Line Integral of a Scalar Function

Let's get an idea of what a scalar line integral could represent. Say we have a surface, $f(x,y)$, and the curve $\mathbf{r}(t)$, where $\mathbf{r}(t)=[x(t),y(t)]$, that's a path through the *xy*-plane. Imagine the path is the floor plan for a wall you're going to build and that $f(x,y)$ is the ceiling of the space. The line integral of a scalar function can help us find the surface area of that wall. You can also think of it as a fence. Whatever horizontal(ish) surface you want.

We can do this in a very Riemann sum kind of way.

- Cut the path up into little arcs, call them Δs_i: $\Delta s_i=\sqrt{\Delta x_i^{\,2}+\Delta y_i^{\,2}}$.
- Pick a point, (x_i,y_i), somewhere along each arc and use it to calculate a representative height, $f(x_i,y_i)$.
- The product $f(x_i,y_i)\Delta s_i$ gives an approximation of the surface area along one arc. (Basically treating it like a rectangle.)
- Add all of these areas to approximate the total surface area.
- Take the limit as $\Delta s_i \to 0$ and we'll get the true surface area.

This limit is the line integral of a *scalar function*. It's really important to note that you're working with a scalar function, not a vector field—do we even really know what a vector field is yet?

In practice here's how you calculate them (also note the new notation):

$$\int_C f(x,y)\,ds=\int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt$$

This whole thing is a function of *t* because $\mathbf{r}(t)=[x(t),y(t)]$ and $ds=\|\mathbf{r}'(t)\|dt$. It works the same in 3D, just with an additional component!
ds is often called the arc length differential or arc length element.
See here: http://commons.wikimedia.org/wiki/File:Line_integral_of_scalar_field.ogv

Problem: Evaluate the line integral $\int_C (1+xy^2)\,ds$ where *C* is the line segment connecting $(0,0)$ and $(1,2)$ in the *xy*-plane. Do this using two different parameterizations for the line segment.

1. From $(0,0)$ to $(1,2)$.

   $\mathbf{r}(t)=$ $\qquad\qquad$ $ds=$

   Evaluate the line integral.

2. From $(1,2)$ to $(0,0)$.

   $\mathbf{r}(t)=$ $\qquad\qquad$ $ds=$

   Evaluate the line integral.

The sign of a scalar line integral with respect to *arc length* (look for the *ds*!) does not depend on the direction in which the path is traced. This makes sense because we make our Δs cuts in the direction of increasing parameter and when evaluating $f\left(x_i, y_i\right)$ it doesn't matter what direction you're going when you do it. With the idea of a fence in mind, people standing on either side of the fence should agree on the surface area of the fence, even if their notions of left and right are switched.

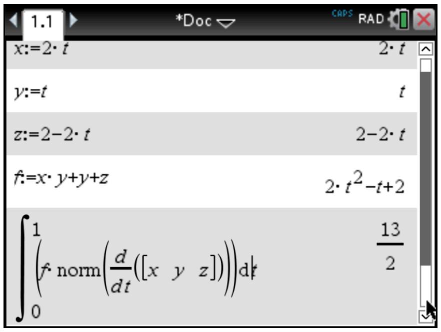
Problem: Evaluate $\int_C f(x,y,z)\,ds$ for $f(x,y,z) = xy + y + z$ where C is given by $\mathbf{c}(t) = \left[2t, t, 2-2t\right]$, and $0 \le t \le 1$.

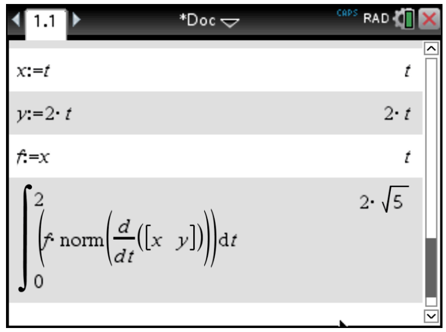
Problem: Evaluate $\int_C x\,ds$ over the given paths.

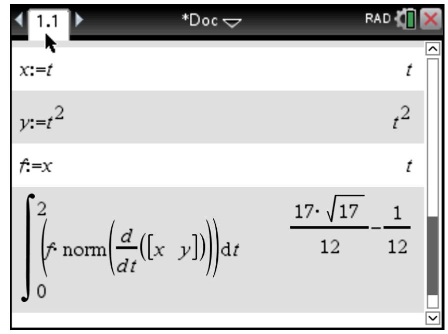
a. The straight-line segment $x = t$, $y = 2t$, from $(0,0)$ to $(2,4)$.

b. The parabolic curve $x = t$, $y = t^2$, from $(0,0)$ to $(2,4)$.

Your TI-Nspire CAS can definitely handle these types of line integrals. Most people just do a lot of the work by hand and then enter the final integral. Sometimes people use it to find derivatives so they can find $\left\|\mathbf{c}'(t)\right\|$. People basically use the calculator at the point where they get stuck. As you look at the screenshots below notice that I'm not putting in any "of *t*" parts. It's way less typing and it's just not necessary. Here are the screenshots.







What if it's not *ds*?

Another form of the line integral of a scalar function is with respect to just x, just y, just z, or some combination of them. Presumably these have applications, but the texts seem somewhat short on them. The Internet wonders as well: http://goo.gl/TIwISb

You calculate these by writing parametric equations for the path, doing substitutions, and evaluating. They look like this:

$$\int_C f(x,y,z)\,dx \text{ or } \int_C f(x,y,z)\,dy \text{ or } \int_C f(x,y,z)\,dz$$

If you pay close attention to the differential, they're just substitution problems.

Problem: Evaluate $\int_C 2xy\,dx$, with C given by $x=\cos t$, $y=\sin t$, $0\le t\le \dfrac{\pi}{2}$.

If your differential is not *ds*, then you're not dealing with a line integral with respect to arc length so don't expect the same things to be true about changing directions along the path. In fact, for these types of line integrals reversing the direction will actually change the sign of the line integral. So we have these:

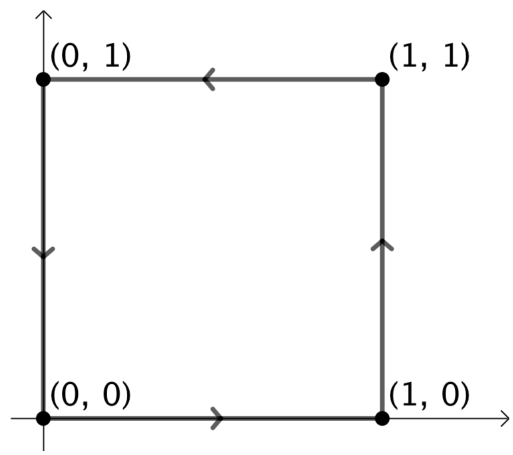
- $\int_{-C} f(x,y,z)\,dx = -\int_C f(x,y,z)\,dx$
- $\int_{-C} f(x,y,z)\,dy = -\int_C f(x,y,z)\,dy$
- $\int_{-C} f(x,y,z)\,dz = -\int_C f(x,y,z)\,dz$
- $\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$

The differential matters—if it's not arc length then switching direction switches the sign.

Line integrals of scalar functions are pretty straightforward. Start with something in lots of dimensions and work your way down to something with one variable and then evaluate.

Let's do one more before moving on to another neat type of thing that's also called a line integral…confusingly.

Problem: $\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is shown in the figure.



Vector Fields

Vector fields are a vector version of a slope field. Instead of showing you what little tiny tangent segments would look like at various points, vector fields show you…a field of vectors, really. Also, they can be—and often are—three-dimensional.

An interesting, dynamic, *almost* vector field can be seen at http://hint.fm/wind/. (It actually might be a vector field…I don't exactly know what that site is plotting, but it's awesome.)

You can basically just take any functions you want, throw them in as component functions, and call it a vector field. Drawing them is, believe it or not, less fun than drawing a slope field since you have to pay attention to magnitude and direction. We're interested in what happens as something moves through a vector field. It leads to some fun and challenging problems that we can do.

Line Integral of a Vector Field

Here's how you calculate the line integral of a vector field along a path *C*, given by the vector-valued function (parameterized path), $\mathbf{r}(t)$, on the interval $a \le t \le b$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

So for a long while you kind of just think of it as: see vector field, do dot product.

Note: The processes of evaluating the line integral of a scalar function, *f*, and evaluating the line integral of a vector field, **F**, are pretty much completely different. Watch out for that! Also there's almost never lines involved for either…

Follow this link to a pretty nice interpretation of line integrals of vector fields:
http://commons.wikimedia.org/wiki/File:Line_integral_of_vector_field.gif

Let's calculate the line integral of a vector field!

Problem: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \left[z, xy, -y^2\right]$ and *C* is given by $\mathbf{r}(t) = \left[t^2, t, \sqrt{t}\right]$ with $0 \le t \le 1$.

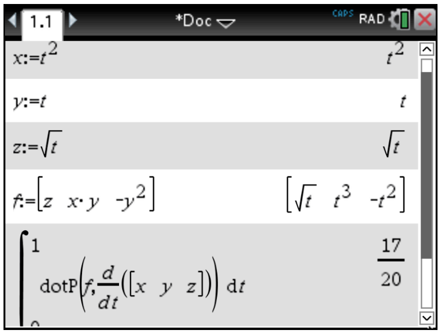
Problem: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y)=\left[x^2,xy\right]$ and $\mathbf{r}(t)=\left[2\cos(t),2\sin(t)\right]$ with $0\le t\le\pi$.

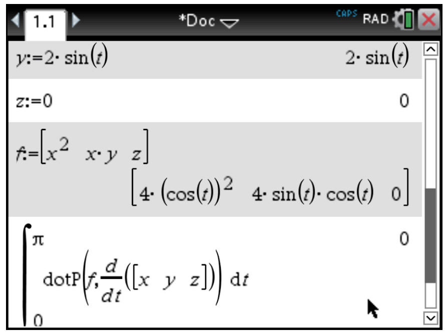
Sometimes vector line integrals will look like $\int_C F_1\,dx+F_2\,dy+F_3\,dz$. Where does this come from? Think of it this way: We know we're supposed to calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$. But $\mathbf{F}=\left[F_1,F_2,F_3\right]$ and $d\mathbf{r}=\left[\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right]dt$. So we can look at $d\mathbf{r}=\left[dx,dy,dz\right]$, so then $\mathbf{F}\cdot d\mathbf{r}=F_1dx+F_2dy+F_3dz$. So we just kind of play around with the integrand and end up with that form of the line integral. (It's quite common to see that form when the vector field is only two-dimensional because…there's a thing we can sometimes do.)

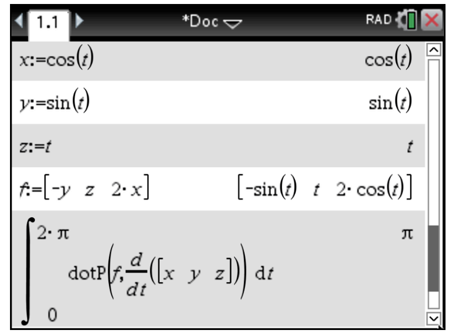
To evaluate $\int_C F_1\,dx+F_2\,dy+F_3\,dz$, just treat it as a straight substitution problem and solve it.

Problem: Evaluate $\int_C -y\,dx+z\,dy+2x\,dz$, where C is the curve described by $\mathbf{r}(t)=\left[\cos(t),\sin(t),t\right]$ and $0\le t\le 2\pi$.

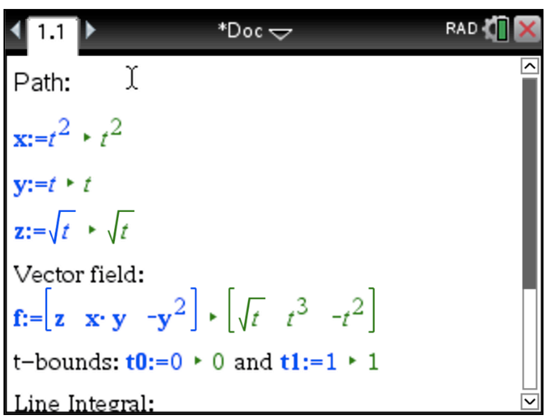
Your TI-Nspire CAS can handle line integrals of vector fields if you put things in correctly. I recommend storing $\mathbf{F}(x,y,z)$ (don't worry about the capital f and don't worry about the bold) and $\mathbf{r}(t)$. For problems like the last one you have to "peel them apart" first. So from $\int_C -y\,dx + z\,dy + 2x\,dz$ we know that $\mathbf{F}(x,y,z) = [-y, z, 2x]$. Below are screenshots for the examples. Unless you're incredibly stuck and all you care about is the final answer, I don't necessarily recommend you do this since it takes a while to set up. Notice again that I don't type in the "of t" part to save time, space, etc.

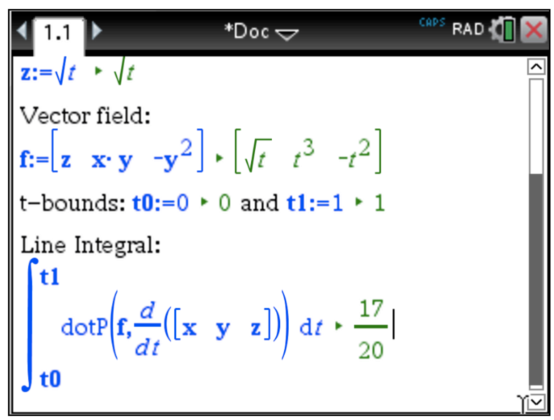






You can also set something up with mathboxes in a Notes page that would be way more worth your while. That I probably do recommend. Here are two screenshots because it doesn't fit in just one.





All you have to do with mathboxes is update the fields and it updates the results dynamically. Don't forget to hit enter after changing things!

Line Integrals have a bunch of properties. Here are some of them (I'm sure there are a lot more obscure ones):

- $\int_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{G}\cdot d\mathbf{r}$
- $\int_C k\mathbf{F}\cdot d\mathbf{r} = k\int_C \mathbf{F}\cdot d\mathbf{r}$
- $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$

And you can break up the curve into lots of smooth little curves and then add up all the line integrals. The last bullet point is very useful when the curve is described, rather than given, and you mess up counterclockwise and clockwise—I assure you, we'll all have been there before this is over.

Let's explore a little more about vector fields and a certain property a vector field can have that will make your life a little easier.

Some vector fields are actually the gradient, ∇f, of a function named *f*, for example. You could also have ∇g, etc. When this situation exists we call *f* the *potential function* of the *gradient vector field*. In notation, this means:

$$\mathbf{F} = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

If you know that **F** is in fact a gradient vector field, you can try to work backwards to find *f*. If a vector field is in fact the gradient of a potential function, we call the vector field *conservative*. So how do we know if a vector field is conservative? There's a test for it!

Test for Conservative Vector Fields
Let $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$ be a (weird) vector.
• If the domain is simply connected, and
• If $\nabla \times \mathbf{F} = 0$, then **F** is a gradient vector field.

If you work it out you'll see that all this calculation is really doing is checking that the mixed-partial derivatives are equal. (For example, that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, etc.) We've seen that before when we switched the order of integration on a double integral! If the mixed partials are equal, this cross product will give the 0 vector.

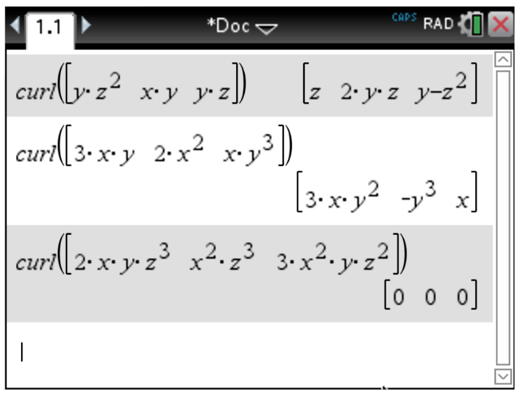
The vector $\nabla \times \mathbf{F}$ is called the *curl* of the vector field and it has applications we won't be discussing but that are in all multivariable calculus textbooks if you want to take a look. For our purposes, to decide if a vector field is the gradient of some function and therefore conservative, you should think, "Check the curl!" and you'll know what to do.

Problem: Find the curl of $\mathbf{F} = \left[3xy, 2x^2, xy^3\right]$

If you're going to find curl often you can define a function on your TI-Nspire CAS to find it. Here's what it looks like:

$$\text{curl}(f)=\left[\frac{d}{dy}(f[1,3])-\frac{d}{dz}(f[1,2]),-\left(\frac{d}{dx}(f[1,3])-\frac{d}{dz}(f[1,1])\right),\frac{d}{dx}(f[1,2])-\frac{d}{dy}(f[1,1])\right]$$

It's pretty annoying to type in, but you only have to do it the one time, store it, and then use it over and over. The argument of the function is a vector field. Make sure you enter it as a vector. Also, multiplication between variables is essential if you want to get the right results. Here's a screenshot of finding curl for a couple of different vector fields.



Problem: Given $f(x,y,z)=\cos(2x)y^2+xz^2$.

a. Find $\mathbf{F}=\nabla f$.

b. Find $\text{curl}(\mathbf{F})$

After showing that a vector field is actually conservative (Check the curl!), you might be interested in finding the potential function—there's no might about it, really. To get that done, first think about what it means for a vector field to be a *gradient vector field*.

Let $\mathbf{F} = [F_1, F_2, F_3] = \nabla f$. This means that each of the following is true:

1. $\int F_1\,dx = f(x,y,z) + h_1(y,z)$
2. $\int F_2\,dy = f(x,y,z) + h_2(x,z)$
3. $\int F_3\,dz = f(x,y,z) + h_3(x,y)$

The functions h_1, h_2, and h_3 are kind of "constants of integration" which, as functions of what they're functions of, would just *disappear* if you took derivatives with respect to the correct variables.

> Two Methods of Finding *Potential* Functions
> Method 1: The Easier Way
> 1. Just do the three integrals $\int F_1\,dx$, $\int F_1\,dy$, and $\int F_1\,dz$. Remember the "constants."
> 2. Compare your results and count everything that shows up but don't double count anything.
>
> Method 2: Probably More Acceptable for Some Reason (Since it's what books say to do)
> 1. Pick whichever antiderivative you want to find and find it. Say you did $\int F_1\,dx$.
> 2. Take the derivative of what you got with respect to *y* and compare it to F_2. Look at what's missing and integrate that with respect to *y*.
> 3. Go back to your antiderivative and take the derivative with respect to *z* and compare it to F_3. Use that to find still more of your function.

This is easier to do than to say. Also, *almost* every example in any book is *almost* trivial and can be done *almost* just by thinking about it.

Problem: Show that $\mathbf{F} = \left[e^x\cos(y) + yz, xz - e^x\sin(y), xy + z\right]$ is conservative and find a potential function.

One more thing to know about conservative vector fields…and it's kind of nuts! But also awesome!

> Conservative Vector Fields and Path Independence
> If **F** is a conservative vector field and points *A* and *B* are in the domain of **F**, then the line integral from *A* to *B* does not depend on the path you take from *A* to *B*—you'll always get the same answer. This is called *path independence*.

Conservative Vector Fields and Line Integrals

We know that:

- A vector field, **F**, is conservative if $\text{curl}(\mathbf{F}) = 0$;
- You find curl by calculating $\nabla \times \mathbf{F}$; and
- $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$; and
- if **F** is conservative, then there exits a function *f* such that $\mathbf{F} = \nabla f$.

Anyway, given that we know all of that, feast your eyes upon this:

> The Fundamental Theorem of Line Integrals
> If $\mathbf{F} = \nabla f$ on a domain, then for every curve *C* in that domain with initial point *A* and terminal point *B*,
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$
> It logically follows that if *C* is closed ($A = B$), then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. Also the symbol $\oint_C$ means that you're doing a line integral on a closed path and is cool.

As mentioned on the previous page, line integrals of conservative vector fields are *path independent*. What's that mean?

> Let $\mathbf{F}$ be a vector field defined on a domain. Line integrals through **F** are path independent if:
> - For any two paths, C_1 and C_2, with the same initial and terminal points, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
> - For all closed paths C, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

So if you find that a vector field is conservative and you need to do a line integral…and you can find a potential function…you should use the Fundamental Theorem for Line Integrals. That's really what we're saying.

Problem: Show that it's a valid method and then apply the Fundamental Theorem of Line Integrals to $\int_C \mathbf{F} \cdot d\mathbf{r}$ given $\mathbf{F} = \left[2xy, x^2 - z^2, -2yz\right]$ and *C* is any path from $(0,0,0)$ to $(3,2,5)$.

The hardest part of these is really just recognizing that you can do it and then finding the potential function.

Because we can never leave well enough alone, there's another way of writing line integrals of vector fields!

Just like with line integrals of scalar functions, we can write our line integrals of vector fields as $\int_C M\,dx + N\,dy + P\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}$. Why? Well, start with the fact that $\mathbf{F} = [M, N, P]$, which is pretty clear. Also, if we say that $d\mathbf{r} = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]dt$, instead of just $\mathbf{r}'(t)\,dt$, you can start to see, symbolically, where it all comes from.

We say the differential form $M dx + N dy + P dz$ is *exact* if $\mathbf{F} = [M, N, P]$ is conservative. Later in life you will solve *exact* differential equations and you will hopefully recall this fact. (It's like maybe half a day in a Differential Equations course.) If you're dealing with an exact differential form, you can use the Fundamental Theorem of Line Integrals.

Problem: Show that $\left(2x\ln(y) - yz\right)dx + \left(\frac{x^2}{y} - xz\right)dy - (xy)\,dz$ is exact and evaluate $\int_C \left(2x\ln(y) - yz\right)dx + \left(\frac{x^2}{y} - xz\right)dy - (xy)\,dz$, where C is a path staying in the simply connected domain of the field traveling from $(1,2,1)$ to $(2,1,1)$.

Okay, so we've done a lot so far…and we're going to do one more thing in these notes. Let's just pause to summarize so far.

An Abundance of Line Integral Options

- If f is a scalar function, we just calculate the line integral: $\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,dt$.
- If **F** is a random vector field, we can calculate the line integral: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$.
- If $\mathbf{F} = \nabla f$, we can calculate the line integral using the FTC of Line Integrals:
  $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$.

and there are some other bonus things in there, like how to know a vector field is conservative, calculating curl, path independence, probably other stuff. It's a lot! Let's do more!

Okay, this time we need to restrict ourselves to the plane. Let's say the *xy*-plane so we all know what to expect. For this discussion our line integrals will generally look like $\int_C P\,dx + Q\,dy$, where we use P and Q for…I don't know why you often switch to those instead of using M and N. We need to define something that will feel pretty obvious.

Let's say that C is the path we're on for our line integral and that C is a *simple closed curve*. A simple curve is one that does not cross itself. A closed curve is one that completely encloses a region, and we'll call that region D, for domain, for reasons that make sense in a bit.

Let's also assume that if you follow C in increasing direction of the parameter we go around the edge of the region counterclockwise. (Keeping the region to our left at all times.) Sometimes if this is the case then instead of seeing C on the integral, like this $\oint_C$, you will see this: $\oint_{\partial D}$. (Those spaces are killing me… but the template I use doesn't allow for them not to be there…)

Okay, so if all of this stuff is true, then we can use Green's Theorem to evaluate our line integral!

> Green's Theorem
> Let D be a domain whose boundary ∂D is a simple closed curve, oriented counterclockwise. If $P(x,y)$ and $Q(x,y)$ are differentiable and have continuous first partial derivatives, then
> $$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Say what??? Our line integral just became a double integral? Yes! That's what it says. According to one textbook I checked a proof of this is beyond the scope of your standard Calc III book…which must be true because it omits a proof and makes the statement on page 1006 of 1045.

How can we easily remember that thing inside the double integral? A determinant, of course!

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

This is the **k** component of the curl of the vector field. (Try calculating it…)

Problem: Use Green's Theorem to evaluate the line integral $\oint_C -y^2\,dx + xy\,dy$ where C is the unit square in QI bounded by the lines $x = 1$ and $y = 1$.

Problem: Use Green's Theorem to evaluate the line integral $\oint_C y^2\,dx + x^2\,dy$ where C is the triangle bounded by $x = 0$, $x + y = 1$, and $y = 0$.

Problem: Use Green's Theorem to evaluate the line integral $\oint_C (6y+x)\,dx + (y+2x)\,dy$ where C is the circle $(x-2)^2 + (y-3)^2 = 4$.