MATH 257 Cheatsheet with LaTeX

Basics

Solve 1st order ODE

Solve 2nd order ODE

Special Case

Cauchy-Euler/Eqidimensional:

$$x^2y'' + axy' + by = 0$$

use $y = x^r$ to solve.

Series solution

Power Series: $f(x) = \sum_{i=0}^{\infty} a_i x^i$
Talyor Series: condition on power series that series is continuous and infinitely differentiable.
$f(x) = \sum_{i=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n$
Maclaurin Series: taylor series where $x_0 = 0$
$f(x) = \sum_{i=0}^{\infty} \frac{f^n(0)}{n!}(x)^n$

Convergence

$|x - x_0| < P$, where P is the radius of convergnece. Use ratio test: $\lim_{i\to\infty} |\frac{Q_{i+1}}{Q_i}| < 1$
Analytic: T-series expansion exist at $x_)$, then the function is analytic at x_0 with the radius of convergence.

Singular point

Take function

$$\mathcal{L}y = P(x)y'' + Q(x)y' + R(x)y = 0$$

Transform function to

$$\mathcal{L}y = y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. if p(x) or q(x) does not converge at x_0, there is a SP at x_0(Also not analytic)
Regular Singular Point (RSP):
Transform function to

$$\mathcal{L}y = (x - x_0)^2y'' + \alpha(x)(x - x_0)y' + \beta(x)y = 0$$

where $\alpha(x) = \frac{Q(x)}{P(x)}(x - x_0)$, $\beta(x) = \frac{R(x)}{P(x)}(x - x_0)^2$. if $lim_{x\to x_0}\alpha(x) = c_1$ and $lim_{x\to x_0}\beta(x) = c_2$, then x_0 is RSP. Other wise it's an Iregular Singuar point(IRSP).

Solve RP

Frobenius Series: $f(x) = (x - x_0)^r \sum_{i=0}^{\infty} a_i(x - x_0)^i$

PDE solving Methods

Boundary Condition

- **Dirichlet:**
  - Heat: $u(0,t) = u(L,t) = 0$
  - Laplace: $\begin{cases} \text{finite :} \\ \text{semi-infinite strip :} \end{cases}$
- **Neumann:** $u_x(0,t) = u_x(L,t) = 0$
- **Periodic:**
  - Heat: $\begin{cases} u(0,t) = u(L,t) \\ u_x(0,t) = u_x(L,t) \end{cases}$
  - Wave
  - Laplace
- **Mixed type A:** $\begin{cases} u(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$
- **Mixed type B:** $\begin{cases} u_x(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

Laplace's Equation

Special steady state of Wave or Heat equations where there is no time invariant: $\frac{\partial u}{\partial t} = \frac{\partial u^2}{\partial^2 t} = 0$

$$\Delta u = \frac{\partial u^2}{\partial^2 t} + \frac{\partial u^2}{\partial^2 x} = 0$$

Types

Circle

Laplace: Transform to polar coordinate
$u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$
where $\begin{cases} r = (x^2 + y^2)^{\frac{1}{2}} \\ \theta = atan(\frac{y}{x}) \end{cases}$

Wedge

characteristics: $u_{xx} + u_{yy} = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0$, where $0 < r < a,\ 0 < \theta < \alpha$

- $u(r,0) = c_1,\ u(r,\alpha) = c_2,\ u(r,\theta)$ bounded as $r \to 0, u(\alpha,\theta) = f(\theta)$
- $u_\theta(r,0) = c_1,\ u_\theta(r,\alpha) = c_2,\ u(r,\theta)$ bounded as $r \to 0, u(\alpha,\theta) = f(\theta)$
- $u_\theta(r,0) = c_1,\ u(r,\alpha) = c_2,\ u(r,\theta)$ bounded as $r \to 0, u(\alpha,\theta) = f(\theta)$
- Neumann on the interior of circle: $u_r(a,\theta) = f(\theta)$, as $r \to \infty,\ u(r,\theta) < \infty$ and periodic: $u(r,\theta) = u(r,\theta + 2\pi)$

Special types

- Wedge with hole: on top of wedge, some edge $u(b,\theta)$ is also defined. $u(r,0) = c_1,\ u(r,\alpha) = c_2\ u(b,\theta) = 0$, $u(r,\theta)$ bounded as $r \to 0, u(\alpha,\theta) = f(\theta)$
- Dirichlet on the interior of circle: $u(a,\theta) = f(\theta)$, as $r \to 0,\ u(r,\theta) < \infty$ and periodic: $u(r,\theta) = u(r,\theta + 2\pi)$
- Dirichlet on the exterior of circle: $u(a,\theta) = f(\theta)$, as $r \to \infty,\ u(r,\theta) < \infty$ and periodic: $u(r,\theta) = u(r,\theta + 2\pi)$

Possion integral forma

Strum-Louisville Boundary Value Problem

Definition 1. Boundary value problem of the form:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0\ 0 < x < \ell$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0\ \beta_1 y(\ell) + \beta_2 y'(\ell) = 0$$

where p, p', q, r are continuous on $0 \le x \le \ell$ and $p(x) \ge 0$ and $r(x) > 0$ on $0 \le x \le \ell$
Then Strum-Liouville eigenvalue problem:

$$\mathcal{L} = \lambda ry = -(py')' + qy$$
$$\alpha_1 y(0) + \alpha_2 y'(0) = 0\ \beta_1 y(\ell) + \beta_2 y'(\ell) = 0$$
$$p(x) > 0, r(x) > 0$$

Properties

- $\int_0^\ell r(x)\phi_j^2(x)\mathrm{d}x = 1$

Robin Boundary Condition

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) &= h_1X(0), \\ X'(\ell) &= -h_2X(\ell),\ h_z \ge 0,\ h_2 \ge 0 \\ X(x) &= A\cos\mu x + B\sin\mu x, \\ X'(x) &= -A\mu\sin\mu x + B\mu\cos\mu x \end{aligned}$$

Solve for $\tan(\mu\ell) = [\frac{\mu(h_1+h_2)}{\mu^2 - h_1h_2}]$

- $h_1,\ h_2 \ne 0$
  $$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x,\ \mu_n \to \frac{n\pi}{\ell}\ as\ n \to \infty$$
- $h_1 \ne 0,\ h_2 = 0$
  $$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x = \frac{\cos\mu_n(\ell - x)}{\sin\mu_n x}$$
- $h_1 \to \infty,\ h_2 \ne 0$
  $$X_n = \sin\mu_n x,\ \mu_n \to [(\frac{2n+1}{2})\frac{\pi}{\ell}]$$

Tools and Useful Equations

Cauchy-Euler/Eqidimensional:

$$x^2y'' + axy' + by = 0$$

Legendre Equation:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

Bessel's Equation:

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

Miscs

Math 257-316 PDE   Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\qquad \sin^2 t + \cos^2 t = 1$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\qquad \sin^2 t = \frac{1}{2}(1 - \cos(2t))$

$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\qquad \cosh^2 t - \sinh^2 t = 1$

$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\qquad \sinh^2 t = \frac{1}{2}(\cosh(2t) - 1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
indicial eq.	$ar^2 + br + c = 0$	$ar(r-1) + br + c = 0$
$r_1 \neq r_2$ real	$y = Ae^{r_1 x} + Be^{r_2 x}$	$y = Ax^{r_1} + Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx} + Bxe^{rx}$	$y = Ax^r + Bx^r \ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x) + B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert) + B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y'' + p(x)y' + q(x)y = 0$ $(\star)$ around $x = x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^\infty a_n(x - x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1) + br + c = 0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^\infty a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^\infty b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_2...$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } \ b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L,L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $\quad Ff(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-) + f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^L |f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^\infty\left(|a_n|^2 + |b_n|^2\right).$$

For $f(x)$ defined in $[0,L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^\infty b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt} = c^2u_{xx},\ -\infty < x < \infty,\ t > 0$ **IC**: $u(x,0) = f(x),\ u_t(x,0) = g(x)$.
SOLUTION: $\quad u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $\quad [p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad a < x < b.$
BC: $\quad \alpha_1y(a) + \alpha_2y'(a) = 0, \quad \beta_1y(b) + \beta_2y'(b) = 0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2 + \alpha_2^2 > 0$. $\beta_1^2 + \beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^\infty c_n\phi_n(x),\ a < x < b\ ,\ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx},\ n = 1, 2, \ldots$$