Physikalisches Anfängerpraktikum
Sommersemester 2023

Versuch 12 Tutor: Goran Stanic

Moment of Inertia

We will calculate the torque of the rotating table and the moment of inertia of the regular and irregular shaped brass plate by using the measuring equipment stop-watch and caliper with a rotating pendulum. We will adopt a static method to determine the center of mass of the irregular shaped plate. To test the law of Steiner, we calculate the moment of inertia of the irregular shaped plate in respect to fives axes which are parallel to the axis passing through the center of mass and the 5 points apart from it.

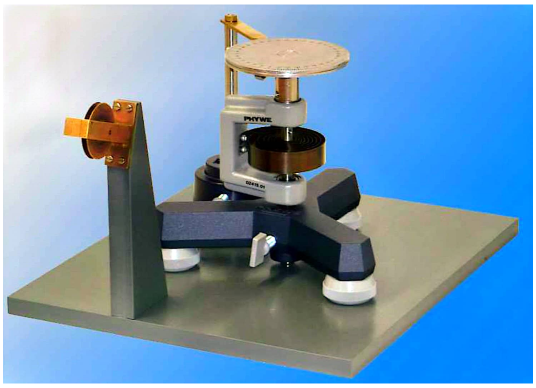


Abbildung 1: Experimental setup

1 Introduction

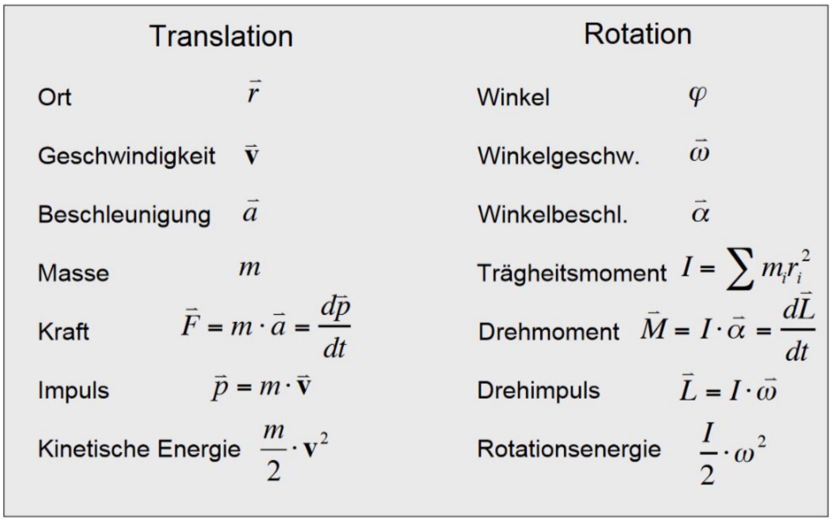
1.1 Physical Principles

In classical mechanics, numerous instances of rotational motion exist, such as a rolling sphere, a spinning windmill, a pendulum in circular motion, and countless others. The physics of rotational motion can be described in a manner entirely analogous to linear translational motion when appropriate quantities are selected. The following table provides an overview of the analogies. (See the Abbildung 2)

Physical description of the linear and rotary motions is based on the completely analogous equations provided that appropriate terms and variables are used. Analogous to the second Newtonian axiom $F = \dot{p}$ and Hooke's Law $F = -kx$, we can derive from these analogies

$$\vec{M} = -D\vec{\varphi} \text{ und } \vec{M} = \dot{\vec{L}} = \frac{\mathrm{d}}{\mathrm{d}t}(\Theta\dot{\vec{\varphi}}) = \Theta\ddot{\vec{\varphi}} \tag{1}$$

$$\Rightarrow -D\vec{\varphi} = \Theta\ddot{\vec{\varphi}} \tag{2}$$



Translation		Rotation	
Ort	$\vec{r}$	Winkel	φ
Geschwindigkeit	$\vec{\mathbf{v}}$	Winkelgeschw.	$\vec{\omega}$
Beschleunigung	$\vec{a}$	Winkelbeschl.	$\vec{\alpha}$
Masse	m	Trägheitsmoment	$I = \sum m_i r_i^2$
Kraft	$\vec{F} = m \cdot \vec{a} = \frac{d\vec{p}}{dt}$	Drehmoment	$\vec{M} = I \cdot \vec{\alpha} = \frac{d\vec{L}}{dt}$
Impuls	$\vec{p} = m \cdot \vec{\mathbf{v}}$	Drehimpuls	$\vec{L} = I \cdot \vec{\omega}$
Kinetische Energie	$\frac{m}{2} \cdot \mathbf{v}^2$	Rotationsenergie	$\frac{I}{2} \cdot \omega^2$

Abbildung 2: Analogies between translational motion and rotational motion

This equation directly leads to a second-order differential equation that describes rotational motion:

$$\Theta\ddot{\vec{\varphi}} + D\vec{\varphi} = 0 \tag{3}$$

In comparison to the kinematic equation $m\ddot{x} + kx = 0$, whose period $T = 2\pi\sqrt{\frac{m}{k}}$ and Energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is. The period for eqution (3) is:

$$T = 2\pi\sqrt{\frac{\Theta}{D}} \tag{4}$$

The differential equation (3) can be solved using an integrating factor $\dot{\vec{\varphi}}$, and directly leads to the energy theorem for harmonic rotational motion:

$$\frac{\Theta}{2}\dot{\vec{\varphi}}^2 + \frac{D}{2}\vec{\varphi}^2 = E = const. \tag{5}$$

In this context, the initial term characterizes rotational energy, while the subsequent term represents the energy arising from the angular deviation of the rotational pendulum by an angle $\vec{\phi}$ from its equilibrium position.

To describe the moment of inertia J_A for rotation around a non-principal axis A, Steiner's theorem proves to be a valuable tool. For an object with a principal moment of inertia J_S about its center of mass and a total mass M, where the axis of rotation is located at a distance a from the center of mass, the expression is as follows:

$$J_A = J_S + Ma^2 \tag{6}$$

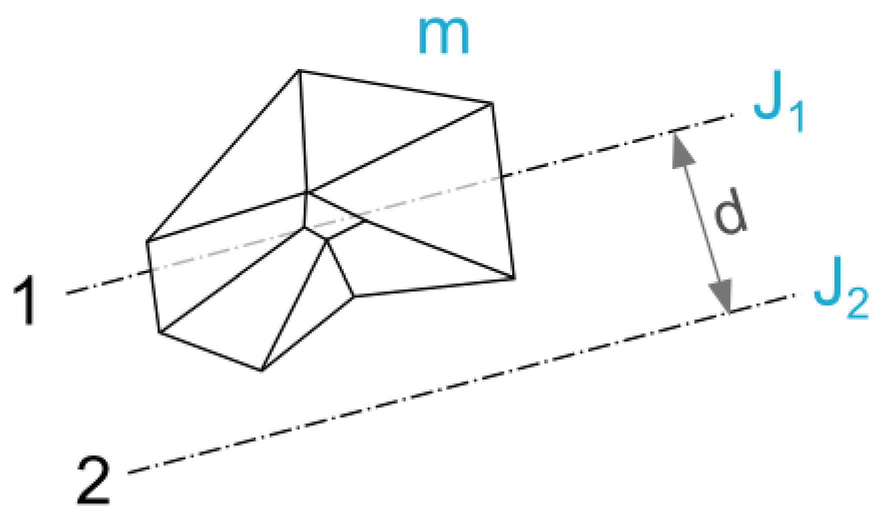


Abbildung 3: To illustration about the law of Steiner

In the case of three-dimensional bodies, their moment of inertia can be expressed as a second-order tensor. There exists a coordinate system in which this tensor takes on a diagonal form. The diagonal components of this tensor are referred to as the principal moments of inertia, and the associated eigenvectors indicate the directions of the principal axes of inertia.

During free rotation (i.e., rotation without the influence of an external torque), the principal axes of inertia are notable because stable rotation can only occur around these axes (otherwise, the body would tumble). The object will tend to rotate about the axis with the greatest moment of inertia

All of these properties will be utilized in the following experiment, aimed at determining the moment of inertia of an unknown body using a rotational pendulum. This approach proves particularly valuable when calculating the moment of inertia becomes challenging due to the body's intricate geometry.

1.2 Relevant formulas

- Firstly, we need two formulas for the torque (torsional momentum) M:

  $$M = -D\varphi \qquad M = -Fr \tag{7}$$

  The deflecting force D of the rotating pendulum can be calculated via the relation between the exerting torque M and the deflection angle ϕ given by the formula. The torque M can be induced in the tangential force F (weight of the loading platform with weight pieces on it) exerts along the perimeter of the disk via the string.

- The moment of inertia J_S of the disk of an extended disk with mass m_S and radius r_S can be simply calculated via the formula:

$$J_S = \frac{1}{2} m_S r_S^2 \tag{8}$$

- The oscillation period T_1 of the rotational pendulum without the disk placed on it is related to the moment of inertia J_T of the turntable (rotating) by the following equation:

$$T_1 = 2\pi \sqrt{\frac{J_T}{D}} \tag{9}$$

- When the disk with the moment of inertia J_S is added, the oscillation period changes to:

$$T_2 = 2\pi \sqrt{\frac{J_T + J_S}{D}} \tag{10}$$

- The moment of inertia J_T can be eliminated from the equations (9), (10) after their squaring and subtraction. As a result on gets:

$$D = \frac{4\pi^2 J_S}{T_2^2 - T_1^2} = \frac{2\pi^2 m_S r_S^2}{T_2^2 - T_1^2} \tag{11}$$

The formula symbols used in **1.1** and **1.2** in this document represent:

M	Torque
D	Deflecting force
φ	Deflection angle
Θ	Tensor of inertia[1]
J	Moment of inertia[2]
m	Mass[2]
r	Radius[2]
E	Energy
F	Force
T	Oscillation duration

[1]: In the two-dimensional case, the inertia tensor Θ simplifies to a scalar J.
[2]: The indices S, T, P respectively represent the disk, the table, and the irregularly shaped plate.

2 Execution of this experiment

Experimental setup, procedure, and measurement protocol: see the following pages.

Messprotokoll PAP I 07.09.2023

Yuting Shi, Yulai Shi 9:00 ~ 12:00

Versuch 12 Trägheitsmoment

Messaufbau:

- Drehpendel mit senkrechter Achse
- Drehgabel und Drehtisch
- Waage gemeinsam für alle Aufbauten
- Handstoppuhr und Messschieber
- Balancierschneide
- Zubehör: Al-Scheibe mit Schnurnut, und Winkelteilung runde Messingscheibe, unregelmäßige Messingscheibe, Gewichtsteller, 6 Auflegegewichte von je 40 g, Selbstklebeetiketten

Aufgabe 1 Bestimmung des Richtmoments

D: Richtmoment M: Drehmoment

ϕ: Winkel der Auslenkung r: Radius der Aluminiumscheibe

Tabelle 1: Winkel bei den verschiedenen Gewichten

Gewicht Nr.	1	2	3	4	5	6 (5.4 g)
Auslenkung ϕ [°]	48	98	148	199	250	~~3~~ 2 98

$\Delta\phi = 1°$

Wir setzen auf der Drehachse eine Aluminiumscheibe mit r und Winkelteilung ~~...~~ auf und schrauben fest. Dann hängen wir den Gewichtsteller an die Schnur und lösen die Schraube. Wir drehen die Scheibe bis die Schnur über den gesamten Umfang der Scheibe liegt. Wir stellen dann 6 Gewichte nacheinander auf und notieren die Winkelauslenkung

Aufgabe 2: Ermittelung des Trägheitsmoments

Wir ermitteln nun D des Drehpendels mit einer Scheibe mit bekanntem Trägheitsmoment J_s aus Schwingungsdauer. Die Al-Scheibe wird abgenommen und der Drehtisch aufgesetzt. Wir messen zuerst die Schwingungsdauer T_1 vom Tisch, dann eine runde Messingscheibe darauf befestigen, sodass ihr Mittelpunkt genau über Achse liegt, und die Schwingungsdauer T_2 messen. Zur Ermittlung wird jeweils 3 mal 10 Schwingungen gemacht.

Tabelle 2: Schwingungsdauer vom Körper

Messung Nr.	1	2	3
T_1 [s]	12.06	11.90	11.92
T_2 [s]	16.18	16.37	16.04

$\Delta T = 0.01$ s

Wir bestimmen weiter den Durchmesser der Scheibe mit Messschieber und die Masse:

$d = (10.410 \pm \cancel{0.00?})\,cm$ $\quad \Delta d = 0.005\,cm$ $\qquad m = (38.6 \pm 0.1)\,g$ ✓ $\qquad m_{Scheibe} = (542.8 \pm 0.1)\,g$ ✓

Jetzt zur Schwerpunktsbestimmung des unregelmäßigen Körpers:

Wir kleben auf die Platte ein neues Etikett und legen sie auf dem Tisch die Schneide. Dann ermitteln wir 2 zueinander liegende Gleichgewichtslagen und kennzeichnen längs der Auflageschneide 2 Striche, die sich im Schwerpunkt kreuzen.

Wir bestimmen das Trägheitsmoment auch aus der Schwingungsdauer (einmal auch 10 Schwingungen). Die Platte wird auf dem Drehtisch befestigt, sodass der SP wieder genau der Zeigerspitze liegt. Wir benutzen dazu das Trägheitsmoment des Drehtisches nach Ausarbeitung von Tabelle 2

Tabelle 3: Schwingdauer unregelmäßiger Platte

Messung Nr.	1	2	3
T_3 [s]	22.78	22.98	22.89

$\Delta T = 0.01\,s$

✓

Masse der Platte:
$M = (685.4 \pm 0.1)\,g$ ✓

Aufgabe 3 Steinersche Satz

Wir ziehen auf dem Etikett eine Gerade in Längsrichtung der Platte durch SP und markieren darauf 5 Punkte im Abstand $a_1, \ldots, a_5$ vom Schwerpunkt. Bestimmen das Trägheitsmoment wie die Ausarbeitung von Tabelle 3

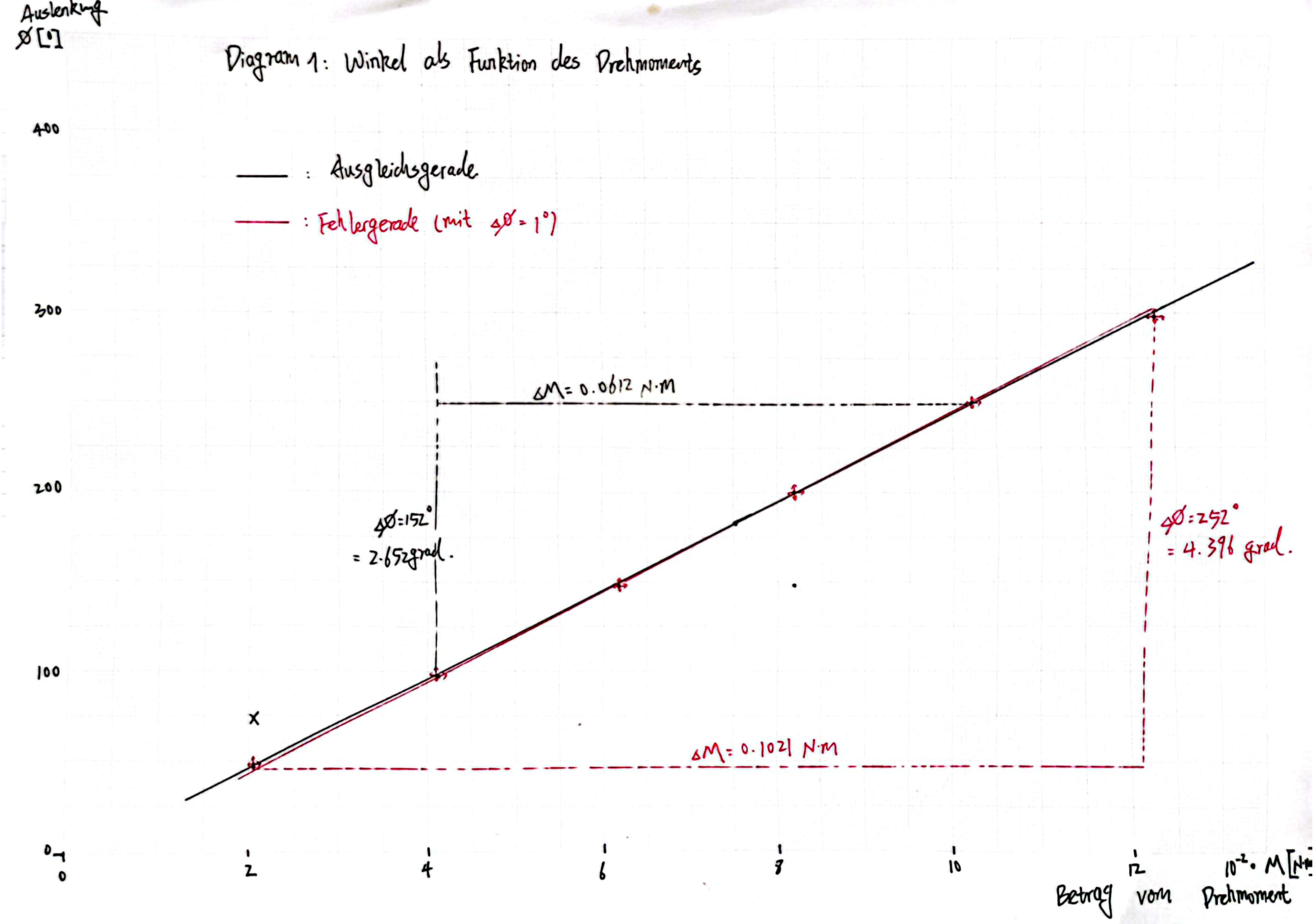
Tabelle 4: Schwingungsdauer bzgl. 5 paralleler Achsen

Abstand vom Schwerpunkt [cm]	1	2	3	4	5	$\Delta s = 0.01$
T_4 [s]	23.06	23.48	24.09	25.39	27.11	$\Delta T = 0.2\,s$

✓

7.9.'23

Diagram 1: Winkel als Funktion des Drehmoments



3 Analysis

3.1 Determination of the deflecting force D of the rotational pendulum by measuring the deflection as a function of the torque.

Tabelle 5: Torque and the error calculate by mass

Torque $M[N \cdot m]$	0.0204	0.0409	0.0613	0.0817	0.1021	0.1225
Winkel ϕ [°]	48	98	148	199	250	298
Error $\Delta M(\cdot 10^{-5})[N \cdot m]$	1	2	3	4	5	6

The deflecting force D can be calculated from the formula (6):

$$M = -D\varphi(Determination) \qquad M = -Fr = -mgr(Errorpropagation) \tag{12}$$

From Diagram 2, we can observe the slope of the regression line as well as the error line: $k_a = \frac{0.0612 N\cdot m}{2.652 grad} = 2.31 \cdot 10^{-2}\ [\frac{N\cdot m}{1 rad}]$ $\qquad k_f = \frac{0{,}1021 N\cdot m}{4.396 grad} = 2.32 \cdot 10^{-2}\ [\frac{N\cdot m}{1 rad}]$

$\Rightarrow \Delta k = 0.01 \cdot 10^{-2}[\frac{N\cdot m}{1 rad}]$

The error of M can be calculated from the Error propagation:

$$\Delta M = \sqrt{\left(\frac{\partial M}{\partial m}\Delta m\right)^2 + \left(\frac{\partial M}{\partial r}\Delta r\right)^2} = \sqrt{(mg\Delta r)^2 + (gr\Delta m)^2} \tag{13}$$

$$\tag{14}$$

$\Rightarrow$ Error: $\Delta m = 0.1g, \Delta\phi = 1$; The acceleration of gravity: $g = 9.8098 m/s^2$, error negligible. $r = (5.2050 \pm 0.0025) cm$, The mean value of D is determined by the slope of the regression line: $D_1 = k_a$

$D_1 = (2.31 \pm 0.01) \cdot 10^{-2}[\frac{N\cdot m}{1 rad}]$

3.2 Determination of the deflecting force D of the rotational pendulum by measuring the oscillation period of a disk.

First, we determine the moment of inertia J_S of the disk by substituting the measured values from Table 2.

$m_S = 542.8g, r_s = 5.205cm$, error $\Delta m_S = 0.1g, \Delta r_S = 0.0025cm$

$$J_S = \frac{1}{2} m_S r_S^2 = \frac{1}{2} \cdot 0{,}5428\ kg \cdot (0{,}052\ m)^2 \approx 7.338 \cdot 10^{-4} kg \cdot m^2 \tag{15}$$

The error ΔJ_S can be calculated as follows:

$$\Delta J_S = \sqrt{\left(\frac{\partial J_S}{\partial m_S}\Delta m_s\right)^2 + \left(\frac{\partial J_S}{\partial r_s}\Delta r_s\right)^2} = \sqrt{(\frac{1}{2}r_s^2\Delta m_s)^2 + (m_s r_s \Delta r_s)^2} \approx 2.6{\cdot}10^{-6} kg{\cdot}m^2 \tag{16}$$

$$\Rightarrow \underline{\underline{J_S = (7.338 \pm 0,026) \cdot 10^{-4} kgm^2}}$$

From table 2, we determine the oscillation period T_1 of the rotational pendulum only with the turntable and T_2 with the additional disk after dividing by 10 and subsequently calculating the mean value:

$$T_1 = \frac{12.06 + 11.90 + 11.92}{10 \times 3} \approx 1.196s;\ T_2 = \frac{16.18 + 16.37 + 16.04}{10 \times 3} \approx 1,620s$$

We calculate the errors of T_1, T_2:
Standard deviation:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} \tag{17}$$

The error of the main value:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n(n-1)}} \tag{18}$$

From our measurement series:

$$\sigma_{T_1} = 0.0072s \qquad \sigma_{\bar{T}_1} = 0.005s$$
$$\sigma_{T_2} = 0.01353s \qquad \sigma_{\bar{T}_2} = 0.010s$$

Incorporating quadratic addition, we also account for the reaction time error that occurs in each individual measurement: $\Delta T = 0.01s$

$$\Longrightarrow \Delta_r T_1 = \sqrt{3(\Delta T)^2} = \Delta_r T_2 = 0.017s$$

By quadratic addition, we calculate the total error of the oscillation periods:

$$\Delta T_1 = T_1\sqrt{\left(\frac{\sigma_{\bar{T}_1}}{T_1}\right)^2 + \left(\frac{\Delta_r T_1}{T_1}\right)^2} \approx 0,0052s \tag{19}$$

$$\Delta T_2 = T_2\sqrt{\left(\frac{\sigma_{\bar{T}_2}}{T_2}\right)^2 + \left(\frac{\Delta_r T_2}{T_2}\right)^2} \approx 0,0101s \tag{20}$$

$$\tag{21}$$

Using equation (11) and substituting $J_S = (7.338 \pm 0.026) \cdot 10^{-4} kg \cdot m^2, T_1 = (1.1961 \pm 0.0052)s, T_2 = (1.6200 \pm 0.0101)s$, we find:

$$D = \frac{4\pi^2 J_S}{T_2^2 - T_1^2} = \frac{4 \cdot \pi^2 \cdot 7.338 \cdot 10^{-4}}{1.62^2 - 1.196^2} \approx 2,424 \cdot 10^{-2} N \cdot m \tag{22}$$

We estimate the error ΔD using the Gaussian error propagation law:

$$\Delta D = \sqrt{\left(\frac{\partial D}{\partial J_S}\Delta J_S\right)^2 + \left(\frac{\partial D}{\partial T_1}\Delta T_1\right)^2 + \left(\frac{\partial D}{\partial T_2}\Delta T_2\right)^2} \tag{23}$$

$$= \sqrt{(\frac{4\pi^2}{T_2^2 - T_1^2}\Delta J_S)^2 + (\frac{8\pi^2 J_S T_1}{(T_2^2 - T_1^2)^2}\Delta T_1)^2 + (\frac{8\pi^2 J_S T_2}{(T_2^2 - T_1^2)^2}\Delta T_2)^2} \tag{24}$$

$$\approx 8.66 \cdot 10^{-5} N \cdot m \tag{25}$$

$$\Rightarrow \underline{\underline{D_2 = (2,424 \pm 0,009) \cdot 10^{-2} N \cdot m}}$$

We now estimate this value to the reference value of the rotational pendulum, which we determine from the slope of the line (D_1):

$$\frac{|D_1 - D_2|}{D_1} = 4.93\% \qquad \frac{|D_1 - D_2|}{D_2} = 4.70\% \tag{26}$$

The deviation of the measured values from the reference value is therefore not significant. In the following, we will use the value determined in section 3.1 because it includes a larger margin of error.

3.3 Calculation of the experimental messured moment of inertia J_S and the Moment of Table J_T

To calculate the moment of inertia of the disk, we first need the moment of inertia of the table. We obtain this from equation (6) and the values from Table 2:

$$T_1 = 2\pi\sqrt{\frac{J_T}{D}} \tag{27}$$

$$\Rightarrow J_T = D\left(\frac{T_1}{2\pi}\right)^2 \tag{28}$$

Here, the error ΔJ_T is given by:

$$\Delta J_T = \sqrt{\left(\frac{\partial J_T}{\partial T_1}\Delta T_1\right)^2 + \left(\frac{\partial J_T}{\partial D}\Delta D\right)^2} = \sqrt{(2D\frac{T_1}{(2\pi)^2}\Delta T_1)^2 + (\left(\frac{T_1}{2\pi}\right)^2 \Delta D)^2} \tag{29}$$

and the values:

$$D_1 = (2.31 \pm 0.01) \cdot 10^{-2} [N \cdot m],\ T_1 = (1.1961 \pm 0.0052)[s],\ T_2 = (1.6200 \pm 0.0101)[s]$$

So we calculated:

$$\Rightarrow \underline{\underline{J_T = (8.378 \pm 0.073) \cdot 10^{-4} kgm^2}}$$

The moment of inertia of the regular shaped disk:

$$T_2 = 2\pi \sqrt{\frac{J_T + J_S}{D}} \tag{30}$$

$$\Rightarrow J_S = D \left(\frac{T_2}{2\pi}\right)^2 - J_T \tag{31}$$

In Following the error of J_S is given by:

$$\Delta J_S = \sqrt{\left(\frac{\partial J_S}{\partial T_2} \Delta T_2\right)^2 + \left(\frac{\partial J_S}{\partial D} \Delta D\right)^2 + \left(\frac{\partial J_S}{\partial J_T} \Delta J_T\right)^2} \tag{32}$$

$$= \sqrt{(2D \frac{T_2}{(2\pi)^2} \Delta T_2)^2 + (\left(\frac{T_2}{2\pi}\right)^2 \Delta D)^2 + (-1 \cdot \Delta J_T)^2} \tag{33}$$

$$\tag{34}$$

We insert all the relevant values in the equations (30)-(33):

$$\Rightarrow \underline{\underline{J_S = (6.994 \pm 0.192) \cdot 10^{-4} kg \cdot m^2}}$$

In our calculations based on experimental data, we found a significant deviation between the measured moment of inertia of the disk and the value calculated by using the theoretical formula (8), and it is noticeably lower than the theoretical value.

Therefore, we have noticed that in the first part, when measuring the torque of the turntable using the pulley and weights, there was significant friction caused by mechanical wear. Consequently, the value D we measured should be obviousliy lower than the actual value, which may result in significant d iscrepancies in the subsequent calculations of the moment of inertia compared to the actual values

Additionally, it could have been considered that there might have been an error in placing the disk and the plate on the table, as the center of mass of each body and the axis of rotation may not have aligned exactly by visual estimation. Let's assume that they were offset by $a \leq 1mm$ from each other. Then, we can calculate using the Steiner's theorem (6), taking into account the quadratic dependence of the additional moment of inertia on the distance between the center of mass and the axis of rotation, and the monotonicity of the quadratic function:

$$\Delta J_x \geq m_x \cdot (1mm)^2 \tag{35}$$
$$\Delta J_S = 1.92 \cdot 10^{-5}\ kg \cdot m^2 \geq m_S \cdot (1mm)^2 = 5.428 \cdot 10^{-7}\ kg \cdot m^2 \tag{36}$$
$$\Delta J_T = 7.3 \cdot 10^{-6}\ kg \cdot m^2 \geq m_T \cdot (1mm)^2 = 3.636 \cdot 10^{-7}\ kg \cdot m^2 \tag{37}$$

These errors for the disk and the plate are on the order of $\cdot 10^{-7}$ and can therefore be neglected since the last significant digit being considered is at $\cdot 10^{-6}$.

3.4 Calculation of the moments of inertia of the irregularly shaped plate during rotation about an axis parallel to the center of mass axis

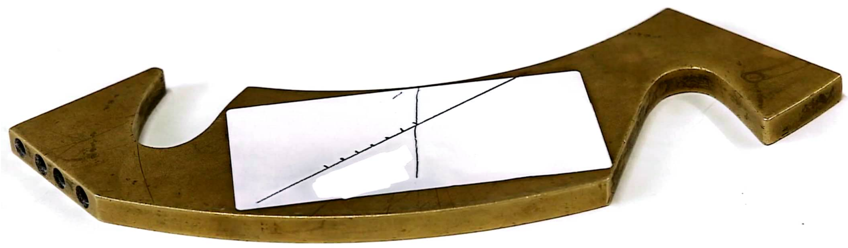


Abbildung 4: Punkte 1 to 5

The moment of inertia of the irregular shaped plate:

$$T_3 = 2\pi\sqrt{\frac{J_T + J_P}{D}} \tag{38}$$
$$\Rightarrow J_P = D\left(\frac{T_3}{2\pi}\right)^2 - J_T \tag{39}$$

$J_T = (8.378 \pm 0.073) \cdot 10^{-4}[kg \cdot m^2]$ This value is calculated by us in **3.3**
In Following the error of J_P is given by:

$$\Delta J_P = \sqrt{\left(\frac{\partial J_P}{\partial T_3}\Delta T_3\right)^2 + \left(\frac{\partial J_P}{\partial D}\Delta D\right)^2 + \left(\frac{\partial J_P}{\partial J_T}\Delta J_T\right)^2} \tag{40}$$
$$= \sqrt{(2D\frac{T_3}{(2\pi)^2}\Delta T_3)^2 + (\left(\frac{T_3}{2\pi}\right)^2 \Delta D)^2 + (-1 \cdot \Delta J_T)^2} \tag{41}$$
$$\tag{42}$$

From our measurement:

$$T_3 = \frac{22.78 + 22.98 + 22.89}{10 \times 3} \approx 2.288s$$

According to the equations (17), (18), We calculate the errors of T_3

$$\sigma_{T_3} = 0.008s \qquad \sigma_{\bar{T}_1} = 0.006s$$

The human reaction time: $\Delta T = 0.01s$
$\implies \Delta_r T_3 = \sqrt{3(\Delta T)^2} = \Delta_r T_1 = \Delta_r T_2 = 0.017s$

$$\Delta T_3 = T_3 \sqrt{\left(\frac{\sigma_{\bar{T}_3}}{T_3}\right)^2 + \left(\frac{\Delta_r T_3}{T_3}\right)^2} \approx 0,0061s \tag{43}$$

Using equation (39),(40) and substituting : $J_P = (2.228 \pm 0.030) \cdot 10^{-3} kg \cdot m^2$,
Now we calculate the moments of inertia of the plate during rotation about an axis parallel to the center of mass.
The following figure illustrates the points around which we rotated the plate.

- Experimental: The moment of inertia $J_{P,x}$ of the irregular shaped plate in different points about the parallel axis:

$$T_4 = 2\pi \sqrt{\frac{J_T + J_{P,x}}{D}} \tag{44}$$

$$\Rightarrow J_{P,x} = D \left(\frac{T_4}{2\pi}\right)^2 - J_T \tag{45}$$

The error of $J_{P,x}$ is given by:

$$\Delta J_{P,x} = \sqrt{\left(\frac{\partial J_{P,x}}{\partial T_4} \Delta T_4\right)^2 + \left(\frac{\partial J_{P,x}}{\partial D} \Delta D\right)^2 + \left(\frac{\partial J_{P,x}}{\partial J_T} \Delta J_T\right)^2} \tag{46}$$

$$= \sqrt{(2D \frac{T_4}{(2\pi)^2} \Delta T_4)^2 + (\left(\frac{T_4}{2\pi}\right)^2 \Delta D)^2 + (-1 \cdot \Delta J_T)^2} \tag{47}$$

$$\tag{48}$$

$\Delta T_4 = 0.2s$ from our protocol of the measurement.

- Theoretical: Using the Steiner's theorem (equation 5), the moment of inertia $J'_{P,x}$ calculated for rotation about the center of mass, the measured perpendicular distances of the points to the center of mass axis, and the corresponding oscillation periods (refer to Table 4), we calculate the theoretical moments of inertia for rotation about the indicated axes as follows:

$$J'_{P,x} = J_P + m_P a_x^2 \tag{49}$$

Error $\Delta J_{P,x}$:

$$\Delta J'_{P,x} = \sqrt{\left(\frac{\partial J'_{P,x}}{\partial J_P}\Delta J_P\right)^2 + \left(\frac{\partial J'_{P,x}}{\partial m_P}\Delta m_P\right)^2 + \left(\frac{\partial J'_{P,x}}{\partial a_x}\Delta a_x\right)^2} \tag{50}$$

$$= \sqrt{(\Delta J_P)^2 + (a_x^2 \Delta m_P)^2 + (2 m_P a_x \Delta a_x)^2} \tag{51}$$

The index x in this case ranges from 1 to 5 and refers to the respective axis (see Fig. 4). For the error in positioning the plate with the points on the rotation axis, we assume: $\Delta a_x = 0.01cm$. The results of the calculations are summarized in the following table.

Tabelle 5: Theoretical and experimental moment of inertia

Point	Distance a_x [cm]	$J'_{P,x} \pm \Delta J'_{P,x} [\cdot 10^{-3} kg \cdot m^2]$	$J_{P,x} \pm \Delta J_{P,x} [\cdot 10^{-3} kg \cdot m^2]$
01	1,0	$2,297 \pm 0,030$	$2,277 \pm 0,540$
02	2,0	$2,506 \pm 0,030$	$2,388 \pm 0,550$
03	3,0	$2,854 \pm 0,031$	$2,558 \pm 0,564$
04	4,0	$3,341 \pm 0,031$	$3,084 \pm 0,606$
05	5,0	$3,966 \pm 0,031$	$3,623 \pm 0,647$

The above table includes the experimentally determined values $J_{P,x}$. These, as well as their errors, can be calculated entirely analogously to **3.3** from the measured values in Table 4 (using equations (45), (46)).

Whether the deviations are significant will now also be determined computationally. For this purpose, it will be calculated whether the deviations between the theoretical and experimental values, taking into account the experimental error and the error of the theoretical values, which is mainly due to the error in the experimentally determined value of J_P, are significant. So, we calculate:

$$y := \frac{|J_{P,x} - J'_{P,x}|}{\sqrt{(\Delta J_{P,x})^2 + (\Delta J'_{P,x})^2}} \tag{52}$$

The results are summarized in the following table. Values below 1 indicate a non-significant deviation.

Tabelle 5: Deviation of theoretical and experimental values

$J'_{P,x} \pm \Delta J'_{P,x} [\cdot 10^{-3} kgm^2]$	$J_{P,x} \pm \Delta J_{P,x} [\cdot 10^{-3} kgm^2]$	y
$2,297 \pm 0,030$	$2,277 \pm 0,540$	0,038
$2,506 \pm 0,030$	$2,388 \pm 0,550$	0,215
$2,854 \pm 0,031$	$2,558 \pm 0,564$	0,526
$3,341 \pm 0,031$	$3,084 \pm 0,606$	0,425
$3,966 \pm 0,031$	$3,623 \pm 0,647$	0,531

Judging from the table all the values of y are significantly smaller than 1, we can argue that none of the deviations are significant.

To establish the Steiner's theorem as well as the linearity between moment of inertia and distance squared, we plotted and analyzed all experimentally and theoretically determined values from Table 5 on a graph.

When entering the values into Graph 2, the errors Δa^2 should also be noticed:

$$\Delta(a^2) = \frac{\partial}{\partial a}(a^2)\Delta a = 2a\Delta a \tag{53}$$

Tabelle 5: Fehler von a_x^2

a_x[cm]	1,0	2,0	3,0	4,0	5,0
a_x^2[cm^2]	1,0	4,0	9,0	16,0	25,0
$\Delta(a_x)^2$[cm]	0,02	0,04	0,06	0,08	0,10

The error in our x-axis is almost negligibly small; therefore, we will simply disregard it.

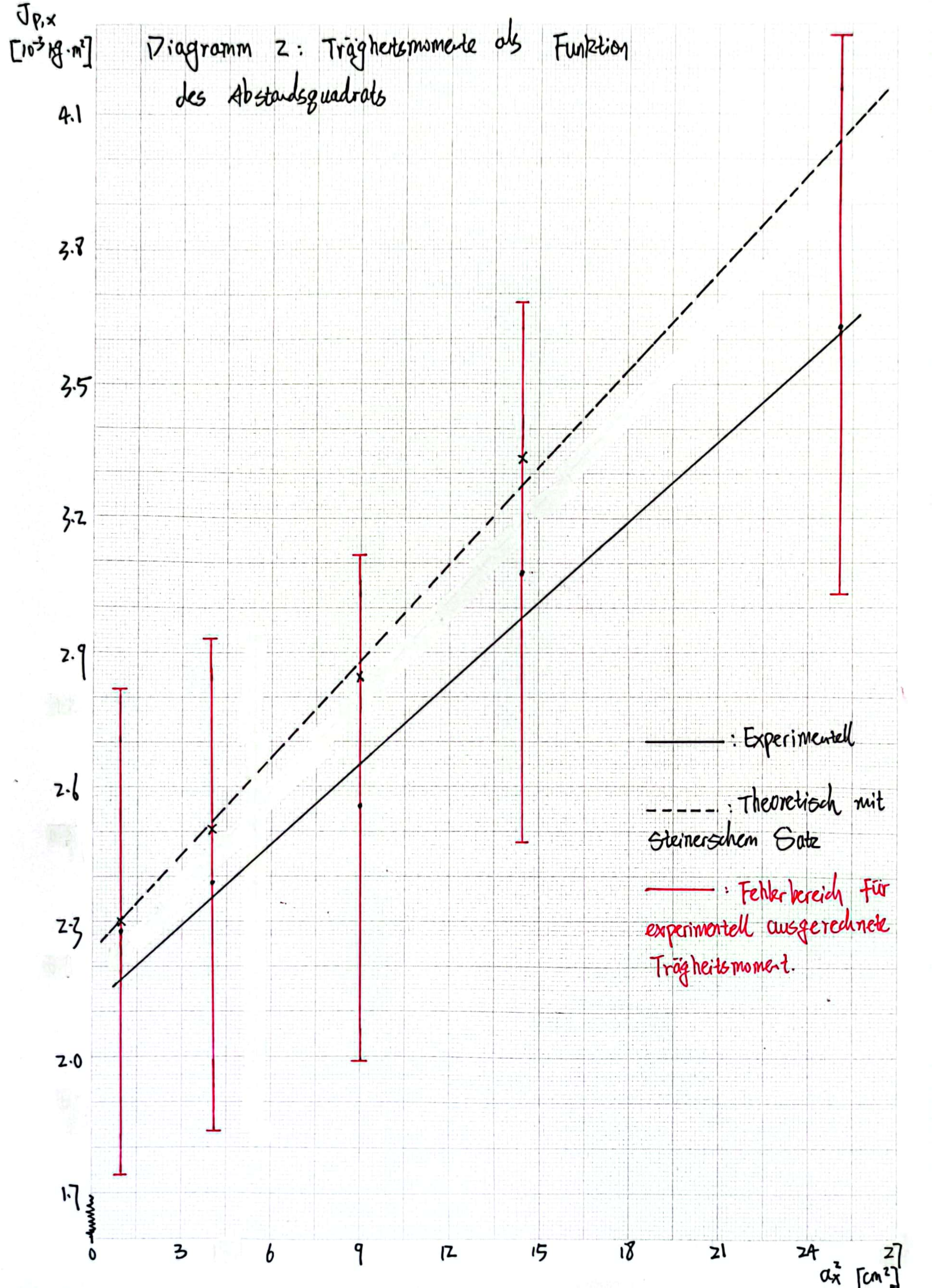


Diagramm 2: Trägheitsmomente als Funktion des Abstandsquadrats
$J_{P,x}$ [10^{-3} kg·m²]
4.1
3.8
3.5
3.2
2.9
2.6
2.3
2.0
1.7
0
3
6
9
12
15
18
21
24
27
a_x^2 [cm²]
: Experimentell
: Theoretisch mit Steinerschem Satz
: Fehlerbereich für experimentell ausgerechnete Trägheitsmoment.

4 Discussion

In this experiment, the torque of a rotational pendulum was determined, and the moments of inertia of a rotating table and an irregularly shaped object were investigated for various positions of the rotational axis within the object. There were no significant deviations observed among the measured values. However, it is noteworthy that these measurements do not necessarily imply high accuracy, as a clear disparity between the experimental curve and the theoretical curve based on Steiner's theorem was observed at the end, as evident in Diagram 2. Additionally, the error bars extend almost halfway up the graph.

Notably, our experimentally determined value for J_S deviated significantly from the theoretical calculation, with the two values not coinciding within the error margin. This discrepancy can be attributed to substantial mechanical wear and increased friction in the apparatus. As a result, a larger value was obtained for the torque due to a smaller angular displacement. The formula $J_T = D \cdot (\frac{T_1}{2\pi})^2$ demonstrates that the moment of inertia of the table increases, while $J_S = D \cdot (\frac{T_2}{2\pi})^2 - J_T$ decreases.

To minimize errors in time measurement, we initiated each period measurement from the maximum displacement. This approach was chosen based on previous experience, as precise determination of the zero-crossing point was error-prone. This is also why we opted for such a short human reaction time of 0.01 seconds. Furthermore, we employed an extremely precise caliper with an error margin of only $0.05mm$, as evident in our protocols displaying the minor significant figures of our measurement errors. Unfortunately, despite the use of precise instruments, significant fluctuations in the results were observed, highlighting the inherent instability of the experiment, as previously discussed regarding the difference in J_S.

This limitation of the experiment could potentially be addressed by using a rotational pendulum with lower friction (e.g.: magnetic levitation) or by conducting a greater number of measurements to reduce the error in the period of oscillation (the error is evidently proportional to $1/n$ for n measurements). However, strong friction inherently imposes a natural limit on the number of observable oscillations.

Considering the examination of the differential equation for a damped harmonic oscillator, it can be inferred that the frequency and thus the period of oscillation are influenced by damping.

$$x(t) = x_0 \exp(-\gamma t)(cos(\omega t) + \frac{\gamma}{\omega} sin(\omega t)), \gamma \leq \omega_0, strongdamping$$
$$x(t) = x_0 \exp(-\gamma t)(cosh(kt) + \frac{\gamma}{\omega} sinh(kt)), \gamma \geq \omega_0, weakdamping$$
$$w = \sqrt{w_0^2 - \gamma^2} \qquad k = \sqrt{\gamma^2 - \omega_0^2}$$

Consequently, equations (8), (9) and (10) and all resulting formulas and results no longer precisely apply, as they were derived from the differential equation (3) for an undamped harmonic oscillator. The presence of significant friction was evident in the fact that the displacement of the rotational pendulum decreased to less than a quarter of its original value during 10 oscillations. Reducing this factor could have been achieved by initiating the pendulum with a smaller displacement at the beginning of each measurement to minimize energy loss. However, this would have worsened the precision of determining the period of oscillation since the pendulum would have been even shorter at maximum displacement, making the precise timing of the maximum displacement more challenging.

It is also possible that a slightly larger displacement (e.g., 160 degrees) caused the restoring torque of the rotational pendulum to no longer be linearly described by Hooke's law. In such a case, the theoretically derived formulas would no longer precisely correspond to the given situation.

In the final part, concerning the determination of the moment of inertia along different axes parallel to the center of mass axis, we obtained an almost perfect theoretical line with a very small margin of error, with nearly all five points lying on this line. In contrast, the experimental curve exhibited a significant margin of error, although all points approximately lay on a straight line, it never matched our Steiner's theorem curve. Fortunately, both lines exhibited acceptable slopes. This suggests that the error is attributed to distortions in the overall experimental data and not to a theoretical conflict caused by local anomalies.

In conclusion, it is worth noting that determining the center of mass was challenging due to difficulties in stably placing the plate on the table-fixed edge and finding two perpendicular equilibrium positions. Therefore, we adopted an alternative approach, suspending the irregular plate with a string, extending the suspensi on line backward, and marking it with a pen to determine the center of mass.