Formulation of optimal control problem requires:

1. Mathematical model of the process to be controlled
2. statement of physic. constraints
3. specification of a performance criterion

① restricted to systems described by ODE with ~~state~~ state variables $x_i(t)$ $i = 1, \dots n$
control inputs $u_i(t)$ $i = 1, \dots, u$

$\Rightarrow \quad \dot{\vec{x}}(t) = \vec{a}(\vec{x}(t), \vec{u}(t), t)$

② constraints: boundary conditions @ certain time points
physical constraints on inputs

state/control that satisfies these constraints is <u>admissible</u> (→ car example)

③ <u>optimal control</u> minimizes/maxim. some specific performance measure J

$$J = \underbrace{h(\vec{x}(t_f), t_f)}_{\text{scalar fct.}} + \int_{t_0 \leftarrow \text{init}}^{t_f \leftarrow \text{final time}} g(\vec{x}(t), \vec{u}(t), t)\, dt$$

← <u>Use Discrete time</u>

assume $\vec{x}_0$ and t_0 are specified

Optimal control Problem

find admissible control u^* which causes system $\dot{x} = \vec{a}(\vec{x}(t), \vec{u}(t), t)$ to follow admissible traj. $\vec{x}^*(t)$ that minimizes $J = \dots$

But: - don't know if optimal control exists
- ~~not~~ maybe not unique opt. control
- we look for global minimum

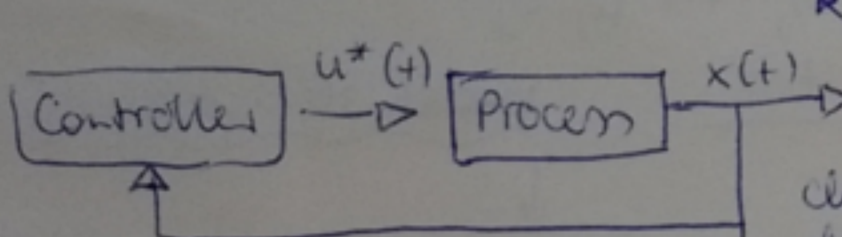
u^*: optimal control
x^*: opt. ~~state~~ trajectory

Optimal control law p 5.3

if found: $\vec{u}^*(t) = \vec{f}(\vec{x}(t), t)$ ← optim. control history from any state can be generated

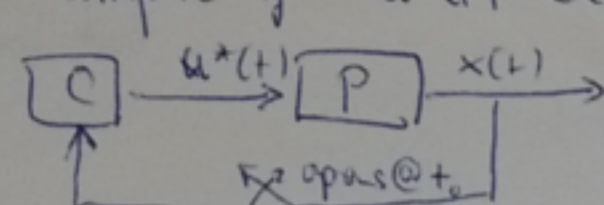
↖ opt. control law

$\vec{u}^* = -K\vec{x}$
$K = K(J, B, A)$



closed loop feedback

compare if $\vec{u}^*(t) = \vec{e}(\vec{x}(t_0), t)$ ("open-loop" opt control)



Linear, time-inv. system

(cont. time, analog for dis. time)

$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$ output $\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$

$d\vec{x} = A\vec{x}(t)\, dt + B\vec{u}(t)\, dt$

Solution: $\vec{x}(t) = \underbrace{\varphi(t, t_0)}_{\text{state transition matrix}} \vec{x}(0) + \int_{t_0}^{t} \varphi(t, \tau) B \vec{u}(\tau)\, d\tau$

$$\boxed{\text{time-inv: } \vec{x}(t) = e^{At}\vec{x}(0) + e^{At}\int_0^t e^{-A\tau} Bu(\tau)\, d\tau}$$

$\vec{x}(0) = -\int_0^t e^{-A\tau} Bu(\tau)\, d\tau$
$\vec{x}(0) = -\sum_{\tau=0}^{t} e^{-A\tau} Bu(\tau)$

Controllability

If there is $t_1 \geq t_0$ and a control $\vec{u}(t)$, $t \in [t_0, t_1]$ which transfers state $\vec{x}_0$ to origin @ t_1, state $\vec{x}_0$ is said to be <u>controllable</u> at time t_0. If all values $\vec{x}_0$ are controllable for all t_0, then system is (completely) <u>controllable</u>

Kalman: LTI system is controllable iff the $n \times mn$ matrix $E \triangleq [B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B]$ has rank n.

$x(1) = Ax(0) + Bu(0)$
$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$
$x(k) = A^k x(0) + (A^{k-1}B \mid \dots \mid A^2B \mid AB \mid B) \begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(k) \end{pmatrix}$

Notions of stability

$\dot{x}(t) = f(x(t), 0, t)$

$x(k+1) = f(x(k), 0, k)$

point $\bar{x}$ is equilibrium point if $f(\bar{x}, 0, t) = 0$

Stability?

Linear systems

$\hat{x}(k) = A^k x(0) = V \cdot \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} W x(0)$

$A^k = (U^{-1} \Lambda U)^k = \overset{V}{U^{-1}} \Lambda^k \overset{W}{U}$

system is asymptotically stable iff $|\lambda_i| < 1 \quad i \in \{1, \dots, n\}$

General unforced sol. @ $\bar{x}=0$ to lin. time-varying system: $x(t) = \Phi(t, t_0)\, x(t_0)$

stable @ $\bar{x}=0$ if $\sup_t \|\Phi(t, t_0)\| = m(t_0) < \infty$

asympt. stable if $\lim_{t \to \infty} \|\Phi(t, t_0)\| \to 0 \quad \forall t_0$

Stabilitation via feedback

State feedback ($x(t)$)

- easier
- can do more with control

$\rightarrow$ eigenvalue placement

output feedback ($y(t) = Cx(t) + Du(t)$)

mention linear-quad. regulator

(use DT here)

$\delta x = Ax + Bu$

$y = x$

$u = Fx + v$

$\delta x = \begin{cases} \dot{x} & \text{CT} \\ x(k+1) & \text{DT} \end{cases}$

closed-loop dyn $\delta x = \underline{(A + BF)}\, x + Bv$

System is stable iff EV of $A + BF$ are in stable region

$\begin{pmatrix} |\sigma_i(A+BF)| < 0 & \forall i & \text{CT} \\ |\sigma_i(A+BF)| < 1 & & \text{DT} \end{pmatrix}$

Can we choose F such that EV are placed @ arbitrary desired locs?

Theorem: There exists matrix F such that $\det(\lambda \mathbb{1} - (A + BF)) = \prod_{i=1}^{n} (\lambda - \mu_i)$ arbitr. $\mu_i \in \mathbb{C}$ iff (A, B) is <u>reachable</u>!

Proof (with $w_i^T B = 0$)

$w_i^T (A + BF) = w_i^T A + (w_i^T B) \cdot F$

$= \lambda_i w_i^T + \sum_{j=1}^{n} c_j w_j^T \cdot F$

$= \lambda_i w_i^T + \sum_j c_j d_j w_j^T$

now $d_j \overset{!}{=} \delta_{ij} d_j$ so $w_j^T \cdot F = \mu_j \cdot w_j^T$

but only if $\underline{w_i^T \cdot B \neq 0}$ — modal reachability

$\downarrow$

if $w^T B = 0$, then $w^T A B = 0$ & $w^T A^k B = 0 \Rightarrow w^T R_n = 0$

$\Rightarrow$ System is unreachable!

assume single-input case ($B = b$):

Application: Stability-optimized-circuits

$\tau \frac{d\vec{x}}{dt} = -\vec{x}(t) + W \Delta \vec{r}(\vec{x}, t) + S(t) + \xi(t)$

What init cond. $a \equiv \Delta\vec{r}(\vec{x}, t=0)$ will give rise to strongest transient response? $\left(\mathcal{E}(a) = \frac{2}{\tau} \int_0^\infty \|\Delta \vec{r}(t)\|^2 dt \right)$

$\rightarrow$ Some assumptions

See Hennequin, Vogels, Gerstner (2014), Neuron.