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We note that Bessel functions of half-integer order are expressible in closed form in terms of trigonometric functions, as illustrated in the following example.

▶ Find the general solution of

$$x^2y^{''} + xy^{'} + (x^2 - \tfrac{1}{4})y = 0$$

This is Bessel's equation with $y = 1/2$, so from (18.80) the general solution is simply

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

However, Bessel functions of half-integer order can be expressed in terms of trigonometric functions. To show this, we note from (18.79) that

$$J_{\pm 1/2}(x) = x^{\pm 1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n \pm 1/2} n! \Gamma(1 + n \pm \frac{1}{2})}$$

Using the fact that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find that, for $v = 1/2$,

$$\begin{aligned} J_{1/2}(x) &= \frac{(\frac{1}{2}x)^{1/2}}{\Gamma(\frac{3}{2})} - \frac{(\frac{1}{2}x)^{5/2}}{1!\Gamma(\frac{5}{2})} + \frac{(\frac{1}{2}x)^{9/2}}{2!\Gamma(\frac{7}{2})} - \cdots \\ &= \frac{(\frac{1}{2}x)^{1/2}}{(\frac{3}{2})\sqrt{\pi}} - \frac{(\frac{5}{2}x)^{1/2}}{1!(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}} + \frac{(\frac{9}{2}x)^{1/2}}{2!(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\sqrt{\pi}} - \cdots \\ &= \frac{(\frac{1}{2}x)^{1/2}}{(\frac{3}{2})\sqrt{\pi}} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = \frac{(\frac{1}{2}x)^{1/2}}{(\frac{3}{2})\sqrt{\pi}} \frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}} \sin x \end{aligned}$$

whereas for $v = -1/2$ we obtain

$$\begin{aligned} J_{1/2}(x) &= \frac{(\frac{1}{2}x)^{-1/2}}{\Gamma(\frac{3}{2})} - \frac{(\frac{1}{2}x)^{3/2}}{1!\Gamma(\frac{5}{2})} + \frac{(\frac{1}{2}x)^{7/2}}{2!\Gamma(\frac{5}{2})} - \cdots \\ &= \frac{(\frac{1}{2}x)^{-1/2}}{\sqrt{\pi}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \sqrt{\frac{2}{\pi x}} \cos x \end{aligned}$$

Therefore the general solution we require is

$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x) = c_1 \sqrt{\frac{2}{\pi x}} \sin x + c_2 \sqrt{\frac{2}{\pi x}} \cos x. \blacktriangleleft$$

18.5.2 Bessel functions for integer v

The definition of the Bessel function $J_v(x)$ given in (18.79) is, of course, valid for all values of v, but, as we all shall see, in the case of integer v the general solution of Bessel's equation cannot be written in the form (18.80). Firstly, let us consider the case $v = 0$, so that the two solutions to the indicial equation are equal, and we clearly obtain only on solution in the form of a Frobenius series. From (18.79),