1819-108-C1-Session-FinalExam

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The original birth-death system gave rise to the equations

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i},$$

which in the $M/M/c$ queue become

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda}{i\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \quad \text{if } 1 \le n \le c,$$

and

$$p_n = p_0 \prod_{i=1}^{c} \frac{\lambda}{i\mu} \prod_{i=c+1}^{n} \frac{\lambda}{c\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!} \left(\frac{1}{c}\right)^{n-c} \quad \text{if } n \ge c$$

We define $\rho = \lambda/(c\mu)$, and in order for the system to be stable, we must have $\rho < 1$. This implies that the mean arrival rate must be less than the maximum potential rate with which customers can be served. This expression for ρ is consistent with our definition in terms of the expected fraction of busy servers, the utilization in an $M/M/c$ queue, since the expected number of busy servers is equal to $c\rho = \lambda/\mu$.

Returning to the previous equation and substituting $c\rho$ for λ/μ, we obtain

$$p_n = p_0 \frac{(c\rho)^n}{n!} \quad \text{for } n \le c,$$

$$p_n = p_0 \frac{(c\rho)^n}{c^{n-c} c!} = p_0 \frac{\rho^n c^c}{c!} \quad \text{for } n \ge c.$$

All that remains is to solve for p_0:

$$1 = \sum_{n=0}^{\infty} p_n = p_0 + \sum_{n=1}^{\infty} p_n = p_0 \left[1 + \sum_{n=1}^{c-1} \frac{(c\rho)^n}{n!} + \sum_{n=c}^{\infty} \frac{\rho^n c^c}{c!}\right],$$

i.e.,

$$p_0 = \left[1 + \sum_{n=1}^{c-1} \frac{(c\rho)^n}{n!} + \sum_{n=c}^{\infty} \frac{\rho^n c^c}{c!}\right]^{-1}.$$

Since

$$\sum_{n=c}^{\infty} \frac{\rho^n c^c}{c!} = \frac{1}{c!} \sum_{n=c}^{\infty} \rho^n c^c = \frac{(c\rho)^c}{c!} \sum_{n=c}^{\infty} \rho^{n-c} = \frac{(c\rho)^c}{c!} \frac{1}{1-\rho},$$

$$p_0 = \left[1 + \sum_{n=1}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!} \frac{1}{1-\rho}\right]^{-1}.$$

This gives us what we need. To summarize, the steady-state distribution of customers in an $M/M/c$ queue is given by

$$p_n = p_0 \frac{(c\rho)^n}{n!} \quad \text{for } n \le c,$$

$$p_n = p_0 \frac{(c\rho)^n}{c^{n-c} c!} = p_0 \frac{\rho^n c^c}{c!} \quad \text{for } n \ge c$$