CHAPTER 1

MATRICES WHICH LEAVE A CONE INVARIANT

1 INTRODUCTION

Let R^n_+ denote the *nonnegative orthant*, the set of all nonnegative vectors in n-dimensional Euclidean space R^n. Let $R^{n \times n}$ denote the set of $n \times n$ real matrices and let $\pi(R^n_+)$ denote the set of $n \times n$ matrices with nonnegative entries. The set R^n_+ is a proper cone (see Section 2) in R^n. Every matrix in $\pi(R^n_+)$ maps R^n_+ into itself. The set $\pi(R^n_+)$ is a proper cone in $R^{n \times n}$ and is closed under matrix multiplication (see Chapter 3). In general, if K is a proper cone in R^n and $\pi(K)$ denotes the set of $n \times n$ matrices which leave K invariant, then $\pi(K)$ is closed under multiplication and is a proper cone in $R^{n \times n}$.

The Perron-Frobenius theorems on nonnegative matrices (see Chapter 2) have been extended to operators which leave a cone invariant in infinite-dimensional spaces. Our interest in this chapter will focus upon finite-dimensional extensions of this sort. Using matrix theory methods we study the spectral properties of matrices in $\pi(K)$, where K is a proper cone in R^n. We combine the use of the Jordan form of a matrix (Birkhoff [1967b], Vandergraft [1968]) and of matrix norms (Rheinboldt and Vandergraft [1973]). The needed background on cones is described, without proofs, in Section 2. In Section 3 we study matrices in $\pi(K)$, in particular K-irreducible matrices. Cone-primitive matrices are discussed in Section 4.

2 CONES

(2.1) Definitions With $S \subseteq R^n$ we associate two sets: S^G, *the set generated* by S, which consists of all finite nonnegative linear combinations of elements of S, and S^*, *the dual of* S, defined by

$$S^* = \{y \in R^n;\ x \in S \to (x, y) \geq 0\},$$

where $(\,,)$ denotes inner product. A set K is defined to be a *cone* if $K = K^{G}$. A set is *convex* if it contains, with any two of its points, the line segment between the points. Dual sets and convex cones are examples of convex sets.

(2.2) Examples of Convex Cones (a) R^n, (b) $\{0\}$, (c) $R^n_+ = \{x \in R^n; x_i \geq 0\}$, (d) $\{0\} \cup \{x \in R^n; x_i > 0\}$ (e) $K_n = \{x \in R^n; (x_2^2 + \cdots + x_n^2)^{1/2} \leq x_1\}$, the *ice cream cone.*

All but example (d) are closed. The dual of a subspace L is its orthogonal complement $L^{\perp}$. Thus the dual of R^n is $\{0\}$ and the dual of $\{0\}$ is R^n. Notice that R^n_+ and K_n are *self-dual.* The dual of (d) is R^n_+. For every S, S^* is a closed convex cone. By the definition of the operator G, $S^{GG} = S^{G}$.

For $*$ we have the following result of Farkas.

(2.3) Exercise

$$S^{**} = \operatorname{cl} S^{G},$$

where cl denotes closure, or equivalently, K is a closed convex cone if and only if $K = K^{**}$ (e.g., Berman [1973]).

(2.4) Definition The cone S^{G} is called a *polyhedral cone* if S is finite.

Thus K is a polyhedral cone if $K = BR^k_+$ for some natural number k and an $n \times k$ matrix B.

The first three examples in (2.2) are of polyhedral cones. We state, without proof, some of the basic properties of such cones.

(2.5) Theorem (a) A nonempty subset K of R^n is a polyhedral cone if and only if it is the intersection of finitely many closed half spaces, each containing the origin on its boundary.

(b) A polyhedral cone is a closed convex cone.

(c) A nonempty subset K of R^n is a polyhedral cone if and only if K^* is a polyhedral cone.

(2.6) Definitions A convex cone is

(a) *pointed* if $K \cap (-K) = \{0\}$,

(b) *solid* if int K, the interior of K, is not empty, and

(c) *reproducing* if $K - K = R^n$.

The proofs of the following statements are left as exercises.

(2.7) Exercise A closed convex cone in R^n is *solid* if and only if it is reproducing (e.g., Krasnoselskii [1964]).

(2.8) Exercise A closed convex cone K is pointed if and only if K^* is solid, (e.g., Krein and Rutman [1948]).

Let K be a pointed closed convex cone. Then the interior of K^* is given by

$$\text{(2.9)} \qquad \operatorname{int} K^* = \{y \in K^*;\ 0 \neq x \in K \rightarrow (x, y) > 0\}.$$

(2.10) Definition A closed, pointed, solid convex cone is called a *proper cone.*

A proper cone induces a partial order in R^n via $y \overset{K}{\leq} x$ if and only if $x - y \in K$. In addition we shall use the notation $y \overset{K}{<} x$ if $x - y \in K$ and $x \neq y$ and $y \overset{K}{\ll} x$ if $x - y \in \operatorname{int} K$.

Of the five cones in (2.2) only R^n_+ and K_n are proper. The entire space R^n is not pointed, $\{0\}$ is not solid, and $\{0\} \cup \operatorname{int} R^n_+$ is not closed.

(2.11) Definition Let K be a closed convex cone. A vector x is an *extremal* of K if $0 \overset{K}{\leq} y \overset{K}{\leq} x$ implies that y is a nonnegative multiple of x.

If K has an extremal vector x then clearly K is pointed and $x \in \operatorname{bd} K$, the boundary of K.

(2.12) Exercise A proper cone is generated by its extremals (e.g., Vandergraft [1968], Loewy and Schneider [1975a]). This is a special case of the Krein–Milman theorem.

(2.13) Definitions If x is an extremal vector of K, then $\{x\}^G$ is called an *extremal ray* of K. A proper cone in R^n which has exactly n extermal rays is called *simplicial*. In other words, $K \subseteq R^n$ is a simplicial cone if $K = BR^n_+$, where B is a nonsingular matrix of order n.

Clearly $(BR^n_+)^* = (B^{-1})^t R^n_+$. In R^2 a polyhedral cone is proper if and only if it is simplicial. In R^3, however, one can construct a polyhedral cone with k extremal rays for every natural number k.

(2.14) Definition Let K and $F \subseteq K$ be pointed closed cones. Then F is called a *face* of K if

$$x \in F, \quad 0 \overset{K}{\leq} y \overset{K}{\leq} x \rightarrow y \in F.$$

The face F is *nontrivial* if $F \neq \{0\}$ and $F \neq K$.

The faces of R^n_+ are of the form

$$F_I = \{x \in R^n_+ : x_i = 0 \text{ if } i \notin I\} \qquad \text{where} \quad I \subseteq \{1, \ldots, n\}.$$

The nontrivial faces of the ice cream cone K_n are of the form $\{x\}^G$, where $x \in \operatorname{bd} K_n$. The *dimension of a face* F is defined to be the dimension of the subspace $F - F$, the *span* of F. Thus the extremal rays of K_n are its one-dimensional faces.

(2.15) Exercise If F is a face of K, then $F = K \cap (F - F)$. If F is nontrivial then $F \subseteq \operatorname{bd} K$ (e.g., Barker [1973]).

Denote the interior of F, relative to its span $F - F$, by $\operatorname{int} F$.

(2.16) Exercise For $x \in K$ let

$$F_x = \{y \in K;\ \text{there exists a positive } \alpha \text{ such that } \alpha y \overset{K}{\leq} x\}.$$

Then

(a) F_x is the smallest face which contains x and it is nontrivial if and only if $0 \neq x \in \operatorname{bd} K$.

(b) F is a face of K and $x \in \operatorname{int} F$ if and only if $F = F_x$, (e.g., Vandergraft [1968]).

The set F_x is called *the face generated by* x.

Let K_1 and K_2 be proper cones, in R^n and R^m, respectively. Denote by $\pi(K_1,K_2)$ the set of matrices $A \in R^{m \times n}$ for which $AK_1 \subseteq K_2$.

(2.17) Exercise The set $\pi(K_1,K_2)$ is a proper cone in $R^{m \times n}$. If K_1 and K_2 are polyhedral then so is $\pi(K_1,K_2)$, (Schneider and Vidyasagar [1970]).

For $K_1 = R^n_+$ and $K_2 = R^m_+$, $\pi(K_1,K_2)$ is the class of $m \times n$ nonnegative matrices. If $A \in \pi(K_1,K_2)$ then $A^t \in \pi(K_2^*,K_1^*)$ by the definitions of the transpose operator t and the dual cone. The interior of $\pi(K_1,K_2)$ can be shown to be the following.

(2.18) Exercise $\operatorname{int} \pi(K_1,K_2) = \{A \in R^{m \times n};\ A(K_1 - \{0\}) \subseteq \operatorname{int} K_2\}$ (e.g., Barker [1972]).

Let

$$(A,B) = \operatorname{tr} AB^t. \tag{2.19}$$

Then (2.19) is an inner product in $R^{m \times n}$.

Let

$$Q = \{uv^t;\ u \in K_2^*,\ v \in K_1\}$$

Then we have the following.

(2.20) Exercise (a) $\pi(K_1,K_2) = Q^*$.

(b) $(\pi(K_1,K_2))^* = Q^G$.

(c) $(\pi(K_1,K_2))^* \subseteq \pi(K_1^*,K_2^*)$ (e.g., Berman and Gaiha [1972], Tam [1977]).

For $m = n$ and $K_1 = K_2 = K$ we use $\pi(K)$ as a short notation for $\pi(K,K)$. Thus $\pi(R^n_+)$ is the class of nonnegative matrices of order n. This set is denoted by $\mathcal{N}_n$ in the literature on semigroups and this notation is used in Chapter 3. The cone $\pi(K_n)$ *contains* the diagonal matrices

$$D = \text{diag}\{d_1, \ldots, d_n\},$$

where $d_1 \geq |d_i|$, $i = 2, \ldots, n$.

(2.21) Definitions The matrices in $\pi(K)$ are called *K-nonnegative* and are said to *leave K invariant*. A matrix A is *K-positive* if

$$A(K - \{0\}) \subseteq \text{int}\, K.$$

It is easy to check that the following is true.

(2.22) A is K-nonnegative if and only if A^t is K^*-nonnegative

and

(2.23) A is K-positive if and only if A^t is K^* positive.

In the next section we shall use norms induced by partial orders.

(2.24) Exercise Let K be a proper cone and let $u \in \text{int}\, K$.

(a) The order interval

$$B_u = \{x \in R^n;\ -u \overset{K}{\leq} x \overset{K}{\leq} u\}$$

is a *convex body* in R^n, i.e., B_u is closed and convex and for any $x \in R^n$, there exists a positive t such that $x \in tB_u$, and

$$x \in B_u, \quad |\alpha| \leq 1 \rightarrow \alpha x \in B_u.$$

(b) B_u being a convex body defines a norm on R^n,

$$\|x\|_u = \inf\{t \geq 0;\ x \in tB_u\}$$

and thus

$$\|u\|_u = 1.$$

(For $u^t = (1, \ldots, 1)$ and $K = R^n_+$ one gets the l_∞ norm.) (See Householder [1964], Rheinboldt and Vandergraft [1973].)

In the next section we shall need the following observations:

$$tu \overset{K}{\geq} y \rightarrow tu - x = tu - y + y - x \in K \qquad \text{and} \qquad tu + x = tu - x + x + x \in K.$$

Thus the norm is monotonic with respect to K; namely,

$$0 \overset{K}{\leq} x \overset{K}{\leq} y \to \|x\|_u \leq \|y\|_u. \tag{2.25}$$

Similarly, the induced operator norm

$$\|A\|_u = \sup_{\|x\|_u = 1} \|Ax\|_u$$

satisfies

$$\|A\|_u = \|Au\|_u \qquad \text{if} \quad A \in \pi(K), \tag{2.26}$$

since $tu - Ax = tu - Au + A(u - x)$ and $tu + Ax = tu - Au + A(u + x)$, if $tu \overset{K}{\geq} Au$, $x \in B_u$ and $A \in \pi(K)$.

3 SPECTRAL PROPERTIES OF MATRICES IN $\pi(K)$

In this section and the next, K denotes a proper cone in R^n, $n > 1$. Let $A \in \pi(K)$. By the (finite-dimensional case of the) Krein–Rutman theorem, A has an eigenvector in K which corresponds to $\rho(A)$, the spectral radius of A. This suggests the following.

(3.1) Question Let A be a matrix such that $\rho(A)$ is an eigenvalue of A. Is there a proper cone which A leaves invariant?

Here we offer a proof, due to Birkhoff, of the finite-dimensional version of the Krein–Rutman theorem. This proof specifies another property of $\rho(A)$, which allows an answer to Question 3.1.

Let λ be an eigenvalue of A. The *degree of* λ, $\deg \lambda$, is the size of the largest diagonal block in the Jordan canonical form of A, which contains λ (the multiplicity of λ in the minimal polynomial of A). With this definition we restate the following theorem.

(3.2) Theorem If $A \in \pi(K)$, then

(a) $\rho(A)$ is an eigenvalue,
(b) if λ is an eigenvalue of A such that $|\lambda| = \rho(A)$, then $\deg \lambda \leq \deg \rho(A)$,
(c) K contains an eigenvector of A corresponding to $\rho(A)$, and
(d) K^* contains an eigenvector of A^t which corresponds to $\rho(A)$.

Proof If $\rho(A) = 0$, A is nilpotent so $A^r = 0$ for some minimal r, and there is $0 \neq x \in K$ such that $w = A^{r-1}x \neq 0$. Clearly $w \in K$ and $Aw = 0$, so that w is the eigenvector in (c).

If $\rho(A) > 0$, let $\{x_{ij}\}$, $i = 1, \ldots, k$, $j = 1, \ldots, m_i$; $\sum_{i=1}^{k} m_i = n$, be a Jordan canonical basis (of C^n); i.e.,

$$Ax_{ij} = \lambda_i x_{ij} + x_{ij-1}, \qquad x_{i0} = 0,$$

where the eigenvalues λ_i are ordered by the following rules:

$$\rho = \rho(A) = |\lambda_1| = \cdots = |\lambda_\nu| > |\lambda_{\nu+1}| \geq \cdots \geq |\lambda_k|,$$

$$m = m_1 = m_2 = \cdots = m_h > m_{h+1} \geq \cdots \geq m_\nu,$$

$$\lambda_l = \rho e^{i\theta_l}, \qquad 0 \leq \theta_l < 2\pi, \qquad l = 1, \ldots, h,$$

$$0 \leq \theta_1 \leq \cdots \leq \theta_h.$$

The principal eigenvectors $\{x_{ij}\}$ are either real or occur in conjugate pairs since A is real and every vector $y \in R^n$ can be written as

$$y = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_{ij} x_{ij}, \qquad \alpha_{ij} = \bar{\alpha}_{pq} \qquad \text{if} \quad x_{ij} = \bar{x}_{pq}.$$

Since K is solid we can choose $y \in \operatorname{int} K$ and a small enough δ such that for all i and j, $c_{ij} = \alpha_{ij} + \delta \neq 0$, and

$$z = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} x_{ij} = y + \delta \sum_{i=1}^{k} \sum_{j=1}^{m_i} x_{ij} \in \operatorname{int} K.$$

Our aim now is to show that K contains a nonzero vector which is a linear combination of the eigenvectors $x_{11}, \ldots, x_{h1}$. To do this we observe that

$$A^r x_{ij} = \sum_{k=0}^{j-1} \binom{r}{k} \lambda_i^{r-k} x_{i,\,j-k} \qquad \text{(induction on } r\text{)}$$

and thus

$$A^r z = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \sum_{s=0}^{j-1} \binom{r}{s} \lambda_i^{r-s} x_{i,\,j-s}.$$

For large values of r the dominant summands will be $c_{im}\binom{r}{m-1}\lambda_i^{r-m+1} x_{i1}$, $i = 1, \ldots, h$, and thus a good approximation of $A^r z$ is

$$A^r z \sim \binom{r}{m-1} \rho^{r-m+1} \sum_{l=1}^{h} c_{lm} e^{i\theta_l} x_{l1} \tag{3.3}$$

The right-hand side of (3.3) is clearly different from zero since the eigenvectors are linearly independent and all the coefficients are nonzero. Thus for every r, $A^r z \neq 0$ and $A^r z \in K$ since $A \in \pi(K)$. The set of rays in K is compact since K is closed, thus the sequence of rays $\{(A^r z)^G\}$ has a convergent subsequence, converging, say, to $\{x_h\}^G$. By (3.3),

$$x_h = \sum_{i=1}^{h} \beta_{ih} x_{i1},$$

and this is a nonzero vector in K.

We now make use of the following lemma whose proof is left as an exercise.

(3.4) Lemma For every complex number α off the nonnegative real axis there exist positive numbers $w_0, \ldots, w_q$ such that $\sum_{p=0}^{q} w_p \alpha^p = 0$.

If $\lambda_h \neq \rho$ then by the lemma there exist positive numbers $w_0, \ldots, w_q$ such that $\sum_{p=0}^{q} w_p \lambda_h^p = 0$. In this case we let

$$x_{h-1} = \sum_{p=0}^{q} w_p A^p x_h = \sum_{p=0}^{q} w_p \sum_{i=1}^{h} \beta_{ih} \lambda_i^p x_{i1}$$
$$= \sum_{i=1}^{h} \beta_{ih} \sum_{p=0}^{q} w_p \lambda_i^p x_{i1} = \sum_{i=1}^{h-1} \beta_{ih-1} x_{i1},$$

where $\beta_{ih-1} = \beta_{ih} \sum_{p=0}^{q} w_p \lambda_i^p$.

The vector x_{h-1} is a nonzero vector in K. This follows from $w_p A^p x_h \in K$ and $w_0 x_h \neq 0$. This proves that $\lambda_1 = \rho$, since otherwise we could use the same process to generate a sequence of nonzero vectors $x_{h-2}, \ldots, x_1, x_0$ but $x_0 = 0$ by Lemma 3.4. If $\lambda_f = \rho$ but $\lambda_{f+1} \neq \rho$, then $x_f = \sum_{i=1}^{f} \beta_{if} x_{i1}$ is a nonzero vector in K and clearly

$$Ax_f = \rho(A) x_f$$

which proves (a), (b), and (c).

Statement (d) follows from (2.22). ■

We now answer Question 3.1.

(3.5) Theorem If $\rho(A)$ is an eigenvalue of A, and if $\deg \rho(A) \geq \deg \lambda$ for every eigenvalue λ such that $|\lambda| = \rho(A)$, then A leaves a proper cone invariant.

Proof In the notation of the proof of Theorem 3.2, $\lambda_i = \rho(A)$ and $m_1 \geq m_i$, $i = 1, \ldots, \nu$. Let the vectors $\{x_{ij}\}$, $j \geq 1$, be normalized so that

$$Ax_{ij} = \lambda_i x_{ij} + \delta x_{ij-1}, \qquad i = 1, \ldots, k, \quad j = 1, \ldots, m_i,$$

where $x_{i0} = 0$ and

$$\delta = \begin{cases} 1 & \text{if} \quad \nu = k, \\ \lambda_1 - |\lambda_{\nu+1}| & \text{if} \quad \nu < k. \end{cases}$$

Let

$$K = \Big\{ x \in R^n;\, x = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_{ij} x_{ij},\ |\alpha_{ij}| \leq \alpha_{1j} \text{ if } j \leq m_1,$$
$$|\alpha_{ij}| \leq \alpha_{1m_1} \text{ if } j \geq m_1,$$
$$\alpha_{ij} = \bar{\alpha}_{pq} \text{ if } x_{ij} = \bar{x}_{pq} \Big\}.$$

We leave it to the reader to complete the proof by checking the following.

(3.6) Exercise K is a proper cone in R^n and $A \in \pi(K)$ (Vandergraft [1968]). ■

Two simple corollaries of Theorem 3.5 are that every strictly triangular matrix has an invariant proper cone and that if all the eigenvalues of the matrix A are real, as in the case when A is real and symmetric, then A or $-A$ is K-nonnegative for some proper cone K.

We now collect some results on order inequalities of the form

$$Bx \overset{K}{\leq} \alpha x, \qquad 0 \neq x \in K,$$

where B is K-nonnegative. As a preparation for these results we start with the following.

(3.7) Exercise Let K_1 and K_2 be proper cones in R^n and R^m, respectively, and let $A \in R^{m \times n}$. Consider the following systems:

(i) $Ax \in \operatorname{int} K_2$, $x \in \operatorname{int} K_1$,
(ii) $A^t y \in K_1^*$, $0 \neq y \in -K_2^*$.
(i_0) $Ax \in K_2$, $0 \neq x \in K_1$,
(ii_0) $A^t y \in \operatorname{int} K_1^*$, $y \in -\operatorname{int} K_2^*$.

Then, exactly one of the systems (i) and (ii) is consistent and exactly one of the systems (i_0) and (ii_0) is consistent (Berman and Ben-Israel [1971]).

(3.8) Notation and Definition The set of matrices for which (i) is consistent is denoted by $S(K_1,K_2)$. The set of matrices for which (i_0) is consistent is denoted by $S_0(K_1,K_2)$. A square matrix A is said to be *K-semipositive* if $A \in S(K,K)$.

A relation between these definitions and positive definiteness is given by the following.

(3.9) Theorem Let A be a square matrix of order n. Then for every proper cone K in R^n:

(a) If $A + A^t$ is positive definite, then $A \in S(K,K^*)$.
(b) If $A + A^t$ is positive semidefinite, then $A \in S_0(K,K^*)$.

Proof (a) Suppose $A \notin S(K,K^*)$. By Exercise 3.7, there exists $0 \neq y \in K$ such that $-A^t y \in K^*$. But then $((A + A^t)y, y) = (A^t y, y) + (y, A^t y) \leq 0$. The proof of (b) is similar. ■

(3.10) Definition A matrix B is said to be *convergent* if $\lim_{k \to \infty} B^k$ exists and is the zero matrix.

(3.11) Exercise (a) Show that B is convergent if and only if $\rho(B) < 1$.

(b) Show that B is convergent if and only if $I - B$ is nonsingular and $(I - B)^{-1} = \sum_{k=0}^{\infty} B^k$ (e.g., Varga [1962], Oldenburger [1940]).

The relation between convergence, semipositivity, and similar properties is now described.

(3.12) Theorem Let $A = \alpha I - B$, where $B \in \pi(K)$. Then

(a) If $Ax \in K$ for some $x \in \operatorname{int} K$, then $\rho(B) \leq \alpha$. If, in addition $\alpha > 0$, then $\lim_{k\to\infty}(\alpha^{-1}B)^k x = x^*$ exists and $x^* \overset{K}{\leq} x$. Moreover $x^* = 0$ if and only if $\alpha^{-1}B$ is convergent, i.e., $\rho(B) < \alpha$.

(b) The matrix A is K-semipositive if and only if $\alpha^{-1}B$ is convergent.

(c) If $\rho(B) \leq \alpha$, then $A \in S_0(K,K)$.

Proof (a) The spectral radius is bounded by all norms. In particular,

$$\rho(B) \leq \|B\|_x = \|Bx\|_x \leq \|\alpha x\|_x = \alpha$$

by (2.26). If α is positive, then the sequence $\{(\alpha^{-1}B)^k x\}$ decreases in the partial order induced by K, and is bounded by $\{0\}$, which assures the existence of x^*, and by which implies $x^* \overset{K}{\leq} x$. If $\rho(\alpha^{-1}B) < 1$, then $\lim_{k\to\infty}(\alpha^{-1}B)^k = 0$. Conversely, by (2.26), $\|(\alpha^{-1}B)^k\|_x = \|(\alpha^{-1}B)^k x\|_x$ and thus if $x^* = 0$, $\|(\alpha^{-1}B)^k\| \to 0$ so $\rho(B) < \alpha$.

(b) If: Let $y \in \operatorname{int} K$. Then

$$x = \alpha A^{-1} y = (I - \alpha^{-1}B)^{-1} y = y + \textstyle\sum_{k=1}^{\infty} (\alpha^{-1}B)^k y \in \operatorname{int} K \quad \text{and} \quad Ax \in \operatorname{int} K.$$

Only if: If A is K-semipositive, then the proof follows by Exercise 3.7.

$$(3.13) \qquad (B^t - \alpha I)y \in K^*, \qquad y \in K^* \to y = 0.$$

Let z be an eigenvector of B^t which lies in K^* and corresponds to $\rho(B)$ (Theorem 3.2(d)). Then if $\rho(B) \geq \alpha$, z is a counterexample to (3.13).

(c) For every natural number k, $\rho(B) < \alpha + (1/k)$. By (b) there exists $x^{(k)} \in K$ such that $Bx^{(k)} \overset{K}{<} (\alpha + (1/k))x^{(k)}$. Since we can normalize the vectors $x^{(k)}$ so that $\|x^{(k)}\| = 1$ for all k, there exists a limit x^* of a converging subsequence, satisfying

$$x^* \in K, \quad \|x^*\| = 1 \quad \text{and thus} \quad x^* \neq 0 \quad \text{and} \quad Bx^* \overset{K}{\leq} \alpha x^*. \quad \blacksquare$$

The assumption $x \in \operatorname{int} K$ cannot be replaced by $x \overset{K}{>} 0$. This can be demonstrated by taking

$$K = R^2_+, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad \alpha = 1, \qquad \text{and} \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A strengthening of the previous results is possible for a subclass of $\pi(K)$ which we now study.

(3.14) Definitions A matrix in $\pi(K)$ is K-irreducible if the only faces of K that it leaves invariant are $\{0\}$ or K itself. A matrix in $\pi(K)$ is K-reducible if it leaves invariant a nontrivial face of K.

Before we state and prove analogues of the previous theorems, we give some characterizations of K-irreduciblity.

(3.15) Theorem A matrix $A \in \pi(K)$ is K-irreducible if and only if no eigenvector of A lies on the boundary of K.

Proof If: Suppose F is a nontrivial face of K. F is a proper cone in $F - F$. Applying Theorem 3.2, part (c), to A_F, the restriction of A to $F - F$, we see that it has an eigenvector $x \in F$, but x is also an eigenvector of A and $x \in \operatorname{bd} K$.

Only if: Let F_x be the face of K generated by x, defined in Lemma 2.16. If x is an eigenvector of A, then $AF_x \subseteq F_x$. If $0 \neq x \in \operatorname{bd} K$, then F_x is nontrivial. ■

(3.16) Theorem A matrix $A \in \pi(K)$ is K-irreducible if and only if A has exactly one (up to scalar multiples) eigenvector in K, and this vector is in $\operatorname{int} K$.

Proof If: The proof follows from Theorem 3.15.

Only if: By Theorem 3.15, A has no eigenvector on $\operatorname{bd} K$. Being K-nonnegative it has an eigenvector in K which has to be in $\operatorname{int} K$. The uniqueness of this eigenvector follows from the first part of the following lemma.

(3.17) Lemma If $A \in \pi(K)$ has two eigenvectors in $\operatorname{int} K$, then A has an eigenvector on the boundary. Furthermore, the corresponding eigenvalues are all equal.

Proof Let

$$Ax_1 = \lambda_1 x_1, \qquad x_1 \in \operatorname{int} K,$$

$$Ax_2 = \lambda_2 x_2, \qquad x_2 \in \operatorname{int} K.$$

The eigenvalues λ_1 and λ_2 are nonnegative since A is K-nonnegative. Assume $\lambda_1 \geq \lambda_2$ and let

$$t_0 = \min\{t > 0 : tx_2 - x_1 \in K\}.$$

This minimum exists since $x_2 \in \operatorname{int} K$. Let $x_3 = t_0x_2 - x_1$. Clearly $x_3 \in \operatorname{bd} K$. If $\lambda_1 = 0$ then so is λ_2 and $Ax_3 = 0$. If $\lambda_1 > 0$ then

$$Ax_3 = t_0\lambda_2 x_2 - \lambda_1 x_1 = \lambda_1\{t_0(\lambda_2/\lambda_1)x_2 - x_1\}.$$

The vector $Ax_3 \in K$. Thus by the definition of t_0, $\lambda_2 \geq \lambda_1$. Thus $\lambda_1 = \lambda_2$ and $Ax_3 = \lambda_1 x_3$, which completes the proof of the lemma and the theorem. ■

The following characterization is given in terms of order inequalities.

(3.18) Theorem A matrix $A \in \pi(K)$ is K-irreducible if and only if

$$Ax \overset{K}{\leq} \alpha x \qquad \text{for some} \quad 0 \neq x \in K \tag{3.19}$$

implies that $x \in \operatorname{int} K$.

Proof If: Suppose A is K-reducible. Then

$$Ax = \lambda x \qquad \text{for some} \quad 0 \neq x \in \operatorname{bd} K.$$

Thus x satisfies (3.19) but is not in int K.

Only if: $AF_x \subseteq F_x$ for every x which satisfies (3.19). ■

Every K-positive matrix is K-irreducible. Conversely we have the following.

(3.20) Theorem An $n \times n$ matrix $A \in \pi(K)$ is K-irreducible if and only if $(I + A)^{n-1}$ is K-positive.

Proof If: Suppose A is K-reducible, so $x \in \operatorname{bd} K$ is an eigenvector of A. Then $(I + A)^{n-1}x \in \operatorname{bd} K$.

Only if: Let y be an arbitrary nonzero element on bd K. By K-irreducibility $y_1 = (I + A)y$ does not lie in $Fy - Fy$ and the dimension of Fy_1 is greater than the dimension of Fy. Repeating this argument shows that $(I + A)^k y \in \operatorname{int} K$ for some $k \leq n - 1$, hence $(I + A)^{n-1}$ is K-positive. ■

As corollaries of Theorem 3.20 and of statements (2.22) and (2.23) we have the following.

(3.21) Corollary If A and B are in $\pi(K)$ and A is K-irreducible, then so is $A + B$.

(3.22) Corollary A K-nonnegative matrix A is K-irreducible if and only if A^t is K^*-irreducible.

We now state the analogs of Theorems 3.2 and 3.5.

(3.23) Theorem If $A \in \pi(K)$ is K-irreducible, then

(a) $\rho(A)$ is a simple eigenvalue and any other eigenvalue with the same modulus has degree 1, and

(b) there is an eigenvector corresponding to $\rho(A)$ in int K, and no other eigenvector (up to scalar multiples) lies in K.

Furthermore, (a) is sufficient for the existence of a proper cone K, for which A is K-nonnegative and K-irreducible.

Proof Part (b) is a restatement of Theorem 3.16. Part (a) follows from Theorem 3.2, provided $\rho(A)$ is simple. Suppose $\rho(A)$ is not simple. Then there exist linearly independent vectors x_1 and x_2, with $x_1 \in \operatorname{int} K$

$$Ax_1 = \rho(A)x_1$$

and either,

$$Ax_2 = \rho(A)x_2 \tag{3.24}$$

or

$$Ax_2 = \rho(A)x_2 + x_1. \tag{3.25}$$

If (3.24) were true, then, for large enough $t > 0$, $x_3 = tx_1 + x_2 \in K$, and x_3 is another eigenvector; this contradicts the uniqueness of the eigenvector. If (3.25) holds, then $-x_2 \notin K$, and we can define

$$t_0 = \min\{t > 0;\ tx_1 - x_2 \in K\}.$$

Then, $\rho(A) \neq 0$ because A is K-irreducible; but $\rho(A) \neq 0$ implies

$$\begin{aligned} A(t_0x_1 - x_2) &= t_0\rho(A)x_1 - \rho(A)x_2 - x_1 \\ &= \rho(A)\left\{\left(t_0 - \frac{1}{\rho(A)}\right)x_1 - x_2\right\} \in K, \end{aligned}$$

which contradicts the definition of t_0. Hence $\rho(A)$ must be simple. To prove the "furthermore" part, we use the proof of Theorem 3.5. The cone K defined there contains only elements of the form $\alpha x_1 + y$, where x_1 is the eigenvector corresponding to $\rho(A)$ and $\alpha = 0$ only if $y = 0$. Hence no other eigenvector can lie in K, so by Theorem 3.16, A is K-irreducible. ■

Part (a) of Theorem 3.23 can be strengthened if A is K-positive.

(3.26) Theorem If A is K-positive, then

(a) $\rho(A)$ is a simple eigenvalue, greater than the magnitude of any other eigenvalue and

(b) an eigenvector corresponding to $\rho(A)$ lies in int K.

Furthermore, condition (a) assures that A is K-positive for some proper cone K.

Proof Part (b) and the simplicity of $\rho(A)$ follow from the previous theorem. Let λ_2 be an eigenvalue of A with eigenvector x_2, and assume $\rho(A) = \rho$ and $\lambda_2 = \rho e^{i\theta}$, $2\pi > \theta > 0$. For any ϕ, either $\operatorname{Re} e^{i\phi} x_2 \in K$ or else one can define a positive number t_ϕ by

$$t_\phi = \min\{t > 0;\ tx_1 + \operatorname{Re} e^{i\phi} x_2 \in K\},$$

where x_1 is the eigenvector in $\operatorname{int} K$ corresponding to $\rho(A)$. The nonzero vector $y = t_\phi x_1 + \operatorname{Re} e^{i\phi} x_2$ lies on $\operatorname{bd} K$, and

$$Ay = \rho(A)(t_\phi x_1 + \operatorname{Re} e^{i(\phi+\theta)} x_2) \in \operatorname{int} K.$$

Hence $\operatorname{Re} e^{i(\phi+\theta)} x_2 \in K$ or $t_\phi > t_{\phi+\theta}$. By repeating this argument it follows that for some ϕ_0,

$$y_0 = \operatorname{Re} e^{i\phi_0} x_2 \in K.$$

By Exercise 3.4, $\theta \neq 0$ implies the existence of a finite set of positive numbers $\{\xi_k\}$ such that

$$\sum_{k=0} \xi_k \rho^k e^{ik\theta} = 0.$$

Hence,

$$\sum_{k=0} \xi_k A^k y_0 = \sum_k \xi_k \rho^k \operatorname{Re}(e^{ik\phi} e^{i\phi_0} x_2) = \operatorname{Re}\left(\sum_k \xi_k \rho^k e^{ik\theta}\right) e^{i\phi_0} x_2 = 0.$$

Thus $y_0 = 0$; i.e., $e^{i(\phi_0 + \pi/2)} x_2 = y_2$ is real. Since $Ay_2 = \lambda_2 y_2$, λ_2 is real. Since $\lambda_2 \neq \rho$, $\lambda_2 = -\rho$. Thus $y_2 \notin K \cup (-K)$.

Let

$$t_0 = \min\{t > 0;\ tx_1 + y_2 \in K\}.$$

Then $0 \neq t_0 x_1 + y_2 \in \operatorname{bd} K$, but

$$A^2(t_0 x_1 + y_2) = \rho^2(t_0 x_1 + y_2) \in \operatorname{int} K,$$

which contradicts the definition of t_0. Hence $|\lambda_2| < \rho(A)$.

To prove the last statement of the theorem, we again use the notation of Theorem 3.5. The cone K becomes

$$K = \left\{x;\ x = \alpha_1 x_1 + \sum_{i=2}^{k} \sum_{j=1}^{m_i} \alpha_{ij} x_{ij},\ |\alpha_{ij}| \leq \alpha_1,\ \alpha_{ij} = \bar{\alpha}_{pq} \text{ if } x_{ij} = \bar{x}_{pq}\right\}.$$

It is easy to check that

(3.27) Exercise $A(K - \{0\}) \subseteq \operatorname{int} K$ (Vandergraft [1968]). ■

K-irreducibility allows a strengthening of the first part of Theorem 3.12.

(3.28) Theorem Let $A \in \pi(K)$ be K-irreducible. Then the existence of a real number α and of $x \in \operatorname{int} K$ such that $0 \neq \alpha x - Ax \in K$ implies that $\rho(A) < \alpha$.

Proof By Theorem 3.12, $\rho(A) \leq \alpha$. If $\rho(A) = 0$ then $\rho(A) < \alpha$. Suppose that $\rho(A) = \alpha \neq 0$. Let x_1 be the eigenvector of A in $\operatorname{int} K$ and

$$z = \|x_1\|_x x - x_1.$$

Then $z \in \operatorname{bd} K$ and

$$0 \overset{K}{\leq} Az = \|x_1\|_x Ax - Ax_1 \overset{K}{\leq} \|x_1\|_x \alpha x - \alpha x_1 = \alpha z,$$

which contradicts the K-irreducibility of A. ■

(3.29) Corollary Let $0 \overset{\pi(K)}{\leq} A \overset{\pi(K)}{\leq} B$, where A is K-irreducible and $A \neq B$. Then $\rho(A) < \rho(B)$.

Proof By (3.21) B too is K-irreducible. Let x be the eigenvector of B in $\operatorname{int} K$. Then

$$Ax \overset{K}{\leq} Bx = \rho(B)x.$$

Since $B \neq A$, $\|B - A\|$ is positive for any norm. In particular $\|B - A\|_x > 0$. Thus, using (2.26)

$$0 < \|B - A\|_x = \|(B - A)x\|_x = \|\rho(B)x - Ax\|_x,$$

so that $\rho(B)x \neq Ax$. Applying (3.28) proves that $\rho(B) > \rho(A)$. ■

(3.30) Corollary If $0 \overset{K}{\leq} A \overset{K}{\leq} B$, then $\rho(A) \leq \rho(B)$.

Proof Let C be a K-positive matrix and define $A_t = A + tC$, $B_t = B + tC$, $t > 0$. Being K-positive, A_t is K-irreducible and by the previous corollary $\rho(A_t) < \rho(B_t)$. Letting $t \to 0$ yields $\rho(A) \leq \rho(B)$. ■

Theorems 3.12 and 3.28 give upper bounds for the spectral radius of K-nonnegative matrices. We now complement them with lower bounds. Here we start in the K-irreducible case.

(3.31) Theorem Let $A \in \pi(K)$ be K-irreducible. If

$$Ax \overset{K}{>} \alpha x, \qquad \text{for some} \quad x \in K, \quad \alpha > 0 \tag{3.32}$$

then $\rho(A) > \alpha$. Conversely, if $\rho(A) > \alpha$, then $Ax \overset{K}{\gg} \alpha x$ for some $x \in \operatorname{int} K$.

Proof Let $\hat{A} = \alpha^{-1}A$. By (3.32) $\hat{A}x \overset{K}{>} x$. Let x_1 be the eigenvector of $\hat{A}$ in int K. Now $\|x\|_{x_1}x_1 - x \in K$, and hence

$$0 \overset{K}{\leq} \hat{A}(\|x\|_{x_1}x_1 - x) \overset{K}{<} \|x\|_{x_1}\rho(\hat{A})x_1 - x \tag{3.33}$$

or

$$x \overset{K}{<} \rho(\hat{A})\|x\|_{x_1}x_1,$$

which, by definition of $\|x\|_{x_1}$, implies that $\rho(\hat{A}) \geq 1$. Equality is impossible because of the K-irreducibility of A and (3.33), thus $\rho(\hat{A}) > 1$; that is, $\rho(A) > \alpha$. Conversely, let x_1 be the eigenvector of A in int K. Then $\rho(A) > \alpha$ implies that $Ax_1 = \rho(A)x_1 \gg \alpha x_1$. ∎

Using the continuity argument of (3.30) we can drop the K-irreducibility.

(3.34) Corollary Let $A \in \pi(K)$. Then $Ax \overset{K}{\geq} \alpha x$ for some $0 \neq x \in K$ with $\alpha > 0$, if and only if $\rho(A) \geq \alpha$.

Combining the lower and upper bounds yields the following.

(3.35) Theorem Let $A \in \pi(K)$. Then

$$\alpha x \overset{K}{\leq} Ax \overset{K}{\leq} \beta x \qquad \text{for some} \quad x \in \operatorname{int} K \tag{3.36}$$

implies that $\alpha \leq \rho(A) \leq \beta$.

If in addition A is K-irreducible, $\alpha x \neq Ax$, and $Ax \neq \beta x$, then $\alpha < \rho(A) < \beta$.

Notice that if A is K-irreducible, then by Theorem 3.18, $x \overset{K}{\gg} 0$ may be replaced by $x \overset{K}{>} 0$ in (3.36).

4 CONE PRIMITIVITY

In this section we study a subclass of the K-irreducible matrices which contain the K-positive ones.

(4.1) Definition A matrix A in $\pi(K)$ is *K-primitive* if the only nonempty subset of bd K which is left invariant by A is $\{0\}$.

The spectral structure of K-primitive matrices is due to the following result.

(4.2) Theorem $A \in \pi(K)$ is K-primitive if and only if there exists a natural number m such that A^m is K-positive.

Proof Let

$$A^m(K - \{0\}) \subseteq \operatorname{int} K \tag{4.3}$$

and let S be a nonempty subset of $\operatorname{bd} K$ which is invariant under A. Then

$$A^m S \subseteq S \subseteq \operatorname{bd} K$$

so by (4.3), $S = \{0\}$.

Conversely, if $x \in \operatorname{int} K$ then $A^m x \in \operatorname{int} K$ for all m. For $x \in \operatorname{bd} K - \{0\}$, consider the sequence $S = \{A^i x\}$, $i = 0,1,2,\ldots$. If A is K-primitive, then there is $m(x)$ such that $A^{m(x)}x \in \operatorname{int} K$, otherwise the nonzero set S, which is contained in $\operatorname{bd} K$, is invariant under A.

Let Q be the compact set $\{x \in K, x^t x = 1\}$. For each $x \in Q$, there is an integer $m(x)$ and a set $U(x)$ open in the relative topology of Q such that

$$A^{m(x)}U(x) \subseteq \operatorname{int} K.$$

The collection $\{U(x); x \in Q\}$ is an open cover of Q from which we may extract a finite subcover, say $U(x_1),\ldots,U(x_n)$, with corresponding exponents $m(x_1),\ldots,m(x_n)$. Let $m = \max\{m(x_1),\ldots,m(x_n)\}$. Let $x \in \operatorname{bd} K - \{0\}$. Then $y = (x^t x)^{-1/2}x \in Q$, and there exists x_i such that $y \in U(x_i)$. Thus

$$(x^t x)^{-1/2}A^m x = A^m y = A^{m-m(x_i)}(A^{m(x_i)}y) \in \operatorname{int} K$$

implying that $A^m x \in \operatorname{int} K$ and thus that A^m is K-positive. ■

(4.4) Corollary A is K-primitive if and only if A^t is K^*-primitive.

Proof The proof follows from statement (2.23). ■

(4.5) Corollary The sum of a K-primitive matrix and a K-nonnegative matrix is K-primitive.

(4.6) Corollary If A is K-primitive and l is a natural number then A^l is K-primitive.

(4.7) Corollary If A is K-primitive then A^l is K-irreducible for every natural number l.

(4.8) Remark The converse of Corollary 4.7 is not true, for let A be a rotation of the ice cream cone K_3 through an irrational multiple of 2π, then A^k is K_3-irreducible for all k but $A(\operatorname{bd} K) = \operatorname{bd} K$, so A is not primitive.

A converse of Corollary (4.7) does exist for polyhedral cones.

(4.9) Exercise Let K be a polyhedral cone having p generators. Then A is K-primitive if and only if the matrices $A, A^2, \ldots, A^{2^p-1}$ are K-irreducible (Barker, [1972]).

We now state a spectral characterization of K-primitive matrices.

(4.10) Theorem A K-irreducible matrix in $\pi(K)$ is K-primitive if and only if $\rho = \rho(A)$ is greater in magnitude than any other eigenvalue.

Proof If: Let v, the eigenvector of A in $\operatorname{int} K$, and ψ, the eigenvector of A^t in $\operatorname{int} K^*$, be normalized so that $\psi^t v = 1$. Define A_1 by

$$A_1 x = Ax - \rho\psi^t x v.$$

It can be shown that the following is true.

(4.11) Exercise λ is an eigenvalue of A_1 if and only if $\lambda \neq \rho$ and λ is an eigenvalue of A.

Let ρ_1 be the spectral radius of A_1. Then

$$\lim_{n\to\infty} (\|A_1\|^n)^{1/n} = \rho_1 < \rho. \tag{4.12}$$

Now, $\psi^t A_1 x = \psi^t A x - \psi^t \rho \psi^t x v = \rho\psi^t x - \rho\psi^t x \psi^t v = 0$, since $\psi^t v = 1$. Thus

$$A^n x = A_1^n x + \rho^n \psi^t x v,$$

so that

$$\|\rho^{-n} A^n x - \psi^t x v\| \leq \rho^{-n} \|A_1^n\| \, \|x\| \to 0.$$

If $0 \neq x \in K$, then $\psi^t x v \in \operatorname{int} K$. Thus for every such x there is an n such that $\rho^{-n} A^n x$, and therefore $A^n x$ is in int K. Using the argument in the proof of Theorem 4.2, this implies that A is K-primitive.

The "only if" part follows from Theorems 3.26 and 4.2. ∎

By the last part of Theorem 3.22, Theorem 4.10 can be restated as follows.

(4.13) Corollary A is K-primitive if and only if it is $\tilde{K}$-positive for some proper cone $\tilde{K}$.

If K is simplicial and $A \in \pi(K)$ is K-irreducible and has m eigenvalues with modulus $\rho = \rho(A)$, then these eigenvalues are ρ times the unit roots of order m and the spectrum of A is invariant under rotation of $2\pi/m$. Since simplicial cones are essentially nonnegative orthants, we shall defer this elegant study to Chapter 2.

5 EXERCISES

(5.1) Prove or give a counterexample:

(a) The sum of proper cones is a proper cone.
(b) The sum of closed convex cones is a closed convex cone.
(c) The sum of polyhedral cones is a polyhedral cone.
(d) The sum of simplicial cones is a simplicial cone.

(5.2) Let K be a closed convex cone. Show that $K \cap K^* = \{0\}$ if and only if K is a subspace (Gaddum, [1952]).

(5.3) Let K be a closed convex cone in R^n.

(a) Show that every point $x \in R^n$ can be represented uniquely as $x = y + z$, where $y \in K$, $z \in -K^*$, and $(y,z) = 0$ (Moreau [1962]).

(b) Show that K contains its dual, K^*, if and only if, for each vector $x \in R^n$ there exist vectors y and t in K such that $x = y - t$, $(y,t) = 0$ (Haynsworth and Hoffman [1969]).

(5.4) Let K be the cone generated by the vectors $(1,1,1)$, $(0,1,1)$, $(-1,0,1)$, $(0,-1,1)$, and $(1,-1,1)$. Show that K is self-dual.

(5.5) Show that every self-dual polyhedral cone in R^3 has an odd number of extremals (Barker and Foran [1976]).

(5.6) Show that $\pi(K)$ is self-dual if and only if K is the image of the nonnegative orthant under an orthogonal transformation (Barker and Loewy [1975]).

(5.7) Let K be the cone generated by the five vectors $(\pm 1,0,1,0)$, $(0,\pm 1,1,0)$, and $(0,0,0,1)$. Let F be the cone generated by $(\pm 1,0,1,0)$. Show that F *is not* a face of K.

(5.8) Let F and G be faces of a proper cone K. Define $F \wedge G = F \cap G$ and let $F \vee G$ be the smallest face of K which contains $F \cup G$. Show that with these definitions $F(K)$, the set of all faces of K, is a complete lattice and that $F(K)$ is distributive if and only if K is simplical (Barker [1973], Birkhoff [1967a]).

(5.9) A cone K is a *direct sum* of K_1 and K_2, $K = K_1 \oplus K_2$, if $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\}$ and $K = K_1 + K_2$. Show that in this case, K_1 and K_2 are faces of K (Loewy and Schneider [1975b]).

(5.10) Let $\operatorname{ext} K$ denote the set of extreme vectors of K and let ΔK be the closure of the convex hull of $\{xy^t;\ x \in K,\ y \in K^*\}$. A cone K is *indecomposable* if $K = K_1 + K_2 \rightarrow K_1 = 0$ or $K_2 = 0$. Let K be a proper cone in R^n. Show that the following are equivalent:

(i) K is indecomposable,
(ii) K^* is indecomposable,
(iii) $\pi(K)$ is indecomposable,
(iv) $\Delta(K)$ is indecomposable,
(v) $I \in \operatorname{ext} \pi(K)$,
(vi) A nonsingular, $A\{\operatorname{ext} K\} \subseteq \operatorname{ext} K \rightarrow A \in \operatorname{ext} \pi(K)$, and
(vii) A nonsingular, $AK = K \rightarrow A \in \operatorname{ext} \pi(K)$ (Barker and Loewy [1975], Loewy and Schneider [1975a]).

(5.11) Show that a proper cone is simplicial if and only if $I \in \Delta(K)$ (Barker and Loewy [1975]).

(5.12) Let H be the space of $n \times n$ hermitian matrices with the inner product $(A,B) = \operatorname{tr} AB$. Show that PSD, the set of positive semidefinite matrices in H, is a self-dual proper cone and that the interior of PSD consists of the positive definite matrices (e.g., Berman and Ben-Israel [1971], Hall [1967]).

(5.13) An $n \times n$ symmetric matrix A is

(a) *copositive if* $x \geq 0 \rightarrow (Ax,x) \geq 0$,
(b) *completely positive* if there are, say, k nonnegative vectors, a_i $(i = 1, \ldots, k)$, such that

$$(Ax,x) = \sum_{i=1}^{k} (a_i,x)^2 \qquad \text{for all} \quad x \in R^n.$$

Let S, CP, and C denote the sets of symmetric, completely positive, and copositive matrices of order n, respectively. Show that with the inner product in S: $(A,B) = \operatorname{tr} AB$, that C and CP are dual cones (Hall [1967]).

(5.14) Let $C^{n \times n}(R)$ be the set of $n \times n$ complex matrices, considered as a real vector space. Which of the following sets is a proper cone in $C^{n \times n}(R)$?

(a) $CDD = \{A \in C^{n \times n};\ |a_{jj}| \geq \sum_{k \neq j} |a_{jk}|,\ j = 1, \ldots, n\}$,
(b) $D_1 = \{A \in C^{n \times n};\ a_{jj} \geq \sum_{k \neq j} |a_{jk}|,\ j = 1, \ldots, n\}$,
(c) $D_2 = \{A \in C^{n \times n};\ \operatorname{Re} a_{jj} \geq \sum_{k \neq j} |a_{jk}|;\ \operatorname{Im} a_{jj} \geq 0,\ j = 1, \ldots, n\}$,
(d) $D_3 = \{A \in C^{n \times n};\ \operatorname{Re} a_{jj} \geq \sum_{k \neq j} |a_{jk}|,\ j = 1, \ldots, n\}$,
(e) $D_H = \{A \in C^{n \times n};\ A = A^H,\ a_{jj} \geq \sum_{k \neq j} |a_{jk}|,\ j = 1, \ldots, n\}$ (Barker and Carlson [1975]).

(5.15) Let K be a proper cone, $u \in \operatorname{int} K$, $v \in \operatorname{int} K^*$. Show that uv^t is K-positive.

(5.16) Let $A \in \pi(K)$ where K is a proper cone. Let $\operatorname{core} A = \bigcap_{m \geq 0} A^m K$. Show that $\operatorname{core} A$ is a pointed closed convex cone. When is $\operatorname{core} A$ a proper cone? (See Pullman [1971].)

(5.17) Let K be a proper polyhedral cone in R^n, and let $A \in \pi(K)$. Show that there exist cones $K_1, \ldots, K_r$ such that $\dim(K_i - K_i) = n_i$, $\sum_{i=1}^r n_i = n$, and A is similar to a block triangular matrix

$$\begin{bmatrix} A_r & & & 0 \\ & A_{r-1} & & \\ & & \ddots & \\ * & & & A_1 \end{bmatrix},$$

where A_j is $n_j \times n_j$ and $A_j = 0$ or A_j is A restricted to $\operatorname{span} K_j$ and is K_j-irreducible (Barker [1974]).

(5.18) Let K be a proper cone in R^n. An $n \times n$ matrix A is called *cross-positive on* K if

$$y \in K, \qquad z \in K^*, \qquad (y,z) = 0 \rightarrow (z,Ay) \geq 0.$$

A is *strongly cross-positive* on K if it is cross-positive on K and for each $0 \neq y \in \operatorname{bd} K$, there exists $z \in K^*$ such that $(y,z) = 0$ and $(z,Ay) > 0$. A is strictly cross-positive on K if

$$0 \neq y \in K, \qquad 0 \neq z \in K^*, \qquad (y,z) = 0 \rightarrow (z,Ay) > 0.$$

Let $\lambda = \max\{\operatorname{Re} \mu;\ \mu \in \operatorname{spectrum}(A)\}$. Prove the following.

(a) A is cross-positive on K if and only if $\alpha x \overset{K}{\gg} Ax$, for some $x \in K$ some α, implies $x \in \operatorname{int} K$.

(b) If A is cross-positive on K, then λ is an eigenvalue of A and a corresponding eigenvector lies in K.

(c) If A is strongly cross-positive on K, then λ is a simple eigenvalue of A, the unique eigenvector of A corresponding to λ lies in $\operatorname{int} K$, and A has no other eigenvector in K.

(d) If A is strictly cross-positive on K, then λ is a simple eigenvalue of A, the unique eigenvector of A corresponding to λ lies in $\operatorname{int} K$, A has no other eigenvector in K and $\lambda > \operatorname{Re} \mu$ for any other eigenvector μ of A (Schneider and Vidyasagar [1970], Tam[1977]).

(5.19) Show that

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

is (R^2_+)-nonnegative and (R^2_+)-reducible. Show that A is K_2-positive (and thus K_2-irreducible), where K_2 is the ice cream cone in R_2.

(5.20) Prove or give a counterexample of the following.

(a) The product of two K-irreducible matrices is K-irreducible.
(b) The product of two K-primitive matrices is K-primitive.

(5.21) Show that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is K_3-irreducible but not K_3-primitive where K_3 is the ice cream cone in R^3.

(5.22) Let $A \in \pi(K)$. Show that the following are equivalent.

(i) A is K-irreducible;
(ii) for some $\lambda > \rho(A)$, $A(\lambda I - A)^{-1}$ is K-positive;
(iii) for every $0 \neq x \in K$ and $0 \neq y \in K^*$ there exists a natural number p such that $y^t A^p x > 0$;
(iv) $\rho(A)$ is simple and A and A^t have corresponding eigenvectors in $\operatorname{int} K$ and $\operatorname{int} K^*$, respectively (Vandergraft [1968], Barker [1972]).

(5.23) $A \in \pi(K)$ is *u-positive* if there exists a vector $0 \neq u \in K$ such that for every $0 \neq x \in K$ there exist positive α, β, k where k is an integer such that

$$\alpha u \overset{K}{\leq} A^k x \overset{K}{\leq} \beta u.$$

Show that

(a) if $A \in \pi(K)$ is u-positive and $u \in \operatorname{int} K$, then A is K-primitive;
(b) if $A \in \pi(K)$ is K-irreducible then u-positivity is equivalent to K-primitivity.

Let K be the nonnegative orthant. Check that

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is K-reducible but not K-primitive, and thus not u-positive, and that the K reducible

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is u-positive for

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(Barker [1972]).

(5.24) Let $A \in \pi(K)$ have a complex eigenvalue $\lambda = \mu + i\nu$, $\nu \neq 0$. If $z = x + iy$ is a corresponding eigenvector, show that x and y are linearly independent and that $\operatorname{span}\{x, y\} \cap K = \{0\}$ (Barker and Turner [1973], Barker and Schneider [1975]).

(5.25) Prove that if K is polyhedral, $A \in \pi(K)$ is K-irreducible and $\rho(A) = 1$, then every eigenvalue of A of modulus 1 is a root of unity (Barker and Turner [1973], Barker [1974]).

(5.26) Let A and B be K-irreducible matrices in $\pi(K)$. Let $\alpha > \rho(A)$. Show that there exist a unique $\lambda > 0$ such that

$$\rho(A + (B/\lambda)) = \alpha.$$

(5.27) A symmetric matrix A is *copositive with respect to a proper cone* K if

$$x \in K \rightarrow x^t A x \geq 0$$

Let A be a symmetric matrix. Prove that $\rho(A)$ is an eigenvalue if and only if A is copositive with respect to a self-dual cone (Haynsworth and Hoffman [1969]).

(5.28) Let $e \in \operatorname{int} K$. $A \in \pi(K)$ is called *K-stochastic* if $y \in K^*$, $y^t e = 1 \rightarrow y^t A e = 1$. Show that if A is K-stochastic then $\rho(A) = 1$ and is an eigenvalue with linear elementary divisors (Marek [1971], Barker [1972]).

6 NOTES

(6.1) Many of the cone-theoretic concepts introduced in this chapter are known by other names. Our convex cone is called a linear semigroup in Krein and Rutman [1948] and a wedge in Varga [a]. Our proper cone is also called cone (Varga [a]), full cone (Berman [1973]), good cone, and positive cone. Dual is polar in Ben-Israel [1969], Haynsworth and Hoffman [1969], and Schneider and Vidyasagar [1970] and conjugate semigroup in Krein and Rutman [1948]. Equivalent terms for polyhedral cone are finite cone (Gale [1960]) and coordinate cone (Smith [1974]). An equivalent term for simplicial cone is minihedral cone (Varga [a]).

(6.2) The ice cream cone K_n is called the circular Minkowski cone in Krein and Rutman [1948]. Usually, K_n is defined as

$$\{x \in R^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq x_n\}.$$

Our slight change of definition makes K_n top heavy. (See Fiedler and Haynsworth [1973].)

(6.3) An n-dimensional space which is partially ordered by a proper cone is called a Kantorovich space of order n in the Russian literature, e.g., Glazman and Ljubic [1974].

(6.4) Theorem 2.5 is borrowed from Klee [1959] and Rockafellar [1970]. Of the many other books on convexity and cones we mention Berman [1973], Fan [1969], Gale [1960], Glazman and Ljubic [1974], Grunbaum [1967], Schaefer [1971, 1974], and Stoer and Witzgal [1970].

(6.5) Many questions are still open, at the writing of this book, concerning the structure of $\pi(K)$ where K is a proper cone. A conjecture of Loewy and Schneider [1975a] states that if $A \in \operatorname{ext} \pi(K)$, the set of extremals of $\pi(K)$, then $A(\operatorname{ext} K) \subseteq \operatorname{ext} K$. The converse is true for a nonsingular A and indecomposable K. (See Exercise 5.9.)

(6.6) The first extension of the Perron [1907] and Frobenius [1908, 1909, and 1912] theorems to operators in partially ordered Banach space is due to Krein and Rutman [1948]. There is an extensive literature on operators that leave a cone invariant in infinite-dimensional spaces. The interested reader is referred to the excellent bibliographies in Barker and Schneider [1975] and in Marek [1970].

(6.7) Theorem 3.2 is due to Birkhoff [1967b]. Theorems 3.5, 3.15, 3.16, 3.20, 3.23, and 3.26 are due to Vandergraft [1968].

(6.8) The concept of irreducibility of nonnegative matrices ($K = R_+^n$) was introduced independently by Frobenius [1912] and Markov [1908]. (See the interesting comparison in Schneider [1977] and Chapter 2.)

(6.9) The definition of a face, given in this chapter, is the one used by Schneider. Vandergraft [1968] defined a face of a cone K to be a subset of bd K which is a pointed closed convex cone generated by extremals of K. Thus K itself is not a face by Vandergraft's definition. Except for K, every face, by Schneider's definition, is a face by Vandergraft's. That the converse is not true is shown in Exercise 5.6. The concepts of K-irreducibility which the definitions of a face yield are, however, the same.

(6.10) Several concepts related or equivalent to K-irreducibility are surveyed in Vandergraft [1968] and Barker [1972]. (See Exercise 5.21.)

(6.11) The sets S and S_0 for nonnegative orthants are discussed in Fiedler and Ptak [1966] and for general cones in Berman and Gaiha [1972]. The concept of semipositivity is studied by Vandergraft [1972].

(6.12) The results on order inequalities and the consequent corollaries are borrowed from Rheinboldt and Vandergraft [1973].

(6.13) Most of Section 4 is based on Barker [1972]. Theorem 4.10 is taken from Krein and Rutman [1948]. Concepts which are equivalent to K-positivity are described in Barker [1972].