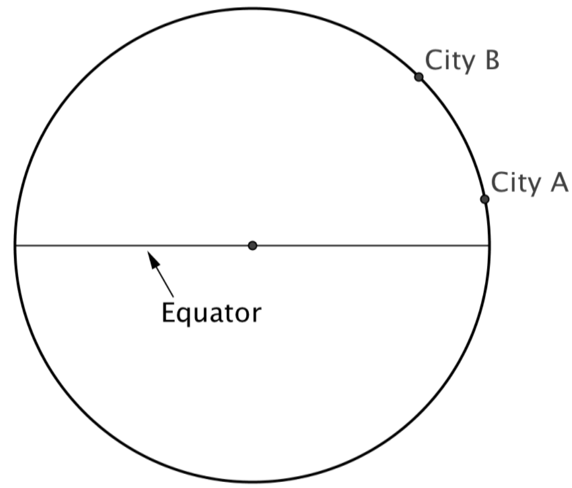
Problem: Go back through the last six examples and confirm your answers (and that you know what you're doing) with your calculator.

Remember: Becoming proficient with your calculator is not optional in this class. Sitting at your seat saying, "I don't know how to use a calculator" means you're not a good student; be a good student.

Here's pretty much my favorite question involving DMS:

Problem: The radius of the earth is approximately 6378.1 km.

a. Find the distance along the surface of the earth between two cities, directly north of each other, whose latitudes are $45°23'14"$ and $11°12'52"$.



b. Suppose a tunnel was dug between the two cities straight through the earth. How long would the tunnel be?

c. At its deepest point below the surface, how far down would the tunnel be?

Problem: The planet Strango has a diameter of 4250 *strangs*. Angles are measured in *dangs* and there are 450 *dangs* in one full revolution.

a. How many earth degrees are in 50 *dangs*?

b. What is a *right angle* on Strango?

c. How many *dangs* are in $65°$?

d. A *great circle* (real thing) is a section of a sphere that contains a diameter of the sphere. If two towns on Strango are 93 *strangs* apart along a great circle, how many *dangs* apart are they in latitude?

e. Why would it be better for everyone involved if Earthlings and Strangers (which is what conspiracy theorists call them…) used radians instead of either degrees or *dangs*?