

CS 87/187, Spring 2016

# RENDERING ALGORITHMS

## Participating Media I



Prof. Wojciech Jarosz

[wojciech.k.jarosz@dartmouth.edu](mailto:wojciech.k.jarosz@dartmouth.edu)

(with slide improvements by Jan Novák)

# Today's Menu

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- Participating media (volumes)
- Theoretical background
  - Interaction coefficients
  - Phase functions
- Volume Rendering Equation
- Ray-marching
- Volumetric path tracing
- Woodcock tracking

# Fog



Steve Lacey

# Clouds & Crepuscular rays



<http://mev.fopf.mipt.ru>

# Fire



# Underwater

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# Surface or Volume?

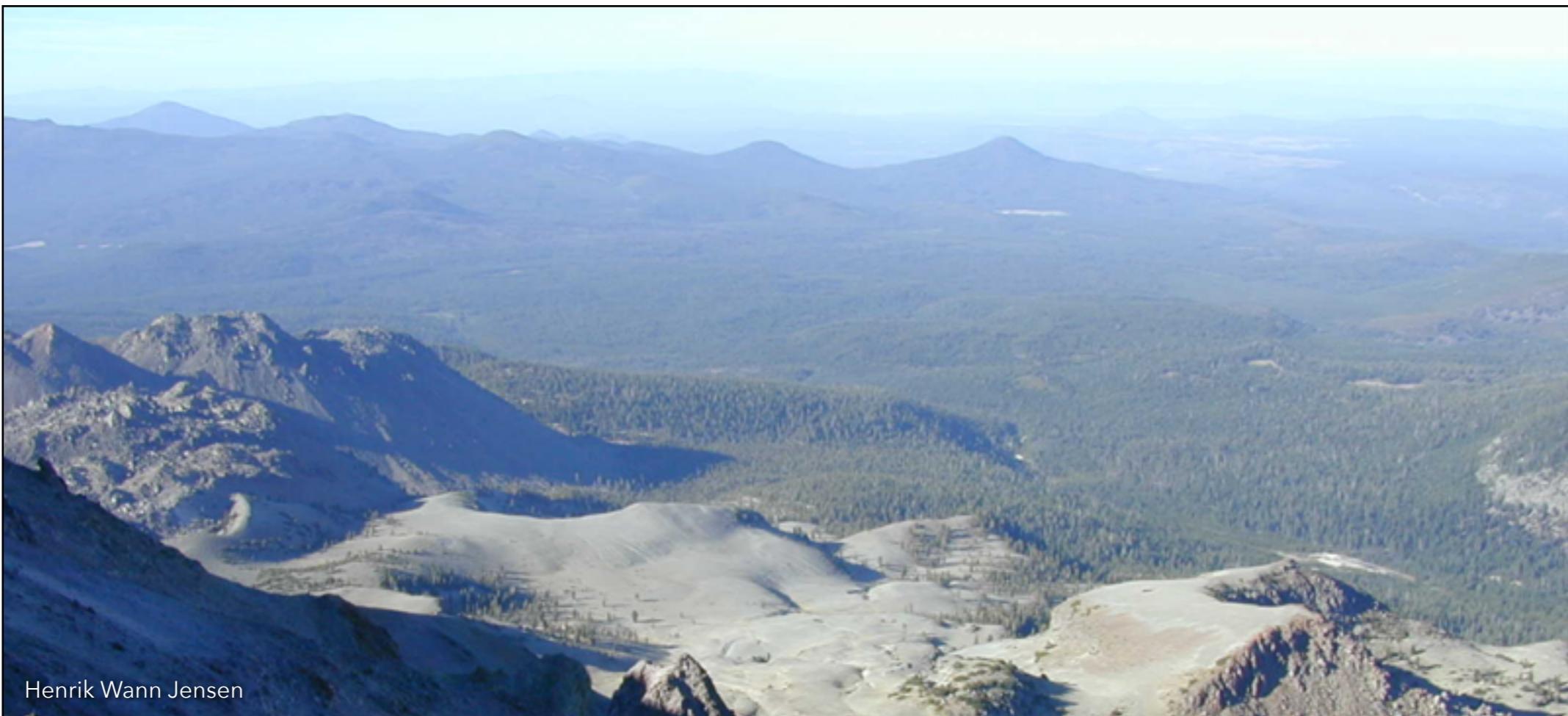
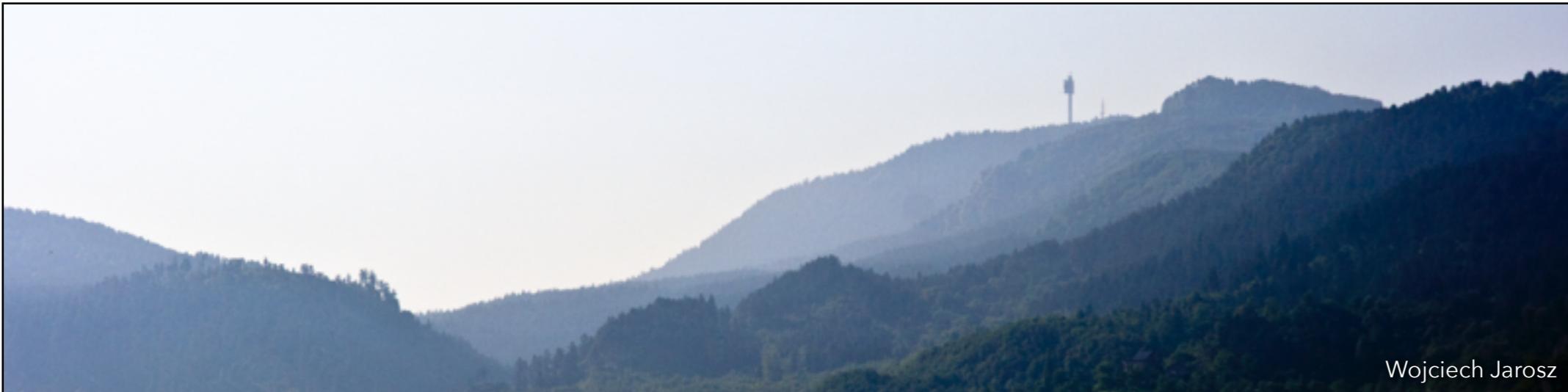


source: Flickr

# Antelope Canyon, Az.



# Aerial (Atmospheric) Perspective



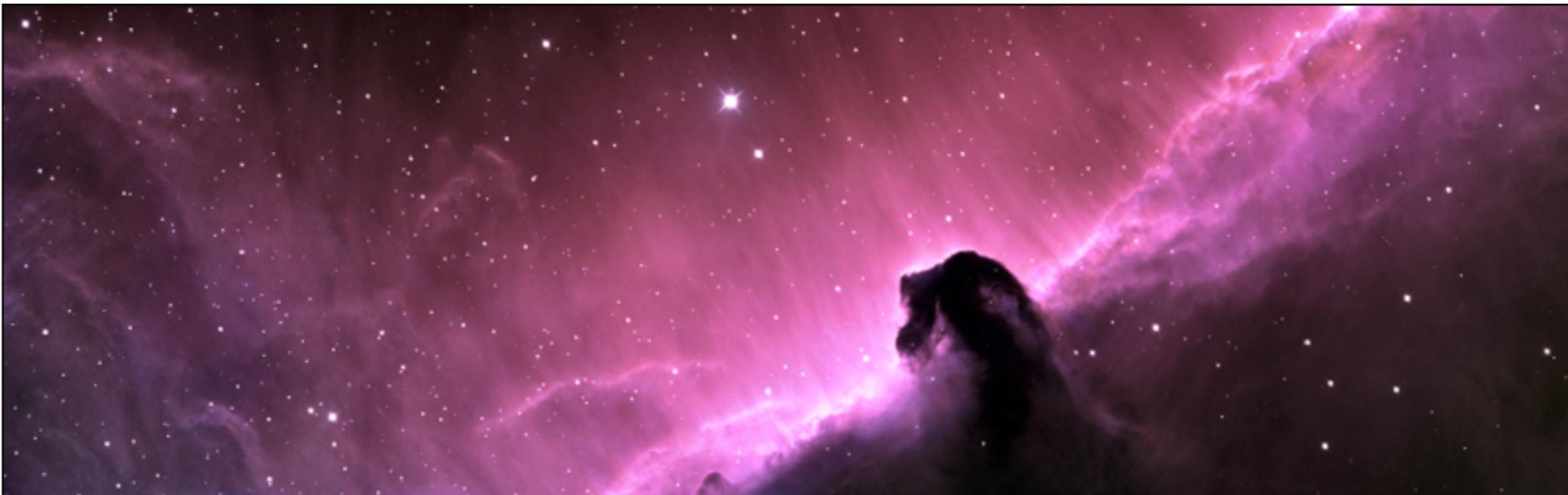
# Leonardo da Vinci (1480)



"Thus, if one is to be five times as distant, make it five times bluer."

—*Treatise on Painting*, Leonardo Da Vinci, pp 295, circa 1480.

# Nebula



T.A.Rector (NOAO/AURA/NSF) and the Hubble Heritage Team (STScI/AURA/NASA)

# Emission



<http://wikipedia.org>

# Absorption



<http://commons.wikimedia.org>

# Scattering



<http://coclouds.com>

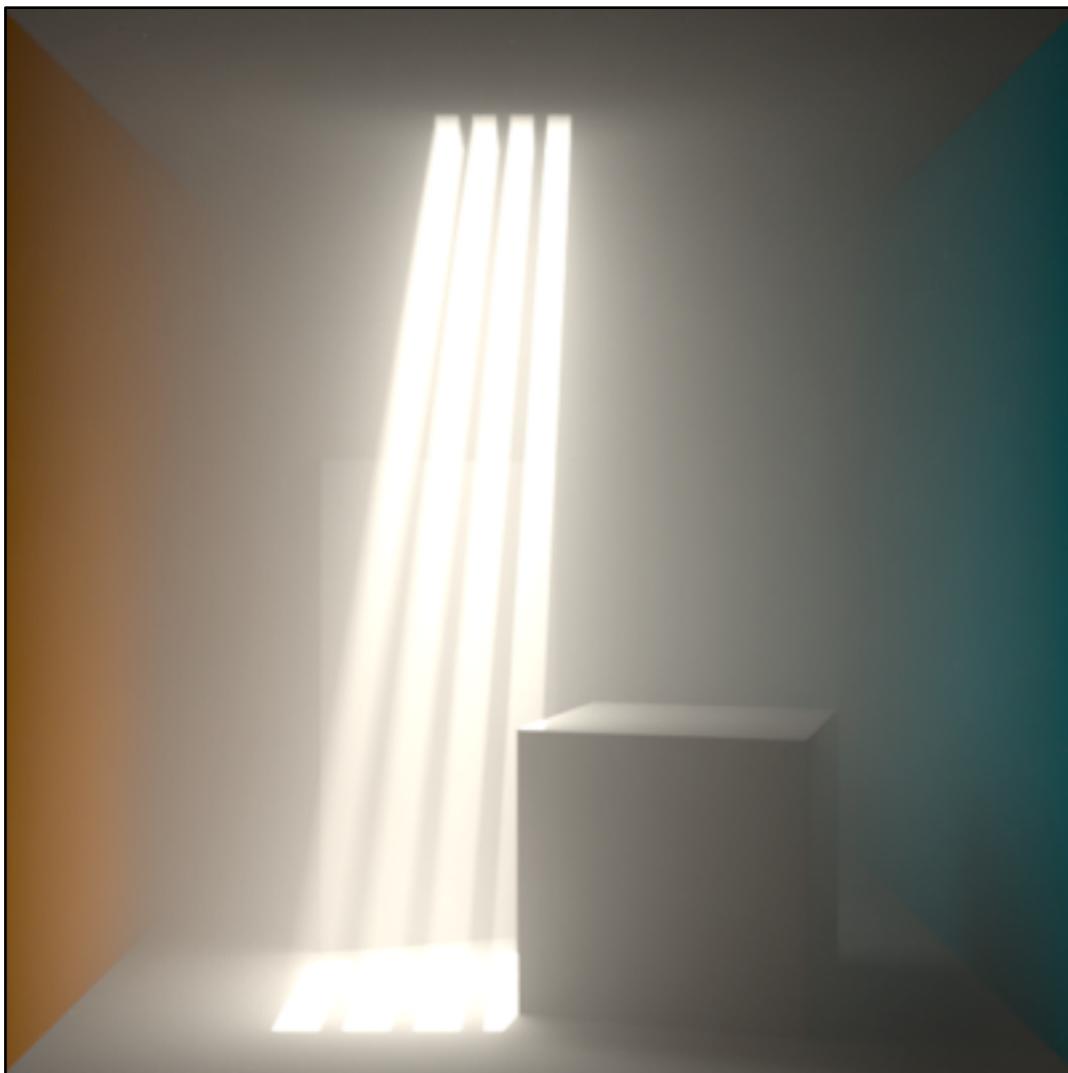
# Defining Participating Media

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- Typically, we do not model particles of a medium explicitly (wouldn't fit in memory)
- The properties are described statistically using various coefficients and densities
  - Conceptually similar idea as microfacet models

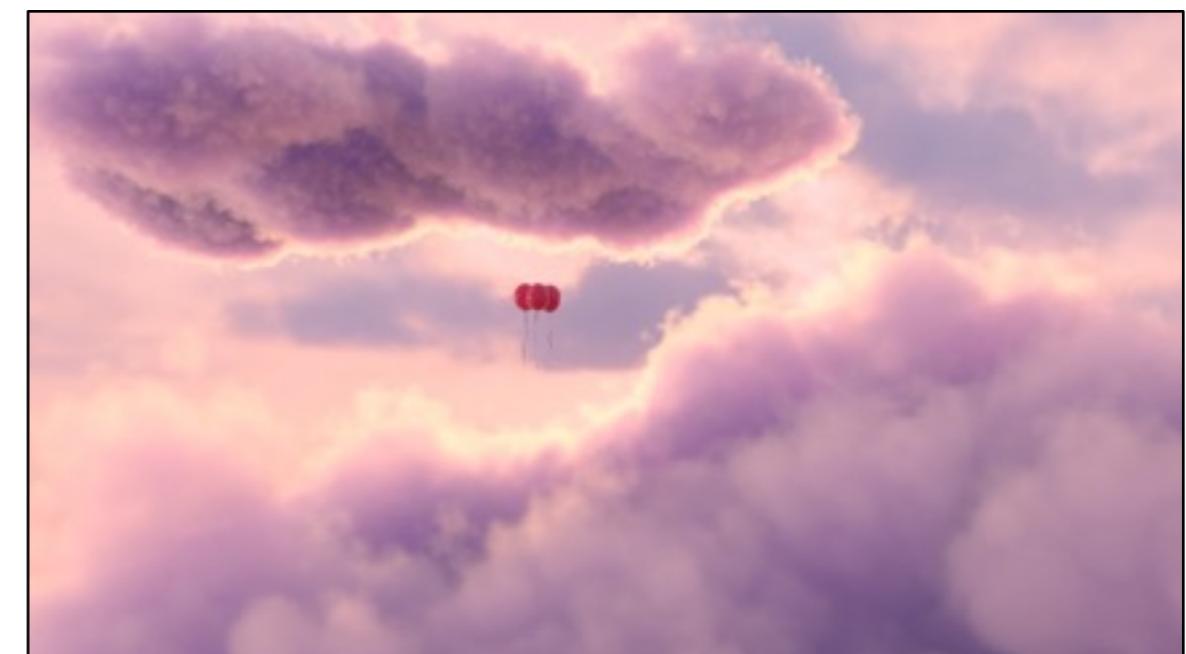
# Defining Participating Media

- Homogeneous:
  - Infinite or bounded by a surface



# Defining Participating Media

- Homogeneous:
  - Infinite or bounded by a surface
- Heterogeneous (spatially varying coefficients):
  - Procedurally e.g. using a noise function
  - Simulation + volume discretization, e.g. a voxel



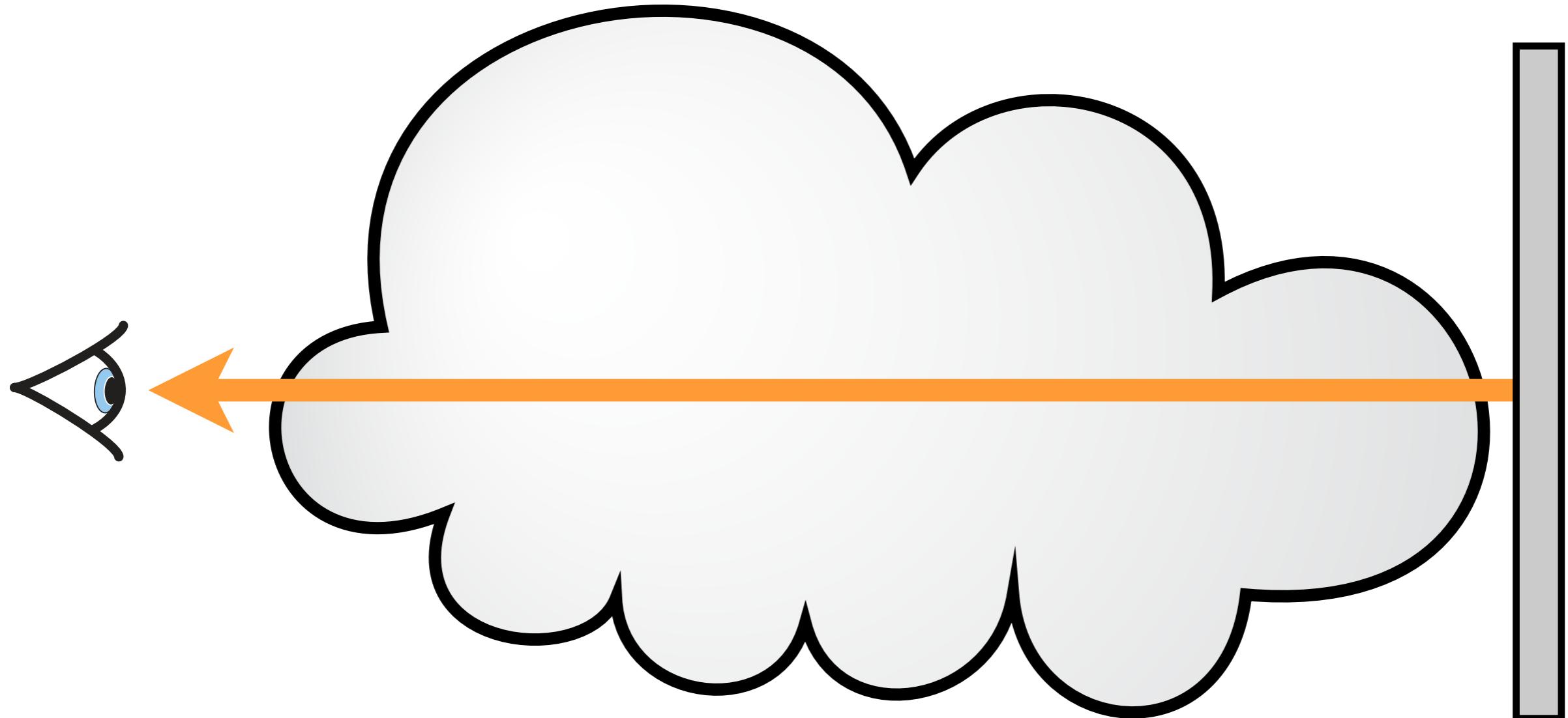
# Radiance

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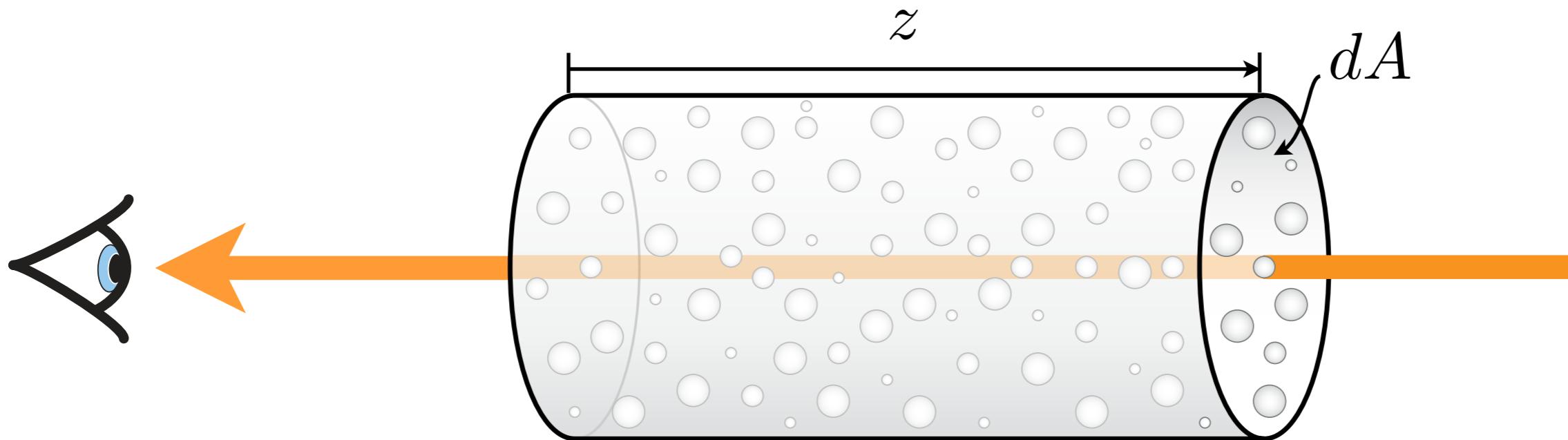
- The main quantity we are interested in for rendering is radiance
- Radiance *changes* along the ray if it goes through a participating medium

$$L_i(\mathbf{x}, \vec{\omega}) \neq L_o(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$$

# Participating Media

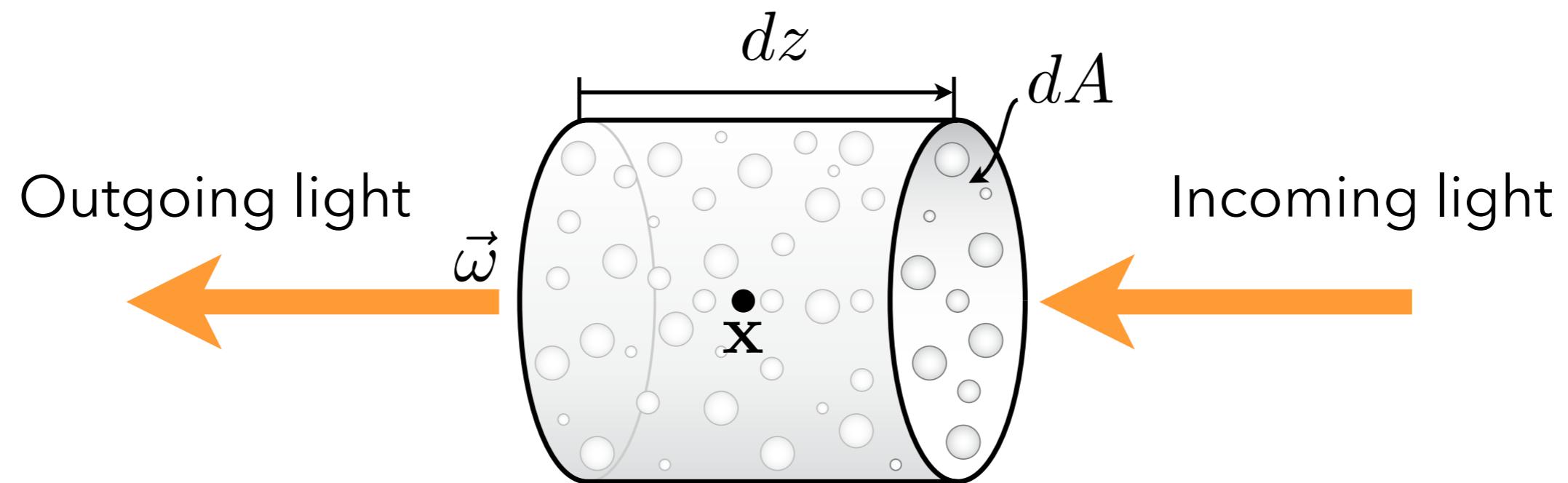


# Differential Beam

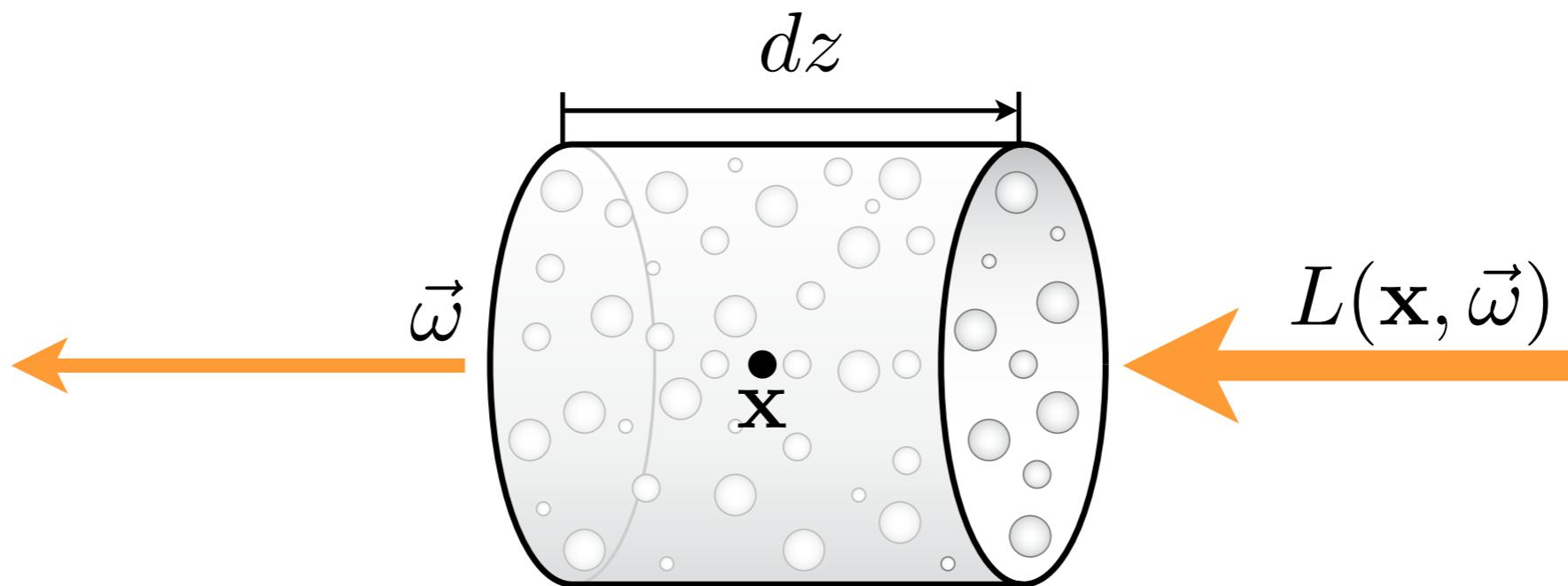


How much light is *lost/gained* along the differential beam due to interactions of light with the medium?

# Differential Beam Segment



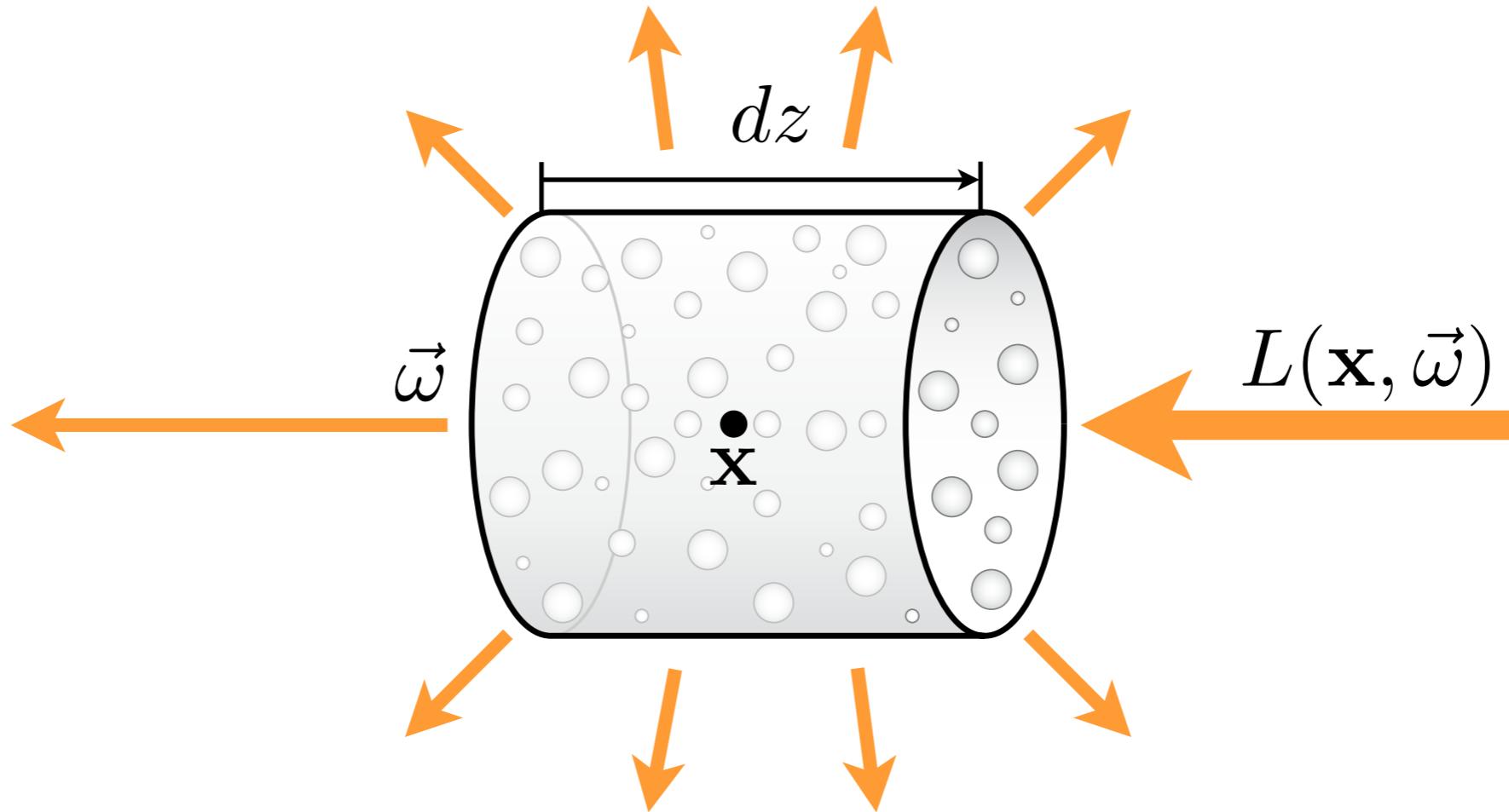
# Absorption



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz$$

$\sigma_a(\mathbf{x})$  : absorption coefficient  $[m^{-1}]$

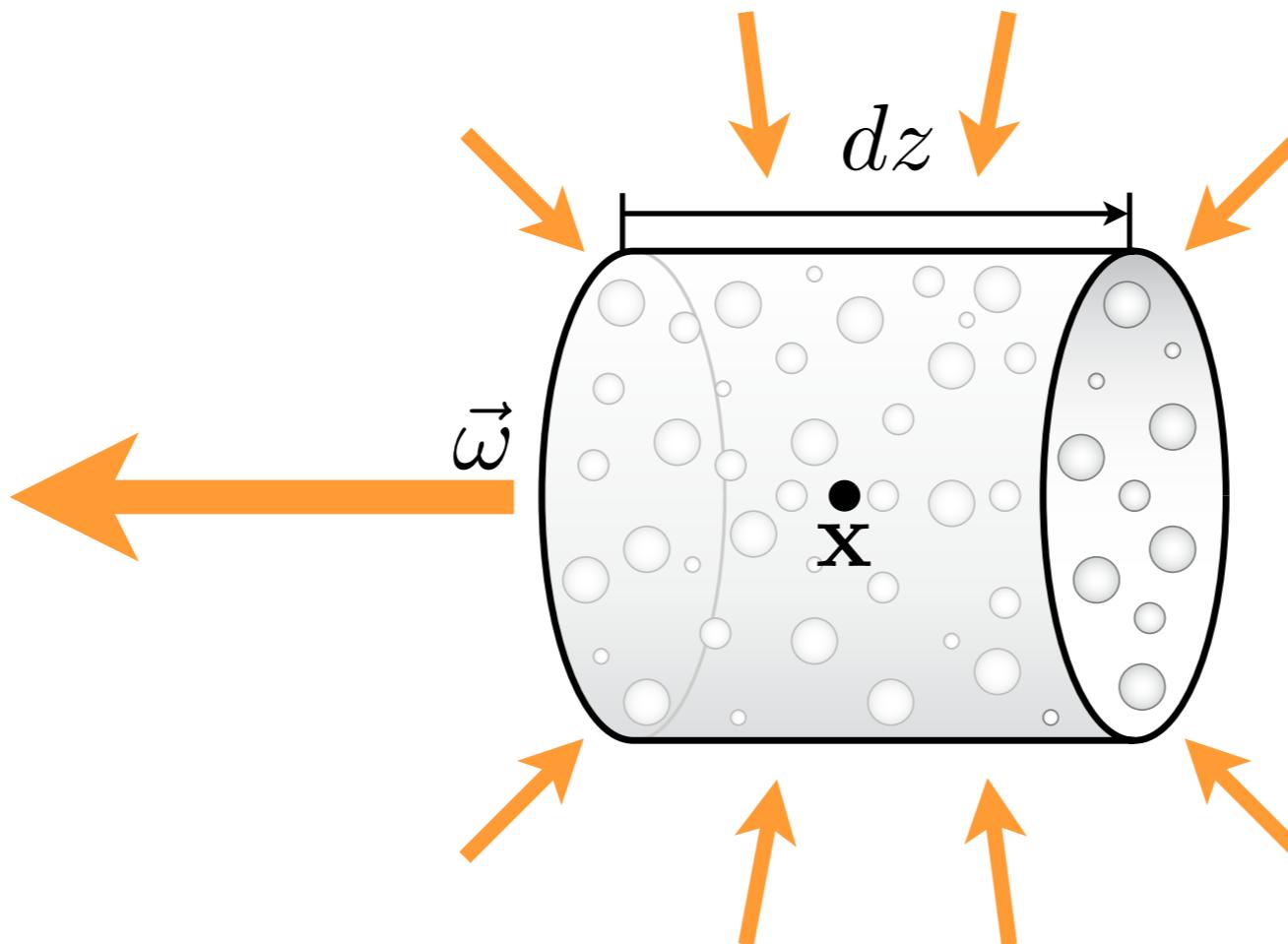
# Out-scattering



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_s(\mathbf{x}) L(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$  : scattering coefficient  $[m^{-1}]$

# In-scattering

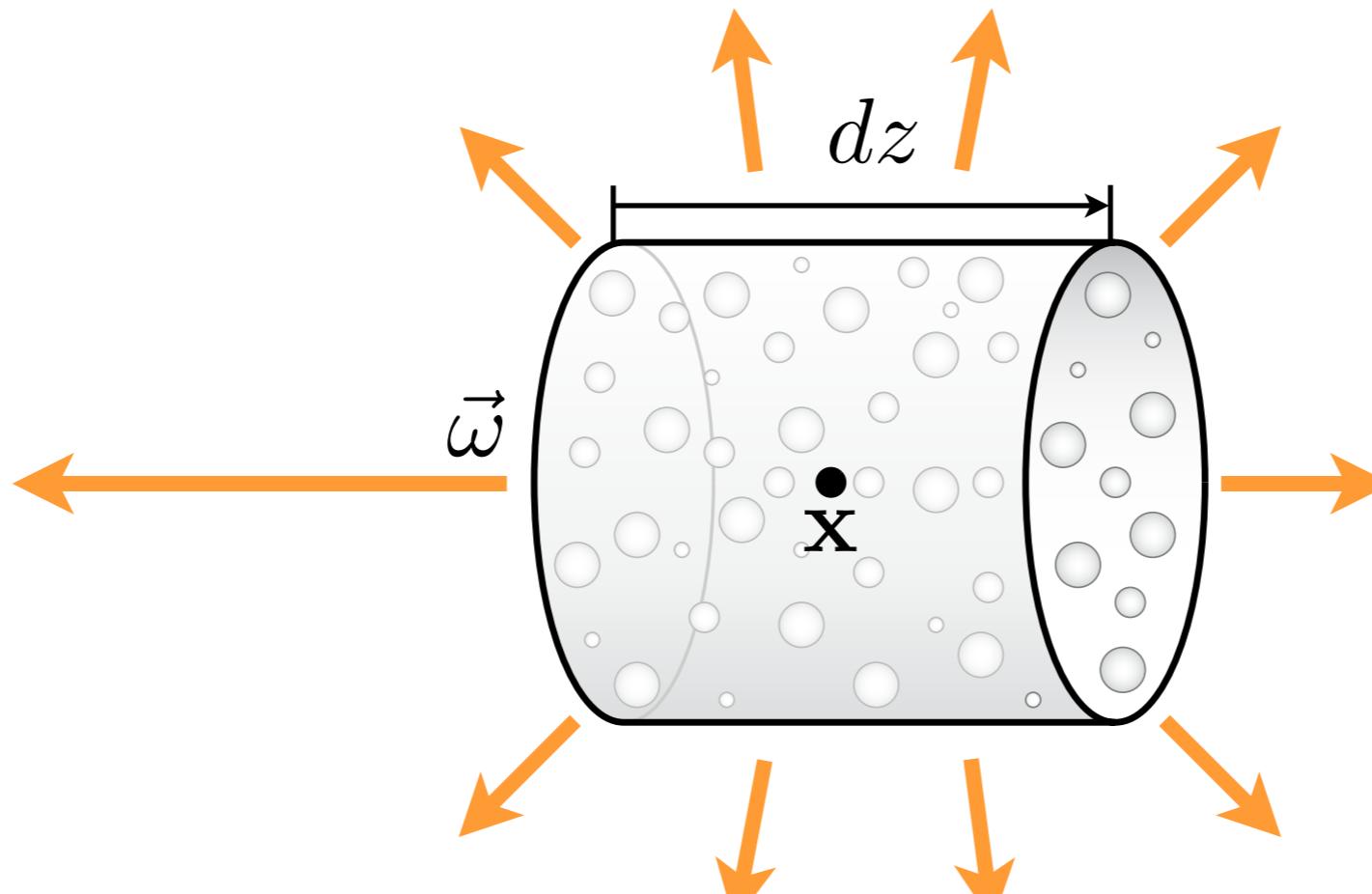


$$dL(\mathbf{x}, \vec{\omega}) = \sigma_s(\mathbf{x}) L_s(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$  : scattering coefficient  $[m^{-1}]$

$L_s(\mathbf{x}, \vec{\omega})$  : in-scattered radiance

# Emission



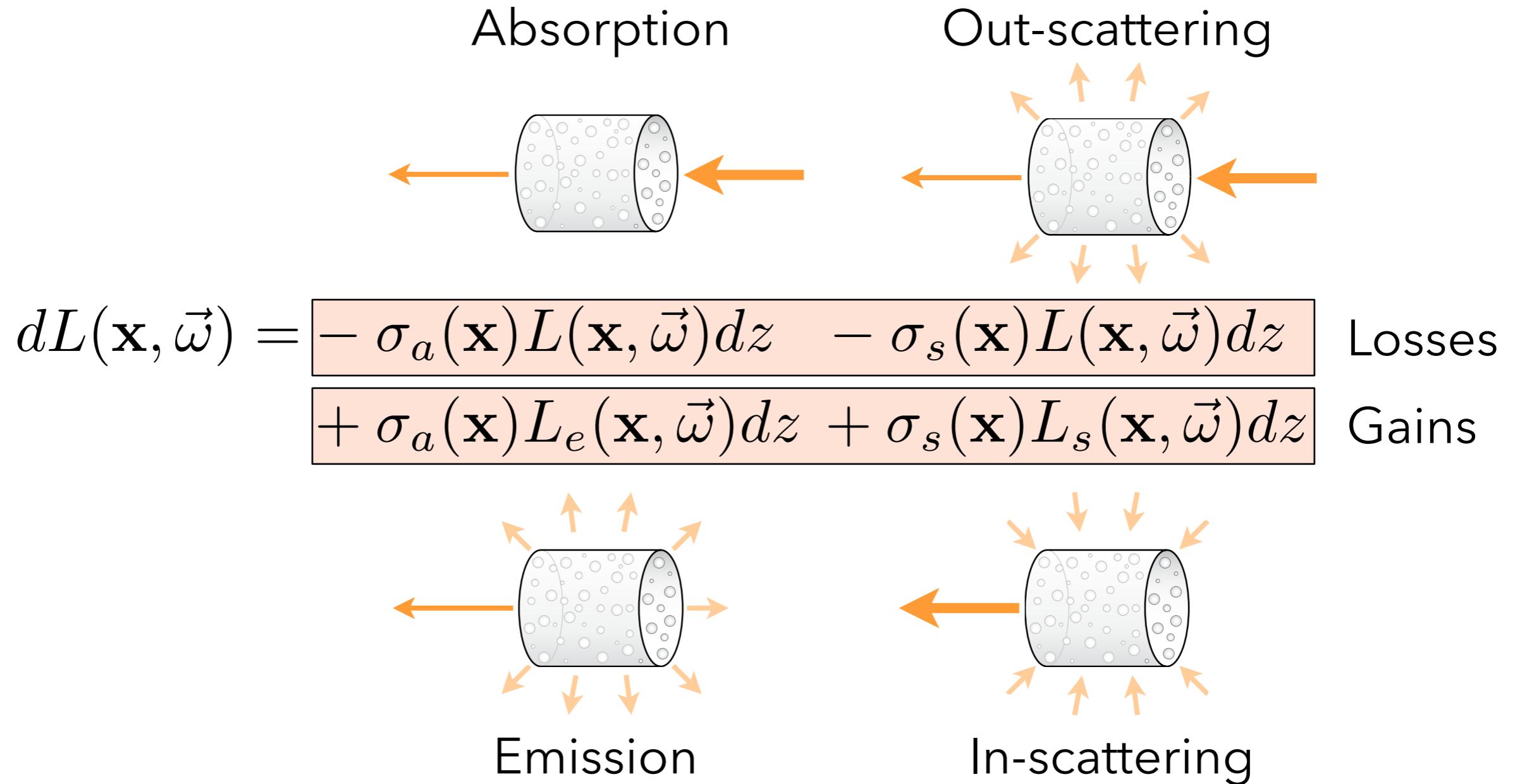
$$dL(\mathbf{x}, \vec{\omega}) = \sigma_a(\mathbf{x}) L_e(\mathbf{x}, \vec{\omega}) dz$$

\*Sometimes modeled without the absorption coefficient just by specifying a “source” term

$\sigma_a(\mathbf{x})$  : absorption coefficient  $[m^{-1}]$

$L_e(\mathbf{x}, \vec{\omega})$  : emitted radiance

# Radiative Transport Equation



What about a beam with a finite length?

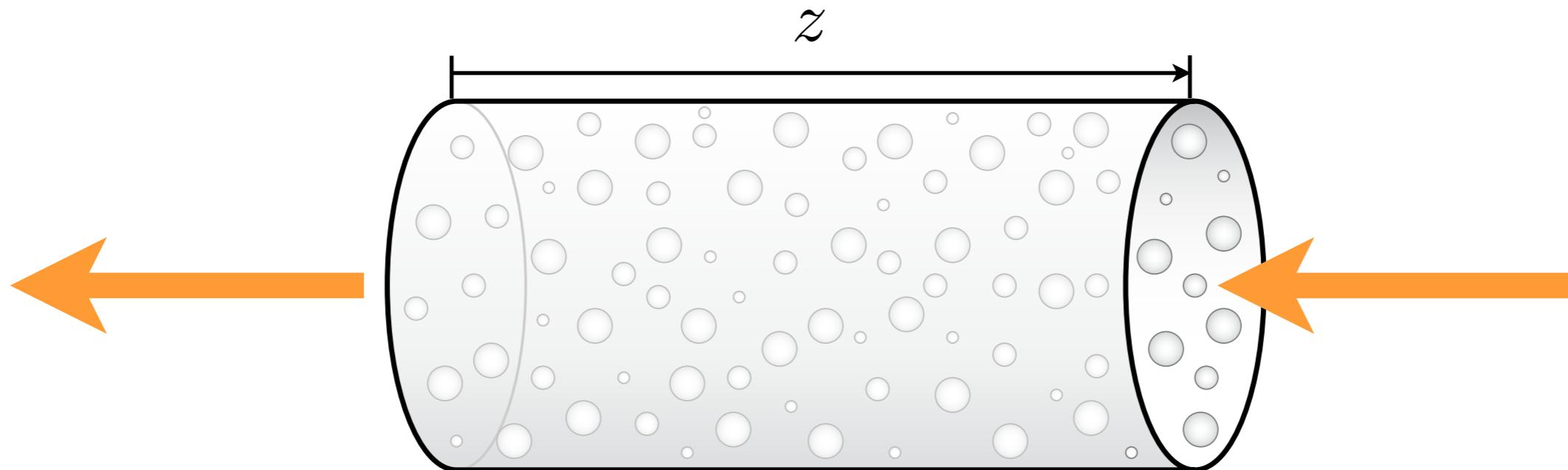
# Losses (Extinction)

Absorption

$$\begin{aligned} dL(\mathbf{x}, \vec{\omega}) &= -\sigma_a(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz - \sigma_s(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \\ &= -\sigma_t(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \end{aligned}$$

Out-scattering

$\sigma_t(\mathbf{x})$  : extinction coefficient [ $m^{-1}$ ]  
: total loss of light per unit distance



# Extinction Along a Finite Beam

$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_t(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz \quad // \text{Assume constant } \sigma_t(\mathbf{x}), \text{ reorganize}$$

$$\frac{dL(\mathbf{x}, \vec{\omega})}{L(\mathbf{x}, \vec{\omega})} = -\sigma_t dz \quad // \text{Integrate along beam from 0 to } z$$

$$\ln(L_z) - \ln(L_0) = -\sigma_t z$$

$$\ln\left(\frac{L_z}{L_0}\right) = -\sigma_t z \quad // \text{Exponentiate}$$

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

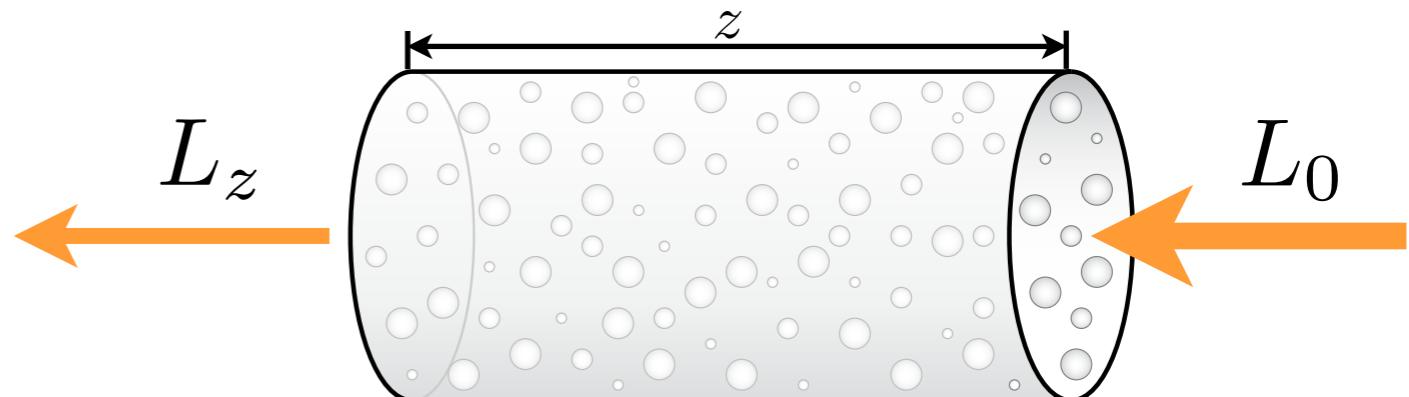
# Beer-Lambert Law

- Expresses the remaining radiance after traveling a finite distance through a medium with constant extinction coefficient
- The fraction is referred to as the *transmittance*
- Think of this as fractional visibility between points

Radiance at distance  $z$

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at the beginning  
of the beam



# Transmittance

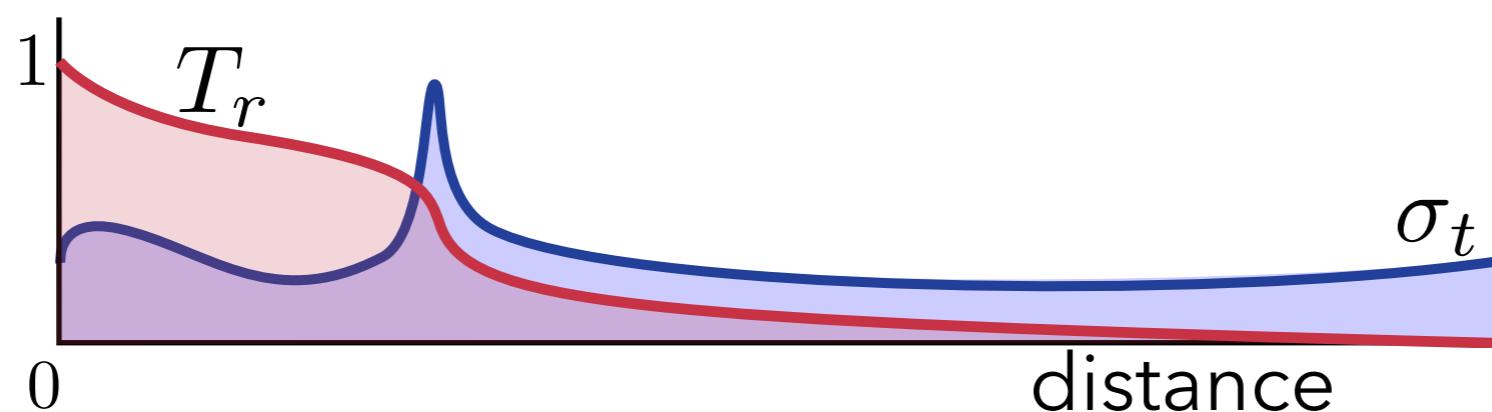
- Homogeneous volume:

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$$

- Heterogeneous volume (spatially varying  $\sigma_t$ ):

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x} - \mathbf{y}\|} \sigma_t(t) dt}$$

↑  
Optical thickness



# Transmittance

- Homogeneous volume:

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$$

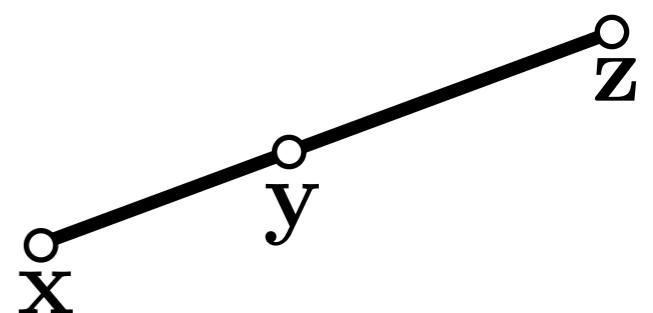
- Heterogeneous volume (spatially varying  $\sigma_t$ ):

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x} - \mathbf{y}\|} \sigma_t(t) dt}$$

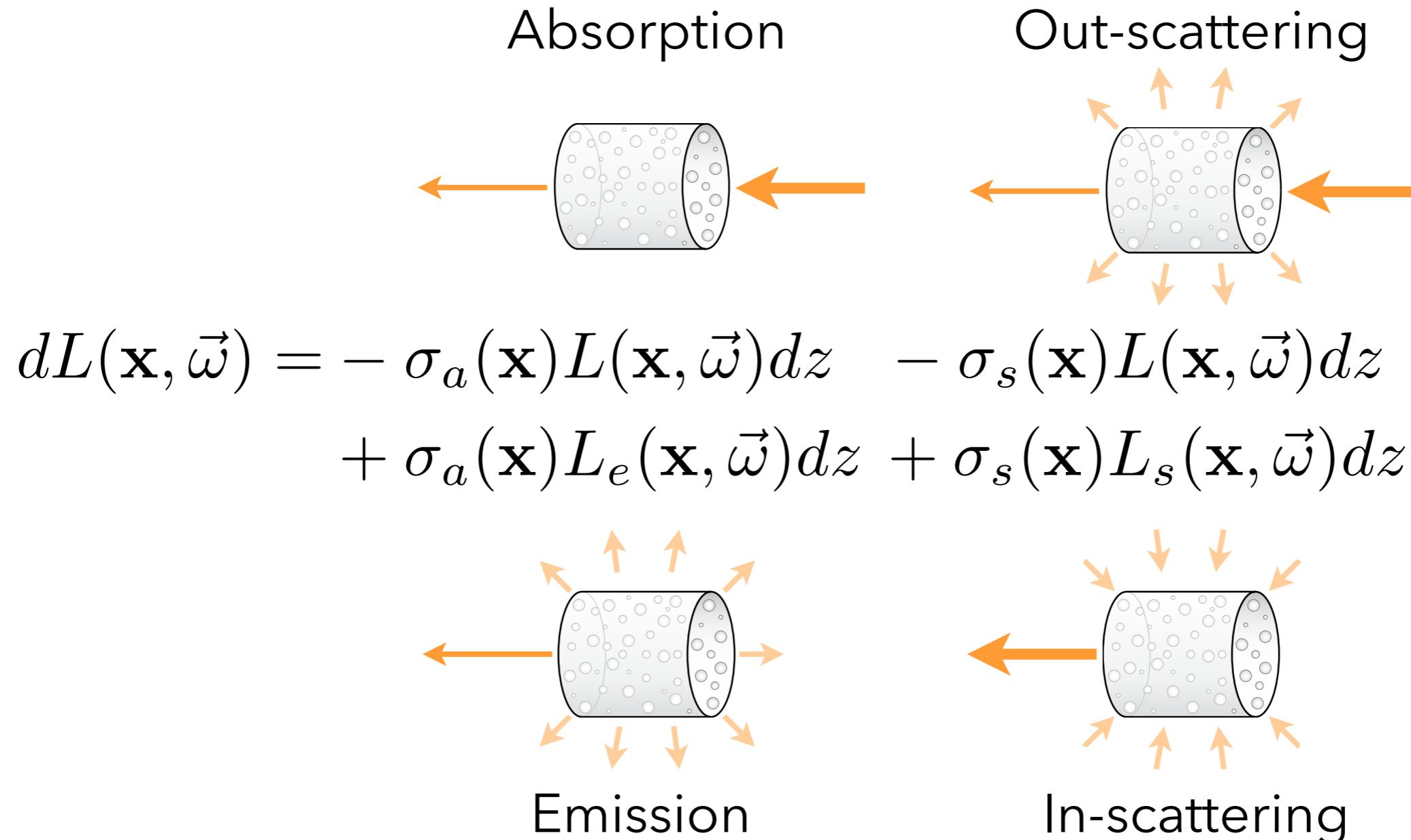
↑  
Optical thickness

- Transmittance is multiplicative:

$$T_r(\mathbf{x}, \mathbf{z}) = T_r(\mathbf{x}, \mathbf{y})T_r(\mathbf{y}, \mathbf{z})$$



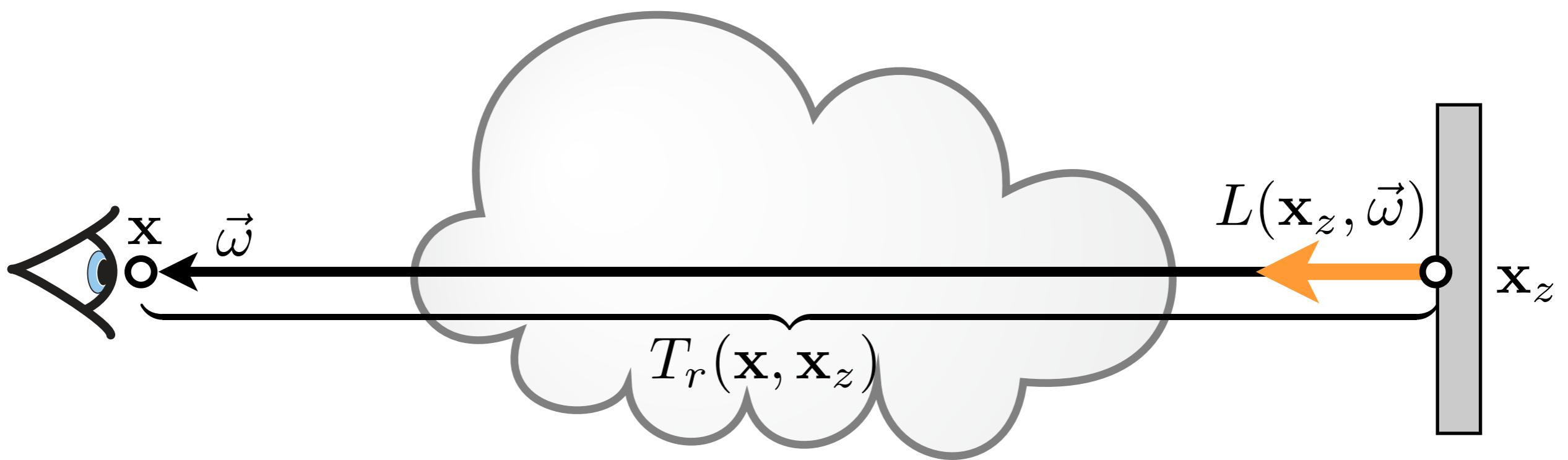
# Radiative Transport Equation



# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

↑  
Reduced (background) surface radiance

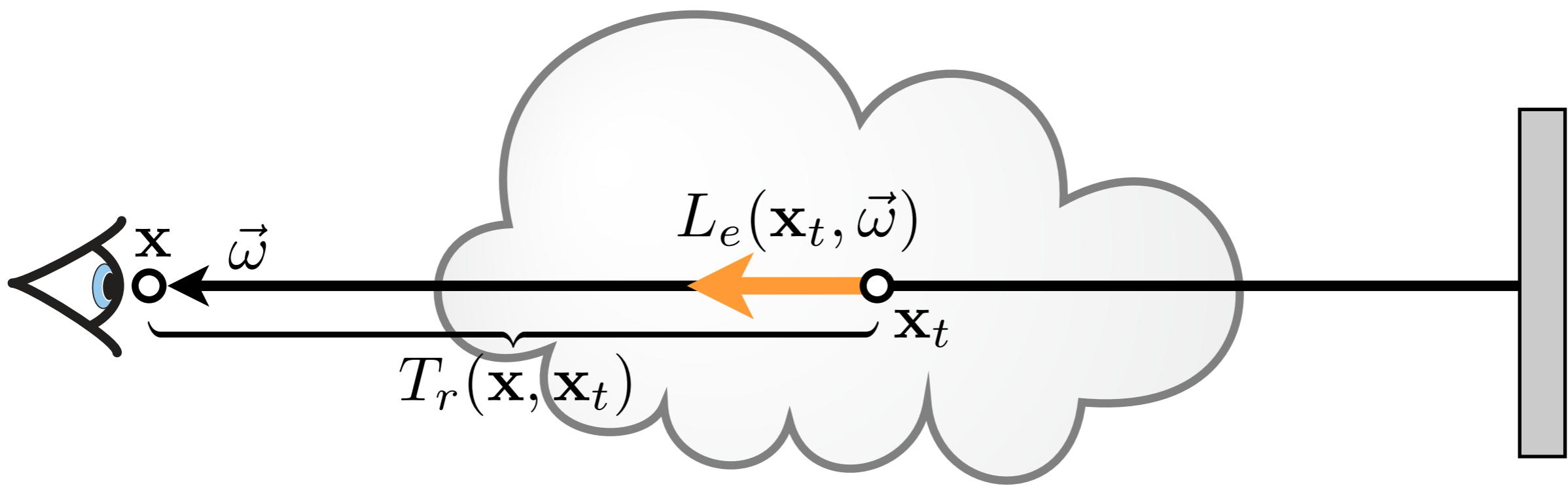


# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega})$$

$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

↑  
Accumulated emitted radiance

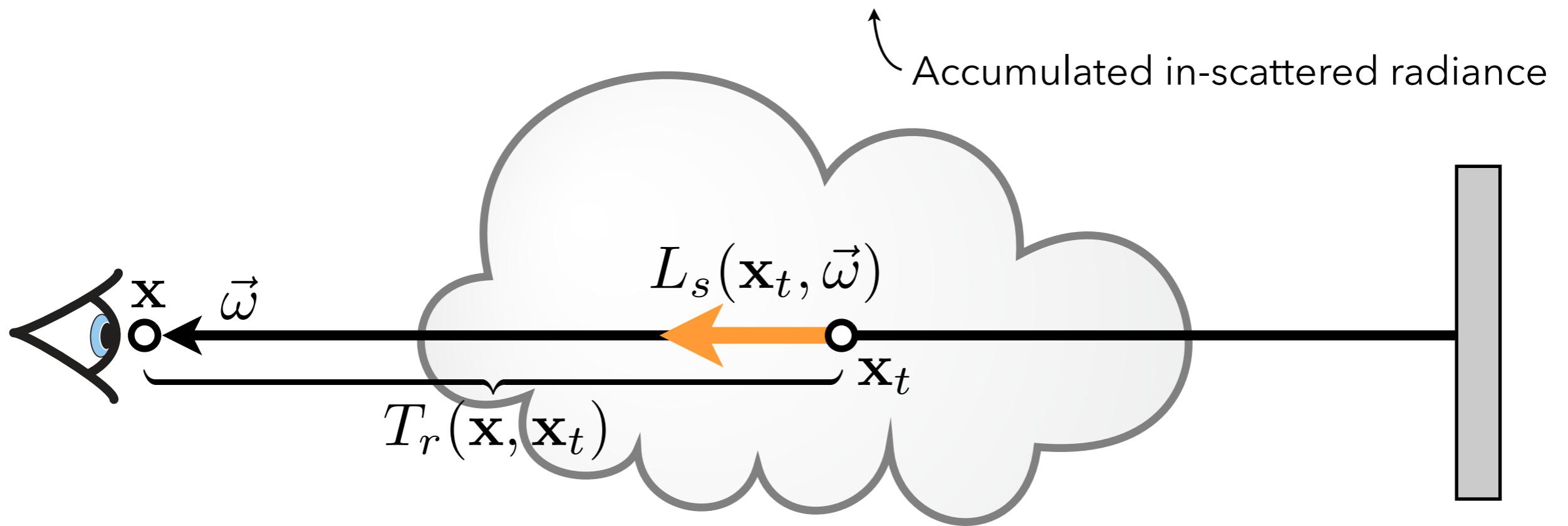


# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega})$$

$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

$$+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt}$$

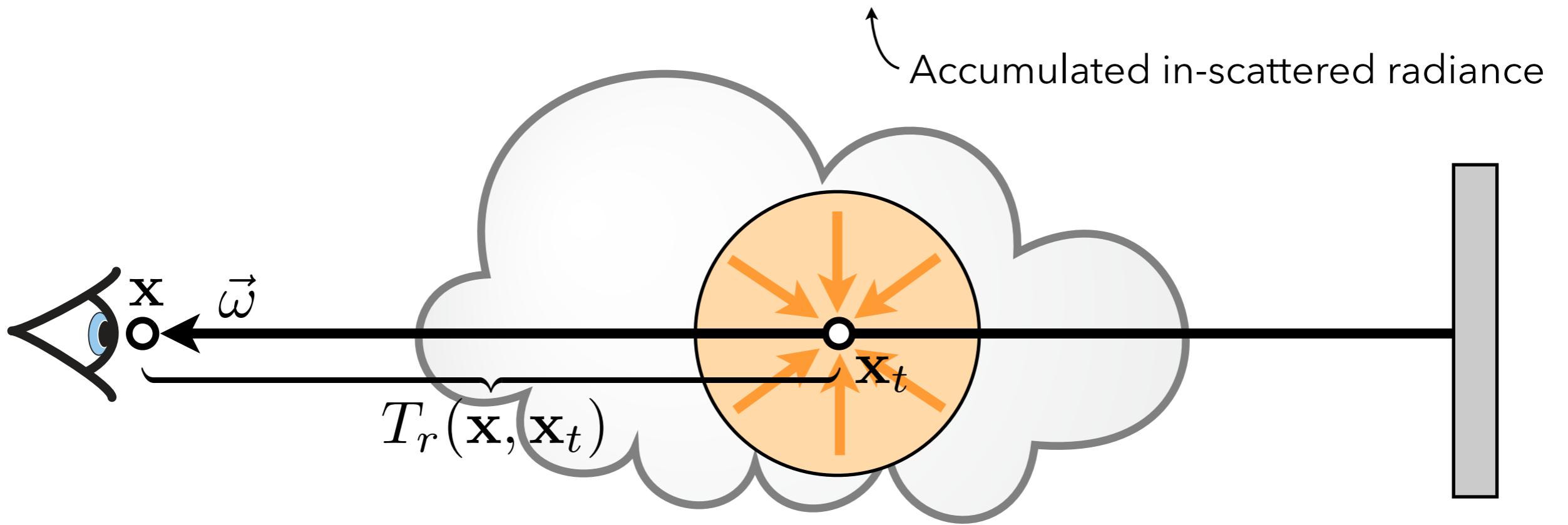


# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega})$$

$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt$$



# Volume Rendering Equation

$$\begin{aligned} L(\mathbf{x}, \vec{\omega}) = & T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\ & + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\ & + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt \end{aligned}$$

# Scattering in Media

# Phase Function $f_p$

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- Describes distribution of scattered light
- Analog of BRDF but for scattering in media
- Integrates to unity (unlike BRDF)

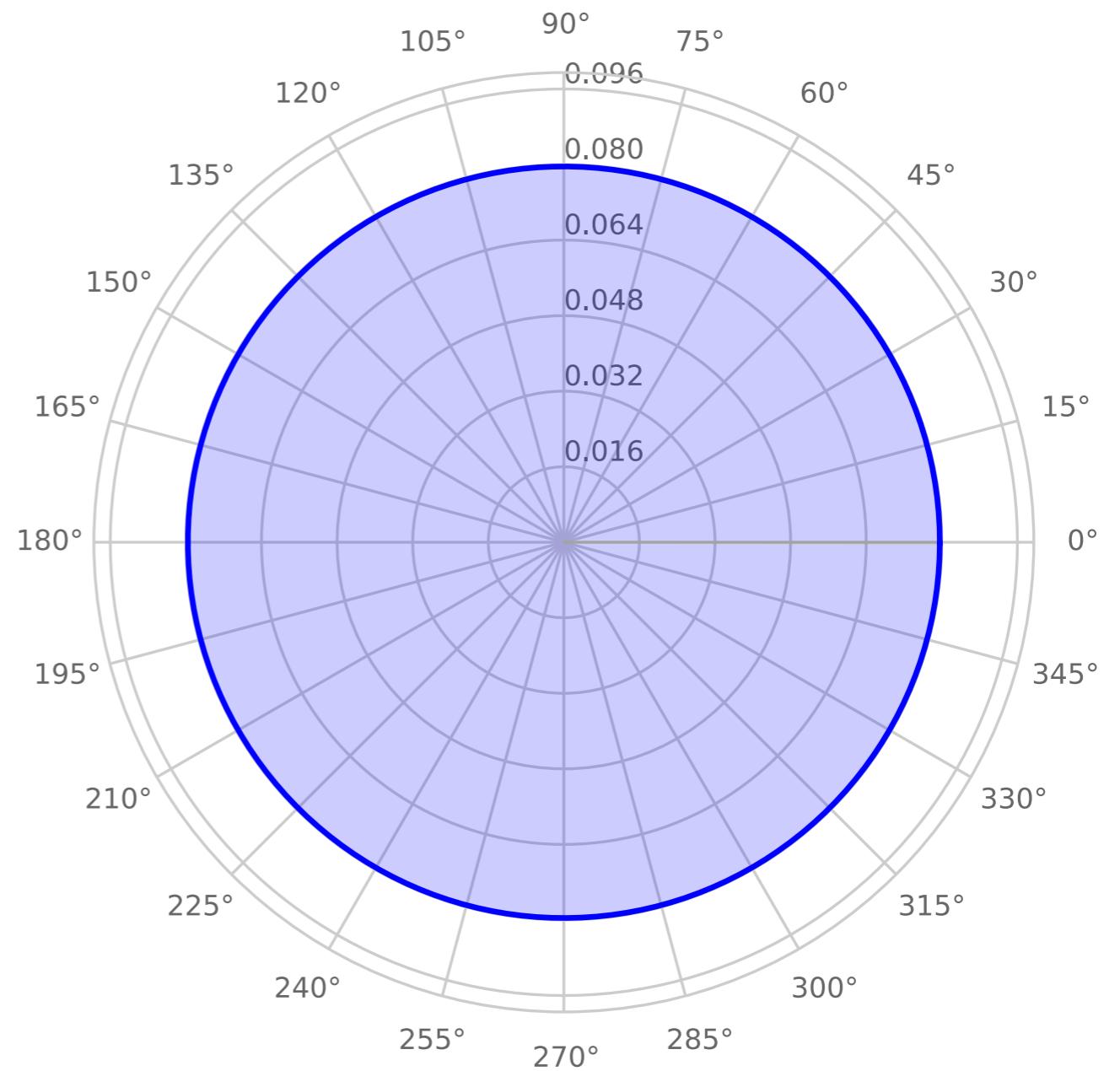
$$\int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) d\vec{\omega}' = 1$$

\*In this course, we use the same convention for phase functions as for BRDFs: direction vectors always *point away* from the shading point  $\mathbf{x}$ . Many publications, however, use a different convention for phase functions, in which direction vectors “follow” the light, i.e. one direction points *towards*  $\mathbf{x}$  and the other away from  $\mathbf{x}$ . When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.

# Isotropic Scattering

- Uniform scattering, analogous to Lambertian BRDF

$$f_p(\vec{\omega}', \vec{\omega}) = \frac{1}{4\pi}$$



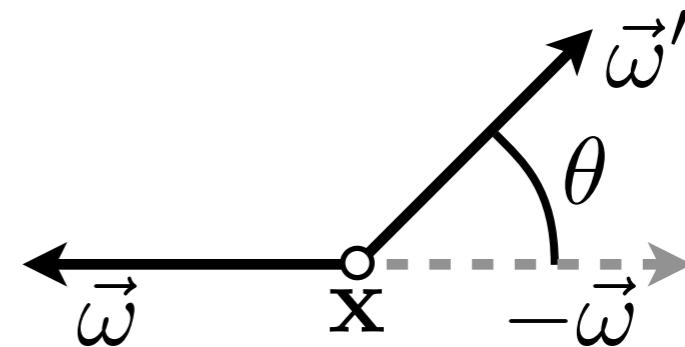
# Anisotropic Scattering

- Quantifying anisotropy ( $g$ , “average cosine”):

$$g = \int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) \cos \theta d\vec{\omega}'$$

- where:

$$\cos \theta = -\vec{\omega} \cdot \vec{\omega}'$$

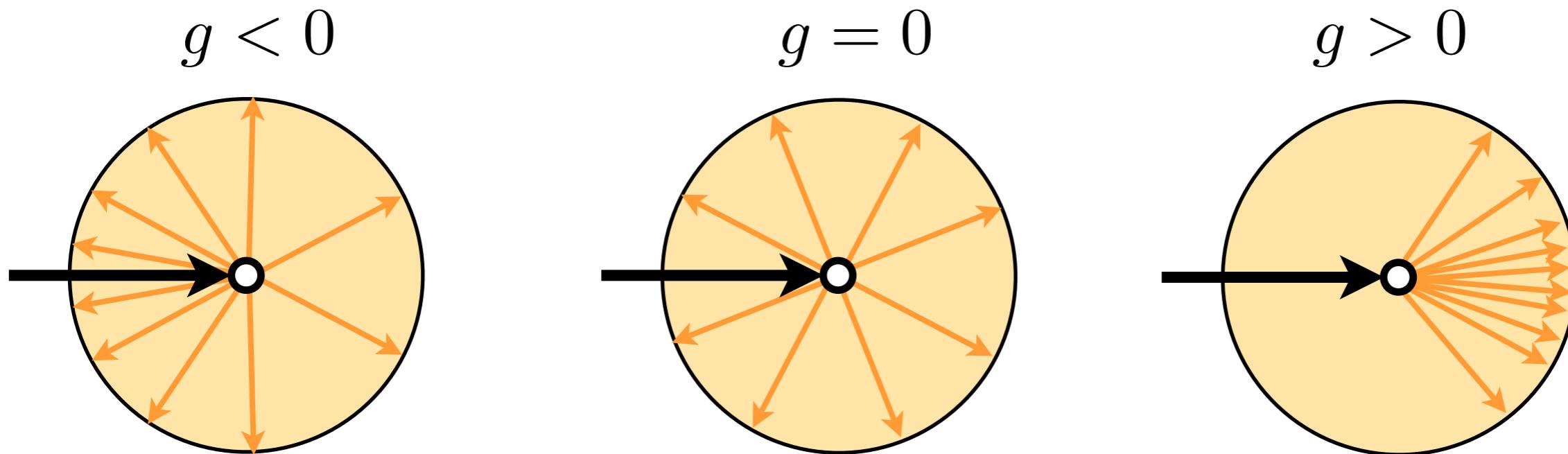


- $g = 0$ : isotropic scattering (on average)
- $g > 0$ : forward scattering
- $g < 0$ : backward scattering

# Henyey-Greenstein Phase Function

- Anisotropic scattering

$$f_{pHG}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}}$$

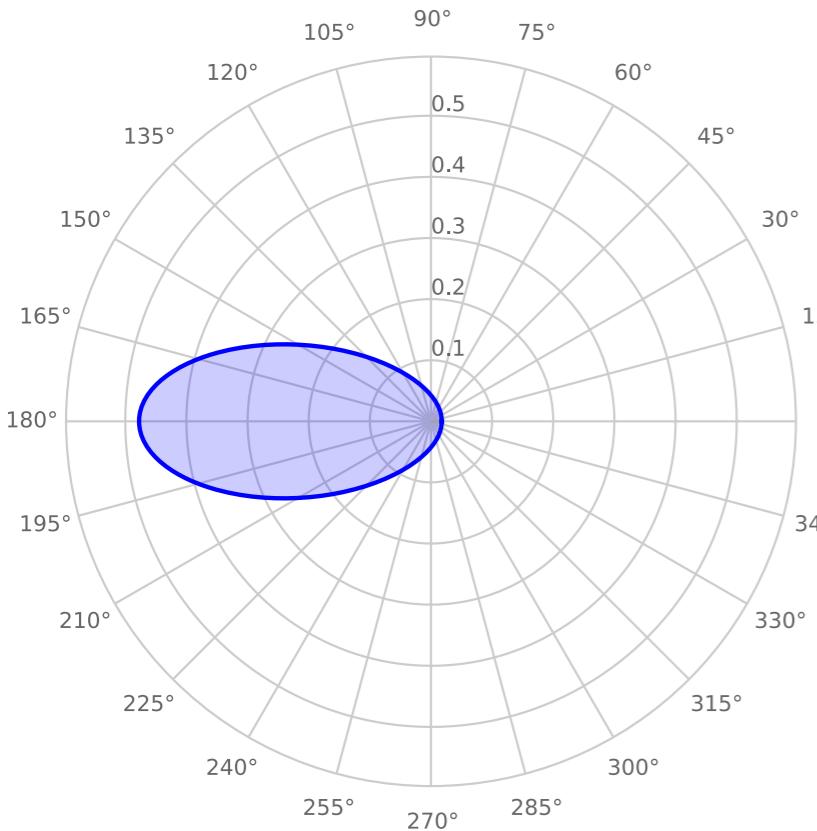


# Henyey-Greenstein Phase Function

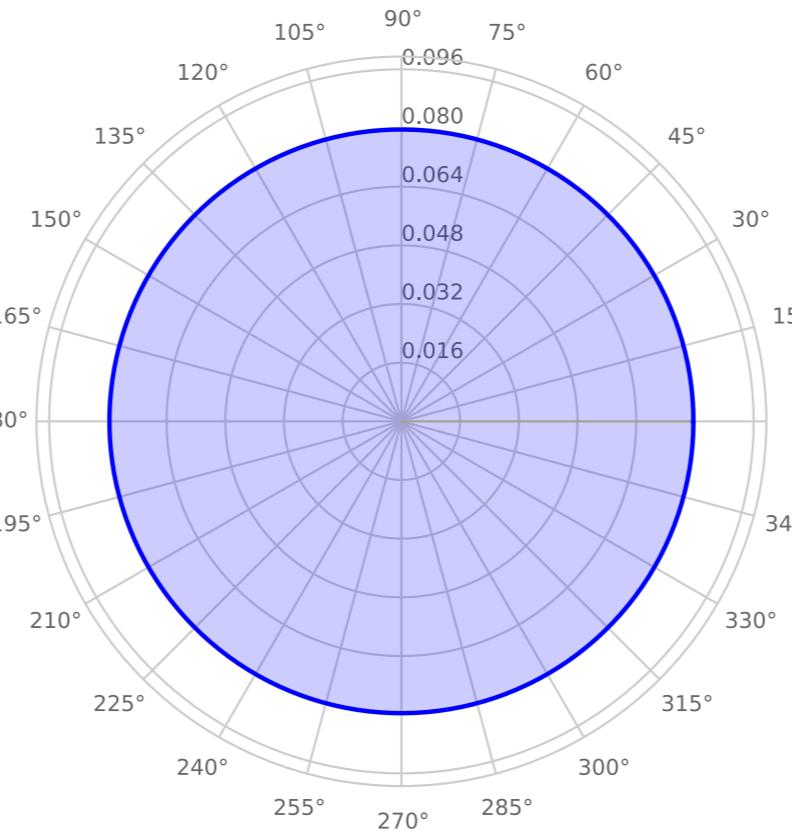
- Anisotropic scattering

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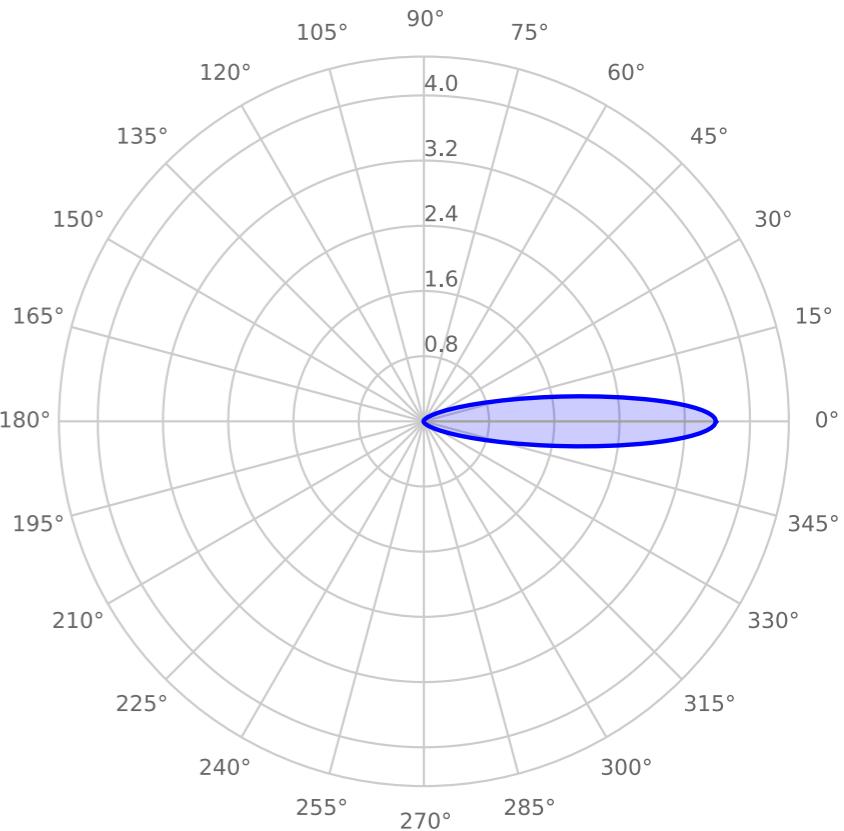
$g = -0.5$



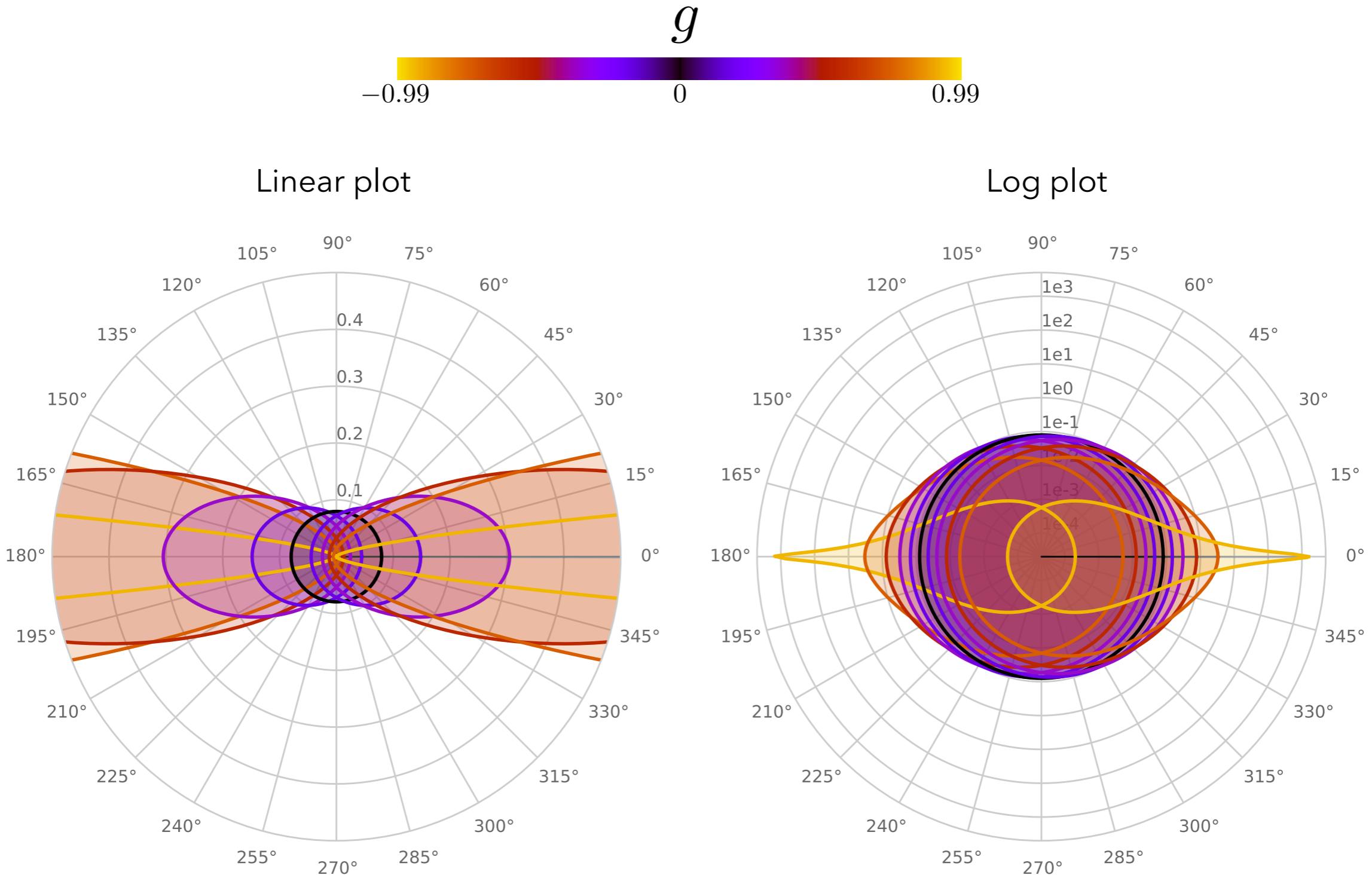
$g = 0$



$g = 0.8$



# Henyey-Greenstein Phase Function



# Henyey-Greenstein Phase Function

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- Empirical phase function
- Introduced for intergalactic dust
- Very popular in graphics and other fields
- Extensions with multiple HG lobes

# Schlick's Phase Function

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- Empirical phase function
- Faster approximation of HG

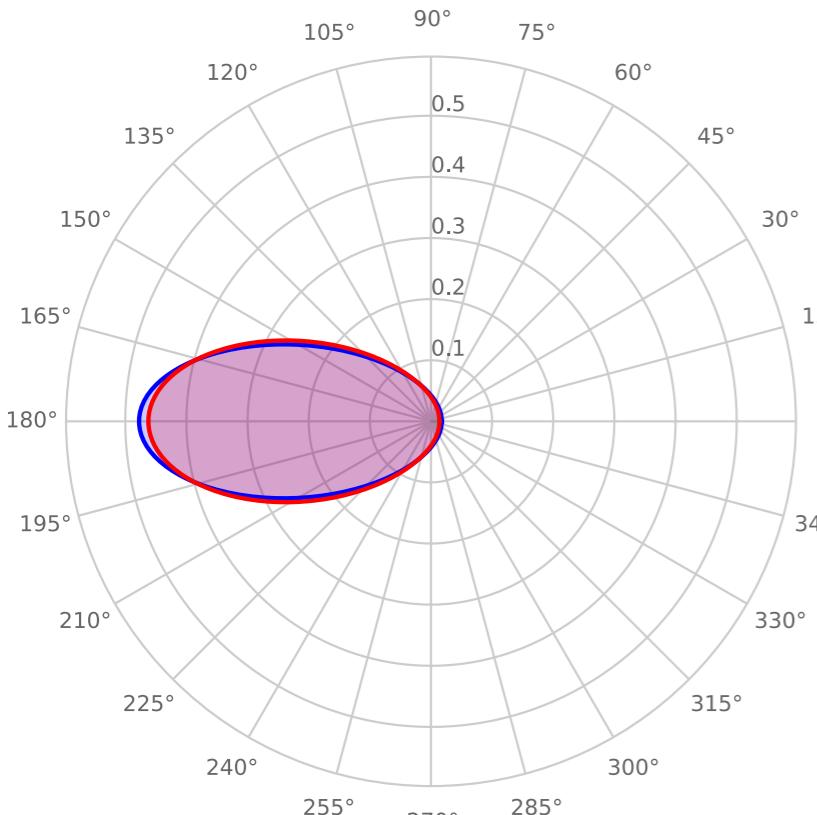
$$f_{p\text{Schlick}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k \cos \theta)^2}$$

$$k = 1.55g - 0.55g^3$$

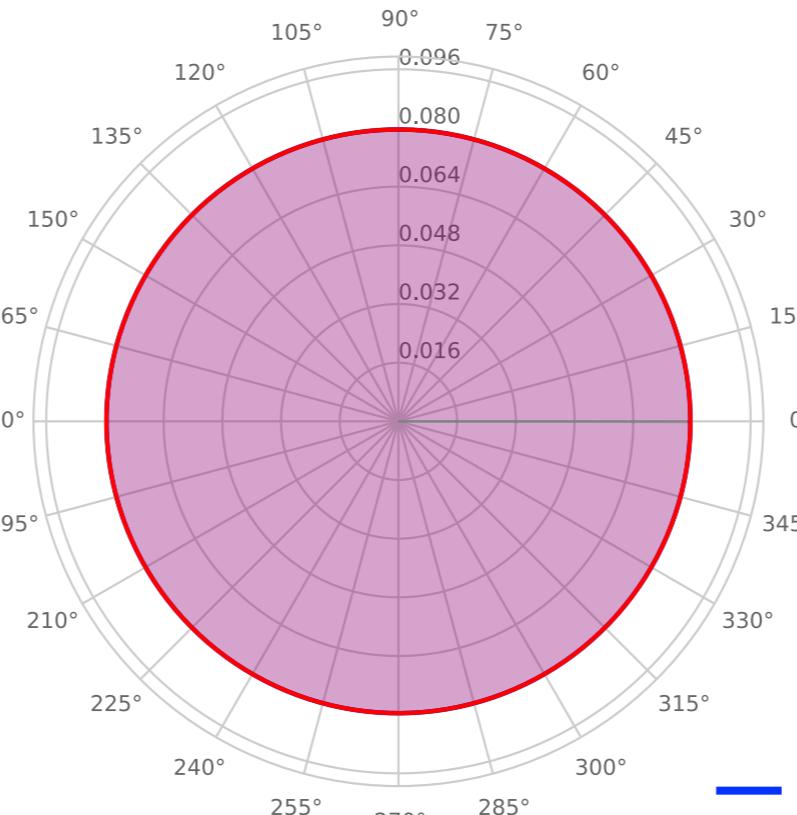
# Schllick's Phase Function

- Empirical phase function
- Cheap approximation of HG

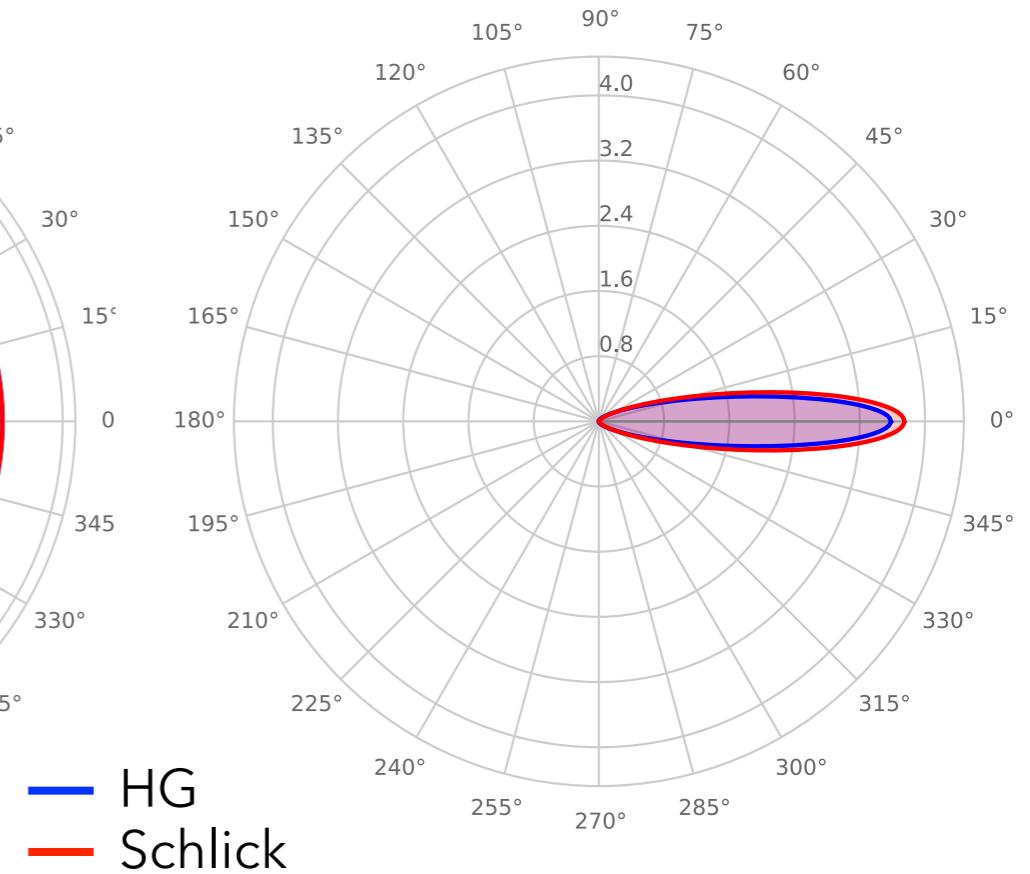
$$g = -0.5 \quad k = -0.706$$



$$g = 0 \quad k = 0$$



$$g = 0.8 \quad k = 0.96$$



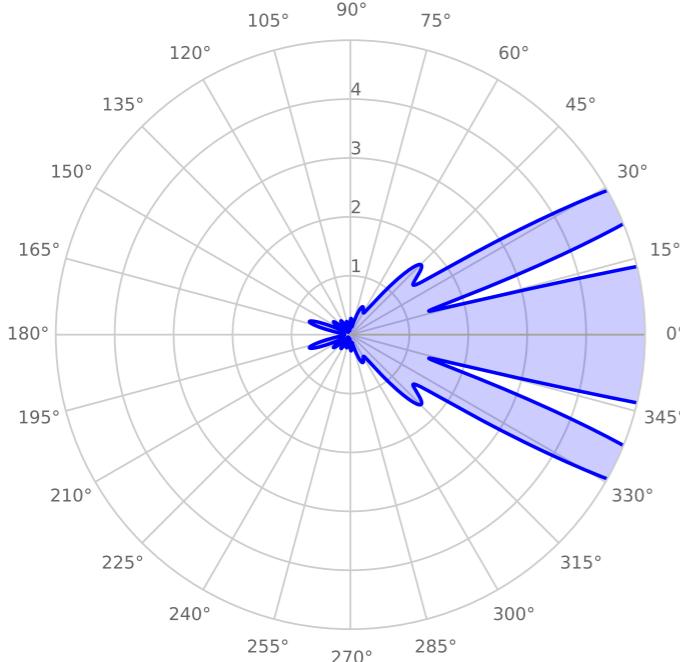
# Lorenz-Mie Scattering

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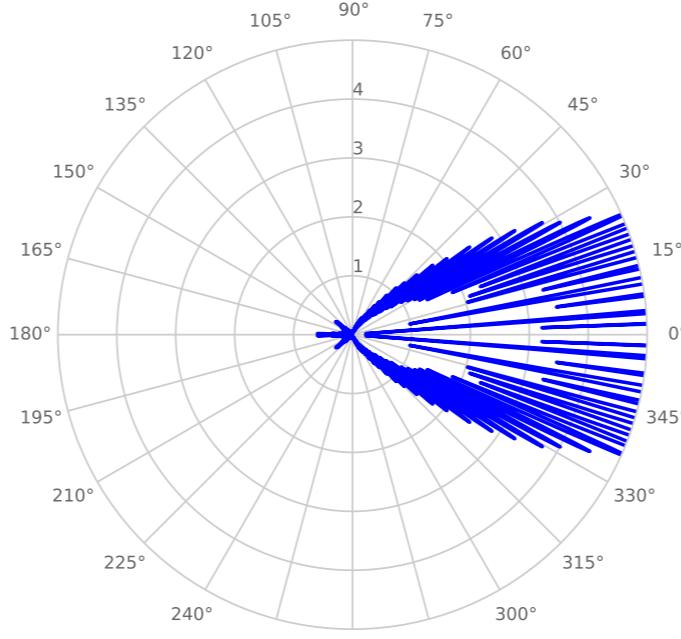
- If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light
- Solution to Maxwell's equations for scattering from any spherical dielectric particle
- Explains many phenomena
- Complicated:
  - Solution is an infinite analytic series

# Lorenz-Mie Phase Function

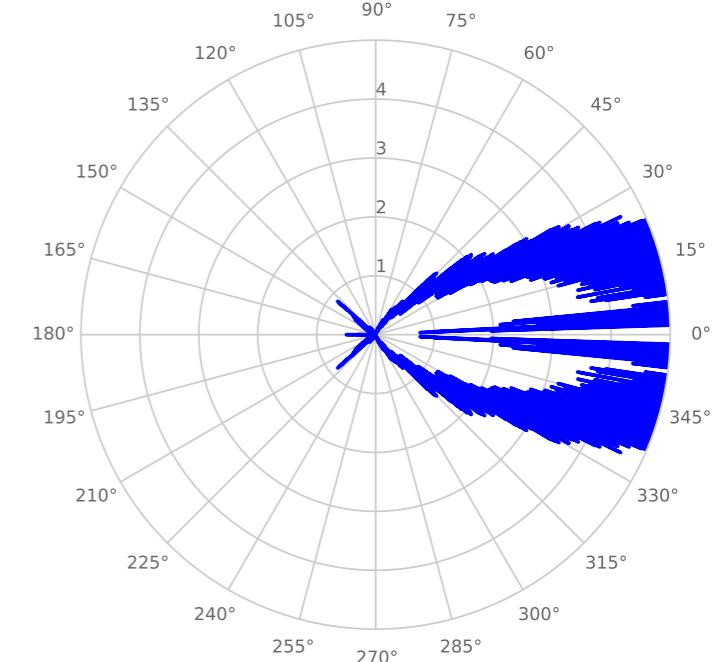
Sphere diameter =  $1\mu m$



Sphere diameter =  $10\mu m$

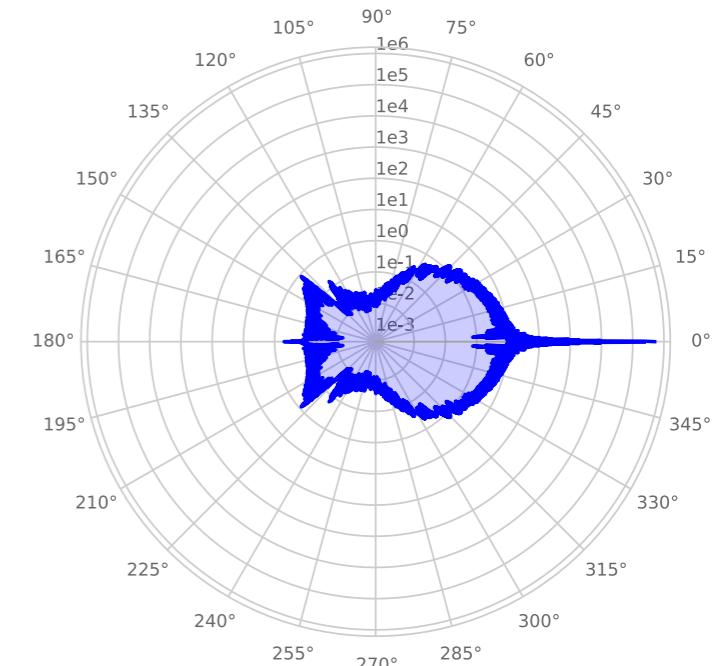
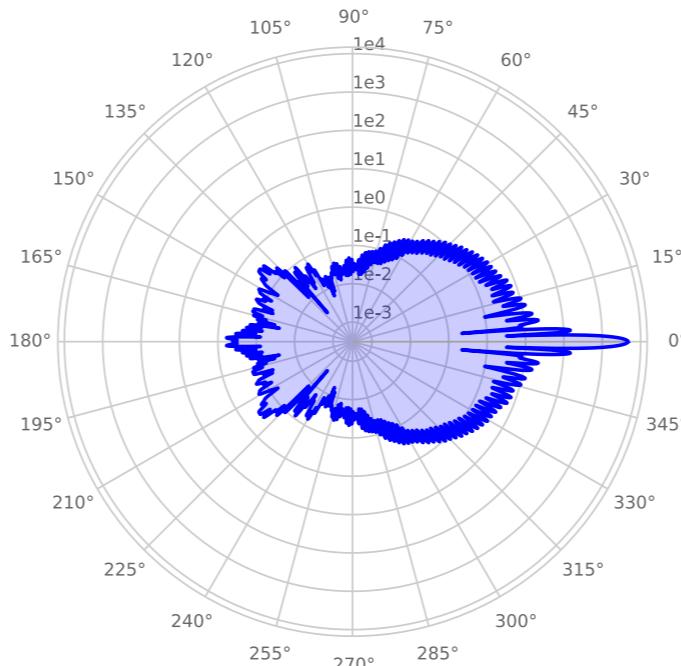
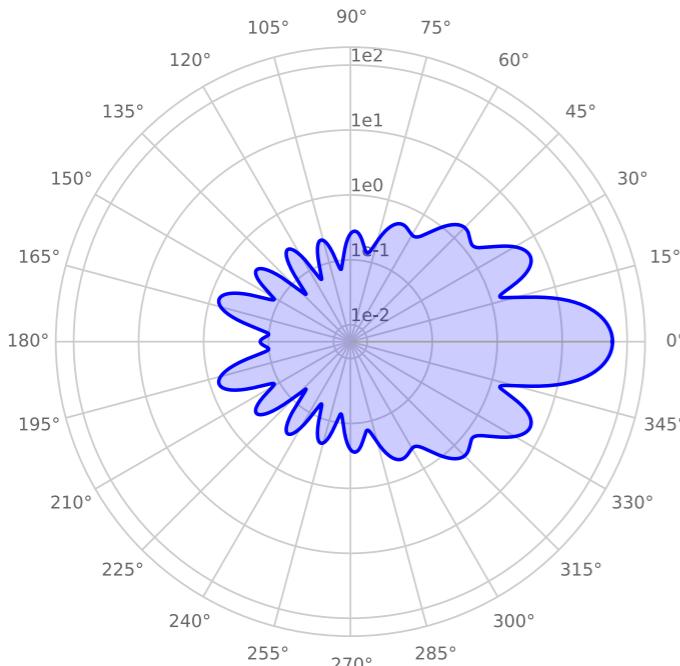


Sphere diameter =  $100\mu m$



Linear plots

Log plots



Data obtained from <http://www.philiplaven.com/mieplot.htm>

# Rainbows

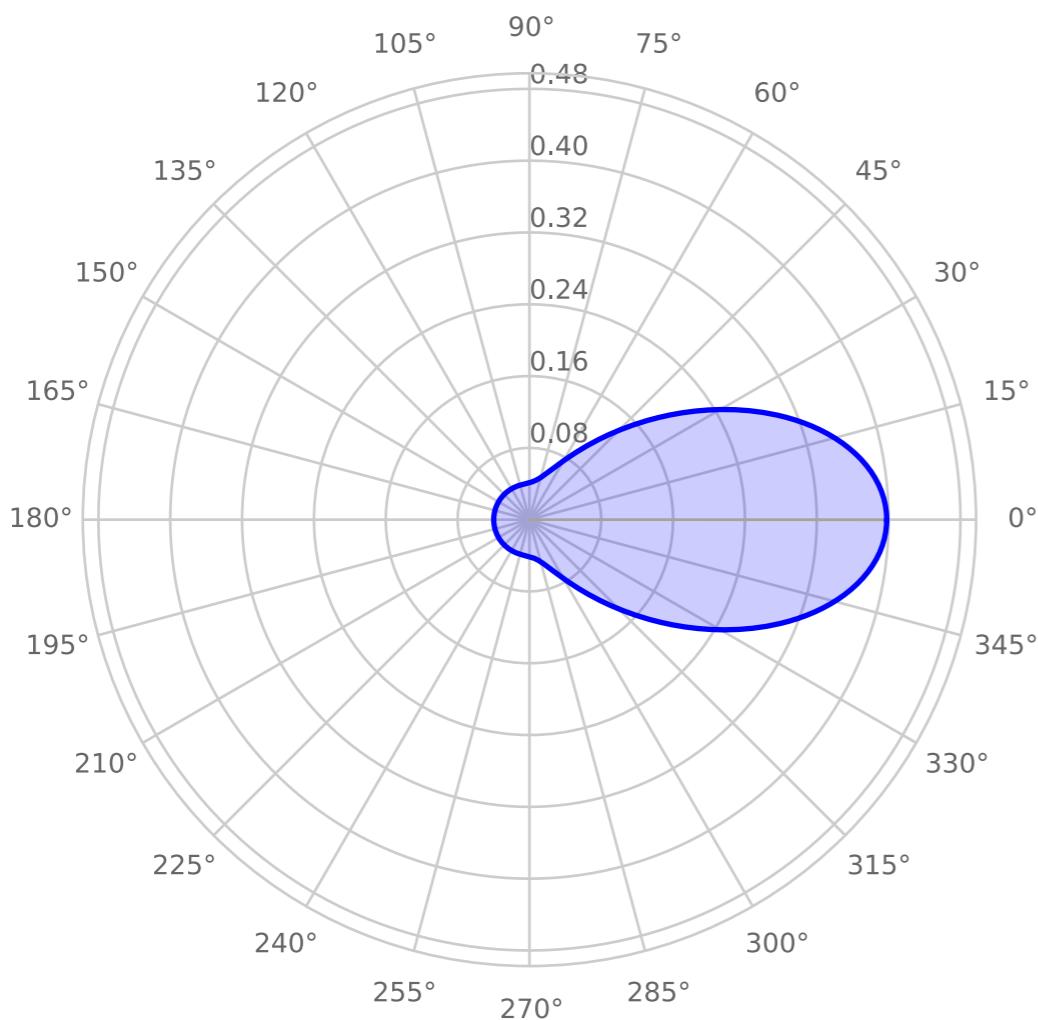


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# Lorenz-Mie Approximations

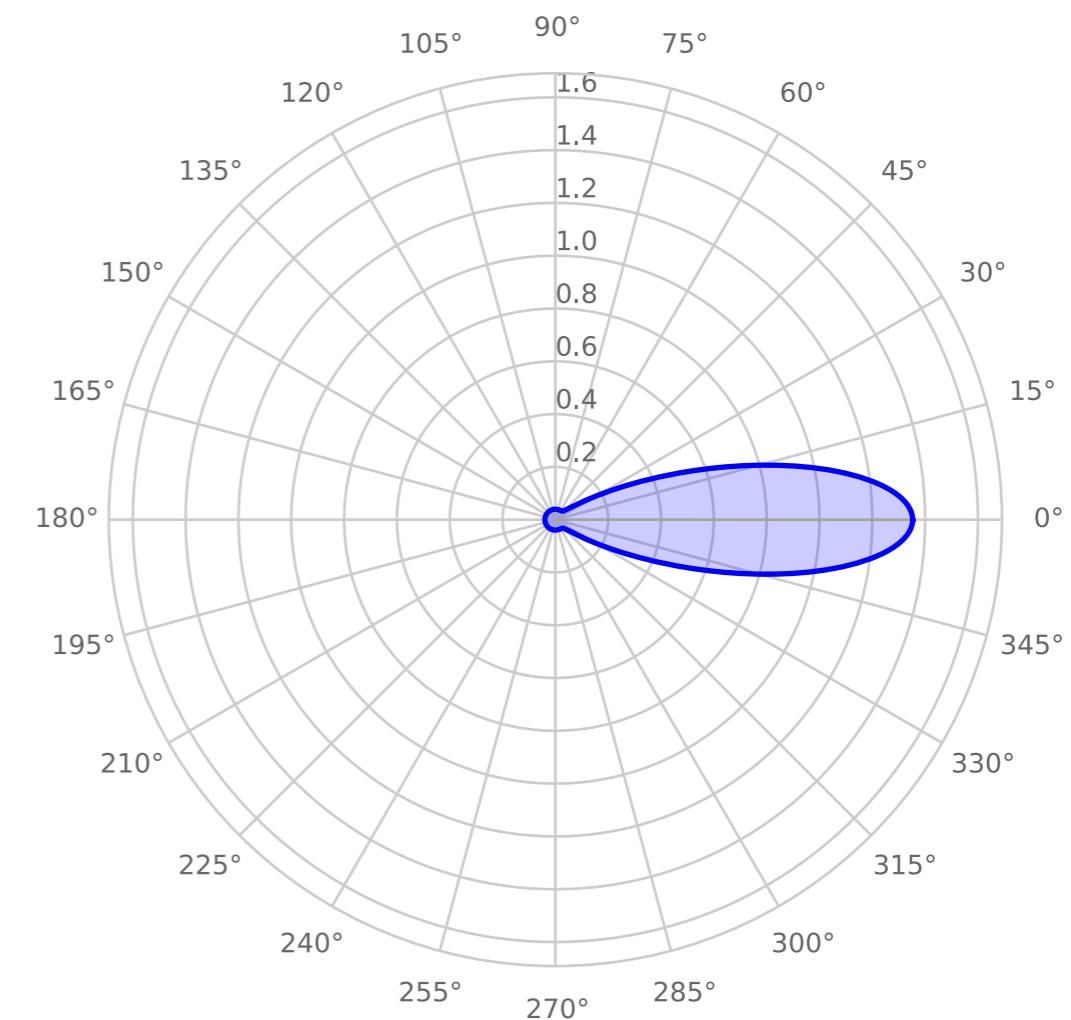
Hazy atmosphere

$$f_{p \text{ hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right)$$



Murky atmosphere

$$f_{p \text{ murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right)$$



# Lorenz-Mie Approximations

Hazy atmosphere

$$f_{p \text{ hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right)$$



[johnib.wordpress.com](http://johnib.wordpress.com)

Murky atmosphere

$$f_{p \text{ murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right)$$



[srollinson.blogspot.com](http://srollinson.blogspot.com)

# Rayleigh Scattering

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- Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light
- Used for atmospheric scattering, gasses, transparent solids
- Highly wavelength dependent

# Rayleigh Scattering

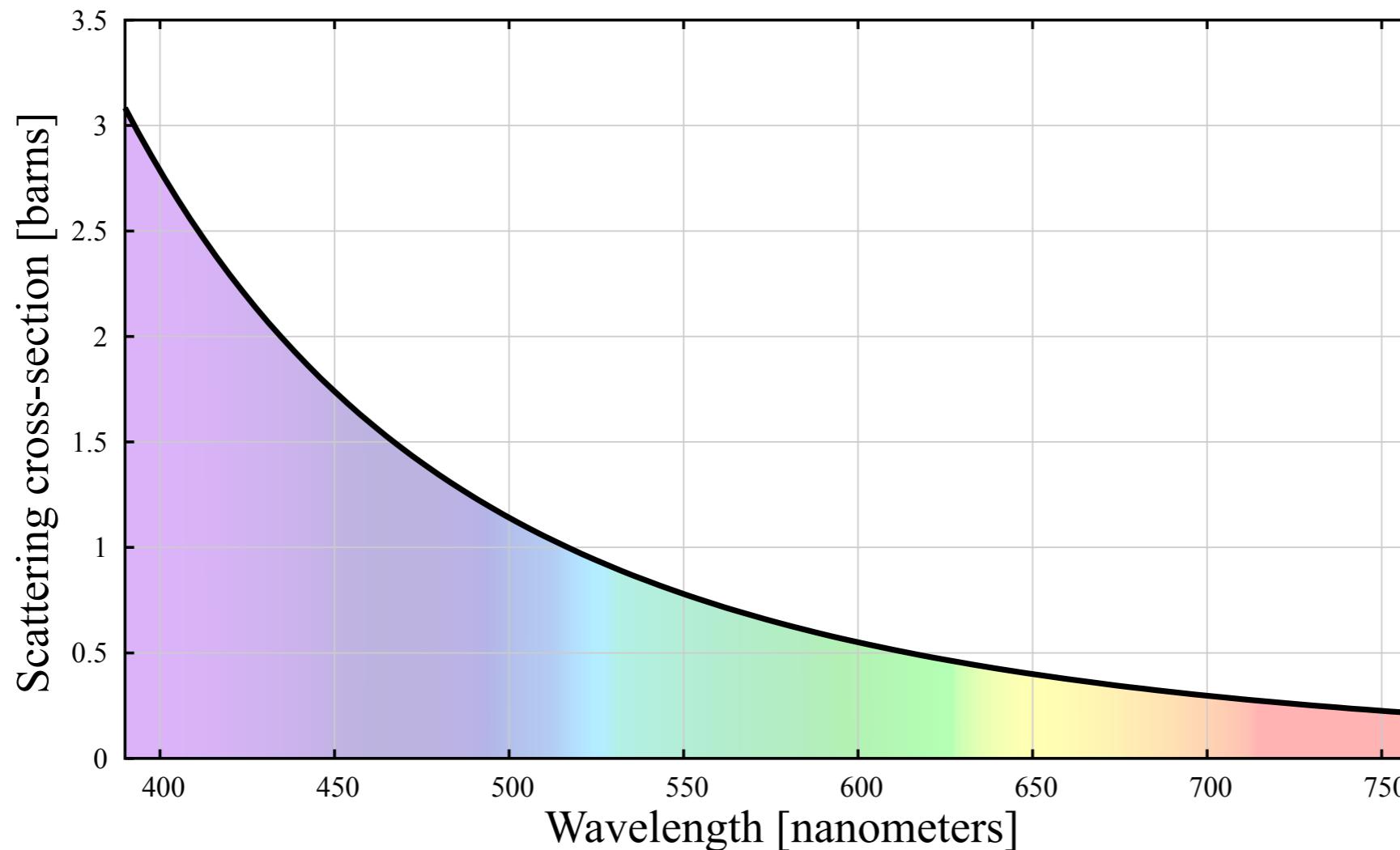
$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

The diagram illustrates the components of the Rayleigh scattering formula. It features four curved arrows pointing from text labels to specific variables in the equation:

- A curved arrow points from "Wavelength" to the variable  $\lambda$ .
- A curved arrow points from "Index of refraction" to the variable  $\eta$ .
- A curved arrow points from "Density of scatterers" to the variable  $\rho$ .
- A curved arrow points from "Diameter of scatterers" to the variable  $d$ .

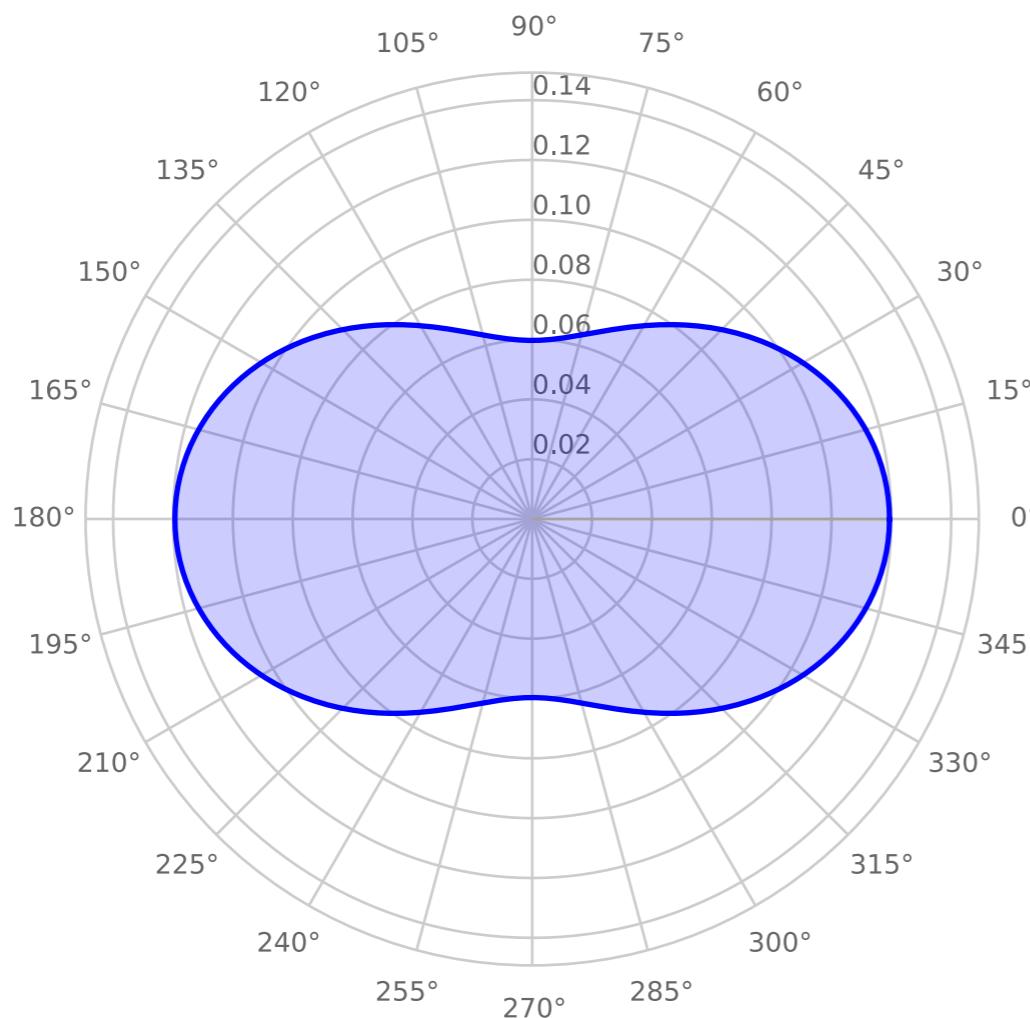
# Rayleigh Scattering

$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$



# Rayleigh Phase Function

$$f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi}(1 + \cos^2 \theta)$$



Scattering at right angles has about half intensity of scattering forward or backward

# Visual Break



Dana Stephenson/Getty Images

# Visual Break

Steam



Smoke



Forward scattering

Backward scattering

# Visual Break

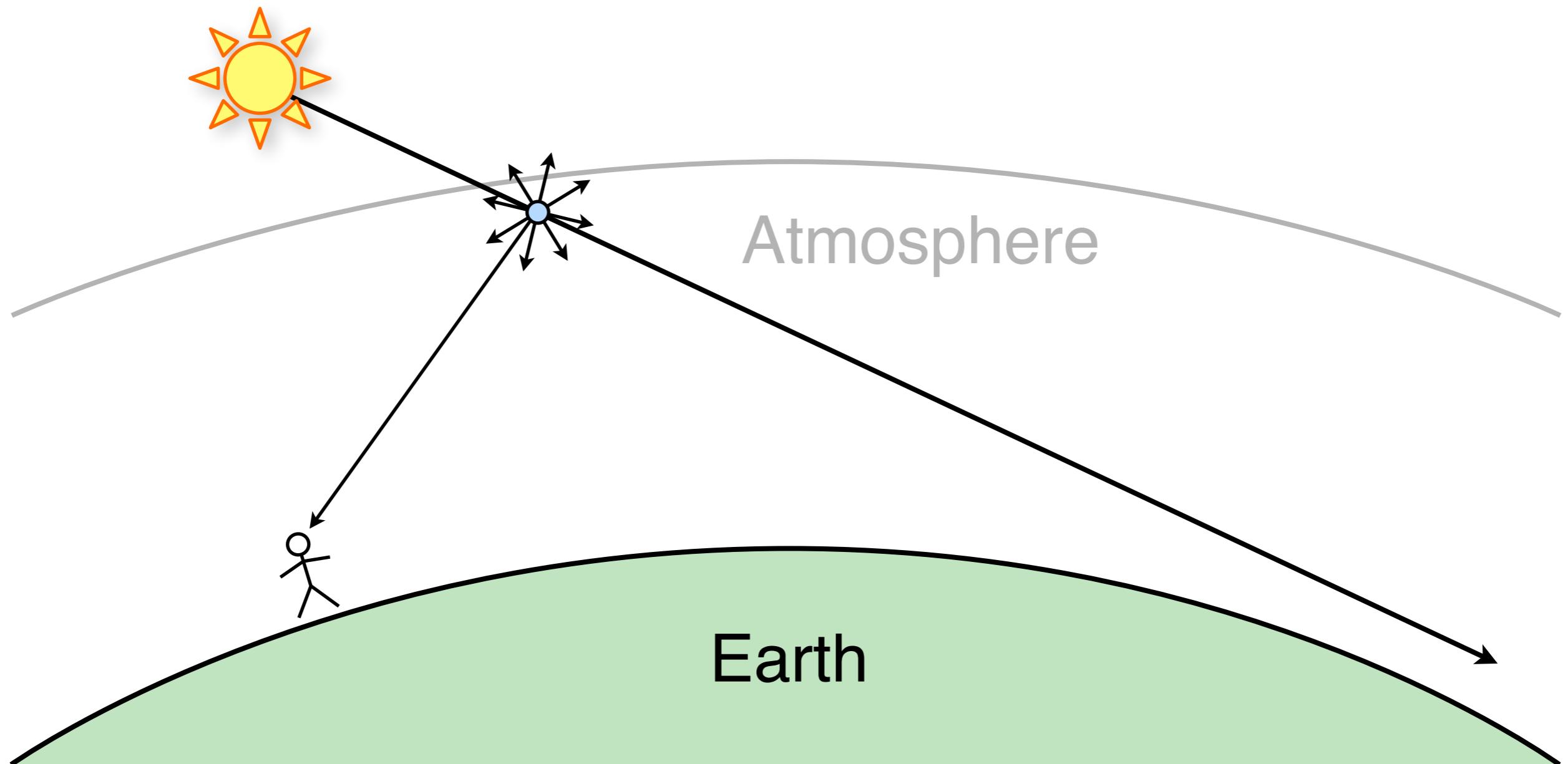


Isotropic scattering

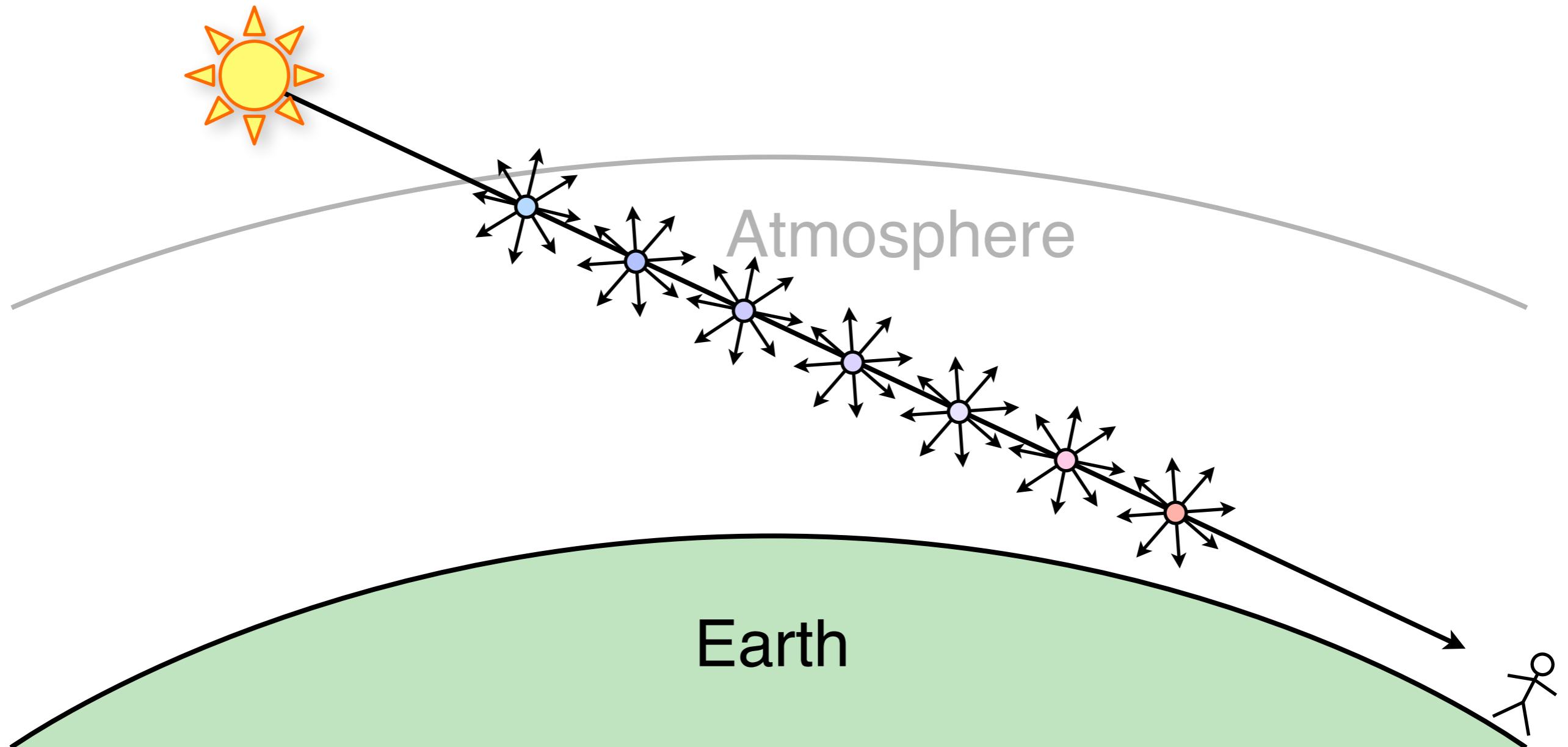


Forward scattering

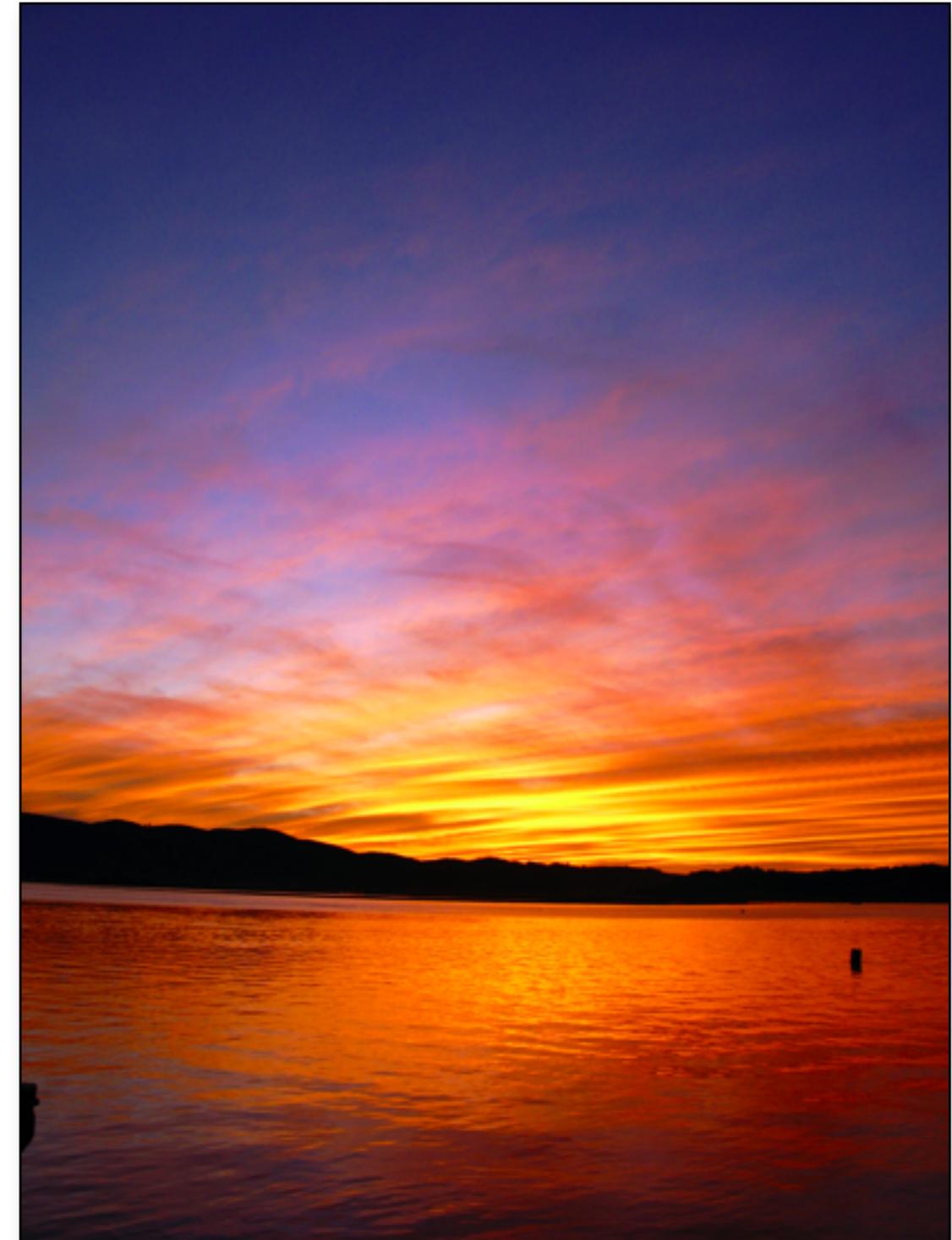
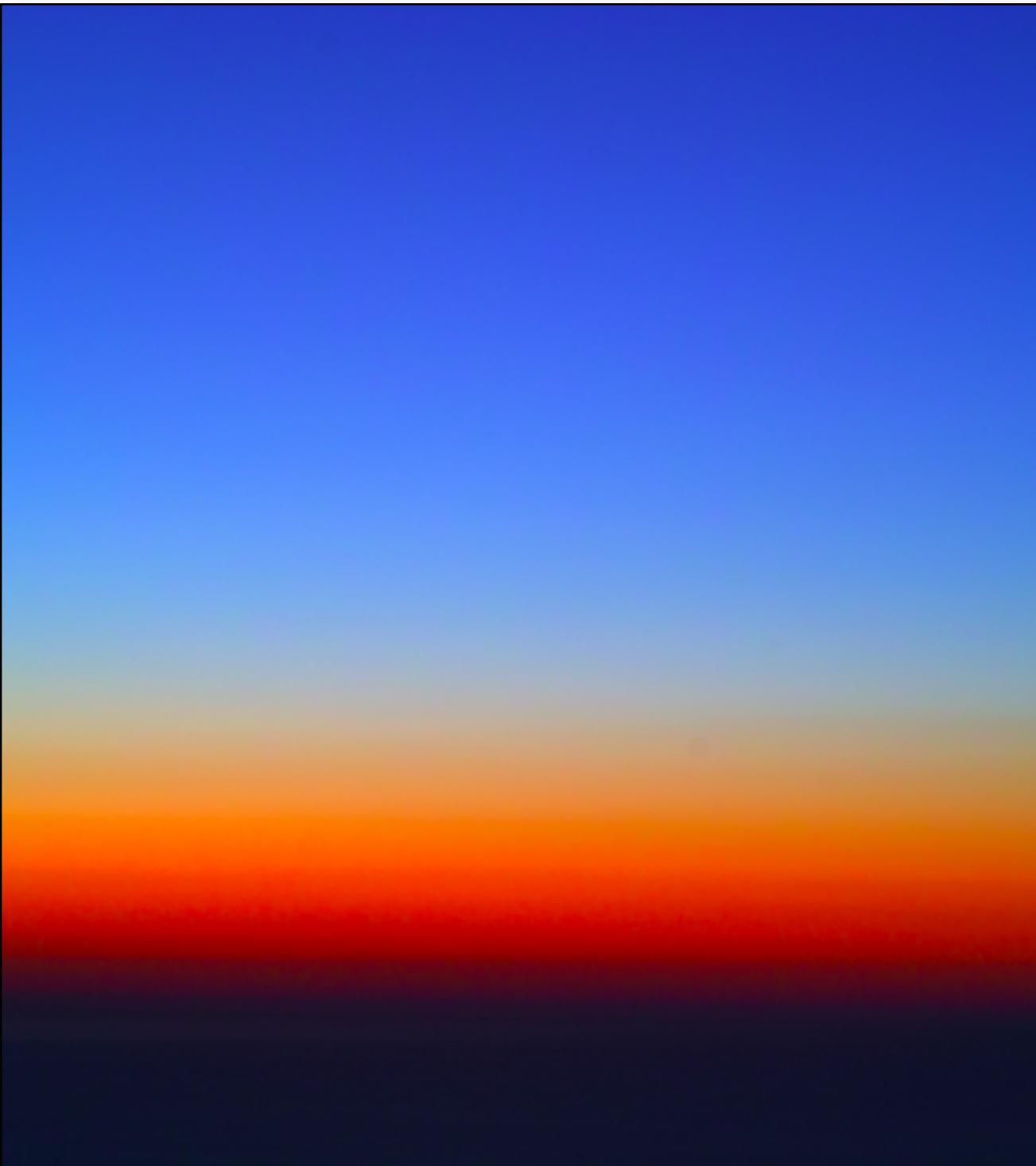
# Why is the Sky Blue?



# Why is the Sunset Red?



# Rayleigh Scattering



# Media Properties (Recap)

- Given:
  - Absorption coefficient  $\sigma_a(\mathbf{x})$
  - Scattering coefficient  $\sigma_s(\mathbf{x})$
  - Phase function  $f_p(\mathbf{x}, \vec{\omega}', \vec{\omega})$
- Derived:
  - Extinction coefficient  $\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$
  - Albedo  $\alpha(\mathbf{x}) = \sigma_s(\mathbf{x}) / \sigma_t(\mathbf{x})$
  - Mean-free path  $E[1/\sigma_t(\mathbf{x})]$
  - Transmittance  $T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t) dt}$

# Homogeneous Isotropic Medium

- Given:
  - Absorption coefficient  $\sigma_a$
  - Scattering coefficient  $\sigma_s$
  - Phase function  $\frac{1}{4\pi}$
- Derived:
  - Extinction coefficient  $\sigma_t = \sigma_a + \sigma_s$
  - Albedo  $\alpha = \sigma_s / \sigma_t$
  - Mean-free path  $1 / \sigma_t$
  - Transmittance  $T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$

# Questions?

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# What is this?

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source: [wikipedia](#)

# Crepuscular Rays



source: [wikipedia](#)

# Anti-Crepuscular Rays



source: [wikipedia](#)

# Solving the Volume Rendering Equation

# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

Attenuated background radiance

Accumulated emitted radiance

Accumulated in-scattered radiance

# Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z)L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t, \vec{\omega})dt + \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t, \vec{\omega})dt}$$

↑  
Accumulated in-scattered radiance

# In-scattered Radiance

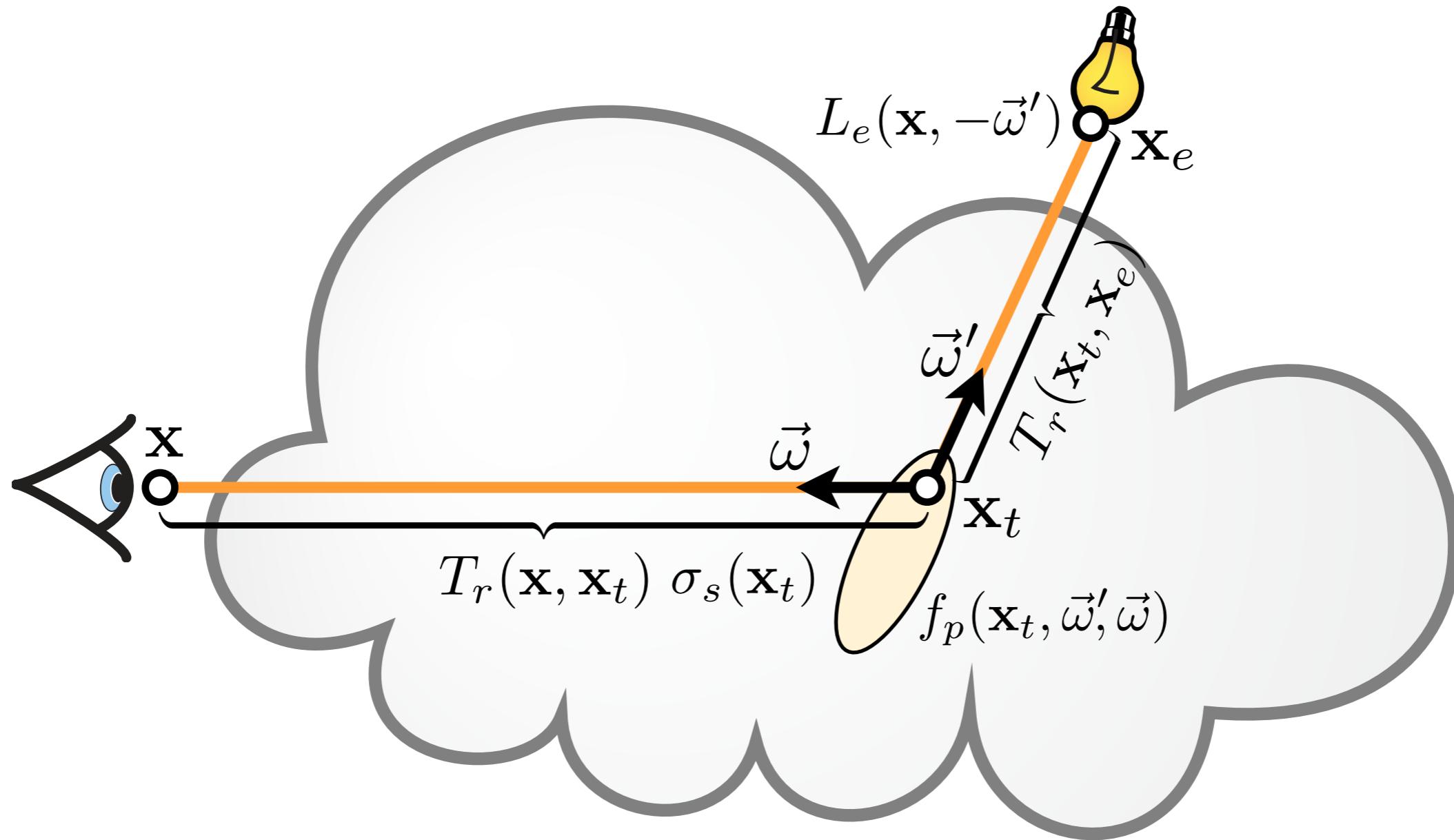
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) [L_s(\mathbf{x}_t, \vec{\omega})] dt$$

$$[L_s(\mathbf{x}_t, \vec{\omega})] = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) [L_i(\mathbf{x}_t, \vec{\omega}')] d\vec{\omega}'$$

- Single scattering
  - $L_i$  arrives directly from a light source (direct illum.)  
i.e.:  $[L_i(\mathbf{x}, \vec{\omega})] = T_r(\mathbf{x}, r(\mathbf{x}, \vec{\omega})) L_e(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$
- Multiple scattering
  - $L_i$  arrives through multiple bounces (indirect illum.)

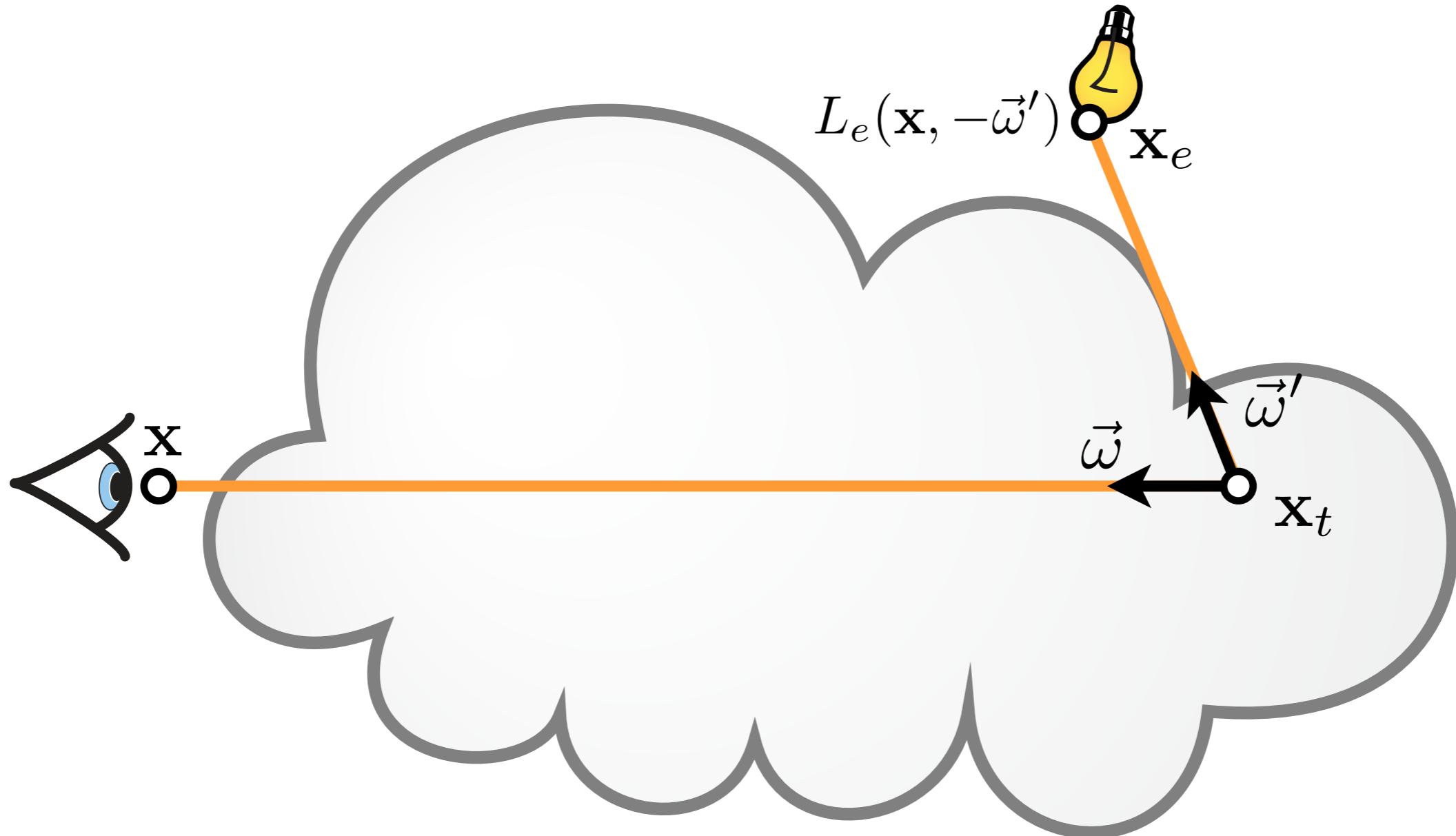
# Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



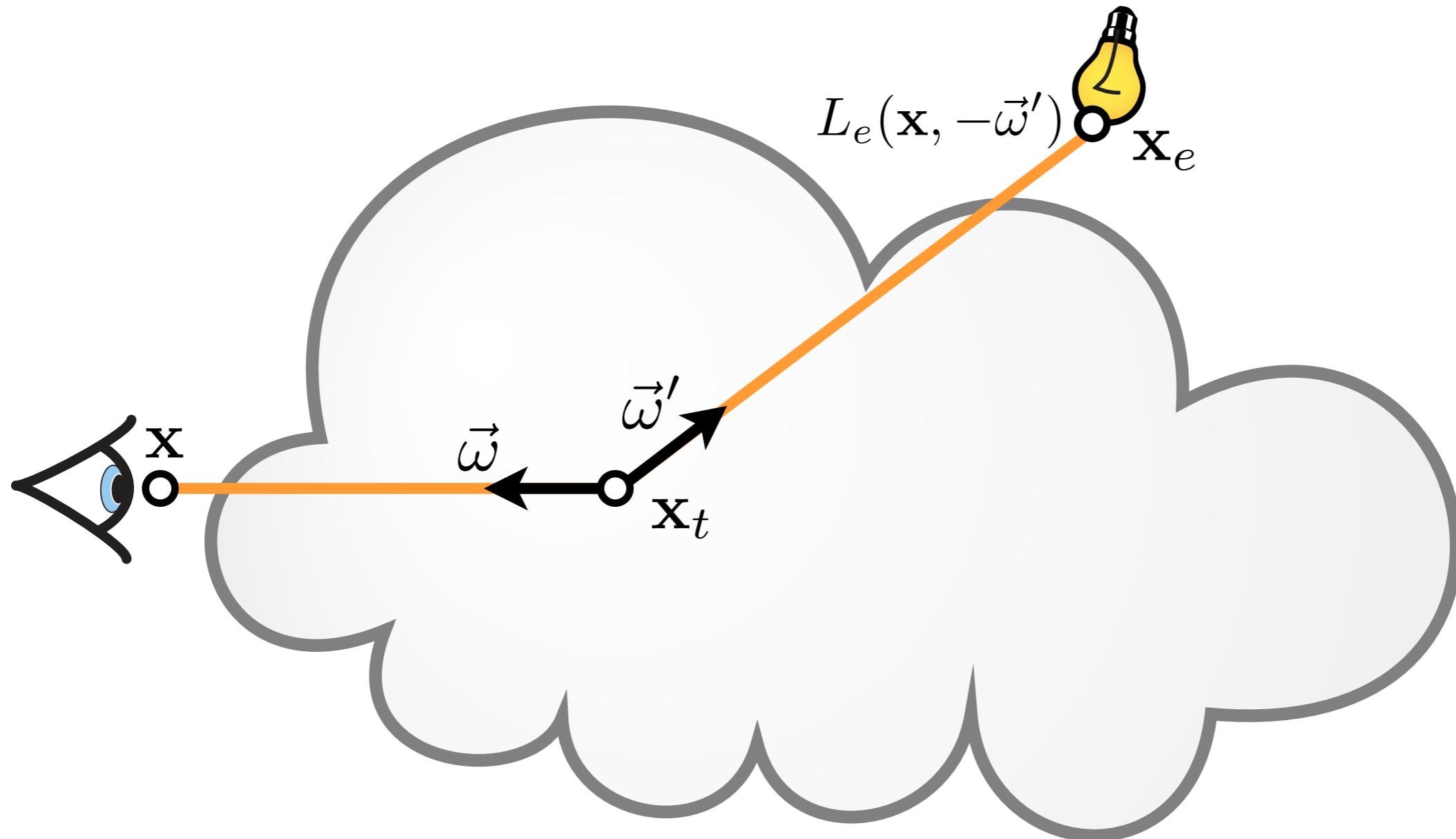
# Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



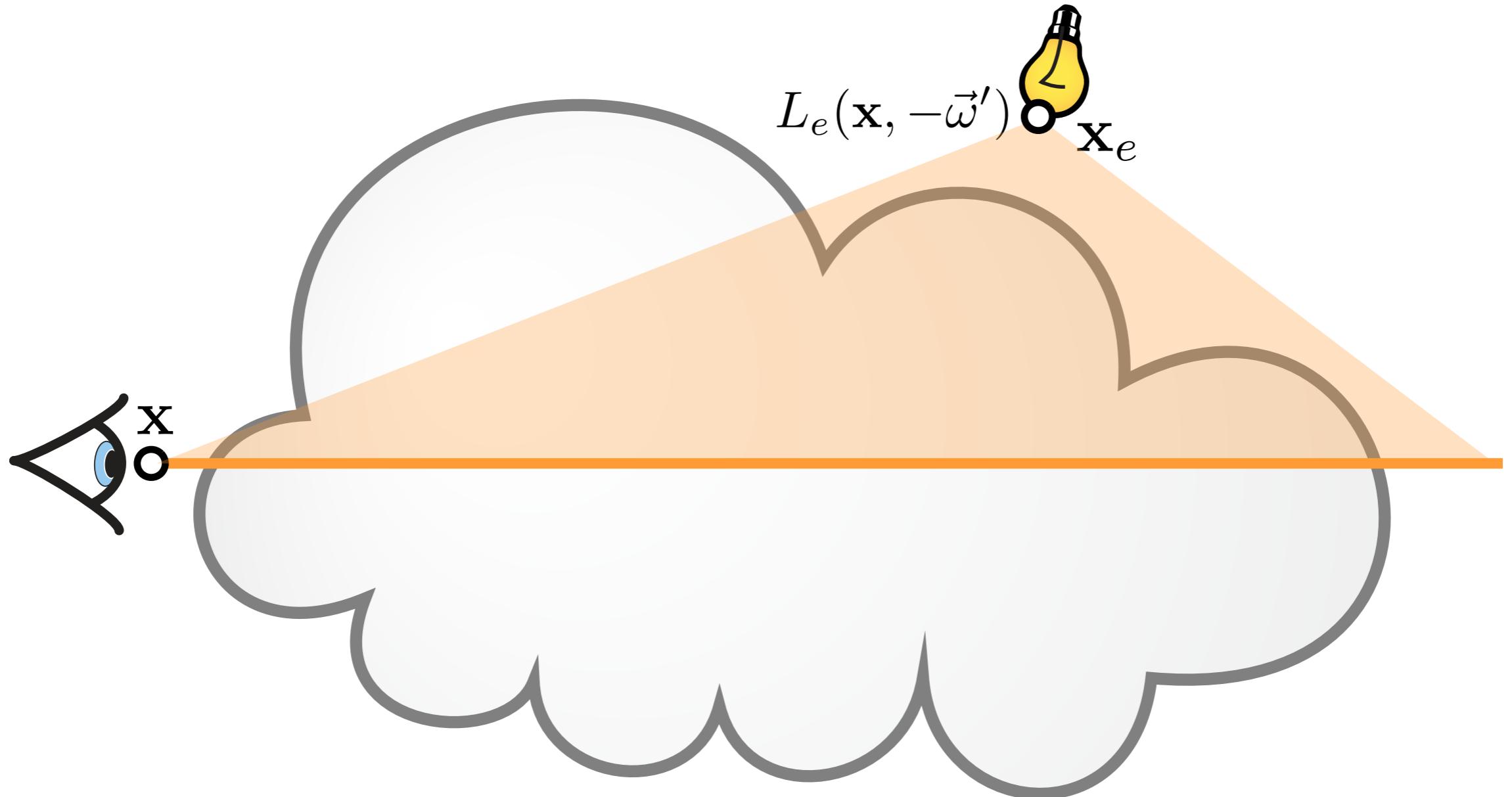
# Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



# Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



# Single Scattering

---

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$

- (Semi-)analytic solutions:
  - Sun et al. [2005]
  - Pegoraro et al. [2009, 2010]
- Numerical solutions:
  - Ray-marching
  - Equiangular sampling

# Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$

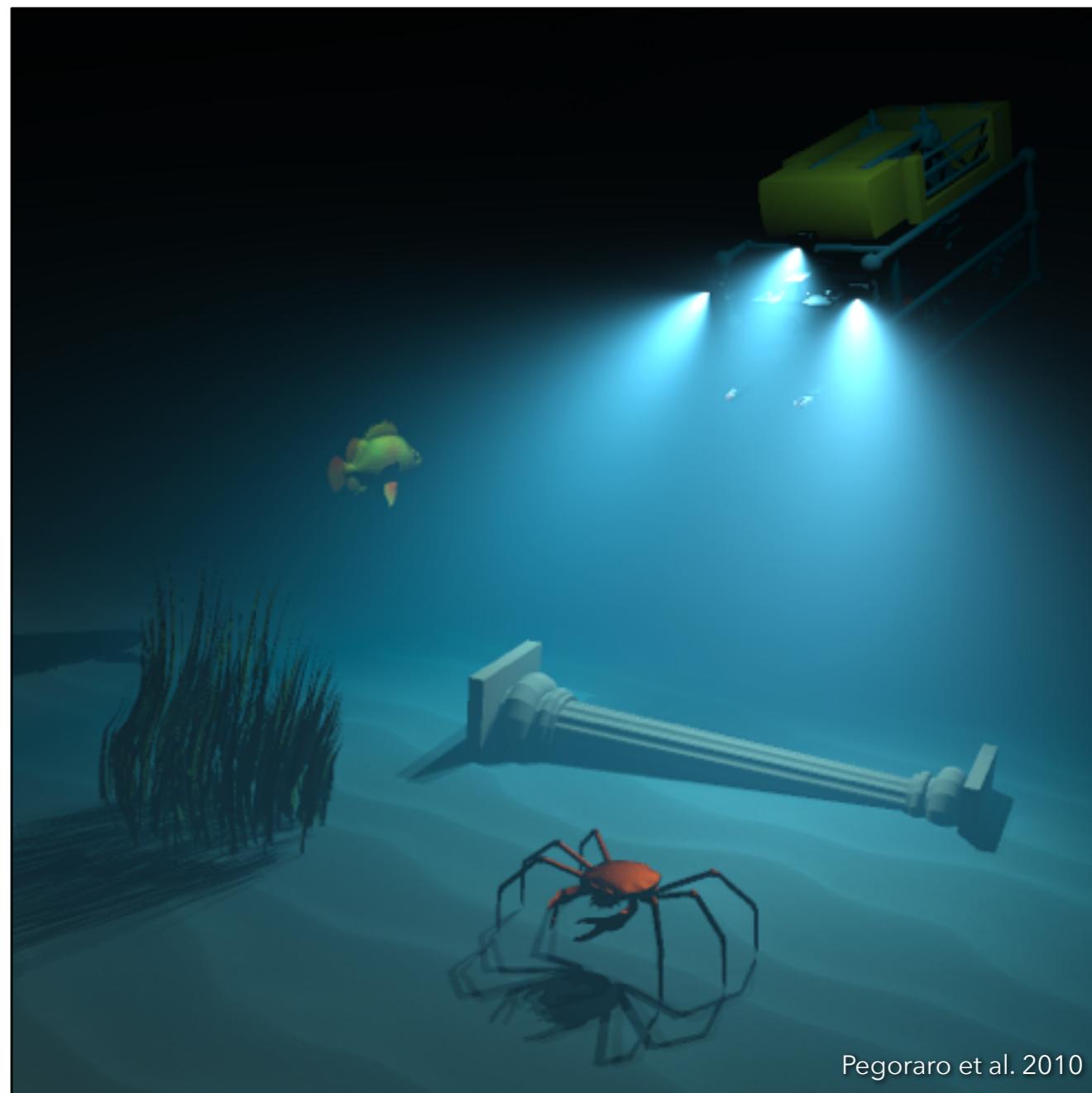
- Assumptions:
  - Homogeneous medium
  - Point or spot light
  - Relatively simple phase function
  - No occlusion

$$L(\mathbf{x}, \vec{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \sigma_s \int_0^z e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_t\|} e^{-\sigma_t \|\mathbf{x}_t, \mathbf{x}_p\|} dt$$

# Analytic Single Scattering

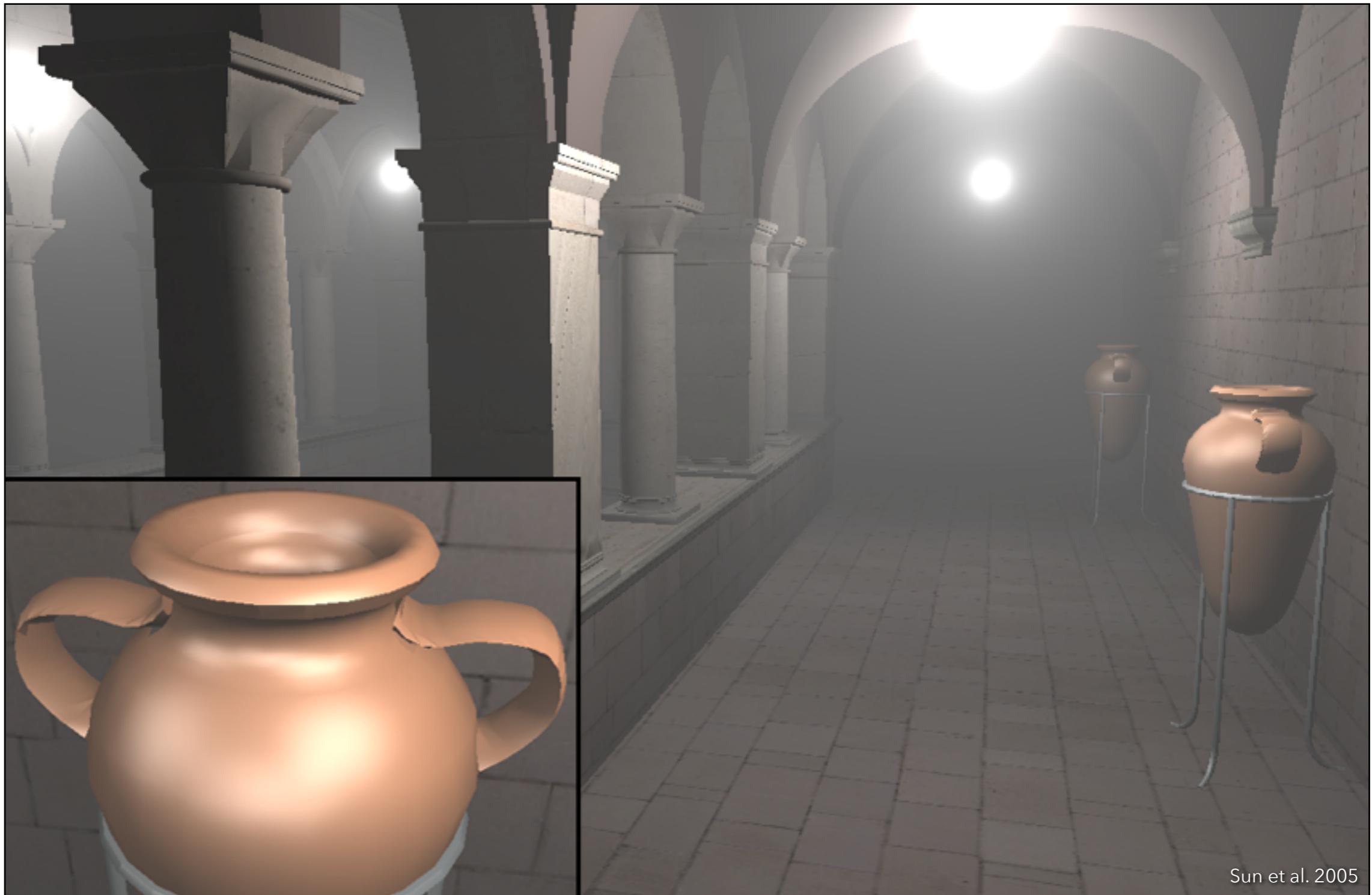


Pegoraro et al. 2009



Pegoraro et al. 2010

# Analytic Single Scattering



Sun et al. 2005



# Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t(x_a - x_b)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n, k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2 + 1)^{n+1}} v^k dv$$

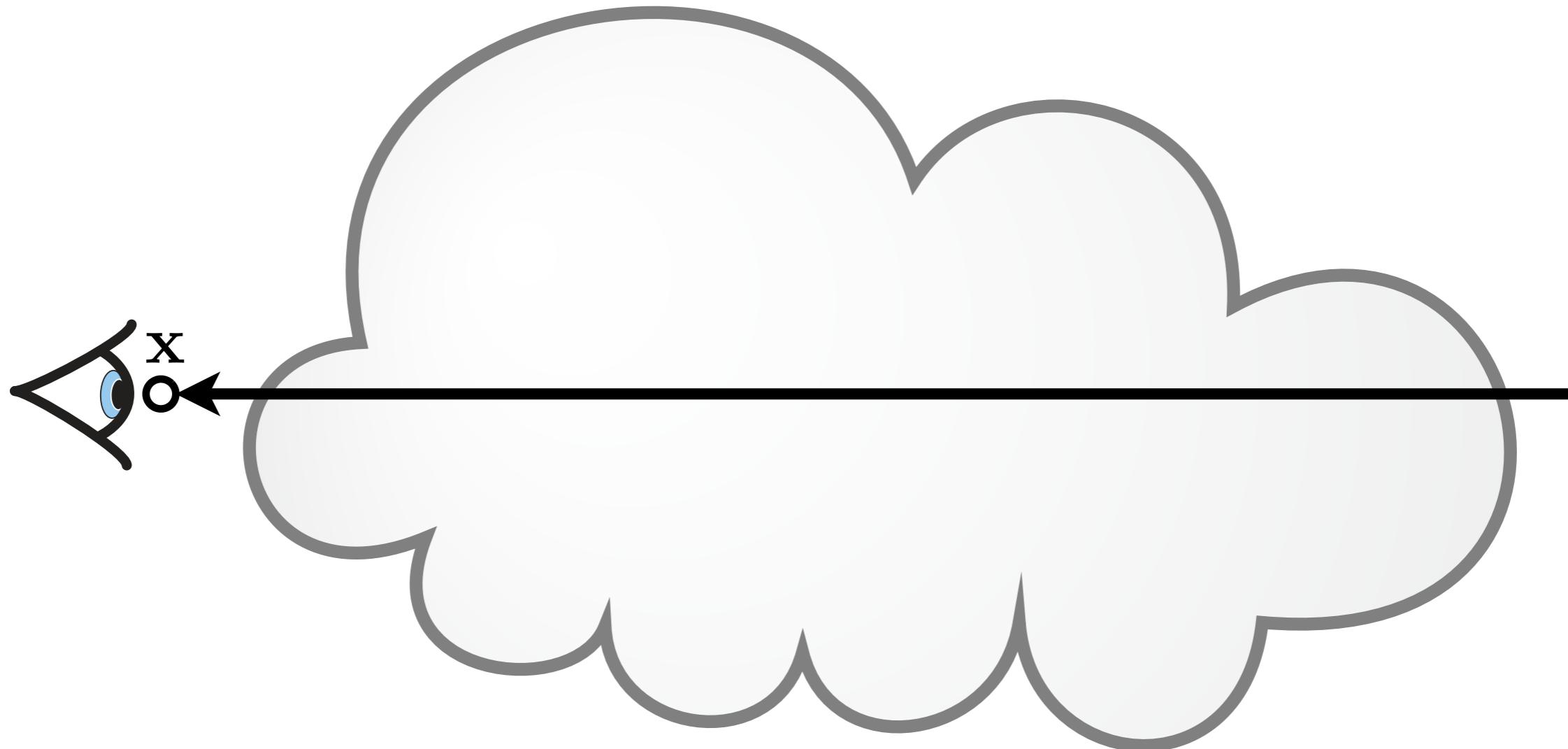
$$\begin{aligned} \int \frac{e^{av}}{(v^2 + 1)^m} v^n dv &= \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \left( \sum_{k=0}^{\min\{m-1-l, n\}} \binom{n}{k} \left( \frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \right. \right. \\ &\quad \left. \left. - e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2 + 1)^j} \sum_{\substack{i=0 \\ i \equiv (m-n-l+k-j) \pmod{2}}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \right) \right) \\ &\quad + \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=0 \\ i \equiv (-m+l+k-j) \pmod{2}}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \end{aligned}$$

No shadows, implementation nightmare, computationally intensive...  
Let's try brute force!

# Ray-Marching

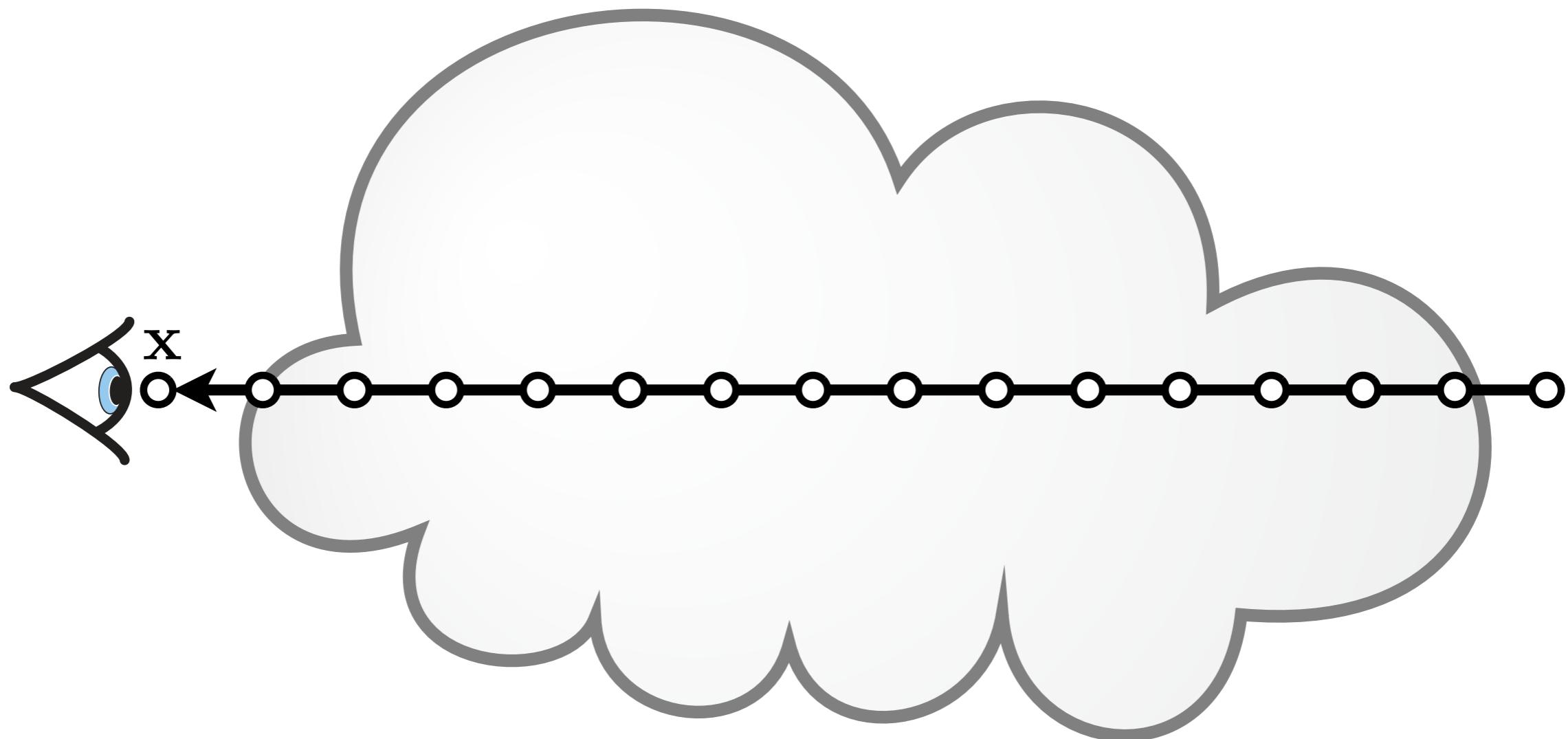
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

↑  
Approximate with Riemann sum



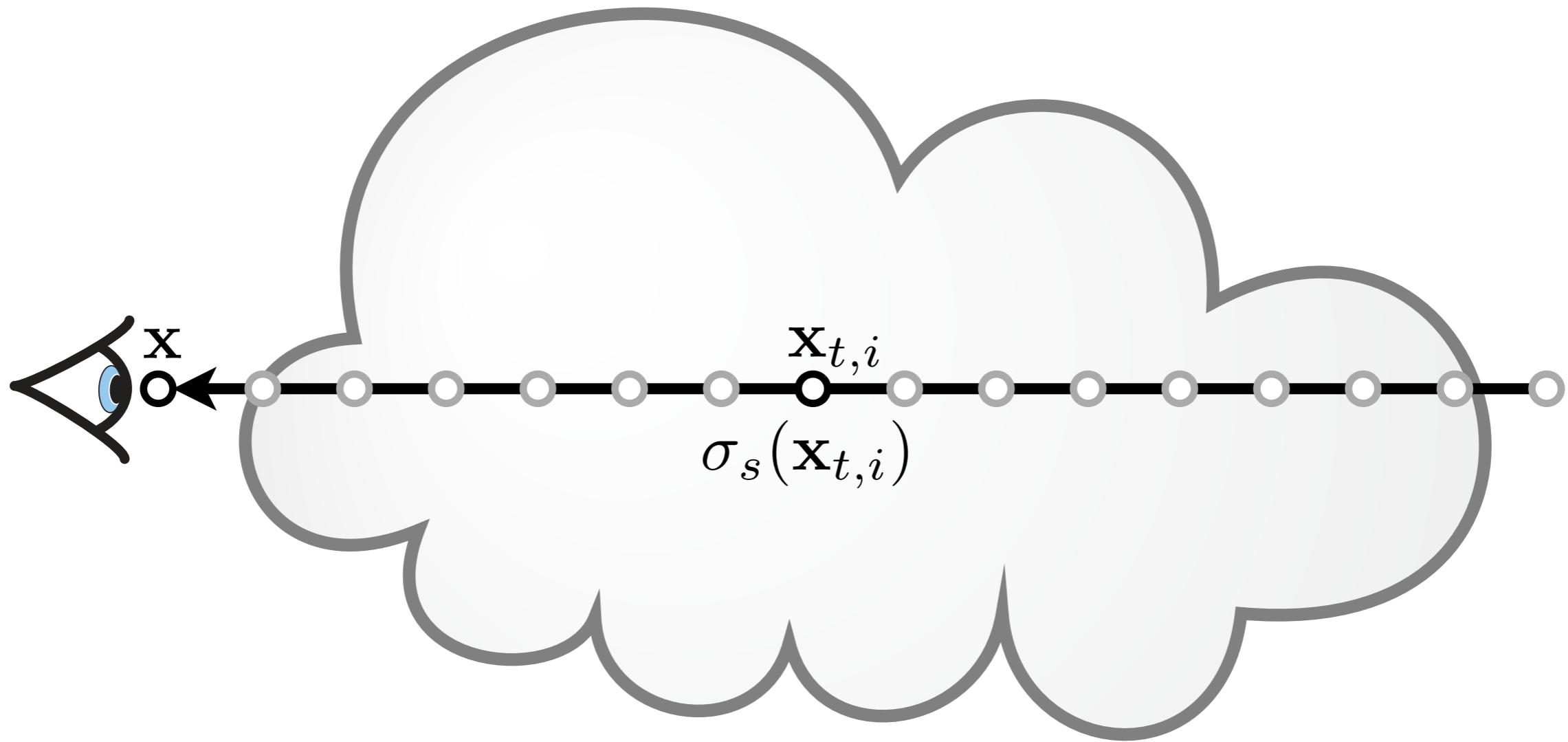
# Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



# Ray-Marching

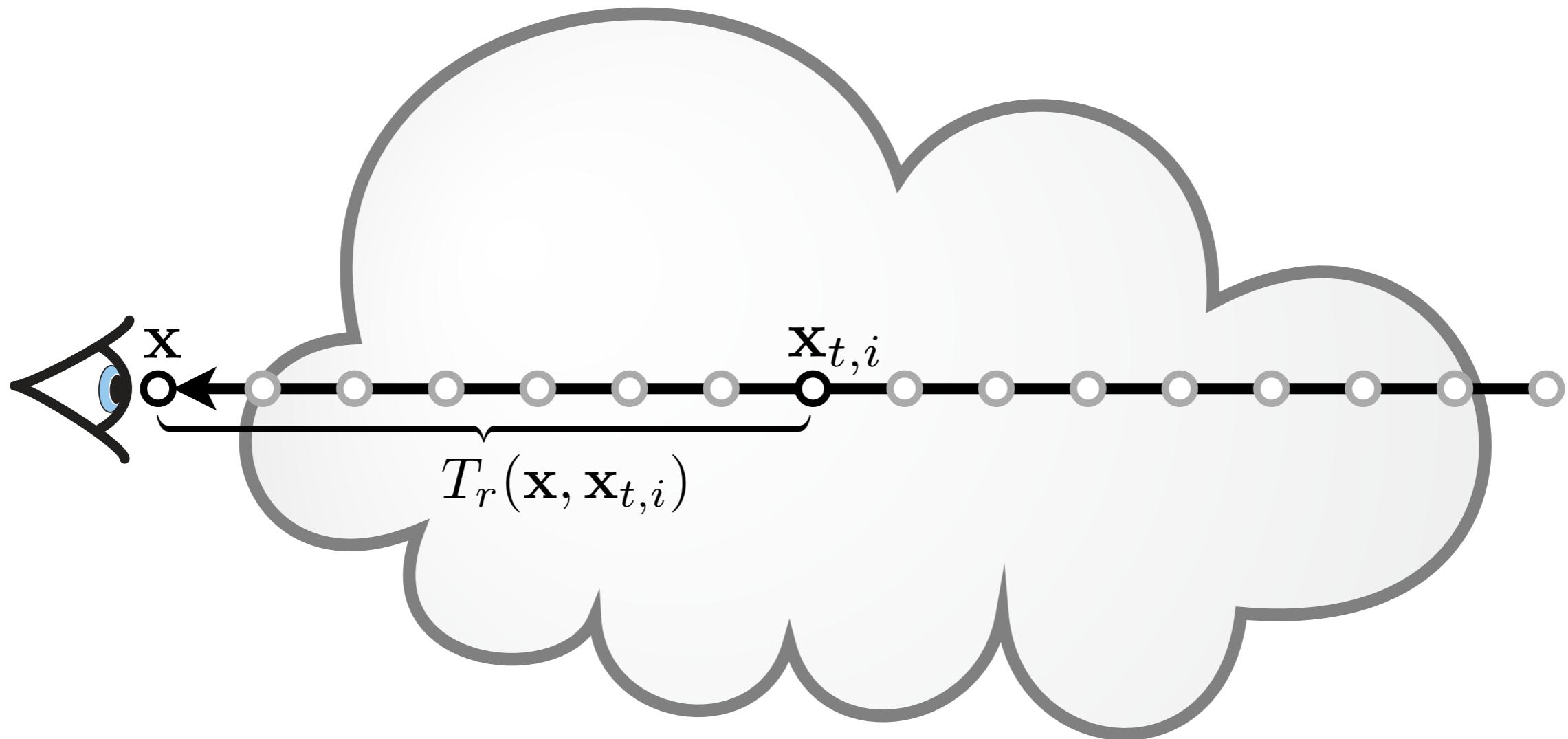
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



# Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$

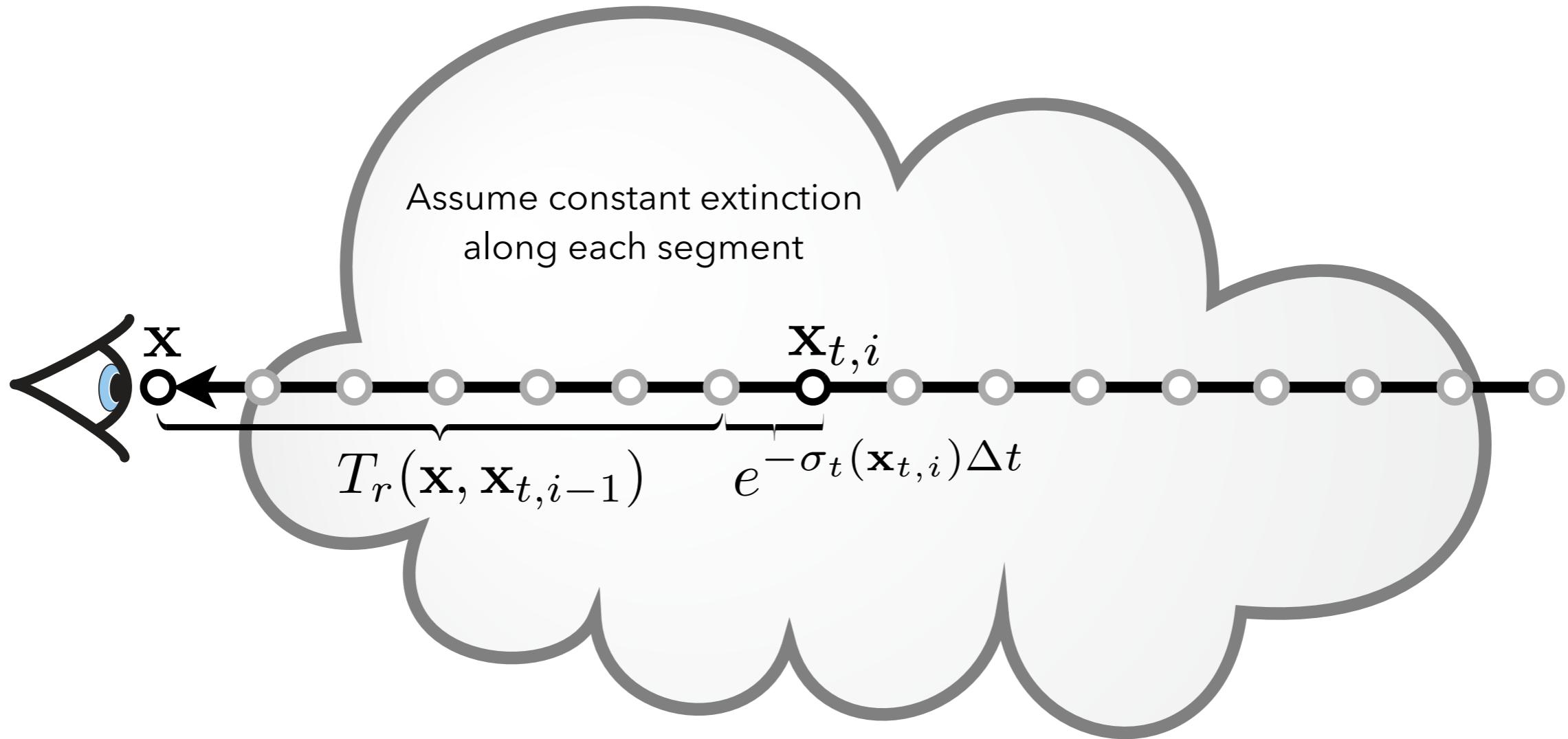
Homogeneous volume:  $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_{t,i}\|}$



# Ray-Marching

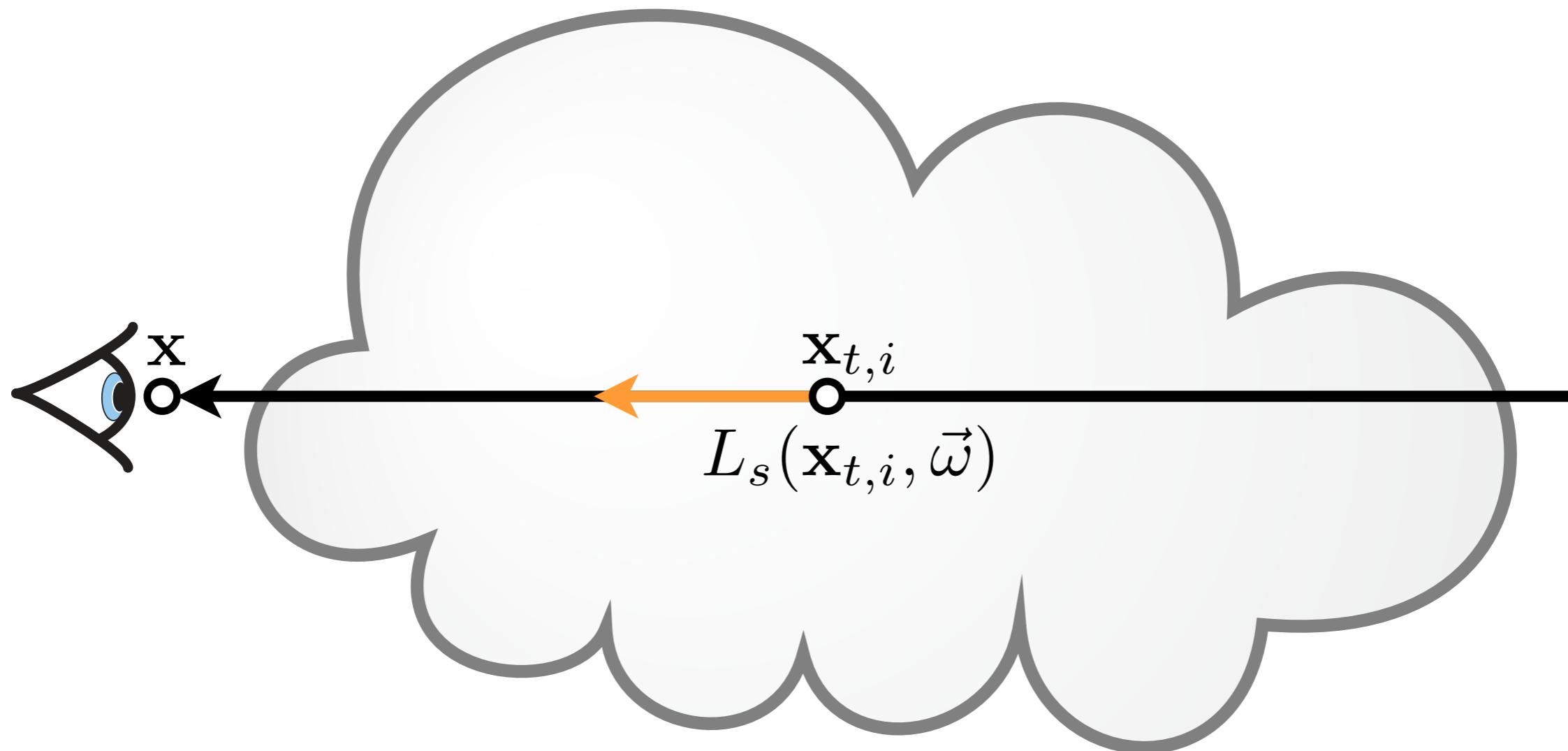
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$

Heterogeneous volume:  $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = T_r(\mathbf{x}, \mathbf{x}_{t,i-1}) e^{-\sigma_t(\mathbf{x}_{t,i}) \Delta t}$



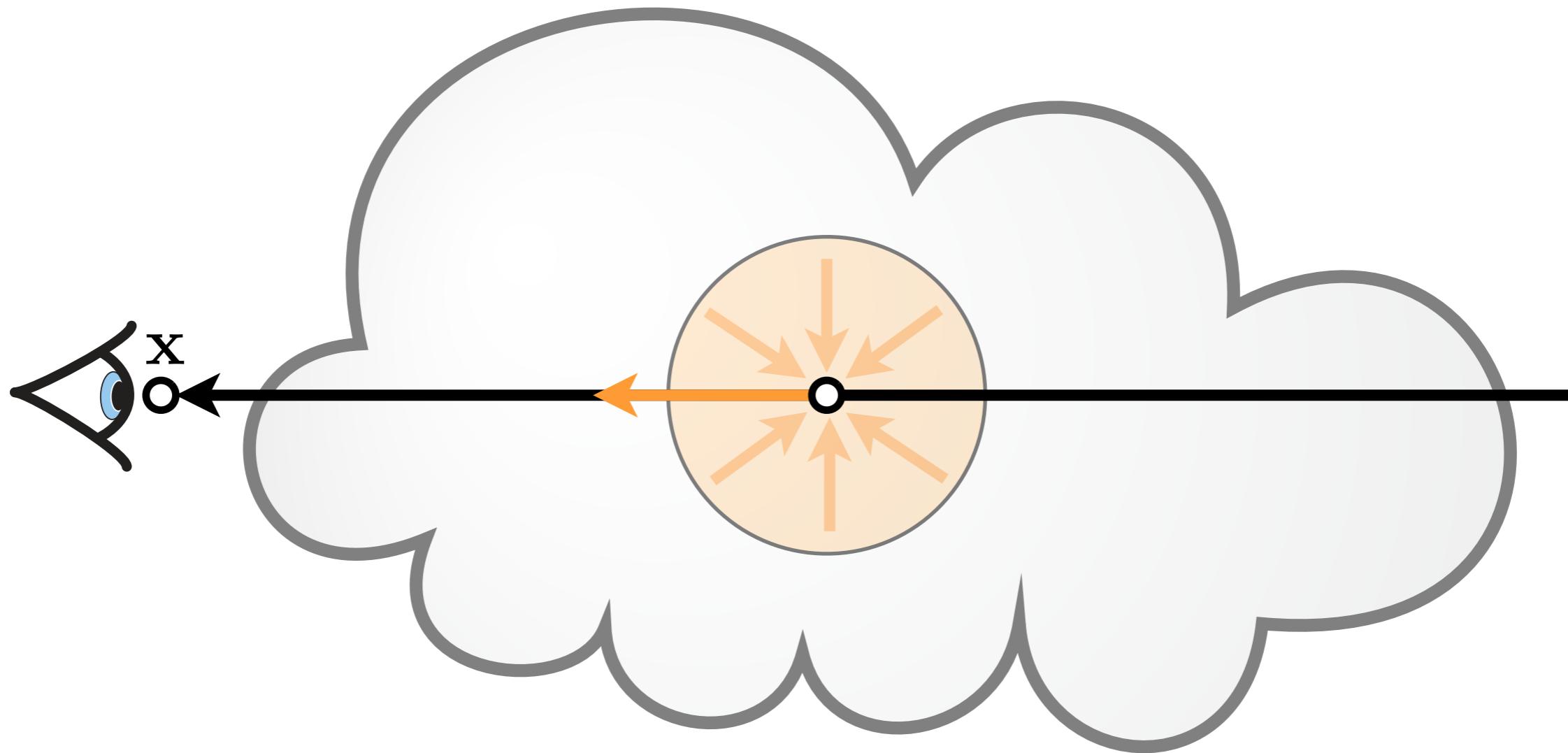
# Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^N T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) [L_s(\mathbf{x}_{t,i}, \vec{\omega})] \Delta t$$



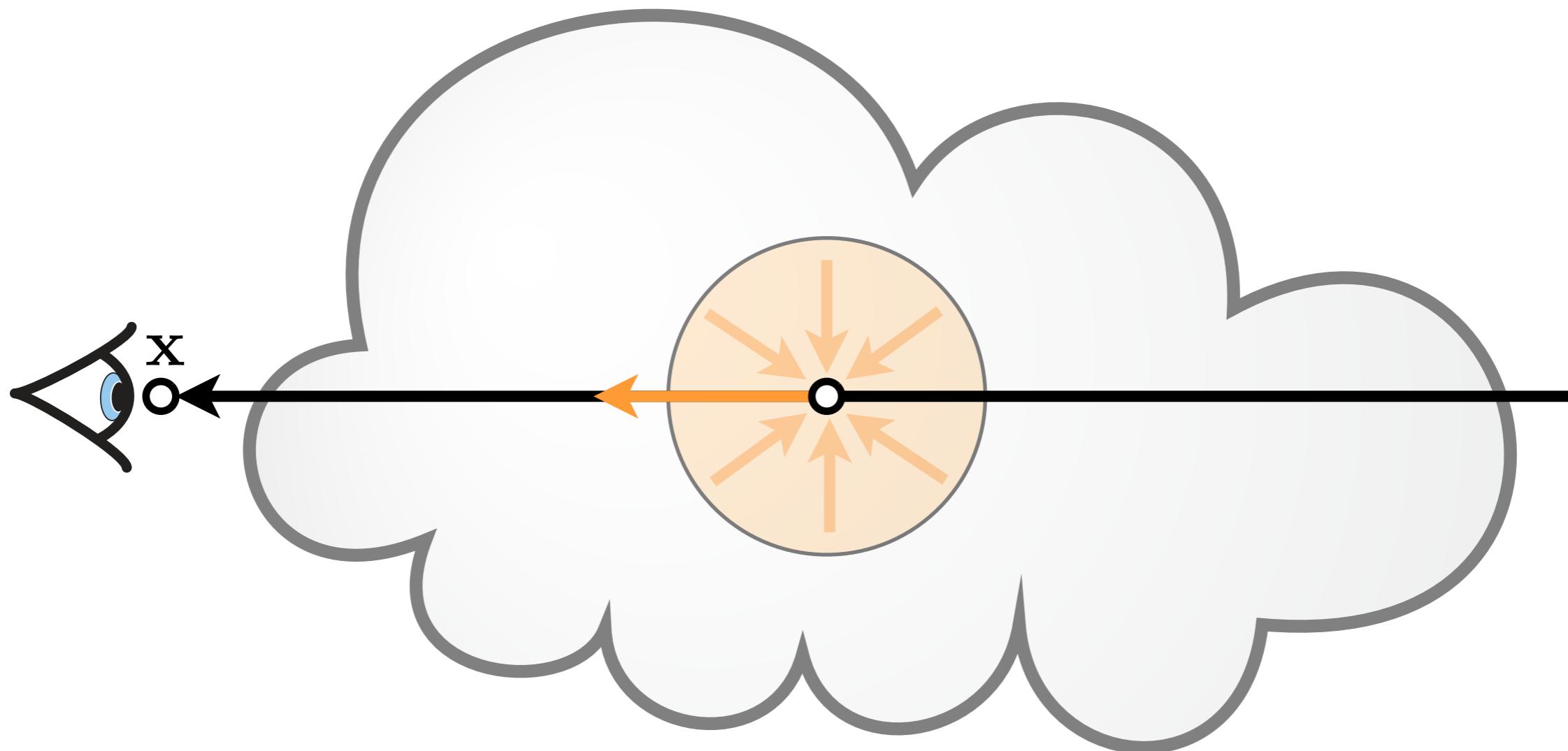
# Ray-Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$$



# Ray-Marching

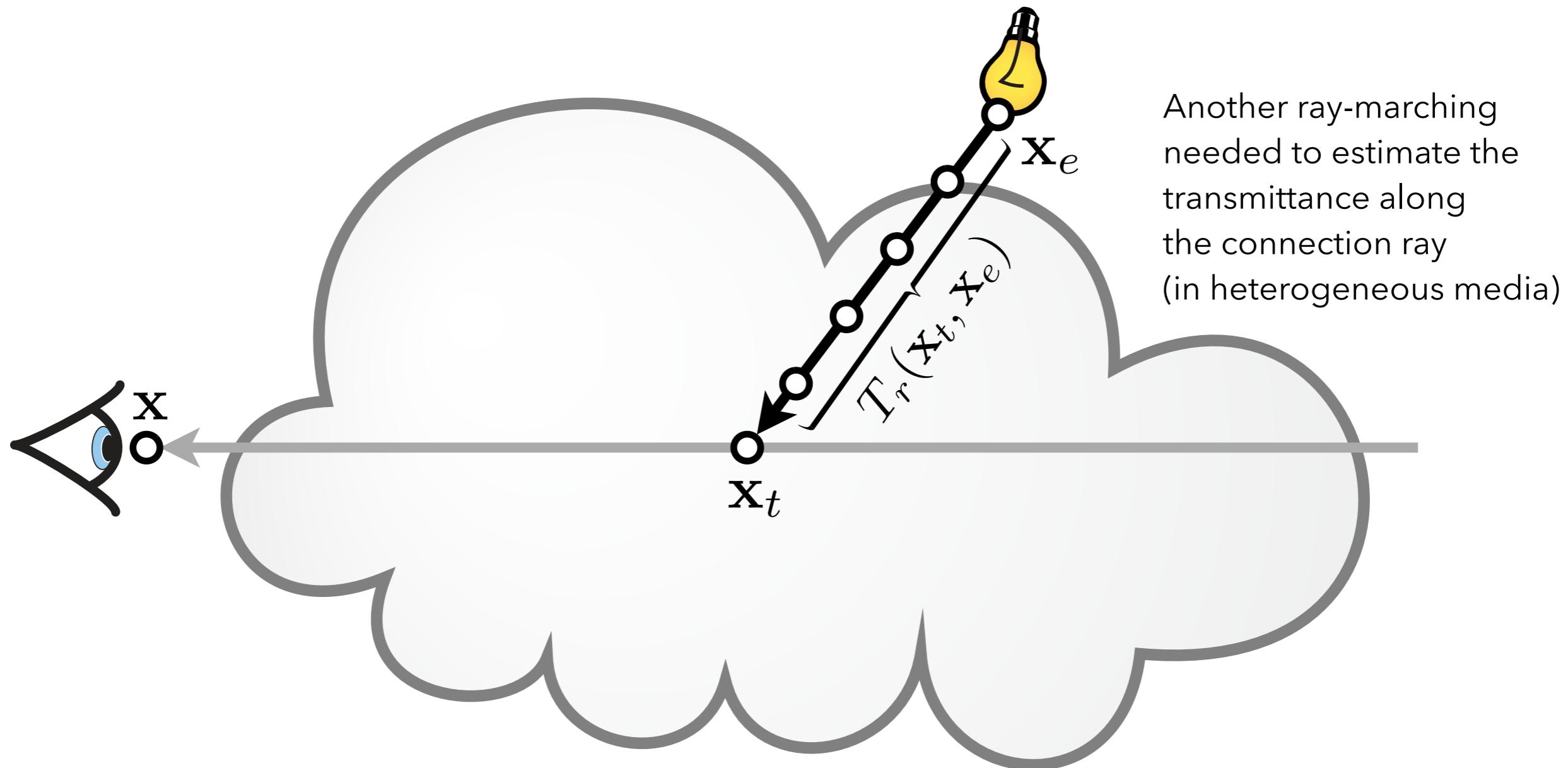
$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=0}^M \frac{f_p(\mathbf{x}_t, \vec{\omega}'_j, \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



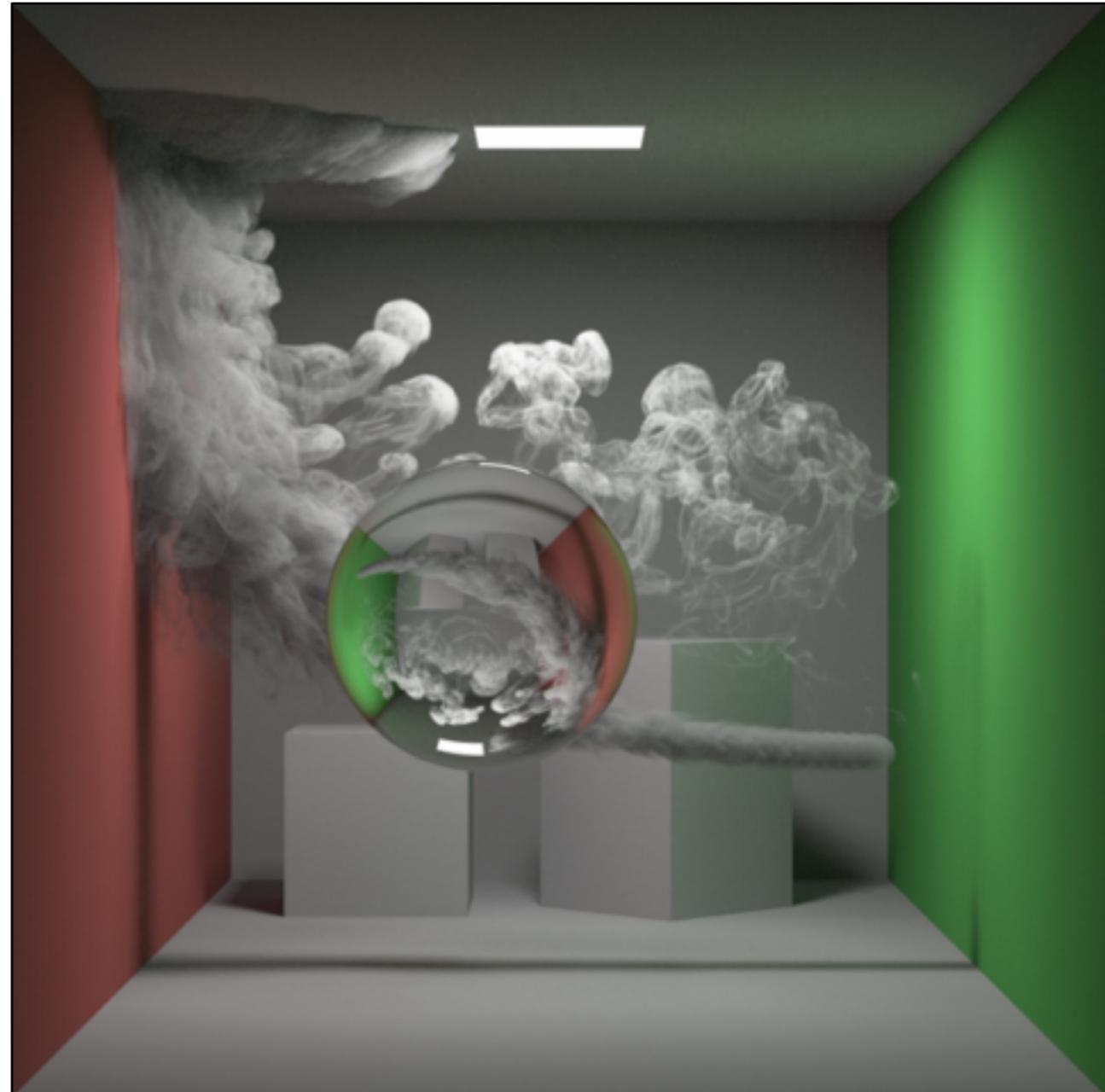
# Ray-Marching

- Single scattering:

$$L_i(\mathbf{x}_t, \vec{\omega}) = T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega})$$

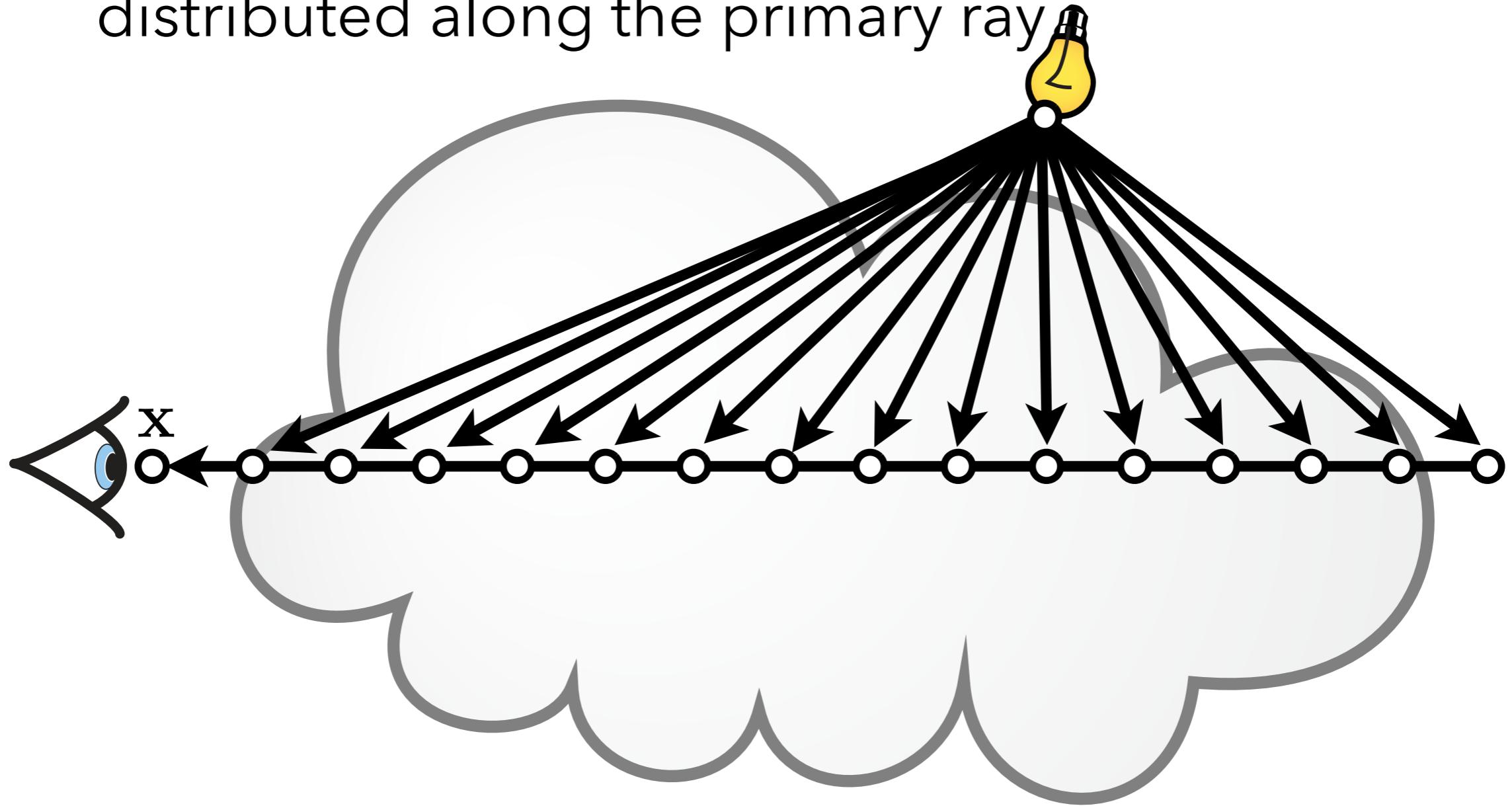


# Visual Break



# Ray-Marching in Heterogeneous Media

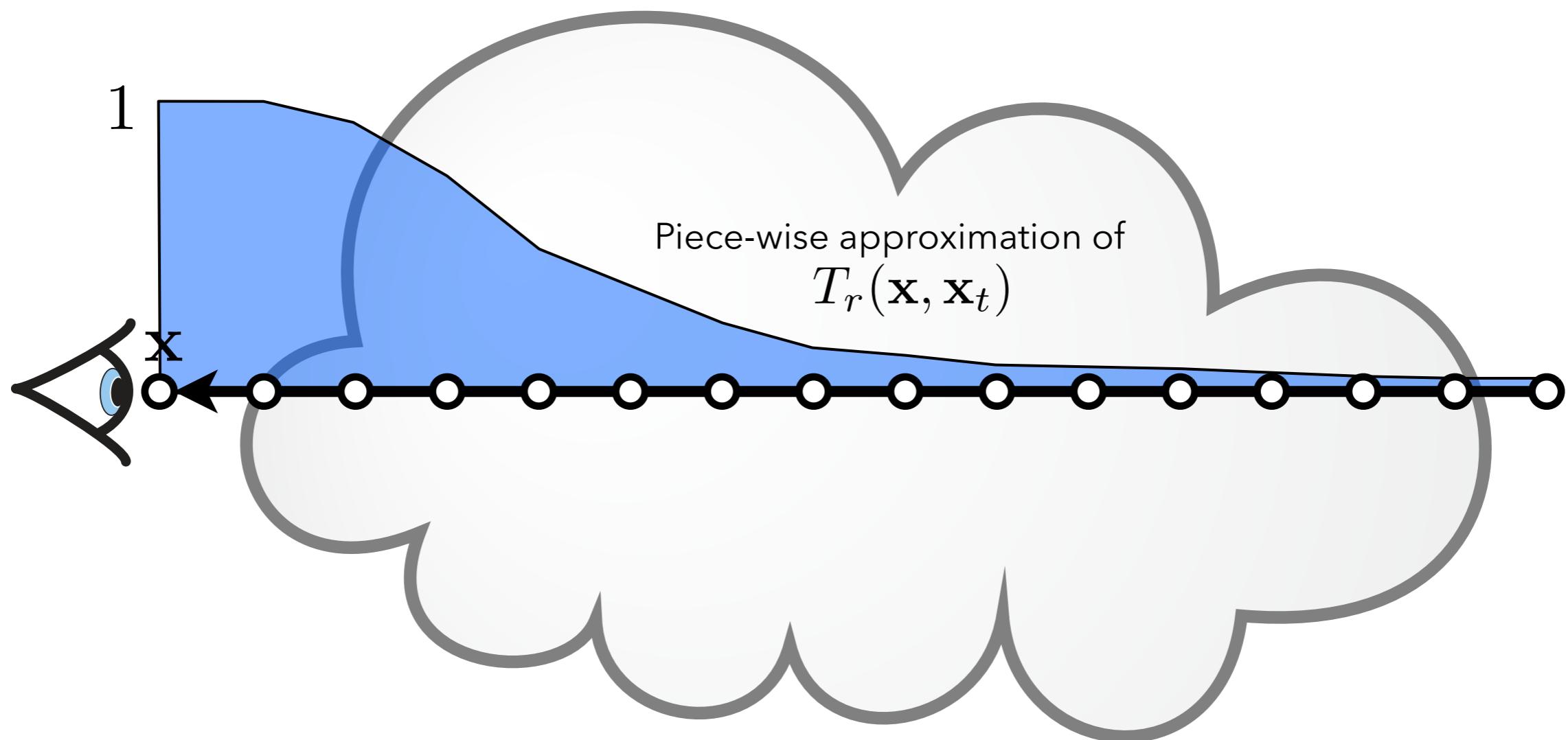
- Marching towards the light source
  - Connections are expensive, many, and uniformly distributed along the primary ray



# Decoupled Transmittance and In-scattering

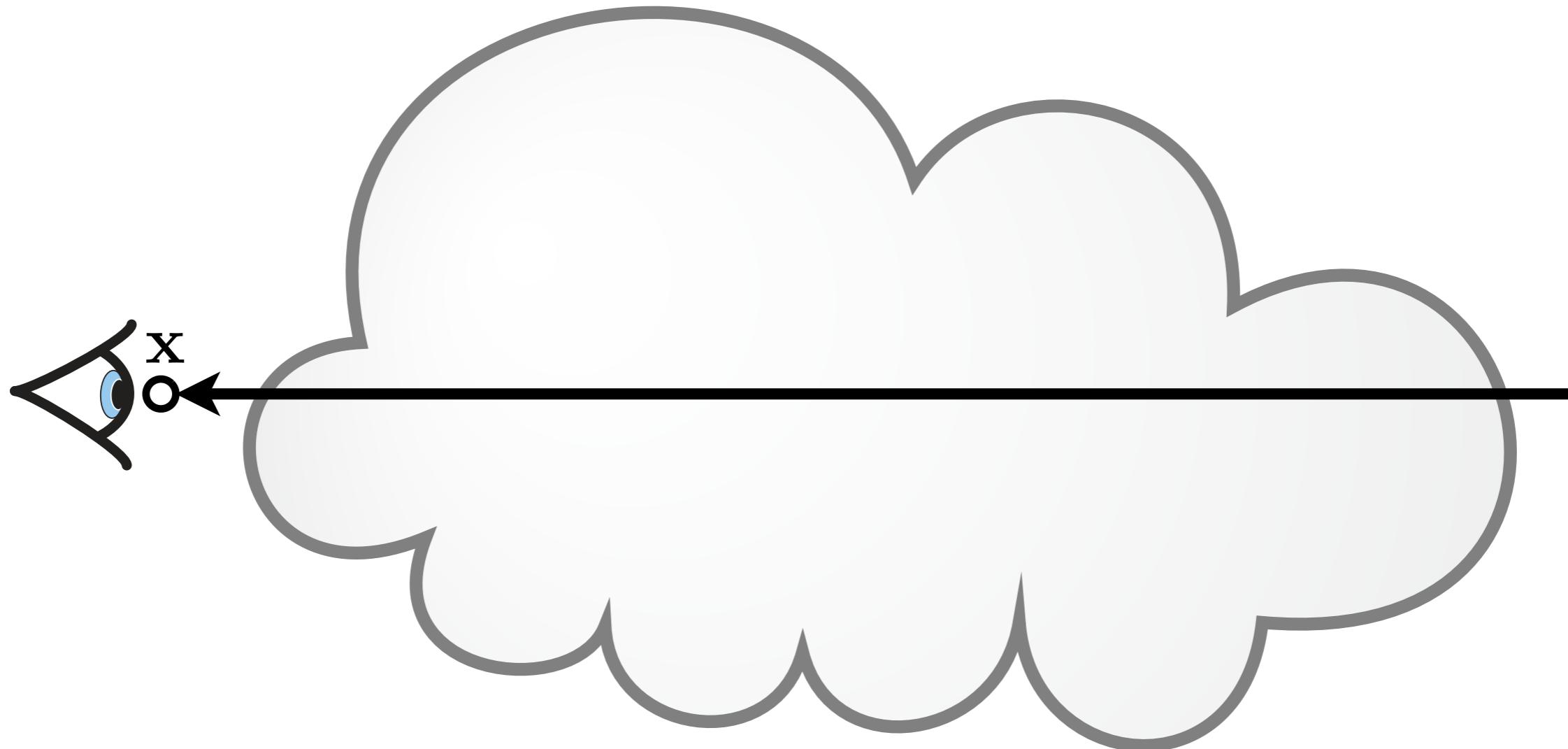
## 1. Ray-march and cache transmittance

- Choose step-size w.r.t. frequency content to accurately capture variations



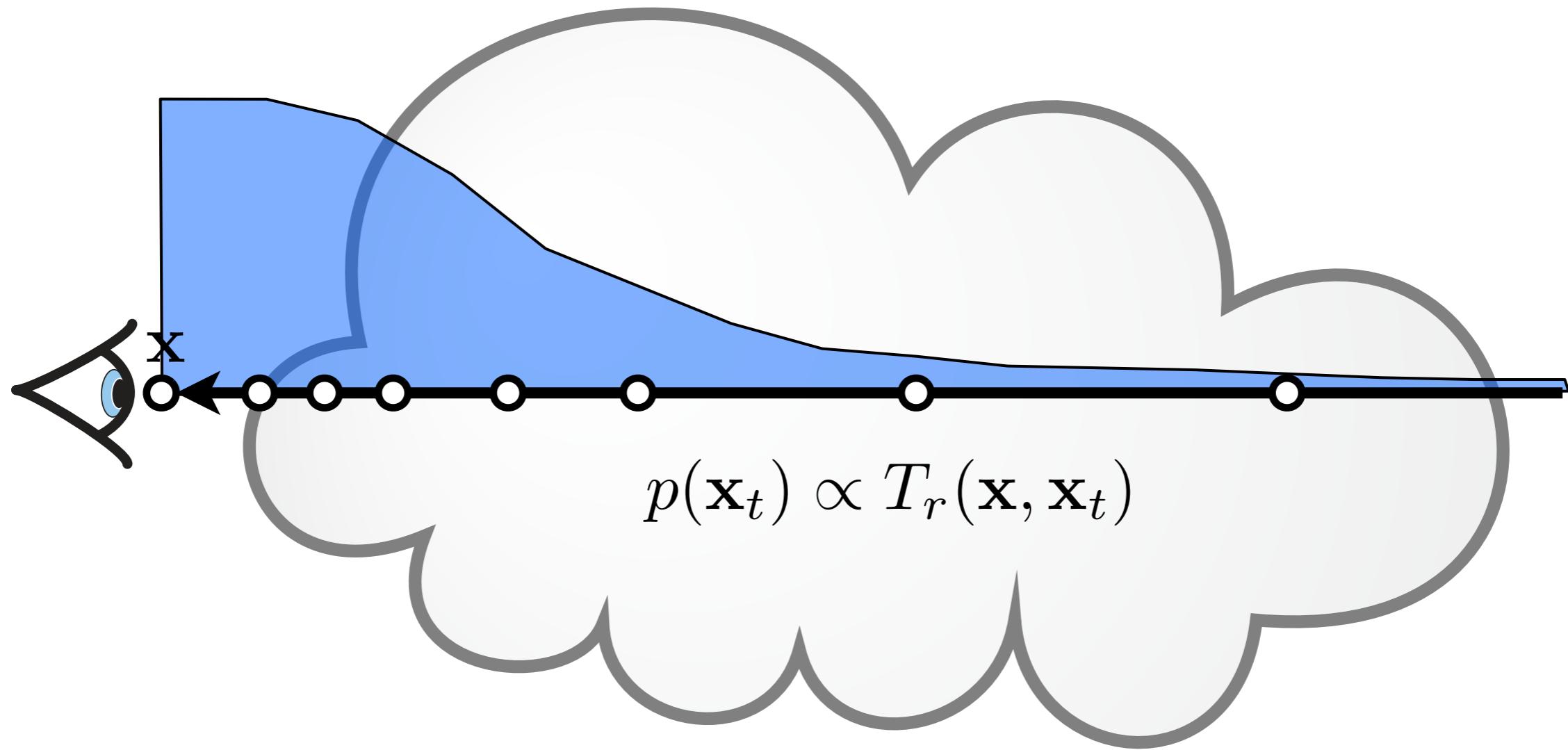
# Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
  - Distribute samples  $\propto$  (part of) the integrand



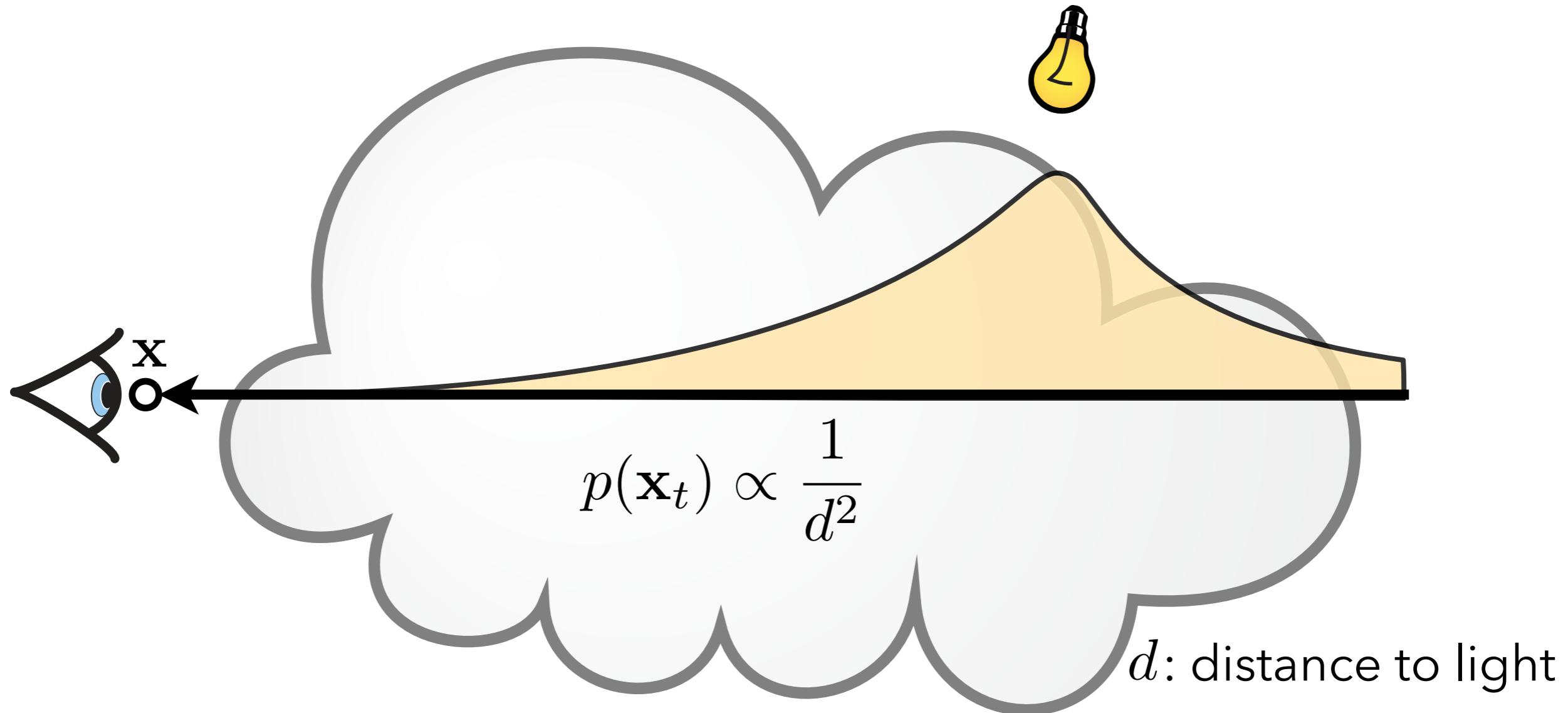
# Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
  - Distribute samples  $\propto$  (part of) the integrand



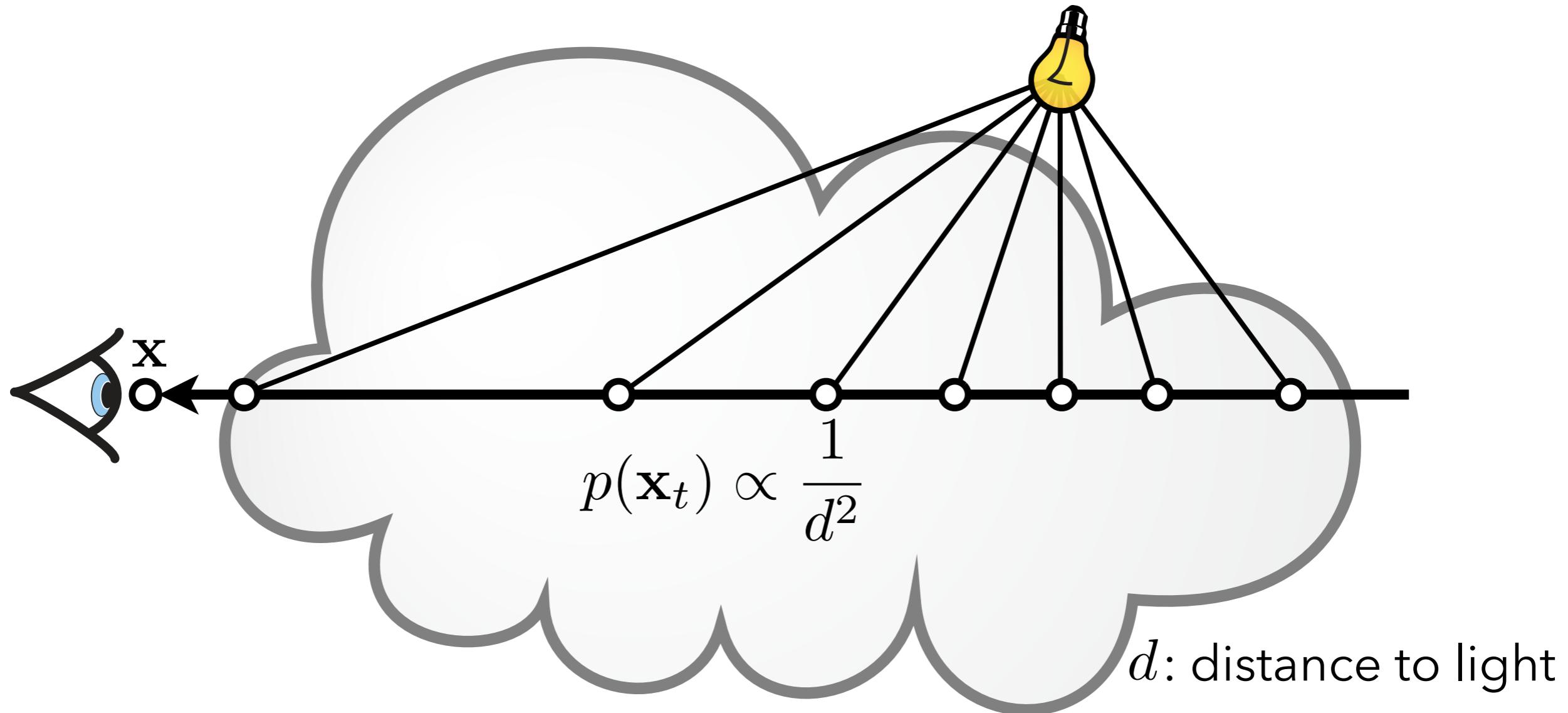
# Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
  - Distribute samples  $\propto$  (part of) the integrand



# Decoupled Transmittance and In-scattering

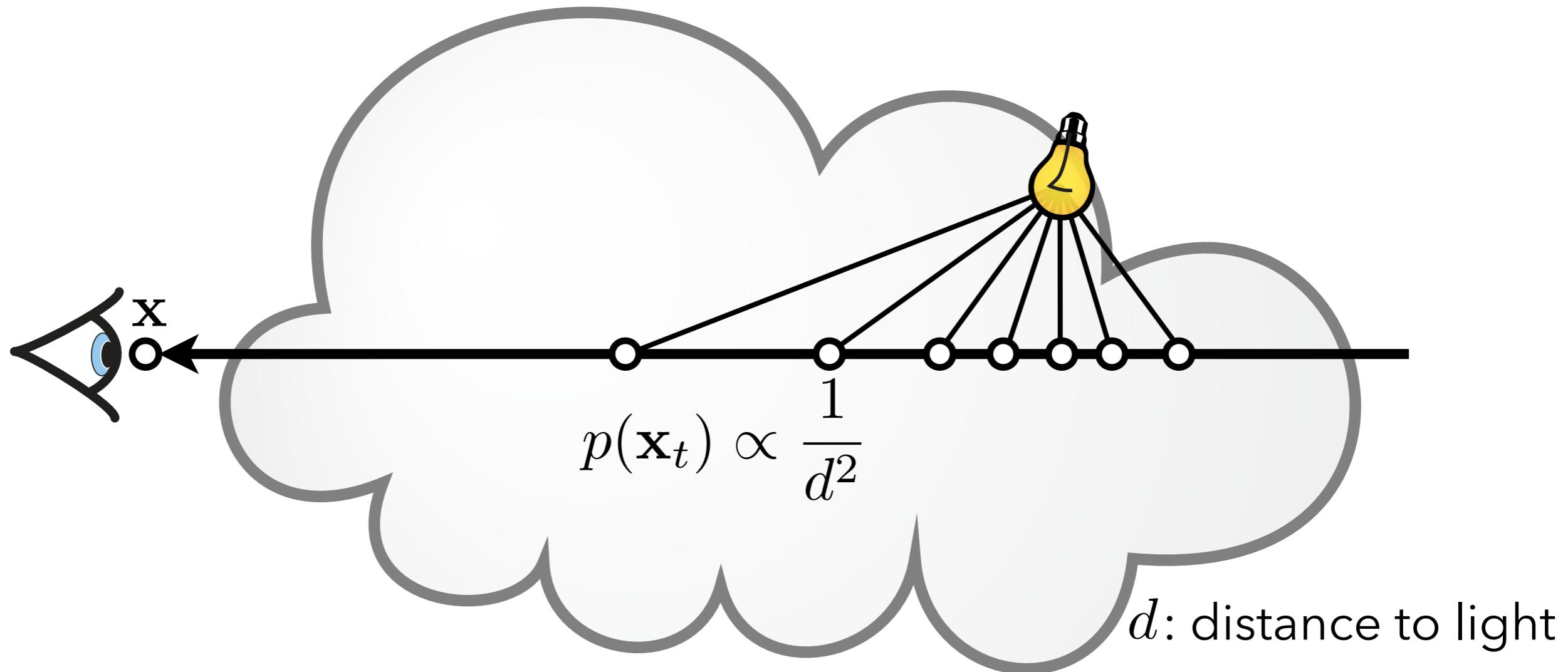
2. Estimate in-scattering using MC integration
  - Distribute samples  $\propto$  (part of) the integrand



# Decoupled Transmittance and In-scattering

## 2. Estimate in-scattering using MC integration

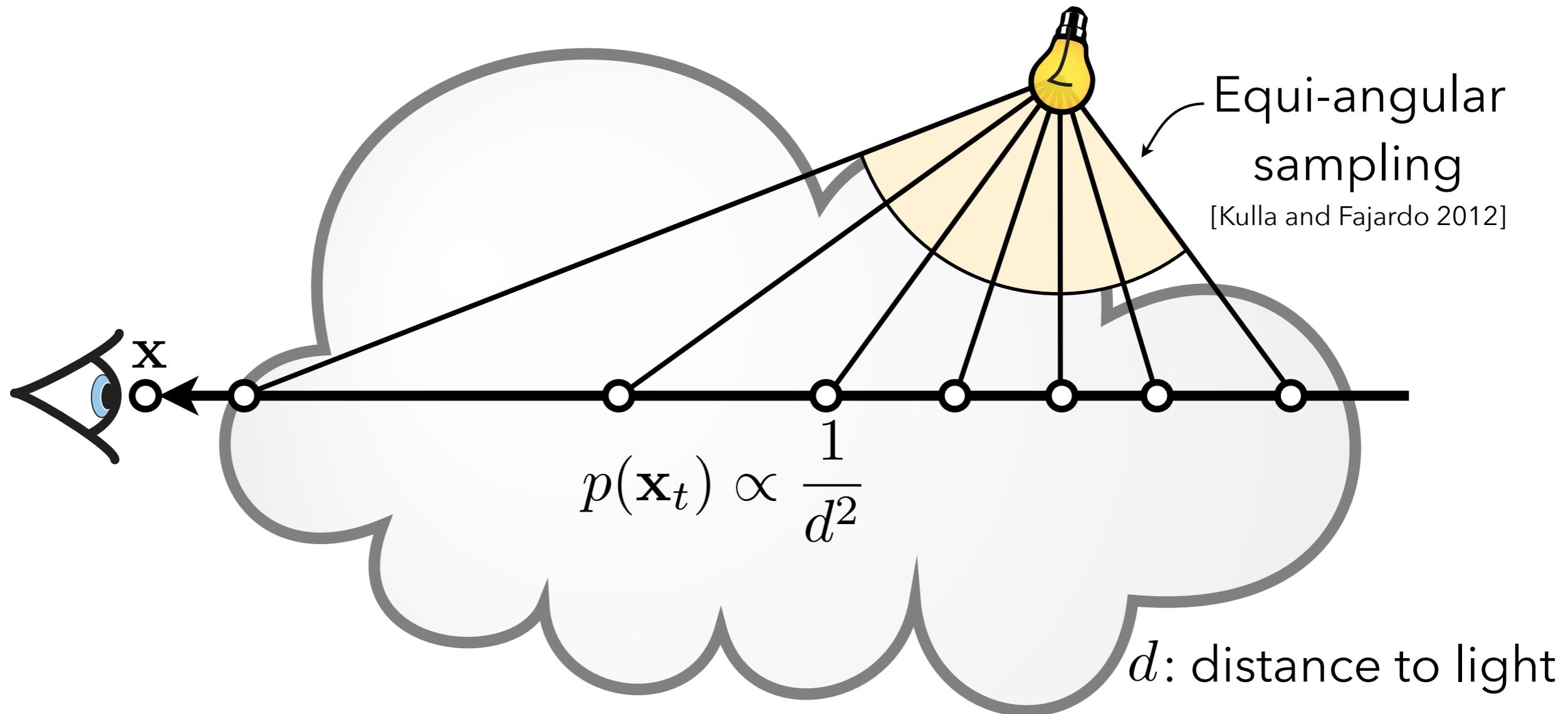
- Distribute samples  $\propto$  (part of) the integrand



# Decoupled Transmittance and In-scattering

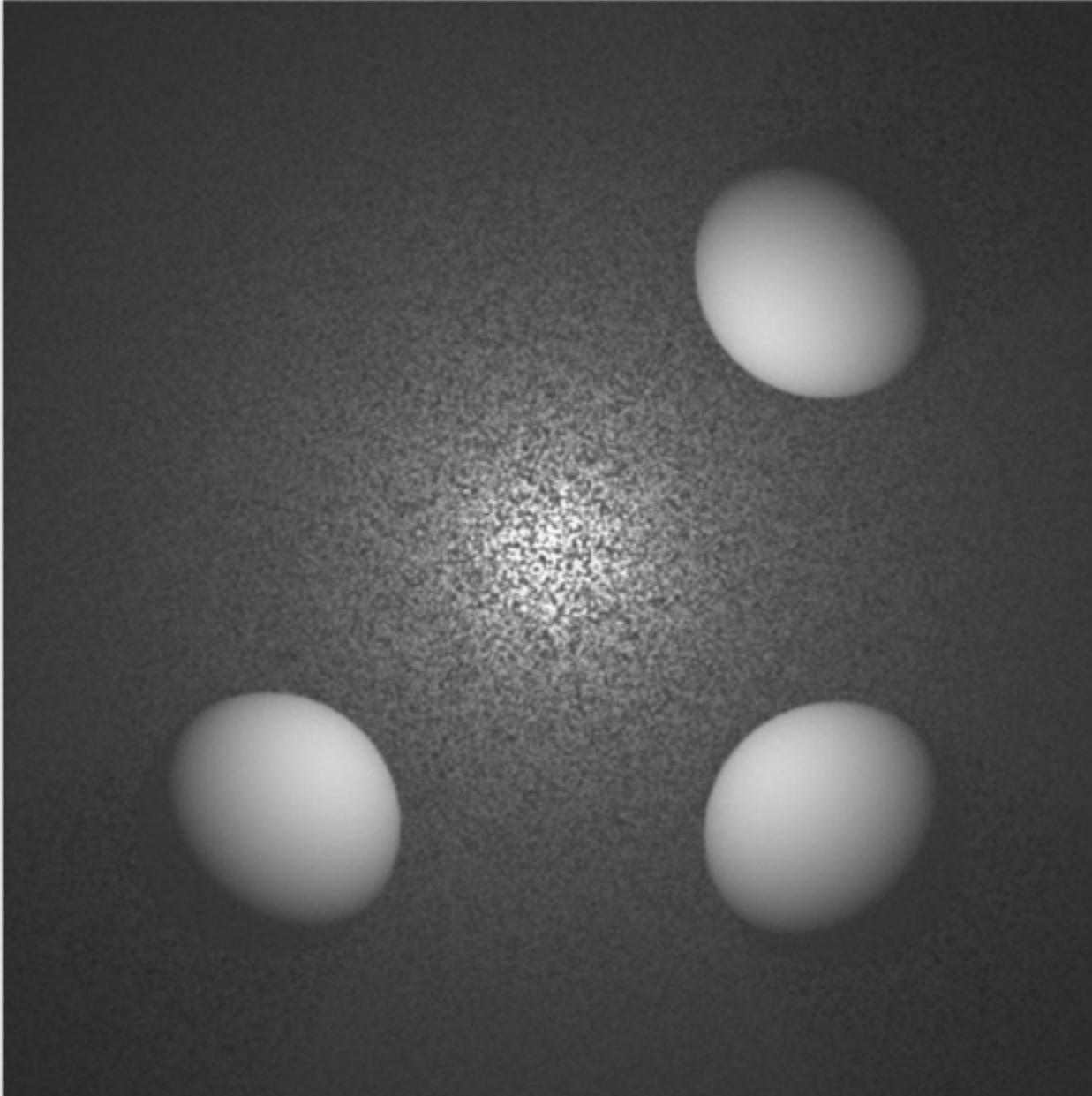
## 2. Estimate in-scattering using MC integration

- Distribute samples  $\propto$  (part of) the integrand

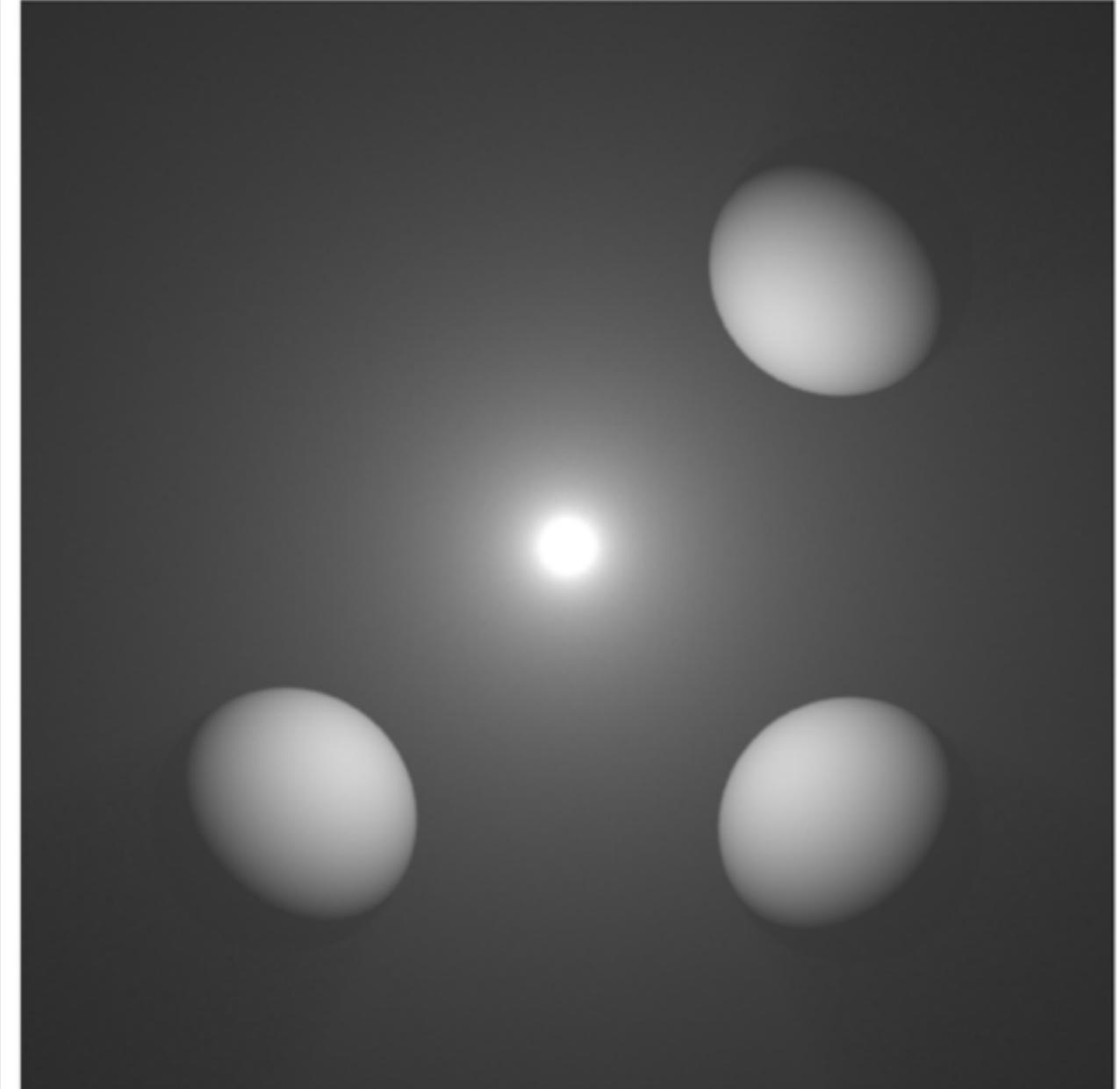


# Decoupled Transmittance and In-scattering

Ray-marching

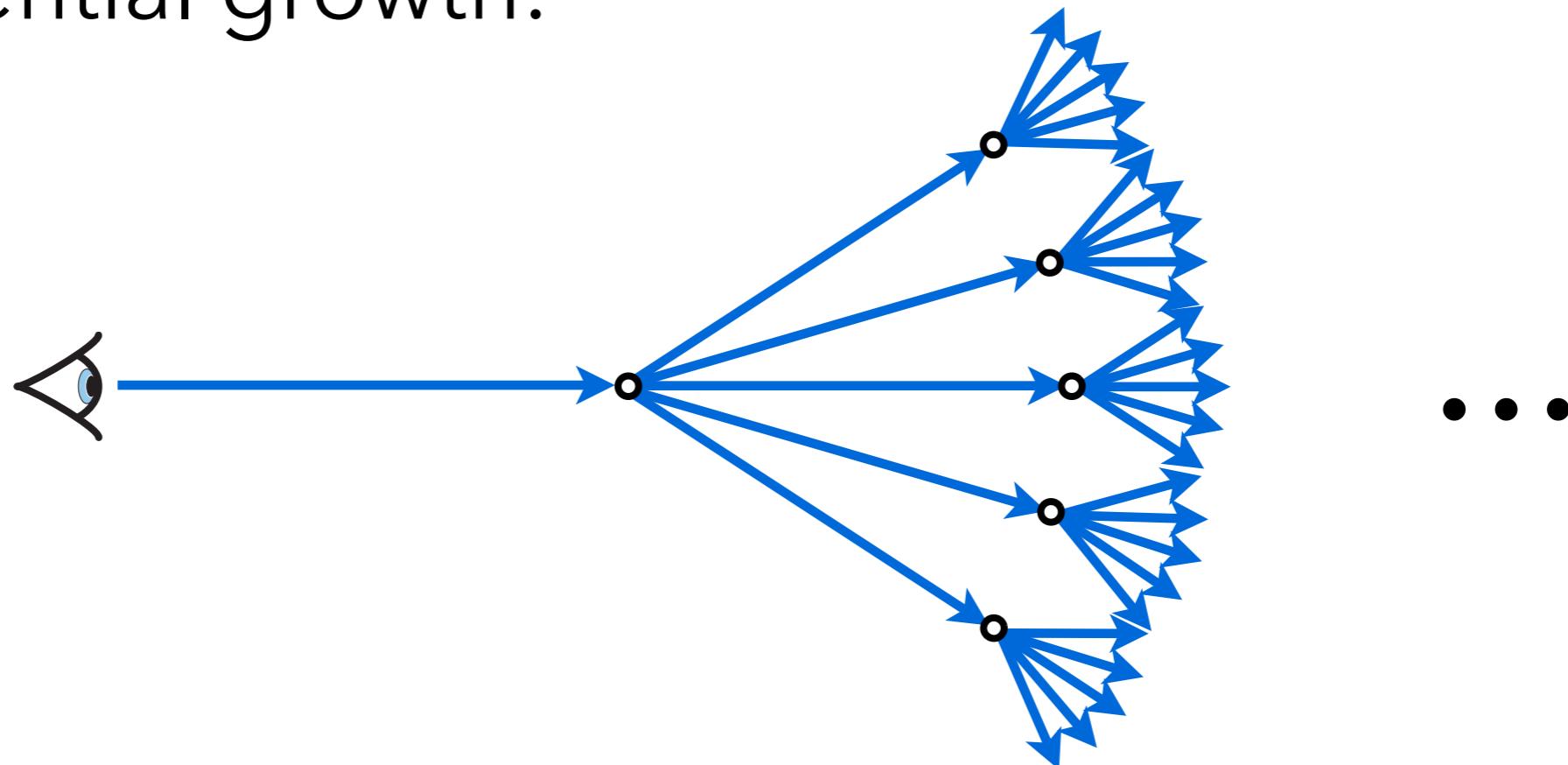


Equiangular sampling



# Multiple Bounces

- Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering
- Exponential growth:



# Visual Break

Single scattering



Multiple scattering



# Volumetric Path Tracing

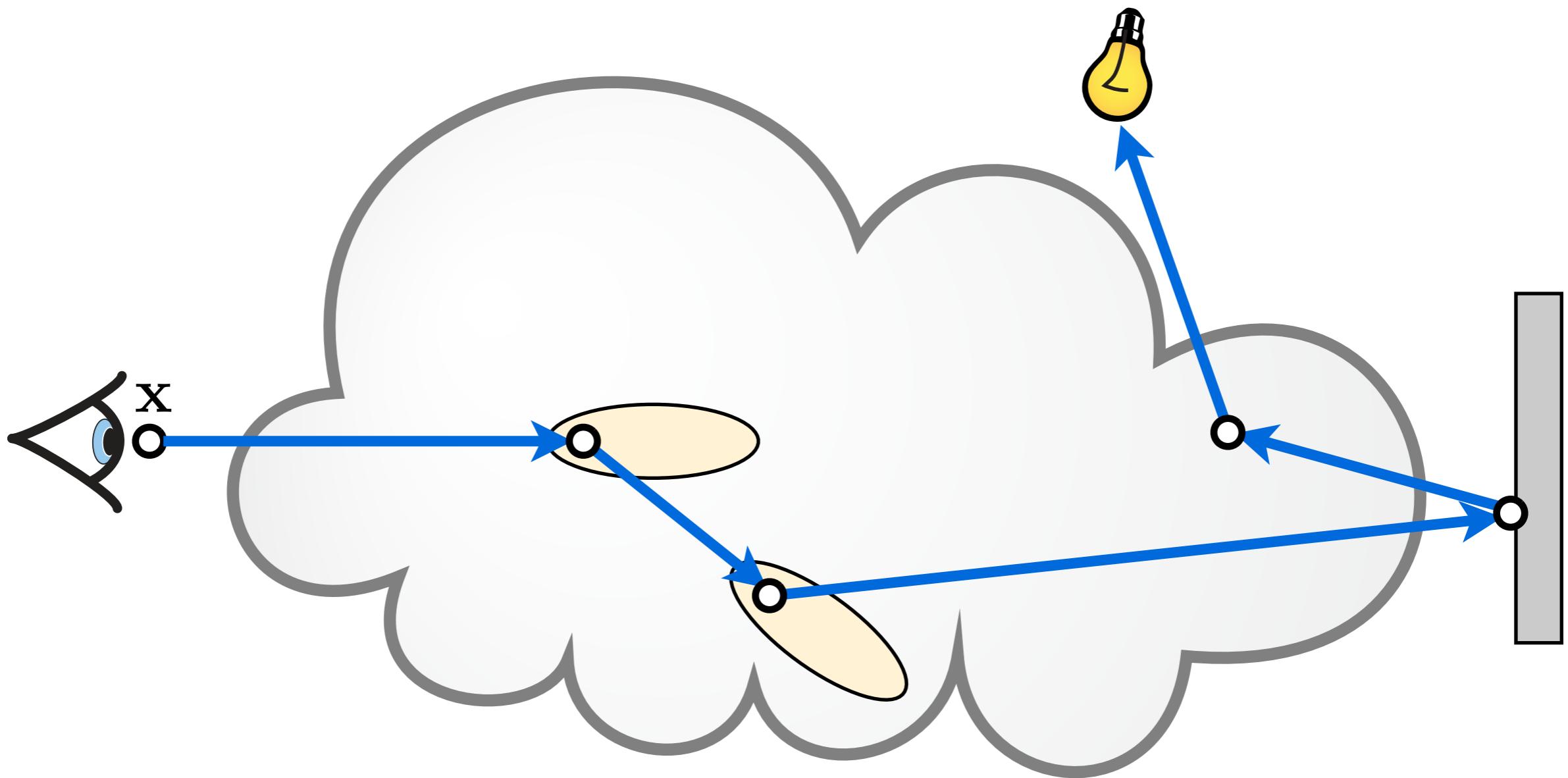
# Volumetric Path Tracing

---

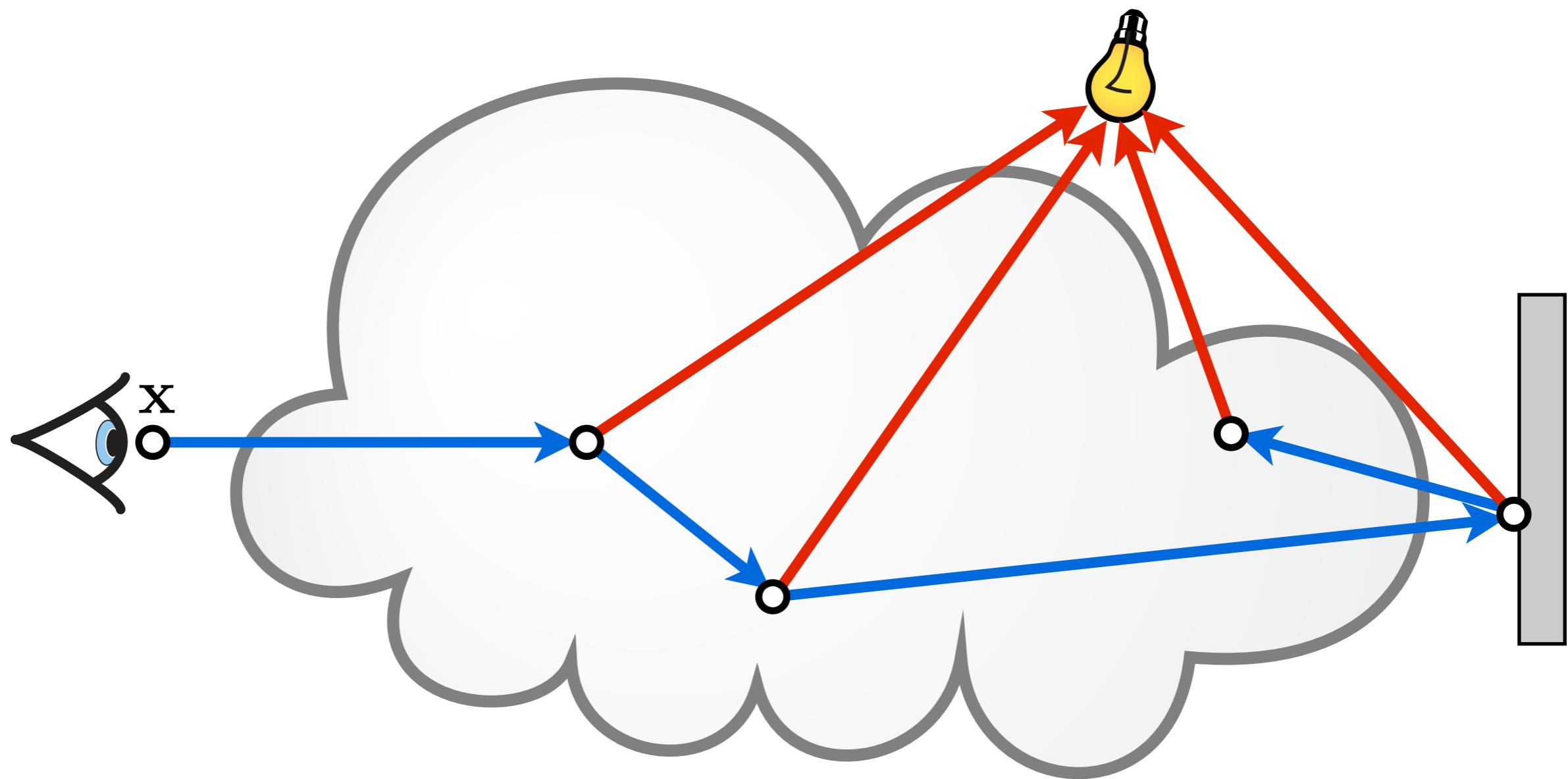
- Motivation:
  - Same as with standard path tracing: avoid the exponential growth
- Paths can:
  - Reflect/refract off surfaces
  - Scatter inside a volume

# Volumetric Path Tracing

1. Sample distance to next interaction
2. Scatter in the volume or bounce off a surface



# Volumetric Path Tracing with NEE



# Sampling the Phase Function

- Isotropic:
  - Uniform sphere sampling
- Henyey-Greenstein:
  - Using the inversion method we can derive

$$\cos \theta = \frac{1}{2g} \left( 1 + g^2 - \left( \frac{1 - g^2}{1 - g + 2g\xi} \right)^2 \right)$$

$$\phi = 2\pi\zeta$$

- PDF is the value of the HG phase function

# Free-path Sampling

- Free-path (or free-flight distance):
  - Distance to the next interaction within the medium
  - Dense media (e.g. milk): short mean-free path
  - Thin media (e.g. atmosphere): long mean-free path
- Ideally, we want to sample proportional to transmittance

$$p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$
$$p(t) \propto T_r(t)$$

simplified notation for brevity

# Free-path Sampling

- Homogeneous media:  $T_r(t) = e^{-\sigma_t t}$

- PDF:  $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

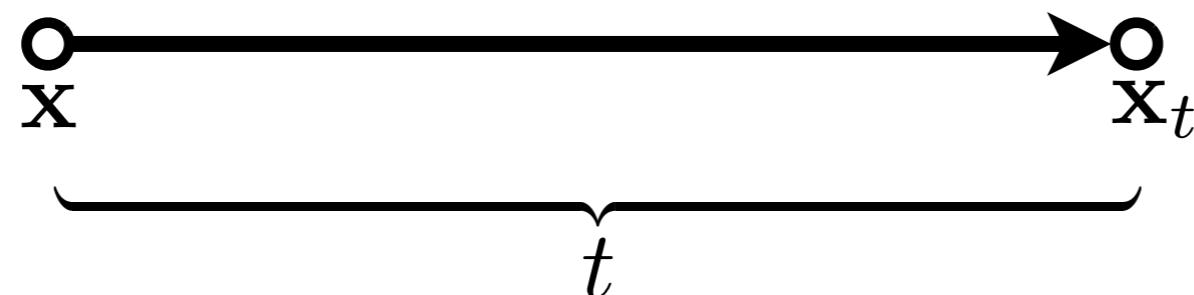
- CDF:  $P(t) = \int_0^t \sigma_t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

- Inverted CDF:

$$P^{-1}(\xi) = -\frac{\ln(1 - \xi)}{\sigma_t}$$

# Free-path Sampling

- Homogeneous media:  $T_r(t) = e^{-\sigma_t t}$
- Recipe:
  - Generate random number  $\xi$
  - Sample distance  $t = -\frac{\ln(1 - \xi)}{\sigma_t}$
  - Compute PDF  $p(t) = \sigma_t e^{-\sigma_t t}$



# Free-path Sampling

- Homogeneous media:  $T_r(t) = e^{-\sigma_t t}$

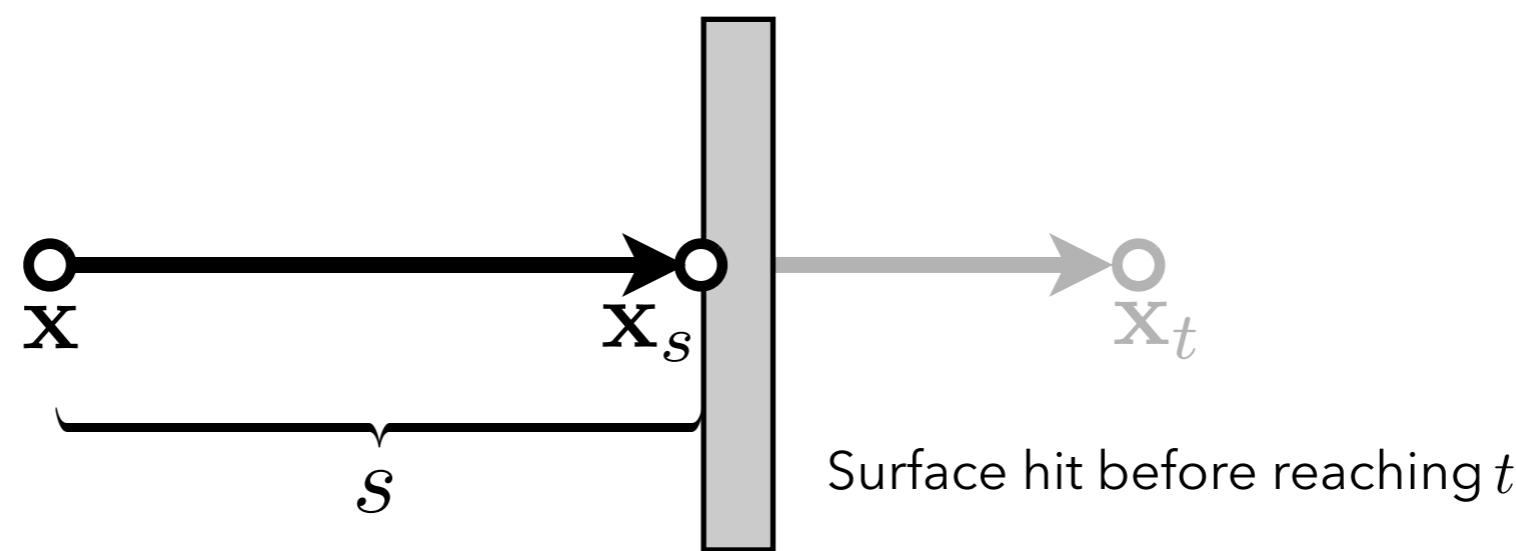
- Recipe:

- Generate random number  $\xi$

- Sample distance  $t = -\frac{\ln(1 - \xi)}{\sigma_t} = s$

- Compute PDF  $p(t) = \sigma_t e^{-\sigma_t t} = e^{-\sigma_t s}$

Note: This is now a probability, not a probability density!



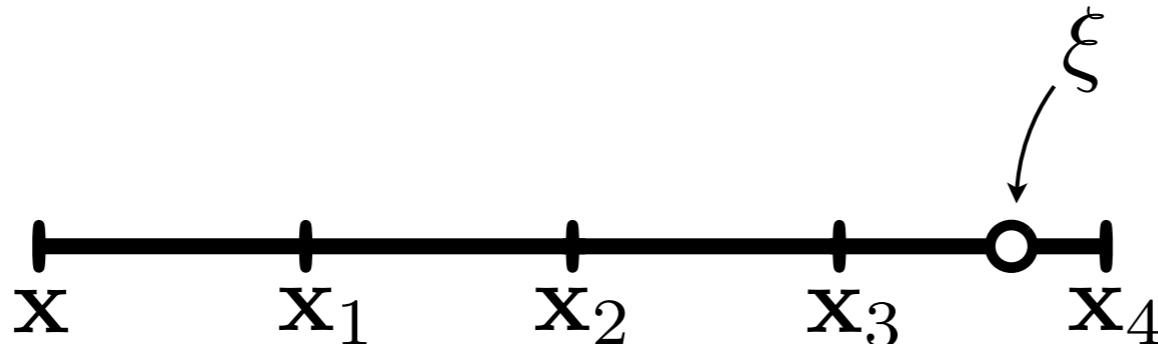
# Free-path Sampling

---

- Heterogeneous media:  $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$ 
  - Closed-form solutions exist only for simple media
    - e.g. linearly or exponentially varying extinction
  - Other solutions:
    - Ray marching
    - Woodcock tracking
    - Regular tracking (3D DDA)

# Ray-Marching

- Assume constant extinction along each step
- Recipe:
  - March until  $T_r(\mathbf{x}_0, \mathbf{x}_i) < \xi$
  - Use closed-form solution within the last segment



# Ray-Marching

---

- Issues:
  - Biased since  $E[e^X] \neq e^{E[X]}$ 
    - The exponentiation skews the noise distribution and the estimation error no longer averages to 0 in the limit
  - Sensitive to resolution of heterogeneous volume
    - To keep bias low, we need to march on the order of the highest frequency (Nyquist-Shannon theorem)
    - Can be expensive

# Volumetric PT for Homogeneous Volumes

```
Color vPT(x, ω)
```

```
tmax = nearestSurface(x, ω)
t = -log(1 - rand()) / σt           // Sample free path
if t < tmax: // Volume interaction
    x += t * ω
    pdf_t = σt * exp(-σt * t)
    (ω', pdf_ω') = samplePF(ω)
    // Note: transmittance and PF cancel out with PDFs
    // except for a constant factor 1/σt
    return Tr(t) * σs * PF(ω) * vPT(x, ω') / (pdf_t * pdf_ω')
else: // Surface interaction
    x += tmax * ω
    Pr_tmax = exp(-σt * tmax)
    (ω', pdf_ω') = sampleBRDF(n, ω)
    // Note: transmittance and prob of sampling the distance
    // cancel out
    return Tr(tmax) * BRDF(ω, ω') * vPT(x, ω') / (Pr_tmax * pdf_ω')
```

# Volumetric PT for Homogeneous Volumes

Color vPT( $x, \omega$ )

```
tmax = nearestSurface(x, ω)
t = -log(1 - rand()) / σt           // Sample free path
if t < tmax: // Volume interaction
    x += t * ω
    pdf_t = σt * exp(-σt * t)
    (ω', pdf_ω') = samplePF(ω)
    // Note: transmittance and PF cancel out with PDFs
    // except for a constant factor 1/σt
    return σs / σt * vPT(x, ω')
else: // Surface interaction
    x += tmax * ω
    Pr_tmax = exp(-σt * tmax)
    (ω', pdf_ω') = sampleBRDF(n, ω)
    // Note: transmittance and prob of sampling the distance
    // cancel out
    return BRDF(ω, ω') / pdf_ω' * vPT(x, ω')
```

# Woodcock Tracking

(a.k.a. pseudo scattering, delta tracking, null-collision algorithm,...)

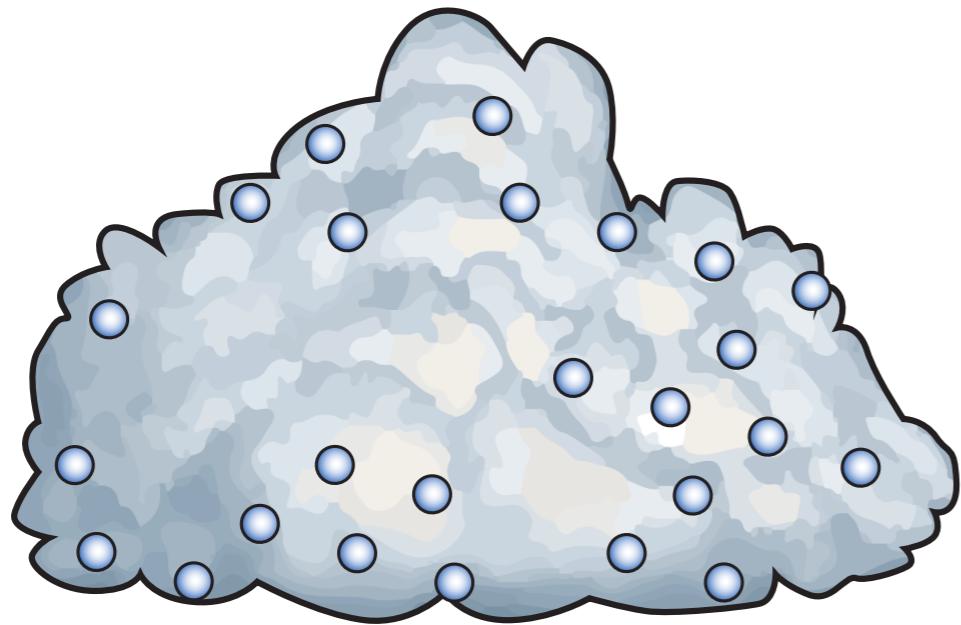
# Woodcock Tracking

---

- Unbiased technique for free-path sampling
- Inspired by *rejection sampling*
- Idea:
  - Add a *fictitious* volume
  - Combined volume (real + fictitious) is homogeneous
  - Generate (*tentative*) free-paths analytically
  - Probabilistically reject/accept collisions based on local concentrations of real vs. fictitious volumes

# Woodcock Tracking

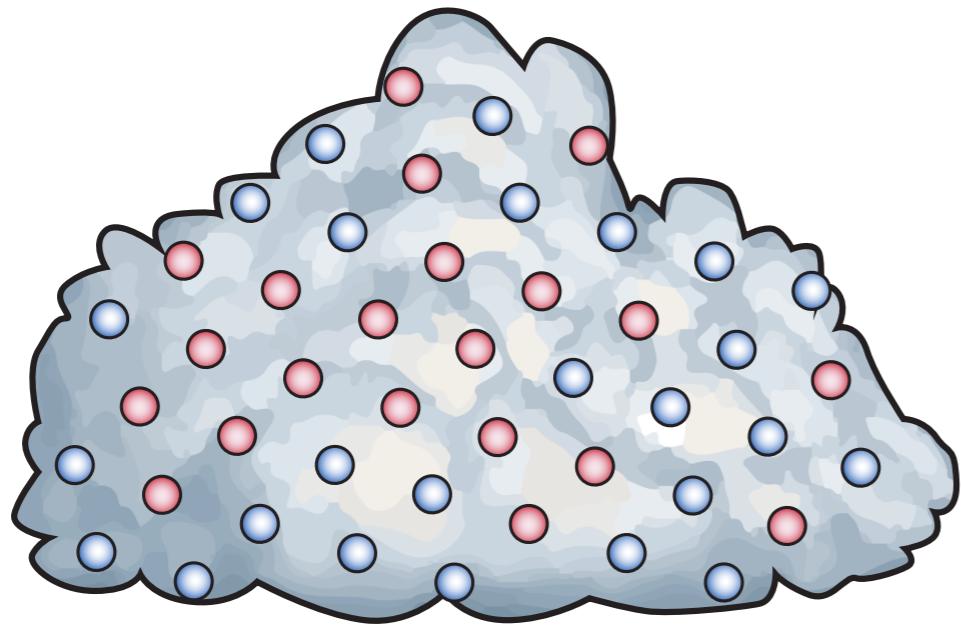
- Add fictitious particles that do not impact transport
- Majorant extinction  $\bar{\sigma}_t = \text{real} + \text{fictitious}$  extinction



\*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

# Woodcock Tracking

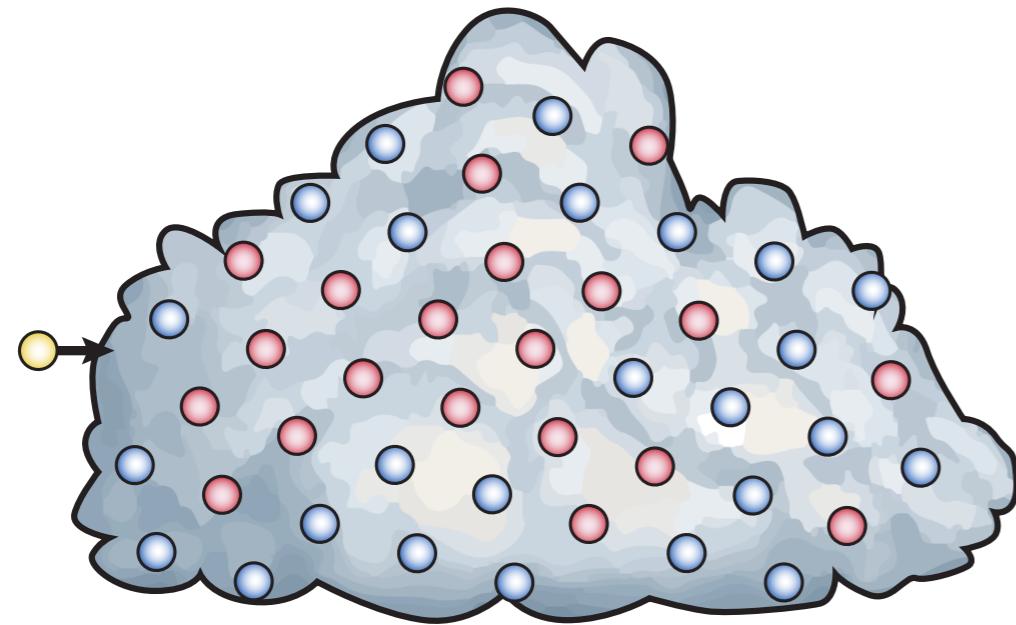
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# Woodcock Tracking

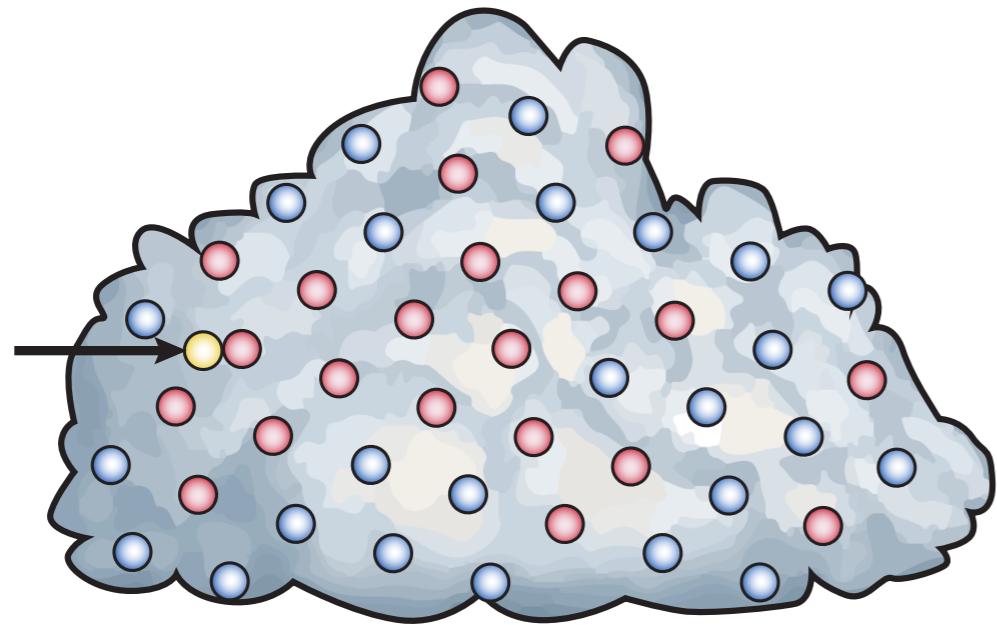
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# Woodcock Tracking

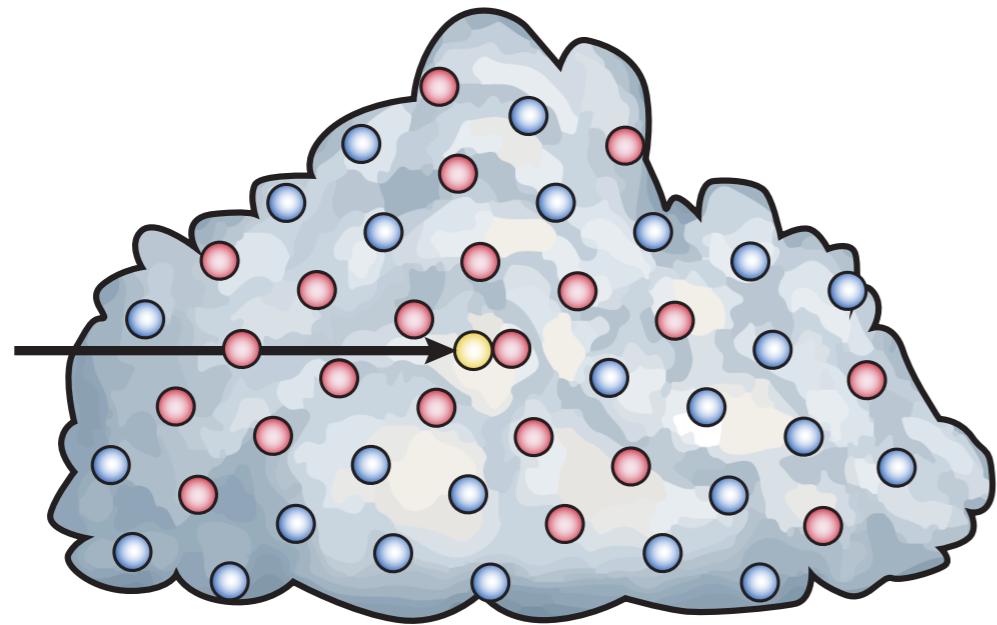
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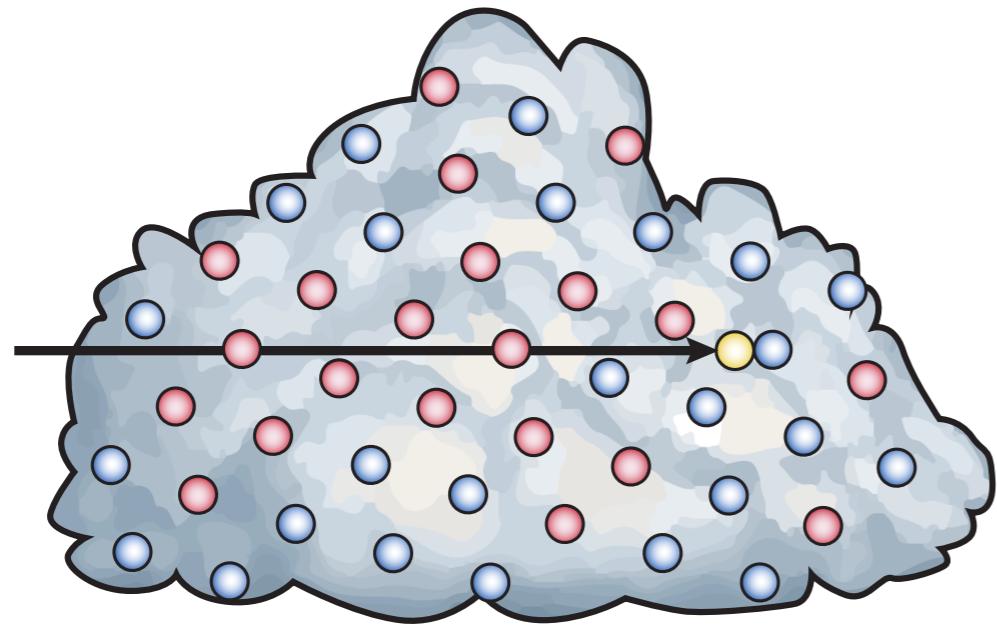
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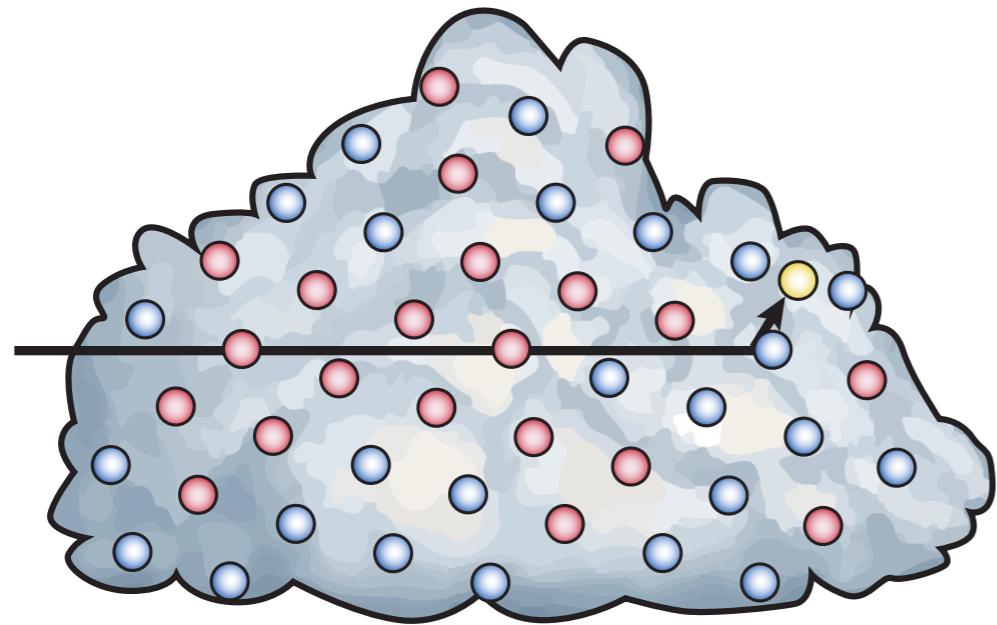
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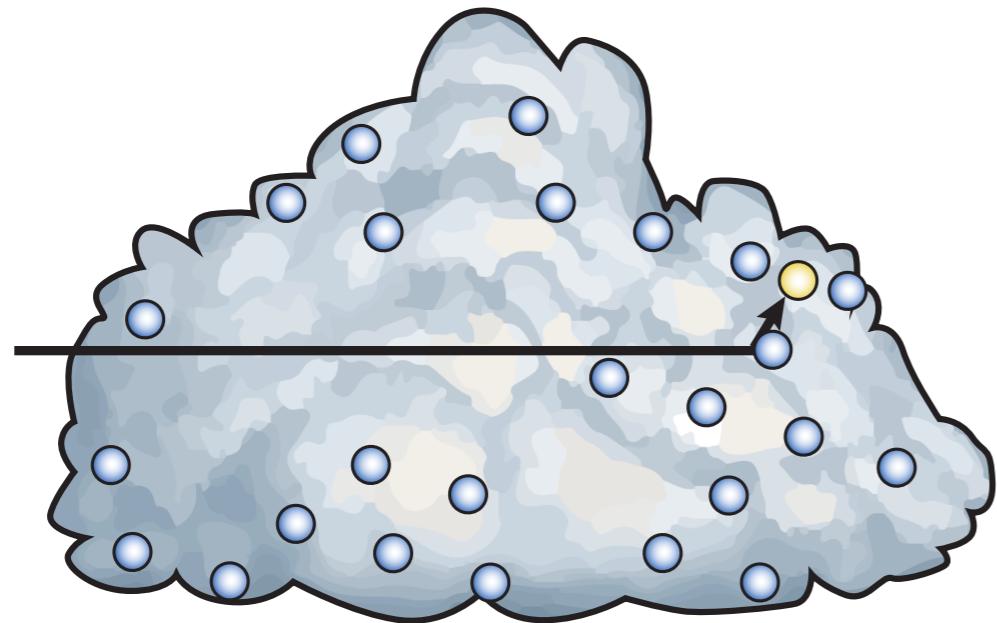
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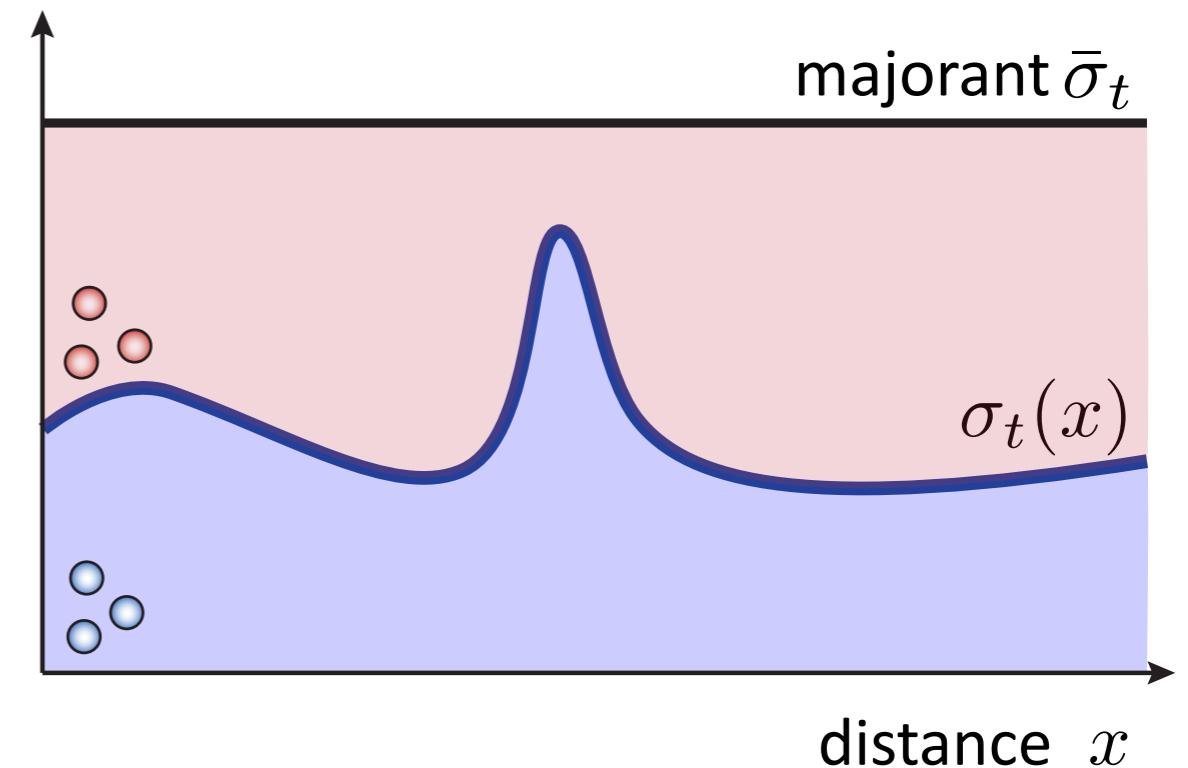
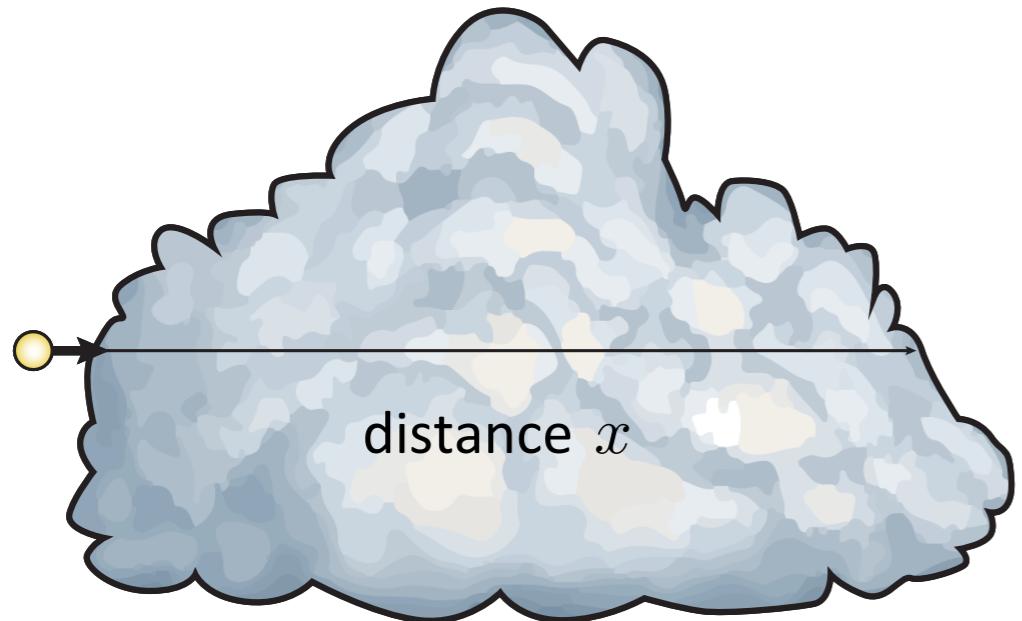


\*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

# Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction  $\bar{\sigma}_t = \text{real} + \text{fictitious extinction}$

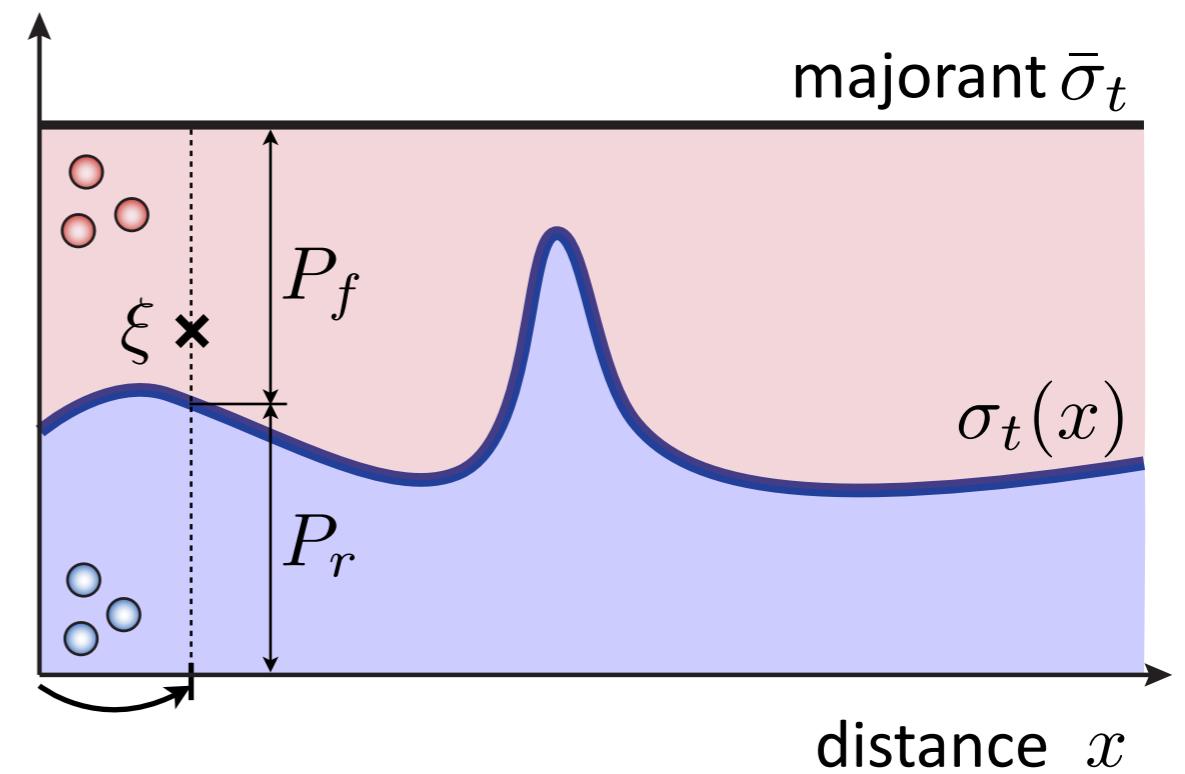
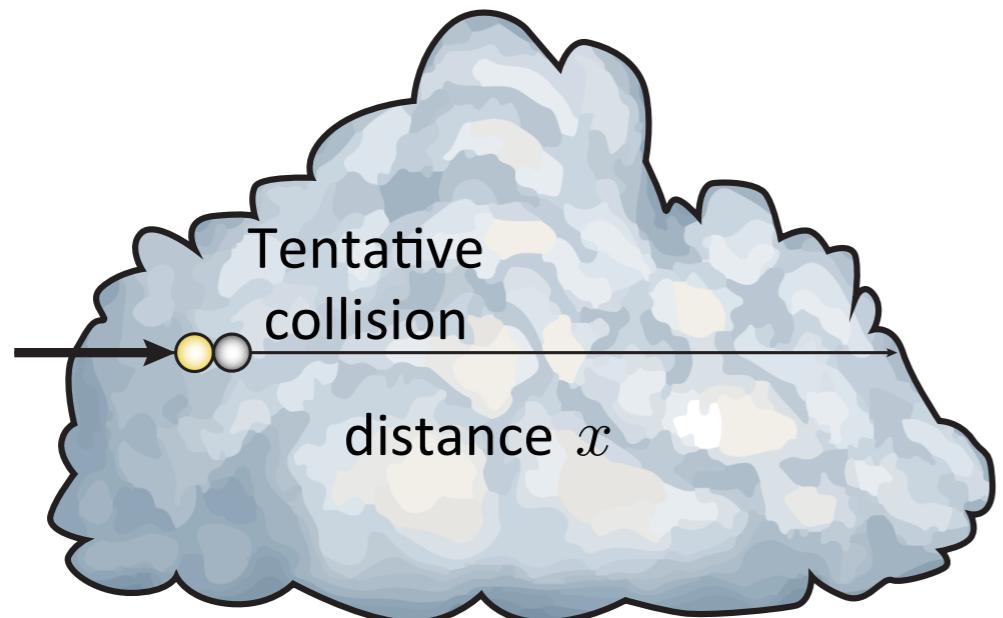
$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \quad P_f = 1 - P_r$$



# Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction  $\bar{\sigma}_t = \text{real} + \text{fictitious extinction}$

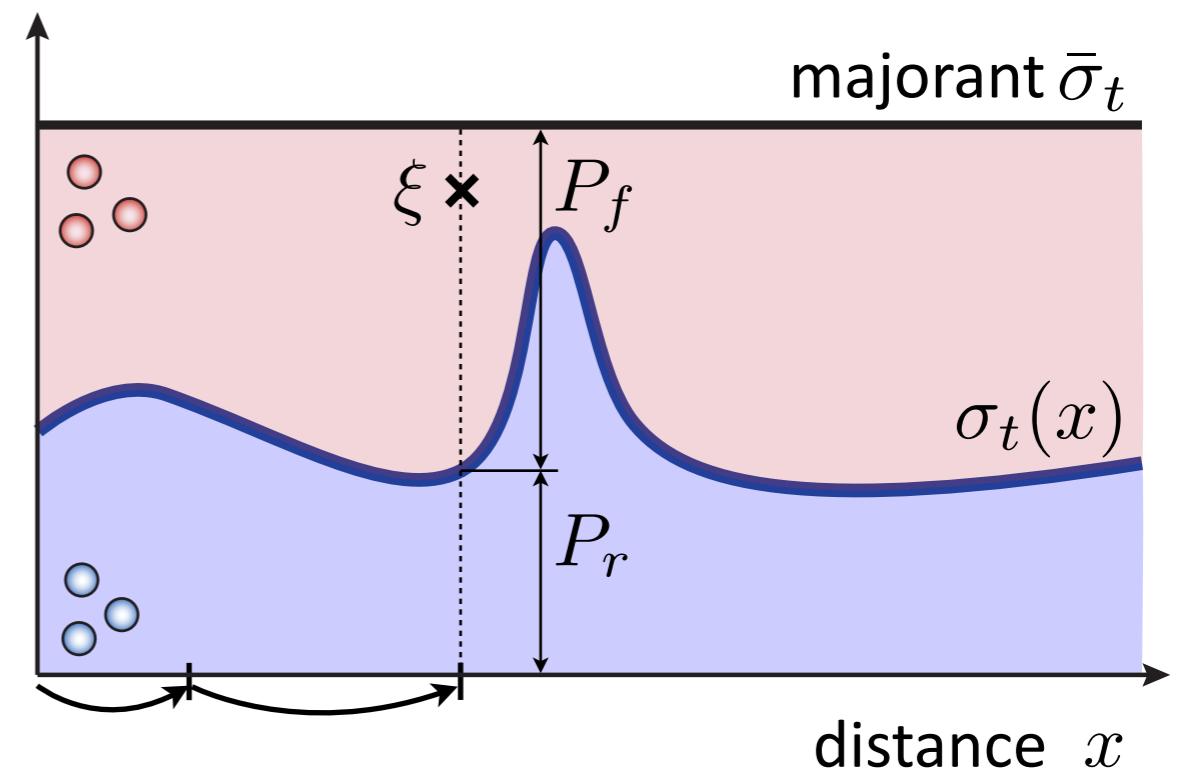
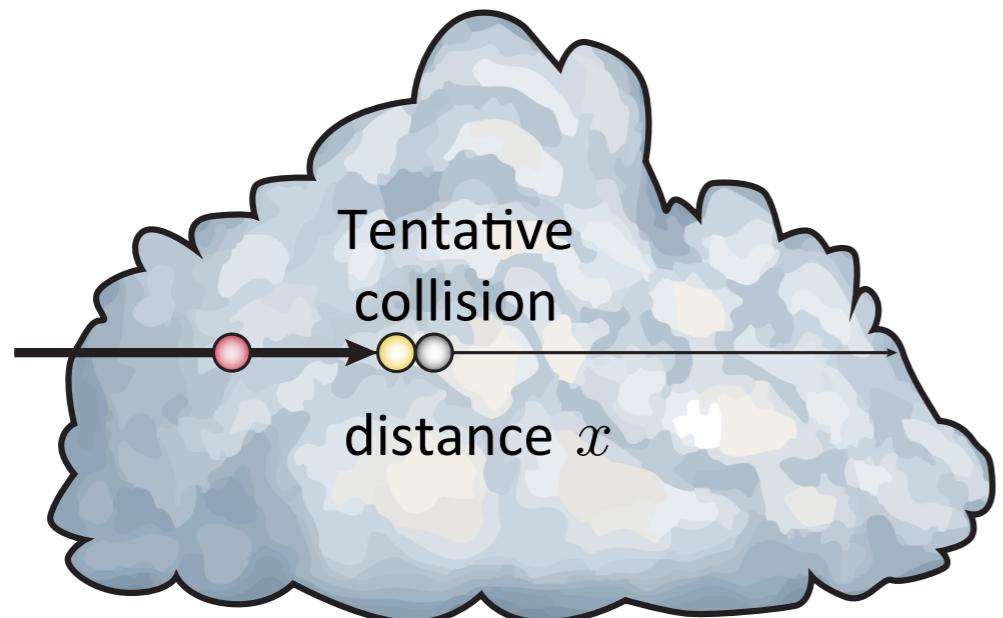
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# Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction  $\bar{\sigma}_t = \text{real} + \text{fictitious extinction}$

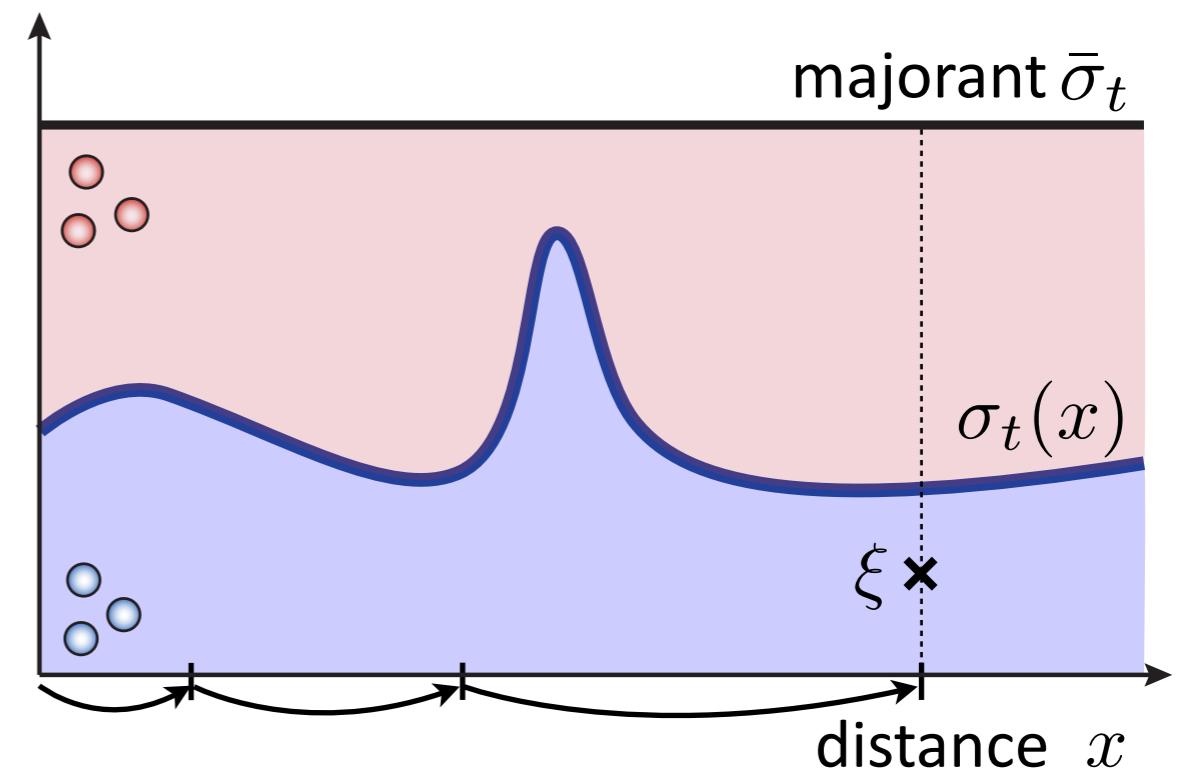
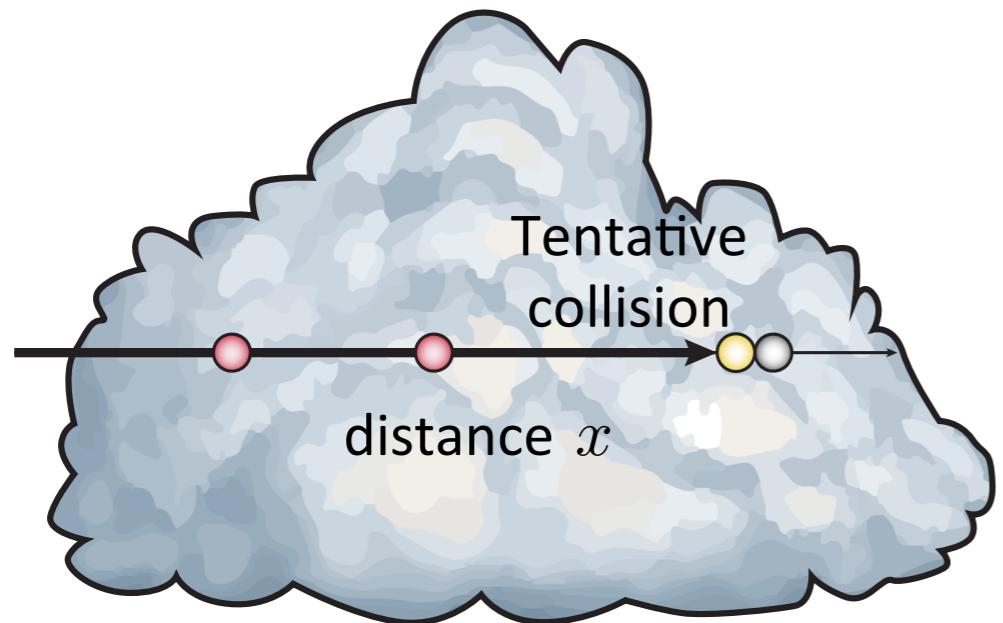
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# Woodcock Tracking

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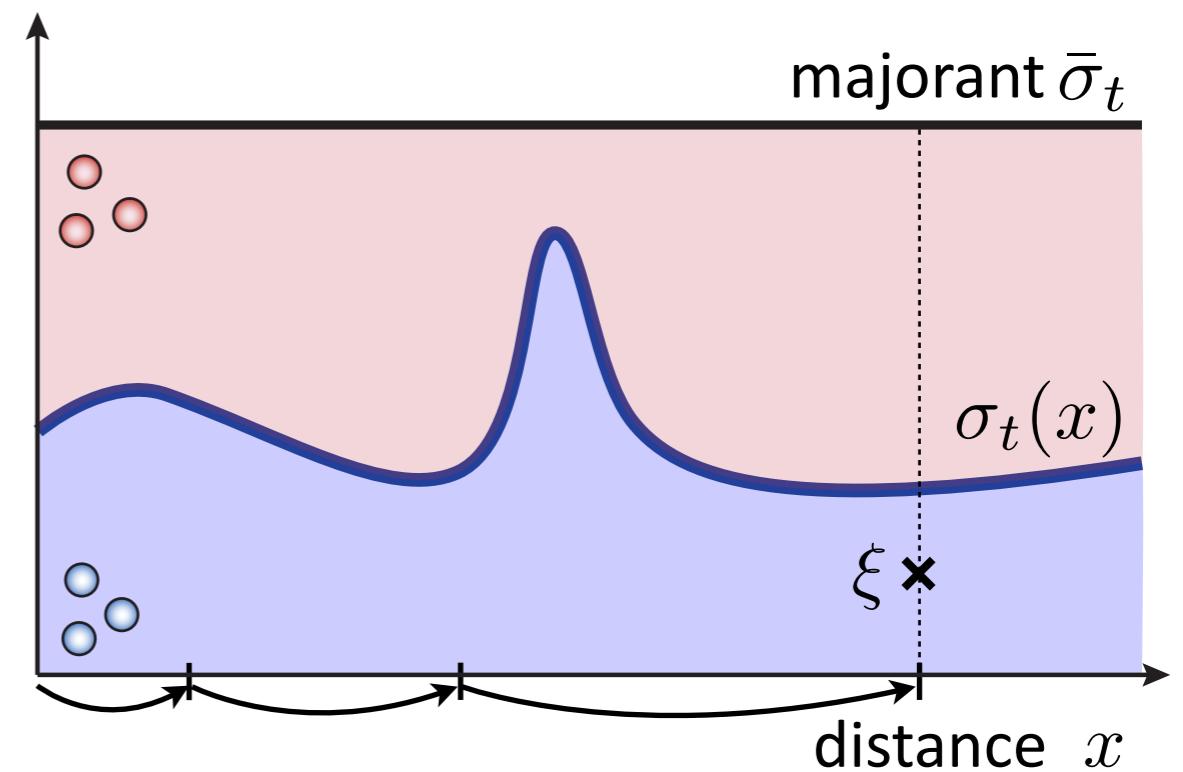
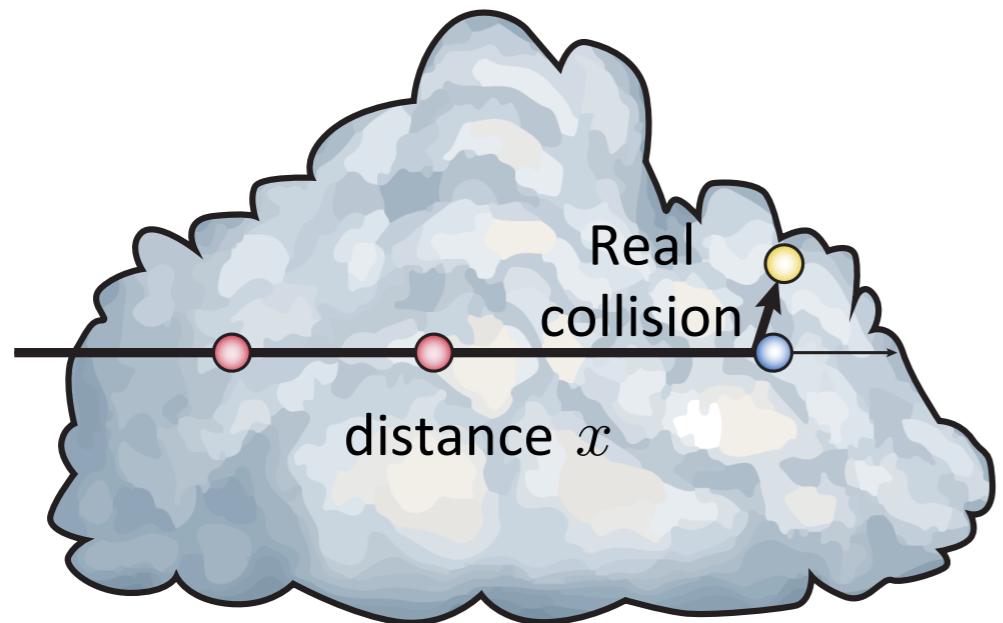
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# Woodcock Tracking

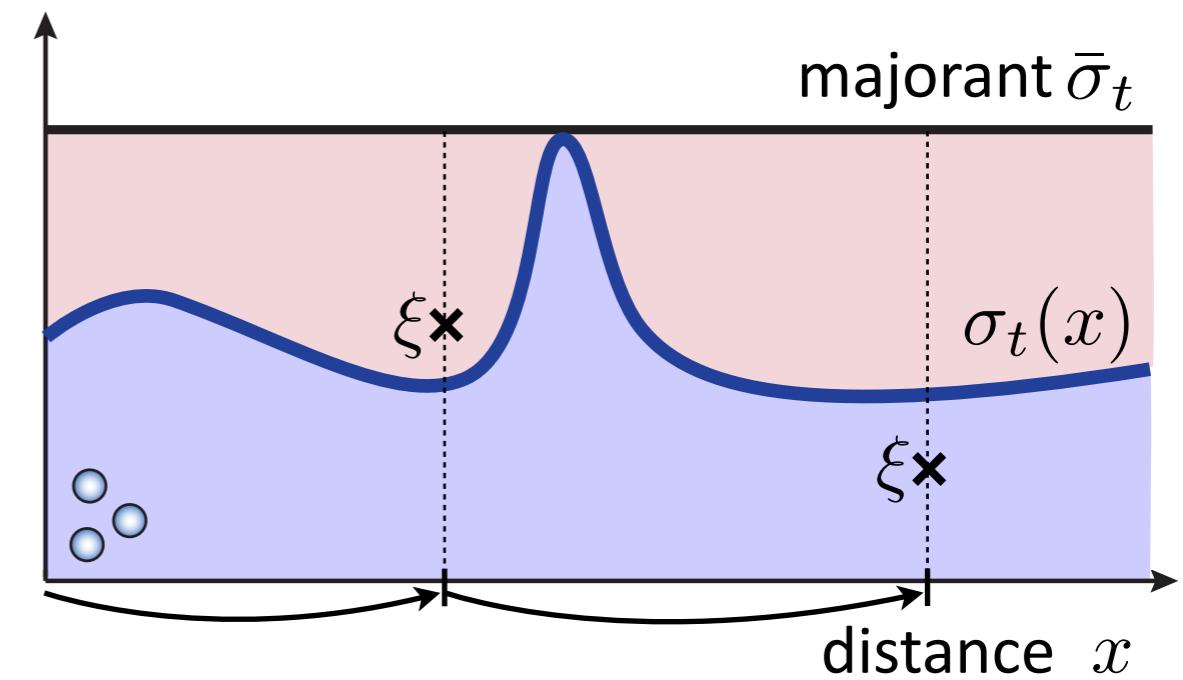
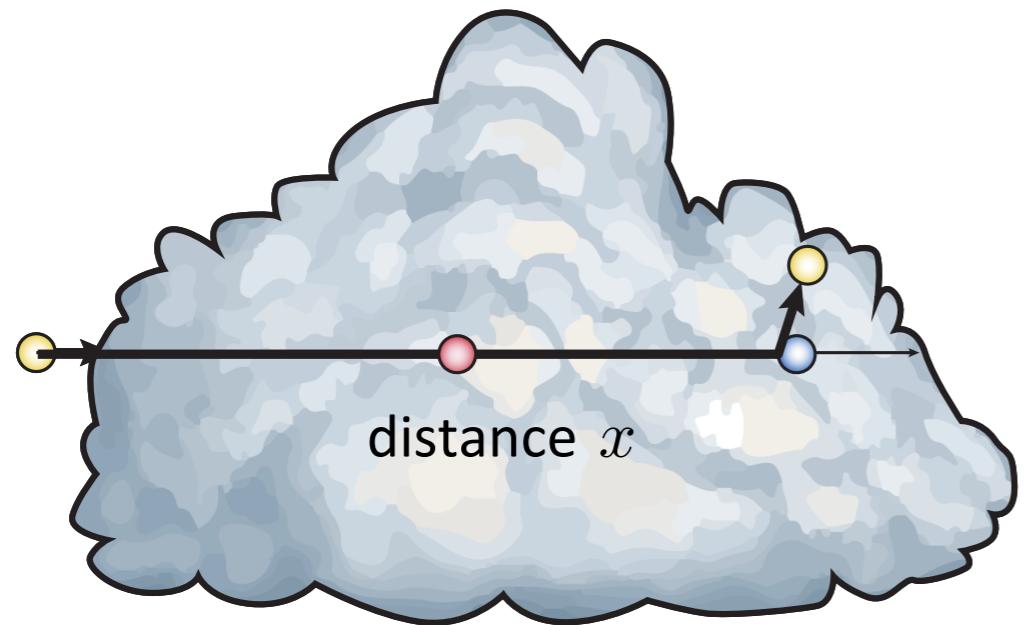
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$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \quad P_f = 1 - P_r$$



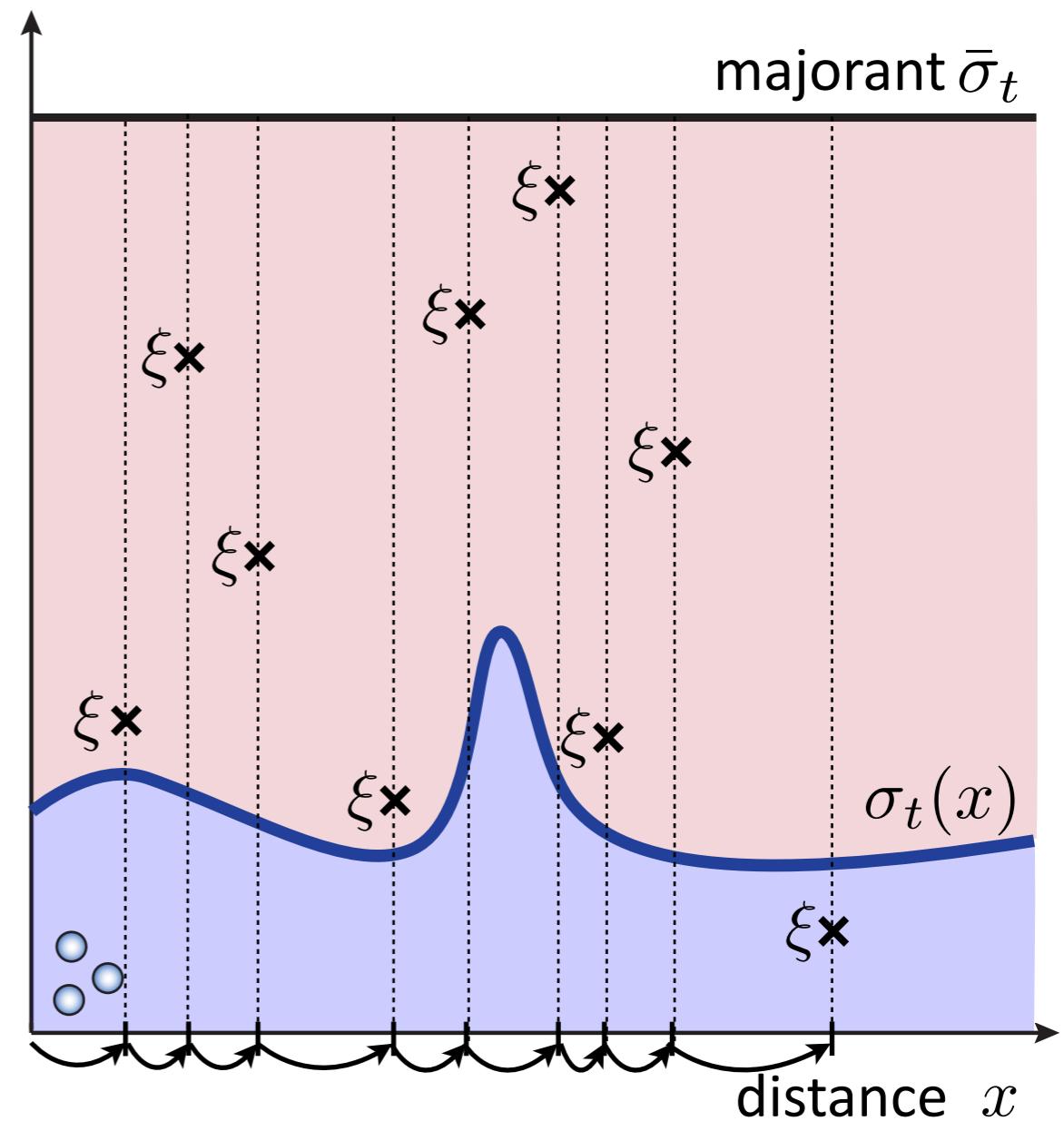
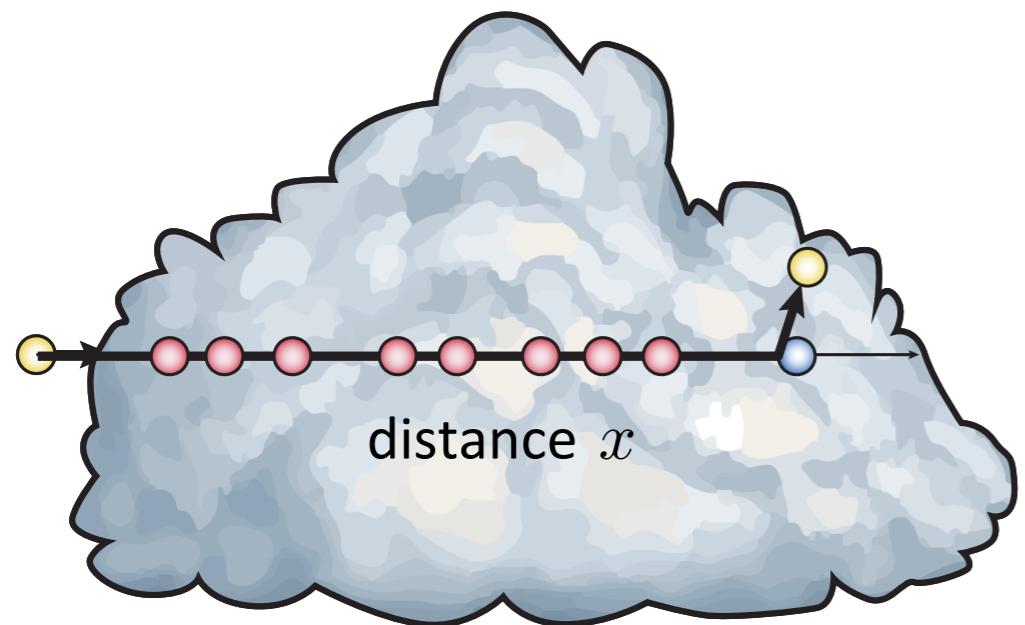
# Woodcock Tracking

- Tight majorant = high efficiency



# Woodcock Tracking

- Loose majorant = many fictitious collisions, high cost



# Woodcock Tracking

```
void preprocess()  
    majorant = findMaximumExtinction()  
  
void sampleFreePath(x, ω)  
    t = 0  
    do:  
        // Sample distance to next tentative collision  
        t += -log(1 - rand()) / majorant  
        // Compute probability of a real collision  
        Pr = getExtinction(x + t*ω) / majorant  
    while Pr < rand()  
    return t
```

# Woodcock Tracking Summary

- Unbiased, see [Coleman 86] for a proof

NUCLEAR SCIENCE AND ENGINEERING, 88, 76-81 (1980)

## Mathematical Verification of a Certain Monte Carlo Sampling Technique and Applications of the Technique to Radiation Transport Problems

W. A. Coleman

Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830

Received September 27, 1980

Revised November 10, 1980

The first section of this paper is a mathematical construction of a certain Monte Carlo procedure for sampling from the distribution

$$P(X) = \int_0^X f(x) \exp(-\int_x^X f(u) du) dx, \quad x \in X.$$

The construction begins by defining a particular random variable  $\lambda$ . The distribution function of  $\lambda$  is developed and found to be identical to  $P(X)$ . The definition of  $\lambda$  describes the sampling procedure. Depending on the behavior of  $f(x)$ , it may be more efficient to sample from  $P(X)$  by obtaining realizations of  $\lambda$  than by the more conventional procedure described in the paper.

Section II is a discussion of applications of the technique to problems in radiation transport where  $P(X)$  is frequently encountered as the distribution function for incident particles. The first application is in charged particle transport where  $f(x)$  is necessarily a continuous function of  $x$ . An application in impulse geometry is also given. The second application concerns the use of the technique to generate a series path, in some cases, directly. It is pointed out that the technique has been used to improve the efficiency of estimating certain quantities, such as the number of absorptions in a material.

### INTRODUCTION

In certain Monte Carlo problems it is necessary to obtain realizations (sample values) of a random variable having a distribution function<sup>1</sup> given by

$$P(X) = \int_0^X f(x) \exp(-\int_x^X f(u) du) dx, \quad 0 \leq X < 1,$$

where  $f(x)$  is any real-valued function having the properties:

(a)  $0 \leq f(x)$  for  $0 \leq x$ ,

(b)  $\lim_{x \rightarrow 0} \int_0^x f(x) dx = \infty$ ,

(c)  $f(x)$  is bounded; there is an  $M > 0$  with  $0 < M < f(x)$  for all  $x$ .

<sup>1</sup> If  $P(X)$  is a distribution function it is nondecreasing,  $P(0) = 0$ , and  $P(1) = 1$ . Many authors refer to such functions as cumulative distribution functions.

where  $\lambda$  will now be proved that:

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The probability that  $\rho_1 < v(z)$  and  $t_n < Z$  may be expressed as the integral of the conditional probability that  $\rho_1 < v(z)$  given  $t_1 = z_1$  with respect to the marginal distribution  $P(t_1)$ :

$$P(Z) = \int_0^Z P(\rho_1 < v(z) | t_1 = z_1) dt_1(z_1) = \int_0^Z v(z) M e^{-Mz} dz_1. \quad (1)$$

Similarly,

$$\begin{aligned} P(Z_n) &= \int_0^Z \int_0^{t_{n-1}} \dots \int_0^{t_{n-2}, z_n} P\left[\rho_1 > v(t_1), \dots, \rho_{n-1} > v\left(\sum_{i=1}^{n-1} t_i\right)\right] \\ &\quad \rho_n < v\left(\sum_{i=1}^n t_i\right) | t_1 = z_1, \dots, t_{n-1} = z_{n-1} dP(t_1, \dots, t_n | z_1, \dots, z_{n-1}). \end{aligned} \quad (2)$$

$P(t_1, \dots, t_n | z_1, \dots, z_{n-1})$  denotes the joint distribution function of the variables  $t_1, \dots, t_n$ . The integral limits in Eq. (2) are determined by first noting that  $0 \leq t_i$ , and, hence,  $t_{i+1} \leq t_i + z_i$ ,  $i = 1, 2, \dots, n-1$ . For the event  $Z_n < Z$  to occur, it is necessary that  $t_n < Z$ , which implies  $t_1 < \dots < t_n < Z$ . In terms of  $t_i$ , it is necessary that  $t_i < Z - \sum_{j=i+1}^n t_j$  for  $i = 1, 2, \dots, n-1$ .

Since  $\rho_1, \rho_2, \dots, \rho_n$  are totally independent, the integrand in Eq. (2) is equal to

$$\begin{aligned} &P(\rho_1 < v(z) | t_1 = z_1) \dots P\left[\rho_{n-1} > v\left(\sum_{i=1}^{n-1} t_i\right) | t_1 = z_1, \dots, t_{n-1} = z_{n-1}\right] \\ &\quad \times P\left[\rho_n < v\left(\sum_{i=1}^n t_i\right) | t_1 = z_1, \dots, t_{n-1} = z_{n-1}\right]. \end{aligned}$$

Also  $t_1, \dots, t_n$  are totally independent and have a common distribution function, so that

$$P(t_1, \dots, t_n | z_1, \dots, z_{n-1}) = P_0(z_1) \dots P_0(z_n) = f(z_1) \dots f(z_n). \quad (3)$$

Substituting these relations into Eq. (2) gives

$$\begin{aligned} P(Z) &= \int_0^Z \int_0^{t_{n-1}} \dots \int_0^{t_{n-2}, z_n} P(\rho_1 < v(z) | t_1 = z_1) \\ &\quad \dots P\left[\rho_{n-1} > v\left(\sum_{i=1}^{n-1} t_i\right) | t_1 = z_1, \dots, t_{n-1} = z_{n-1}\right] \\ &\quad \times \left[ P_0(z_1) \dots P_0\left(\sum_{i=1}^n t_i\right) | t_1 = z_1, \dots, t_n = z_n \right] dP_0(z_1) \dots dP_0(z_n) \\ &\quad \times \int_0^Z \int_0^{t_{n-1}} \dots \int_0^{t_{n-2}, z_n} \left[ 1 - v(z) \right] \\ &\quad \dots \left[ 1 - v\left(\sum_{i=1}^n t_i\right) \right] dP_0(z_1) \dots dP_0(z_n) M e^{-Mz} \dots M e^{-Mz_n}. \end{aligned}$$

It is convenient to proceed with the probabilities expressed in terms of the variables  $t_1, \dots, t_n$ . The transformations from  $t_1, \dots, t_n$  are direct. Introducing  $v(x)$  for brevity, the expressions for  $P(Z)$  and  $P(Z_n)$ ,  $n \geq 1$ , become

$$P(Z) = \int_0^Z v(z) M e^{-Mz} dz_1, \quad (4)$$

$$\begin{aligned} \text{and} \quad P(Z_n) &= \int_0^Z dz_1 M^2 v(z_1) e^{-Mz_1} \int_0^{z_1} dz_{n-1} v(z_{n-1}) \int_0^{z_{n-1}} dz_{n-2} \dots \int_0^{z_2} dz_1 v(z_1) \\ &\quad + \int_0^Z dz_1 v(z_1) M^2 e^{-Mz_1} \int_0^{z_1} dz_2 v(z_2) \int_0^{z_2} dz_3 \dots \int_0^{z_{n-1}} dz_1 v(z_1). \end{aligned} \quad (5)$$

<sup>2</sup> See for example, WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, Vol. II, p. 206 (1966).

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$$\int_0^z dz_1 v(z_1) \int_0^{z_1} dz_2 \dots \int_0^{z_{n-1}} dz_n v(z_n) = \frac{\left[ \int_0^z v(z) dz \right]^{n+1}}{(n+1)!} \quad z \in \mathbb{R}.$$

Equation (2) is true for  $n = 2$  by inspection. For  $n = 3$ ,

$$\int_0^z dz_1 v(z_1) \int_0^{z_1} dz_2 v(z_2) \int_0^{z_2} dz_3 v(z_3) = \int_0^z dz_1 \frac{d}{dz_1} \left[ \frac{\left[ \int_0^{z_1} v(z) dz \right]^2}{2} \right] = \frac{\left[ \int_0^z v(z) dz \right]^3}{3!}.$$

Assuming Eq. (2) to be true for arbitrary  $n$ , it can be shown to hold for  $n+1$  by multiplying Eq. (2) by  $v(z)$  and integrating from 0 to  $z$ .

$$\begin{aligned} &\int_0^z dz_1 v(z_1) \int_0^{z_1} dz_2 \dots \int_0^{z_{n-1}} dz_n v(z_n) + \int_0^z dz_1 v(z_1) \frac{\left[ \int_0^z v(z) dz \right]^{n+1}}{(n+1)!} \\ &\quad \times \int_0^z dz_1 \frac{d}{dz_1} \left[ \frac{\left[ \int_0^{z_1} v(z) dz \right]^2}{2} \right] = \frac{\left[ \int_0^z v(z) dz \right]^{n+2}}{n!}. \end{aligned}$$

It follows that Eq. (2) holds for arbitrary  $n \geq 1$ .

Substituting the identity [Eq. (2)] into Eq. (4) gives

$$P(Z) = \int_0^Z dz_1 v(z_1) M e^{-Mz_1} \frac{\left[ \int_0^z v(z) dz \right]^{n+1}}{(n+1)!} \quad z \in \mathbb{R}.$$

Equation (4) becomes

$$\begin{aligned} P(0 < Z) &= \sum_{n=1}^{\infty} P(Z_n) = P(Z) + \sum_{n=2}^{\infty} \int_0^Z dz_1 v(z_1) M^2 e^{-Mz_1} \frac{\left[ \int_0^z v(z) dz \right]^{n+1}}{(n+1)!} \\ &\quad + \sum_{n=3}^{\infty} \int_0^Z dz_1 v(z_1) M^3 e^{-Mz_1} \frac{\left[ M \int_0^z v(z) dz \right]^2}{n!} \\ &\quad + \int_0^Z dz_1 v(z_1) M e^{-Mz_1} \exp \left[ M \int_0^z v(z) dz \right] + \int_0^Z dz_1 v(z_1) \exp \left[ - \int_0^z M v(z) dz \right] \\ &\quad + \int_0^Z dz_1 v(z_1) \exp \left[ \int_0^z M v(z) dz \right] dz_1. \end{aligned}$$

Hence  $Z$  has the distribution function given in Eq. (1).

### II. APPLICATIONS OF THE TECHNIQUE TO RADIATION TRANSPORT PROBLEMS

and the kinetic energy  $E$  of the incident particle. The kinetic energy of a charged particle varies between nuclear events due to interactions with electrons. For the more massive charged particles, such as protons and alphas, the kinetic energy is usually assumed to be a continuous, decreasing function of position. Consider a material composed uniformly of one nuclear species  $N$ . Assume the kinetic energy  $E_0$  of a particular type of particle  $p$  at a position  $x_0 > 0$  is known. The kinetic energy  $E$  of  $p$  at an arbitrary point  $x > x_0$  is a function of  $x$ ,

$$E = f_{N,N_0}(x).$$

Each of the variables  $p$ ,  $N$ , and  $x_0$  has been fixed. Denote the macroscopic cross section for a particle of type  $p$  undergoing a nonelastic collision with a stationary nucleus of type  $N$  depends upon  $p$ ,  $N$ ,

$$X = f_{N,N_0}(x).$$

# Woodcock Tracking Summary

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- Unbiased, see [Coleman 86] for a proof
- Majorant extinction
  - defines the combined homogeneous volume
  - must bound the real extinction
  - loose majorants lead to many fictitious collisions

# Transmittance via Woodcock Tracking

- Can be used to estimate transmittance  $T_r(\mathbf{x}, \mathbf{y})$
- Recipe:
  - Sample free path from  $\mathbf{x}$  towards  $\mathbf{y}$
  - If a real collision occurs before  $\mathbf{y}$ , then  $T_r(\mathbf{x}, \mathbf{y}) = 0$
  - else  $T_r(\mathbf{x}, \mathbf{y}) = 1$
- To get a finer estimate, sample multiple free paths and count the fraction that made it beyond  $\mathbf{y}$

# Next Time

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- Fast(er) multiple scattering
  - volumetric photon mapping
  - photon beams
  - virtual ray lights...

# Questions?

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# Errata

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