SIAM J. SCI. STAT. COMPUT.
Vol. 9, No. 5, September 1988

A BISECTION METHOD FOR MEASURING THE DISTANCE OF A STABLE MATRIX TO THE UNSTABLE MATRICES*

RALPH BYERS†

Abstract. We describe a bisection method to determine the 2-norm and Frobenius norm distances from a given matrix A to the nearest matrix with an eigenvalue on the imaginary axis. If A is stable in the sense that its eigenvalues lie in the open-left half plane, then this distance measures how "nearly unstable" A is. Each step provides either a rigorous upper bound or a rigorous lower bound on the distance. A few bisection steps can bracket the distance within an order of magnitude. Bisection avoids the difficulties associated with nonlinear minimization techniques and the occasional failures associated with heuristic estimates. We show how the method might be used to estimate the distance to the nearest matrix with an eigenvalue on the unit circle.

Key words. stable matrix, eigenvalues, rounding errors, Hamiltonian matrix

AMS subject classifications. 65F15, 65G05, 93B35, 93D20

1. Introduction. Suppose $A \in \mathbf{C}^{n\times n}$ has no eigenvalue on the imaginary axis. Let $U \subset \mathbf{C}^{n\times n}$ be the set of matrices with at least one eigenvalue on the imaginary axis. The distance from A to U is defined by [10]

$$\beta(A) = \min \{\|E\| \mid A + E \in U\}.$$

If A is *stable* in the sense that all its eigenvalues have negative real part, then $\beta(A)$ is the distance to the set of unstable matrices [10]. It is a measure of how "nearly unstable" is the stable matrix A.

In this paper we describe a bisection algorithm for calculating $\beta(A)$. A few steps can detect that A is within a small tolerance of being unstable or bracket $\beta(A)$ within an order of magnitude by nonheuristic upper and lower bounds.

It can be shown that

$$\beta(A) = \min_{\omega\in\mathbf{R}} \sigma_{\min}(A - \omega iI),$$

where $\sigma_{\min}(A-\omega iI)$ is the smallest singular value of $A-\omega iL$. So, for any real ω, an upper bound on $\beta(A)$ is

$$\beta(A) \leqq \sigma_{\min}(A - \omega iI). \tag{1}$$

The bisection method described in this paper gives upper and lower bounds on the global minimum of $f(\omega) = \sigma_{\min}(A-\omega iI)$. It is not affected by the number of local minima $f(\omega)$ may have, nor does it require any starting value in order to begin. The technique can be used to bracket $\beta(A)$ to any accuracy, but it converges too slowly to be used to calculate $\beta(A)$ many significant digits. Fortunately, $\beta(A)$ rarely needs to be calculated more accurately than within an order of magnitude.

Throughout, $\lambda(A)$ represents the set of eigenvalues of A. A superscript "T" represents a matrix transpose, and a supercript "H" represents a complex (Hermitian) transpose. The norm $\|\cdot\|$ may represent either the operator 2-norm $\|A\| = \max_{x\neq 0}(\|Ax\|_2/\|x\|_2)$ or the Frobenius norm $\|A\| = \sqrt{\text{trace}\,(A^H A)}$.

* Received by the editors December 1, 1986; accepted for publication (in revised form) February 4, 1988. Part of this work was done while the author was visiting the Computer Science Department, Stanford University, Stanford, California 94305 during July 1986 and part of this work was conducted while he was a faculty member at North Carolina State University.

† Department of Mathematics, University of Kansas, Lawrence, KS 66045.

2. Bisection method. Given $\sigma \geqq 0$ and an n-by-n matrix A, define the $2n$-by-$2n$ matrix $H = H(\sigma)$ by

$$H(\sigma) = \begin{bmatrix} A & -\sigma I_n \\ \sigma I_n & -A^H \end{bmatrix}. \tag{2}$$

Here I_n denotes the n-by-n identity matrix. Theorem 1 shows how the eigenvalues of $H(\sigma)$ distinguish the case $\sigma \geqq \beta(A)$ from the case $\sigma < \beta(A)$.

THEOREM 1. *$H(\sigma)$ has an eigenvalue whose real part is zero if and only if $\sigma \geqq \beta(A)$.*

Proof. If for some $\omega \in \mathbf{R}$, $\omega i \in \lambda(H)$, then there are nonzero vectors u, $v \in \mathbf{C}^n$ such that

$$\begin{bmatrix} A & -\sigma I_n \\ \sigma I_n & -A^H \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \omega i \begin{bmatrix} v \\ u \end{bmatrix}. \tag{3}$$

Multiplying out the left-hand side gives

$$(A - \omega i I)v = \sigma u \tag{4}$$

and

$$(A - \omega i I)^H u = \sigma v. \tag{5}$$

Thus σ is a singular value of $A - \omega i I$ and (1) implies $\sigma \geqq \beta(A)$.

Conversely, suppose that $\sigma \geqq \beta(A)$. Let $f: \mathbf{R} \to \mathbf{R}$ be the function

$$f(\alpha) = \sigma_{\min}(A - \alpha i I).$$

The function f is continuous and $\lim_{\alpha \to \infty} f(\alpha) = \infty$. So, at some $\alpha \in \mathbf{R}$, f attains its minimum value $f(\alpha) = \beta(A) \leqq \sigma$. The intermediate value theorem implies that for some $\omega \in \mathbf{R}$, $f(\omega) = \sigma$. So σ is a singular value of $A - \omega i I$ and there are unit vectors $u \in \mathbf{C}^n$ and $v \in \mathbf{C}^n$ satisfying (4) and (5). But this implies (3) and $\omega i \in \lambda(H)$. □

Suppose that α is a lower bound and γ is an upper bound on $\beta(A)$. The bounds can be improved by choosing a number σ between α and γ and checking to see if $H(\sigma)$ has an eigenvalue with zero real part. Theoretically, any accuracy can be obtained, but the bisection method converges only linearly, so highly accurate calculations of $\beta(A)$ may be expensive. Fortunately, all that is usually needed is an estimate of $\beta(A)$ that is correct to within a factor of 10. The following algorithm estimates $\beta(A)$ to within a factor of 10 or indicates that $\beta(A)$ is less than a small tolerance. The algorithm makes use of the naive upper bound $\beta(A) \leqq \frac{1}{2}\|A + A^H\|$.

ALGORITHM 1.
INPUT: an n-by-n matrix A and tolerance $\tau > 0$
OUTPUT: $\alpha \in \mathbf{R}$ and $\gamma \in \mathbf{R}$ such that either $\gamma/10 \leqq \alpha \leqq \beta(A) \leqq \gamma$ or
$0 = \alpha \leqq \beta(A) \leqq \gamma \leqq 10\tau$

$\alpha := 0$; $\gamma := \frac{1}{2}\|(A + A^H)\|$
WHILE $\gamma > 10\,\mathbf{MAX}(\tau, \alpha)$
  $\sigma := \sqrt{\gamma\,\mathbf{MAX}(\tau, \alpha)}$
  IF $H(\sigma)$ has an eigenvalue with zero real part **THEN** $\gamma := \sigma$ **ELSE** $\alpha := \sigma$.

If $\tau = \frac{1}{2}10^{-p}\|A + A^H\|$, then at most $\lceil \log_2 p \rceil$ bisection steps are required. For the choice $\tau = \frac{1}{2}10^{-8}\|A + A^H\|$, at most three bisection steps are required.

The bulk of the work in Algorithm 1 is deciding whether $H(\sigma)$ has an eigenvalue with zero real part. One straightforward way of doing this is to calculate all the eigenvalues. An algorithm that takes advantage of the special structure of $H(\sigma)$ is square reduction [3], [11]. For $A \in \mathbf{R}^{n \times n}$, it uses approximately $18n^3$ flops and storage

for approximately $2n^2$ floating-point numbers in addition to the original n-by-n matrix A. A flop is the computational effort required to execute the FORTRAN statement

$$A(I, J) = A(I, J) + B^* A(K, J).$$

The square-reduced algorithm applies only to real matrices and uses only real arithmetic. A complex matrix $A \in \mathbf{C}^{n\times n}$ must be presented as a $2n$-by-$2n$ real matrix.

3. Rounding. In practice, the success of Algorithm 1 depends on its being able to decide, *despite rounding errors*, whether $H(\sigma)$ has an eigenvalue with zero real part. As long as the decision is made correctly, the upper and lower bounds obtained by the algorithm are correct. In this section we show that if the decision is made in a numerically stable way, then it is correct except possibly when σ is within the rounding error of $\beta(A)$. So, in the worst case, $\beta(A)$ may lie outside the bounds given by the algorithm by an amount proportional to the precision of the arithmetic.

Define $J \in \mathbf{R}^{2n\times 2n}$ by

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}, \tag{6}$$

where I_n is the n-by-n identity matrix and 0_n is the n-by-n zero matrix. A matrix $H \in \mathbf{C}^{2n\times 2n}$ is said to be *Hamiltonian* if JH is Hermitian or equivalently if $JHJ = H^H$. A matrix $S \in \mathbf{C}^{2n\times 2n}$ is said to be *symplectic* if $S^H JS = J$. Hamiltonian and symplectic matrices are important in control theory and have been studied extensively. A few references are [1]-[3], [7], [11].

The following theorem generalizes Theorem 1 to all Hamiltonian matrices.

THEOREM 2. *Let J be as in* (6), *$H \in \mathbf{C}^{2n\times 2n}$ be Hamiltonian and $\sigma \in \mathbf{R}$. If $H - \sigma J$ has an eigenvalue on the imaginary axis then $|\sigma| \geqq \beta(H)$. If $|\sigma| \geqq \beta(H)$ then at least one of $H + \sigma J$ and $H - \sigma J$ has an eigenvalue on the imaginary axis.*

Proof. If $\omega \in \mathbf{R}$ and $\omega i \in \lambda(H - \sigma J)$, then

$$0 = \det(H - \sigma J - \omega i I) = \det(-J)\det(JH - \omega i J + \sigma I).$$

Now $\det(-J) = -1$, so $-\sigma \in \lambda(JH - \omega i J)$. Because H is Hamiltonian, $JH - \omega i J$ is Hermitian. The singular values of a Hermitian matrix are the absolute values of its eigenvalues. So, $|\sigma|$ is a singular value of $H - \omega i I$ and (1) implies $|\sigma| \geqq \beta(H)$.

Conversely, suppose that $|\sigma| \geqq \beta(H)$. As in the proof of Theorem 1, an application of the intermediate value theorem to $f(\alpha) = \sigma_{\min}(H - \alpha i I)$ guarantees the existence of a number $\omega \in \mathbf{R}$ such that $|\sigma|$ is a singular value $H - \omega i I$. Since J is unitary, $|\sigma|$ is also a singular value of the Hermitian matrix $JH - \omega i J$ and $\pm\sigma \in \lambda(JH - \omega i J)$. So,

$$0 = \det(JH - \omega i J \pm \sigma I) = \det(J)\det(H - \omega i I \pm \sigma J).$$

Hence $\omega i \in \lambda(H \pm \sigma J)$. □

The matrix $H(\sigma)$ in (2) and in Algorithm 1 is Hamiltonian. The next theorem shows that, despite rounding errors, the decision that $H(\sigma)$ does or does not have an eigenvalue on the imaginary axis is essentially correct, providing that Hamiltonian structure is preserved.

THEOREM 3. *Let $\sigma \in \mathbf{R}$ be nonnegative and let $H(\sigma)$ be as in* (2). *Suppose $E \in \mathbf{C}^{2n\times 2n}$ is Hamiltonian and $K(\sigma) = H(\sigma) + E$. If $K(\sigma)$ has an eigenvalue with zero real part, then $\beta(A) \leqq \sigma + 2\|E\|$. If $K(\sigma)$ does not have an eigenvalue with zero real part, then $\beta(A) \geqq \sigma - 2\|E\|$.*

Proof. Let $\omega \in \mathbf{R}$ be such that $\omega i \in \lambda(H(\beta(A)))$. So

$$0 = \det(J)\det(H(\beta(A)) - \omega i I) = \det(JH(0) - \omega i J + \beta(A) I), \tag{7}$$

i.e., $-\beta(A) \in \lambda(JH(0) - \omega iJ)$. The Wielandt-Hoffman Theorem [6] implies that there is an eigenvalue $-\gamma \in \mathbf{R}$ of $JH(0) - \omega iJ + JE = JK(0) - \omega iJ$ such that $\gamma - \beta(A) \leqq \|E\|$. Theorem 2 implies that $|\gamma| \geqq \beta(K(0))$. Hence $\beta(K(0)) \leqq \beta(A) + \|E\|$. A similar argument with the roles of $H(0)$ and $K(0)$ reversed shows that $\beta(A) \leqq \beta(K(0)) + \|E\|$, so

$$\beta(A) - \|E\| \leqq \beta(K(0)) \leqq \beta(A) + \|E\|.$$

If $K(\sigma)$ has an eigenvalue on the imaginary axis, then

$$\sigma \geqq \beta(K(0)) \geqq \beta(A) - \|E\|.$$

If neither $K(\sigma)$ nor $H(\sigma)$ has an eigenvalue on the imginary axis, then

$$\sigma < \beta(A) < \beta(A) + 2\|E\|.$$

If $K(\sigma)$ has no eigenvalue on the imaginary axis and $H(\sigma)$ does, then there is a number $\omega \in \mathbf{R}$ such that $H(\sigma) - \omega iI$ is singular. So,

$$\sigma - \beta(K(0)) \leqq \sigma_{\min}(K(0) - \omega iI - \sigma J) = \sigma_{\min}(H(\sigma) - \omega iI + E)$$

$$\leqq \sigma_{\min}(H(\sigma) - \omega iI) + \|E\| = \|E\|.$$

Since $\beta(K(0)) \leqq \beta(A) + \|E\|$, we have $\sigma \leqq \beta(A) + 2\|E\|$. □

Using t-digit base-b arithmetic, the square-reduced algorithm [11] applied to a Hamiltonian matrix H delivers the eigenvalues of a Hamiltonian matrix $K = H + E$, where $\|E\| = O(b^{t/2}\|H\|)$. Theorem 3 shows that if the square-reduced algorithm is used to test $H(\sigma)$ for a pure imginary eigenvalue in Algorithm 1, then the results may be "taken at face value."

4. Generalizations. The bisection method extends to estimating

$$\gamma(A) = \min\{\|E\| \mid \text{for some } \theta \in \mathbf{R};\ e^{i\theta} \in \lambda(A + E)\}, \tag{8}$$

i.e., the distance from A to the nearest matrix with an eigenvalue on the unit circle. If A is stable in the sense that all its eigenvalues lie inside the unit circle, then $\gamma(A)$ is a measure of how "nearly unstable" A is. By applying the method to simple modifications of A, the unit circle can be replaced by an arbitrary circle and the imaginary axis can be replaced by an arbitrary line.

A *matrix pencil* is the set of all matrices of the form $G - \lambda H$, where G and H are given n-by-n matrices and $\lambda \in \mathbf{C}$. A number $\mu \in \mathbf{C}$ is a *generalized eigenvalue* of the pencil $G - \lambda H$ if μ is a root of the polynomial $\psi(\lambda) = \det(G - \lambda H)$. Effective software for calculating generalized eigenvalues can be found in EISPACK [9]. The following matrix pencil theorem is an analogue of Theorem 1.

THEOREM 4. *For $A \in \mathbf{C}^{n\times n}$, there is a number $\Gamma(A) \in \mathbf{R}$ such that $\Gamma(A) \geqq \gamma(A)$ and for $\Gamma(A) \geqq \sigma \geqq \gamma(A)$, the $2n$-by-$2n$ matrix pencil*

$$F(\sigma) - \lambda G(\sigma) = \begin{bmatrix} -\sigma I_n & A \\ I_n & 0_n \end{bmatrix} - \lambda \begin{bmatrix} 0_n & I_n \\ A^T & -\sigma I_n \end{bmatrix} \tag{9}$$

has a generalized eigenvalue of magnitude 1. *Furthermore, if $\sigma < \gamma(A)$, then* (9) *has no generalized eigenvalue of magnitude* 1.

Proof. Define the function $g : [0, 2\pi) \to R$ by

$$g(\theta) = \sigma_{\min}(A - e^{i\theta} I_n).$$

This is a continuous function on a compact set, so it attains a maximum and a minimum value. Let $\Gamma(A)$ be the maximum value. An easy singular-value argument shows the minimum value is $\gamma(A)$.

The intermediate value theorem implies that for every σ such that $\Gamma(A) \geqq \sigma \geqq \gamma(A)$, there is a $\theta \in [0, 2\pi]$ such that $\sigma = g(\theta)$. It follows that $\Gamma(A) \geqq \sigma \geqq \gamma(A)$ if and only if there is a number $\theta \in [0, 2\pi]$ such that σ is an eigenvalue of

$$K(\theta) = \begin{bmatrix} 0_n & (A - e^{i\theta} I_n) \\ (A^H - e^{i\theta} I_n) & 0_n \end{bmatrix}. \tag{10}$$

Now $\sigma \in \mathbf{R}$ is an eigenvalue of $K(\theta)$ if and only if there is a vector $u \in \mathbf{C}^{2n}$ such that

$$K(\theta)u = \sigma u.$$

But this is easily rearranged to give

$$F(\sigma)u = e^{i\theta} G(\sigma)u. \qquad \square$$

The following algorithm estimates $\gamma(A)$ within a factor of 10 or indicates that $\gamma(A)$ is less than a small tolerance. It makes use of the naive bound $\Gamma(A) \geqq \sigma_{\min}(A - I_n)$.

ALGORITHM 2.
INPUT: an n-by-n matrix A and tolerance $\tau > 0$
OUTPUT: $\alpha \in \mathbf{R}$ and $\delta \in \mathbf{R}$ such that $\delta/10 \leqq \alpha \leqq \gamma(A) \leqq \delta$ or $0 = \alpha \leqq \gamma(A) \leqq \delta \leqq 10\tau$

$\alpha := 0$; $\delta := \sigma_{\min}(A - I_n)$
WHILE $\delta > 10\,\mathbf{MAX}(\tau, \alpha)$
  $\sigma := \sqrt{\delta\,\mathbf{MAX}(\tau, \alpha)}$
  IF (9) has a generalized eigenvalue of magnitude 1, **THEN** $\delta := \sigma$ **ELSE** $\alpha := \sigma$.

Let $\mathrm{Re}(z)$ represent the real part of $z \in \mathbf{C}$ and $\mathrm{Arg}(z)$ represent the argument of z. It is easy to show that for $r > 0$ and $\rho \in \mathbf{C}$, $r^{-1}\gamma(r(A + \rho I))$ is the distance from A to the nearest matrix with an eigenvalue on the circle $\{z \in \mathbf{C} \,|\, |z - \rho| = r^{-1}\}$. Algorithm 2 applied to $rA + \rho I$ estimates this quantity. Similarly, if $\rho \in \mathbf{C}$ and $\theta \in \mathbf{R}$, then $\beta(e^{i\theta}(A - \rho I))$ is the distance from A to the nearest matrix with an eigenvalue on the line $\mathrm{Arg}(z - \rho) = \frac{1}{2}\pi - \theta$. Note that all lines in the complex plane take this form. Furthermore, if all eigenvalues of A lie in a convex region R bounded by the p lines, $\mathrm{Arg}(z - \rho_j) = \frac{1}{2}\pi - \theta_j$, $j = 1, 2, 3, \cdots, p$, then the magnitude of the smallest perturbation that drives an eigenvalue outside R is $\min_j \beta(e^{i\theta_j} A - \rho_j I)$. This quantity is estimated by p applications of Algorithm 1.

As with Algorithm 1, the success of Algorithm 2 depends on its being able to decide, despite rounding errors, if a generalized eigenvalue of (9) has magnitude 1. If $F(\sigma)$ is nonsingular, then $G(\sigma)F(\sigma)^{-1}$ is symplectic. Certainly, an argument similar to that in § 3 would show that if this symplectic structure is preserved in finite precision arithmetic, then it is safe to accept the magnitude of a computed generalized eigenvalue as equal to one exactly when it is within a small tolerance of one. Unfortunately, at this writing, there is no known algorithm that preserves this symplectic structure. The analogue of the square-reduced algorithm requires inverting $F(\sigma)$, but if A is singular, then so is $F(\sigma)$. The QZ-like algorithm of [8] preserves the structure of a similar pencil, but does not apply here.

5. Conclusions. This section presents a brief comparison of the bisection method to two other approaches in estimating $\beta(A)$ suggested in [10]. One is the heuristic estimate

$$\beta(A) \approx \min \{\sigma_{\min}(A - \mathrm{Re}(\lambda) iI) \,|\, \lambda \in \lambda(A)\}. \tag{11}$$

The other method consists of applying a general nonlinear minimization algorithm to

$$f(\omega) = \sigma_{\min}(A - \omega i I). \tag{12}$$

The heuristic (11) uses some complex arithmetic, but with the use of a condition number estimator, the work and storage requirements are about one third of the bisection method. It is more economical, but the cost of the nonheuristic bisection method is still of the same order of magnitude. (In contrast, other heuristic algorithms, such as heuristic matrix condition number estimators [4], are an order of magnitude cheaper than corresponding nonheuristic algorithms.) The heuristic gives an upper bound that is often within an order of magnitude of $\beta(A)$. However, there are examples for which the bound is greater than $\beta(A)$ by an arbitrary amount [5]. The bisection method does not fail to approximate $\beta(A)$ to within an order of magnitude. Furthermore, it delivers both an upper and a lower bound on $\beta(A)$.

In the absence of a good initial guess, a nonlinear minimization algorithm applied to (12) may start slowly. Accurate function values using a singular value decomposition cost $O(n^3)$ flops each [6] and so are too expensive except for low-dimensional problems. The function values may be economically estimated with a condition estimator [10], but this can destroy the smoothness of $f(\omega)$ on which many minimization algorithms depend. It is not always the case that $f(\omega)$ is an "easy" function to minimize. When $\beta(A) = 0$, $f(\omega)$ may not be smooth; the minimum lies on a cusp. Furthermore, for an n-by-n matrix A, $f(\omega)$ may have as many as n local minima and as few as one. A nonlinear minimization algorithm must be used $O(n)$ times to search for all possible local minima. If fewer than n local minima are discovered, there remains some doubt that all were found. In contrast, the bisection algorithm does not require good starting values. Cusps in $f(\omega)$ do not affect the bisection algorithm nor does the number of local minima.

Bisection is competitive with the heuristic (11) and nonlinear minimization (12). Bisection has no possibility of failure. It produces upper and lower bounds that differ by no more than a factor of 10. It enjoys a favorable rounding error analysis.

Acknowledgments. I am grateful to Stephen Campbell and Robert Hartwig for helpful conversations.

REFERENCES

[1] N. BURGOYNE AND R. CUSHMAN, *Normal forms for real linear Hamiltonian systems*, in The 1976 Ames Research Center (NASA) Conference of Geometric Control Theory, C. F. Martin and R. Hermann, eds., Mathematical Sciences Press, Brookline, MA, 1976, pp. 483-529.

[2] R. BYERS, *A Hamiltonian* QR *algorithm*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 212-229.

[3] ———, *Hamiltonian and symplectic algorithms for the algebraic Riccati equation*, Ph.D. dissertation, Cornell University, Ithaca, NY, 1983.

[4] A. K. CLINE, A. R. CONN, AND C. F. VAN LOAN, *Generalizing the* LINPACK *condition estimator*, in Numerical Analysis, J. P. Hennart, ed., Lecture Notes in Mathematics 909, Springer-Verlag, New York, Berlin, 1982, pp. 73-83.

[5] J. W. DEMMEL, *A counterexample for two conjectures about stability*, IEEE Trans. Automat. Control, 32 (1987), pp. 340-342.

[6] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, The Johns Hopkins Press, Baltimore, MD, 1983.

[7] A. J. LAUB AND K. R. MEYER, *Canonical forms for symplectic and Hamiltonian matrices*, Celestial Mech., 9 (1974), pp. 213-238.

[8] V. MEHRMANN, *A symplectic-orthogonal method for single input or single output discrete time optimal linear quadratic control problems*, Fakultat für Mathematik, Universität Bielefed, 4800 Bielefeld 1, West Germany, 1986.

[9] B. T. SMITH, J. M. BOYLE, J. J. DONGARRA, B. S. GARBOW, Y. IKEBE, V. C. KEMA, AND C. B. MOLER, *Matrix Eigensystem Routines*—EISPACK *Guide*, Lecture Notes in Computer Science 6, Springer-Verlag, New York, Berlin, 1976.

[10] C. F. VAN LOAN, *How near is a stable matrix to an unstable matrix*, Contemporary Math., 47 (1985), pp. 465-477.

[11] ———, *A symplectic method for approximating all eigenvalues of a Hamiltonian matrix*, Linear Algebra Appl., 61 (1984), pp. 233-251.