3. $$J^*_{m-1,m} = \frac{1}{2}\vec{x}^T(m-1)\underbrace{\left[(A^T + F^T_{m-1}B^T)P(0)(A+BF_{m-1}) + Q + F^T_{m-1}RF_{m-1}\right]}_{P(1)\ \text{symmetric}}\vec{x}(m-1)$$

* $$J_{m-2,m} = J^*_{m-1,m}(\underbrace{\vec{x}(m-1)}_{A\vec{x}(m-2)+B\vec{u}(m-2)}) + \frac{1}{2}\left(\vec{x}^T(m-2)\,Q\,\vec{x}(m-2) + \vec{u}^T(m-2)\,R\,\vec{u}(m-2)\right)$$

$\Rightarrow$ same form as $J_{m-1,m}$ $\Rightarrow$ $$J^*_{m-2,m} = \frac{1}{2}\vec{x}^T(m-2)\underbrace{\left[(A+BF_{m-2})^T P(1)(A+BF_{m-2}) + Q + F^T_{m-2}RF_{m-2}\right]}_{P(2)}\vec{x}(m-2)$$

$\Longrightarrow \forall K,$

$$\begin{cases} (2)\ J^*_{m-K,m} = \frac{1}{2}\vec{x}^T(m-K)\underbrace{\left[(A+BF_{m-K})^T P(K-1)(A+BF_{m-K}) + Q + F^T_{m-K}RF_{m-K}\right]}_{P(K)}\vec{x}(m-K) \\ (1)\ F_{m-K} = -(B^T P(K-1)B+R)^{-1}B^T P(K-1)A \\ u^*_{m-K,m} = F_{m-K}\,\vec{x}(m-K) \end{cases}$$

$\Rightarrow$ Iterate

* If the system is controllable & time-invariant, $H=0$, R & Q are C^t

$$\Rightarrow \lim_{m\to\infty}(F_{m-K}) \Rightarrow F_\infty$$

Find time-invariant control by solving the algebraic Riccati equation

$$P_\infty = \left(A + B\left[-(B^T P_\infty B + R)^{-1}B^T P_\infty A\right]\right)^T P_\infty \left(A + B\left[-(B^T P_\infty B + R)^{-1}B^T P_\infty A\right]\right) + Q$$
$$+ \left[-(B^T P_\infty B + R)^{-1}B^T P_\infty A\right]^T R \left[-(B^T P_\infty B + R)^{-1}B^T P_\infty A\right]$$

Donald E KIRK

Optimal Control Theory - an intro 3.10

Linear quadratic regulator in discrete time

→ dynamics: discrete time: $\vec{x}(k+1) = A\vec{x}(k) + B\vec{u}(k)$

→ cost: $J = \frac{1}{2}\vec{x}^T(n) H \vec{x}(n) + \frac{1}{2}\sum_{k=0}^{n-1}\left[\vec{x}^T(k) Q \vec{x}(k) + \vec{u}^T(k) R \vec{u}(k)\right]$

H: real symmetric semi + definite; Q, R: real symmetric positive definite

→ Initializing: cost of being in the final state: $J_{nn} = \frac{1}{2}\vec{x}^T(n) H \vec{x}(n) \equiv J^*_{NN} \equiv \frac{1}{2}\vec{x}^T(n) P(0) \vec{x}(n)$

$P(0) \equiv H$

Cost of passing from state $(n-1)$ to state n:

$$J_{n-1,n} = J^*_{n,n} + \frac{1}{2}\left(\vec{x}^T_{(n-1)} Q \vec{x}(n-1) + \vec{u}^T(n-1) R \vec{u}(n-1)\right)$$

$$= \frac{1}{2}\left(A\vec{x}(n-1) + B\vec{u}(n-1)\right)^T P(0)\left(A\vec{x}(n-1) + B\vec{u}(n-1)\right) + \frac{1}{2}\left[\vec{x}^T(n-1) Q \vec{x}(n-1) + \vec{u}^T(n-1) R \vec{u}(n-1)\right]$$

$$J^*_{n-1,n} = \min_{\vec{u}_{n-1}} J_{n-1,n}\left(\vec{x}(n-1), \vec{u}(n-1)\right)$$

1. Find $\frac{\partial J_{n-1,n}}{\partial \vec{u}(n-1)} = \begin{bmatrix} \frac{\partial J_{n-1,n}}{\partial u_1(n-1)} \\ \vdots \\ \frac{\partial J_{n-1,n}}{\partial u_m(n-1)} \end{bmatrix} = \vec{0}$

terms linear in $\vec{u}$: $\left(\vec{u}^T(n-1) B^T P(0) A\vec{x}(n-1)\right)\frac{1}{2}$
$\left(\vec{x}^T A^T(n-1) P(0) B\vec{u}(n-1)\right)\frac{1}{2}$ can be transposed

terms quadratic: $\left(\vec{u}^T(n-1)\left(B^T P(0) B + R\right)\vec{u}(n-1)\right)\frac{1}{2}$

* R symmetric by assumption

* $\left(B^T P(0) B\right)^T = B^T P(0)^T B = B^T P(0) B$ — $P(0)$ symmetric

⇒ symmetric

$$[u_1\ u_2\ u_3]\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} =$$

$$[u_1\ u_2\ u_3]\begin{bmatrix} S_{11}u_1 + S_{12}u_2 + S_{13}u_3 \\ S_{12}u_1 + S_{22}u_2 + S_{23}u_3 \\ S_{13}u_1 + S_{23}u_2 + S_{33}u_3\end{bmatrix} =$$

$$\left[S_{11}u_1^2 + S_{12}u_1u_2 + S_{13}u_1u_3 + S_{12}u_1u_2 + S_{22}u_2^2 + S_{23}u_2u_3 + S_{13}u_1u_3 + S_{23}u_2u_3 + S_{33}u_3^2\right]$$

$$\Rightarrow \frac{\partial(\vec{u}^T S \vec{u})}{\partial \vec{u}} = 2S\vec{u}$$

$$\frac{\partial J_{n-1,n}}{\partial \vec{u}(n-1)} = \left(B^T P(0) B + R\right)\vec{u}(n-1) + B^T P(0) A\vec{x}(n-1)$$

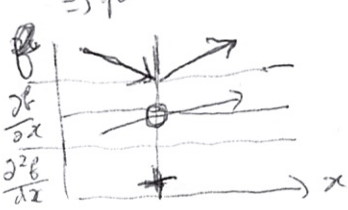
2. $\frac{\partial^2 J_{n-1,n}}{\partial^2 \vec{u}(n-1)} = \begin{bmatrix} \frac{\partial^2 J_{n-1,n}}{\partial^2 u_1} & \frac{\partial^2 J_{n-1,n}}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 J_{n-1,n}}{\partial u_1 \partial u_m} \\ \frac{\partial^2 J_{n-1,n}}{\partial u_2 \partial u_1} & \frac{\partial^2 J_{n-1,n}}{\partial^2 u_2} & \cdots & \frac{\partial^2 J_{n-1,n}}{\partial u_2 \partial u_m} \\ \vdots & & & \\ \frac{\partial^2 J_{n-1,n}}{\partial u_m \partial u_1} & \frac{\partial^2 J_{n-1,n}}{\partial u_m \partial u_2} & \cdots & \frac{\partial^2 J_{n-1,n}}{\partial^2 u_m}\end{bmatrix}$

$= B^T P(0) B + R$ — R: positive definite; $B^T P(0) B$: positive semi-definite

$\forall z,\ z^T P(0) z \geq 0$; take $z = By\ \forall y \Rightarrow y^T \underbrace{B^T P(0) B}_{\text{positive semi def}} y \geq 0$

⇒ positive definite

One-d case



$$\Rightarrow \vec{u}^*_{(n-1)} = -\left(B^T P(0) B + R\right)^{-1} B^T P(0) A\vec{x}(n-1)$$

$$= F(n-1)\,\vec{x}(n-1)$$

F: time-dependent feedback matrix