# Policy Optimization over Submanifolds for Constrained Feedback Synthesis

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#### **Abstract**

In this paper, we study linearly constrained policy optimizations over the manifold of Schur stabilizing controllers, equipped with a Riemannian metric that emerges naturally in the context of optimal control problems. We provide extrinsic analysis of a generic constrained smooth cost function, that subsequently facilitates subsuming any such constrained problem into this framework. By studying the second order geometry of this manifold, we provide a Newton-type algorithm with local convergence guarantees that exploits this inherent geometry without relying on the exponential mapping nor a retraction. The algorithm hinges instead upon the developed stability certificate and the linear structure of the constraints. We then apply our methodology to two well-known constrained optimal control problems. Finally, several numerical examples showcase the performance of the proposed algorithm.

#### **Index Terms**

Optimization over Submanifolds; Structured LQR Control; Output-feedback LQR control; Constrained Stabilizing Controllers

#### I. Introduction

In recent years, direct Policy Optimization (PO) for different variants of the Linear Quadratic Regulators (LQR) problems have attracted considerable attention in the literature. In the meantime, PO for linearly constrained LQR, e.g., state-feedback Structured LQR (SLQR) and Output-feedback LQR (OLQR), has been less explored due to the intricate geometry of the respective feasible sets and the non-convexity of the cost function. While reparameterization of the LQR problem to a convex setup is possible for unconstrained cases [1], in general, trivial constraints directly on the policy quickly become nontrivial and non-convex after such reparameterizations.<sup>1</sup> Furthermore, the domain of the

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<sup>1</sup>There are exceptions to this statement-like when conditions such as quadratic invariance can be invoked [2].

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optimization problems for constrained LQR (and its variants) are generally non-convex [3] and even disconnected [4]. As such, there are no guarantees that first-order stationary points are necessarily local minima.

Finding the linear output-feedback policy directly for the OLQR problem was first addressed in [5], a procedure that involves solving nonlinear matrix equations at each iteration. Since then, there has been on-going research efforts to address this problem adopting distinct perspectives [6]–[13]. Various complexities involved in different aspect of this problem has been studied for decades [14]–[16]. Due to these complications, developing iteration numerical procedures are left as the most effective approach whose performance may depend on the problem instance/setting. In particular, first and second order methods have been adopted for solving SLQR and OLQR problems (see e.g., [7], [13] and references therein). However, these methods 1) often utilize backtracking line-search techniques at each iteration—which may be computationally expensive/unavailable (depending on the problem setup), 2) do not provide explicit guarantees for convergence, 2) do not exploit the inherent non-Euclidean geometry of the problem, and finally, 4) do not offer a unifying setup that can handle general linear constraints in feedback synthesis.

Recently, state-feedback LQR problems have been studied through the lens of first order methods, in both discrete-time [17] and continuous-time [18] setups. This point of view was initiated when the LQR cost was shown to be gradient dominant [19], facilitating a global convergence guarantee of first order methods for this problem—despite of its non-convexity. Since then, PO using first order methods has been investigated for variants of LQR problem, such as OLQR [20], model-free setup [21], and riskconstrained LQR [22] just to name a few. The gradient dominant property, however, is only known to be valid with respect to the global optimum of the unconstrained case, and is not necessarily expected for the general constrained LQR problems. By merely using the first order information of the cost function, Projected Gradient Descent (PGD) techniques—whenever the projection is possible—can be shown to converge to first order stationary points but with a sublinear convergence rate (e.g., see [17] and [20] for SLQR and OLQR problems, respectively). A sublinear rate is generally unfavorable from a practical point of view, particularly as second order information of the LQR cost can be utilized. Despite the inevitable complications arising from the non-convexity and even non-connectedness of the feasible set for general constrained optimal control problems, one may consider developing faster convergent algorithms that could facilitate an accelerated exploration of the feasible set for local optima. Note that the structure on the policy can be enforced through regularization; however, this approach merely promotes the constraints and does not address the problems considered here because the constraints are prescribed as a hard requirement for feasibility of the solution (e.g.see [23] for merely promoting

sparsity for SLQR problem). Here, we aim to utilize the second order information of the cost function to improve the convergence rate, and at the same time, provide a general approach that can address *other* linear constraints from a unifying perspective; as such, the proposed approach can be adopted for problems such as SLQR and OLQR. Indeed, any such approach is still affected by the inevitable complexities (e.g. existence/uniqueness or hardness) inherent to the problem [14], [15].

By ignoring the geometry of the problem, one may aim to optimize for the *linearly* constrained LQR cost by directly utilizing first or second order methods. Since the domain is non-convex, this might be possible—depending on the problem scenario,—by incorporating an Armijo-type backtracking line-search or requiring that the initial condition be close to the local optima. Here, we preclude from incorporating any line-search in order to systematically exploit the geometry inherent in the LQR cost. This inherent geometry will be precisely captured by the Riemannian metric defined later in §III (see Lemma III.2). Incorporating any back-tracking technique is then considered as an immediate extension to our setup. Furthermore, by adopting a geometric perspective in designing direct policy optimization, we aim to pave the road for future consideration of interesting system theoretic criteria that result in *nonlinear* constraints directly on the feedback policy; handling such constraints can significantly benefit from the intrinsic geometry of the problem as investigated in this work (see §VII for an example).

Generally, the second order behavior of the cost function can be utilized in order to obtain a descent direction—as in the Newton method—as long as the "Hessian" stays positive definite. However, this second order information can be obtained in a variety of ways; for example, with respect to the usual Euclidean geometry (especially for linearly constrained problems) or more interestingly, with respect to the non-Euclidean geometry inherent to the cost function itself. A simple—yet relevant—example of the above statement is depicted in Figure 1 in relation to the set of "diagonally" constrained stabilizing controllers (denoted by  $\widetilde{S}$ )—which turns out to be non-convex even for this simple example consisting of two inputs and two states. More specifically, this set is the intersection of the 4-dimensional set of stabilizing controllers with a 2-dimensional plane defining the diagonal constraint. Note how the "Euclidean Hessian" of the constrained LQR cost (denoted by  $\overline{\text{Hess}} h$ ) is positive definite on a smaller subset of  $\widetilde{S}$ —especially on the vicinity of the minimum (denoted by minimum). Therefore, one would expect that the neighborhood of minimum, on which the Newton updates using the Euclidean geometry is guaranteed to converge, should be relatively small. In the meantime, if the second order behavior of the LQR cost is considered through the lens of Riemannian geometry, its "Riemannian Hessian" (denoted by Hess h) captures the behavior of the cost function more effectively, such that it remains positive definite on a larger domain—compared with  $\overline{\text{Hess}} h$ . Hence, one expects a significant difference in the performance of second order optimization algorithms utilizing these two distinct geometries. To

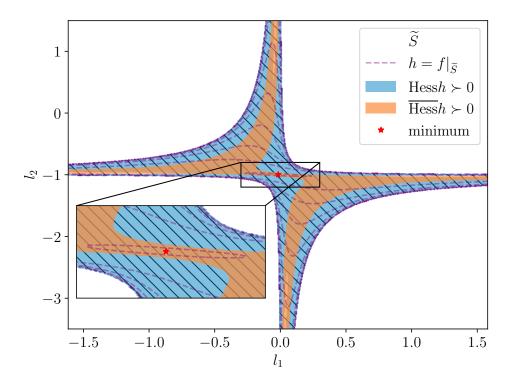


Fig. 1: The submanifold  $\widetilde{\mathcal{S}}$  of diagonal state-feedback stabilizing controllers  $(K = \operatorname{diag}(l_1, l_2))$  for a  $2 \times 2$  system; superimposed with the level-sets of the constrained LQR cost (h), and the regions on which its "Hessian" is positive definite with respect to the inherent Riemannian geometry (Hess h) and the Euclidean geometry (Hess h), respectively. Note that  $\widetilde{\mathcal{S}}$  is covered by the region on which Hess  $h \succ 0$ . See Example 1 for more details.

proceed further with this example rigorously, we need to defined  $\operatorname{Hess} h$  and devise a second order algorithm that iteratively optimize this constrained problem in vicinity of any local optimum. This machinery is developed in the rest of this paper, after which we come back to this example (see Example 1 in  $\S VI$ ).

The machinery developed in the literature for optimization over manifolds heavily depends on having access to either the exponential mapping [24], [25] or a *retraction*<sup>2</sup> from its tangent bundle onto the manifold itself [26]. However, due to the intricate geometry of the manifold of Schur stabilizing controllers, the exponential mapping is computationally expensive and a suitable retraction is generally not available. Furthermore, many useful constraints for optimal LQR problems (such as OLQR or SLQR) inherit a linear structure. Therefore, it is pertinent to ask whether we can still exploit the intrinsic non-Euclidean geometry—induced by, say, the quadratic cost and linear dynamics, in optimal

<sup>&</sup>lt;sup>2</sup>A better terminology here is *graph projection*, however we stick with *retraction* as used in other manifold optimization literature.

control problems as illustrated in Figure 1— circumventing the absence of a computationally feasible retraction.

In this paper, we consider a general optimization problem over the set of linearly constrained stabilizing feedback gains; this setup can easily be tailored to other class of constrained control synthesis problems. We introduce a Newton-type algorithm that utilizes both the inherent Riemannian geometry as well as the linear structure of the constraints, and provide its convergence analysis to the local minima. Here, in the absence of any computationally feasible (global) retraction from the tangent bundle to the manifold, we obtain the so-called *stability certificate* that—together with the linear structure of the constraints—substitute the role that a retraction would generally play, ensuring the feasibility of the next iterate. Finally, as the unit stepsize for the proposed iterates may not be possible in general, we guarantee a linear convergence rate- that eventually becomes quadratic as the iterates converge. Finally, we provide applications of the proposed methodology to the well-known state-feedback SLQR and OLQR problems, followed by numerical examples.

Our contributions can thus be summarized as follows: (i) We study the second order geometry of the manifold of stabilizing controllers induced by a pertinent Riemannian metric and its associated connection (a generalization of directional derivatives [27]), in order to understand the second order information of a cost by defining the Riemannian Hessian. (ii) We provide extrinsic analysis for first and second-order behavior of a generic smooth cost function constrained to a Riemannian submanifold. This, in turn, allows for a generic treatment of constrained optimization problems on the manifold of stabilizing controllers. (iii) We introduce a Newton-type algorithm with convergence guarantees that exploits the inherent Riemannian geometry in the absence of the exponential mapping or any retraction, effectively providing a stability certificate for linearly constrained feedback gains. (iv) We apply our methodology to SLQR and OLQR problems by first, computing the second order behavior of the LQR cost with respect to the Riemannian connection, and then, explicating the solution to Newton equation for each case using this geometry. (v) While our approach allows for considering any choice of connection, here, we focus on the associated Riemannian connection- we also make a comparison to the ordinary Euclidean connection. (vi) Finally, we provide several numerical examples to showcase the performance and advantages of the proposed methodology that exploits the intrinsic geometry of constrained feedback stabilization.

The rest of the paper is organized as follows. In §II, we introduce the generic (stabilizing) feedback synthesis problem. We then provide the analysis of this problem through the lens of differential geometry in §III. In §IV, we present the algorithm and its convergence analysis for optimization on submanifolds of stabilizing controllers endowed with a linear structure. Applications to SLQR and

OLQR problems are then presented in §V. Finally, numerical examples are provided in §VI, followed by concluding remarks in §VII. The appendix contains proofs of the results discussed here.

**Notation:** The space of  $m \times n$  matrices over the reals is denoted by  $M(m \times n, \mathbb{R})$  with the trivial smooth structure determined by the atlas consisting of the single chart  $(M(m \times n, \mathbb{R}), \text{vec})$ , where  $\text{vec}: M(m \times n, \mathbb{R}) \to \mathbb{R}^{mn}$  denotes the operator that returns a vector obtained by (vertically) stacking the columns of a matrix—from left to right. We denote the transpose operator and the spectral norm of a matrix by  $(\cdot)^{\intercal}$  and  $\|\cdot\|_2$ , respectively. The trace and spectral radius of a square matrix is denoted by  $\text{tr}\,[\cdot]$  and  $\rho(\cdot)$ , respectively. The *Loewner* partial order of symmetric positive (semi-)definite matrices is denoted by  $\succ (\succcurlyeq)$ ; we use the same notation to denote positive (semi-)definiteness of 2-tensor fields. The maximum and minimum eigenvalues of symmetric matrices will be designated by  $\overline{\lambda}$  and  $\underline{\lambda}$ , respectively. The set of positive integers less than or equal to m is denoted by [m]. By  $\mathcal{M} := \{A \in M(n \times n, \mathbb{R}) \mid \rho(A) < 1\}$ , we denote the set of (Schur) stable matrices, and define the "Lyapunov map"

$$\mathbb{L}: \mathcal{M} \times \mathbf{M}(n \times n, \mathbb{R}) \to \mathbf{M}(n \times n, \mathbb{R}),$$

that sends (A, Z) to the unique solution X of

$$X = AXA^{\mathsf{T}} + Z,\tag{1}$$

which has the representation  $X = \sum_{i=0}^{\infty} A^i Z(A^{\dagger})^i$ . If  $Z \succeq 0 \ (\succ 0)$ , then  $X \succeq 0 \ (\succ 0)$ . Furthermore, when  $Z \succeq 0$ , then  $X \succ 0$  if and only if  $(A, Z^{1/2})$  is controllable (see [28] and references therein). Finally, for manifolds we follow the notation and results in [29] and [27] unless stated otherwise.

# II. PROBLEM STATEMENT

Given a stabilizable pair (A, B) with  $A \in \mathbf{M}(n \times n, \mathbb{R})$  and  $B \in \mathbf{M}(n \times m, \mathbb{R})$ , we define

$$\mathcal{S} \coloneqq \{ K \in \mathbf{M}(m \times n, \mathbb{R}) \mid \rho(A + BK) < 1 \},\$$

as the set of stabilizing feedback gains. Subsequently, we will introduce a non-Euclidean geometry over  $\mathcal S$  using a metric arising naturally in the context of optimal control problems. We are often interested in controller gains K that lie in a relatively "simple" subset K of  $M(m \times n, \mathbb R)$ , such that  $\widetilde{\mathcal S} := K \cap \mathcal S$  is an embedded submanifold of  $\mathcal S$  (see §III-A). A common example of this is a linear subspace of  $M(m \times n, \mathbb R)$  characterizing a prescribed sparsity pattern for the admissible controller gains. Another example is the optimal output-feedback synthesis considered in §V-D.

Herein, we are concerned with the optimization problem:

$$\min_{K} f(K) \qquad \text{s.t.} \quad K \in \widetilde{\mathcal{S}}, \tag{2}$$

where  $f \in C^{\infty}(\mathcal{S}, \mathbb{R}) = C^{\infty}(\mathcal{S})$ —non-convex in general—,  $\widetilde{\mathcal{S}}$  is an embedded submanifold of  $\mathcal{S}$ , and especially when it is endowed with a linear structure. Our approach involves using this linear structure with an appropriate Riemannian geometry of  $\mathcal{S}$  to circumvent the absence of a (computationally feasible) global retraction from  $T\mathcal{S}$  onto  $\mathcal{S}$  (or  $\widetilde{\mathcal{S}}$ )—due to the intricate geometry of  $\mathcal{S}$ . In this paper, the existence of the local (or global) optima for (2) (hence the minimization) is assumed. See [30], [31] for applications of this problem.

In order to handle a generic embedded submanifold  $\widetilde{\mathcal{S}}$ , we study the behavior of the restricted function  $h \coloneqq f|_{\widetilde{\mathcal{S}}}$  from an extrinsic point of view, an approach that can be generalized to any such submanifold. We note that in general, the function f is not convex and the constraint submanifold  $\widetilde{\mathcal{S}}$  might be disconnected. Thus, here we focus on local convergence results that aim to exploit the inherent geometry of the problem in order to achieve fast convergence rates—with relatively reasonable computational complexity.

#### III. GEOMETRY OF THE SYNTHESIS PROBLEM

In order to examine (2), we analyze the domain manifold using machinery borrowed from differential geometry. Note that embedded submanifolds  $\widetilde{\mathcal{S}}$  endowed with a linear structure can certainly be investigated without using such a machinery. However, neither the corresponding results can be generalized to submanifolds with a "nonlinear" structure, nor the geometry induced by the cost function can be exploited for developing the corresponding optimization algorithms.

Before we proceed, it is worth noting that if we were to directly apply the results developed for optimization over the manifolds (such as [26]), it would have been necessary to access a retraction from the tangent bundle  $T\widetilde{\mathcal{S}}$  onto  $\widetilde{\mathcal{S}}$ . Unfortunately, due to the intricate geometry of  $\mathcal{S}$ , such a mapping is generally not available. Additionally, we will see that the Riemannian exponential map, with respect to the inherent geometry associated with optimal control problems, involves a system of Ordinary Differential Equations (ODE) whose coefficients are solutions to different Lyapunov equations. Therefore, even though it is possible to compute the exponential mapping, in general, it is hard to justify the associated computational overhead. Nonetheless, we show how we can circumvent this issue if the Riemannian tangential projection onto  $T\widetilde{\mathcal{S}}$  is available— an operation that is more streamlined in general.

# A. Analysis of the domain manifold

It is known that S is contractible [32], and unbounded when  $m \ge 2$  with the topological boundary  $\partial S = \{K \in M(m \times n, \mathbb{R}) \mid \rho(A + BK) = 1\}$  as a subset of  $M(m \times n, \mathbb{R})$ . Furthermore, S is open in

 $M(m \times n, \mathbb{R})$  (by continuity of eigenvalues for matrices with smooth entries [33, Theorem 5.2] and passing to the quotient [34, Theorem 3.73]); as such S is a submanifold without boundary.

In this paper we focus on S as a manifold on its own. Since S can be covered by a single smooth chart, the tangent bundle of S, denoted by TS, is diffeomorphic to  $S \times \mathbb{R}^{mn}$ , which in turn is diffeomorphic phic to  $S \times M(m \times n, \mathbb{R})$  under the map  $\mathrm{Id}_S \times \mathrm{vec}^{-1}$ . We refer to this composition of diffeomorphisms as "the usual identification of the tangent bundle" (or  $T_KS \cong \mathbf{M}(m \times n, \mathbb{R})$  at any point  $K \in \mathcal{S}$ ) if we need to identify any element of TS (or  $T_KS$ ). In particular, let us denote the coordinates of this global chart by  $(x^{i,j})$  for S, its associated global coordinate frame by  $(\frac{\partial}{\partial x^{i,j}})$  or simply  $(\partial_{i,j})$ , and its dual coframe by  $(dx^{i,j})$ , where  $i=1,\ldots,m$  and  $j=1,\ldots,n$ . Moreover, the  $(k,\ell)$ th element of any matrix  $A \in \mathbf{M}(m \times n, \mathbb{R})$  is denoted by  $[A]_{k,\ell}$  or  $[A]^{k,\ell}$  depending on viewing A as a point or a tangent vector, respectively. Then, for example, under the usual identification of tangent bundle, for any fixed i and j, we identify  $\partial_{i,j}$  as a matrix in  $\mathbf{M}(m \times n, \mathbb{R})$  whose elements are  $[\partial_{i,j}]^{k,\ell} = 1$  if k = i and  $\ell=j$ , and otherwise  $[\partial_{i,j}]^{k,\ell}=0$ . We also use the Einstein summation convention as explained in [27] for double indices; for example, we write  $x^{i,j}\partial_{i,j}$  to denote  $\sum_{i=1}^m \sum_{j=1}^n x^{i,j}\partial_{i,j}$ . A vector field V on S is a smooth map  $V: S \to TS$ , usually written as  $K \to V_K$ , with the property that  $V_K \in T_KS$  for all  $K \in \mathcal{S}$ . A covariant 2-tensor field is a smooth real-valued multilinear function of 2 vector fields. We denote the set of all vector fields over S by  $\mathfrak{X}(S)$ , and the bundle of covariant 2-tensor fields on  $\mathcal{S}$  by  $T^2(T^*\mathcal{S})$ . Finally, for any general mapping  $P:\mathcal{S}\to\star$ , we use  $P_K,P|_K$  or P(K) to denote the element in  $\star$  that  $K \in \mathcal{S}$  has been mapped to.

Before we proceed with the analysis, we present a technical lemma that will be used frequently throughout the paper.

**Lemma III.1.** The subset  $\mathcal{M}$  is an open submanifold of  $M(n \times n, \mathbb{R})$ , the Lyapunov map  $\mathbb{L} : \mathcal{M} \times M(n \times n, \mathbb{R}) \to M(n \times n, \mathbb{R})$  is smooth, and its differential acts as,

$$d \mathbb{L}_{(A,Q)}[E,F] = \mathbb{L}(A, E \mathbb{L}(A,Q)A^{\mathsf{T}} + A \mathbb{L}(A,Q)E^{\mathsf{T}} + F),$$

for any  $(E,F) \in T_{(A,Q)}(\mathcal{M} \times M(n \times n, \mathbb{R}))$  with the identification that follows by  $T_{(A,Q)}(\mathcal{M} \times M(n \times n, \mathbb{R})) \cong T_A \mathcal{M} \oplus T_Q M(n \times n, \mathbb{R}) \cong M(n \times n, \mathbb{R}) \oplus M(n \times n, \mathbb{R})$ . Furthermore, for any  $A \in \mathcal{M}$  and  $Q, \Sigma \in M(n \times n, \mathbb{R})$  we have, the so-called "Lyapunov-trace" property,

$$\operatorname{tr}\left[\mathbb{L}(A^{\mathsf{T}}, Q)\Sigma\right] = \operatorname{tr}\left[\mathbb{L}(A, \Sigma)Q\right].$$

<sup>&</sup>lt;sup>3</sup>Cf. [35] for a parallel study of Hurwitz stabilizing controllers.

<sup>&</sup>lt;sup>4</sup>We emphasize that the vectorization operator vec does not preserve the structure of its matrix input, e.g., vec(AB) is not a simple function of vec(A) and vec(B); as such, we are using double indices in order to maintain the *matrix* structure of points on S.

Next, we note (see (9) in §V) that many optimal control problems such as SLQR, OLQR and even Linear Quadratic Gaussian (LQG) share a similar cost structure, as  $f(K) = \frac{1}{2} \text{tr} \left[ P_K \Sigma_K \right]$ , where mappings  $P, \Sigma : \mathcal{S} \to \mathbf{M}(n \times n, \mathbb{R})$  send K to

$$P_K := \mathbb{L}(A_{\text{cl}}^{\mathsf{T}}, Q + K^{\mathsf{T}}RK), \quad \Sigma_K := \Sigma_1 + K^{\mathsf{T}}\Sigma_2K,$$
 (3)

respectively, with  $A_{\rm cl} := A + BK$  and  $\Sigma_1, \Sigma_2 \succeq 0$  as some prescribed matrices with appropriate dimensions. Motivated by this observation, we define a covariant 2-tensor field on S which will be proved to be a Riemannain metric. Similar metrics have been defined in literature in order to efficiently capture the geometry of the problem at hand; e.g. see [36] that computes the dominant subspace of a matrix by optimizing the Rayleigh quotient on the Grassmann manifold.

**Lemma III.2.** Let  $\langle \cdot, \cdot \rangle : \mathfrak{X}(S) \times \mathfrak{X}(S) \to C^{\infty}(S)$  denote the mapping that, under the usual identification of the tangent bundle, for any  $V, W \in \mathfrak{X}(S)$  sets<sup>5</sup>

$$\langle V, W \rangle \big|_{K} := \operatorname{tr} \left[ (V_K)^{\mathsf{T}} W_K \ \mathbb{L}(A_{\operatorname{cl}}, \Sigma_K) \right], \quad \forall K \in \mathcal{S}.$$

Then, this map is well-defined, induced by a smooth symmetric covariant 2-tensor field.

Now, let  $g: \mathcal{S} \to T^2(T^*\mathcal{S})$  be a smooth section of the bundle  $T^2(T^*\mathcal{S})$  that sends K to  $\langle \cdot, \cdot \rangle \big|_K$ . Then, g is in fact a Riemannian metric under mild conditions formalized below.

**Proposition III.3.** If  $(A_{\operatorname{cl}}, \Sigma_K^{1/2})$  is controllable and  $\Sigma_K \succeq 0, \forall K \in \mathcal{S}$ , then  $(\mathcal{S}, g)$  is a Riemannian manifold. Moreover, if we define the mapping  $Y : \mathcal{S} \to M(n \times n, \mathbb{R})$  sends K to

$$Y_K := \mathbb{L}(A_{\mathrm{cl}}, \Sigma_K),$$

then, with respect to the dual coframe  $(dx^{i,j})$ ,  $g = g_{(i,j)(k,\ell)} dx^{i,j} \otimes dx^{k,\ell}$  where each  $g_{(i,j)(k,\ell)} \in C^{\infty}(\mathcal{S})$  satisfies  $g_{(i,j)(k,\ell)}(K) = [Y_K]_{\ell,j}$  if i = k, and 0 otherwise. Furthermore, the inverse "matrix"  $g^{(i,j)(k,\ell)}$  satisfies  $g^{(i,j)(k,\ell)}(K) = [Y_K^{-1}]_{\ell,j}$  if i = k, and 0 otherwise.

**Remark 1.** The premise of Proposition III.3 is satisfied if  $\Sigma_K \succ 0$  for all  $K \in \mathcal{S}$ ; e.g., when  $\Sigma_1 \succ 0$  and  $\Sigma_2 \succeq 0$ . Also, by some algebraic manipulation, the well-known Hewer's algorithm [37] can be viewed as a Riemannian quasi-Newton iteration with respect to this "Riemannian metric" but with the "Euclidean connection"—see §III-D.

<sup>&</sup>lt;sup>5</sup>The notation  $\langle \cdot, \cdot \rangle$  should not be confused with the (ordinary) inner product in inner-product spaces as it is varying over S.

# B. Riemannian Connection on TS

First, consider a Riemannian submanifold  $(\widetilde{\mathcal{S}}, \widetilde{g})$  with  $\widetilde{g} := \iota_{\widetilde{\mathcal{S}}}^* g$ , where  $\iota_{\widetilde{\mathcal{S}}}^*$  denotes the pull-back by inclusion. In order to understand the second order behavior—i.e., the "Hessian"—of a smooth function on  $\widetilde{\mathcal{S}}$ , we need to study the notion of connection in  $T\mathcal{S}$ , and how it relates to analogous construct in the tangent bundle  $T\widetilde{\mathcal{S}}$ —see [27] for further details on connection. Recall that, by Fundamental Theorem of Riemannian Geometry, there exists a unique connection  $\nabla:\mathfrak{X}(\mathcal{S})\times\mathfrak{X}(\mathcal{S})\to\mathfrak{X}(\mathcal{S})$  in  $T\mathcal{S}$  that is compatible with g and symmetric, i.e., for all  $U,V,W\in\mathfrak{X}(\mathcal{S})$  we have:

- $\nabla_U \langle V, W \rangle = \langle \nabla_U V, W \rangle + \langle V, \nabla_U W \rangle$ ,
- $\nabla_U V \nabla_V U \equiv [U, V],$

where  $[V, W] \in \mathfrak{X}(\mathcal{S})$  denotes the Lie bracket of V and W.

Note that, the restriction of  $\nabla$  to  $\mathfrak{X}(\widetilde{\mathcal{S}}) \times \mathfrak{X}(\widetilde{\mathcal{S}})$  would not be a connection in  $T\widetilde{\mathcal{S}}$  as its range does not necessary lie in  $\mathfrak{X}(\widetilde{\mathcal{S}})$ . However, we can denote the (Riemannian) tangential and normal projections by  $\pi^{\top}: T\mathcal{S}|_{\widetilde{\mathcal{S}}} \to T\widetilde{\mathcal{S}}$  and  $\pi^{\perp}: T\mathcal{S}|_{\widetilde{\mathcal{S}}} \to N\widetilde{\mathcal{S}}$ , respectively, with  $N\widetilde{\mathcal{S}}$  indicating the normal bundle of  $\widetilde{\mathcal{S}}$ . Subsequently, by Gauss Formula, if  $\widetilde{\nabla}: \mathfrak{X}(\widetilde{\mathcal{S}}) \times \mathfrak{X}(\widetilde{\mathcal{S}}) \to \mathfrak{X}(\widetilde{\mathcal{S}})$  denotes the Riemannian connection in the tangent bundle  $T\widetilde{\mathcal{S}}$ , then we can compute it as follows:

$$\widetilde{\nabla}_U V = \pi^\top \nabla_U V,\tag{4}$$

for any  $U, V \in \mathfrak{X}(\widetilde{\mathcal{S}})$ , where they are extended arbitrarily to vector fields on a neighborhood of  $\widetilde{\mathcal{S}}$  in  $\mathcal{S}$ .

For computational purposes, we also obtain the Christoffel symbols associated with g (denoted by  $\Gamma^{(i,j)}_{(k,\ell)(p,q)}$ ) in the global coordinate frame. This completely characterizes the connection  $\nabla$  and enables us to compute it in this frame.

**Proposition III.4.** Consider a point  $K \in \mathcal{S}$  and under the usual identification of  $T\mathcal{S}$ , define<sup>6</sup>

$$dY_K(p,q) \coloneqq \mathbb{L}\Big(A_{\mathrm{cl}}, \ B\partial_{(p,q)}Y_KA_{\mathrm{cl}}^\intercal + A_{\mathrm{cl}}Y_K\partial_{(p,q)}^\intercal B^\intercal + \partial_{(p,q)}^\intercal \Sigma_2K + K^\intercal \Sigma_2\partial_{(p,q)}\Big),$$

<sup>&</sup>lt;sup>6</sup>This coincides with the action of  $\partial_{(p,q)}|_K$  on the mapping  $K \to Y_K$ .

for each  $(p,q) \in [m] \times [n]$  where  $Y_K = \mathbb{L}(A_{\operatorname{cl}}, \Sigma_K)$ . Then, the Christoffel symbols associated with the metric g in the global coordinate frame  $(\partial_{(i,j)})$  satisfies

$$\Gamma_{(k,\ell)(p,q)}^{(i,j)}(K) = \begin{cases} \frac{1}{2} \left[ dY_K(p,q) Y_K^{-1} \right]_{(\ell,j)}, & \text{if } k = i \neq p, \\ \frac{1}{2} \left[ dY_K(k,\ell) Y_K^{-1} \right]_{(q,j)}, & \text{if } p = i \neq k, \\ -\frac{1}{2} \sum_s \left[ dY_K(i,s) \right]_{(q,\ell)} \left[ Y_K^{-1} \right]_{(s,j)}, & \text{if } p = k \neq i, \\ \frac{1}{2} \sum_s \left( \left[ dY_K(i,\ell) \right]_{(q,s)} + \left[ dY_K(i,q) \right]_{(\ell,s)} \\ - \left[ dY_K(i,s) \right]_{(q,\ell)} \right) \left[ Y_K^{-1} \right]_{(s,j)}, & \text{if } p = k = i, \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 2.** Note that with respect to the global coordinates of  $(S, g, \nabla)$ , the *geodesic equation* is a system of (mn) second-order ordinary differential equations whose varying coefficients involve  $(mn)^3$  Christoffel symbols  $\Gamma^{(i,j)}_{(i,\ell)(i,q)}$  as obtained above. Therefore, computing the Riemannian Exponential mapping is computationally burdensome; as such we avoid using it as a retraction in this work.

# C. Extrinsic analysis of a smooth function constrained on a Riemannian submanifold

In this subsection, we study the the gradient and Hessian operators of a constrained smooth function from an extrinsic point of view, which is yet to be defined. In other words, we consider  $(\widetilde{\mathcal{S}}, \widetilde{g})$  as a Riemannian submanifold of  $(\mathcal{S}, g)$ , where  $\widetilde{g} = \iota_{\widetilde{\mathcal{S}}}^* g$ , with  $\iota_{\widetilde{\mathcal{S}}}^*$  denoting the pull-back by inclusion of  $\widetilde{\mathcal{S}}$  into  $\mathcal{S}$ . Then, by considering any smooth function f on  $\mathcal{S}$ , we can define its restriction to  $\widetilde{\mathcal{S}}$  as

$$h \coloneqq f|_{\widetilde{\mathcal{S}}},$$

and examine how its gradient and Hessian operator are related to those of f. In order to answer this question, we utilize the Riemannian connection in order to analyze the second order behavior of f (or that of h).

Recall from [27] that the gradient of f with respect to the Riemannian metric g, denoted by grad  $f \in \mathfrak{X}(S)$ , is the unique vector field satisfying

$$\langle V, \operatorname{grad} f \rangle = Vf,$$

for any  $V \in \mathfrak{X}(\mathcal{S})$ . Additionally, for any normal vector field N (i.e., a smooth section of  $N\widetilde{\mathcal{S}}$ ), the "Weingarten map in the direction of N" is a self-adjoint linear map denoted by  $\mathbb{W}_N : \mathfrak{X}(\widetilde{\mathcal{S}}) \to \mathfrak{X}(\widetilde{\mathcal{S}})$ , which defines a smooth bundle homomorphism from  $T\widetilde{\mathcal{S}}$  to itself (which is linear on each tangent space), characterized by [27]:

$$\langle \mathbb{W}_N[V], W \rangle = \langle N, \pi^{\perp}(\nabla_V W) \rangle, \quad \forall V, W \in \mathfrak{X}(\mathcal{S}).$$

Finally, define the "Hessian operator" of  $f \in C^{\infty}(\mathcal{S})$  as the map  $\operatorname{Hess} f : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$  defined by

$$\operatorname{Hess} f[U] := \nabla_U \operatorname{grad} f,$$

for any  $U \in \mathfrak{X}(\mathcal{S})$ . Note that we use the same notation to denote the gradient and Hessian operators defined on the submanifold  $\widetilde{\mathcal{S}}$ ; for further discussions regarding the Hessian operator refer to §A. Next, we formalize this abstract extrinsic analysis as follows.

**Proposition III.5.** Suppose  $\widetilde{S}$  is an embedded Riemannian submanifold of S both equipped with their respective Riemannian connections. Let  $f \in C^{\infty}(S)$  be any smooth function; then  $h := f|_{\widetilde{S}}$  is smooth on  $\widetilde{S}$  and we have

$$\operatorname{grad} h = \pi^{\top}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}}).$$

Furthermore, under the usual identification of  $T\widetilde{\mathcal{S}} \subset T\mathcal{S}$ , for any  $V \in \mathfrak{X}(\widetilde{\mathcal{S}})$  we have,

$$\operatorname{Hess} h[V] = \pi^{\top}(\operatorname{Hess} f[V]|_{\widetilde{\mathcal{S}}}) + \mathbb{W}_{\pi^{\perp}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}})}[V],$$

where V is arbitrarily extended to vector fields on a neighborhood of  $\widetilde{S}$  in S.

# D. On the choice of connection

On the manifold  $(S, g, \nabla)$ , computing the exponential map requires finding solution to a system of ODEs of dimension mn. This approach is not only computationally demanding but also does not necessarily provide an exponential map of the submanifold  $\widetilde{S}$  (unless it happens to be totally geodesic). In order to avoid the computation of the Riemannian exponential map, it seems reasonable to perform updates by using simpler "retractions" from the tangent bundle to the manifold (cf. [26]); however, in general, we do not have access to such a retraction in our setup. Another computational overhead of utilizing the Riemannian connection associated with the Riemannian metric g pertains to the  $(mn)^2$ -number of Lyapunov equations involved in obtaining the Christoffel symbols at each point.

On the other hand, for applications in which the submanifold appears as  $\widetilde{\mathcal{S}} = \mathcal{S} \cap \mathcal{K}$ , where  $\mathcal{K}$  is an affine subspace of  $\mathrm{M}(m \times n, \mathbb{R})$ , it might seem reasonable to consider the ambient manifold  $(\mathcal{S}, g, \overline{\nabla})$ , where  $\overline{\nabla}$  refers to the so-called "Euclidean connection;" i.e., the connection whose coefficients (with respect to the global coordinates) all vanish  $(\overline{\Gamma}_{(k,\ell)(p,q)}^{i,j} \equiv 0 \text{ on } \mathcal{S})$ . This results in a simpler "Hessian" operator which, however, does not respect the geometry of  $(\mathcal{S},g)$  simply because  $\overline{\nabla}$  is not compatible with the metric g—in contrast to its associated Riemannian connection. Nonetheless, for completeness, we also define the "Euclidean Hessian operator" of  $f \in C^{\infty}(\mathcal{S})$  as the map  $\overline{\mathrm{Hess}} f : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$  defined by

$$\overline{\mathrm{Hess}}\,f[U]\coloneqq\overline{\nabla}_U\,\mathrm{grad}\,f,$$

for any  $U \in \mathfrak{X}(S)$ . This operator enjoys similar properties as that of Hess f, but contains different second order information about f (e.g., see Figure 1 for a comparison).

# IV. OPTIMIZATION ON SUBMANIFOLDS OF ${\mathcal S}$ WITH LINEAR STRUCTURE

In this section, we propose an optimization algorithm for smooth cost functions, constrained to submanifolds of S that are endowed with a linear structure; that is,  $\widetilde{S} = S \cap K$ , where K entails a linear structure in  $M(m \times n, \mathbb{R})$ . The proposed algorithm, does not involve the exponential mapping (due to its computational complexity); note that no other retraction from the tangent space onto the manifold S is known. Instead, we exploit this linear structure together with a geometrically-induced stability certificate that guarantees stability of the iterates by adjusting the respective stepsize.

In what follows, we first introduce this stability certificate and then propose the algorithm. We then show how this certificate can be utilized to choose stepsizes that guarantee a linear convergence rate; furthermore, we will discuss existence of neighborhoods—containing a local minima—on which the algorithm achieves a quadratic rate of convergence.

# A. Stability certificate and (direct) policy optimization

Recall that S is open in  $M(m \times n, \mathbb{R})$ ; nonetheless, we provide the following result that quantifies this fact with respect to the problem parameters, an observation that has an immediate utility for analyzing iterative algorithms on S.

**Lemma IV.1.** Consider a smooth mapping  $Q: S \to M(n \times n, \mathbb{R})$  that sends K to any  $Q_K \succ 0$ . For any direction  $G \in T_K S \cong M(m \times n, \mathbb{R})$  at any point  $K \in S$ , if

$$0 \le \eta \le s_K := \frac{\underline{\lambda}(\mathcal{Q}_K)}{\left(2\,\overline{\lambda}\,(\,\mathbb{L}(A_{cl}^\mathsf{T},\mathcal{Q}_K))\,\|BG\|_2\right)},$$

then  $K^+ := K + \eta G \in \mathcal{S}$ ;  $s_K$  will be referred to as the "stability certificate" at K.

**Proof.** As K is stabilizing, for any such  $\mathcal{Q}_K$ , there exists a matrix  $P = \mathbb{L}(A_{\operatorname{cl}}^{\mathsf{T}}, \mathcal{Q}_K) \succ 0$  satisfying  $P = A_{\operatorname{cl}}^{\mathsf{T}} P A_{\operatorname{cl}}^{\mathsf{T}} + L$  with  $A_{\operatorname{cl}}^{\mathsf{T}} \coloneqq A + B K^{\mathsf{T}}$  and  $L \coloneqq \mathcal{Q}_K + A_{\operatorname{cl}}^{\mathsf{T}} P A_{\operatorname{cl}}^{\mathsf{T}} - A_{\operatorname{cl}}^{\mathsf{T}} P A_{\operatorname{cl}}^{\mathsf{T}}$ . Therefore, in order to establish that  $K^+$  is stabilizing, as  $P \succ 0$ —by the Lyapunov Stability Criterion [38, Theorem8.4]—it suffices to show that  $L \succ 0$ . Next,

$$L = \mathcal{Q}_{K} - \eta G^{\mathsf{T}} B^{\mathsf{T}} P A_{\mathsf{cl}} - \eta A_{\mathsf{cl}}^{\mathsf{T}} P B G - \eta^{2} G^{\mathsf{T}} B^{\mathsf{T}} P B G$$

$$\succcurlyeq \mathcal{Q}_{K} - a A_{\mathsf{cl}}^{\mathsf{T}} P A_{\mathsf{cl}} - (1 + \frac{1}{a}) \eta^{2} G^{\mathsf{T}} B^{\mathsf{T}} P B G$$

$$= (1 + a) \mathcal{Q}_{K} - a P - (1 + \frac{1}{a}) \eta^{2} G^{\mathsf{T}} B^{\mathsf{T}} P B G$$

$$(5)$$

because for any a > 0,  $P \succ 0$  implies

$$aA_{\mathrm{cl}}^{\mathsf{T}}PA_{\mathrm{cl}} + (\frac{\eta^2}{a})G^{\mathsf{T}}B^{\mathsf{T}}PBG \succcurlyeq \eta G^{\mathsf{T}}B^{\mathsf{T}}PA_{\mathrm{cl}} + \eta A_{\mathrm{cl}}^{\mathsf{T}}PBG.$$

Now, by recalling the infinite-sum representation of P and the fact that  $\mathcal{Q}_K \succeq 0$ , we conclude that  $\overline{\lambda}(P) \geq \overline{\lambda}(\mathcal{Q}_K) \geq \underline{\lambda}(\mathcal{Q}_K)$ . Then, we proceed as follows: if  $\overline{\lambda}(P) > \underline{\lambda}(\mathcal{Q}_K)$  then we choose  $a = \frac{\underline{\lambda}(\mathcal{Q}_K)}{(2\overline{\lambda}(P) - 2\underline{\lambda}(\mathcal{Q}_K))} > 0$ ; otherwise, we choose  $a > 2\eta^2 \|BG\|_2^2$ . Either way, by comparing the minimum eigenvalues of both sides in (5):

$$\underline{\lambda}(L) \ge \underline{\lambda}(\mathcal{Q}_K)/2 - \left\lceil \frac{2\overline{\lambda}^2(P)}{\underline{\lambda}(\mathcal{Q}_K)} - \overline{\lambda}(P) \right\rceil \eta^2 \|BG\|_2^2.$$

Therefore, if  $|\eta| \leq \frac{\underline{\lambda}(\mathcal{Q}_K)}{(2\overline{\lambda}(P)\|BG\|_2)}$ , then  $L \succ 0$ , which completes the proof.

**Remark 3.** The proceeding lemma also provides a "conditioning" of the optimization problem in terms of system parameters A, B. In other words, for any choice of  $\mathcal{Q}_K \succ 0$  at any  $K \in \mathcal{S}$ , the ratio  $\overline{\lambda}(\mathbb{L}(A_{\operatorname{cl}}^{\mathsf{T}}, \mathcal{Q}_K))/\underline{\lambda}(\mathcal{Q}_K)$  represents a condition number revealing geometric information on the manifold at K. In a sense, this ratio reflects the "Riemannian curvature" of  $(\mathcal{S}, g, \nabla)$ ; this connection will be further explored in our future work.

Next, we propose an algorithm with convergence guarantees with at least a linear rate (when the iterates are far away from the local optima) and eventually a Q-quadratic rate (when the iterates are close enough to the local optima). The complication here is that we do not have access to a retraction with a reasonable computational complexity (see Remark 2). We claim that, starting close enough to a local minimum, a Newton-type method using Riemannian metric and the Euclidean/Riemannian connection must converge quadratically if one could have used the stepsize  $\eta=1$ . This is in fact due to the exponential mapping with respect to the Euclidean connection that serves as a retraction with desirable properties. However, the stability certificate suggests that at least away from the local minimum, it might not be possible to use such a large stepsize. Therefore, a stepsize rule has to be deduced—that in turn, hinges upon the stability certificate—the algorithm is summarized in Algorithm 1. Hereafter, we refer to the solution  $G \in T_K \widetilde{\mathcal{S}}$  of the following equation as the *Newton direction* on  $\widetilde{\mathcal{S}}$ :

$$\operatorname{Hess} h_K[G] = -\operatorname{grad} h_K,$$

where  $h = f|_{\tilde{S}}$ ; similarly, it is referred to as the *Euclidean Newton direction* if Hess h is replaced by  $\overline{\text{Hess}} h$ .

# **Algorithm 1:** Riemannian Newton-type Policy Optimization (RNPO) for Constrained Problems on $\mathcal S$

- 1: Initialization: Problem parameters (A, B),
  the linear constraint  $\mathcal{K}$  and an initial feasible stabilizing controller  $K_0 \in \widetilde{\mathcal{S}} = \mathcal{S} \cap \mathcal{K}$  and set t = 0
- 2: Choose a smooth mapping  $K \to \mathcal{Q}_K \succ 0$
- 3: Until stopping criteria are met, do
- 4: Find the Newton direction  $G_t$  on  $\widetilde{\mathcal{S}}$  satisfying

$$\operatorname{Hess} h_{K_t}[G_t] = -\operatorname{grad} h_{K_t}$$

- 5: Use  $Q_{K_t}$  to obtain a stability certificate  $s_{K_t}$
- 6: Compute step-size  $\eta_t = \min\{s_{K_t}, 1\}$
- 7: Update:  $K_{t+1} = K_t + \eta_t G_t$
- 8:  $t \leftarrow t + 1$

For the examples of the mapping Q in Line 2 see Remark 6. In Line 4, Hess h can be replaced by its Euclidean counterpart  $\overline{\text{Hess}} h$ . The update in Line 7 is possible due to the linear structure of  $\widetilde{S}$  induced by K.

# B. Linear-quadratic convergence of RNPO

In this section, we establish the local linear-quadratic convergence of RNPO algorithm on the submanifold  $\widetilde{S}$  using differential geometric techniques [24], [26], [27]. Herein, avoiding the exponential map induced by the Riemannian connection for updating the iterates, and instead relying the stability certificate, add another layer of complications for the convergence analysis. To proceed, we say that  $K^*$  is a critical point of h if  $\operatorname{grad} h_{K^*} = 0$ , and additionally, it is "nondegenerate" if  $\operatorname{Hess} h_{K^*}$  is nondegenerate, i.e.,

$$\langle \operatorname{Hess} h_{K^*}[G_1], G_2 \rangle = 0, \ \forall G_2 \in T_{K^*} \widetilde{\mathcal{S}} \implies G_1 = 0 \in T_{K^*} \widetilde{\mathcal{S}}$$

**Lemma IV.2.** Suppose  $K^*$  is a nondegenerate local minimum of  $h := f|_{\widetilde{S}}$ . Then, it is isolated, grad  $h_{K^*} = 0$  and there exists a neighborhood of  $K^*$  on which Hess h is positive definite. Furthermore, Hess  $h_{K^*} = \overline{\text{Hess}} h_{K^*}$ .

**Theorem IV.3.** Suppose  $K^*$  is a nondegenerate local minimum of  $h := f|_{\widetilde{S}}$  over the submanifold  $\widetilde{S} = S \cap K$  for some linear constraint K. Then, there exists a neighborhood  $U^* \subset \widetilde{S}$  of  $K^*$  with the following property: whenever  $K_0 \in U^*$ , the sequence  $\{K_t\}$  generated by RNPO remains in  $U^*$  (therefore, it is stabilizing), and it converges to  $K^*$  at least at a linear rate— and eventually—with a quadratic one.

**Remark 4.** The above result implies that there exist neighborhoods containing each nondegenerate local minimum of the constrained cost function on which the convergence of RNPO is guaranteed. The usefulness of this result is that the initial iterate  $K_0$  is not required to be in a (small) neighborhood of the optimum on which the step-size  $\eta=1$  is feasible. Instead, by carefully incorporating the stability certificate (Lemma IV.1), we can obtain a larger basin of attraction for the iterates (see Figure 3 in  $\S VI$ ). Finally, even though the convergence rate is initially linear, as the algorithm proceeds, a quadratic convergence rate is achieved.

## V. APPLICATIONS

In this section, we discuss applications of the developed methodology for optimizing the LQR cost over two distinct submanifolds, namely those induced by Structured LQR (SLQR) and Output-feedback LQR (OLQR) problems. Consider a discrete-time linear time-invariant dynamics

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k, \qquad \boldsymbol{y}_k = C\boldsymbol{x}_k, \tag{6}$$

where  $A \in M(n \times n, \mathbb{R})$ ,  $B \in M(n \times m, \mathbb{R})$  and  $C \in M(d \times n, \mathbb{R})$  are the system parameters for some integers n, m and d;  $\boldsymbol{x}_k$ ,  $\boldsymbol{y}_k$  and  $\boldsymbol{u}_k$  denote the states, output and inputs vectors at time k, respectively, and  $\boldsymbol{x}_0$  is given. Conventionally, the Linear Quadratic Regulators (LQR) problem is to design a sequence of input signals  $\boldsymbol{u} = (\boldsymbol{u}_k)_0^{\infty} \in \ell_2$  that minimizes the following quadratic cost

$$J_{\boldsymbol{x}_0}(\boldsymbol{u}) = \frac{1}{2} \sum_{k=0}^{\infty} \boldsymbol{x}_k^{\mathsf{T}} Q \boldsymbol{x}_k + \boldsymbol{u}_k^{\mathsf{T}} R \boldsymbol{u}_k, \tag{7}$$

subject to the dynamics in (6), where  $Q = Q^{\mathsf{T}} \succcurlyeq 0$  and  $R = R^{\mathsf{T}} \succ 0$  are prescribed cost parameters. Using Dynamic Programming or Calculus of Variations, it is well known (see e.g., §22.7 in [39]) that the optimal state-feedback solution  $\boldsymbol{u}^*$  to this problem reduces to solving the Discrete-time Algebraic Riccati Equation (DARE) for the optimal cost matrix  $P_{\mathrm{LQR}}$ . This results in a linear state-feedback optimal control  $\boldsymbol{u}_k^* = K_{\mathrm{LQR}} \boldsymbol{x}_k$ , where  $K_{\mathrm{LQR}} \in \mathbf{M}(m \times n, \mathbb{R})$  is the optimal LQR gain (policy) obtained from  $P_{\mathrm{LQR}}$ . Also, the associated optimal cost can be obtained as  $J_{\boldsymbol{x}_0}(\boldsymbol{u}^*) = \frac{1}{2} \boldsymbol{x}_0^{\mathsf{T}} P_{\mathrm{LQR}} \boldsymbol{x}_0$ .

Naturally, one could think of the LQR cost as a map  $K \mapsto J_{x_0}(u)|_{u=Kx}$ , however, this would depend on  $x_0$  and generally, its value can still be finite while K is not necessarily stabilizing (i.e., when  $K \notin \mathcal{S}$  is not feasible). Instead, in order to avoid the dependency on the initial state while considering the constraints on the policy directly, we pose the following constrained optimization problem,

$$\min_{K} f(K) := \mathop{\mathbb{E}}_{\boldsymbol{x}_0 \sim \mathcal{D}} J_{\boldsymbol{x}_0}(\boldsymbol{u}) \tag{8}$$

s.t. 
$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k, \ \forall k \geq 0, \ \boldsymbol{u}_k = K\boldsymbol{x}_k, \ K \in \widetilde{\mathcal{S}},$$

where  $\widetilde{\mathcal{S}}$  is an embedded submanifold of  $\mathcal{S}$ , and  $\mathcal{D}$  denotes a distribution of zero-mean multivariate random variables of dimension n with covariance matrix  $\Sigma_1$  so that  $0 \prec \Sigma_1 = \Sigma_1^{\mathsf{T}} \in \mathbf{M}(n \times n, \mathbb{R})$ .

Next, we can reformulate (8) as follows. For each stabilizing controller  $K \in \mathcal{S}$ , from (6) and (7) we have that

$$J_{\boldsymbol{x}_0}(K\boldsymbol{x}) = \frac{1}{2} \sum_{k=0}^{\infty} \boldsymbol{x}_0^{\intercal} (A_{\mathrm{cl}}^k)^{\intercal} [Q + K^{\intercal} R K] A_{\mathrm{cl}}^k \boldsymbol{x}_0,$$

where  $A_{\rm cl} \coloneqq A + BK$ . Since,  $A_{\rm cl}$  is a stability matrix, the sum  $\sum_{k=0}^{\infty} (A_{\rm cl}^k)^{\mathsf{T}} [Q + K^{\mathsf{T}} R K] A_{\rm cl}^k$  converges, which is equal to the unique solution  $P_K \coloneqq \mathbb{L}(A_{\rm cl}^{\mathsf{T}}, K^{\mathsf{T}} R K + Q)$ . Therefore,  $f(K) = \frac{1}{2} \mathbb{E}_{\boldsymbol{x}_0 \sim \mathcal{D}} \operatorname{tr} [P_K \boldsymbol{x}_0 \boldsymbol{x}_0^{\mathsf{T}}] = \frac{1}{2} \operatorname{tr} [P_K \Sigma_1]$ , and thus the problem in (8) reduces to

$$\min_{K} f(K) = \frac{1}{2} \operatorname{tr} \left[ P_K \Sigma_1 \right] \quad \text{s.t.} \quad K \in \widetilde{\mathcal{S}}. \tag{9}$$

This reformulation of the LQR cost function has been adopted before (see e.g., [17], [19], [40]) but the inherent geometry of the submanifold  $\widetilde{S}$  has generally been overlooked. If there is no constraint on the controller, i.e.,  $\widetilde{S} = S$ , then a well-known quasi-Newton algorithm—due to Hewer—converges to the optimal state-feedback control at a quadratic rate [37]. Otherwise,  $\widetilde{S}$  may have disconnected components, and in general, the constrained cost function may introduce stationary points that are not local minima. Nonetheless, in this section, we apply the techniques developed in §III and Algorithm 1 to the constraint arising in the well-known SLQR and OLQR problems. Note that both of these problems can be cast as an optimization in (9) with  $\widetilde{S}$  denoting a specific submanifold of S that will be detailed in §V-C and §V-D, respectively.

# A. Solving for the Newton direction

In order to solve for the Newton direction at any  $K \in \widetilde{\mathcal{S}}$ , suppose that the tuple  $(\widetilde{\partial}_{(p,q)}|_{(p,q)\in D})$  denotes a smooth local frame for  $\widetilde{\mathcal{S}}$  on a neighborhood of K, where D is a subset of  $[m] \times [n]$  depending on the dimension of  $\widetilde{\mathcal{S}}$ . In fact, by Proposition III.5, for any  $G = [G]^{k,\ell} \widetilde{\partial}_{(k,\ell)}|_K \in T_K \widetilde{\mathcal{S}}$  (interpreted as a subspace of  $T_K \mathcal{S}$ ), finding the Newton direction on  $\widetilde{\mathcal{S}}$  reduces to solving the following system of |D|-linear equations (for each index  $(p,q) \in D$ ),

$$\sum_{(k,\ell)\in D} [G]^{k,\ell} h_{;(k,\ell)(p,q)}(K) = -\left\langle \pi^{\top}(\operatorname{grad} f|_K), \widetilde{\partial}_{(p,q)}|_K \right\rangle,$$

where  $h_{;(k,\ell)(p,q)}$  denote the coordinates of  $\widetilde{\nabla}^2 h$  with respect to the local coframe dual to  $(\widetilde{\partial}_{(p,q)}|_{(p,q)\in D})$ . Thus, by (12),  $h_{;(k,\ell)(p,q)}(K) = \left\langle \operatorname{Hess} h_K[\widetilde{\partial}_{(k,\ell)}|_K], \widetilde{\partial}_{(p,q)}|_K \right\rangle$ ; or with  $\operatorname{Hess} h$  replaced by  $\overline{\operatorname{Hess}} h$ , depending on the connection.

<sup>&</sup>lt;sup>7</sup>Each  $T_K\widetilde{\mathcal{S}}$  can be viewed as a subspace of  $T_K\mathcal{S}$  as  $\widetilde{\mathcal{S}}\subset\mathcal{S}$  is embedded.

# B. Analysis for the special cost function

We now turn our attention towards analysis of the following cost function specific to optimal control problems. In order to specialize the results obtained so far to this case, we set  $\Sigma_2 = 0$  in the definition of  $\Sigma_K$  in (3).

**Proposition V.1.** On the Riemannian manifold  $(S, g, \nabla)$ , define  $f \in C^{\infty}(S)$  with  $f(K) = \frac{1}{2} \text{tr} [P_K \Sigma_1]$ , where  $P_K = \mathbb{L}(A_{\text{cl}}^{\mathsf{T}}, K^{\mathsf{T}}RK + Q)$ . Then, f is smooth and under the usual identification of the tangent bundle

grad 
$$f_K = RK + B^{\dagger} P_K A_{cl}$$
.

Furthermore, Hess f and  $\overline{\text{Hess}}$  f are both self-adjoint operators such that, for any  $E, F \in T_K S$ ,

$$\langle \operatorname{Hess} f_K[E], F \rangle = \langle B^{\mathsf{T}}(S_K|_F) A_{\operatorname{cl}}, E \rangle + \langle (R + B^{\mathsf{T}} P_K B) E + B^{\mathsf{T}}(S_K|_E) A_{\operatorname{cl}}, F \rangle - \left\langle \operatorname{grad} f_K, [E]^{k,\ell} [F]^{p,q} \Gamma^{i,j}_{(k,\ell)(p,q)}(K) \partial_{i,j} \right\rangle,$$

$$\langle \overline{\text{Hess}} f[E], F \rangle = \langle B^{\mathsf{T}}(S_K|_F) A_{\text{cl}}, E \rangle + \langle (R + B^{\mathsf{T}} P_K B) E + B^{\mathsf{T}}(S_K|_E) A_{\text{cl}}, F \rangle$$

with  $\Gamma^{i,j}_{(k,\ell)(p,q)}$  denoting the Christoffel symbols of g and

$$S_K|_E := \mathbb{L}(A_{\operatorname{cl}}^{\mathsf{T}}, E^{\mathsf{T}}\operatorname{grad} f_K + (\operatorname{grad} f_K)^{\mathsf{T}} E).$$

**Remark 5.** For comparison in the absence of any constraint (i.e. when  $\widetilde{S} = S$ ), the Hewer's update  $K_{t+1} = -(B^{\mathsf{T}}P_{K_t}B + R)^{-1}B^{\mathsf{T}}P_{K_t}A$  in [37] can be written as

$$K_{t+1} = K_t + 1\widehat{G}_t$$

with the Riemannian quasi-Newton direction  $\widehat{G}_t$  satisfying:

$$\widehat{H}_{K_t}[\widehat{G}_t] = -\operatorname{grad} f_{K_t}$$

where  $\widehat{H}_{K_t} := (R + B^{\mathsf{T}} P_{K_t} B)$  is a positive definite approximation of both  $\operatorname{Hess} f_{K_t}$  and  $\operatorname{\overline{Hess}} f_{K_t}$ . The main "algebraic luck" here is that the unit stepsize will remain stabilizing throughout this quasi-Newton updates. In general, specially on constrained submanifolds  $\widetilde{\mathcal{S}}$  and without having such a luck, we use the stability certificate as developed in Lemma IV.1

The next corollary provides a similar result for the LQR cost restricted to an embedded submanifold of S, which is a consequence of Proposition III.5 and Proposition V.1.

**Corollary V.2.** Under the premise of Proposition V.1, let  $h = f|_{\widetilde{S}}$ , where  $\widetilde{S} \subset S$  is an embedded Riemannian submanifold with the induced connection. Then, h is smooth and under the usual identification of tangent bundle

$$\operatorname{grad} h_K = \pi^{\top} (RK + B^{\dagger} P_K A_{\operatorname{cl}}).$$

Furthermore, Hess h is a self-adjoint operator and can be characterized as follows: for any  $E, F \in T_K \widetilde{S} \subset T_K S$ ,

$$\langle \operatorname{Hess} h_K[E], F \rangle = \langle B^{\mathsf{T}}(S_K|_F) A_{\operatorname{cl}}, E \rangle + \langle (R + B^{\mathsf{T}} P_K B) E + B^{\mathsf{T}}(S_K|_E) A_{\operatorname{cl}}, F \rangle$$
$$- \left\langle \operatorname{grad} h_K, [E]^{k,\ell} [F]^{p,q} \Gamma^{i,j}_{(k,\ell)(p,q)}(K) \partial_{i,j} \right\rangle,$$

where  $\Gamma^{i,j}_{(k,\ell)(p,q)}$  are the Christoffel symbols of g and

$$S_K|_E := \mathbb{L}(A_{\operatorname{cl}}^{\mathsf{T}}, E^{\mathsf{T}}\operatorname{grad} f_K + (\operatorname{grad} f_K)^{\mathsf{T}} E).$$

**Remark 6.** If  $Q \succ 0$ , then we can choose the mapping  $Q : K \to Q_K$  to be  $Q_K = Q + K^T R K$ , and thus the stability certificate  $s_K$  as defined in Lemma IV.1 must satisfy

$$s_K \ge \frac{\underline{\lambda}(Q)\,\underline{\lambda}(\Sigma_1)}{(4f(K)\,\|BG\|_2)},$$

where f denotes the LQR cost. This is due to the fact that  $R,Q,\Sigma_1\succ 0$  and so is  $P_K\succ 0$ ; hence, by the trace inequality,  $f(K)\geq (\frac{1}{2})\,\underline{\lambda}(\Sigma_1)\,\overline{\lambda}(P_K)$ . The claimed lower-bound on the stability certificate then follows by combining the last inequality with the definition of the stability certificate. Otherwise if  $Q\succeq 0$ , one can leverage on the observability of the pair  $(A,Q^{1/2})$  to obtain derive analogous results.

#### C. State-feedback SLQR

Any desired sparsity pattern on the controller gain K imposes a linear constraint set, denoted by  $\mathcal{K}_D$ , which indicates a linear subspace of  $\mathbf{M}(m \times n, \mathbb{R})$  with nonzero entries only for a prescribed subset D of entries, i.e., for any  $K \in \mathcal{K}_D$  and  $(i,j) \notin D$  we must have  $[K]_{i,j} = 0$ . Let the tuple  $(x^{(i,j)}|_{(i,j)\in[m]\times[n]})$  denote the component functions of the global smooth chart  $(\mathbf{M}(m \times n, \mathbb{R}), \mathrm{vec})$ , and define  $\Phi: \mathbf{M}(m \times n, \mathbb{R}) \to \mathbb{R}^{mn-|D|}$  with

$$\Phi(K) = \sum_{(i,j) \notin D} [K]_{i,j} x^{(i,j)}.$$

Then, it is easy to see that  $\Phi$  is a smooth submersion, and so is  $\Phi|_{\mathcal{S}}$  as  $\mathcal{S}$  is an open submanifold of  $\mathbf{M}(m \times n, \mathbb{R})$ . Therefore, as  $\widetilde{\mathcal{S}} = \mathcal{S} \cap \mathcal{K}_D = (\Phi|_{\mathcal{S}})^{-1}(0)$ , by Submersion Level Set Theorem we conclude that  $\widetilde{\mathcal{S}}$  is a properly embedded submanifold of dimension |D|.

Furthermore, at any point  $K \in \widetilde{\mathcal{S}}$  and for any tangent vector  $E \in T_K \mathcal{S}$ , we can compute the tangential projection  $\pi^{\top}: T_K \mathcal{S} \to T_K \widetilde{\mathcal{S}}$  as follows:

$$\pi^{\top} E = \arg \min_{\widetilde{E} \in T_K \widetilde{\mathcal{S}}} \left\langle E - \widetilde{E}, E - \widetilde{E} \right\rangle. \tag{10}$$

As  $\mathcal{K}_D$  is a linear subspace of  $\mathbf{M}(m \times n, \mathbb{R})$ , we can identify  $T_K \widetilde{\mathcal{S}}$  with  $\mathcal{K}_D$  itself (due to dimensional reasons). Then, it is not hard to show that the unique solution  $\widetilde{E}^*$  to the minimization above (with linear constraint and strongly convex cost function, as  $Y_K \succ 0$ ), must satisfy  $E - \widetilde{E}^* \perp \mathcal{K}_D$  with respect to the Riemannian metric at K; or equivalently,

$$\operatorname{Proj}_{\mathcal{K}_D}\left[(E - \widetilde{E}^*)Y_K\right] = 0,$$

where  $\operatorname{Proj}_{\mathcal{K}_D}$  denotes the Euclidean projection onto the sparsity pattern  $\mathcal{K}_D$ . Note that at each  $K \in \widetilde{\mathcal{S}}$ , the last equality consists of |D| nontrivial linear equations involving |D| unknowns (as the nonzero entries of  $\widetilde{E}^*$ ), which can be solved efficiently.

Finally, if  $\widetilde{\partial}_{(i,j)}$  (as described in §V-A) is taken to be  $\widetilde{\partial}_{(i,j)} = \partial_{(i,j)}$  for  $(i,j) \in D$ , then  $(\widetilde{\partial}_{(i,j)}|_{(i,j)\in D})$  forms a global smooth frame for  $T\widetilde{\mathcal{S}}$ . Thus, for each  $(k,\ell), (p,q)\in D$ , the coordinates  $h_{;(k,\ell)(p,q)}(K)$  simplifies to

$$h_{;(k,\ell)(p,q)}(K) = \left\langle B^{\mathsf{T}}(S_K|_{\partial_{(p,q)}}) A_{\mathsf{cl}}, \partial_{(k,\ell)} \right\rangle + \left\langle (R + B^{\mathsf{T}} P_K B) \partial_{(k,\ell)} + B^{\mathsf{T}}(S_K|_{\partial_{(k,\ell)}}) A_{\mathsf{cl}}, \partial_{(p,q)} \right\rangle \\ - \left\langle \pi^{\mathsf{T}} \operatorname{grad} f_K, \Gamma^{i,j}_{(k,\ell)(p,q)}(K) \partial_{i,j} \right\rangle.$$

# D. Output-feedback LQR (OLQR)

The OLQR problem can be formulated as the optimization problem in (8) with the submanifold  $\widetilde{S} = S \cap \mathcal{K}_C$  with the constraint set  $\mathcal{K}_C$  defined as,

$$\mathcal{K}_C := \{ K \in \mathbf{M}(m \times n, \mathbb{R}) \mid K = LC, \ L \in \mathbf{M}(m \times d, \mathbb{R}) \},$$

where  $C \in M(d \times n, \mathbb{R})$  is the prescribed output matrix. Note that  $\mathcal{K}_C$  is a linear subspace of  $M(m \times n, \mathbb{R})$  whose dimension depends on the rank of C. For simplicity of presentation, we suppose C has full rank equal to  $d \leq n$ . Now, define  $\Psi : M(m \times n, \mathbb{R}) \to M(m \times n, \mathbb{R})$  as,

$$\Psi(K) = K(I_n - C^{\dagger}C),$$

where  $\dagger$  denotes the Moore-Penrose inverse. Note that  $\Psi$  is a linear map that is surjective onto its range, denoted by  $\mathcal{R}$ , which is an m(n-d) dimensional linear subspace of  $M(m \times n, \mathbb{R})$ . Therefore,  $\Phi: \mathcal{S} \to \mathcal{R}$  defined as the restriction of  $\Psi$  both in domain and codomain is a smooth submersion.

Finally, as  $CC^{\dagger}C = C$ , we can observe that  $Ker(\Psi) = \mathcal{K}_C$ . Therefore,  $\widetilde{\mathcal{S}} = \mathcal{K}_C \cap \mathcal{S} = \Phi^{-1}(0)$  and thus, by Submersion Level Set Theorem,  $\widetilde{\mathcal{S}}$  is a properly embedded submanifold of  $\mathcal{S}$  with dimension md. We also conclude that at each  $K \in \widetilde{\mathcal{S}}$ , we can canonically identify the tangent space at K as follows:

$$T_K \widetilde{\mathcal{S}} = \operatorname{Ker}(d\Phi_K) \cong \mathcal{K}_C.$$

Next, at any  $K \in \widetilde{\mathcal{S}}$  and for any  $E \in T_K \mathcal{S}$ , the tangential projection of E, denoted by  $\widetilde{E}^* = \pi^\top E$ , is the unique solution of a minimization similar to (10). Moreover, under the above identification,  $E - \widetilde{E}^* \perp \mathcal{K}_C$  (with respect to the Riemannian metric) must be satisfied, or equivalently,

$$\operatorname{tr}\left[C^{\mathsf{T}}L^{\mathsf{T}}(E-\widetilde{E}^*)Y_K\right]=0, \quad \forall L \in \mathbf{M}(m \times d, \mathbb{R}).$$

Here,  $Y_K = \mathbb{L}(A_{\text{cl}}, \Sigma_1)$  is positive definite and since C is assumed to be full-rank,  $CY_KC^{\top}$  is positive definite. Hence, we conclude that  $\pi^{\top} E = L^*C$  with  $L^* \in M(m \times d, \mathbb{R})$  being the unique solution of the following linear equation

$$L^*CY_KC^{\top} = EY_KC^{\top}.$$

Finally, at each point  $K \in \widetilde{\mathcal{S}}$ , we denote the global coordinate functions of  $\mathrm{M}(m \times d, \mathbb{R})$  by the tuple  $(\bar{x}^{i,j})$  for  $(i,j) \in D := [m] \times [d]$  and its corresponding global coordinate frame by  $(\bar{\partial}_{(i,j)})$ . Under the identification of  $T_K \widetilde{\mathcal{S}} \cong \mathcal{K}_C$  explained above  $((\widetilde{\partial}_{(i,j)} = \bar{\partial}_{(i,j)}C))$ , with  $(i,j) \in D$ , these global vector fields form a global smooth frame for  $T\widetilde{\mathcal{S}}$  as they are linearly independent on  $\widetilde{\mathcal{S}}$  (recall that C has full-rank). But then, the coordinates of the covariant Hessian  $h_{;(k,\ell)(p,q)}(K)$  with respect to this frame can be computed by substituting  $E = \bar{\partial}_{(k,\ell)}C$  and  $F = \bar{\partial}_{(p,q)}C$  in Corollary V.2 for each  $(k,\ell), (p,q) \in D$ —similar to the SLQR case. It is worth noting that the sparsity pattern in E, F and Christoffel symbols can simplify the computations; we will not further delve into this issue due to brevity.

# VI. EXAMPLES

In this section, we provide numerical examples for optimizing the LQR cost over submanifolds induced by SLQR and OLQR problems. Recall that, for each of these problems, we can compute the coordinate functions of the covariant Hessian  $h_{;(k,\ell)(p,q)}(K)$  with respect to the corresponding coordinate frame described in the previous subsections. Therefore, finding the Newton direction G at any point  $K \in \widetilde{\mathcal{S}}$  reduces to solving the system of linear equations for the unknowns  $[G]^{k,\ell}$ , as described in §V-A, and forming the Newton direction as  $G = [G]^{k,\ell} \widetilde{\partial}_{(k,\ell)}|_{K} \in T_K \widetilde{\mathcal{S}}$ .

For each of SLQR and OLQR problems, we have simulated three different algorithms, the first two are the variants of Algorithm 1 where we use Riemannian connection or Euclidean connection to compute  $\operatorname{Hess} h$  or  $\operatorname{\overline{Hess}} h$ , respectively. Note, despite the fact that  $\operatorname{Hess} h_{K^*} = \operatorname{\overline{Hess}} h_{K^*}$  whenever  $\operatorname{grad} h_{K^*} = 0$  (as shown in Lemma IV.2), it is not necessarily the case where  $\operatorname{grad} h$  does not vanish; therefore, we expect  $\operatorname{Hess} h$  and  $\operatorname{\overline{Hess}} h$  to contain very different information on neighborhoods of isolated local minima which directly influence the performance of RNPO as will be discussed below. The third algorithm is the Projected Gradient Descent (PGD) as studied in [17]. This is particularly possible for these constraints as, under relevant assumptions, one is able to perform PGD updates by having access to merely the projection onto linear subspace of matrices—see e.g. [17, Theorem 7.1]. Here, the step size for the PGD algorithm is a constant value that guarantees the iterates stay stabilizing as suggested therein.

**Example 1** (Trajectories of RNPO using Hess versus  $\overline{\text{Hess}}$ ). In order to illustrate how the performance of RNPO is different in terms of using the Riemannian connection (Hess h) versus Euclidean connection ( $\overline{\text{Hess}} h$ ), we consider an example with system parameters  $(A|B|Q|R|\Sigma) =$ 

We run RNPO and PGD algorithms for both OLQR and SLQR problems involving two decision variables, so that we can plot the trajectories of iterates over the level curves of the associated cost function from different initial points (as illustrated in Figure 3 and Figure 2, respectively). In this example, the stopping criterion for Algorithm 1 is when the error of iterates from optimality is below a small tolerance level  $(10^{-12})$ ; unless the hessian fails to be positive definite for Algorithm 1 using  $\overline{\text{Hess}}$  where we run the algorithm for a fixed number of iterations. Since PGD does not practically converges even with large number of iterates due to its sublinear convergence rate, we terminate it when Algorithm 1 using Hess is converged.

In Figure 2, first, note that RNPO with  $\overline{\text{Hess}}$  does not converge if initialized away from the local minimum (and away from the line  $l_2=-1$ ) simply because the Euclidean Hessian fails to be positive definite therein (see Figure 1). On the other hand, RNPO with Hess successfully captures the inherent geometry of the problem and converges from all initializations. These observations exemplify how RNPO can exploit the connection compatible with the metric (inherent to the cost function) in order to provide more effective iterate updates. Second, the square marker on each trajectory of RNPO indicates the first time stepsize  $\eta_t=1$  is guaranteed to be stabilizing (i.e.,  $s_{K_t}\geq 1$ ). It can be seen that the neighborhood of the local minimum (zoomed in)—on which the identity stepsize is possible—is relatively small. Whereas, by using the stability certificate, the specific choice of stepsize adopted here enables RNPO to handle initialization further away from the local minimum. Finally, PGD algorithm

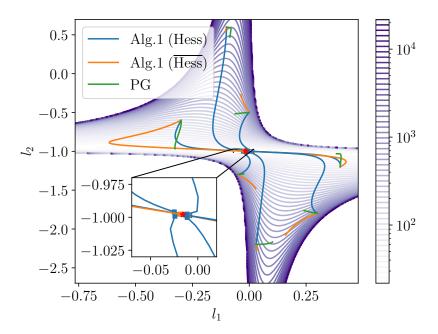


Fig. 2: The trajectories of iterates  $K = \operatorname{diag}(l_1, l_2)$  generated by RNPO (with  $\operatorname{Hess} h$  and  $\operatorname{\overline{Hess}} h$ ) and PGD—from different initial points—for the SLQR problem with constraint  $D = \{(1, 1), (2, 2)\}$ , over the level curves of f in (8).

has a sublinear rate; as such, despite its progress at the onset of the iterates, it becomes rather slow over time and not practically convergent. Next, notice that in both Figure 2 and Figure 3, the trajectories of RNPO with Hess are much more favorable (with faster convergence) in comparison to RNPO with Hess, particularly, when initialized from points further away from the local minimum and closer to the boundary. Additionally, similar to Figure 2, the region on which the unit stepsize is guaranteed to be stabilizing is relatively small in Figure 3.

Example 2 (Randomly selected system parameters). Next, we consider an example with n=6 number of states and m=3 number of inputs, and simulate the behavior of RNPO and PGD for 100 randomly sampled system parameters. Particularly, the parameters (A,B) are sampled from a zero-mean unit-variance normal distribution, where A is scaled so that the open-loop system is stable, i.e.,  $K_0=0$  is stabilizing, and the pair is controllable. Also, we choose  $Q=\Sigma=I_n$  and  $R=I_m$  in order to consistently compare the convergence behaviors across different samples. For the SLQR problem, we randomly sample for the sparsity pattern D so that at least half of the entries are zero and all of them have converged from  $K_0=0$  in less than 30 iterations. For the OLQR problem, we also randomly sample the output matrix C with d=2, where 98% and 92% of them have converged from  $K_0=0$  in less than 50 iterations using Hess and Hess, respectively.

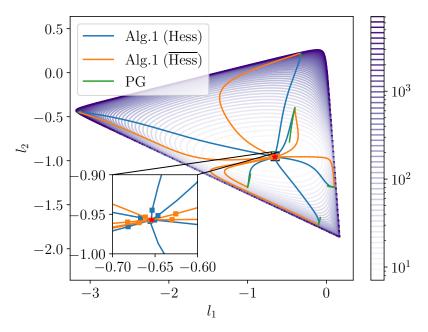


Fig. 3: The trajectories of iterates  $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^{\mathsf{T}}$  generated by RNPO (with Hess h and  $\overline{\mathrm{Hess}}\,h$ ) and PGD—from different initial points—for the OLQR problem with output matrix  $C = \begin{pmatrix} 1.0 & 1.0 \end{pmatrix}$ , over the level curves of f in (8).

The minimum, maximum and median progress of the three algorithms for both SLQR and OLQR problems are illustrated in Figure 4a and Figure 4b, respectively. As guaranteed in the Theorem IV.3, the linear-quadratic convergence behavior of RNPO is observed in these problems. In both cases, Algorithm 1 (blue curves) built upon the Riemannian connection has a superior convergence rate compared with the case of using the Euclidean connection (orange curves); this was expected as the Riemannian connection is compatible with the metric induced by the geometry inherent to the cost function itself. This superior performance of Algorithm 1, in the meantime, comes hand-in-hand with the required computation of the Christoffel symbols.

## VII. CONCLUSIONS AND FUTURE DIRECTIONS

In this work, we considered the problem of optimizing a smooth function over submanifolds of Schur stabilizing controllers S. In order to treat this problem in a more general setting, we studied the first and second order behavior of a smooth function when constrained to an embedded submanifold from an extrinsic point of view. Subsequently, using the second order information of the restricted function, we developed an algorithm that guarantees convergence to local minima—at least with a linear rate—and eventually with a quadratic rate. Combining this approach with backtracking line-search techniques

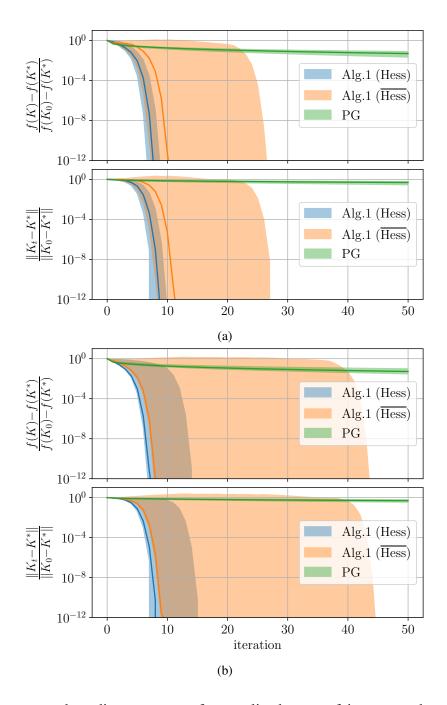


Fig. 4: The min, max and median progress of normalized error of iterates and cost values at each iteration of Algorithm 1, for the (a) SLQR and (b) OLQR problems with 100 different randomly sampled system parameters, sparsity patterns and output matrices.

or positive definite modifications of the Hessian operator [41], [42] can be considered as immediate future directions for a global convergence analysis.

Even though the proposed algorithm depends on the linear structure of  $\widetilde{\mathcal{S}}$ , the machinery developed here can be utilized for more involved submanifolds, a topic that will be considered in our future

works. As an example, and in contrast to the other two (SLQR and OLQR) problems considered, we can explore how a constraint on the "average input energy" can be translated to a *nonlinear* constraint that pertains to the inherent geometry of the LQR problem. In this direction, we define the *average* input energy, denote by  $E_u$ , as a measure of average energy consumption, namely,

$$E_{\boldsymbol{u}} := \underset{\boldsymbol{x}_0 \sim \mathcal{D}}{\mathbb{E}} \|\boldsymbol{u}\|_{\ell_2}^2,$$

where  $\|.\|_{\ell_2}$  refers to  $\ell_2$ -norm. If a static linear policy, i.e., u = Kx for  $K \in \mathcal{S}$ , is desired for this problem setup, then the closed-loop system assumes the form  $x_k = (A_{\rm cl})^k x_0$ ; one can now show that under the usual identification of  $T_K\mathcal{S}$  with  $M(m \times n, \mathbb{R})$ , we have  $E_u(K) = |K|_{g_K}^2$ . But then,  $E_u : \mathcal{S} \to \mathbb{R}$  is smooth by composition, and as such, every regular level set of  $E_u$  translates to an upperbound on the average input energy. Regular Level Set Theorem now implies that the average input energy optimal control synthesis can be pursued via an embedded submanifold of  $\mathcal{S}$  which has a nonlinear but simple structure whenever considered in the associated Riemannian geometry. Solving this problem still requires an efficient retraction that would substitute the linear updates possible in SLQR and OLQR problems. The framework discussed in this work allows the integration of such retractions in the synthesis procedure. As such, the proposed work opens up a new approach for solving a wide range of constrained optimization problems over the manifold of Schur stabilizing controllers.

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#### **APPENDIX**

On the Hessian Operator. The Hessian operator (denoted by  $\operatorname{Hess} f$ ) as introduced in §III-C is well-defined and the value of  $\operatorname{Hess} f[U]$  at any  $K \in \mathcal{S}$  depends only on  $U_K$ ; this is due to the property for the connection. Note that

$$\langle \operatorname{Hess} f[U], W \rangle = U \langle \operatorname{grad} f, W \rangle - \langle \operatorname{grad} f, \nabla_U W \rangle = U(Wf) - (\nabla_U W)f = W(Uf) - (\nabla_W U)f,$$
(11)

 $\forall U, W \in \mathfrak{X}(\mathcal{S})$ , where the first equality is the consequence of having the Riemannian connection compatible with the metric, the second one is by the definition of grad f, and the last one is due to symmetry of the Riemannian connection. Thus, by (11), the Hessian operator is self-adjoint, i.e.,

$$\langle \operatorname{Hess} f[U], W \rangle = \langle U, \operatorname{Hess} f[W] \rangle$$
.

Similarly, we can define  $\operatorname{Hess} h$  for any smooth function  $h \in C^{\infty}(\widetilde{\mathcal{S}})$ , where we consider the submanifold  $\widetilde{\mathcal{S}} \subset \mathcal{S}$  with the induced Riemannian metric and the associated Riemannian connection of  $\mathcal{S}$ .

Next, for the computational purposes, we would like to introduce the "covariant Hessian of f" with respect to g, denoted by  $\nabla^2 f$ . It is a 2-tensor field obtained by taking total covariant derivative of f twice. The Riemannian connection is symmetric, and so is the covariant Hessian. Furthermore, the covariant Hessian and Hessian operator are related as,

$$\nabla^2 f[W, U] = \nabla_U \nabla_W f - (\nabla_U W) f = \langle \text{Hess } f[U], W \rangle, \tag{12}$$

where the last equality follows by (11).

Recall now that  $\operatorname{grad} f = (df)^{\sharp}$ , where  $\sharp$  denotes the index raising operator, referred to as the "sharp" operator. Also,  $\operatorname{Hess} f : \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$  can be viewed as the total covariant derivative of  $\operatorname{grad} f$ , i.e.,  $\operatorname{Hess} f = \nabla \operatorname{grad} f$ . But then

$$\operatorname{Hess} f = \nabla (df)^{\sharp} = (\nabla (df))^{\sharp} = (\nabla^2 f)^{\sharp}, \tag{13}$$

where the equality in the middle follows by the fact the index raising operator commute with the covariant derivative operator, and the last equality is due to the definition of connection for a smooth function  $f \in C^{\infty}(\mathcal{S})$ . Note that in (13), the index raising refer to the second argument of  $\nabla^2 f$ . However, as the covariant Hessian of any smooth function is a symmetric 2-tensor field, the index raising could be with respect to any of the entries. Finally, similar definitions and relations as discussed above are available for h as a smooth function on the embedded Riemannian submanifold  $\widetilde{\mathcal{S}}$  with the induced metric and corresponding connection; these are omitted for brevity.

**Proof of Lemma III.1.** Since  $\rho: M(n \times n, \mathbb{R}) \to \mathbb{R}$  is a continuous map,  $\mathcal{M}$  is an open subset of  $M(n \times n, \mathbb{R})$  and thus an open submanifold. For each  $A \in \mathcal{M}$ , by Lyapunov Stability Criterion, there exists a unique solution X to (1) which has the infinite-sum representation. But, as for each  $A \in \mathcal{M}$ , the series converges, each matrix entry of X can be written as a convergent power series of elements of A and A. Therefore, each matrix entry of A is a real analytic function of several variables (as defined in [43]) on the open subset A0 A1 A2. Hence, we conclude that A3 A4 well-defined smooth map. Next, under the identification in the premise, it follows that,

$$d \mathbb{L}_{(A,Q)}[E,F] = d \mathbb{L}_{(A,Q)}[E,0] + d \mathbb{L}_{(A,Q)}[0,F].$$

<sup>&</sup>lt;sup>8</sup>An alternative argument can be provided by the closed form solution of (1) and its "vectorization" involving rational functions of several variables with non-vanishing denominators—cf. Lemma 3.6 in [17].

However,  $\mathbb{L}$  is linear in the second variable, so  $d\mathbb{L}_{(A,Q)}[0,F]=\mathbb{L}(A,F)$ . Also since  $\mathcal{M}$  is open, for small enough  $\varepsilon$ ,  $\gamma:[0,\varepsilon]\to\mathcal{M}\times M(n\times n,\mathbb{R})$  with  $\gamma(t)=(A+tE,Q)$  is a well-defined smooth curve starting at (A,Q) whose initial velocity is (E,0). Then,

$$d \mathbb{L}_{(A,Q)}[E,0] = d/dt \big|_{t=0} \mathbb{L} \circ \gamma(t).$$

Let  $X_t := \mathbb{L} \circ \gamma(t)$  and  $X := \mathbb{L} \circ \gamma(0)$ ; then we obtain,

$$X_t - X = \mathbb{L} \left( A, t(EXA^{\mathsf{T}} + AXE^{\mathsf{T}}) + \mathcal{O}(t^2) \right)$$
$$= t \, \mathbb{L}(A, EXA^{\mathsf{T}} + AXE^{\mathsf{T}}) + \mathcal{O}(t^2),$$

where the first equality is by direct algebraic manipulation and the second one follows by linearity of  $\mathbb{L}$  in the second coordinate. Therefore,  $d\mathbb{L}_{(A,Q)}[E,0] = \mathbb{L}(A,EXA^{\mathsf{T}} + AXE^{\mathsf{T}})$ , and the first claim follows by adding the two computed differentials and using linearity of  $\mathbb{L}$  in the second coordinate again. Finally, note that any square matrix has a spectrum identical to its transpose; therefore if  $A \in \mathcal{M}$  then  $A^{\mathsf{T}} \in \mathcal{M}$ , and thus the last property follows by the convergent series representations of  $\mathbb{L}(A^{\mathsf{T}},Q)$  and  $\mathbb{L}(A,\Sigma)$ , as well as the cyclic permutation property of trace.

**Proof of Lemma III.2.** By Lemma III.1 for each  $K \in \mathcal{S}$ ,  $\mathbb{L}(A_{\operatorname{cl}}, \Sigma_K)$  is uniquely determined, symmetric and smooth in K, since  $A_{\operatorname{cl}}$  is stabilizing. Also,  $\mathbb{L}(A_{\operatorname{cl}}, \Sigma_K) \in M(n \times n, \mathbb{R})$  is positive semidefinite definite by observing the infinite-sum representation of solution to Lyapunov equation and the fact that  $\Sigma_K \succeq 0$ . Next,  $\operatorname{tr}[(V_K)^{\mathsf{T}}W_K \mathbb{L}(A_{\operatorname{cl}}, \Sigma_K)]$  is a smooth function of elements of  $V_K, W_K$  and  $\mathbb{L}(A_{\operatorname{cl}}, \Sigma_K)$ . Therefore, for any  $V, W \in \mathfrak{X}(\mathcal{S})$ , the function  $\langle V, W \rangle$  as defined in the premise is well-defined and smooth on  $\mathcal{S}$ . Additionally, by linearity of trace, we observe that  $\langle \cdot, \cdot \rangle$  is multilinear over  $C^{\infty}(\mathcal{S})$ , i.e.,

$$\left\langle fU+hV,W\right\rangle =f\left\langle U,W\right\rangle +h\left\langle V,W\right\rangle ,$$

for any  $f, h \in C^{\infty}(\mathcal{S})$  and  $U, V \in \mathfrak{X}(\mathcal{S})$ , and similarly for the second entry. Therefore, by Tensor Characterization Lemma [29, Lemma 12.24], it is induced by a smooth covariant 2-tensor field. Finally, symmetry follows from that the fact that for any  $K \in \mathcal{S}$ ,

$$\langle V, W \rangle \big|_{K} = \operatorname{tr} \left[ \left( \mathbb{L}(A_{\operatorname{cl}}, \Sigma_{K}) \right)^{\intercal} (W_{K})^{\intercal} V_{K} \right] = \operatorname{tr} \left[ (W_{K})^{\intercal} V_{K} \left( \mathbb{L}(A_{\operatorname{cl}}, \Sigma_{K}) \right)^{\intercal} \right] = \langle W, V \rangle \big|_{K},$$

where we have used the symmetry of  $\mathbb{L}(A_{\mathrm{cl}}, \Sigma_K)$ .

**Proof of Proposition III.3.** We know that S is a smooth manifold, and by Lemma III.2 and Smoothness Criteria for Tensor Fields [29, Proposition 12.19], g is a smooth symmetric covariant 2-tensor field. Hence, it suffices to show that it is positive definite at each point  $K \in S$ . But  $\Sigma_K \succeq 0$  and  $(A_{\rm cl}, \Sigma_K)$  is controllable; therefore  $Y_K = \mathbb{L}(A_{\rm cl}, \Sigma_K)$  is a positive definite matrix, implying that

 $g_K(E,E) = \operatorname{tr}\left[(EY_K^{\frac{1}{2}})^{\intercal}EY_K^{\frac{1}{2}}\right] \geq 0$ , for any  $E \in T_K\mathcal{S}$  with equality if and only if E is the zero element. Next, to compute the coordinate representation of g, for each coordinate pairs (i,j) and  $(k,\ell)$  we have,

$$g_{(i,j)(k,\ell)}(K) = g_K(\partial_{i,j}|_K, \partial_{k,\ell}|_K) = \operatorname{tr}\left[(\partial_{i,j}|_K)^{\mathsf{T}}\partial_{k,\ell}|_K Y_K\right],$$

under the usual identification of  $T_K\mathcal{S}$ . Since under this identification  $\partial_{i,j}$  corresponds to the element of  $M(m \times n, \mathbb{R})$  with entry 1 in (i,j)-th coordinate and zero elsewhere, the expression for  $g_{(i,j)(k,\ell)}$  follows by direct computation of the last equality. Finally, by definition of "inverse matrix", for each (i,j) and  $(k,\ell)$ , we must have  $\sum_{r,s} g^{(i,j)(r,s)} g_{(r,s)(k,\ell)} = 1$  if  $(i,j) = (k,\ell)$  and 0 otherwise. Next, for each i=k, let  $\left[g^{(k,\cdot)(k,\cdot)}\right]$  denote the matrix with  $g^{(k,j)(k,s)}$  as its (j,s)th entry. Then, by the expression for  $g_{(i,j)(k,\ell)}$ , it must satisfy  $\left[g^{(k,\cdot)(k,\cdot)}\right]Y_K = I_n$ , and therefore  $\left[g^{(k,\cdot)(k,\cdot)}\right] = Y_K^{-1}$  as  $Y_K \succ 0$ . The rest follows by doing a similar computation for each  $i \neq k$  and noting the zero pattern in the expression for  $g_{(i,j)(k,\ell)}$ .

# Proof of Proposition III.4. We know that

$$\Gamma_{(k,\ell)(p,q)}^{(i,j)} = \sum_{r,s} (g^{(i,j)(r,s)}/2) \left( \partial_{(k,\ell)} g_{(r,s)(p,q)} + \partial_{(p,q)} g_{(r,s)(k,\ell)} - \partial_{(r,s)} g_{(k,\ell)(p,q)} \right), \tag{14}$$

for any  $(i,j), (k,\ell), (p,q) \in [m] \times [n]$ , where  $g^{(i,j)(r,s)}$  denotes the inverse matrix of  $g_{(i,j)(r,s)}$ . If  $k \neq i \neq p \neq k$ , then by sparsity pattern in the expression for  $g_{(i,j)(k,\ell)}$  in Proposition III.3, we obtain  $\Gamma^{(i,j)}_{(k,\ell)(p,q)} = 0$ . Next, if  $k = i \neq p$ , then (14) simplifies to

$$\sum_{a} \frac{g^{(i,j)(i,s)}}{2} \left( \partial_{(p,q)} g_{(i,s)(i,\ell)} \right) = \frac{1}{2} \left( \partial_{(p,q)} [Y_K]_{(\ell,.)} \right) [Y_K^{-1}]_{(.,j)}.$$

Next, for any fix i, p and q, let  $\Gamma^{(i,.)}_{(i,.)(p,q)}$  denote the  $n \times n$  matrix with  $\Gamma^{(i,j)}_{(i,\ell)(p,q)}$  as its  $(\ell,j)$  entry. Then it must satisfy  $\Gamma^{(i,.)}_{(i,.)(p,q)} = \frac{1}{2} \left( \partial_{(p,q)} Y_K \right) Y_K^{-1}$ , where  $\partial_{(p,q)} Y_K$  indicates the action of tangent vector  $\partial_{(p,q)}$  on the composite map  $K \to (A_{\operatorname{cl}}, \Sigma_K) \stackrel{\mathbb{L}}{\to} Y_K$ . By Lemma III.1, we can compute,

$$\partial_{(p,q)} Y_K = d \, \mathbb{L}_{(A_{\mathrm{cl}}, \Sigma_K)} \left[ B \partial_{(p,q)}, \ \partial_{(p,q)}^\intercal \Sigma_2 K + K^\intercal \Sigma_2 \partial_{(p,q)} \right]$$

under the usual identification of  $T_K\mathcal{S}$ . Thus,  $\partial_{(p,q)}Y_K=dY_K(p,q)$  which proves the second case. The third case follows by the symmetry of the Riemannian connection, i.e.,  $\Gamma^{(i,j)}_{(k,\ell)(p,q)}=\Gamma^{(i,j)}_{(p,q)(k,\ell)}$ . Next, if  $p=k\neq i$ , then by Proposition III.3, (14) simplifies to

$$\sum_{s} \frac{g^{(i,j)(i,s)}}{2} \left( \partial_{(i,s)} g_{(k,\ell)(k,q)} \right) = \frac{1}{2} \sum_{s} \left[ \partial_{(i,s)} Y_K \right]_{(q,\ell)} [Y_K^{-1}]_{(s,j)},$$

with  $\partial_{(i,s)}Y_K=dY_K(i,s)$  computed similarly. Finally, if k=i=p then similarly (14) simplifies to

$$\sum_{s} \frac{g^{(i,j)(i,s)}}{2} \Big( \partial_{(i,\ell)} g_{(i,s)(i,q)} + \partial_{(i,q)} g_{(i,s)(i,\ell)} - \partial_{(i,s)} g_{(i,\ell)(i,q)} \Big),$$

and substituting each term similarly completes the proof.

**Proof of Proposition III.5.** Note that f is smooth and  $\widetilde{\mathcal{S}}$  is an embedded submanifold of  $\mathcal{S}$ . Therefore,  $h:\widetilde{\mathcal{S}}\to\mathbb{R}$  is smooth by restriction and we can define  $\operatorname{grad} h$  and  $\operatorname{Hess} h$  on  $\widetilde{\mathcal{S}}$ . But,  $\operatorname{grad} h\in\mathfrak{X}(\widetilde{\mathcal{S}})$  is the unique vector field on  $\widetilde{\mathcal{S}}$  such that  $\widetilde{g}(W,\operatorname{grad} h)=Wh$  for any  $W\in\mathfrak{X}(\widetilde{\mathcal{S}})$ . Unraveling the definition implies that for any  $K\in\widetilde{\mathcal{S}}\subset\mathcal{S}$ ,

$$dh_K(W_K) = df_K(d\iota_{\widetilde{S}}(W_K)) = g_K \left( d\iota_{\widetilde{S}}(W_K), \operatorname{grad} f_K \right)$$
$$= g_K \left( d\iota_{\widetilde{S}}(W_K), d\iota_{\widetilde{S}}(\pi^{\top} \operatorname{grad} f_K) \right),$$

as  $h = f \circ \iota_{\widetilde{S}}$  and thus  $dh = df \circ d\iota_{\widetilde{S}}$ , where the last equality follows by the fact that  $\iota_{\widetilde{S}}(W_K) \in T_K S$  is tangent to  $\widetilde{S}$ . By definition of tangential projection,  $\pi^{\top}(\operatorname{grad} f|_{\widetilde{S}})$  is then a vector field on  $\widetilde{S}$  that satisfies

$$Wh = \widetilde{g}(W, \pi^{\top} \operatorname{grad} f|_{\widetilde{\mathcal{S}}}),$$

for any  $W \in \mathfrak{X}(\widetilde{S})$ . Therefore, the first claim follows by uniqueness of the gradient. Next, note that the Hessian operator of  $h \in C^{\infty}(\widetilde{S})$  is defined as  $\operatorname{Hess} h[V] := \widetilde{\nabla}_V \operatorname{grad} h$ , for any  $V \in \mathfrak{X}(\widetilde{S})$ . But then, the first claim together with (4) and the linearity of connection imply that

$$\begin{aligned} \operatorname{Hess} h[V] &= \pi^{\top} \nabla_{V} (\pi^{\top} (\operatorname{grad} f|_{\widetilde{\mathcal{S}}})) \\ &= \pi^{\top} (\operatorname{Hess} f[V]|_{\widetilde{\mathcal{S}}}) - \pi^{\top} \nabla_{V} (\pi^{\perp} (\operatorname{grad} f|_{\widetilde{\mathcal{S}}})), \end{aligned}$$

where all V,  $\pi^{\top}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}})$  and  $\pi^{\perp}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}})$  are extended arbitrarily to vector fields on a neighborhood of  $\widetilde{\mathcal{S}}$  in  $\mathcal{S}$ . Finally, the extrinsic expression of  $\operatorname{Hess} h$  follows by The Weingarten Equation [27, Proposition 8.4], indicating that  $\pi^{\top} \nabla_{V}(\pi^{\perp}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}})) = -\mathbb{W}_{\pi^{\perp}(\operatorname{grad} f|_{\widetilde{\mathcal{S}}})}[V]$ .

**Proof of Lemma IV.2.** Let  $\widetilde{\gamma}: (-\varepsilon, \varepsilon) \to \widetilde{\mathcal{S}}$  denote the smooth geodesic curve on the submanifold  $\widetilde{\mathcal{S}}$  with  $\widetilde{\gamma}(0) = K^*$  and  $\widetilde{\gamma}'(0) = F$  for an arbitrary  $F \in T_{K^*}\widetilde{\mathcal{S}}$ . Define  $\ell(t) := h \circ \widetilde{\gamma}(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$  which it is smooth by composition. Then,  $K^*$  is a local minimum for h, so is t = 0 for  $\ell(t)$  following by smoothness of  $\widetilde{\gamma}$ . Therefore,

$$0 = \ell'(0) = dh_{\widetilde{\gamma}(0)} \circ \widetilde{\gamma}'(0) = \langle \operatorname{grad} h_{K^*}, F \rangle,$$

and as F was an arbitrary tangent vector, we conclude that  $\operatorname{grad} h_{K^*} = 0$ . Now, recall that non-degenerate critical points are isolated [44, Corollary 2.3]. Next, Taylor's formula for  $\ell$  at t = 0 yields

$$\ell(t) = \ell(0) + t \langle \operatorname{grad} h_{K^*}, F \rangle + \frac{1}{2} \ell''(s) t^2,$$

for some  $s \in (0, t)$ . As grad  $h_{K^*} = 0$  and t = 0 is a local minimum of  $\ell(t)$ , we must have  $\ell''(s) \ge 0$ ; by tending  $t \to 0$ , smoothness of  $\ell$  implies that  $\ell''(0) \ge 0$ . But,

$$\ell''(t) = \widetilde{D}_t \left\langle \operatorname{grad} h_{\widetilde{\gamma}(t)}, \widetilde{\gamma}'(t) \right\rangle = \left\langle \widetilde{D}_t \operatorname{grad} h_{\widetilde{\gamma}(t)}, \widetilde{\gamma}'(t) \right\rangle$$

where  $\widetilde{D}_t$  denotes the covariant derivative along  $\widetilde{\gamma}$  on  $\widetilde{\mathcal{S}}$ , and the last equality follows by its compatibility with the metric and the fact that  $\widetilde{\gamma}$  is a geodesic (so that  $\widetilde{D}_t\widetilde{\gamma}'(t)\equiv 0$ ). As  $\operatorname{grad} h|_{\widetilde{\gamma}(t)}\in\mathfrak{X}(\gamma)$  is clearly extendable, we conclude that

$$\ell''(t) = \left\langle \nabla_{\widetilde{\gamma}'(t)} \operatorname{grad} h \middle|_{\widetilde{\gamma}(t)}, \widetilde{\gamma}'(t) \right\rangle = \left\langle \operatorname{Hess} h_{\widetilde{\gamma}(t)} [\widetilde{\gamma}'(t)], \widetilde{\gamma}'(t) \right\rangle,$$

and thus particularly  $\ell''(0) = \langle \operatorname{Hess} h_{K^*}[F], F \rangle$ . Since F was arbitrary and  $K^*$  is nondegenerate,  $\ell''(0) \geq 0$  implies that  $\operatorname{Hess} h_{K^*}$  is positive definite. Next, existence of a neighborhood at  $K^*$  on which  $\operatorname{Hess} h$  is positive definite follows by smoothness—in particular continuity—of the operator  $\operatorname{Hess} h_K$  in K. Finally, let  $\widetilde{\nabla}$  and  $\widetilde{\overline{\nabla}}$  denote the connections on  $T\widetilde{\mathcal{S}}$  induced, respectively, by the connections  $\nabla$  and  $\overline{\nabla}$  on  $T\mathcal{S}$ . Then by The Difference Tensor Lemma, the difference tensor between  $\widetilde{\nabla}$  and  $\widetilde{\overline{\nabla}}$ —defined as  $D(U,V) := \widetilde{\nabla}_U V - \widetilde{\overline{\nabla}}_U V$  for any  $U,V \in \mathfrak{X}(\widetilde{\mathcal{S}})$ —is indeed a (1,2)-tensor field. That means, as  $\operatorname{grad} h_{K^*} = 0$ ,

$$\operatorname{Hess} h_{K^*}[U_{K^*}] - \overline{\operatorname{Hess}} h_{K^*}[U_{K^*}] = D(U, \operatorname{grad} h)|_{K^*} = 0.$$

The last claim then follows as  $U \in \mathfrak{X}(\widetilde{\mathcal{S}})$  was arbitrary.

**Proof of Theorem IV.3.** By Lemma IV.2,  $\operatorname{grad} h_{K^*} = 0$  and there exists a neighborhood  $\mathcal{U}$  of  $K^*$  on which  $\operatorname{Hess} h_K$  is positive. Furthermore, by continuity of  $\operatorname{Hess} h$  (and, if necessary, shrinking  $\mathcal{U}$ ) we can obtain constant positive scalars m and M such that for all  $K \in \mathcal{U}$  and  $G \in T_K \widetilde{\mathcal{S}}$ ,

$$m |G|_{g_K}^2 \le \langle \operatorname{Hess} h_K[G], G \rangle \le M |G|_{g_K}^2,$$
 (15)

where  $|\cdot|g_K$  denotes the norm induced by g at K. In particular, if  $G_t \in T_{K_t}\widetilde{\mathcal{S}}$  is the Newton direction at some point  $K_t \in \mathcal{U}$ , then (by Cauchy-Schwartz inequality at  $K_t$ )

$$|G_t|_{g_{K_t}} \le \left(\frac{1}{m}\right)|\operatorname{grad} h_{K_t}|_{g_{K_t}}. \tag{16}$$

Next, define the curve  $\gamma:[0,s_{K_t}]\to\widetilde{\mathcal{S}}$  with  $\gamma(\eta)=K_t+\eta G_t$ , and consider a smooth parallel vector field (with respect to the Riemannian connection)  $E(\eta)$  along  $\gamma$ —refer to [27] for "parallel vector fields along curves" and "parallel transport". Also, define  $\phi:[0,s_{K_t}]\to\mathbb{R}$  with  $\phi(\eta)\coloneqq \left\langle \operatorname{grad} h_{\gamma(\eta)}, E(\eta) \right\rangle$ . Notice that  $\operatorname{grad} h$  is smooth, so is  $\phi$  and by compatibility with the metric and that  $\operatorname{grad} h_{\gamma(\eta)}$  is clearly extendable, we have

$$\phi'(\eta) = \langle D_{\eta} \operatorname{grad} h_{\gamma(\eta)}, E(\eta) \rangle = \langle \operatorname{Hess} h_{\gamma(\eta)}[G_t], E(\eta) \rangle,$$

where  $D_{\eta}$  is the covariant derivative along  $\gamma$  and  $G_t$  is extended to the vector field along  $\gamma$  with constant coordinates in the global coordinate frame. Thus, as

$$\phi(\eta) = \phi(0) + \eta \phi'(0) + \int_0^{\eta} [\phi'(\tau) - \phi'(0)] d\tau,$$

by direct substitution and the fact that  $G_t$  is the Newton direction at iteration t, we obtain that

$$\phi(\eta) = (\eta - 1) \langle \operatorname{Hess} h_{K_t}[G_t], E(0) \rangle + \int_0^{\eta} \langle [\operatorname{Hess} h_{\gamma(\tau)} - \mathcal{P}_{0,\tau}^{\gamma} \operatorname{Hess} h_{\gamma(0)}] G_t, E(\tau)) \rangle d\tau,$$

where  $\mathcal{P}_{0,\tau}^{\gamma}$  denotes the parallel transport from 0 to  $\tau$  along  $\gamma$ . Again, as every parallel transport map along  $\gamma$  is a linear isometry we claim that

$$\left\langle \operatorname{grad} h_{K_{t+1}}, E(\eta_t) \right\rangle = (\eta_t - 1) \left\langle \mathcal{P}_{0,\eta_t}^{\gamma} \operatorname{Hess} h_{K_t}[G_t], E(\eta_t) \right\rangle + \int_0^{\eta_t} \left\langle \mathcal{P}_{\tau,\eta_t}^{\gamma} [\operatorname{Hess} h_{\gamma(\tau)} - \mathcal{P}_{0,\tau}^{\gamma} \operatorname{Hess} h_{\gamma(0)}] G_t, E(\eta_t) \right\rangle d\tau.$$

Note that, for each  $\tau \in [0, s_{K_t}]$ ,  $\text{Hess } h_{\gamma(\tau)}$  is a self-adjoint operator that is smooth in  $\tau$  as  $\gamma$  is. So, by (15), we obtain

$$|\mathcal{P}_{0,\eta_t}^{\gamma} \operatorname{Hess} h_{K_t}[G_t]|_{g_{K_{t+1}}} \le M |G_t|_{g_{K_t}}$$

and by smoothness there exist a constant L > 0 such that

$$|\mathcal{P}_{\tau,\eta_t}^{\gamma}[\operatorname{Hess} h_{\gamma(\tau)} - \mathcal{P}_{0,\tau}^{\gamma} \operatorname{Hess} h_{\gamma(0)}]G_t|_{g_{K_{t+1}}} \leq \tau L |G_t|_{g_{K_t}}^2$$

where we used the isometry of parallel transport again in obtaining the bounds. Therefore, by choosing the parallel vector field  $E(\eta)$  along  $\gamma$  such that  $E(\eta_t) = \operatorname{grad} h_{K_{t+1}}$  we obtain that

$$\left|\operatorname{grad} h_{K_{t+1}}\right|_{g_{K_{t+1}}} \leq M|1 - \eta_t| \left|G_t\right|_{g_{K_t}} + \left(\frac{\eta_t L}{2}\right) \left|G_t\right|_{g_{K_t}}^2 \leq \frac{M|1 - \eta_t|}{m} \left|\operatorname{grad} h_{K_t}\right|_{g_{K_t}} + \frac{L\eta_t}{2m^2} \left|\operatorname{grad} h_{K_t}\right|_{g_{K_t}}^2$$
(17)

where the last inequality follows by (16). Next, let  $F_{t+1} \in T_{K_{t+1}}\widetilde{S}$  be tangent vector that  $\xi(\eta) = \widetilde{\exp}_{K_{t+1}}[\eta F_{t+1}]$  is the minimum-length geodesic in  $\widetilde{S}$  joining  $\xi(0) = K_{t+1}$  to  $\xi(1) = K^*$ , where  $\widetilde{\exp}_{K_{t+1}}[\eta F_{t+1}]$  denotes the exponential map on  $\widetilde{S}$ . This is certainly possible (by shrinking  $\mathcal{U}$  if necessary) because geodesics are locally-minimizing [27]. Similar to the function  $\phi$ , define  $\psi:[0,1] \to \mathbb{R}$  with

$$\psi(\eta) := \langle \operatorname{grad} h_{\xi(\eta)}, E(\eta) \rangle,$$

for some parallel vector  $E(\eta)$  along  $\xi$ . Then, similarly

$$\psi'(\eta) = \langle D_{\eta} \operatorname{grad} h_{\xi(\eta)}, E(\eta) \rangle = \langle \operatorname{Hess} h_{\xi(\eta)} [\xi'(\eta)], E(\eta) \rangle.$$

The velocity of any geodesic is a parallel vector field along itself, so by choosing  $E(\eta) = \xi'(\eta)$  and using the fundamental lemma of calculus for  $\psi$  we obtain that

$$\psi(1) = \left\langle \operatorname{grad} h_{K_{t+1}}, F_{t+1} \right\rangle + \int_0^1 \left\langle \operatorname{Hess} h_{\xi(\tau)}[\xi'(\tau)], \xi'(\tau) \right\rangle d\tau$$

Note that  $\psi(1)=0$  and  $|\xi'(\tau)|g_{\xi(\tau)}=|F_{t+1}|g_{K_{t+1}}$  for all  $\tau$  as  $\xi$  is a geodesic. Thus, by using (15), we conclude that

$$m |F_{t+1}|_{g_{K_{t+1}}} \le |\operatorname{grad} h_{K_{t+1}}|_{g_{K_{t+1}}} \le M |F_{t+1}|_{g_{K_{t+1}}}$$
 (18)

Finally, combining (17) and (18) at two iterations t+1 and t, and noticing  $\operatorname{dist}(K_{t+1}, K^*) = |F_{t+1}|g_{K_{t+1}}$  imply that

$$\operatorname{dist}(K_{t+1}, K^*) \le \left(\frac{|1 - \eta_t| M^2}{m^2}\right) \operatorname{dist}(K_t, K^*) + \left(\frac{\eta_t L M^2}{2m^3}\right) \operatorname{dist}(K_t, K^*)^2, \tag{19}$$

where  $\operatorname{dist}(\cdot,\cdot)$  denotes the Riemannian distant function between two points. Next, note that the mapping  $K \to \mathcal{Q}_K$  is chosen to be smooth such that  $\mathcal{Q}_K \succ 0$ , therefore as a result of Lemma III.1, the mapping  $K \to \mathbb{L}(A_{\operatorname{cl}}^\intercal, \mathcal{Q}_K)$  is smooth by composition. By smoothness (in particular continuity) of this mapping and the continuity of the maximum eigenvalue (utilized in the definition of stability certificate  $s_K$  in Lemma IV.1), we can shrink  $\mathcal{U}$ —if necessary—to obtain a positive constant c > 0 such that

$$s_{K_t} \ge \frac{c}{|G_t|_{g_{K_t}}} \ge \frac{cm}{M \operatorname{dist}(K_t, K^*)},\tag{20}$$

where the last inequality follows by combining (16), (18) and the fact that  $\operatorname{dist}(K_t, K^*) = |F_t|g_{K_t}$ . Now, pick  $r \in (0,1)$ ; if we set  $\mathcal{U}^* \subset \mathcal{U} \subset \widetilde{\mathcal{S}}$  such that for any  $K_0 \in \mathcal{U}^*$  we have

$$\operatorname{dist}(K_0, K^*) < \min\{\frac{cm}{M(1 - r/2)}, \frac{m^3 r}{LM^2}\},$$

then by the choice of stepsize  $\eta_t = \min\{s_{K_t}, 1\}$  and the lower-bound in (20), we can claim that

$$\frac{|1 - \eta_0| M^2}{m^2} + (\frac{\eta_0 L M^2}{2m^3}) \operatorname{dist}(K_0, K^*) < r.$$

But then, (19) implies that  $\operatorname{dist}(K_1, K^*) \leq r \operatorname{dist}(K_0, K^*)$ . Therefore,  $K_1 \in \mathcal{U}^*$  as r < 1, and thus by induction we conclude a linear convergence rate to  $K^*$ . Consequently, (20) implies that  $s_{K_t} \geq 1$  for large enough t, and thus by the choice of step-size (19) simplifies to

$$\operatorname{dist}(K_{t+1}, K^*) \le (\frac{LM^2}{2m^3}) \operatorname{dist}(K_t, K^*)^2,$$

guaranteeing a quadratic convergence rate. Finally, Lemma IV.2 implies that a critical point is nondegenerate with respect to the induced Riemannian connection on  $T\widetilde{\mathcal{S}}$  if and only if it is so with respect to the Euclidean one. The proof for RNPO with  $\overline{\text{Hess}}$  then follows similarly by redefining  $\phi$  and  $\psi$  using the Euclidean metric under the usual identification of the tangent bundle.

**Proof of Proposition V.1.** By definition,  $f: \mathcal{S} \to \mathbb{R}$  can be viewed as the composition:

$$f: K \xrightarrow{\Phi} (A_{\operatorname{cl}}^{\mathsf{T}}, K^{\mathsf{T}}RK + Q) \xrightarrow{\mathbb{L}} P_K \xrightarrow{\Psi} \frac{1}{2} \operatorname{tr} \left[ P_K \Sigma_1 \right].$$
 (21)

Since the first and last maps are smooth (i.e., linear or quadratic in K), we conclude that  $f \in C^{\infty}(S)$  by composition and Lemma III.1. For any  $K \in S$ , we can compute its differential at K, denoted by  $df_K$ , using the chain rule:

$$df_K(E) = d\Psi_{P_K} \circ d \mathbb{L}_{(A_{c_1}^\intercal, K^\intercal RK + Q)} \circ d\Phi_K(E),$$

for any  $E \in T_K S$ . But  $\Psi$  is a linear map, and under the usual identification of the tangent bundle we obtain

$$d\Phi_K(E) = (E^{\dagger}B^{\dagger}, E^{\dagger}RK + K^{\dagger}RE).$$

Therefore, by Lemma III.1 we claim the followings

$$d(\mathbb{L} \circ \Phi)_K(E) = \mathbb{L} \left( A_{\text{cl}}^{\mathsf{T}}, \ E^{\mathsf{T}} (B^{\mathsf{T}} P_K A_{\text{cl}} + RK) + (K^{\mathsf{T}} R + A_{\text{cl}}^{\mathsf{T}} P_K B) E \right), \tag{22}$$

$$\implies df_K(E) = \Psi \circ \mathbb{L} \left( A_{\text{cl}}^{\mathsf{T}}, E^{\mathsf{T}} (B^{\mathsf{T}} P_K A_{\text{cl}} + RK) + (K^{\mathsf{T}} R + A_{\text{cl}}^{\mathsf{T}} P_K B) E \right).$$

Thus,  $df_K(E) = \langle E, RK + B^\intercal P_K A_{\operatorname{cl}} \rangle$  with  $Y_K = \mathbb{L}(A_{\operatorname{cl}}, \Sigma_1)$ —by Lyapunov-trace property—which is well-defined and unique as  $A_{\operatorname{cl}}$  is a stability matrix. As  $df_K(E) = Ef$ , the expression for grad  $f \in \mathfrak{X}(\mathcal{S})$  then follows by its definition. Next, as the Hessian operator is self-adjoint (see §A), in order to obtain Hess f we can compute (11) for any  $U, W \in \mathfrak{X}(\mathcal{S})$ . As Hess  $f[U]|_K$  only depends on the value of U at K, it suffices to obtain Hess  $f_K[U_K]$  at each  $K \in \mathcal{S}$  with  $U_K = E$  for arbitrary  $E \in T_K \mathcal{S}$ . To do so, we compute  $\langle \operatorname{Hess} f_K[E], F \rangle$  for an arbitrary vector  $F \in T_K \mathcal{S}$  by extending F to the vector field W along the curve  $\gamma: t \to K + tE$  with constant coordinates with respect to the global coordinate frame  $(\partial_{(i,j)})$ . As  $\langle \operatorname{Hess} f[U], W \rangle |_K$  only depends on the value of  $W_K = F$  and  $W_K = E$ , how these vector fields have been extended is arbitrary. By properties of the Riemannian connection and the fact that W can be extended with constant coordinates, here we can compute  $\nabla_U W|_K$  in the global coordinate frame  $(\partial_{(i,j)})$  and obtain,

$$\nabla_{U}W|_{K} = [E]^{k,\ell} [F]^{p,q} \Gamma^{i,j}_{(k,\ell)(p,q)}(K) \partial_{i,j},$$
(23)

where  $\Gamma^{i,j}_{(k,\ell)(p,q)}(K)$  denotes the Christoffel symbols associated with the Riemannian metric g at the point  $K \in \mathcal{S}$ . Therefore, from (11) we have that

$$\langle \operatorname{Hess} f[U], W \rangle |_{K} = Er - \langle \operatorname{grad} f_{K}, \nabla_{U} W |_{K} \rangle,$$
 (24)

where  $r := \langle \operatorname{grad} f, W \rangle \in C^{\infty}(\mathcal{S})$ . By the expression obtained for  $\operatorname{grad} f$  and that W has constant coordinates, the mapping  $K \to r(K)$  can be decomposed as:

$$K \xrightarrow{\operatorname{Id} \times \operatorname{Id}} (K, K) \xrightarrow{\operatorname{Id} \times (\mathbb{L} \circ \Phi)} (K, P_K) \xrightarrow{\Xi} (A_{\operatorname{cl}}^{\mathsf{T}}, (\operatorname{grad} f_K)^{\mathsf{T}} F + F^{\mathsf{T}} \operatorname{grad} f_K) \xrightarrow{\Psi \circ \mathbb{L}} r(K),$$

where we used the Lyapunov-trace property and invariance of trace under transpose to justify the last mapping. Also note that  $\Phi$  and  $\Psi$  are defined in (21) and  $\Xi: \mathbf{M}(m \times n, \mathbb{R}) \times \mathbf{M}(n \times n, \mathbb{R}) \to \mathbf{M}(n \times n, \mathbb{R}) \times \mathbf{M}(n \times n, \mathbb{R})$  is defined as above. Therefore, under the usual identification of tangent bundle, for any  $(E, G) \in T_{(K, P_K)}(\mathbf{M}(m \times n, \mathbb{R}) \times \mathbf{M}(n \times n, \mathbb{R}))$  we can compute that

$$d\Xi_{(K,P_K)}[E,G] = \left(E^{\dagger}B^{\dagger}, \ E^{\dagger}(R+B^{\dagger}P_KB)F + A_{\mathrm{cl}}^{\dagger}GBF + F^{\dagger}(R+B^{\dagger}P_KB)E + F^{\dagger}B^{\dagger}G^{\dagger}A_{\mathrm{cl}}\right). \tag{25}$$

Therefore, by the chain rule, for any  $E \in T_K S$  we have

$$dr_K(E) = \Psi \circ d \mathbb{L} \circ d\Xi \circ (E, d(\mathbb{L} \circ \Phi)_K(E)),$$

where the base points of differentials are understood and dropped for brevity. But then, by (22) and (25) we get that

$$d\Xi\left[E,d(\mathbb{L}\circ\Phi)_K(E)\right] = \left(E^{\mathsf{T}}B^{\mathsf{T}},E^{\mathsf{T}}(R+B^{\mathsf{T}}P_KB)F + A_{\mathrm{cl}}^{\mathsf{T}}(S_K|_E)BF + F^{\mathsf{T}}(R+B^{\mathsf{T}}P_KB)E + F^{\mathsf{T}}B^{\mathsf{T}}(S_K|_E)A_{\mathrm{cl}}\right),$$
 where  $S_K|_E$  is defined in the premise. Therefore,

$$dr_K(E) = \Psi \circ \mathbb{L}\Big(A_{\mathrm{cl}}^{\mathsf{T}}, \ E^{\mathsf{T}}(R + B^{\mathsf{T}}P_KB)F + A_{\mathrm{cl}}^{\mathsf{T}}(S_K|_E)BF + A_{\mathrm{cl}}^{\mathsf{T}}(S_K|_F)BE + E^{\mathsf{T}}B^{\mathsf{T}}(S_K|_F)A_{\mathrm{cl}}\Big),$$

and by Lyapunov-trace property we can simplify it as follows

$$2dr_K(E) = \operatorname{tr} \left[ E^{\mathsf{T}} B^{\mathsf{T}}(S_K|_F) A_{\mathsf{cl}} Y_K + A_{\mathsf{cl}}^{\mathsf{T}}(S_K|_F) B E Y_K \right]$$
$$+ \operatorname{tr} \left[ (E^{\mathsf{T}}(R + B^{\mathsf{T}} P_K B) + A_{\mathsf{cl}}^{\mathsf{T}}(S_K|_E) B) F Y_K \right]$$
$$+ \operatorname{tr} \left[ F^{\mathsf{T}}((R + B^{\mathsf{T}} P_K B) E + B^{\mathsf{T}}(S_K|_E) A_{\mathsf{cl}}) Y_K \right],$$

where  $Y_K = \mathbb{L}(A_{\text{cl}}, \Sigma_1)$ . Noting that  $Y_K$ ,  $P_K$ ,  $S_K|_E$  and  $S_K|_F$  are all symmetric, using the cyclic permutation property of trace, we get that

$$dr_K(E) = \langle (R + B^{\mathsf{T}} P_K B) E + B^{\mathsf{T}} (S_K|_E) A_{\mathrm{cl}}), F \rangle + \langle B^{\mathsf{T}} (S_K|_F) A_{\mathrm{cl}}, E \rangle.$$
 (26)

Then, the expression for Hess f follows by substituting (26) and (23) in (24). Finally, the expression of  $\overline{\text{Hess}} f$  can be obtained similarly by threading through the definitions.

**Proof of Corollary V.2.** Smoothness of h and the expression of its gradient follows immediately by Proposition III.5 and Proposition V.1. In order to compute  $\operatorname{Hess} h_K$ , we can combine its extrinsic representation as obtained in Proposition III.5 with (24), and use the definition of Weingarten map to obtain that

$$\langle \operatorname{Hess} h_K[E], F \rangle = \left\langle \pi^\top (\operatorname{Hess} f[U]|_{\widetilde{S}}), W \right\rangle \Big|_K + \left\langle \mathbb{W}_{\pi^\perp (\operatorname{grad} f|_{\widetilde{S}})}(U), W \right\rangle \Big|_K$$
$$= Er - \left\langle \operatorname{grad} f_K, \nabla_U W|_K \right\rangle + \left\langle \pi^\perp \operatorname{grad} f_K, \pi^\perp \nabla_U W|_K \right\rangle$$
$$= Er - \left\langle \pi^\top \operatorname{grad} f_K, \nabla_U W|_K \right\rangle,$$

for any  $E, F \in T_K \widetilde{\mathcal{S}} \subset \mathcal{S}$ , which are extended to vector fields on a neighborhood in  $\mathcal{S}$  with constant coordinates with respect to the global coordinate frame. The claimed expression of  $\operatorname{Hess} h_K$  then follows by substituting (23) and (26) into the last expression.

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