CS 87/187, Spring 2016

RENDERING ALGORITHMS

Participating Media I

Prof. Wojciech Jarosz

wojciech.k.jarosz@dartmouth.edu

(with slide improvements by Jan Novák)

Dartmouth

Today's Menu

- Participating media (volumes)
- Theoretical background
  - Interaction coefficients
  - Phase functions
- Volume Rendering Equation
- Ray-marching
- Volumetric path tracing
- Woodcock tracking

Fog

Clouds & Crepuscular rays

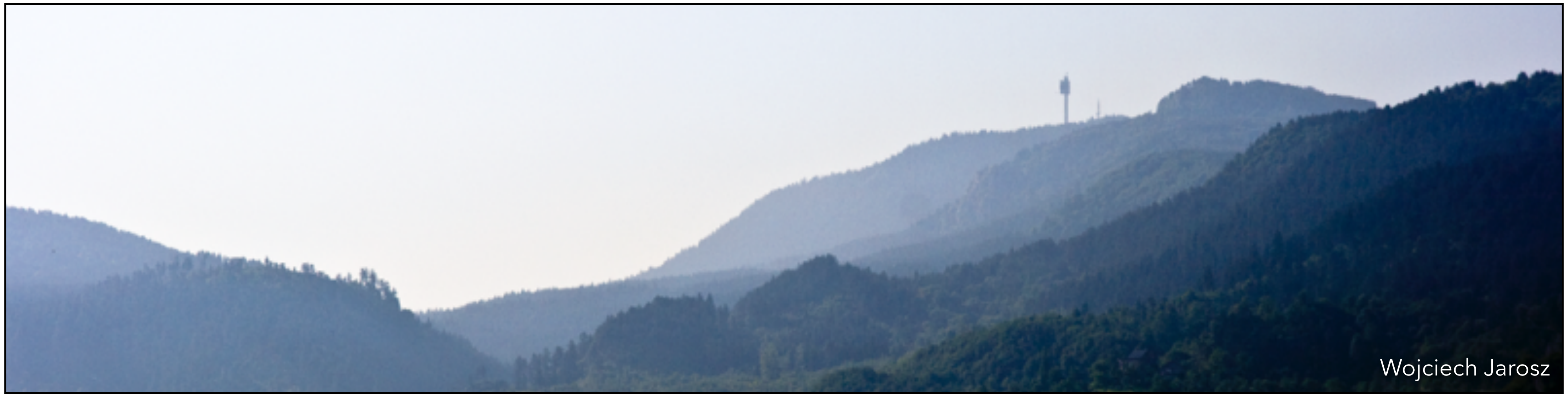
Fire

Underwater

Surface or Volume?

Antelope Canyon, Az.

Aerial (Atmospheric) Perspective



Leonardo da Vinci (1480)

"Thus, if one is to be five times as distant, make it five times bluer."

—*Treatise on Painting*, Leonardo Da Vinci, pp 295, circa 1480.

Nebula

T.A.Rector (NOAO/AURA/NSF) and the Hubble Heritage Team (STScI/AURA/NASA)

Emission

Absorption

Scattering

Defining Participating Media

- Typically, we do not model particles of a medium explicitly (wouldn't fit in memory)
- The properties are described statistically using various coefficients and densities
  - Conceptually similar idea as microfacet models

Defining Participating Media

- Homogeneous:
  - Infinite or bounded by a surface

Defining Participating Media

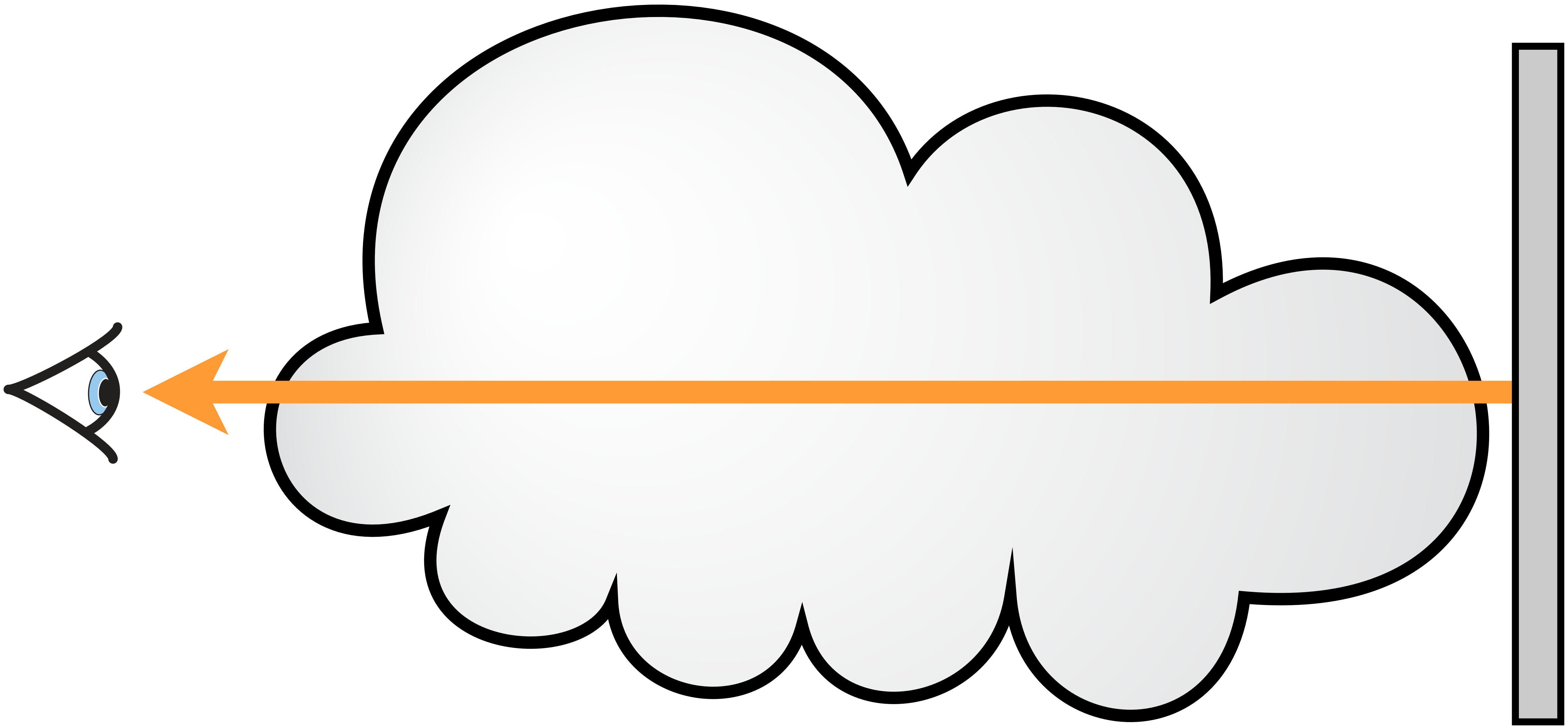
- Homogeneous:
  - Infinite or bounded by a surface
- Heterogeneous (spatially varying coefficients):
  - Procedurally e.g. using a noise function
  - Simulation + volume discretization, e.g. a voxel

Radiance

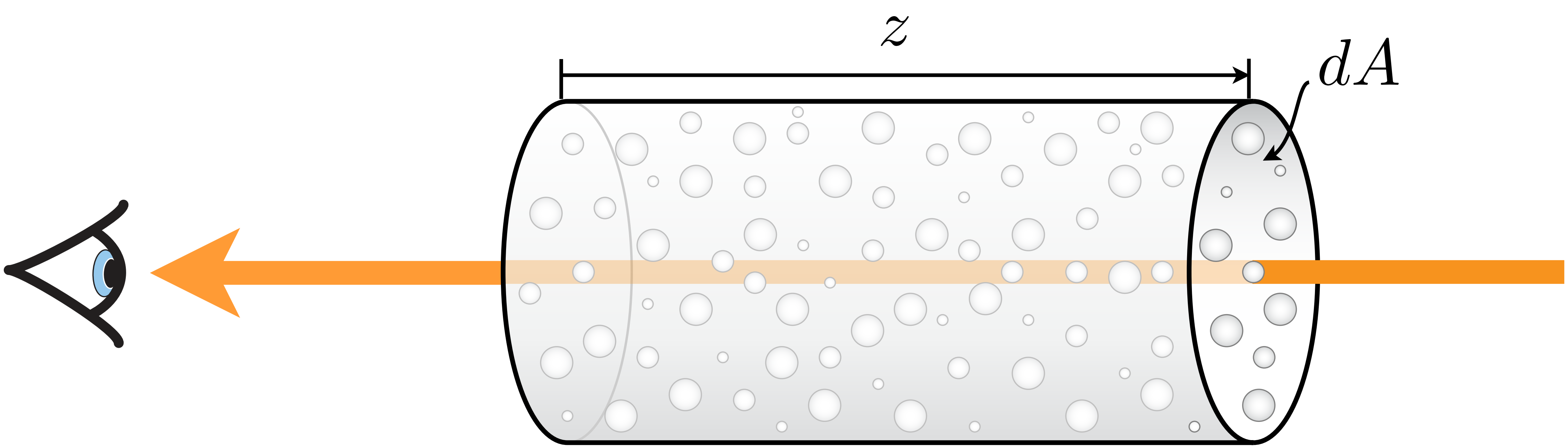
- The main quantity we are interested in for rendering is radiance
- Radiance *changes* along the ray if it goes through a participating medium

$$L_i(\mathbf{x}, \vec{\omega}) \neq L_o(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$$

Participating Media

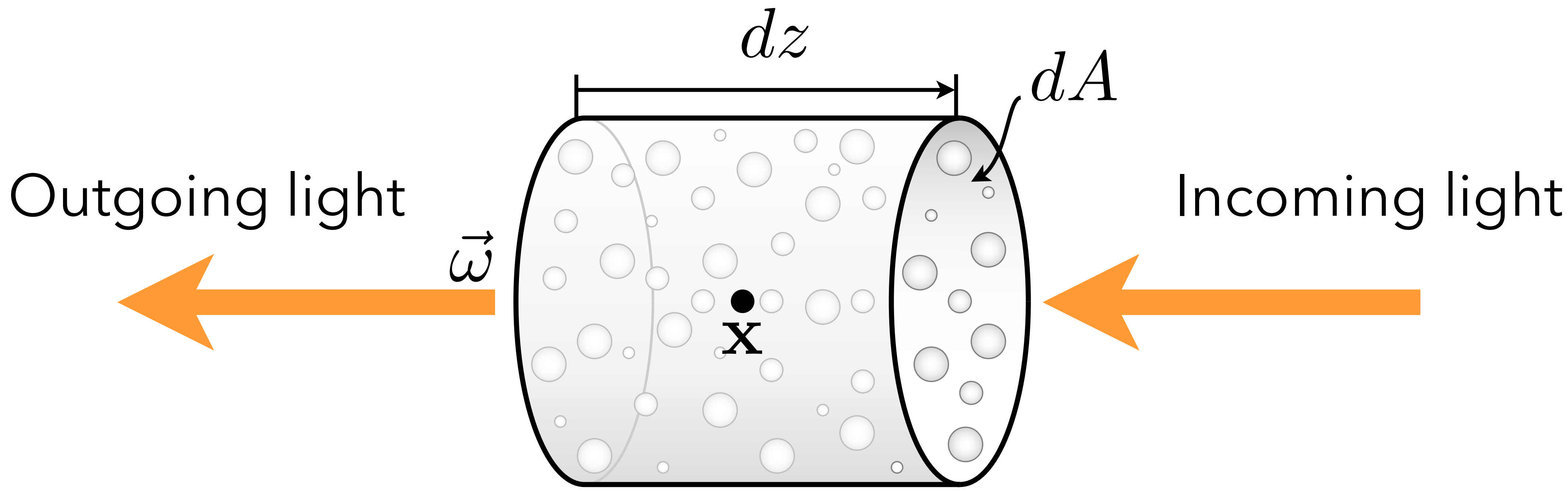


Differential Beam

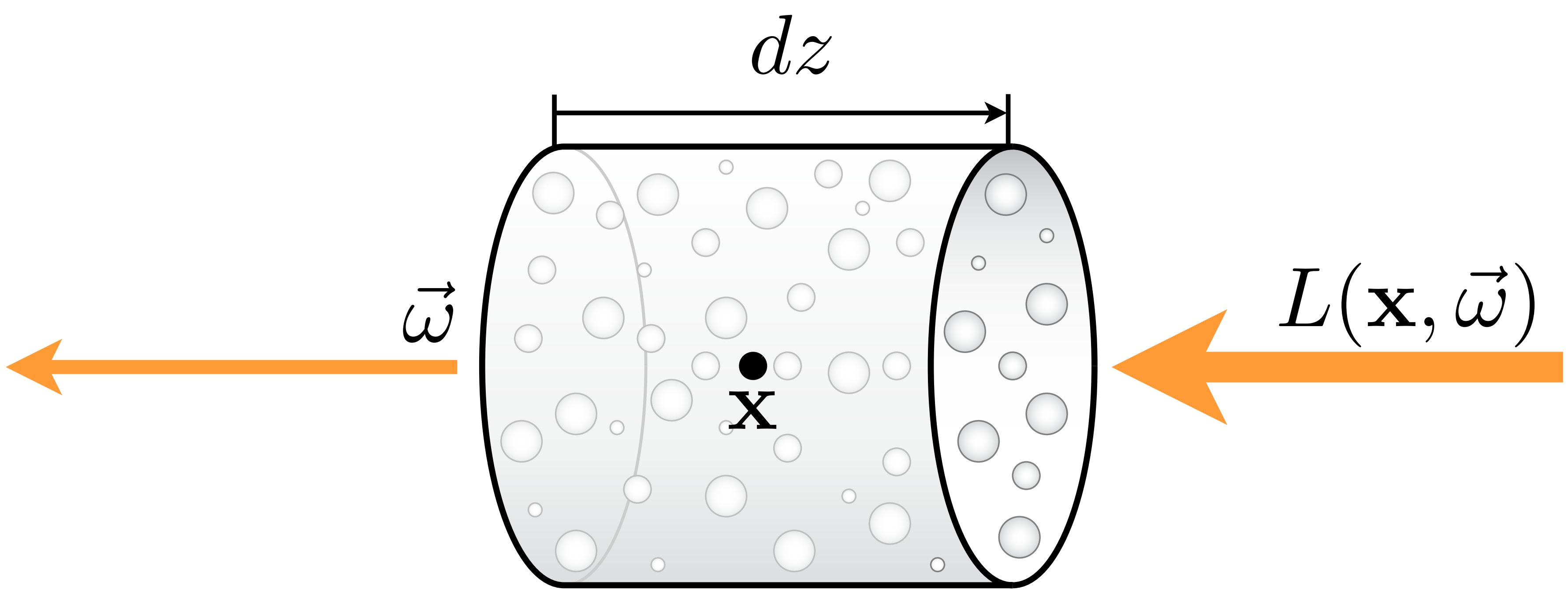


How much light is *lost/gained* along the differential beam due to interactions of light with the medium?

Differential Beam Segment



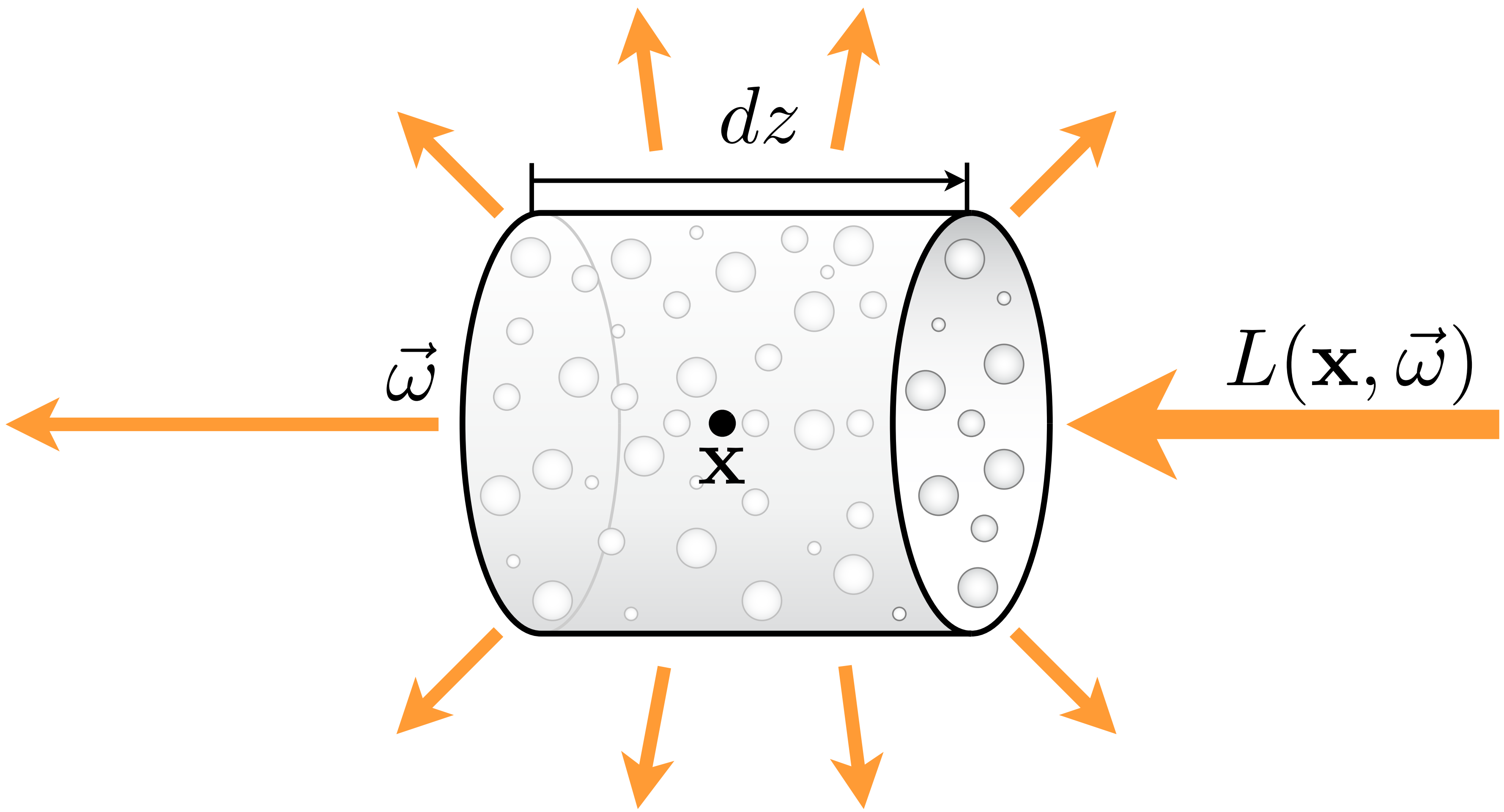
Absorption



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_a(\mathbf{x}) L(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$

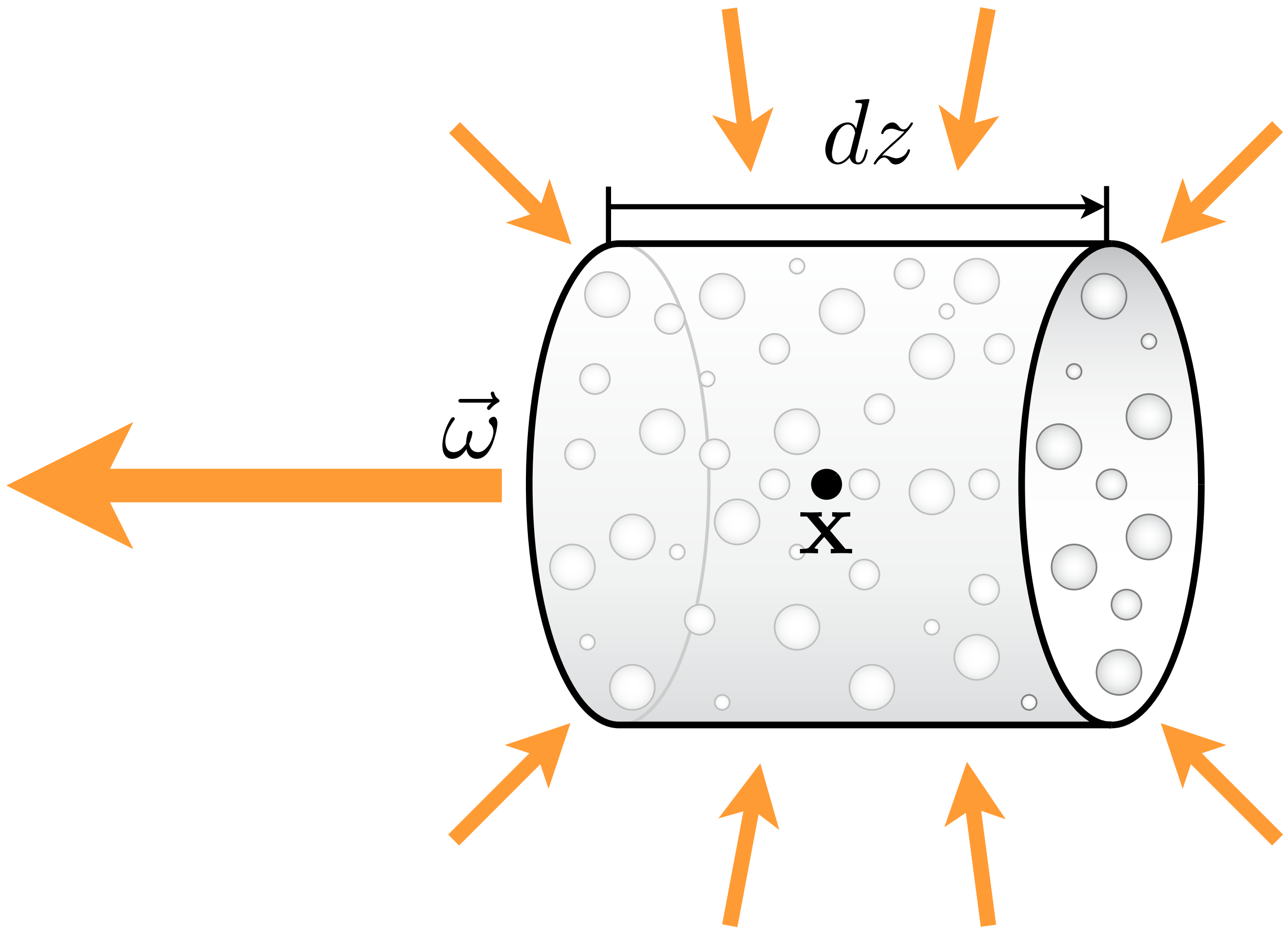
Out-scattering



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_s(\mathbf{x}) L(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

In-scattering

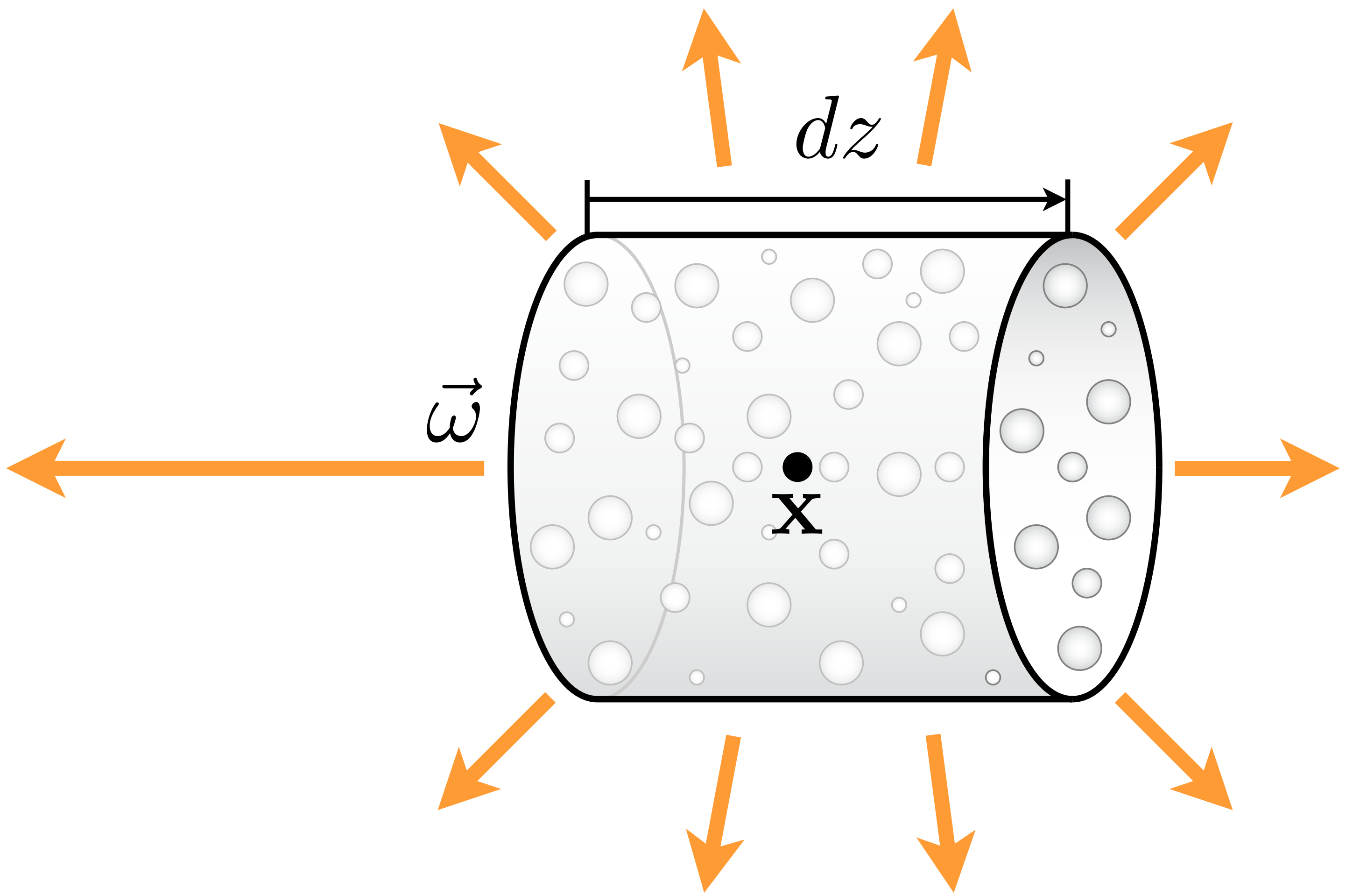


$$dL(\mathbf{x}, \vec{\omega}) = \sigma_s(\mathbf{x}) L_s(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

$L_s(\mathbf{x}, \vec{\omega})$: in-scattered radiance

Emission



$$dL(\mathbf{x}, \vec{\omega}) = \sigma_a(\mathbf{x}) L_e(\mathbf{x}, \vec{\omega}) dz$$

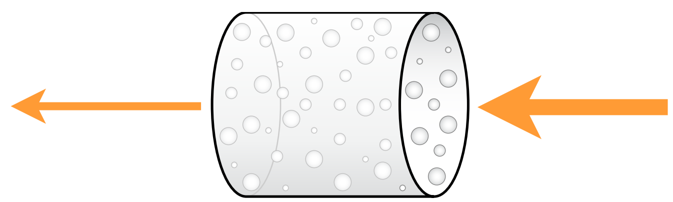
*Sometimes modeled without the absorption coefficient just by specifying a "source" term

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$

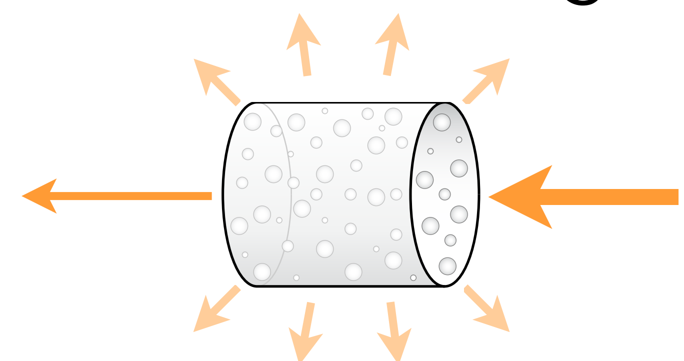
$L_e(\mathbf{x}, \vec{\omega})$: emitted radiance

Radiative Transport Equation

Absorption



Out-scattering

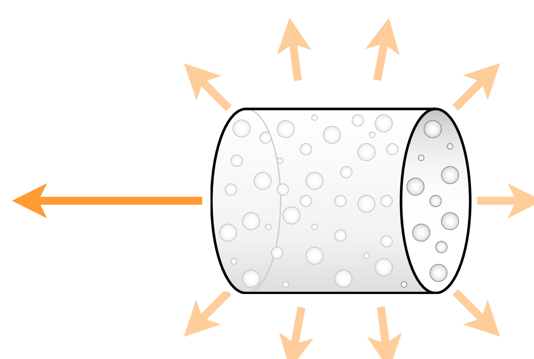


$$dL(\mathbf{x},\vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$

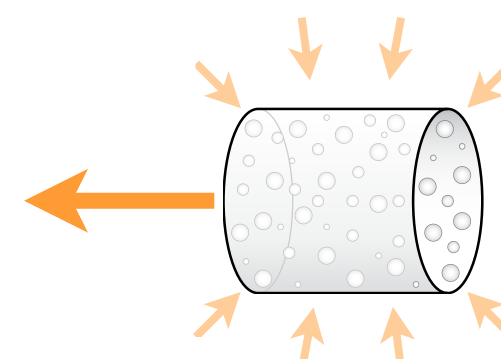
Losses

$$+\sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})dz \; +\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})dz$$

Gains



Emission



In-scattering

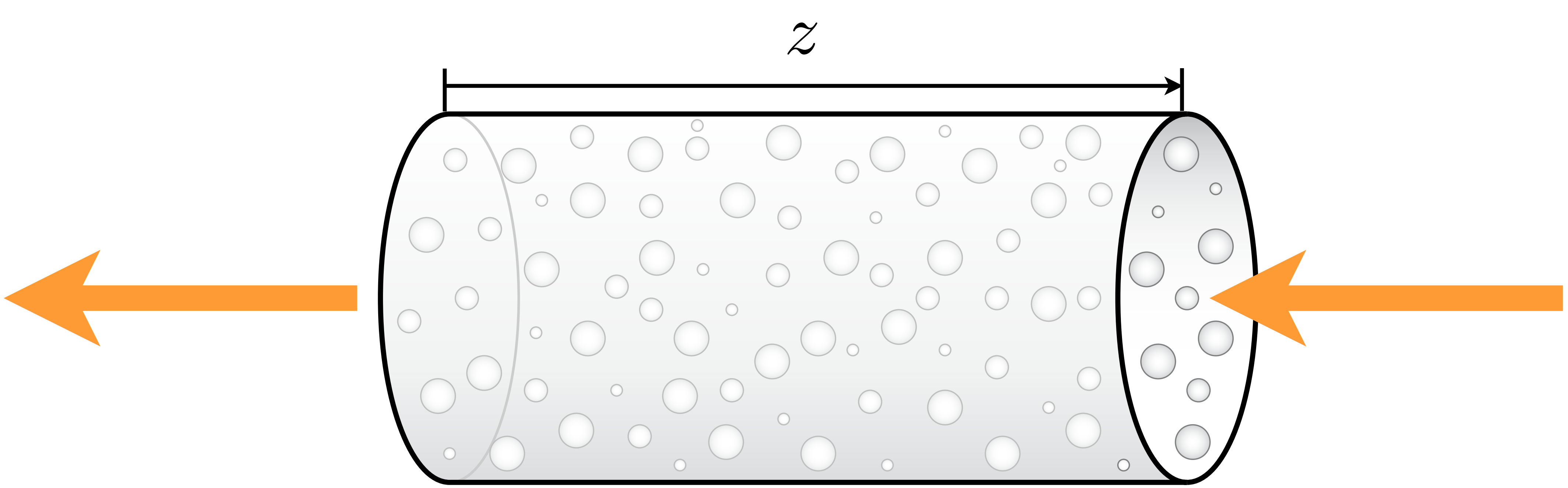
What about a beam with a finite length?

Losses (Extinction)

Absorption — Out-scattering

$$dL(\mathbf{x},\vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz - \sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$
$$= -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$

$\sigma_t(\mathbf{x})$: extinction coefficient $[m^{-1}]$
: total loss of light per unit distance



Extinction Along a Finite Beam

$$dL(\mathbf{x},\vec{\omega}) = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$ // Assume constant $\sigma_t(\mathbf{x})$, reorganize

$$\frac{dL(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t dz$$ // Integrate along beam from 0 to z

$$\ln(L_z) - \ln(L_0) = -\sigma_t z$$

$$\ln\left(\frac{L_z}{L_0}\right) = -\sigma_t z$$ // Exponentiate

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

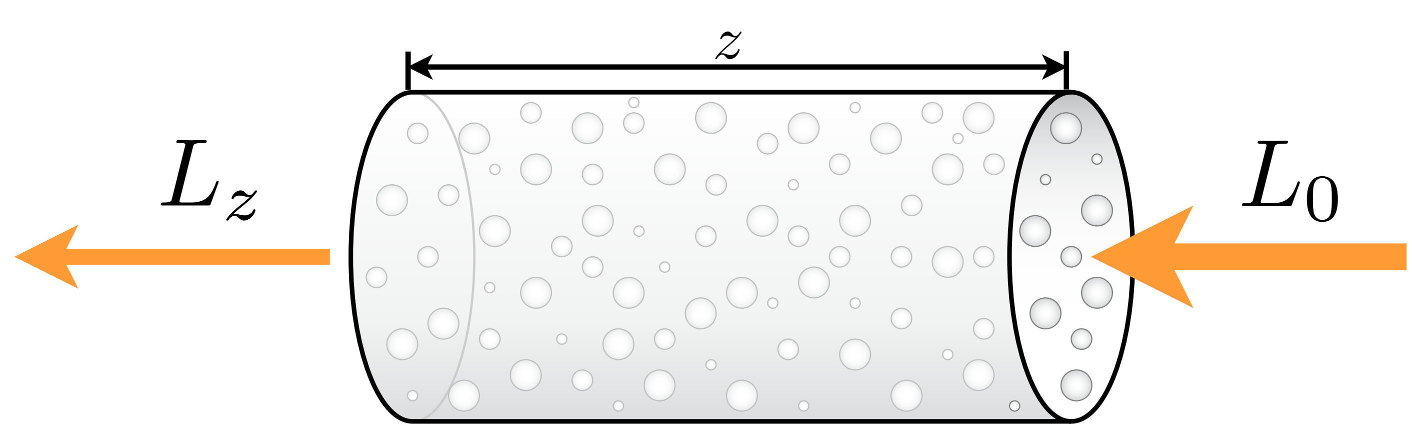
Beer-Lambert Law

- Expresses the remaining radiance after traveling a finite distance through a medium with constant extinction coefficient
- The fraction is referred to as the ***transmittance***
- Think of this as fractional visibility between points

Radiance at distance z

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at the beginning of the beam



Transmittance

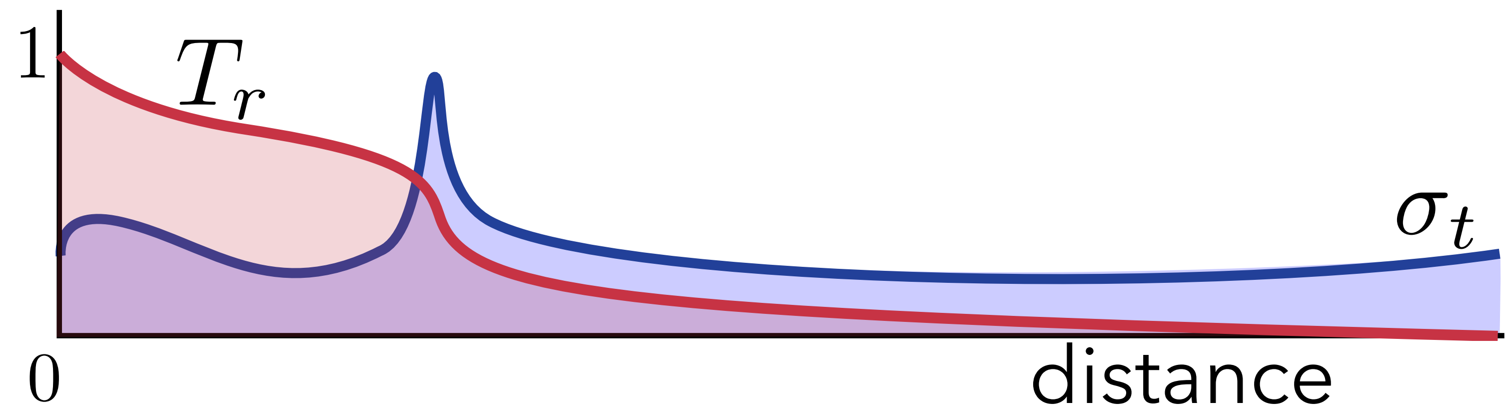
- Homogeneous volume:

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\sigma_t \|\mathbf{x}-\mathbf{y}\|}}$$

- Heterogeneous volume (spatially varying σ_t):

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t)dt}}$$

Optical thickness



Transmittance

- Homogeneous volume:

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\sigma_t \|\mathbf{x}-\mathbf{y}\|}}$$

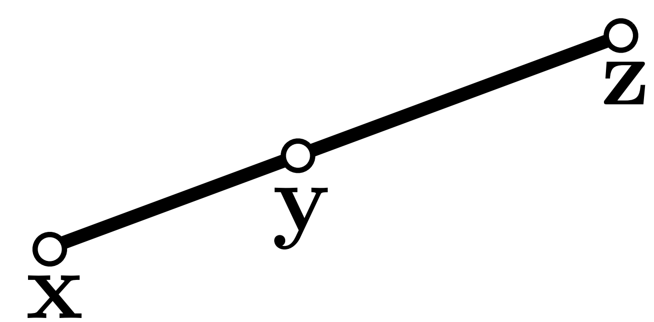
- Heterogeneous volume (spatially varying σ_t):

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t)\,dt}}$$

Optical thickness

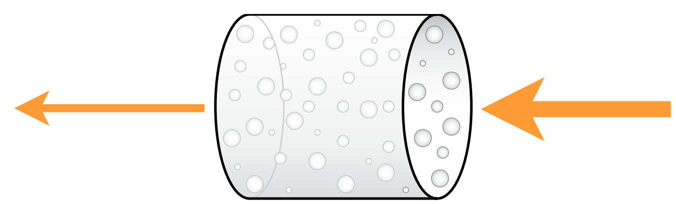
- Transmittance is multiplicative:

$$T_r(\mathbf{x},\mathbf{z}) = T_r(\mathbf{x},\mathbf{y})T_r(\mathbf{y},\mathbf{z})$$

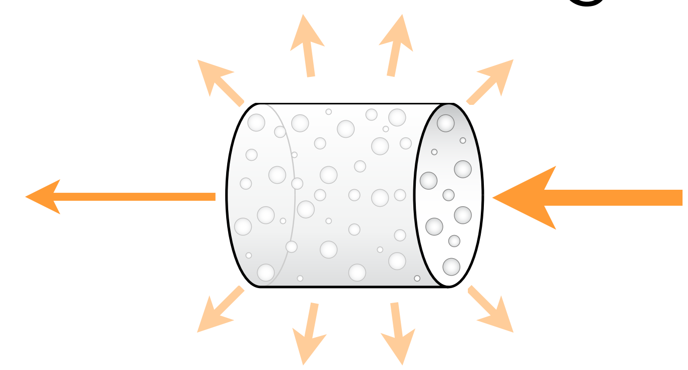


Radiative Transport Equation

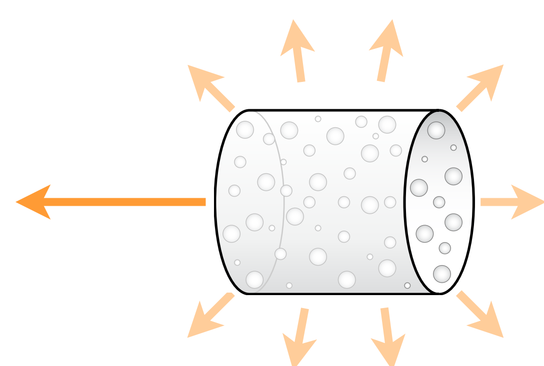
Absorption



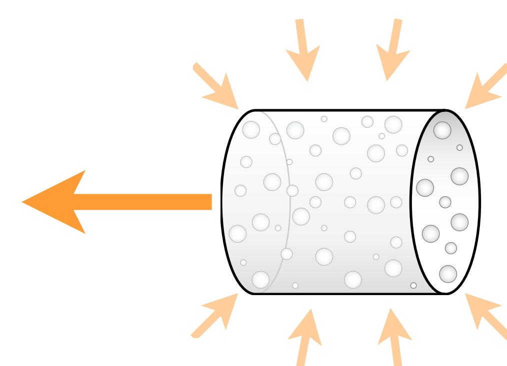
Out-scattering



$$\begin{aligned} dL(\mathbf{x},\vec{\omega}) = &- \sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \quad - \sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \\ &+ \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})dz + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})dz \end{aligned}$$



Emission

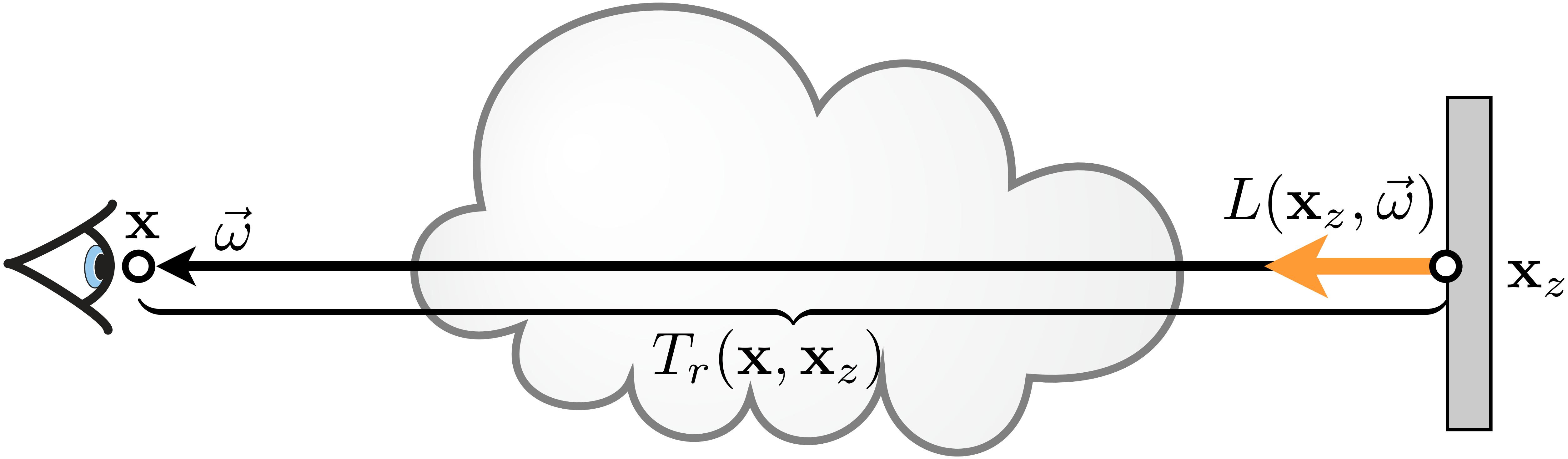


In-scattering

Volume Rendering Equation

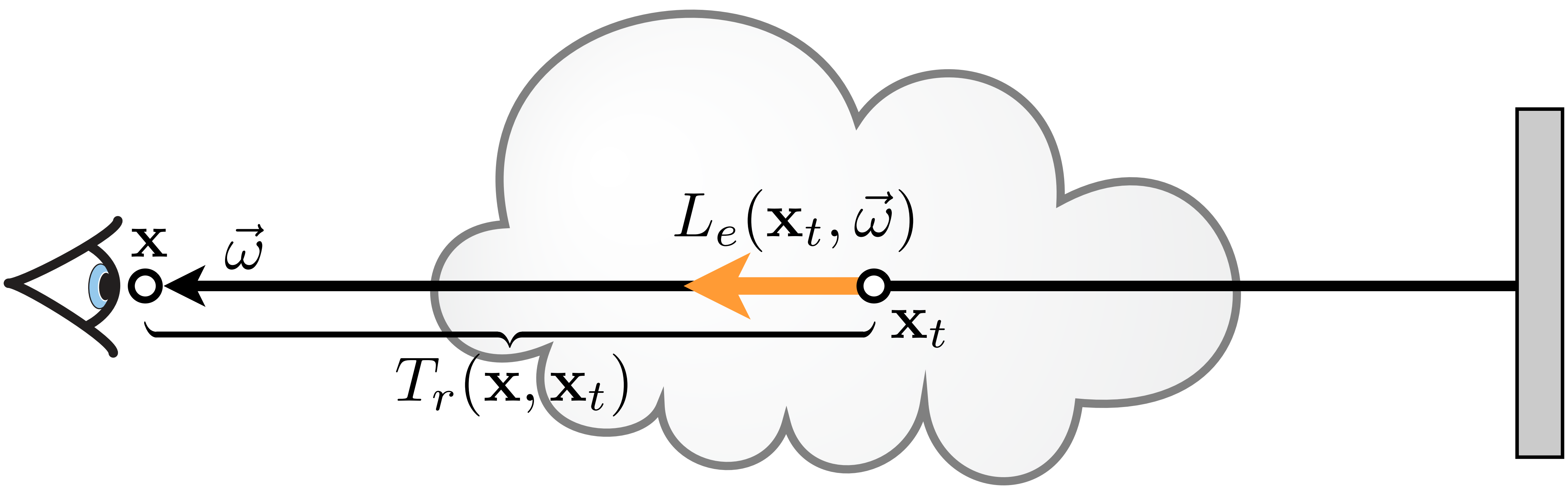
$$L(\mathbf{x},\vec{\omega}) = \boxed{T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})}$$

Reduced (background) surface radiance



Volume Rendering Equation

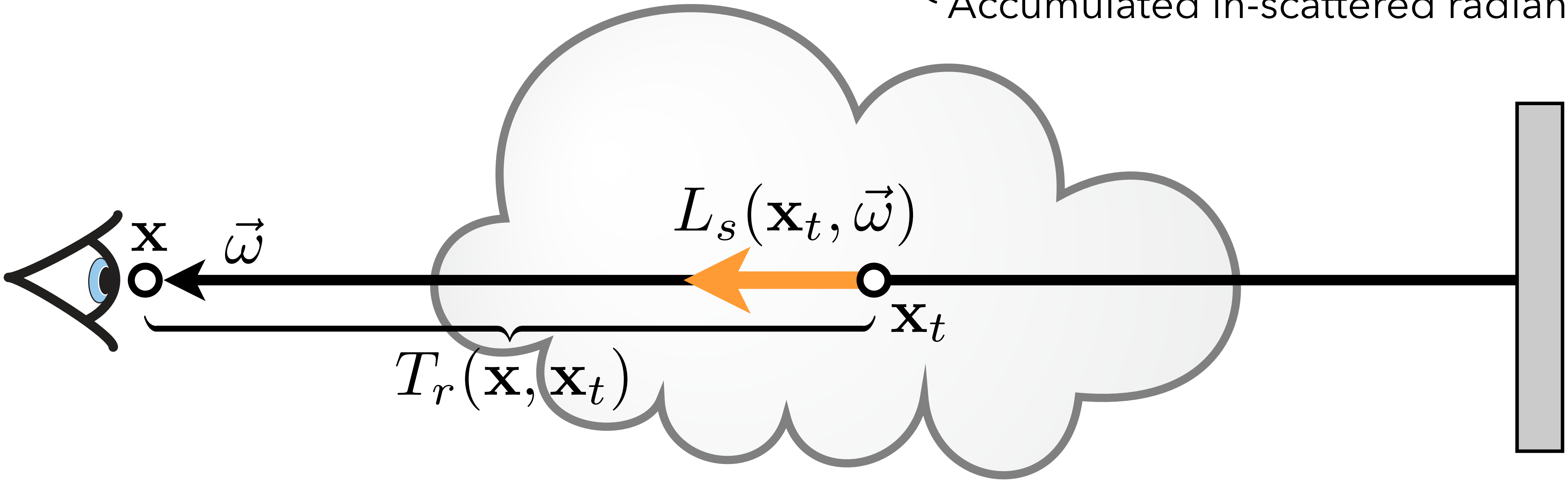
$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \underbrace{\boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt}}_{\text{Accumulated emitted radiance}}$$



Volume Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) = &\, T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
&+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
&+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt}
\end{aligned}
$$

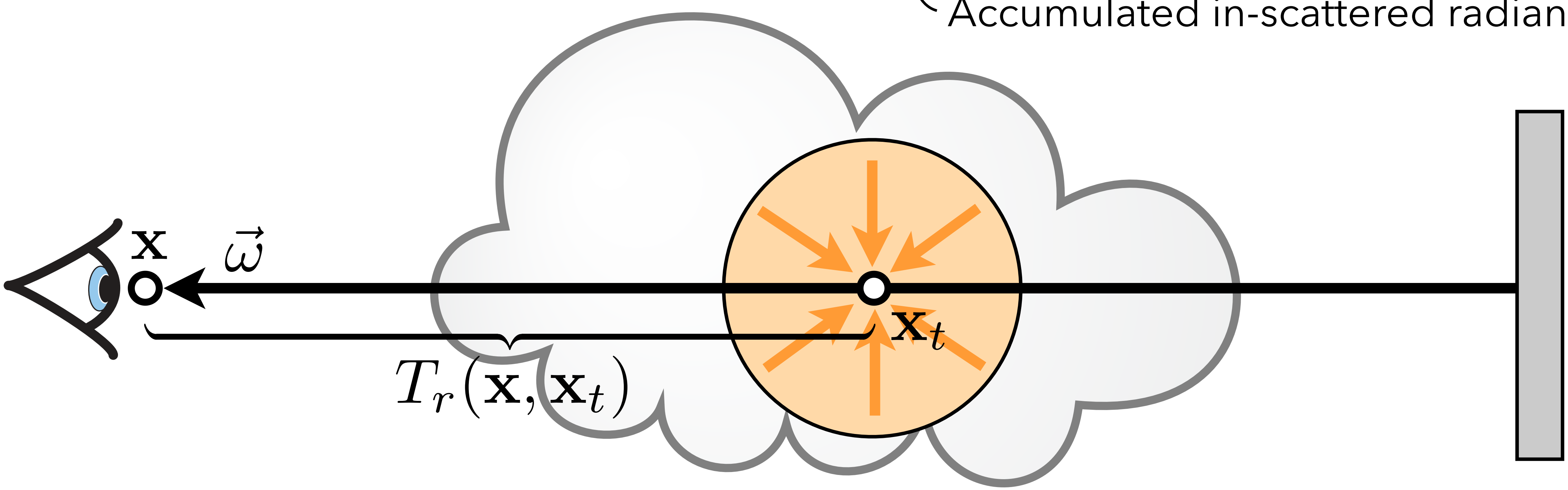
Accumulated in-scattered radiance



Volume Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) = {} & T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
& + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
& + \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt}
\end{aligned}
$$

Accumulated in-scattered radiance



Volume Rendering Equation

$$
\begin{aligned}
L(\mathbf{x},\vec{\omega}) = {} & T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) \\
& + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt \\
& + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})L_i(\mathbf{x}_t,\vec{\omega}')d\vec{\omega}'dt
\end{aligned}
$$

Scattering in Media

Phase Function f_p

- Describes distribution of scattered light
- Analog of BRDF but for scattering in media
- Integrates to unity (unlike BRDF)

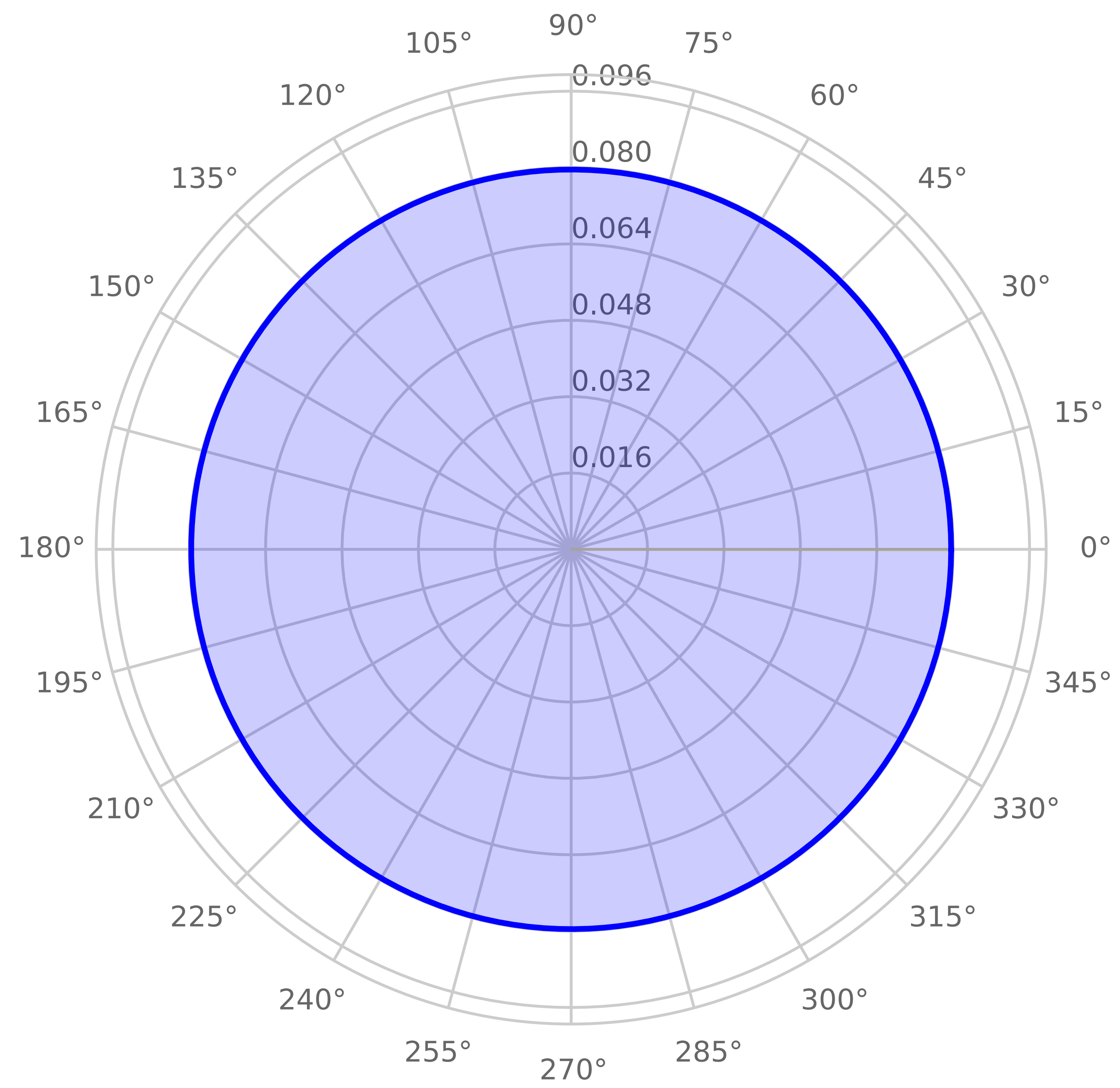
$$\int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) d\vec{\omega}' = 1$$

*In this course, we use the same convention for phase functions as for BRDFs: direction vectors always *point away* from the shading point **x**. Many publications, however, use a different convention for phase functions, in which direction vectors "follow" the light, i.e. one direction points *towards* **x** and the other *away* from **x**. When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.

Isotropic Scattering

- Uniform scattering, analogous to Lambertian BRDF

$$f_p(\vec{\omega}', \vec{\omega}) = \frac{1}{4\pi}$$



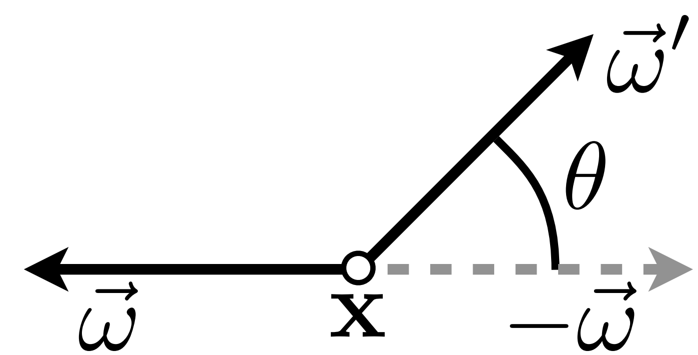
Anisotropic Scattering

- Quantifying anisotropy (g, "average cosine"):

$$g = \int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) \cos\theta \, d\vec{\omega}'$$

- where:

$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

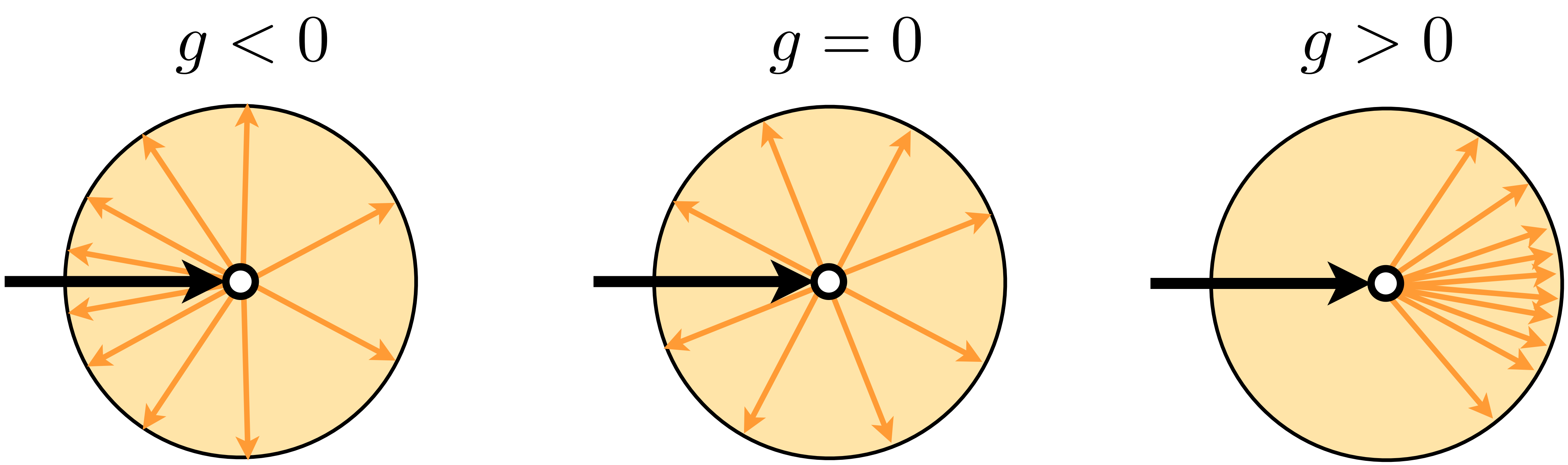


- $g = 0$: isotropic scattering (on average)
  $g > 0$: forward scattering
  $g < 0$: backward scattering

Henyey-Greenstein Phase Function

- Anisotropic scattering

$$f_{p\mathrm{HG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}$$

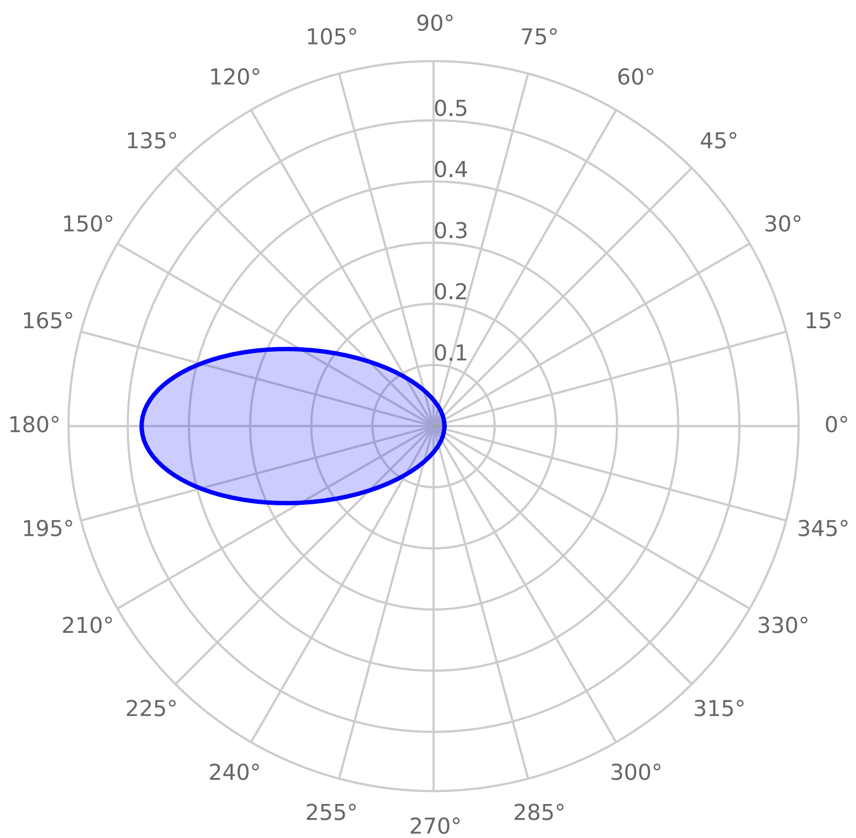


Henyey-Greenstein Phase Function

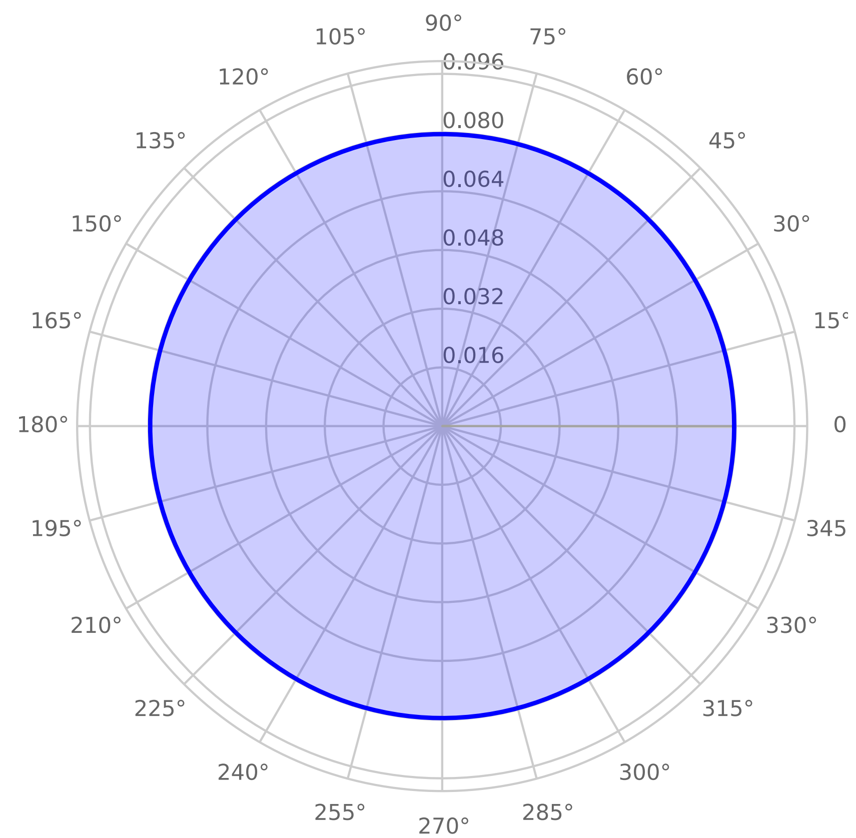
- Anisotropic scattering

$$f_{p\mathrm{HG}}(\theta) = \frac{1}{4\pi} \frac{1-g^2}{(1+g^2-2g\cos\theta)^{3/2}}$$

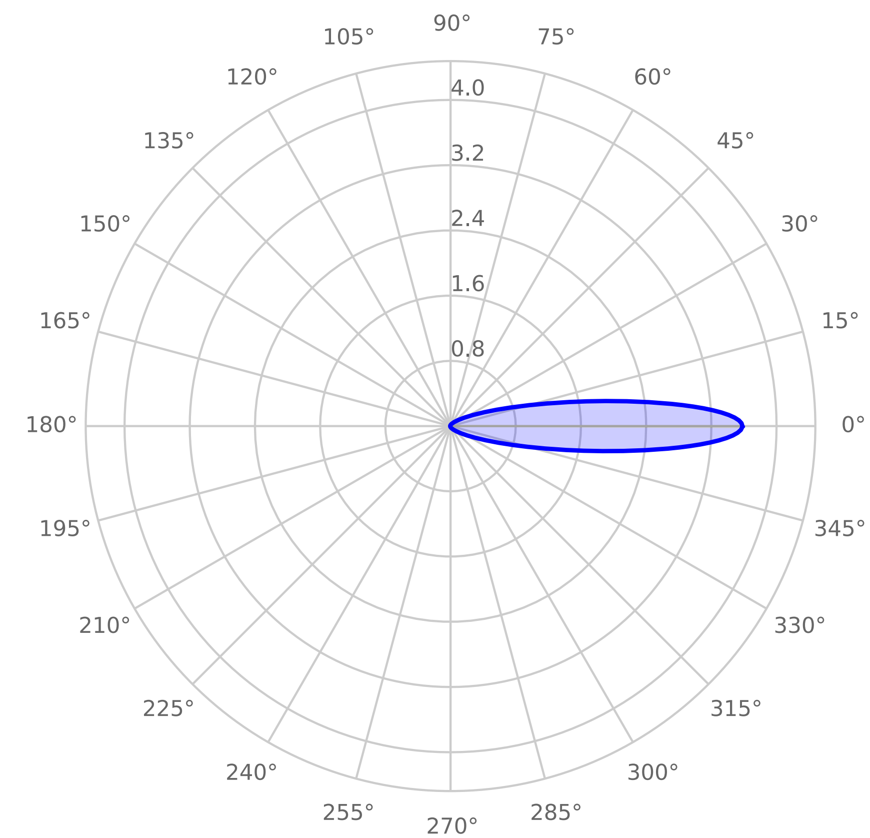
$g = -0.5$



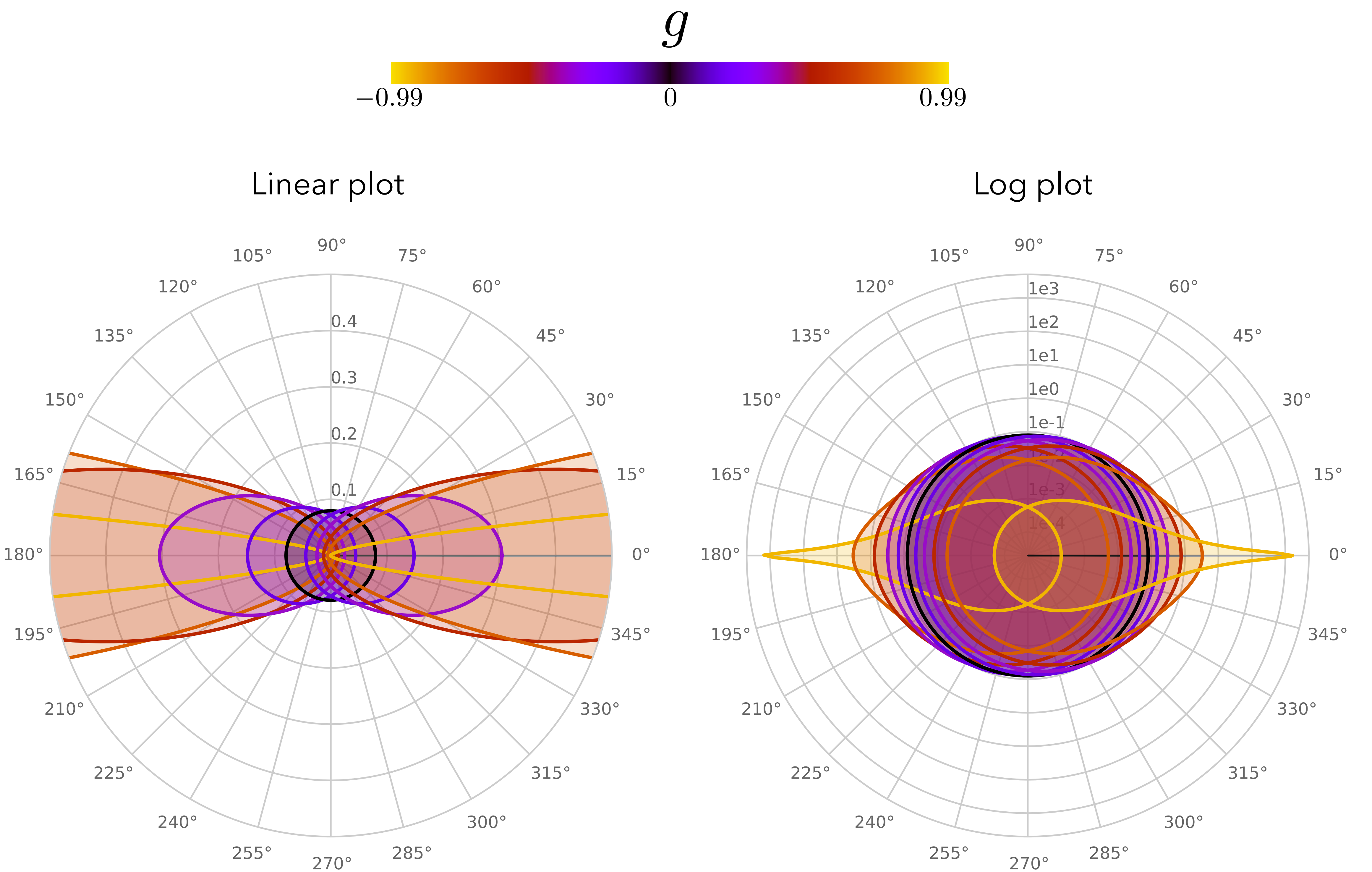
$g = 0$



$g = 0.8$



Henyey-Greenstein Phase Function



Henyey-Greenstein Phase Function

- Empirical phase function
- Introduced for intergalactic dust
- Very popular in graphics and other fields
- Extensions with multiple HG lobes

Schlick's Phase Function

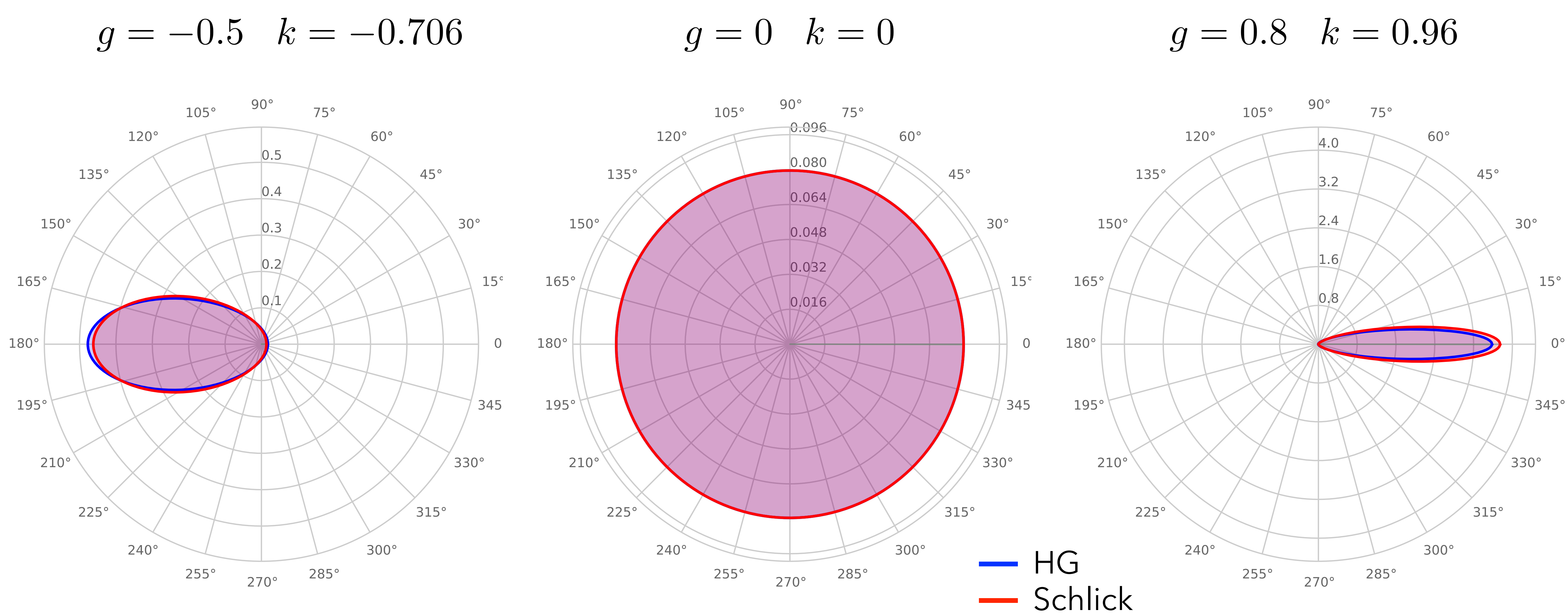
- Empirical phase function
- Faster approximation of HG

$$f_{p\text{Schlick}}(\theta) = \frac{1}{4\pi} \frac{1-k^2}{(1-k\cos\theta)^2}$$

$$k = 1.55g - 0.55g^3$$

Schlick's Phase Function

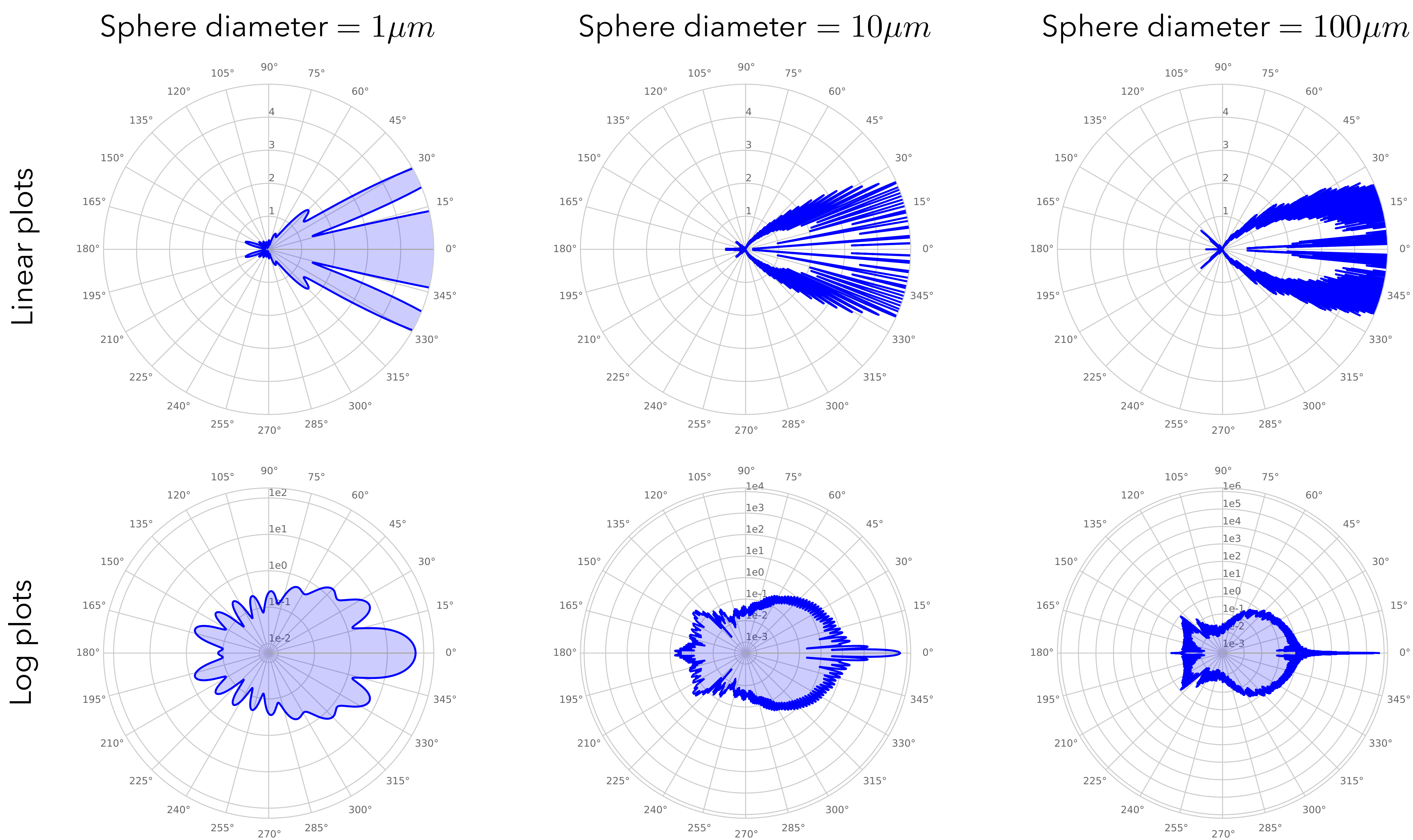
- Empirical phase function
- Cheap approximation of HG



Lorenz-Mie Scattering

- If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light
- Solution to Maxwell's equations for scattering from any spherical dielectric particle
- Explains many phenomena
- Complicated:
  - Solution is an infinite analytic series

Lorenz-Mie Phase Function



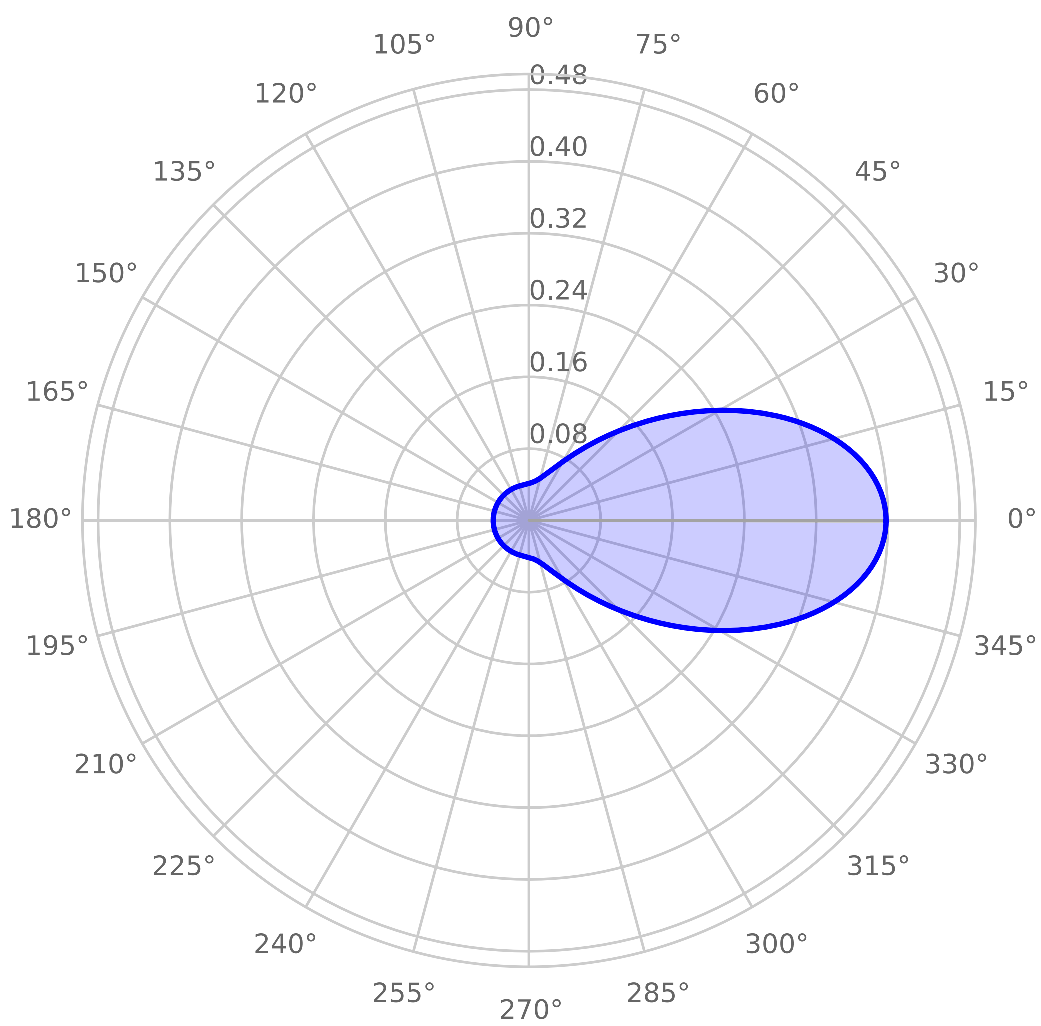
Data obtained from http://www.philiplaven.com/mieplot.htm

Rainbows

Lorenz-Mie Approximations

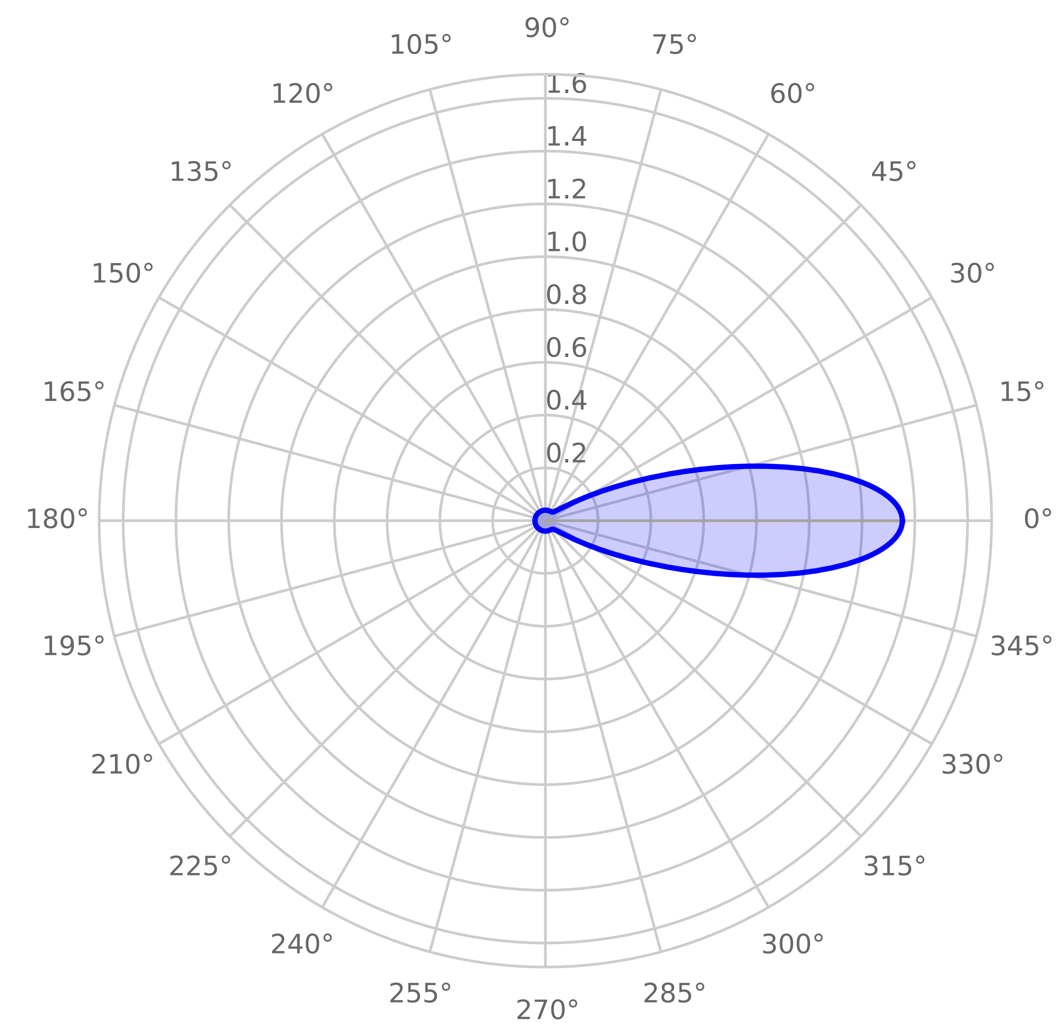
Hazy atmosphere

$$f_{p\,\mathrm{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^{8}\right)$$



Murky atmosphere

$$f_{p\,\mathrm{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$



Lorenz-Mie Approximations

Hazy atmosphere

$$f_{p\,\mathrm{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^{8}\right)$$

Murky atmosphere

$$f_{p\,\mathrm{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$

Rayleigh Scattering

- Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light
- Used for atmospheric scattering, gasses, transparent solids
- Highly wavelength dependent

Rayleigh Scattering

Wavelength

Index of refraction

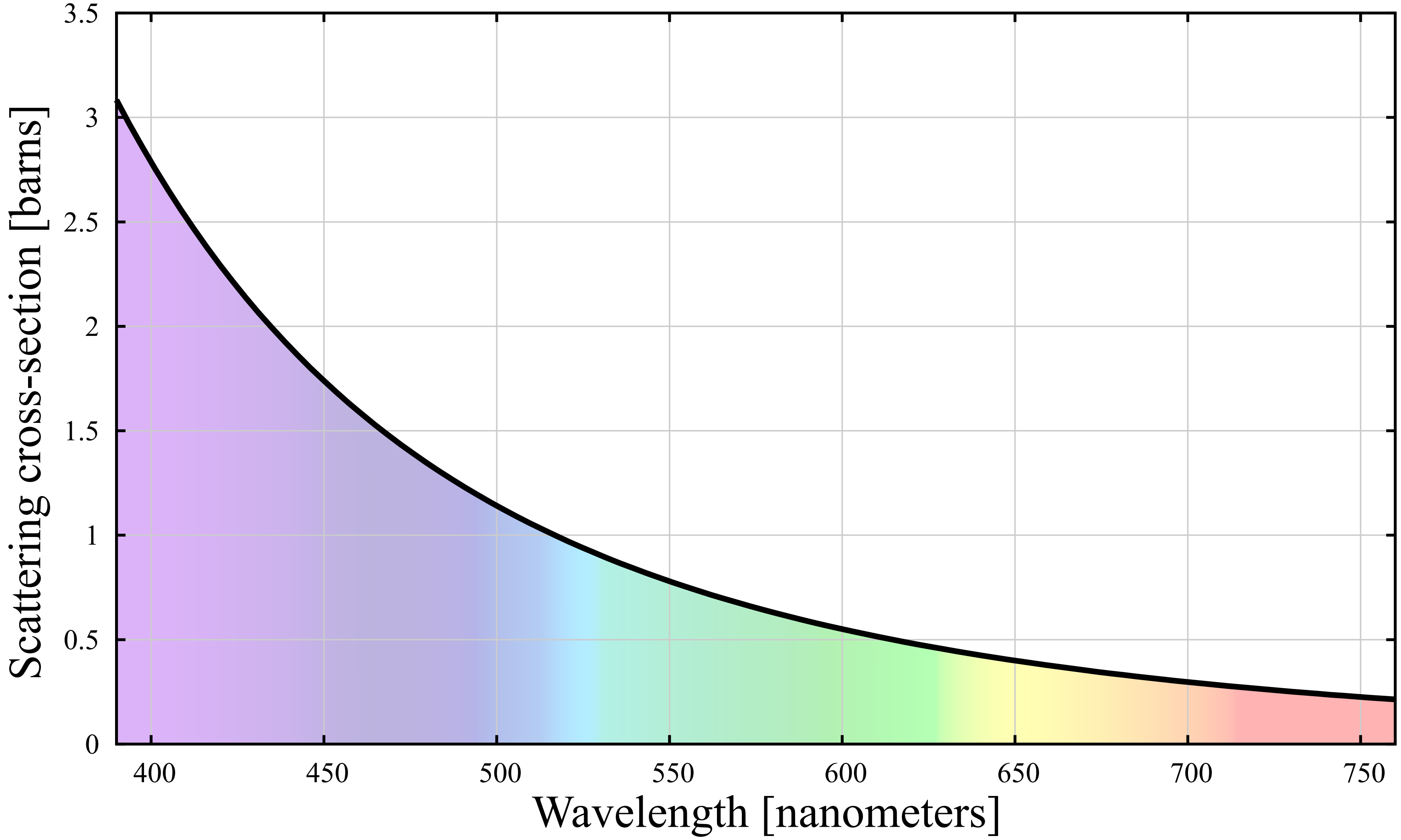
$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho\,\frac{2\pi^5 d^6}{3\lambda^4}\left(\frac{\eta^2 - 1}{\eta^2 + 2}\right)^2$$

Density of scatterers

Diameter of scatterers

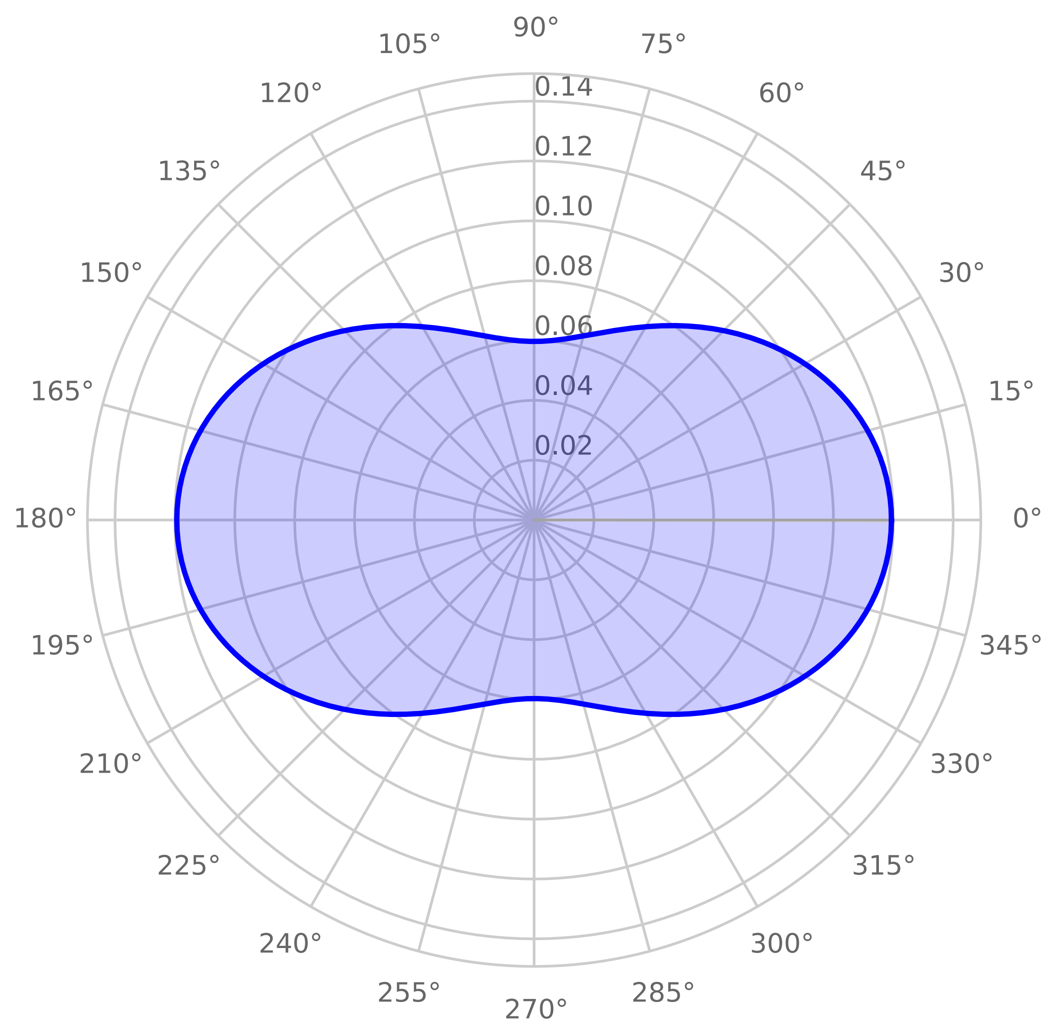
Rayleigh Scattering

$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \, \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$



Rayleigh Phase Function

$$f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi}(1 + \cos^2\theta)$$



Scattering at right angles has about half intensity of scattering forward or backward

Visual Break

Visual Break

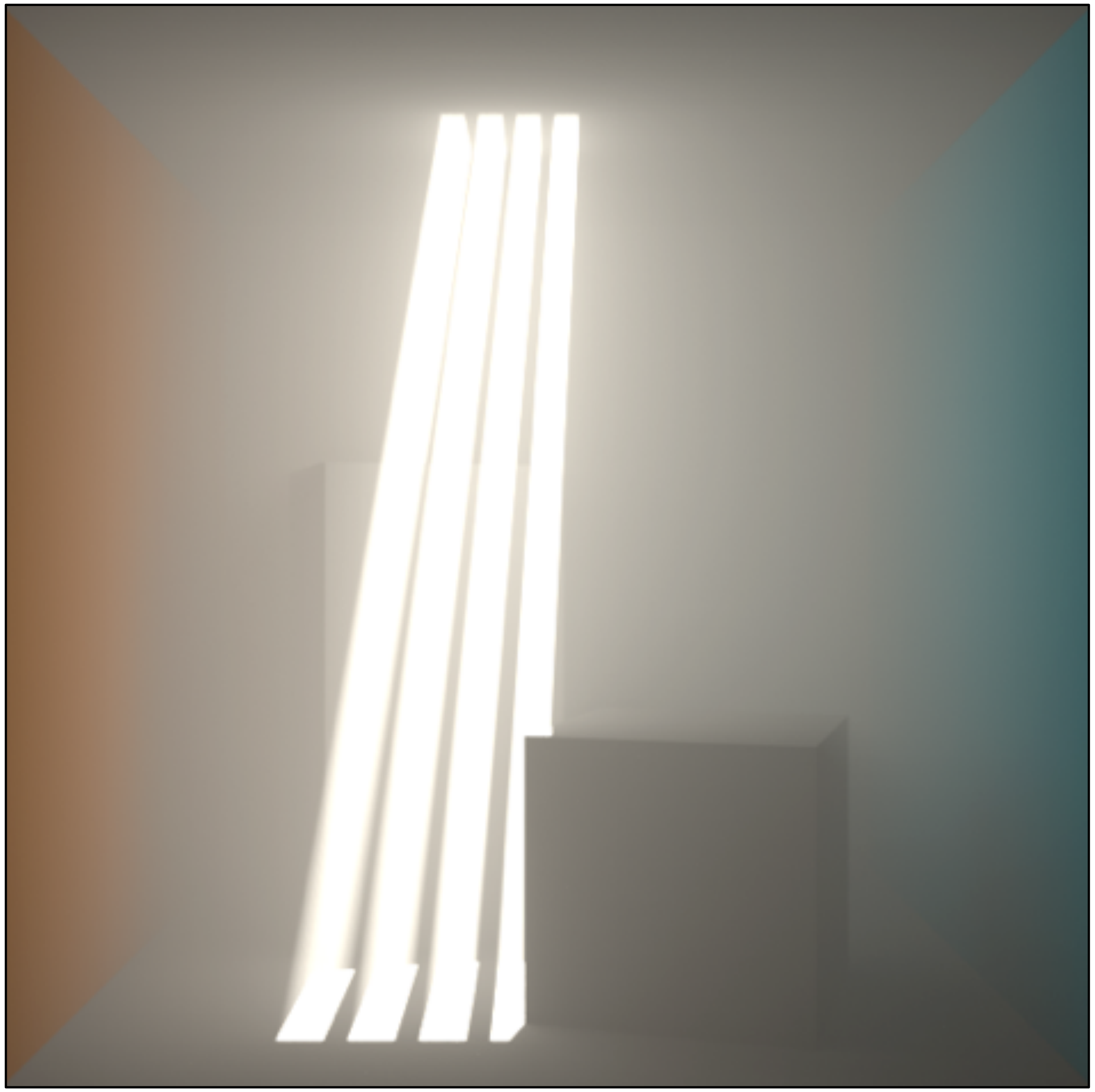
Steam

Forward scattering

Smoke

Backward scattering

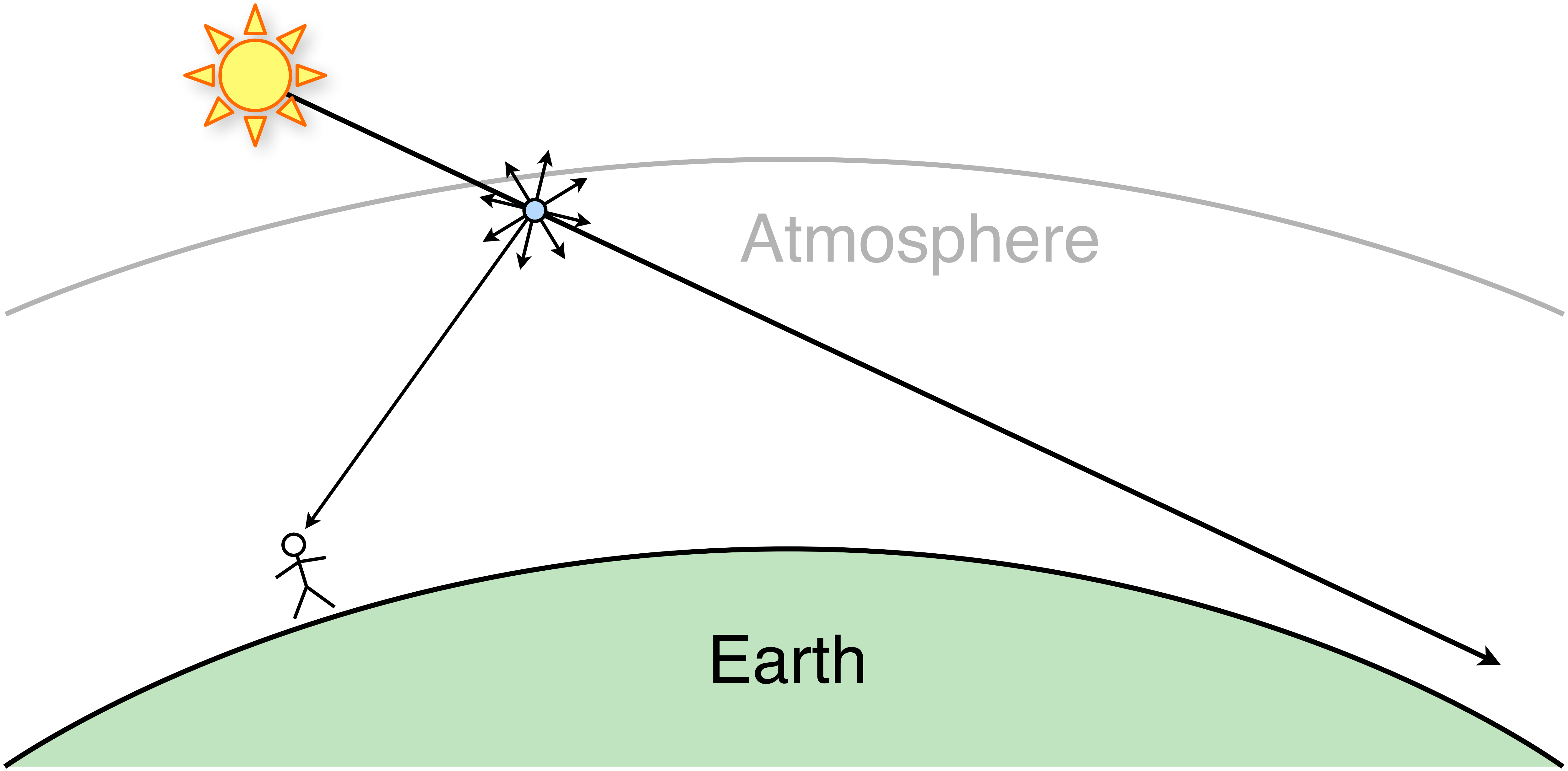
Visual Break



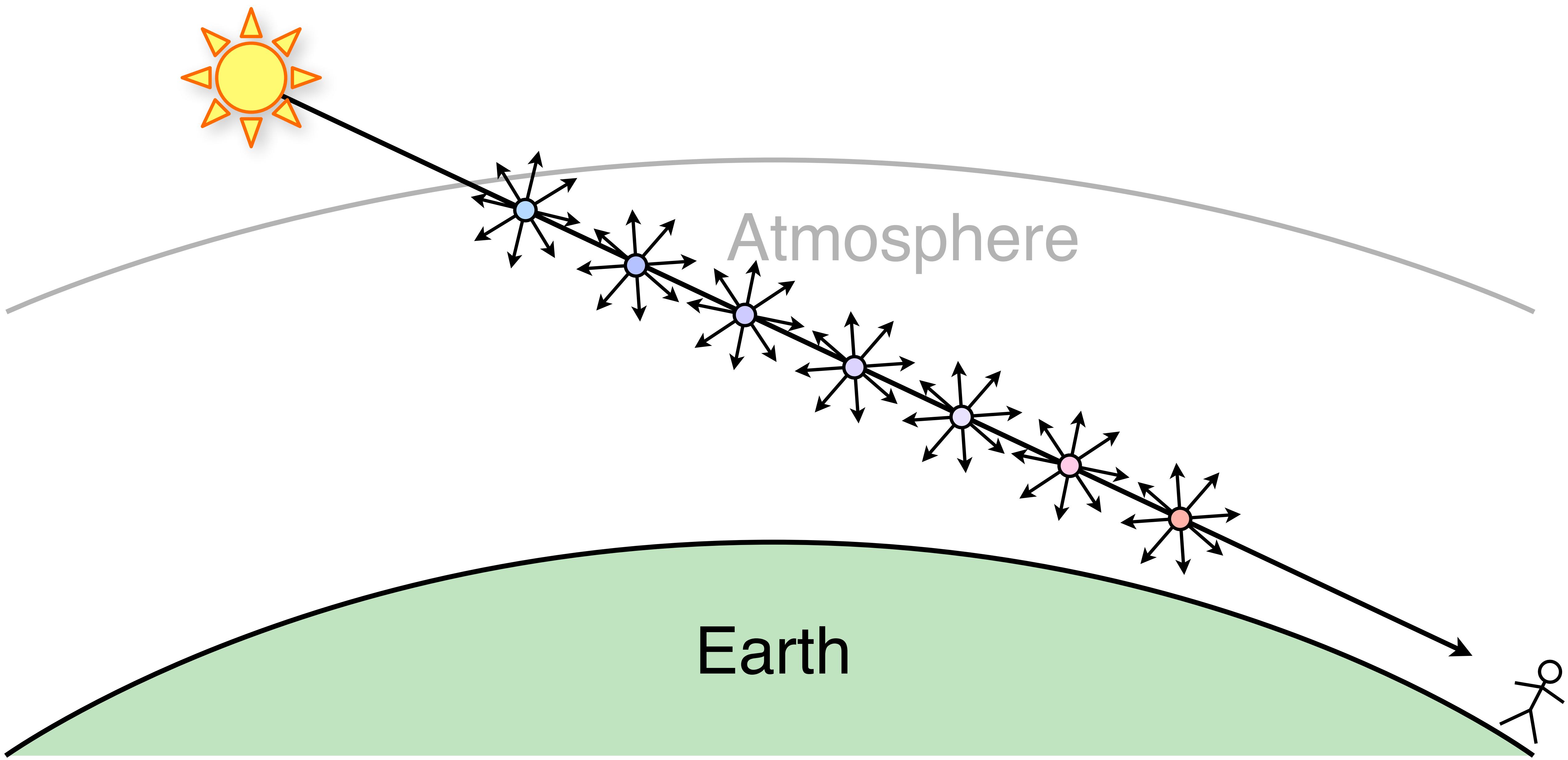
Isotropic scattering

Forward scattering

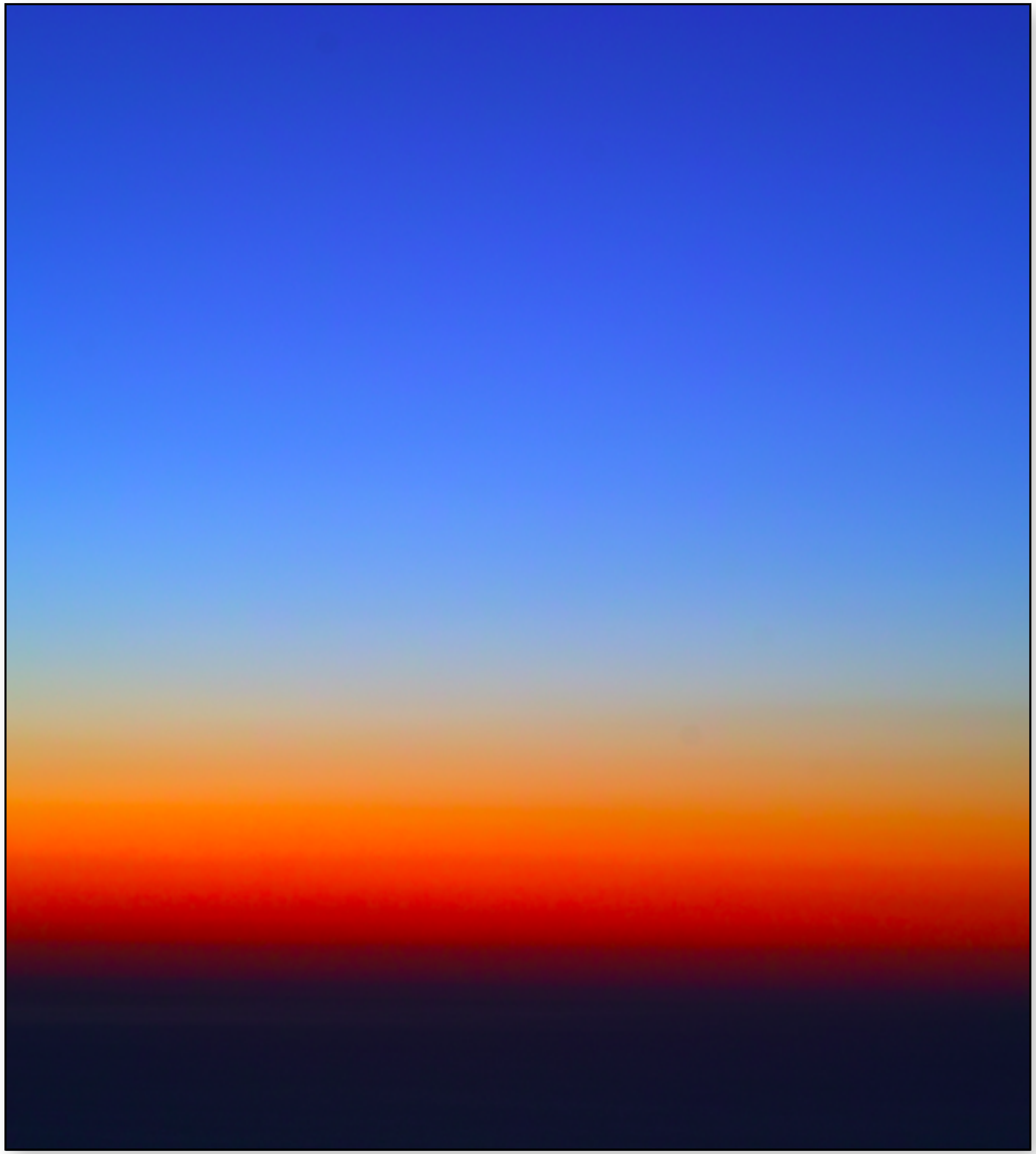
Why is the Sky Blue?

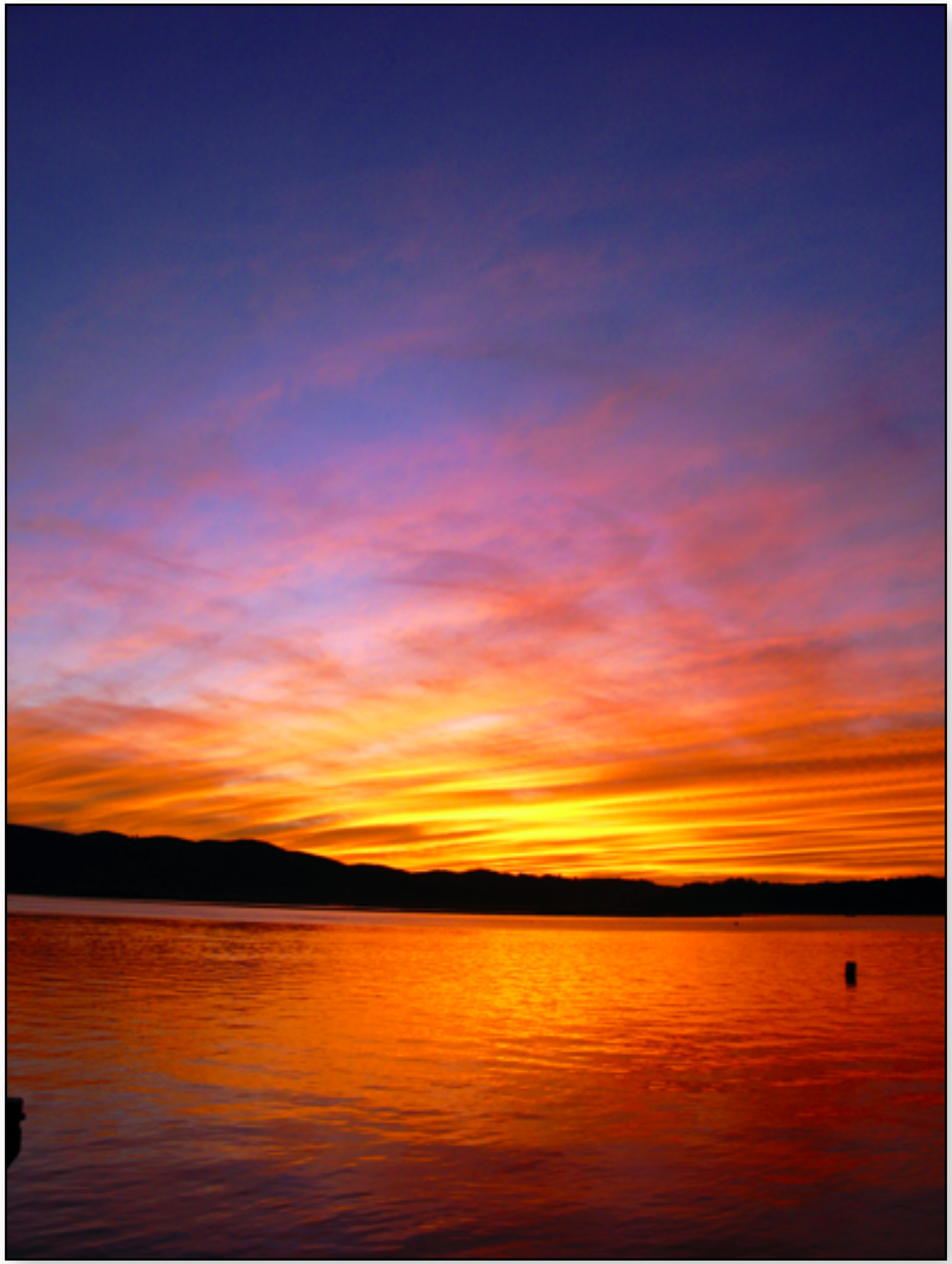


Why is the Sunset Red?



Rayleigh Scattering





Media Properties (Recap)

- Given:
  - Absorption coefficient $\sigma_a(\mathbf{x})$
  - Scattering coefficient $\sigma_s(\mathbf{x})$
  - Phase function $f_p(\mathbf{x}, \vec{\omega}', \vec{\omega})$
- Derived:
  - Extinction coefficient $\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$
  - Albedo $\alpha(\mathbf{x}) = \sigma_s(\mathbf{x})/\sigma_t(\mathbf{x})$
  - Mean-free path $E[1/\sigma_t(\mathbf{x})]$
  - Transmittance $T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t)dt}$

Homogeneous Isotropic Medium

- Given:
  - Absorption coefficient σ_a
  - Scattering coefficient σ_s
  - Phase function $\frac{1}{4\pi}$
- Derived:
  - Extinction coefficient $\sigma_t = \sigma_a + \sigma_s$
  - Albedo $\alpha = \sigma_s / \sigma_t$
  - Mean-free path $1/\sigma_t$
  - Transmittance $T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x}-\mathbf{y}\|}$

Questions?

What is this?

source: wikipedia

Crepuscular Rays

Anti-Crepuscular Rays

Solving the Volume Rendering Equation

Volume Rendering Equation

Attenuated background radiance

Accumulated emitted radiance

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) = &\boxed{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})} \\
&+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt} \\
&+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt}
\end{aligned}
$$

Accumulated in-scattered radiance

Volume Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) = {} & T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
& + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
& + \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt}
\end{aligned}
$$

Accumulated in-scattered radiance

In-scattered Radiance

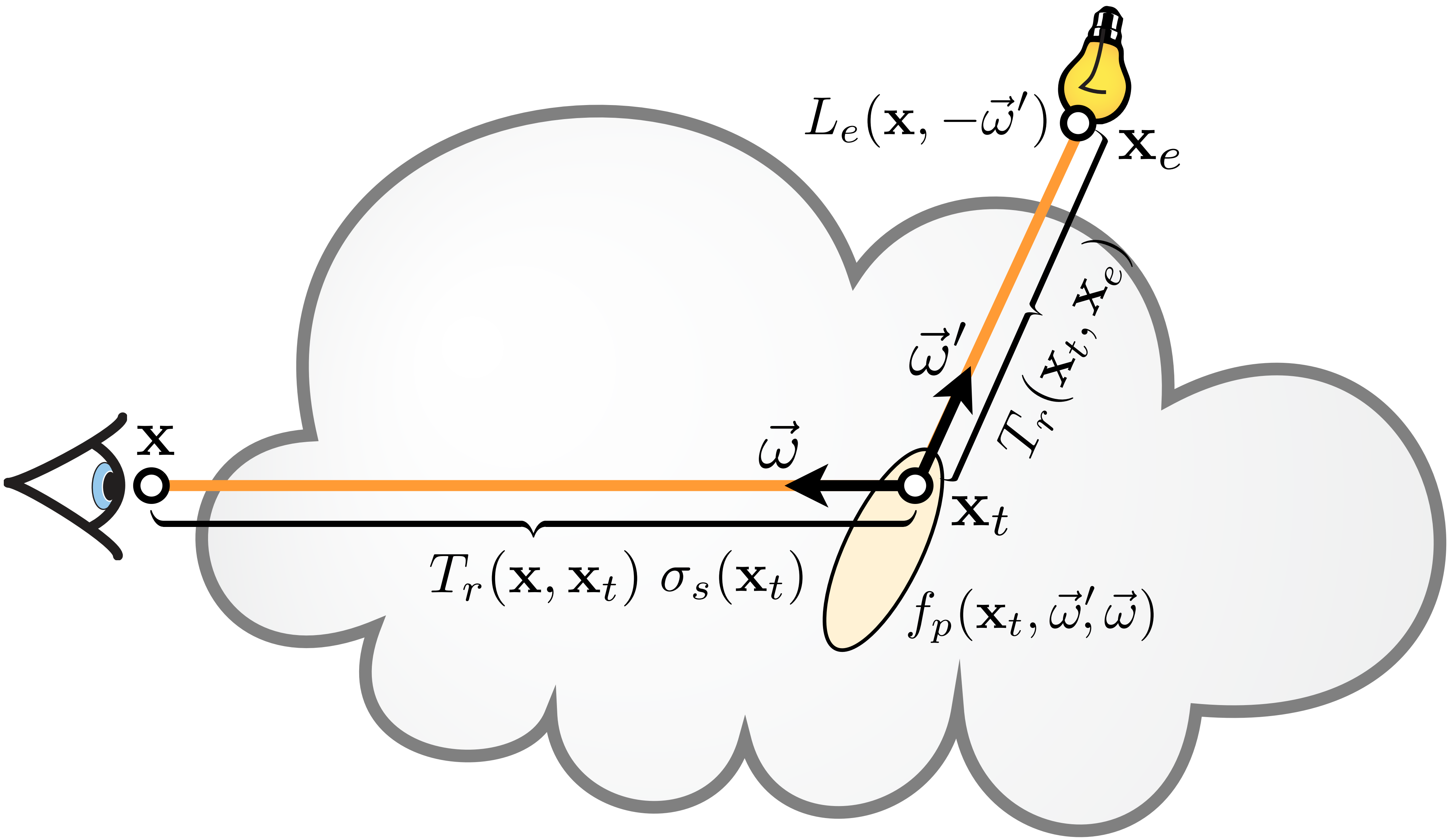
$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\boxed{L_s(\mathbf{x}_t,\vec{\omega})}dt$$

$$\boxed{L_s(\mathbf{x}_t,\vec{\omega})} = \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})\boxed{L_i(\mathbf{x}_t,\vec{\omega}')}d\vec{\omega}'$$

- Single scattering
  - L_i arrives directly from a light source (direct illum.)
    i.e.: $\boxed{L_i(\mathbf{x},\vec{\omega})} = T_r(\mathbf{x},r(\mathbf{x},\vec{\omega}))\, L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$
- Multiple scattering
  - L_i arrives through multiple bounces (indirect illum.)

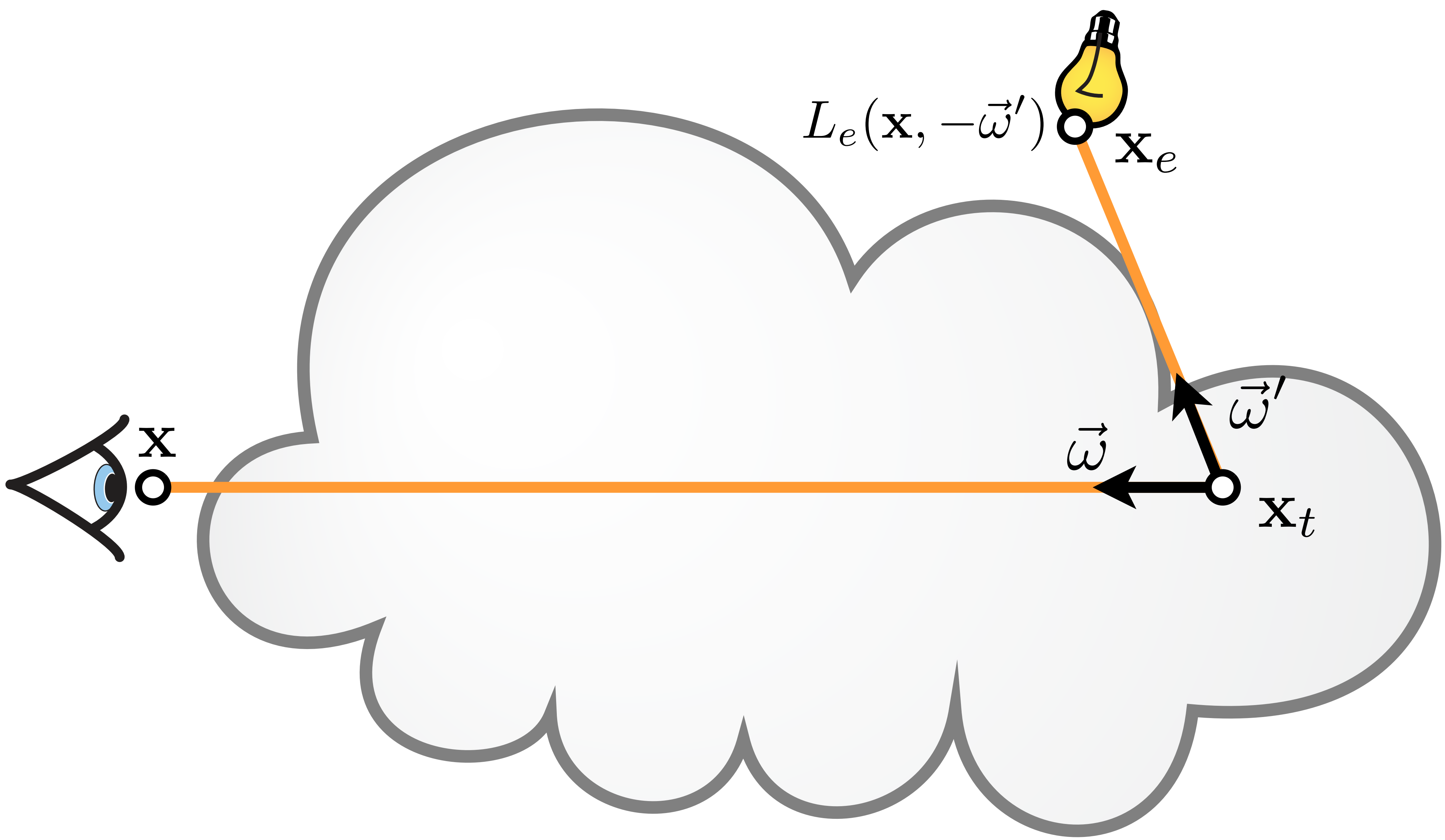
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



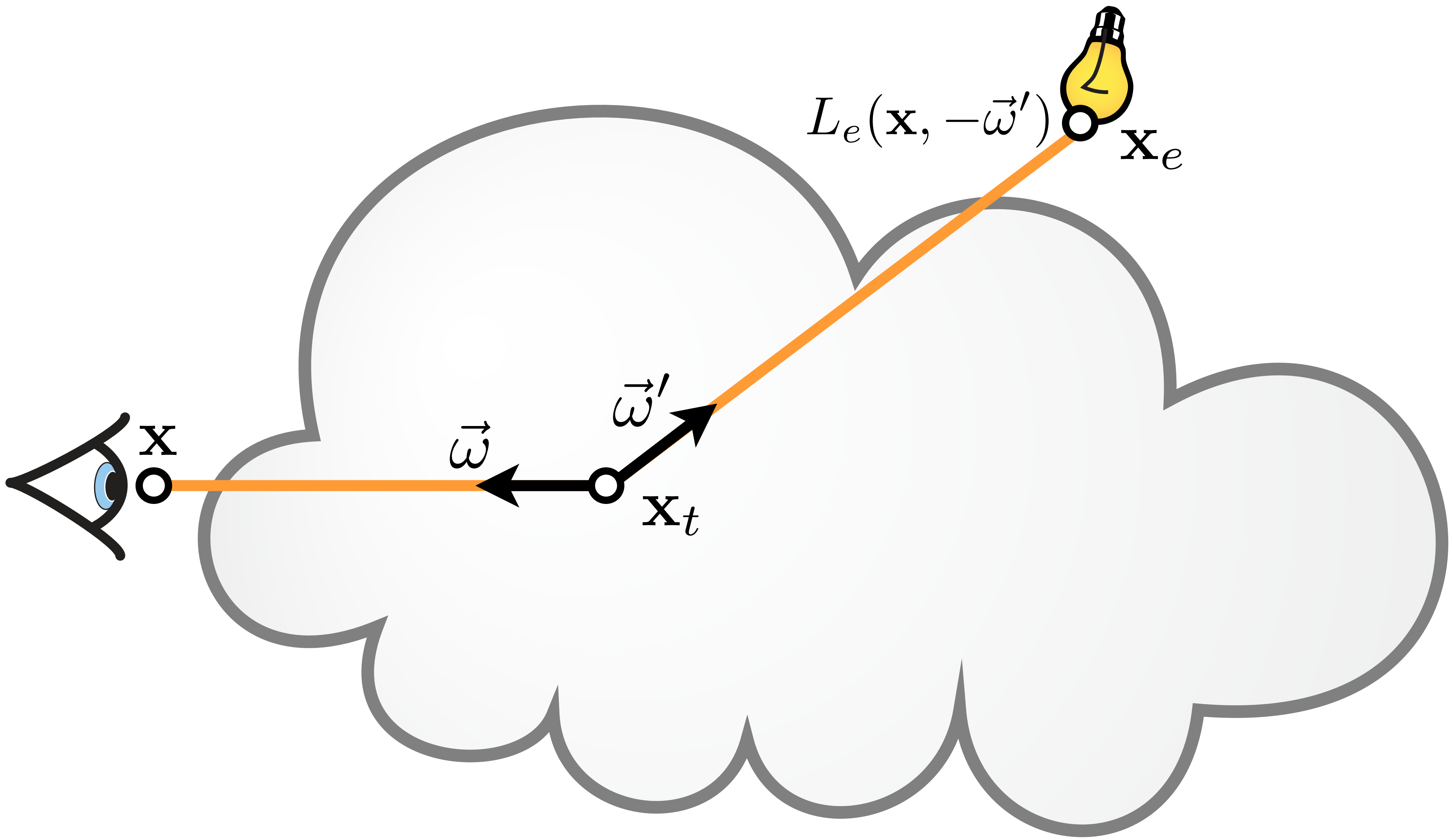
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



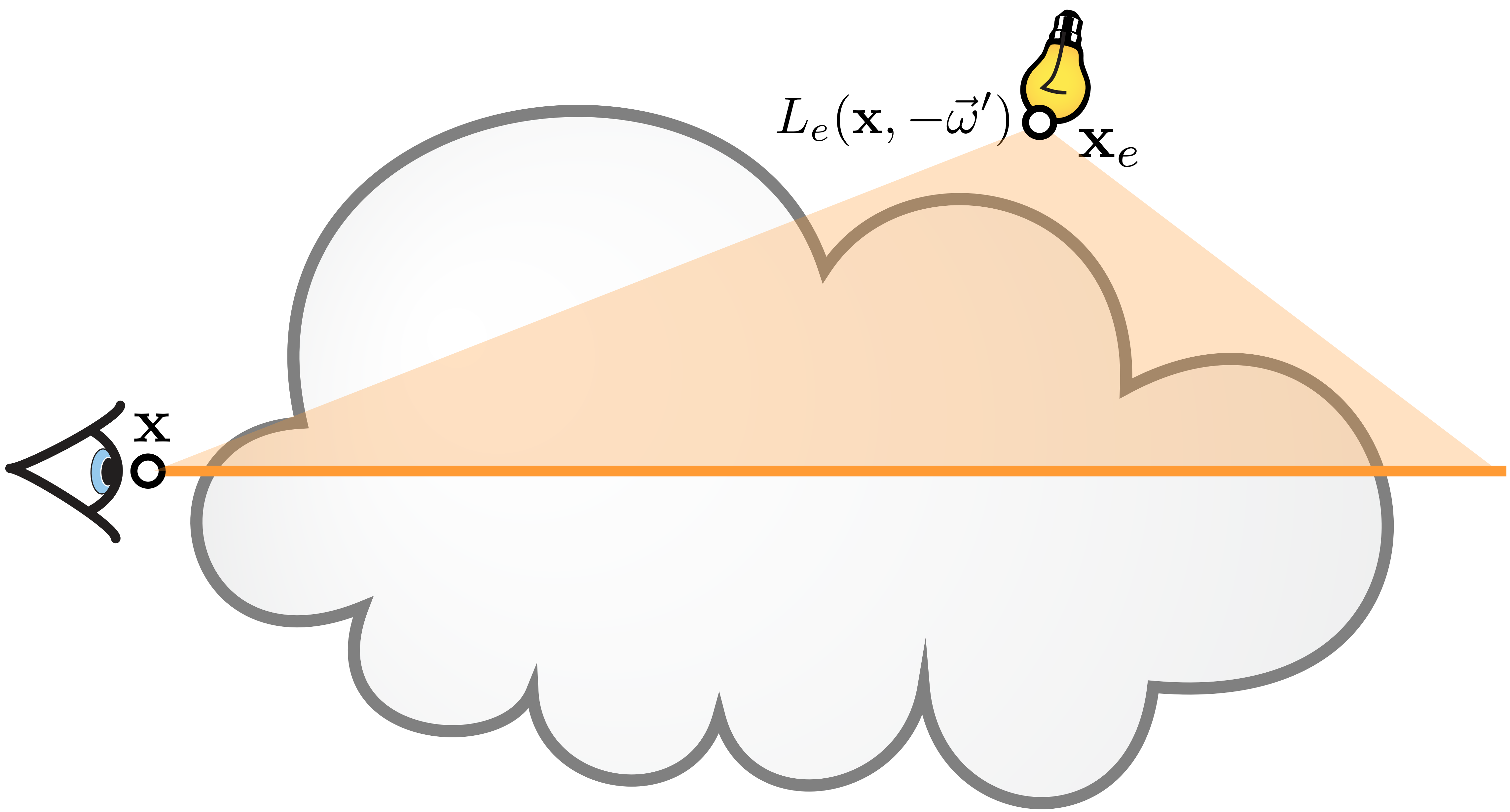
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \boxed{\int_0^z} T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$

- (Semi-)analytic solutions:
  - Sun et al. [2005]
  - Pegoraro et al. [2009, 2010]
- Numerical solutions:
  - Ray-marching
  - Equiangular sampling

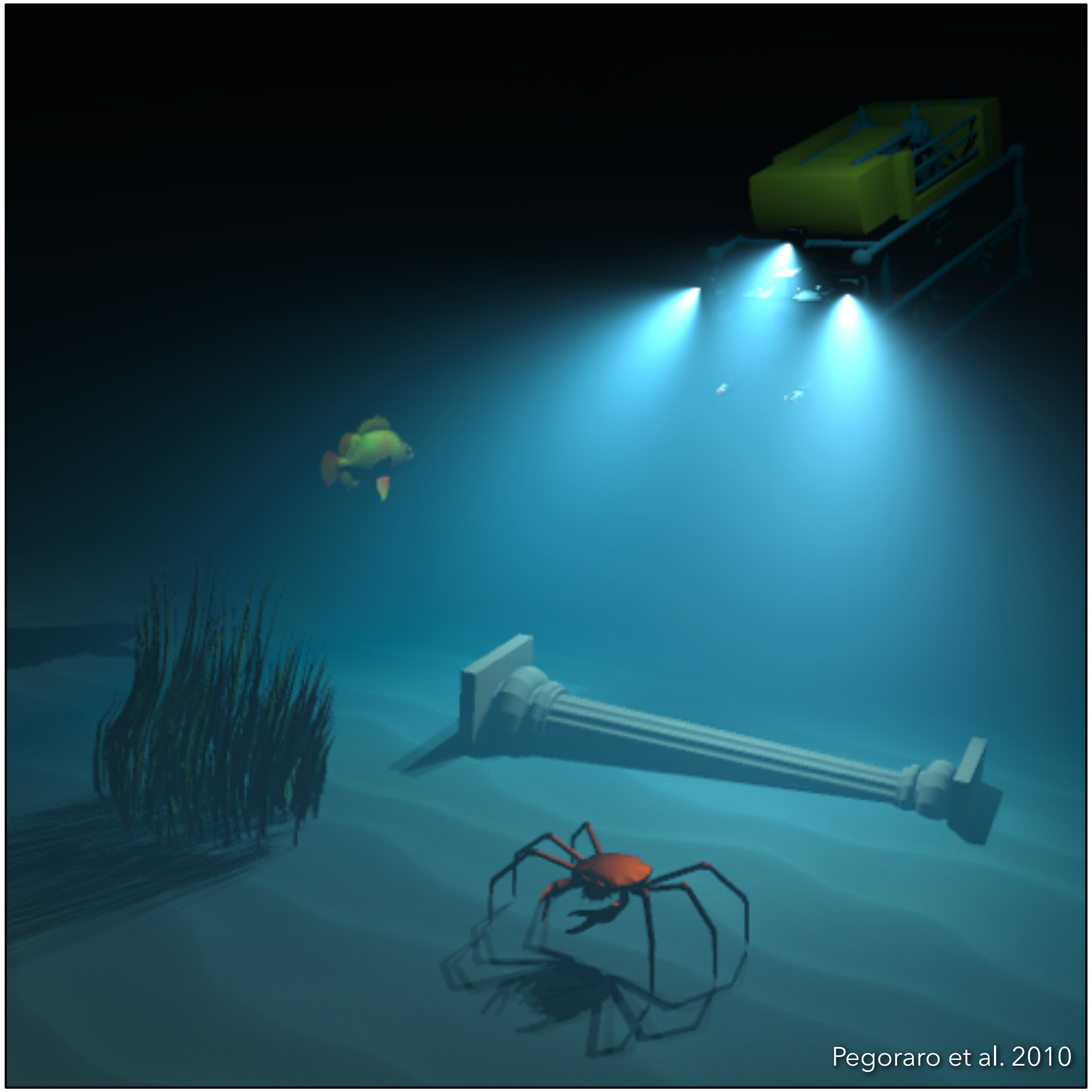
Analytic Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$

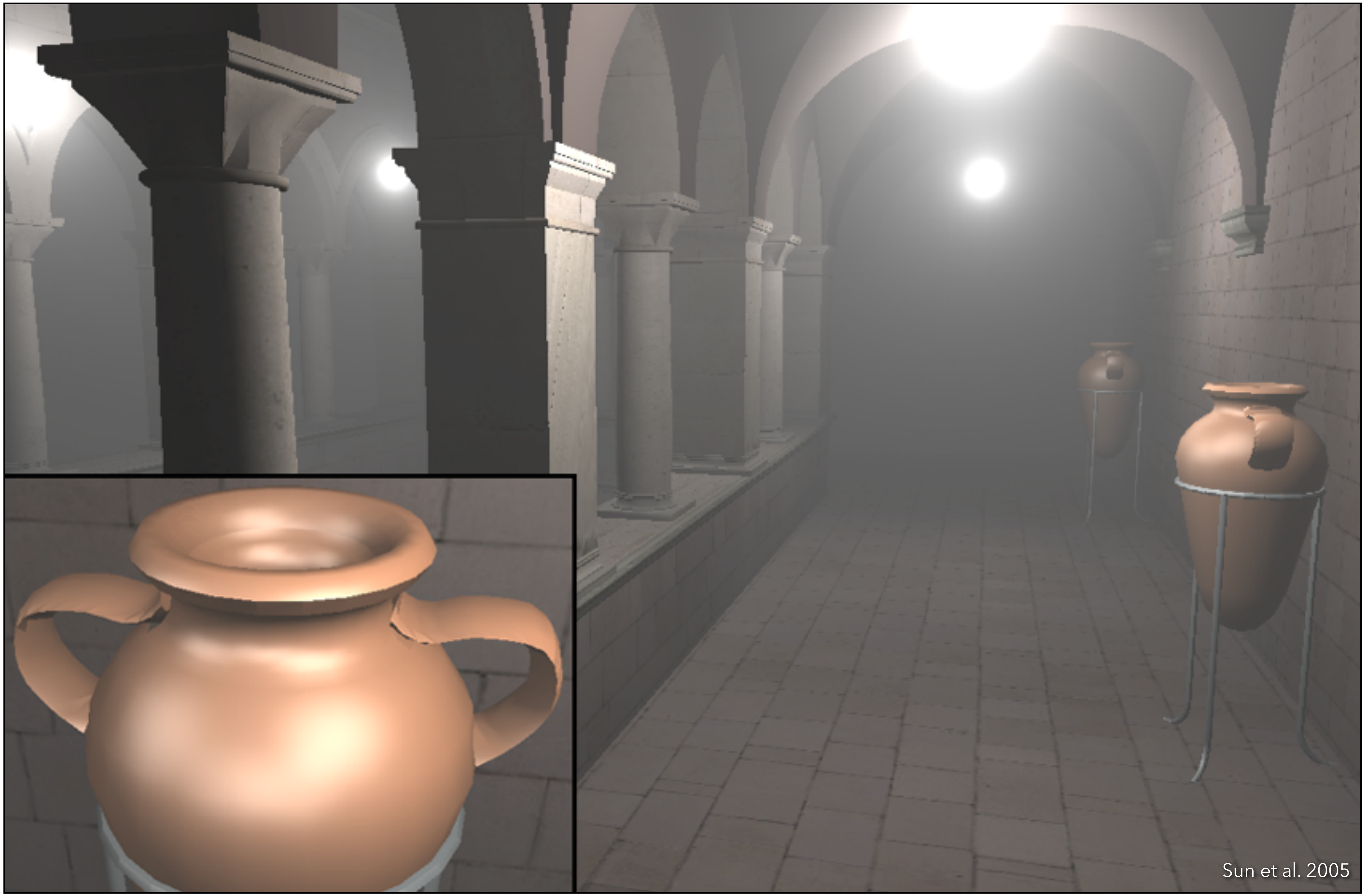
- Assumptions:
  - Homogeneous medium
  - Point or spot light
  - Relatively simple phase function
  - No occlusion

$$L(\mathbf{x}, \vec{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \sigma_s \int_0^z e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_t\|} e^{-\sigma_t \|\mathbf{x}_t, \mathbf{x}_p\|} dt$$

Analytic Single Scattering



Analytic Single Scattering



Sun, B., Ramamoorthi, R., Narasimhan, S. G., and Nayar, S. K.
A practical analytic single scattering model for real-time rendering.
SIGGRAPH 2005.

Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2+1)^{n+1}} v^k \mathrm{d}v$$

$$\begin{aligned}
\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \Bigg(& \sum_{k=0}^{\min\{m-1-l,n\}} \binom{n}{k} \Bigg(\frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \\
& - e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2+1)^j} \sum_{\substack{i=(m-n-l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \Bigg) \\
& + \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=(-m+l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \Bigg)
\end{aligned}$$

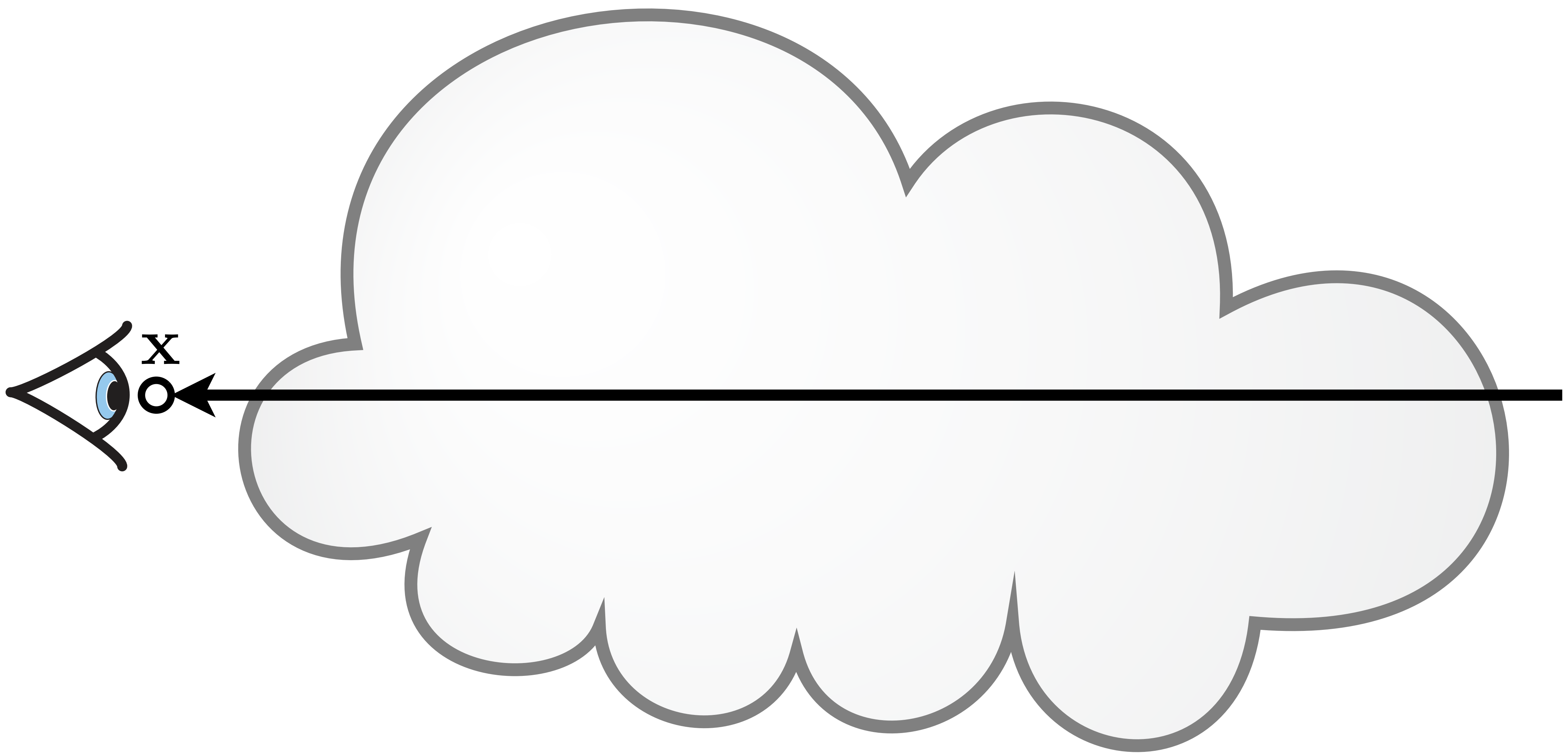
No shadows, implementation nightmare, computationally intensive…

Let's try brute force!

Ray-Marching

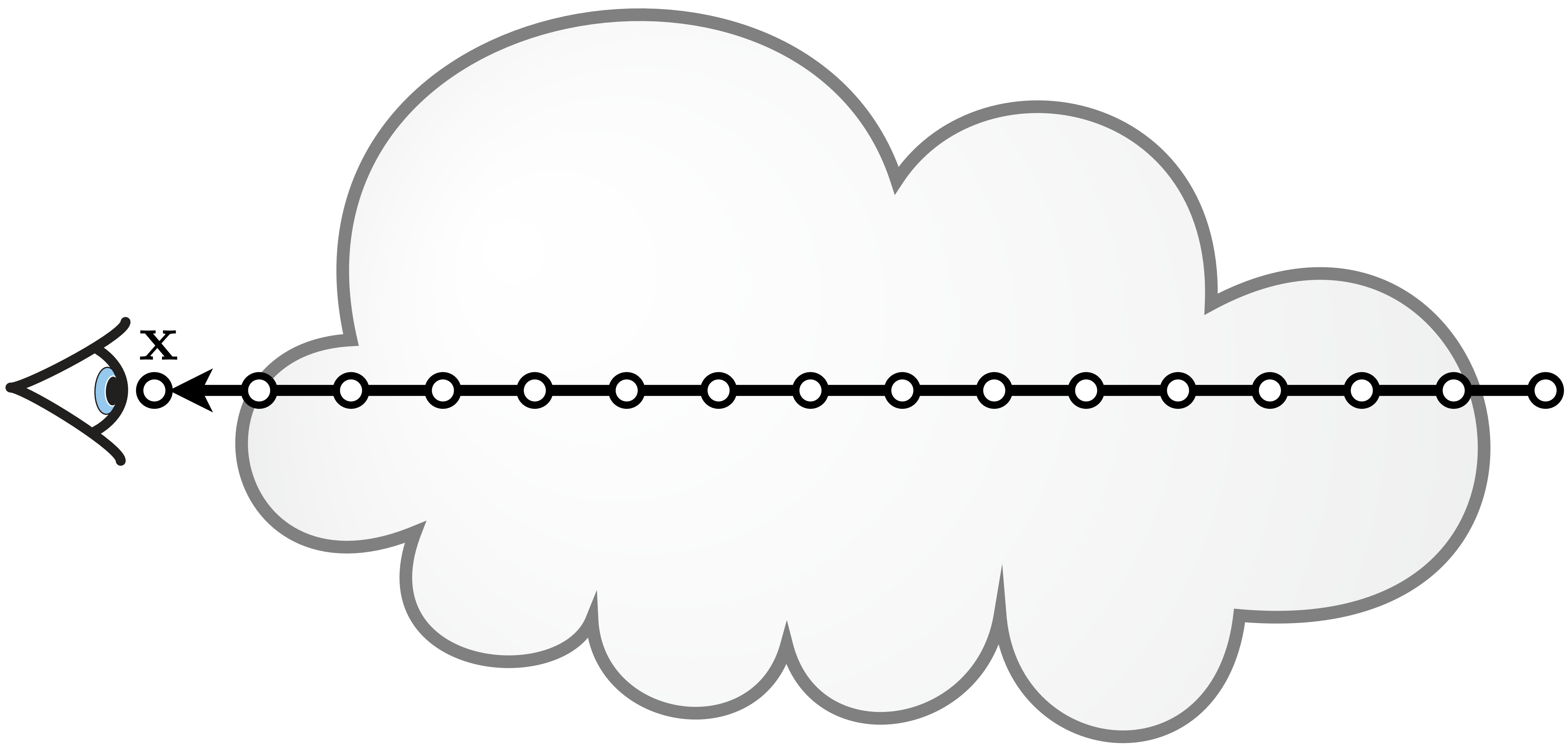
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t, \vec{\omega})dt$$

Approximate with Riemann sum



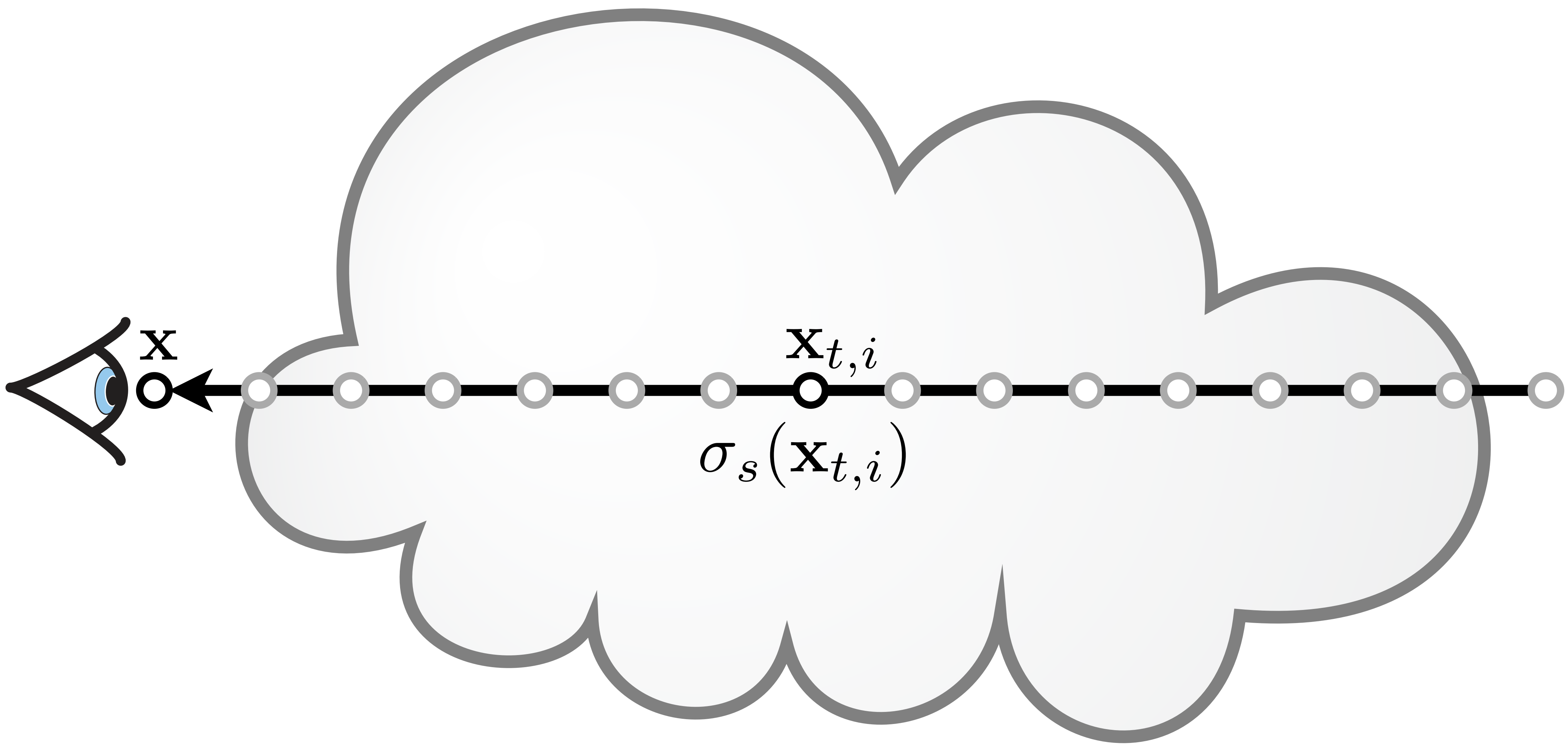
Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

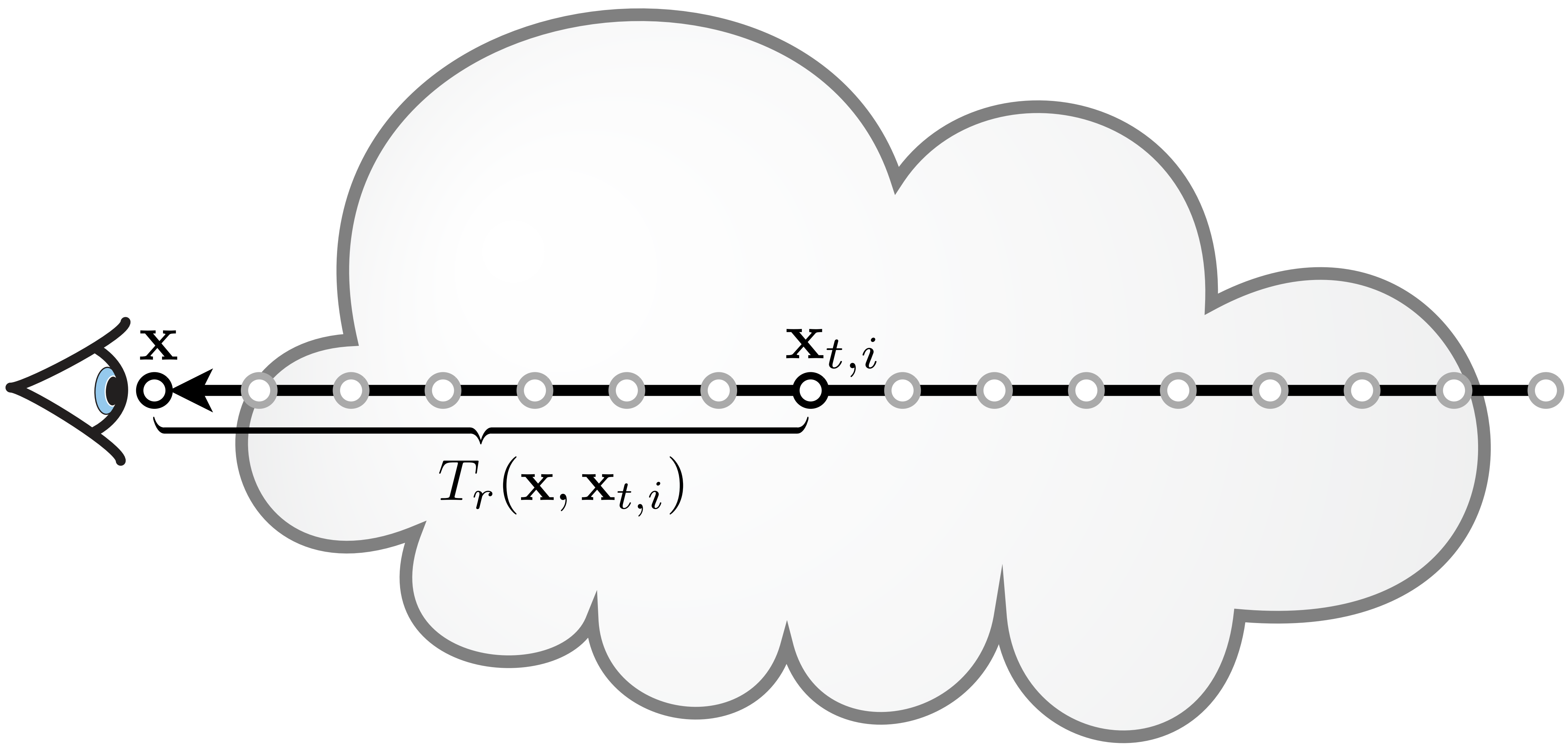
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,i}) \boxed{\sigma_s(\mathbf{x}_{t,i})} L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} \boxed{T_r(\mathbf{x}, \mathbf{x}_{t,i})} \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$

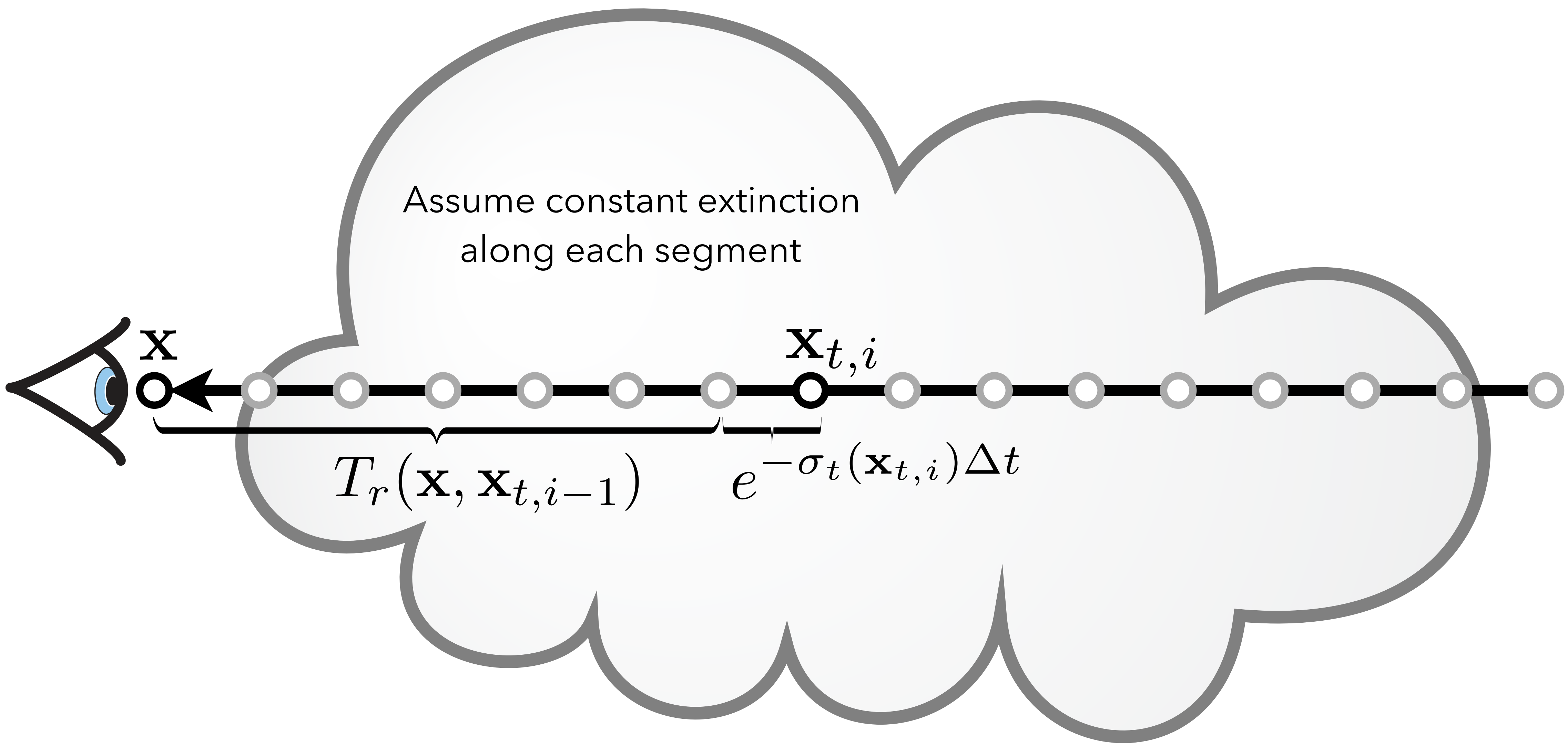
Homogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = e^{-\sigma_t \|\mathbf{x}, \mathbf{x}_{t,i}\|}$



Ray-Marching

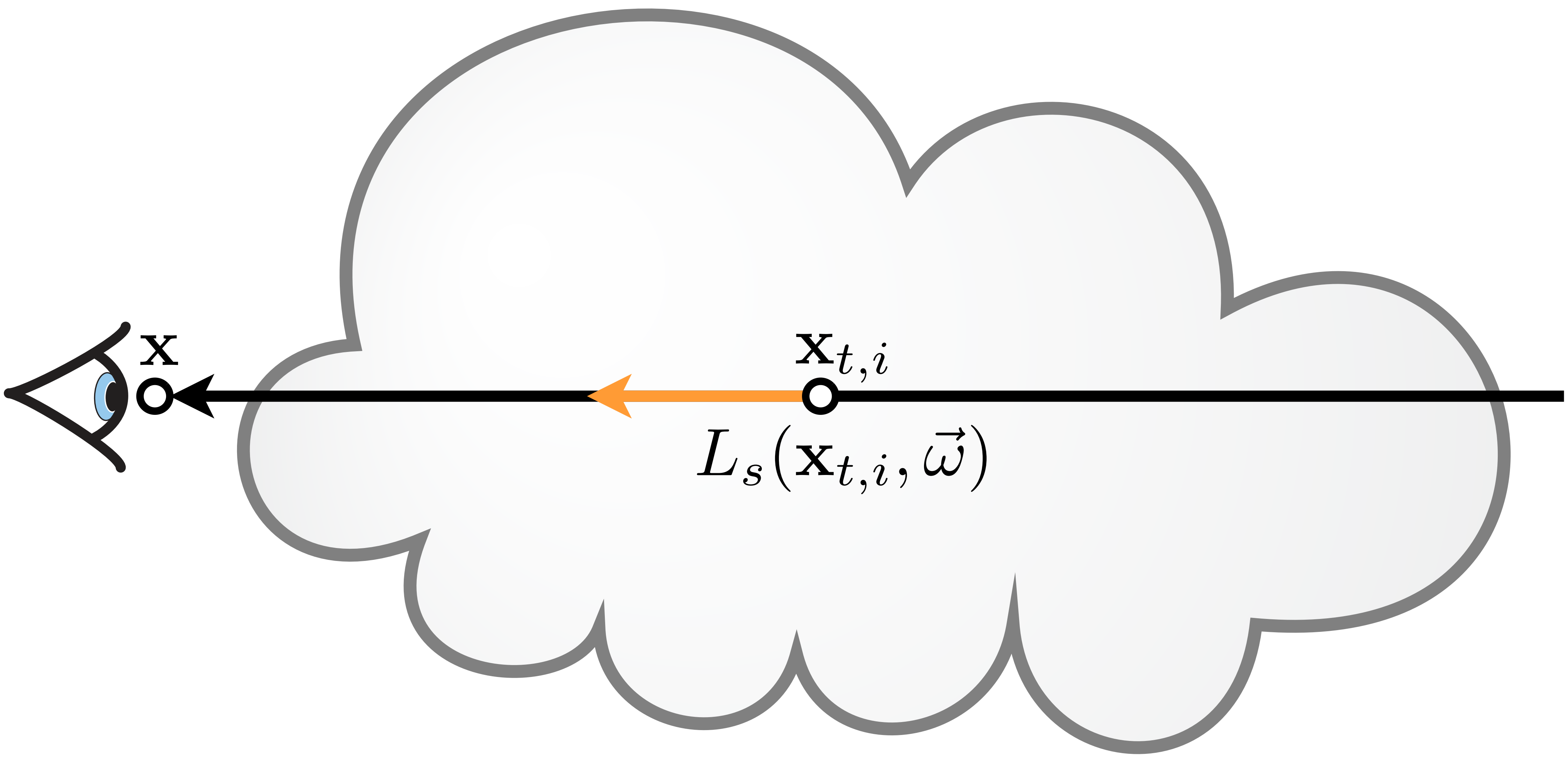
$$L(\mathbf{x},\vec{\omega}) \approx \sum_{i=1}^{N} \boxed{T_r(\mathbf{x},\mathbf{x}_{t,i})}\,\sigma_s(\mathbf{x}_{t,i})L_s(\mathbf{x}_{t,i},\vec{\omega})\Delta t$$

Heterogeneous volume: $T_r(\mathbf{x},\mathbf{x}_{t,i}) = T_r(\mathbf{x},\mathbf{x}_{t,i-1})\, e^{-\sigma_t(\mathbf{x}_{t,i})\Delta t}$



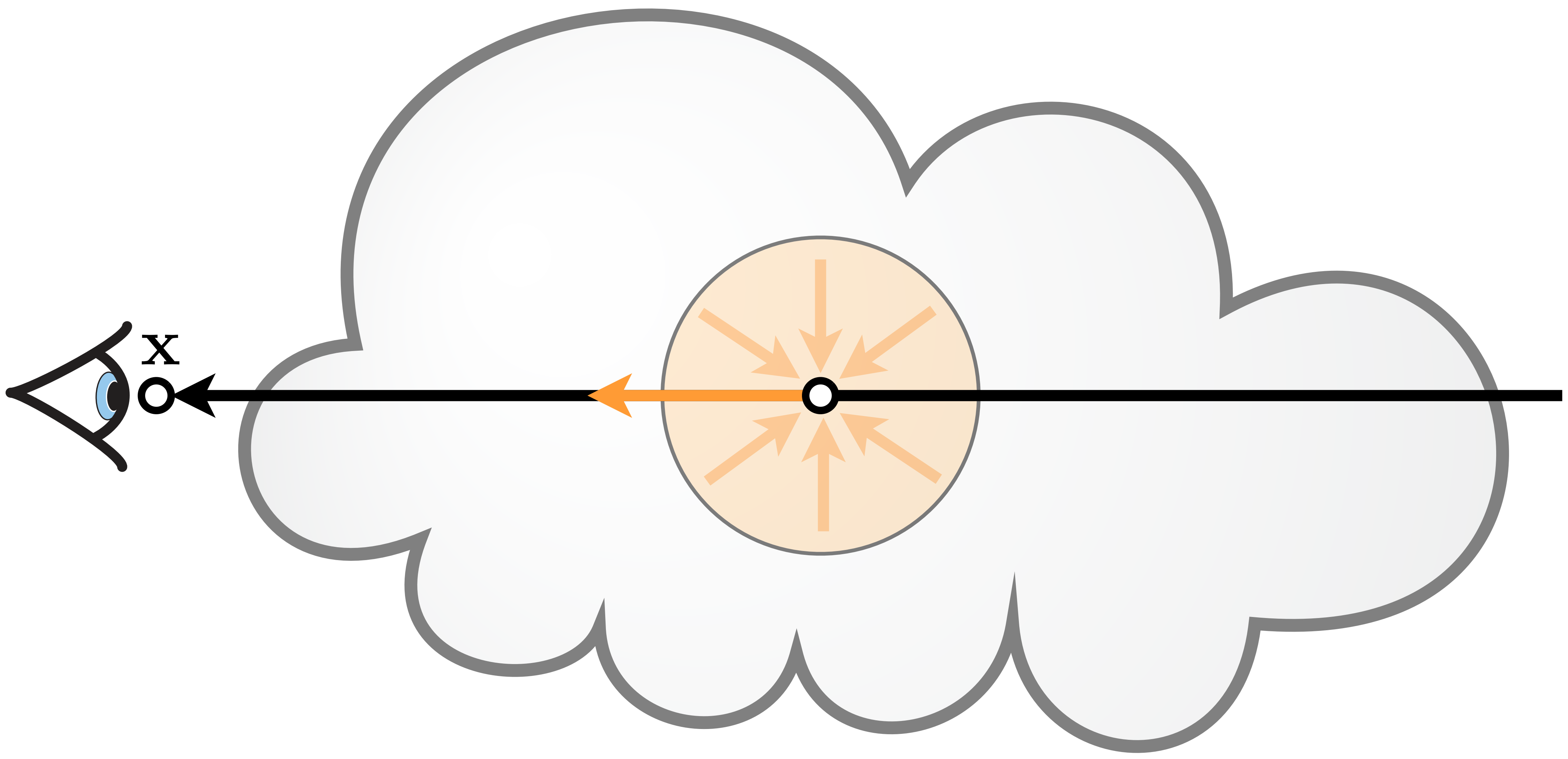
Ray-Marching

$$L(\mathbf{x},\vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x},\mathbf{x}_{t,i})\sigma_s(\mathbf{x}_{t,i})\boxed{L_s(\mathbf{x}_{t,i},\vec{\omega})}\Delta t$$



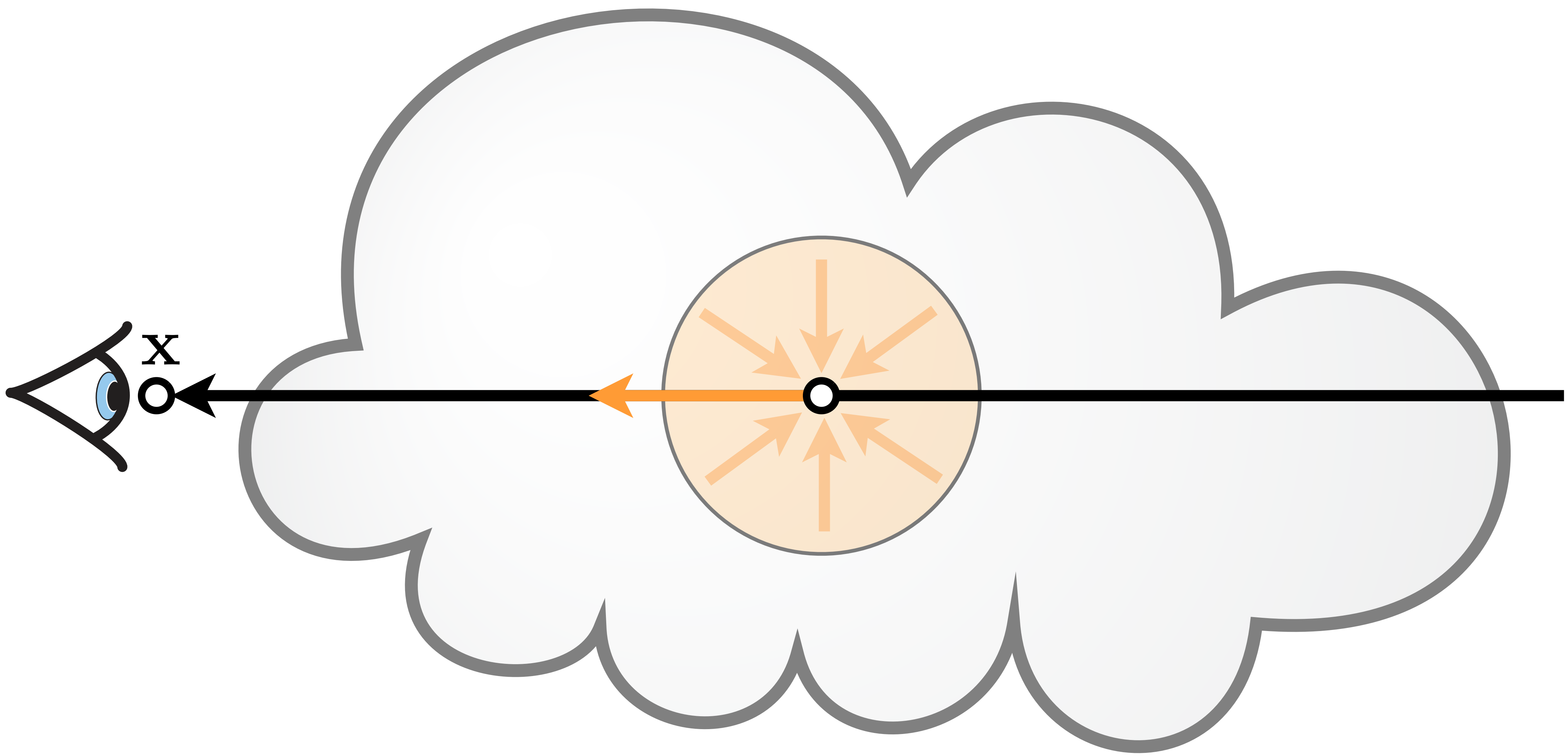
Ray-Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \boxed{\int_{S^2}} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$$



Ray-Marching

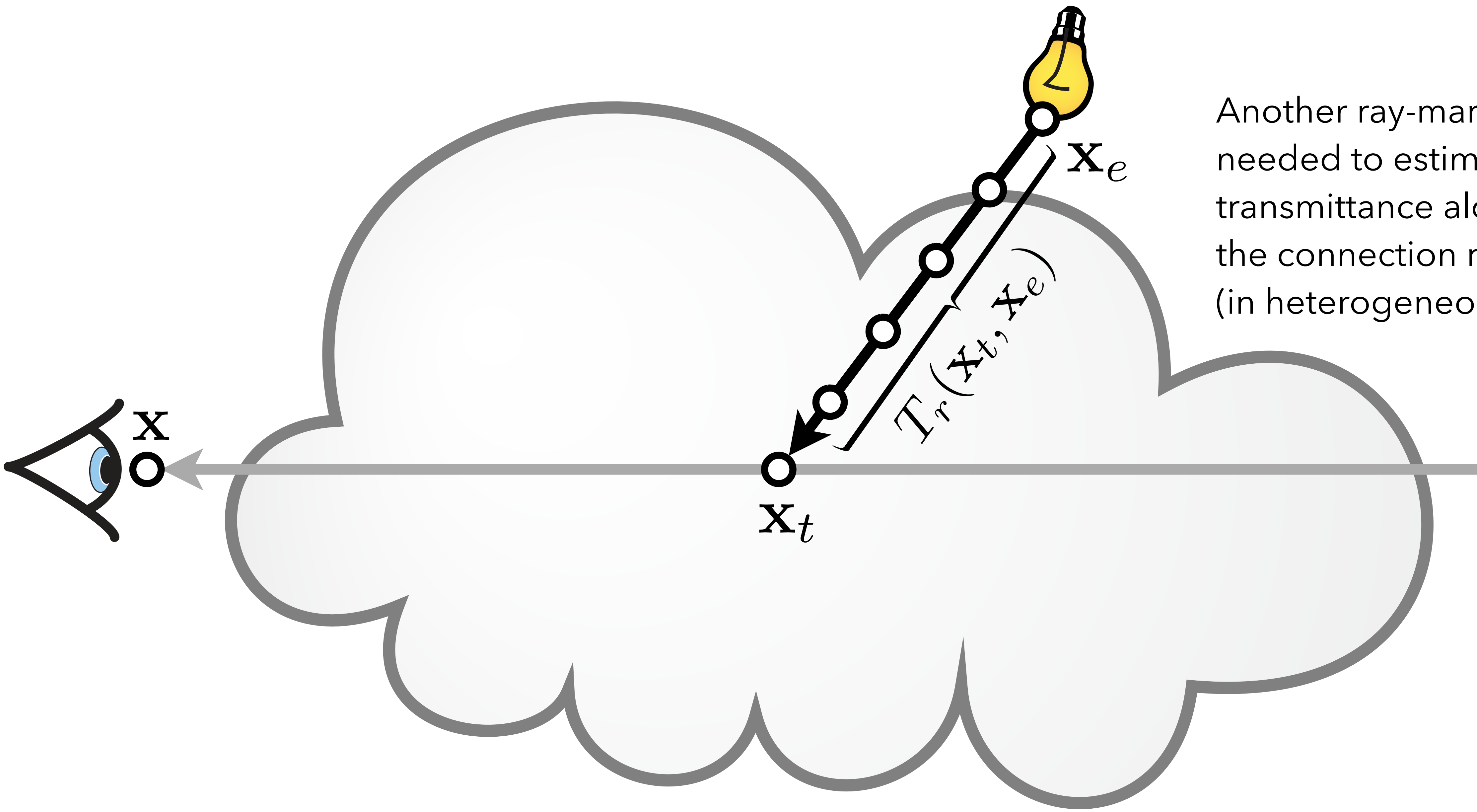
$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=0}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}'_j, \vec{\omega}) \boxed{L_i(\mathbf{x}_t, \vec{\omega}'_j)}}{p(\vec{\omega}'_j)}$$



Ray-Marching

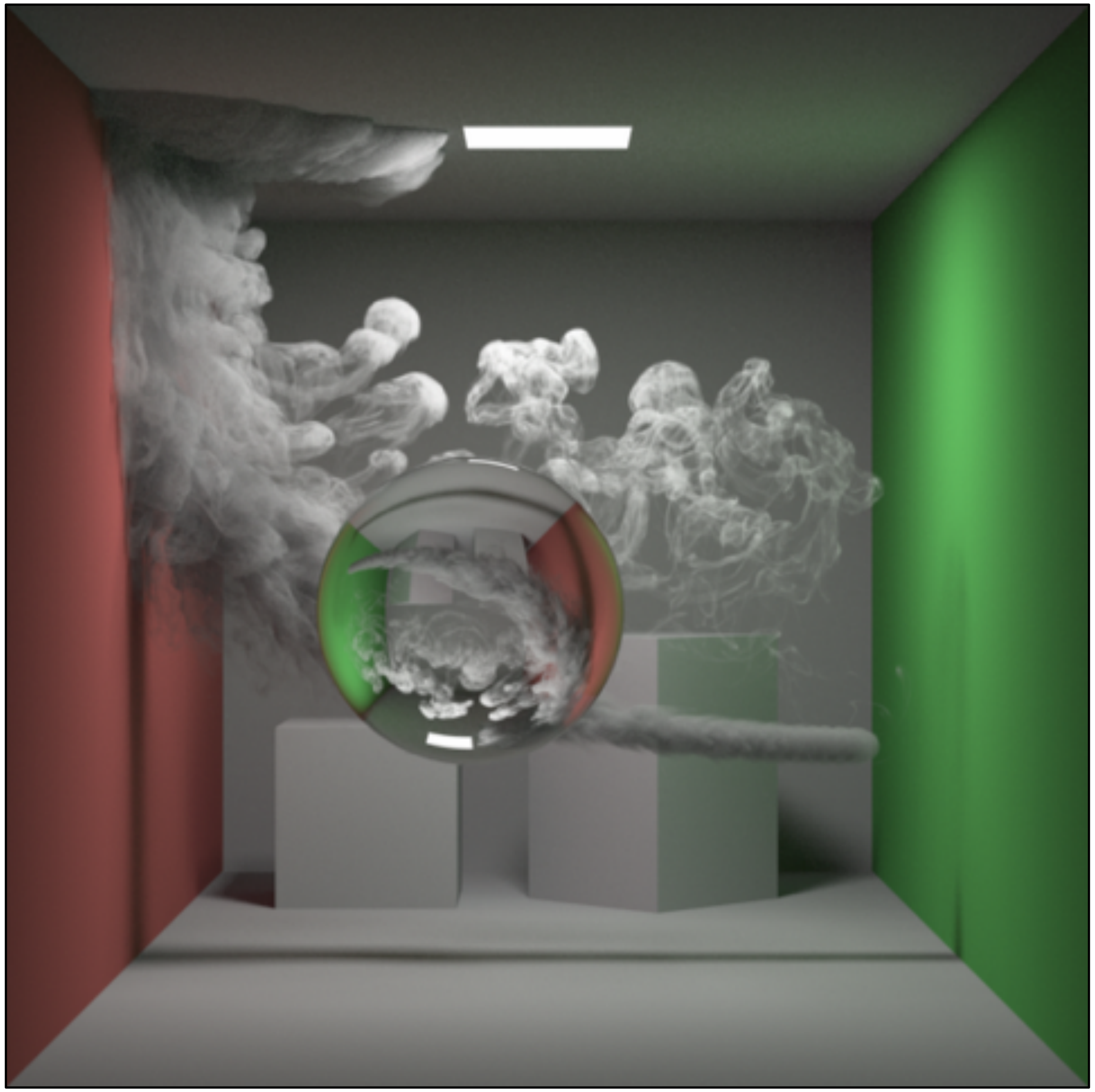
- Single scattering:

$$L_i(\mathbf{x}_t, \vec{\omega}) = T_r(\mathbf{x}_t, \mathbf{x}_e)\, L_e(\mathbf{x}_e, -\vec{\omega})$$



Another ray-marching needed to estimate the transmittance along the connection ray (in heterogeneous media)

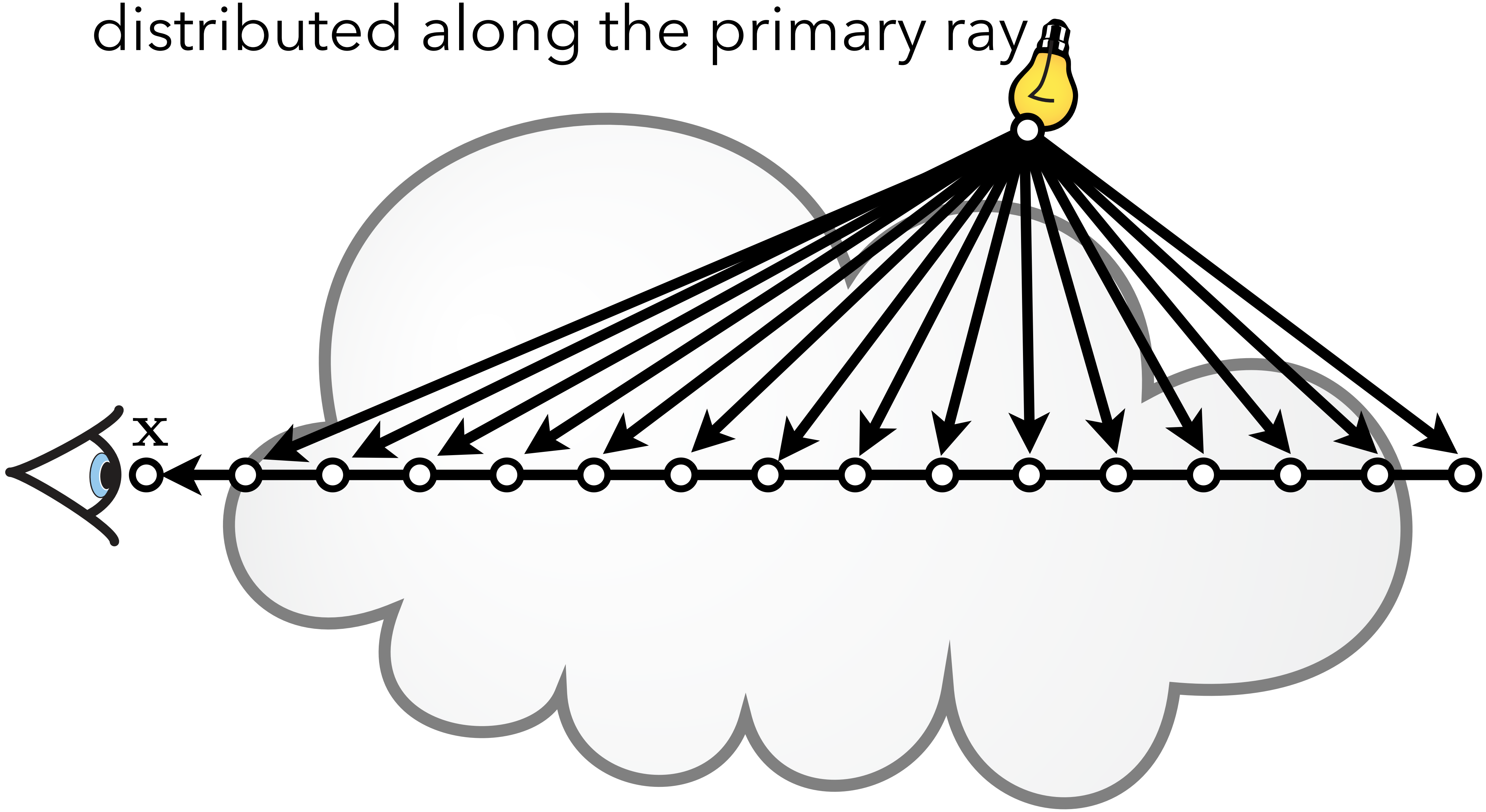
Visual Break



 Images courtesy of Kulla and Fajardo

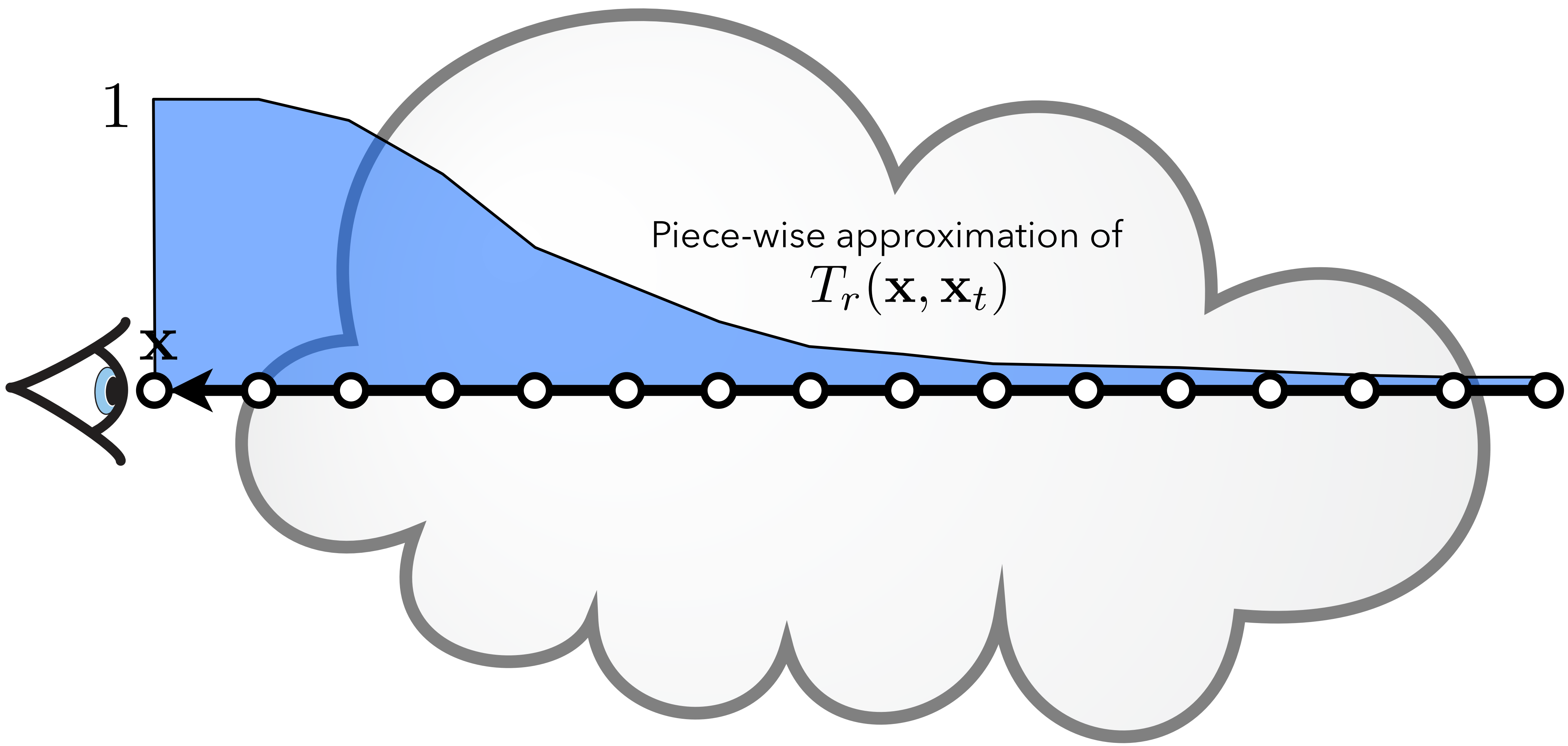
Ray-Marching in Heterogeneous Media

- Marching towards the light source
  - Connections are expensive, many, and uniformly distributed along the primary ray



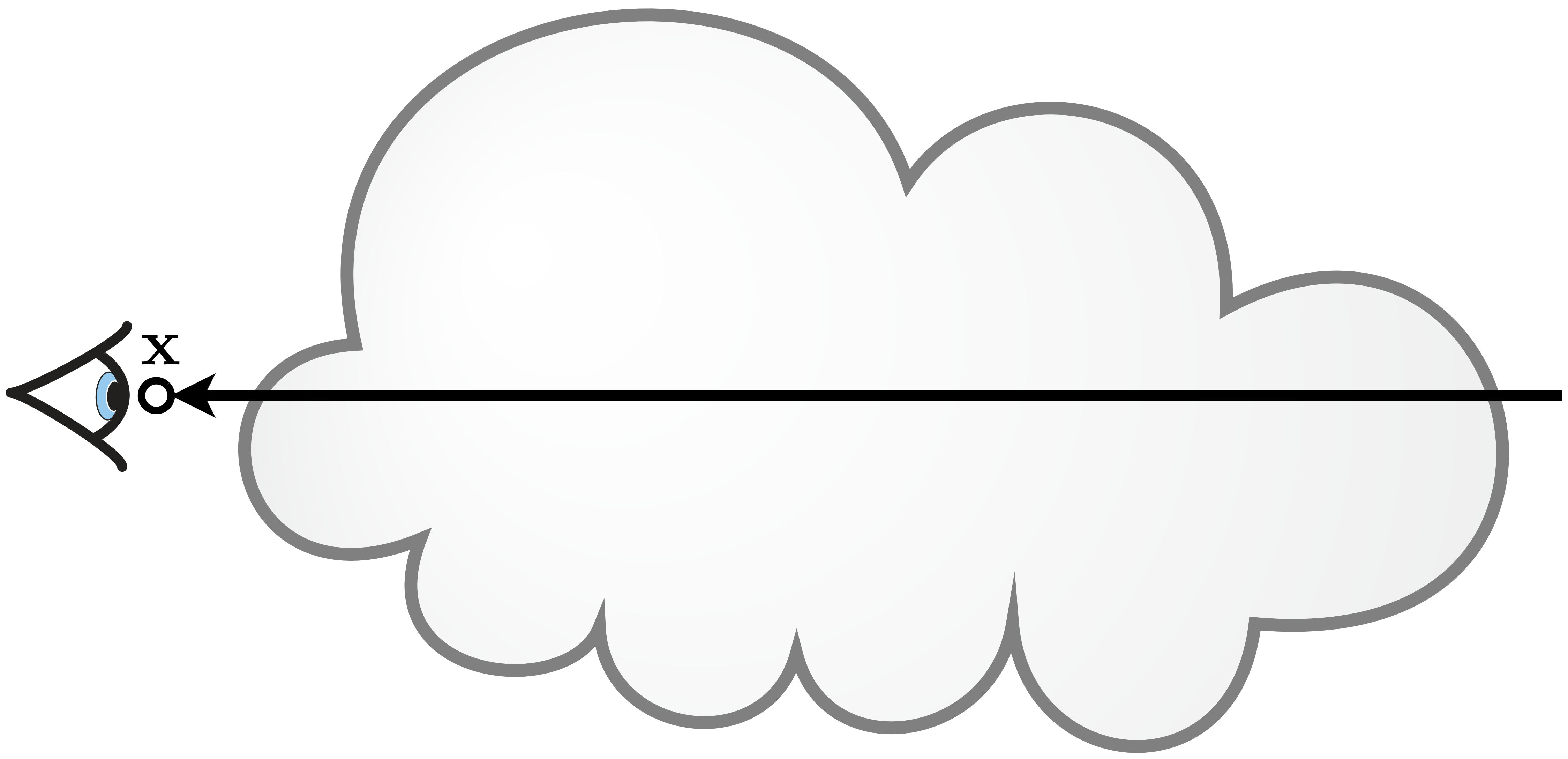
Decoupled Transmittance and In-scattering

1. Ray-march and cache transmittance
   - Choose step-size w.r.t. frequency content to accurately capture variations



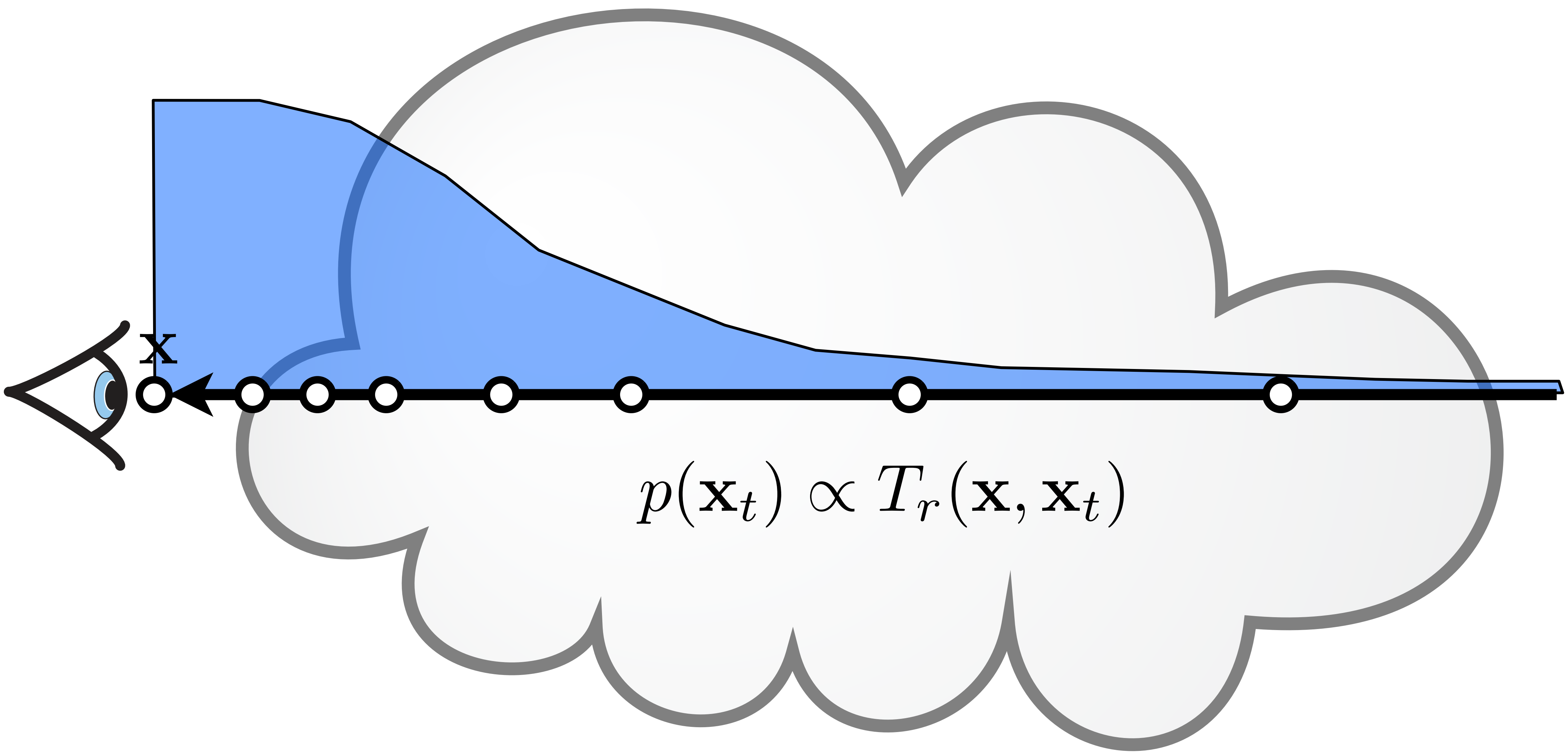
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
   - Distribute samples ∝ (part of) the integrand



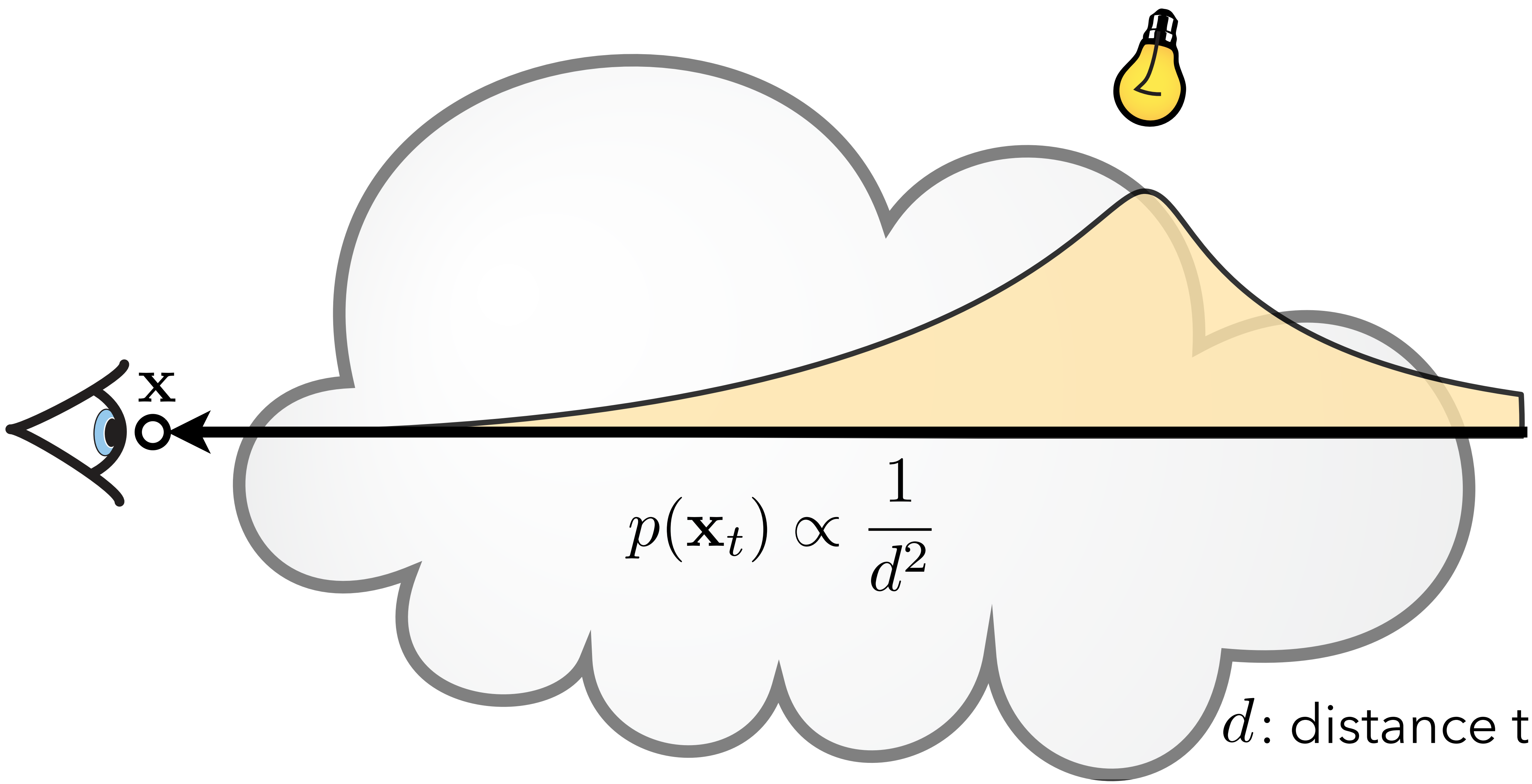
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
   - Distribute samples ∝ (part of) the integrand



Decoupled Transmittance and In-scattering

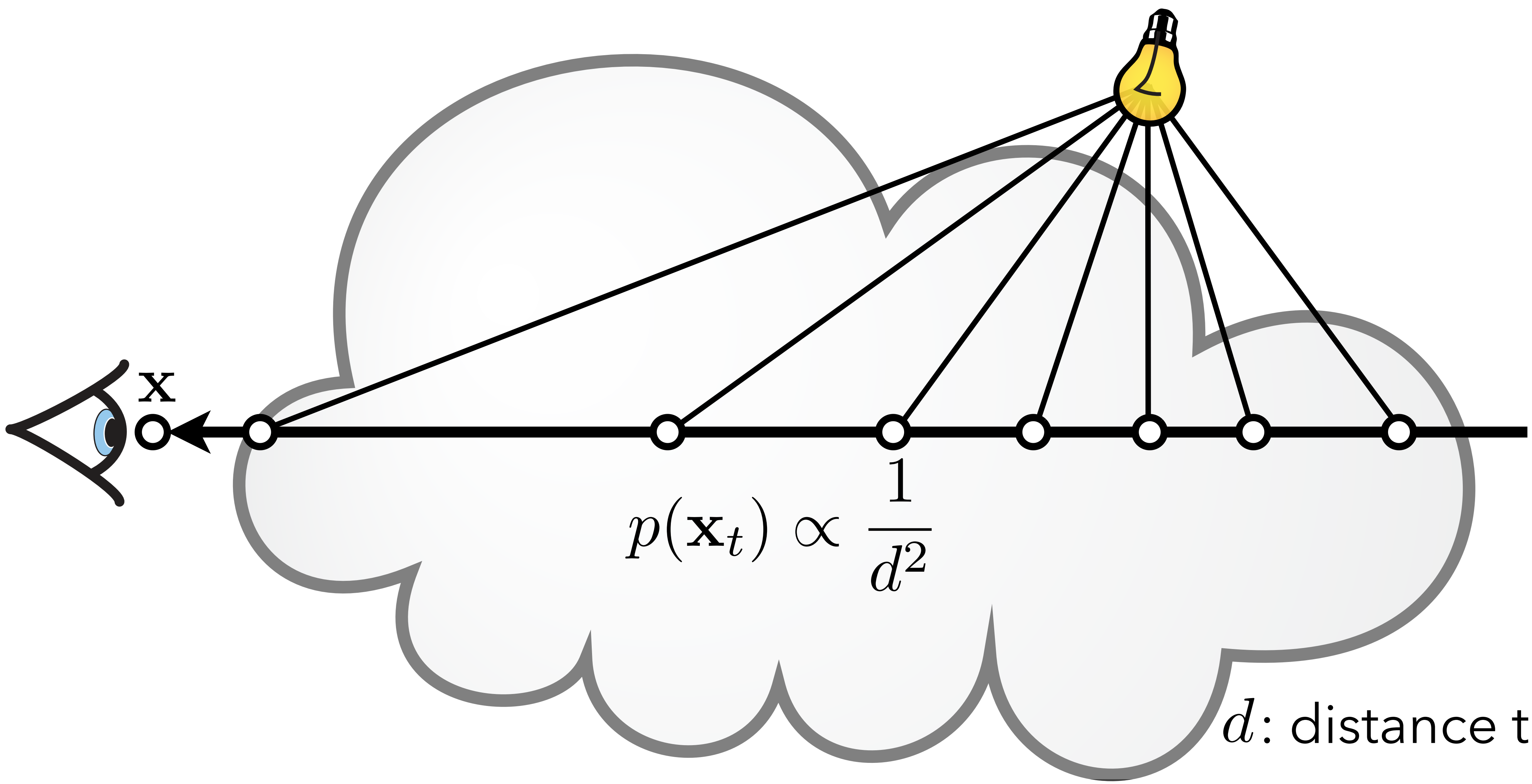
2. Estimate in-scattering using MC integration
   - Distribute samples ∝ (part of) the integrand



d: distance to light

Decoupled Transmittance and In-scattering

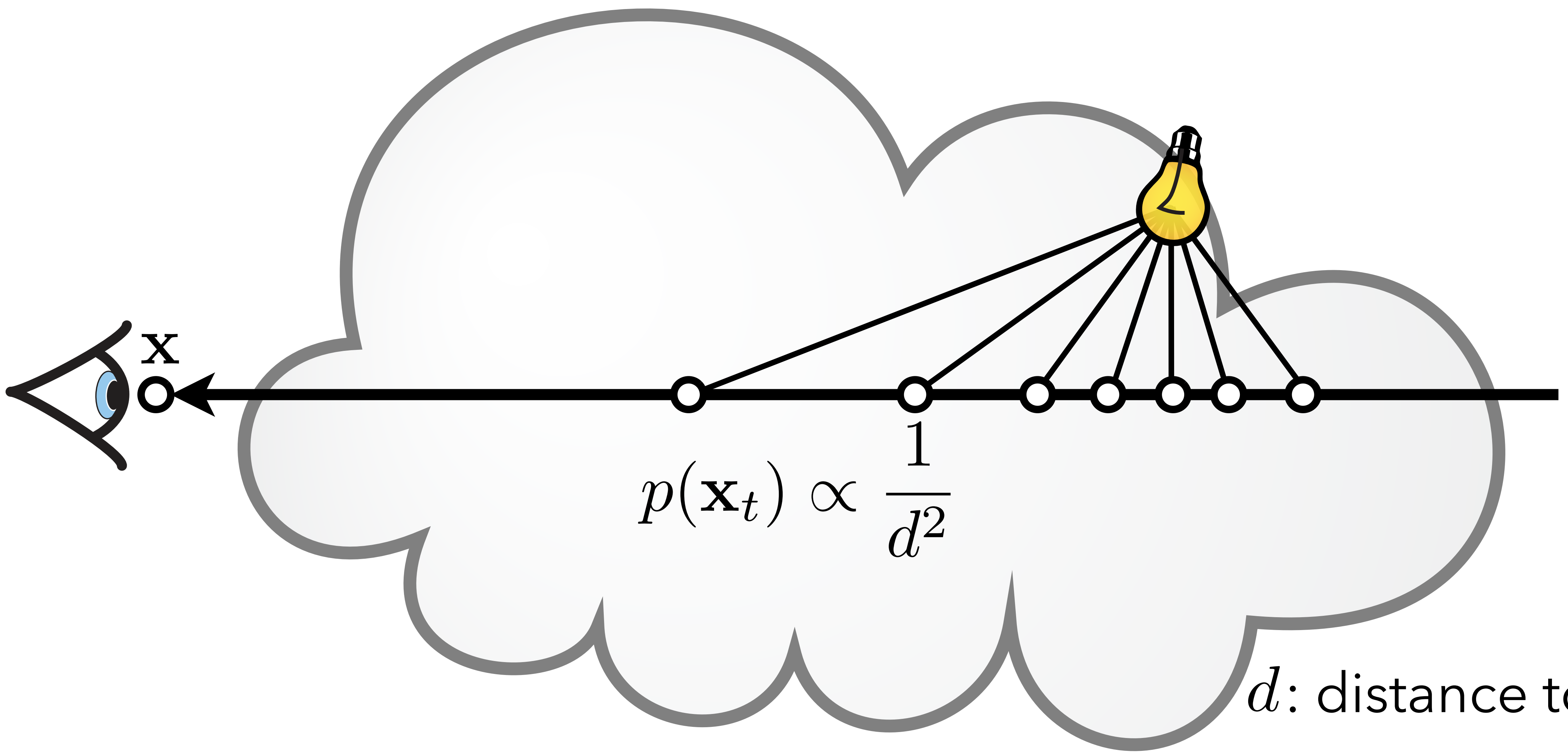
2. Estimate in-scattering using MC integration
   - Distribute samples $\propto$ (part of) the integrand



d: distance to light

Decoupled Transmittance and In-scattering

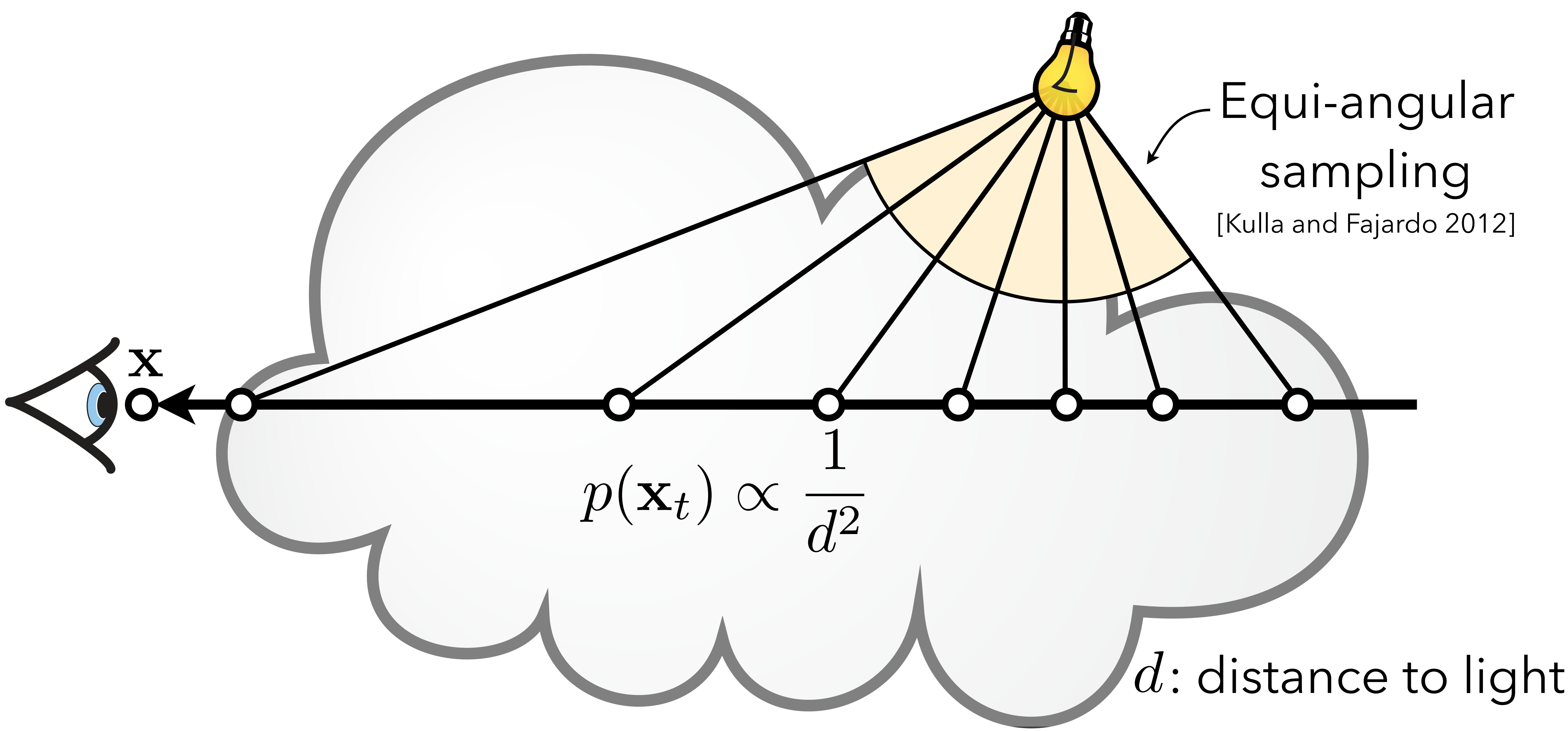
2. Estimate in-scattering using MC integration
   - Distribute samples ∝ (part of) the integrand



d: distance to light

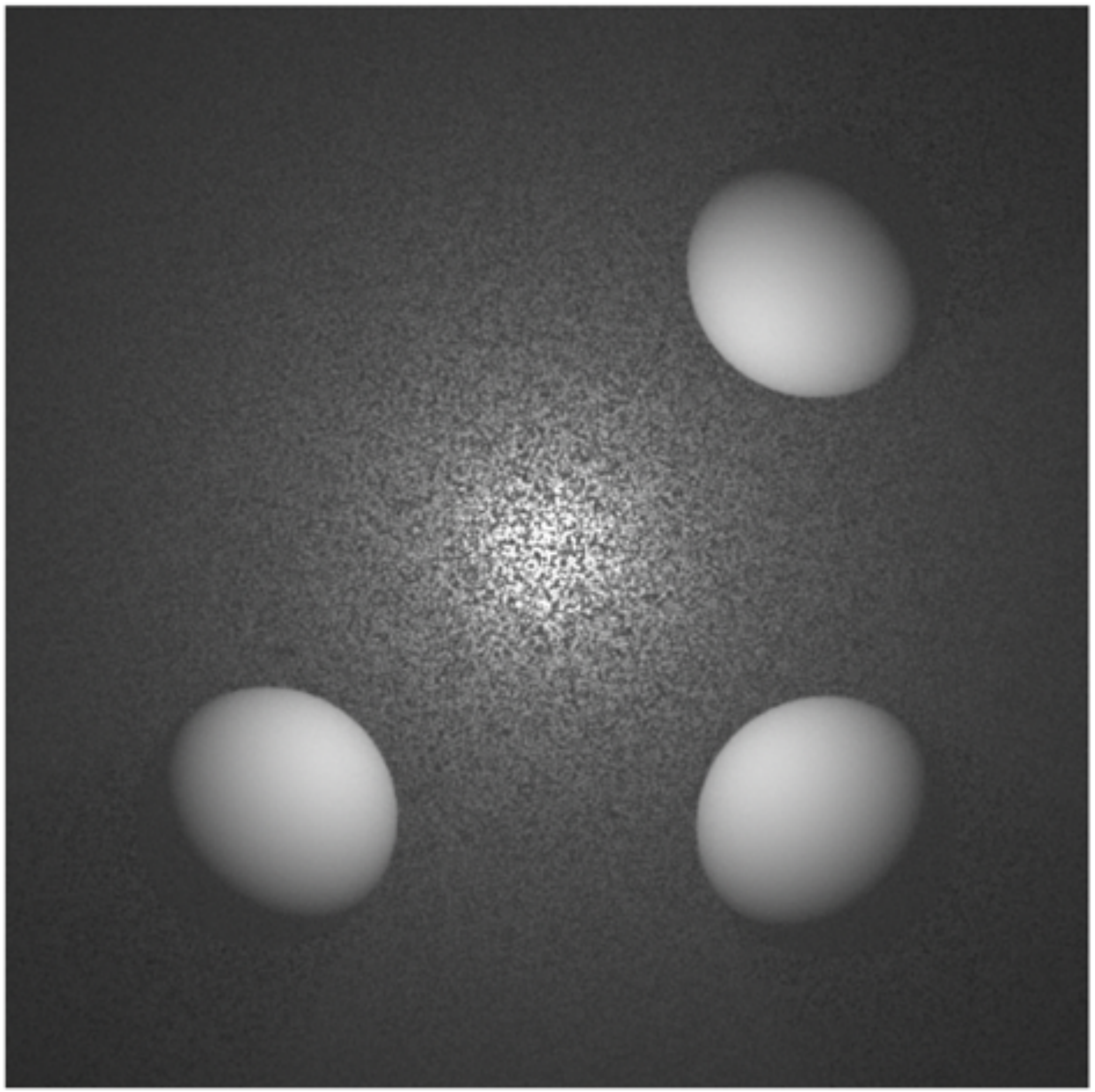
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration
   - Distribute samples $\propto$ (part of) the integrand

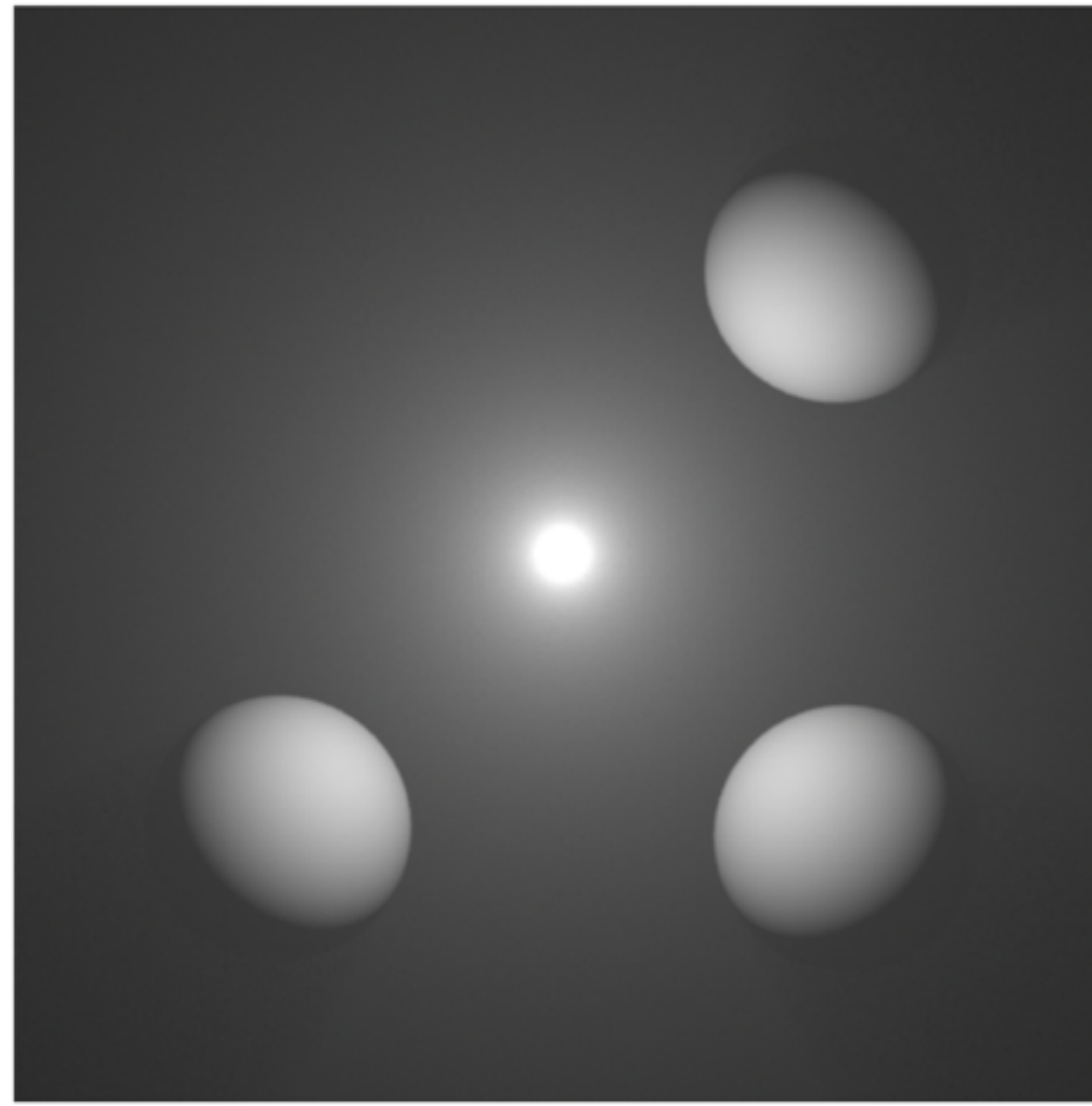


Decoupled Transmittance and In-scattering

Ray-marching



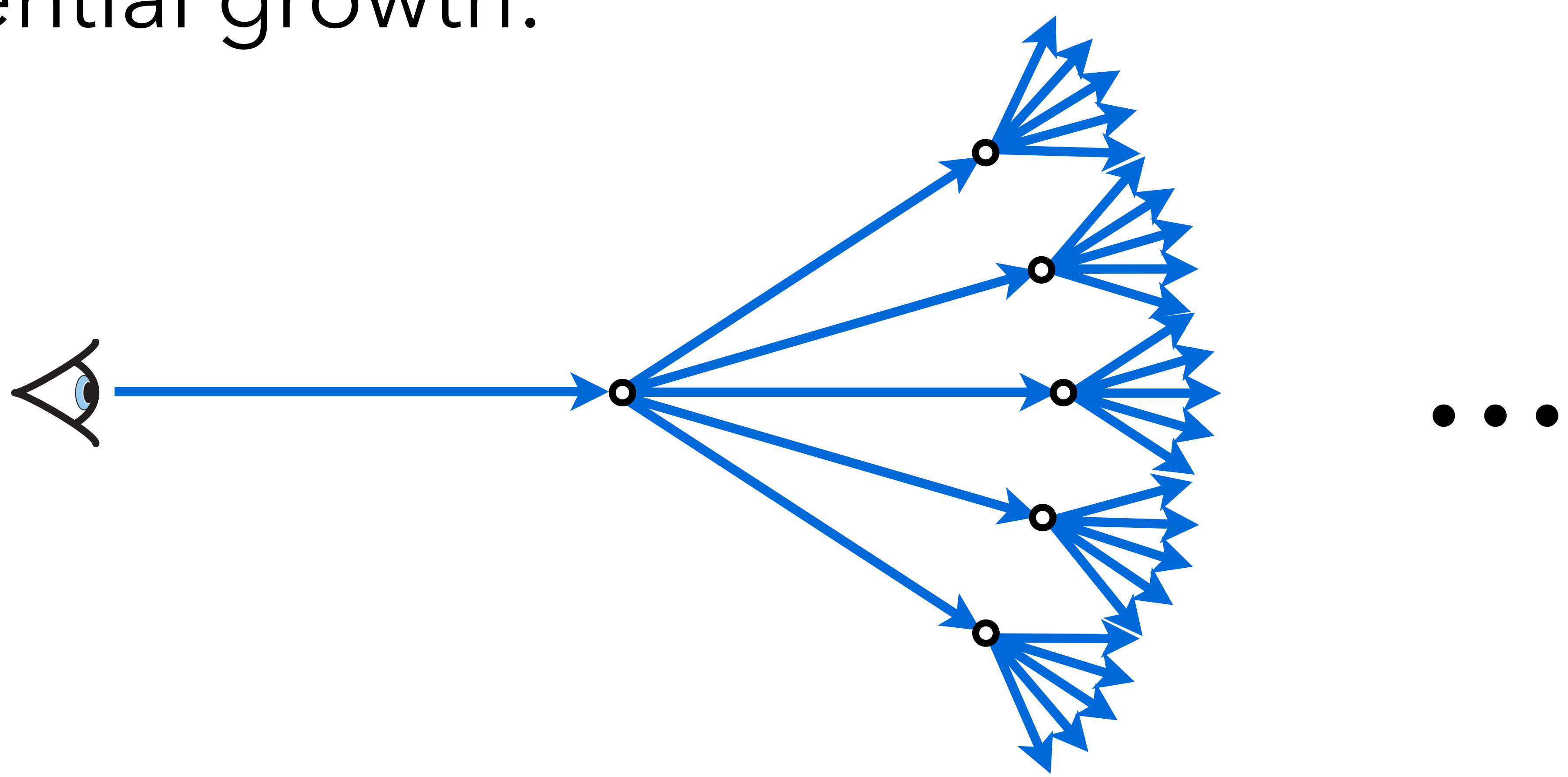
Equiangular sampling



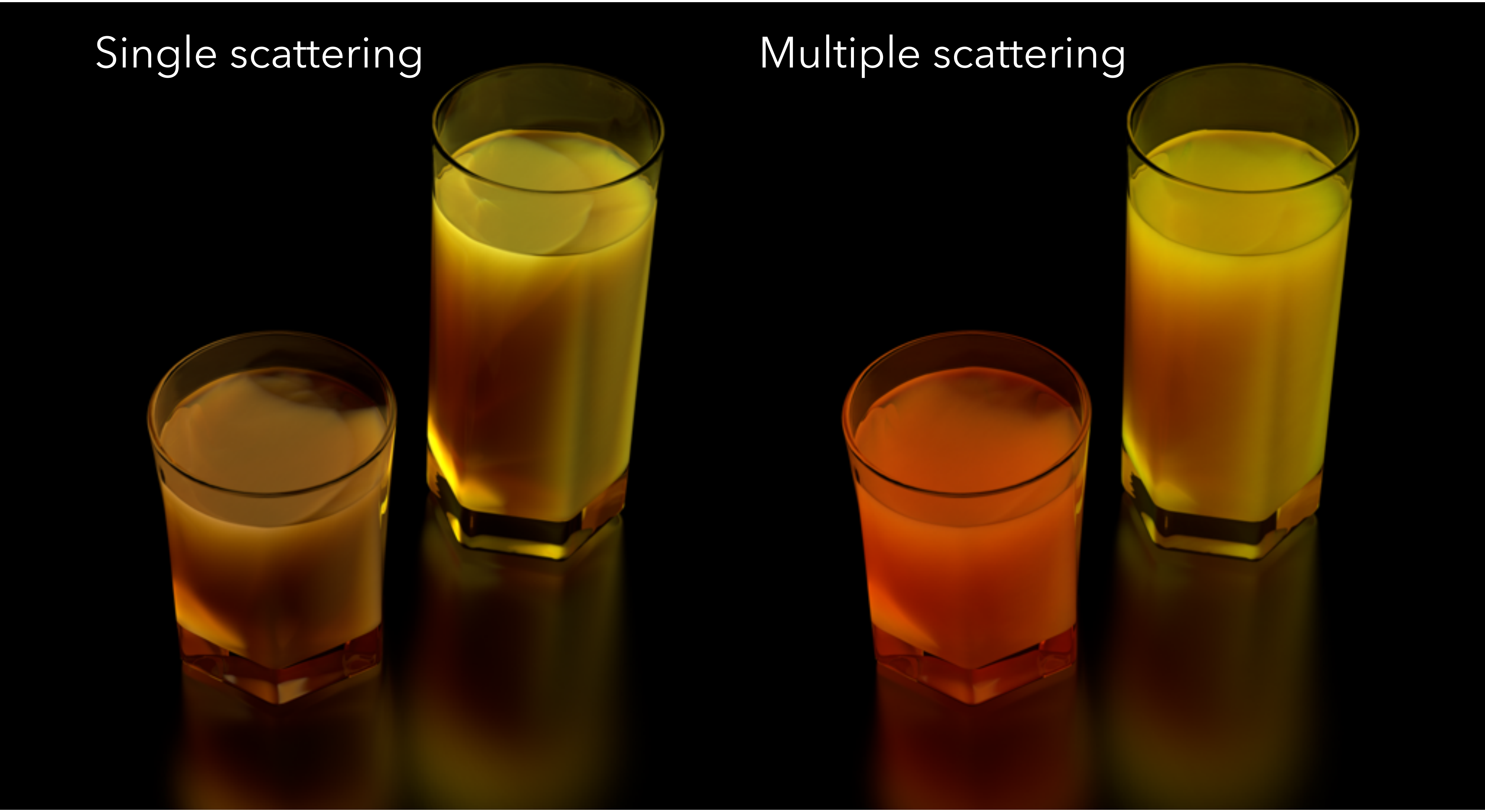
 Images courtesy of Kulla and Fajardo

Multiple Bounces

- Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering
- Exponential growth:



Visual Break



Volumetric Path Tracing

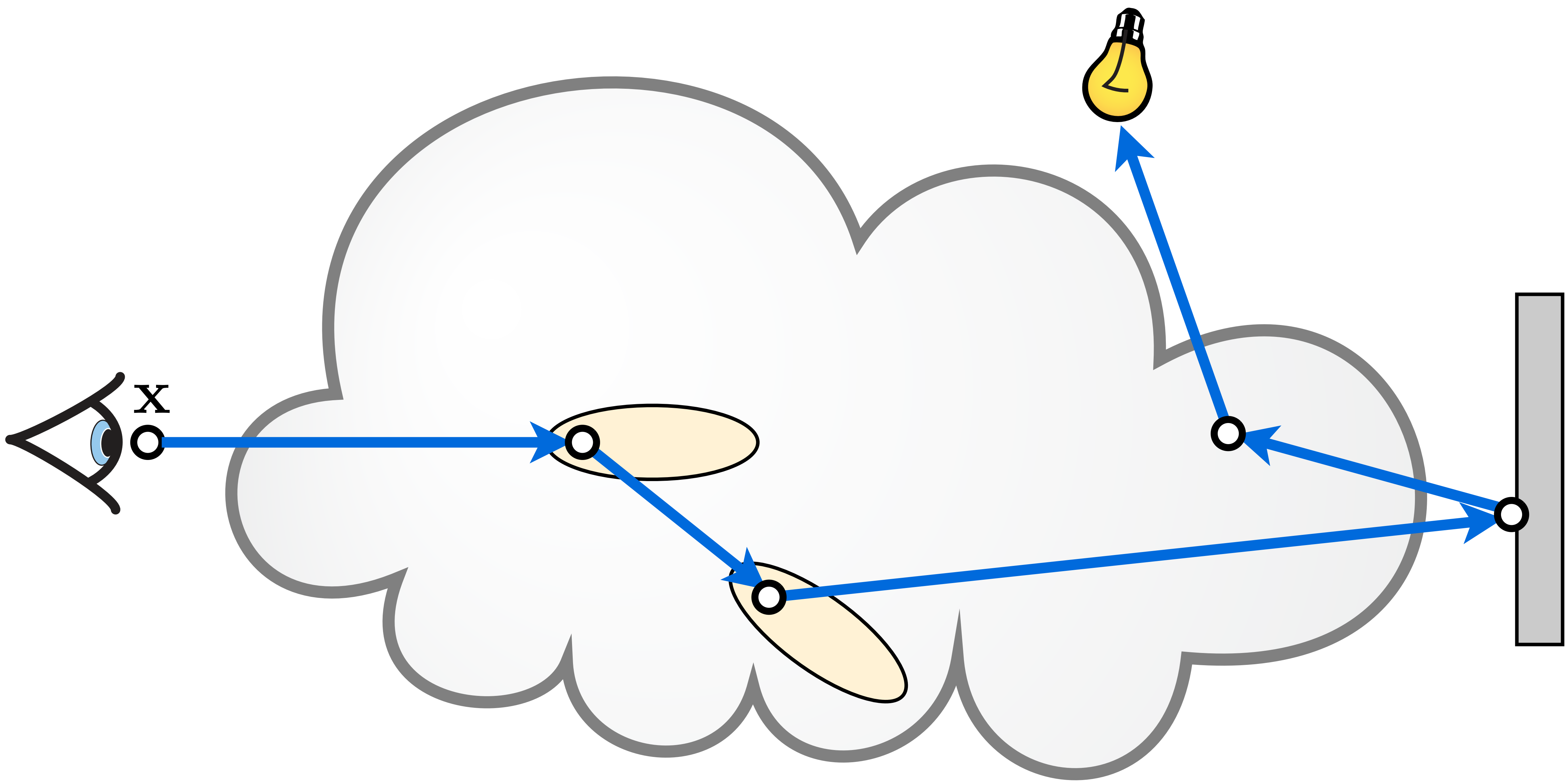
Volumetric Path Tracing

- Motivation:
  - Same as with standard path tracing: avoid the exponential growth
- Paths can:
  - Reflect/refract off surfaces
  - Scatter inside a volume

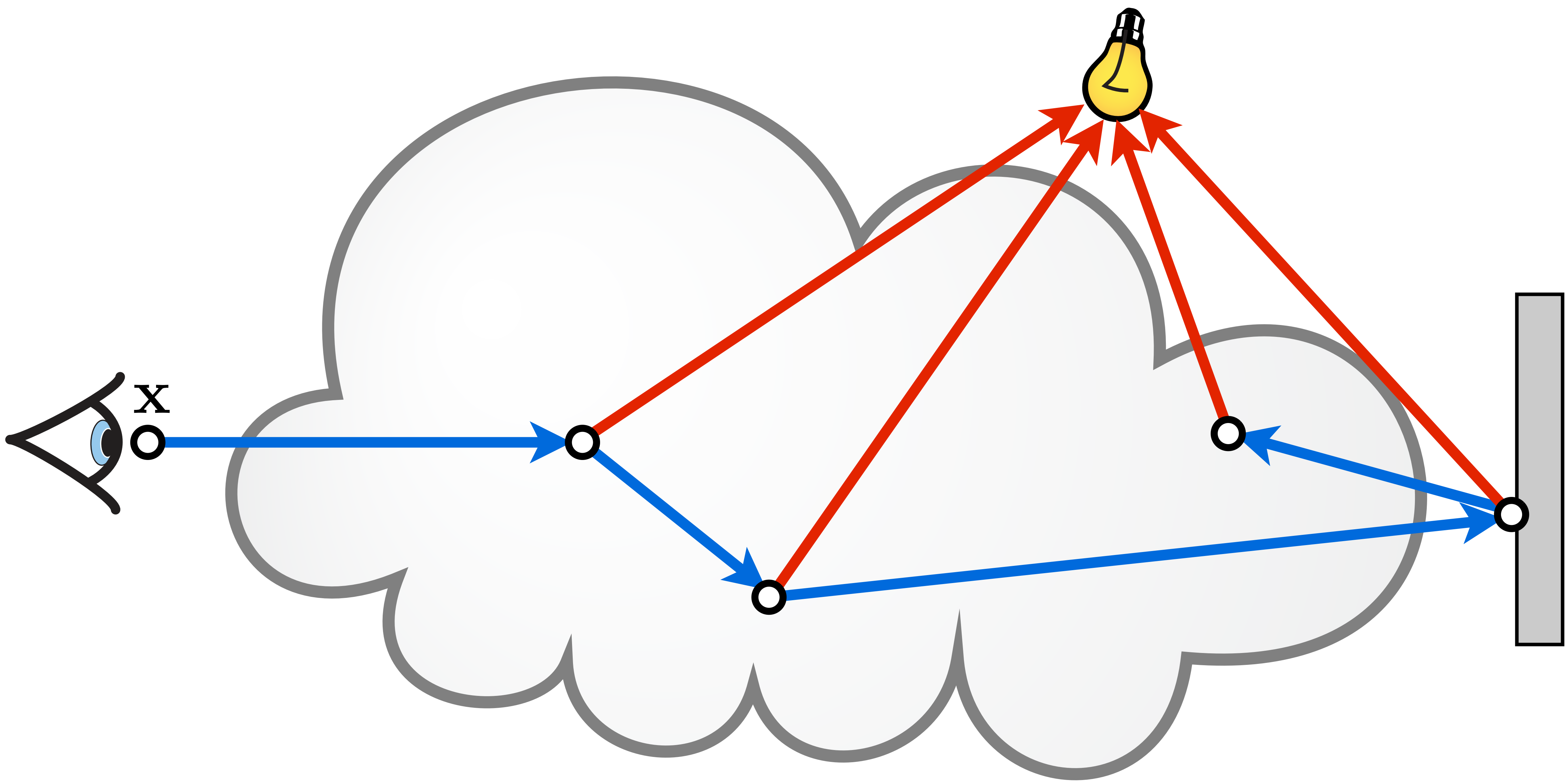
Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface



Volumetric Path Tracing with NEE



Sampling the Phase Function

- Isotropic:
  - Uniform sphere sampling
- Henyey-Greenstein:
  - Using the inversion method we can derive

$$\cos\theta = \frac{1}{2g}\left(1 + g^2 - \left(\frac{1-g^2}{1-g+2g\xi)}\right)^2\right)$$

$$\phi = 2\pi\zeta$$

  - PDF is the value of the HG phase function

Free-path Sampling

- Free-path (or free-flight distance):
  - Distance to the next interaction within the medium
  - Dense media (e.g. milk): short mean-free path
  - Thin media (e.g. atmosphere): long mean-free path
- Ideally, we want to sample proportional to transmittance

$$p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$

$$p(t) \propto T_r(t)$$

simplified notation for brevity

Free-path Sampling

- Homogeneous media: $T_r(t) = e^{-\sigma_t t}$
  - PDF: $p(t) \propto e^{-\sigma_t t}$

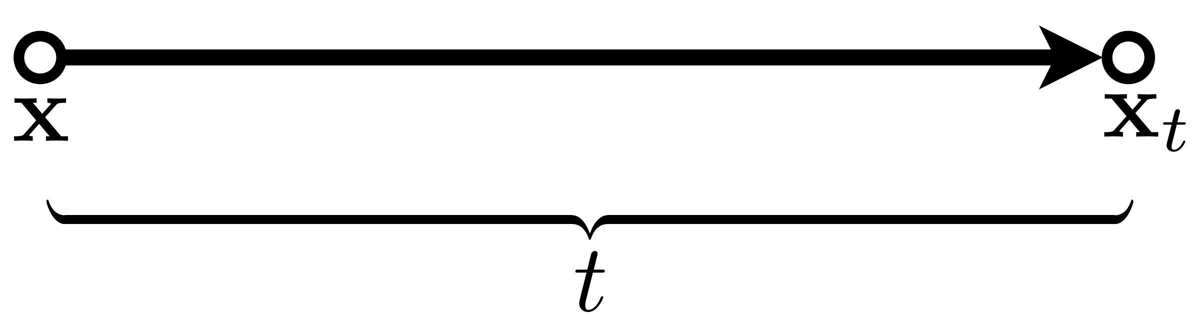
$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

  - CDF: $P(t) = \int_0^t \sigma_t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$
  - Inverted CDF:

$$P^{-1}(\xi) = -\frac{\ln(1-\xi)}{\sigma_t}$$

Free-path Sampling

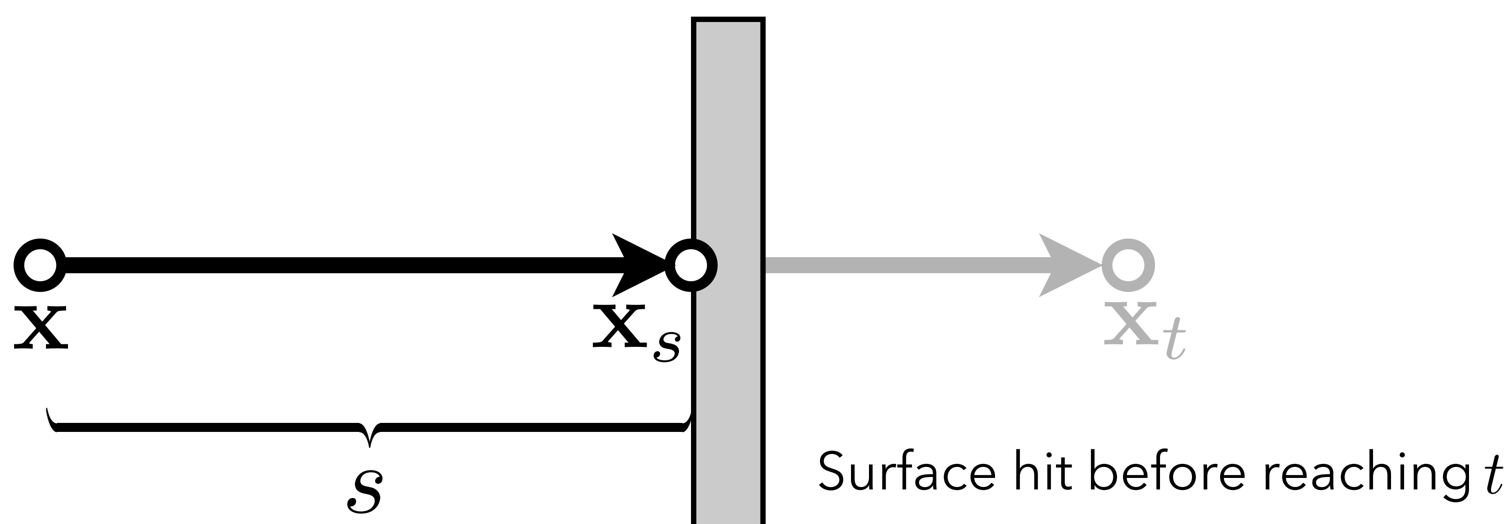
- Homogeneous media: $T_r(t) = e^{-\sigma_t t}$
- Recipe:
  - Generate random number ξ
  - Sample distance $t = -\frac{\ln(1-\xi)}{\sigma_t}$
  - Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



Free-path Sampling

- Homogeneous media: $T_r(t) = e^{-\sigma_t t}$
- Recipe:
  - Generate random number ξ
  - Sample distance $t = \cancel{-\frac{\ln(1-\xi)}{\sigma_t}} = s$
  - Compute PDF $p(t) = \cancel{\sigma_t e^{-\sigma_t t}} = e^{-\sigma_t s}$

Note: This is now a probability, not a probability density!

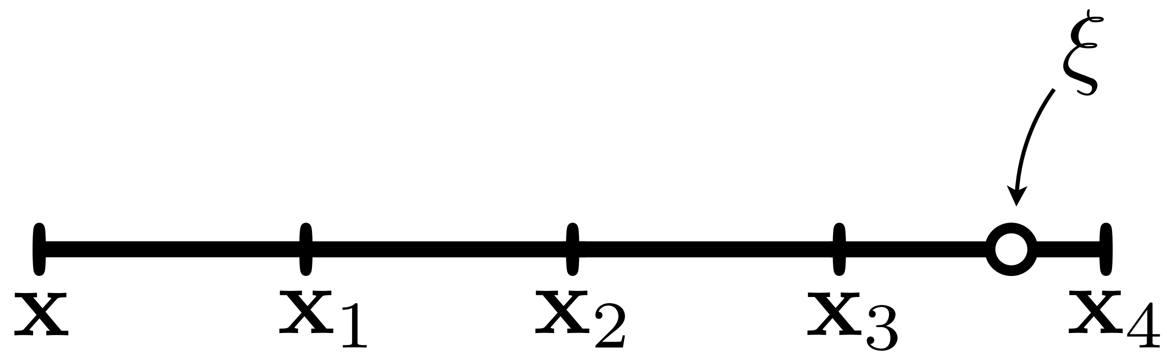


Free-path Sampling

- Heterogeneous media: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$
  - Closed-form solutions exist only for simple media
    - e.g. linearly or exponentially varying extinction
  - Other solutions:
    - Ray marching
    - Woodcock tracking
    - Regular tracking (3D DDA)

Ray-Marching

- Assume constant extinction along each step
- Recipe:
  - March until $T_r(\mathbf{x}_0, \mathbf{x}_i) < \xi$
  - Use closed-form solution within the last segment



Ray-Marching

- Issues:
  - Biased since $E[e^X] \neq e^{E[X]}$
    - The exponentiation skews the noise distribution and the estimation error no longer averages to 0 in the limit
  - Sensitive to resolution of heterogeneous volume
    - To keep bias low, we need to march on the order of the highest frequency (Nyquyist-Shannon theorem)
    - Can be expensive

Volumetric PT for Homogeneous Volumes

```
Color vPT(x, ω)
    tmax = nearestSurface(x, ω)
    t = -log(1 - rand()) / σt        // Sample free path
    if t < tmax: // Volume interaction
        x += t * ω
        pdf_t = σt * exp(-σt * t)
        (ω', pdf_ω') = samplePF(ω)
        // Note: transmittance and PF cancel out with PDFs
        // except for a constant factor 1/σt
        return Tr(t) * σs * PF(ω) * vPT(x,ω') / (pdf_t * pdf_ω')
    else: // Surface interaction
        x += tmax * ω
        Pr_tmax = exp(-σt * tmax)
        (ω', pdf_ω') = sampleBRDF(n, ω)
        // Note: transmittance and prob of sampling the distance
        // cancel out
        return Tr(tmax) * BRDF(ω, ω') * vPT(x, ω') / (Pr_tmax * pdf_ω')
```

Volumetric PT for Homogeneous Volumes

```
Color vPT(x, ω)
    tmax = nearestSurface(x, ω)
    t = -log(1 - rand()) / σ_t        // Sample free path
    if t < tmax: // Volume interaction
        x += t * ω
        pdf_t = σ_t * exp(-σ_t * t)
        (ω', pdf_ω') = samplePF(ω)
        // Note: transmittance and PF cancel out with PDFs
        // except for a constant factor 1/σ_t
        return σ_s / σ_t * vPT(x,ω')
    else: // Surface interaction
        x += tmax * ω
        Pr_tmax = exp(-σ_t * tmax)
        (ω', pdf_ω') = sampleBRDF(n, ω)
        // Note: transmittance and prob of sampling the distance
        // cancel out
        return BRDF(ω, ω') / pdf_ω' * vPT(x, ω')
```

Woodcock Tracking

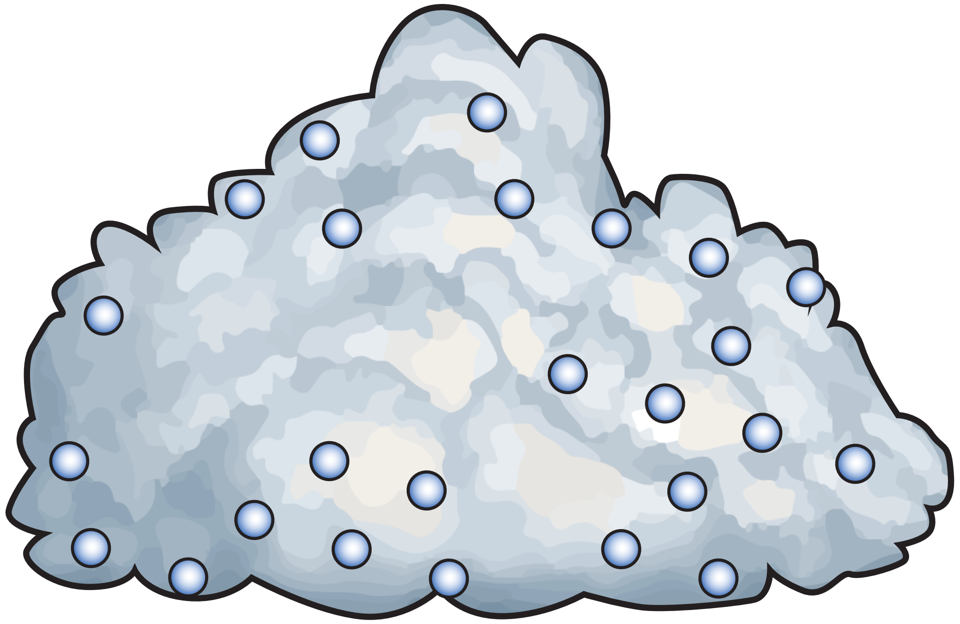
(a.k.a. pseudo scattering, delta tracking, null-collision algorithm,…)

Woodcock Tracking

- Unbiased technique for free-path sampling
- Inspired by *rejection sampling*
- Idea:
  - Add a *fictitious* volume
  - Combined volume (real + fictitious) is homogeneous
  - Generate (*tentative*) free-paths analytically
  - Probabilistically reject/accept collisions based on local concentrations of real vs. fictitious volumes

Woodcock Tracking

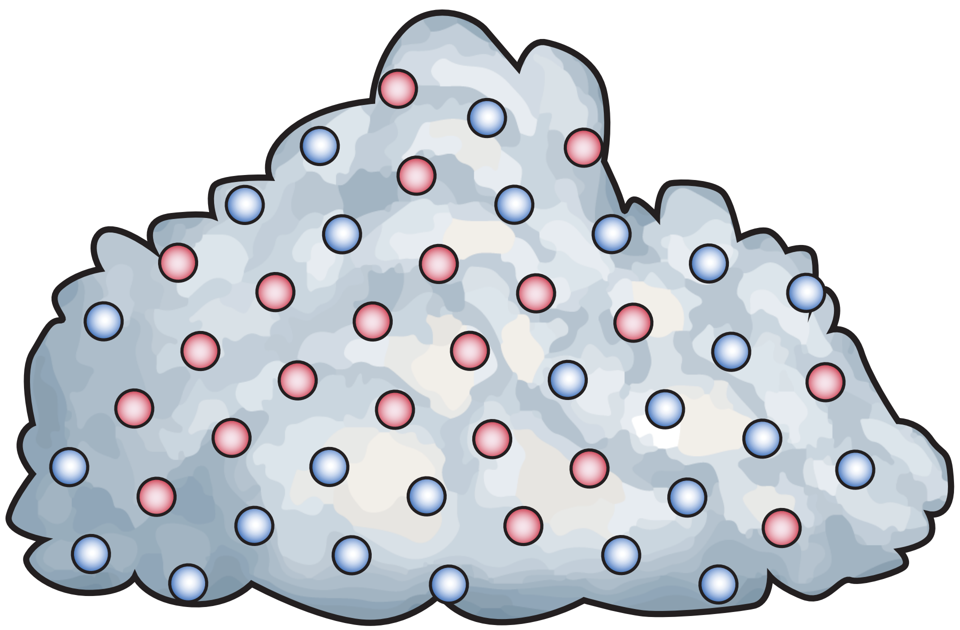
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

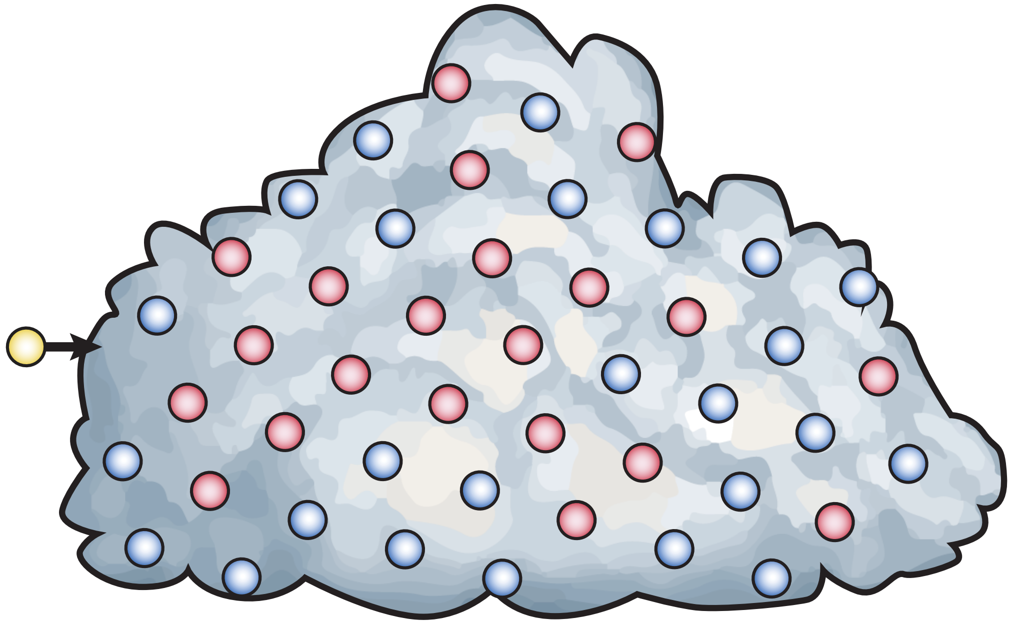
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

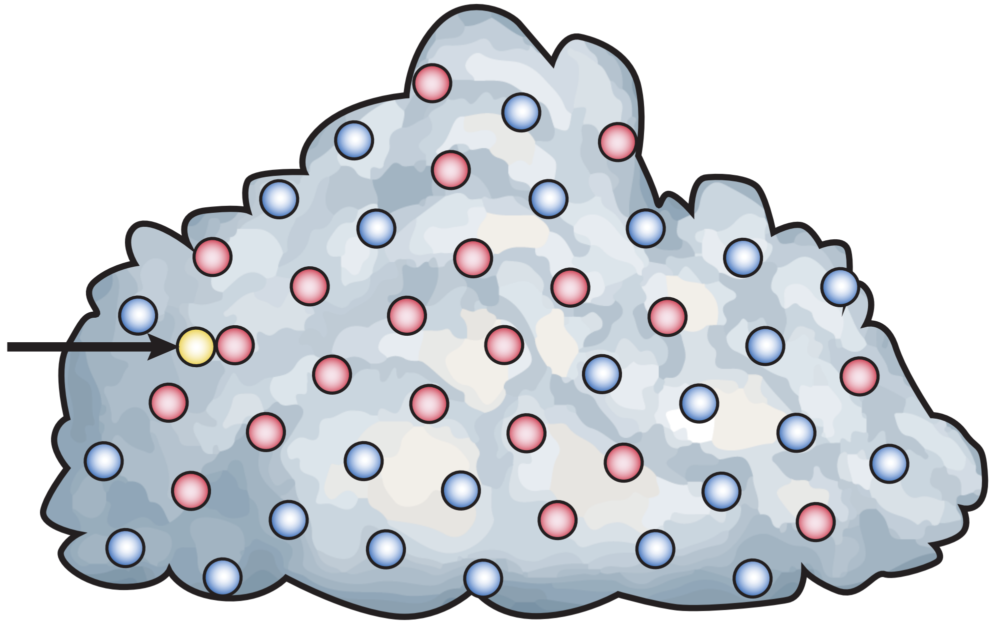
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

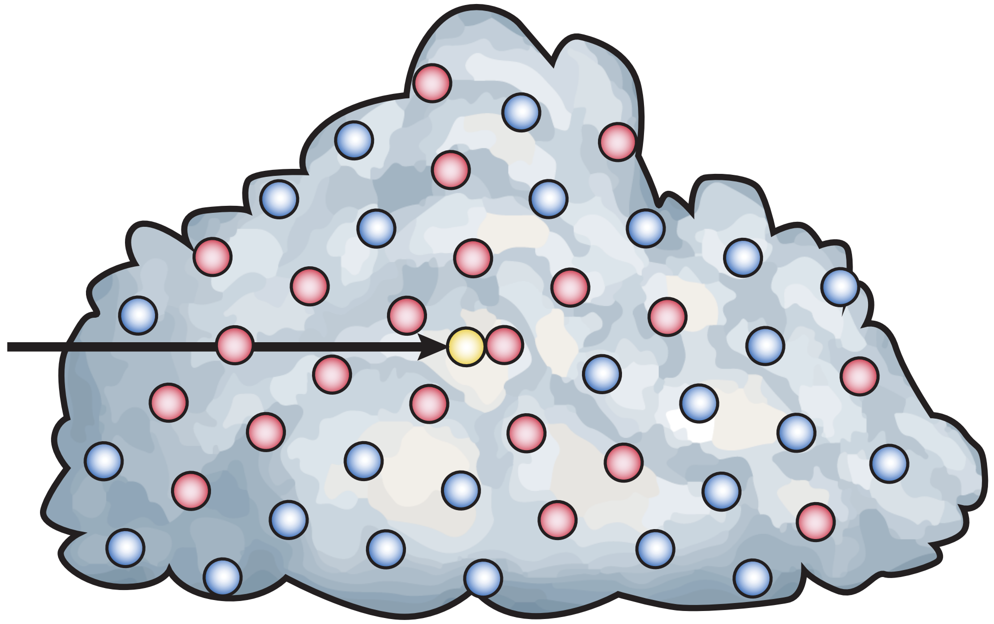
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

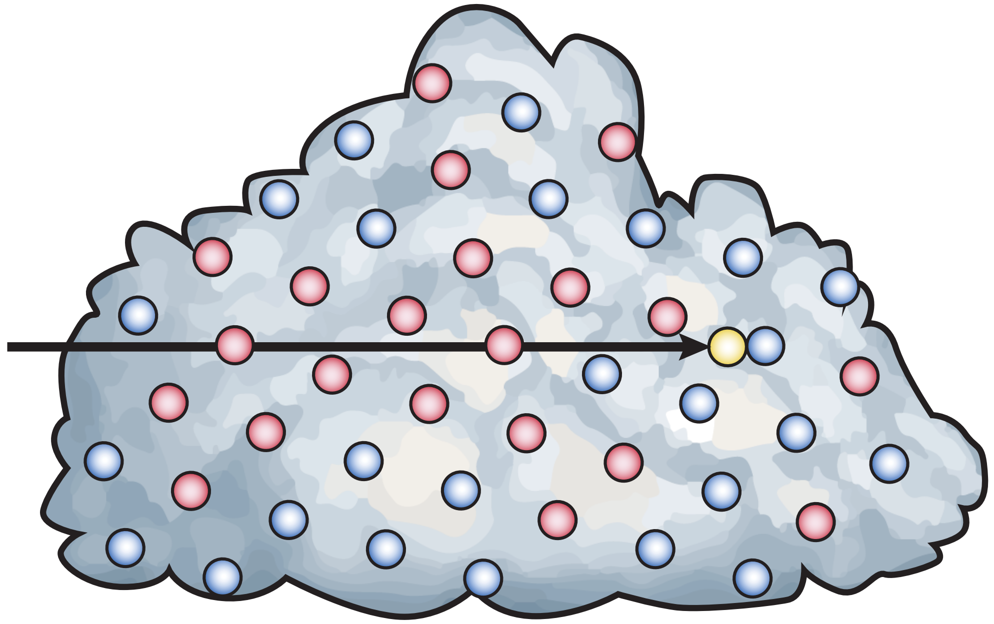
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

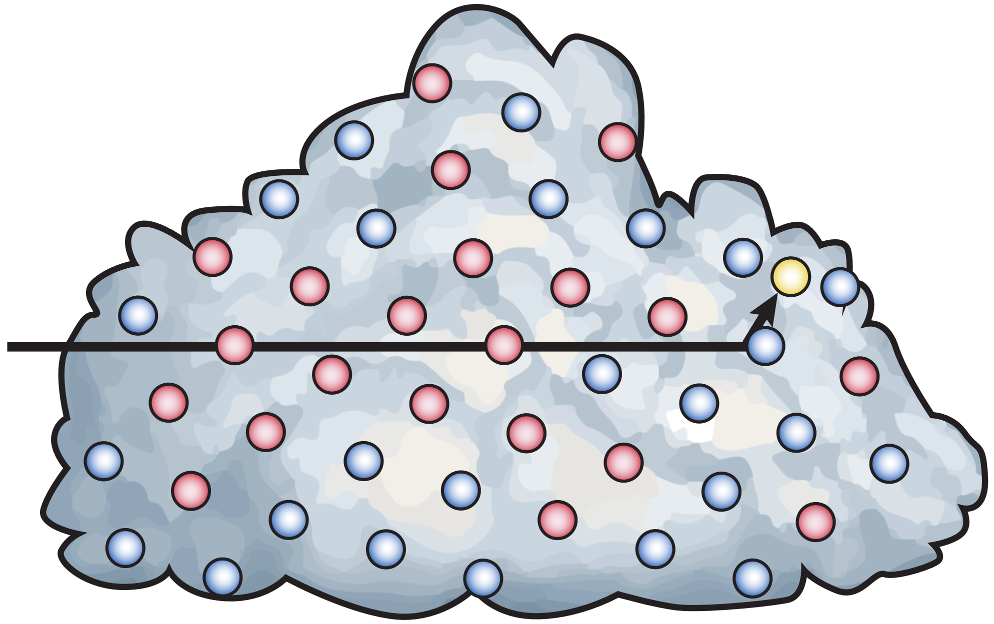
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

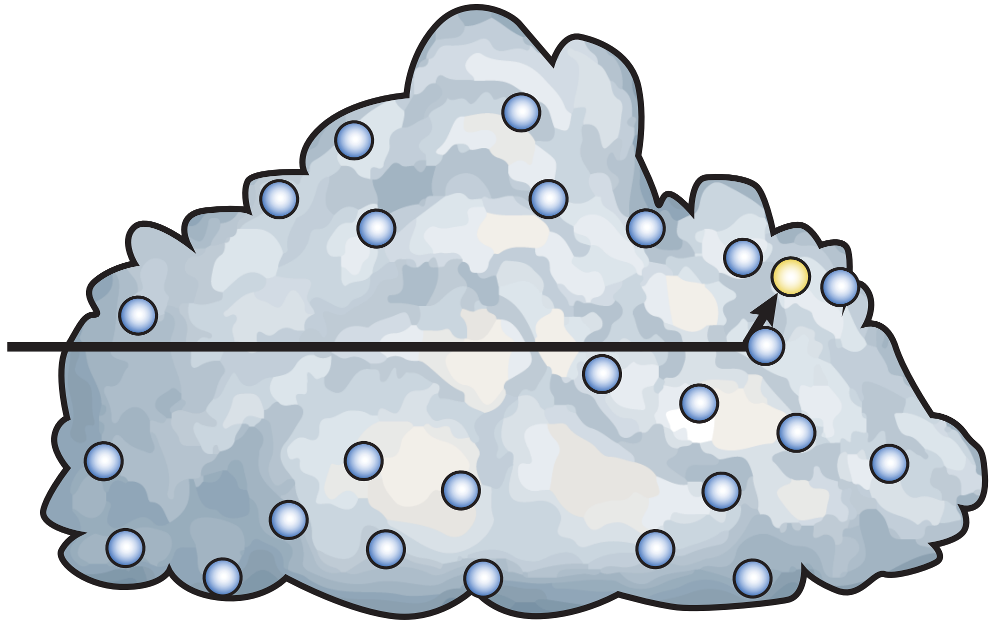
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

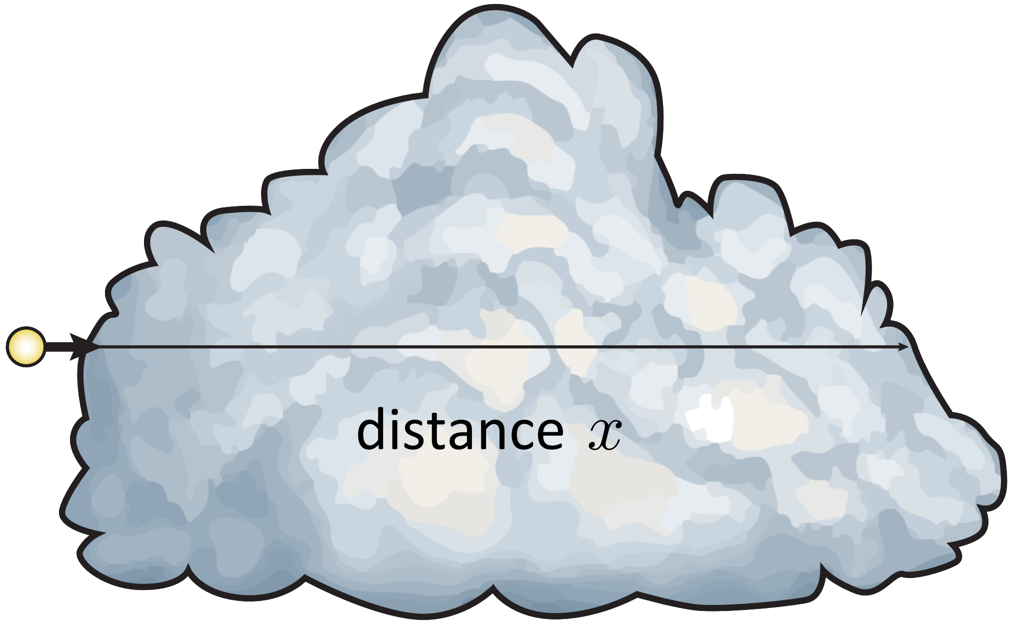
- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$= real + fictitious extinction



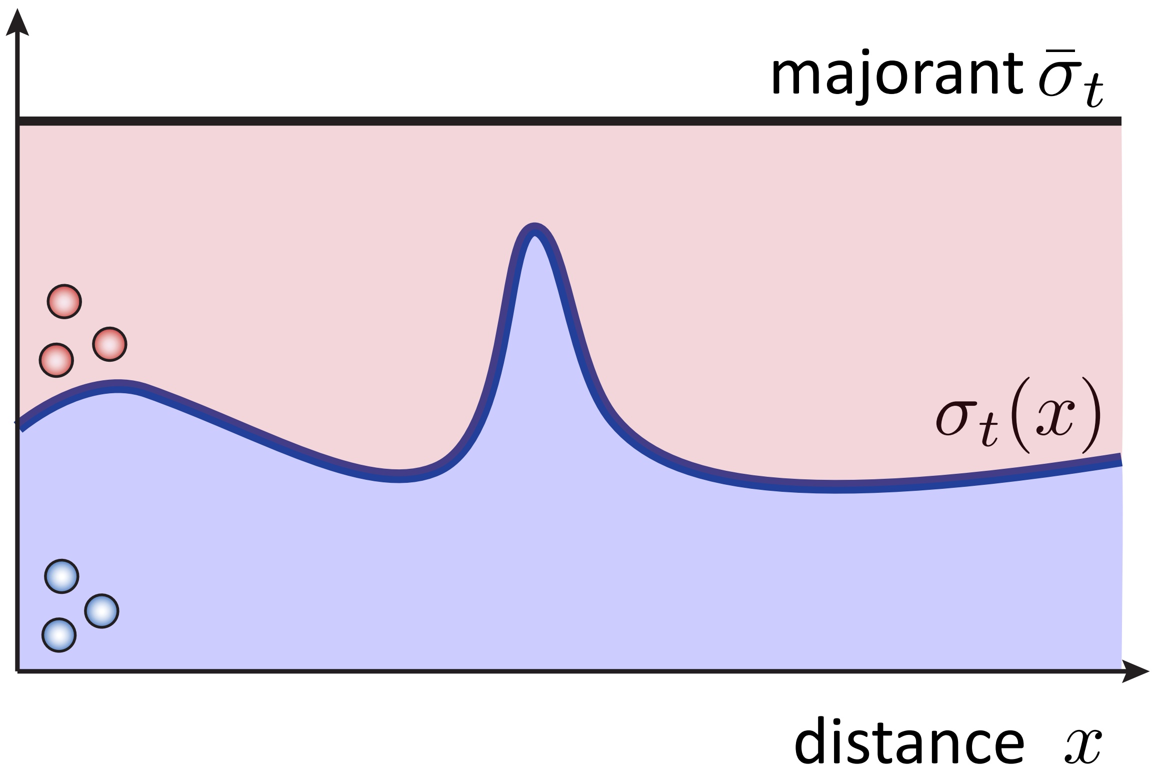
*Particles are used only for illustration, the two volumes are modeled using their extinction coefficients.

Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction

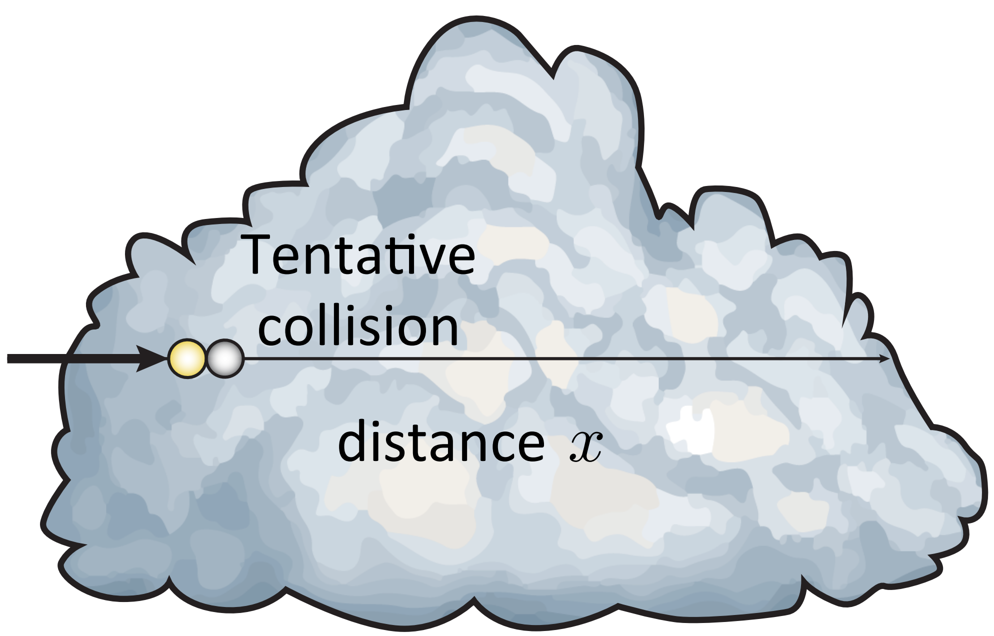


$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \qquad P_f = 1 - P_r$$

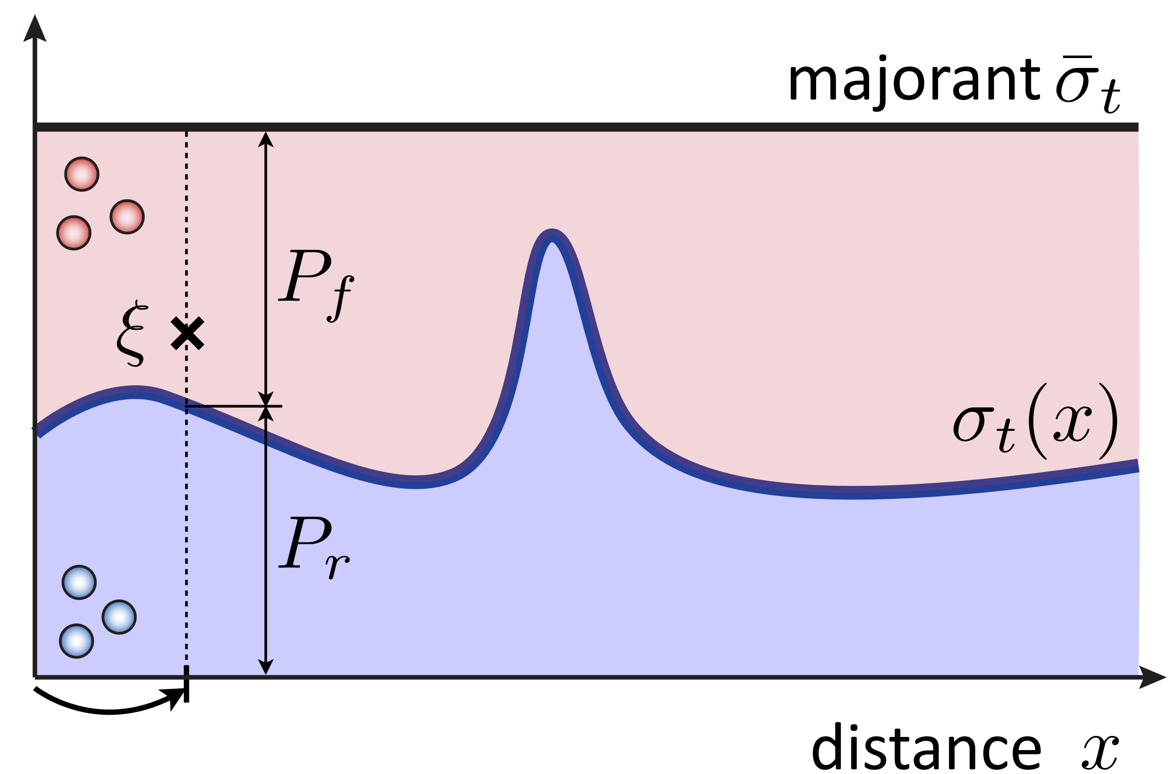


Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction

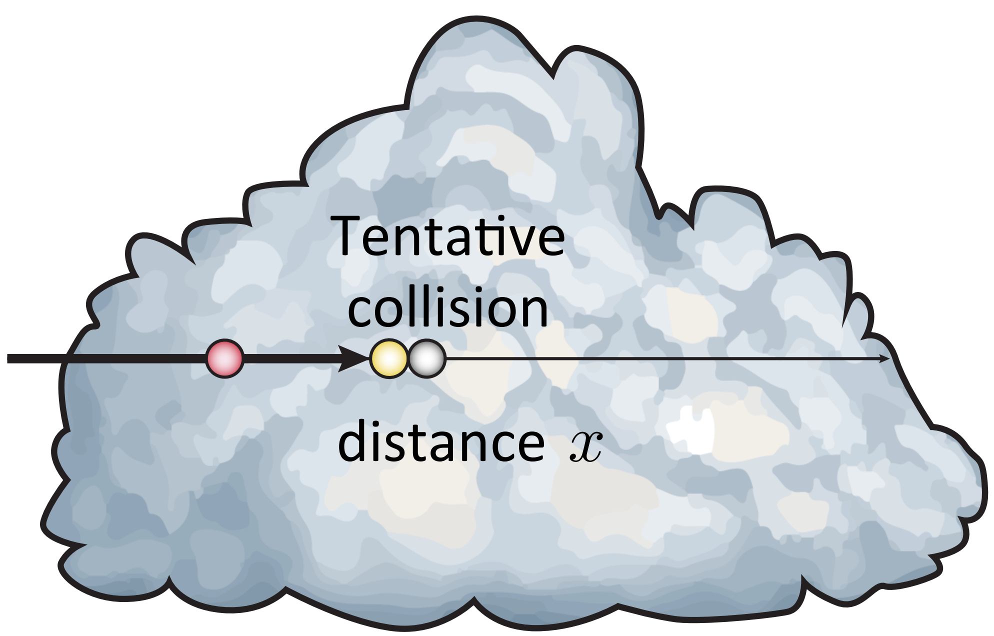


$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \qquad P_f = 1 - P_r$$

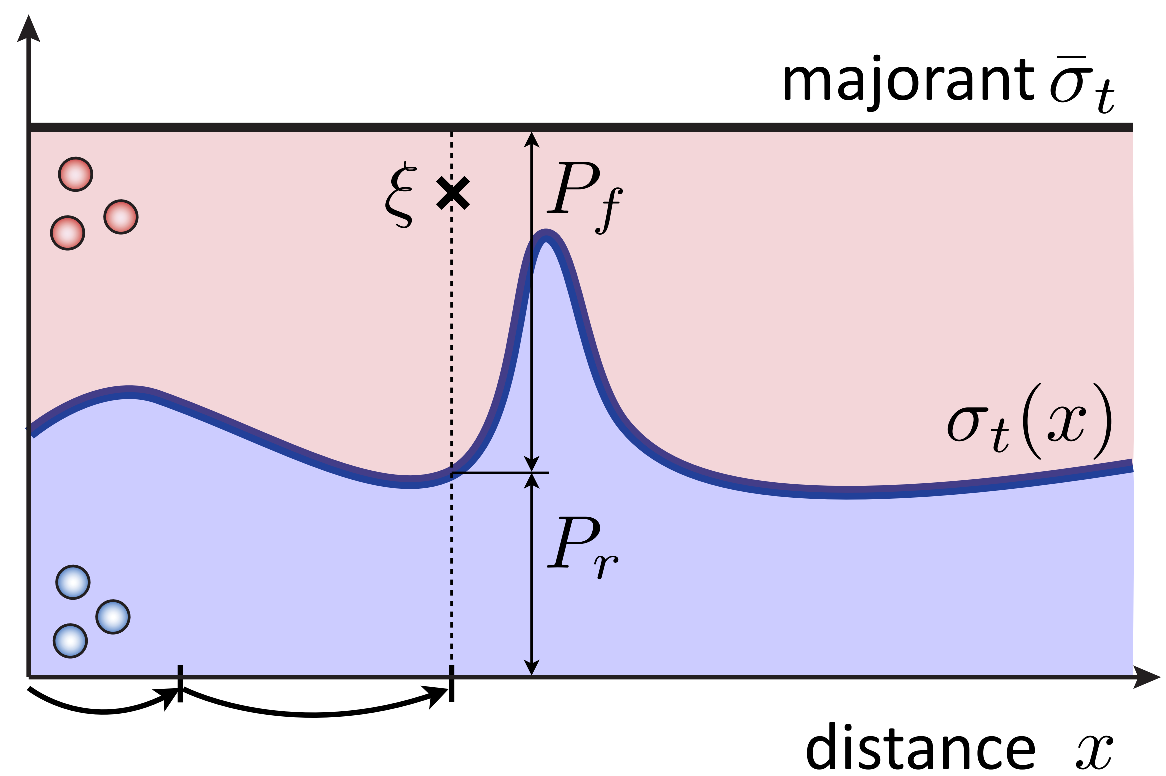


Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction



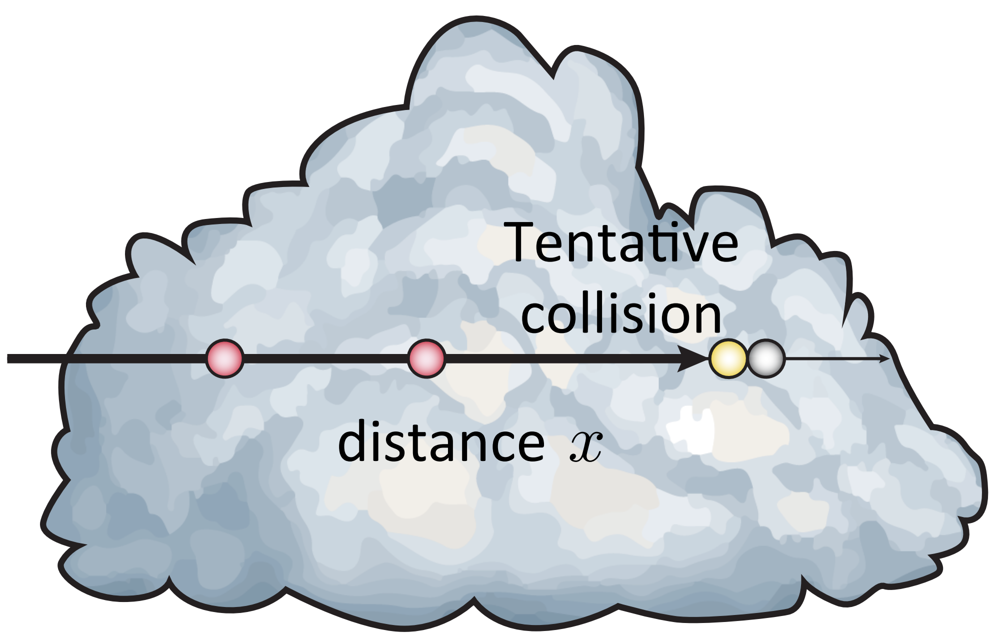
$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \qquad P_f = 1 - P_r$$

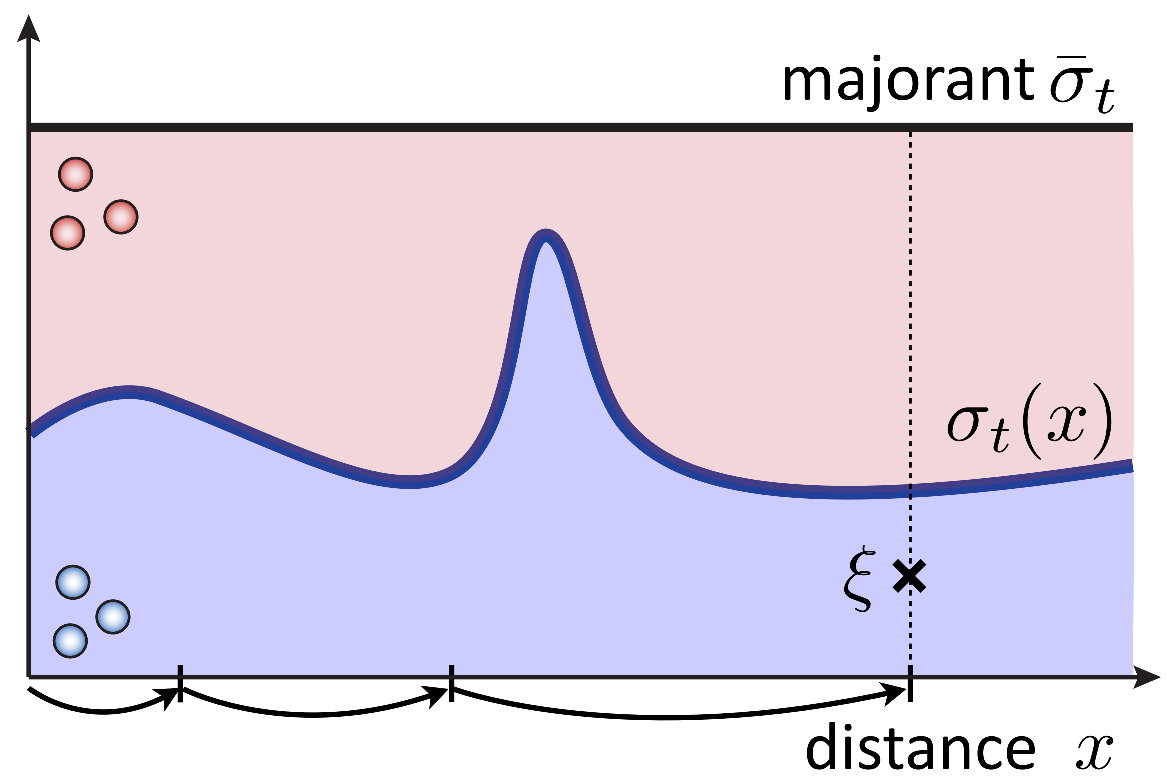


Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction

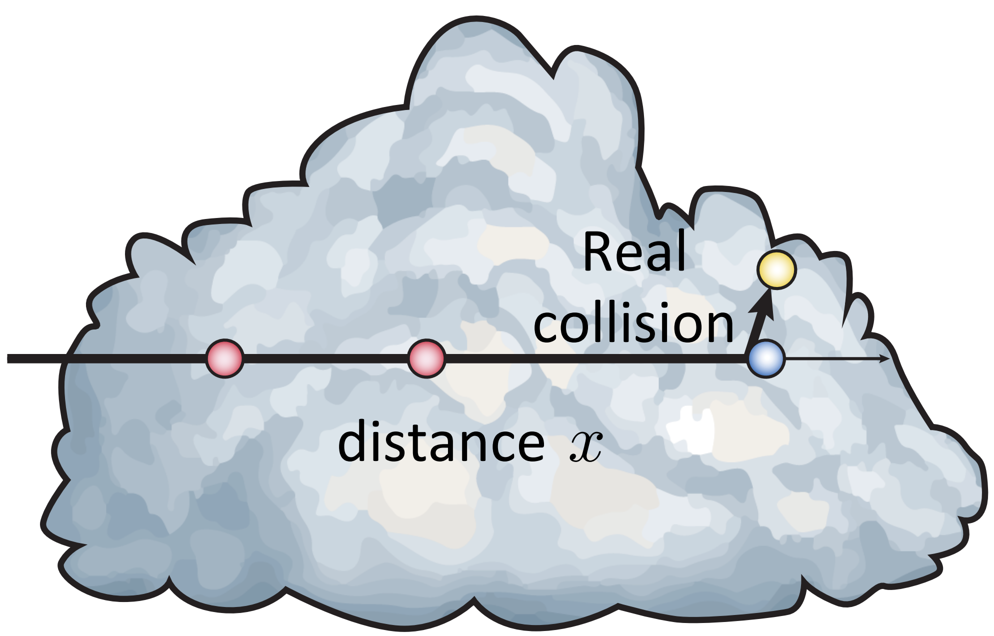
$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \qquad P_f = 1 - P_r$$



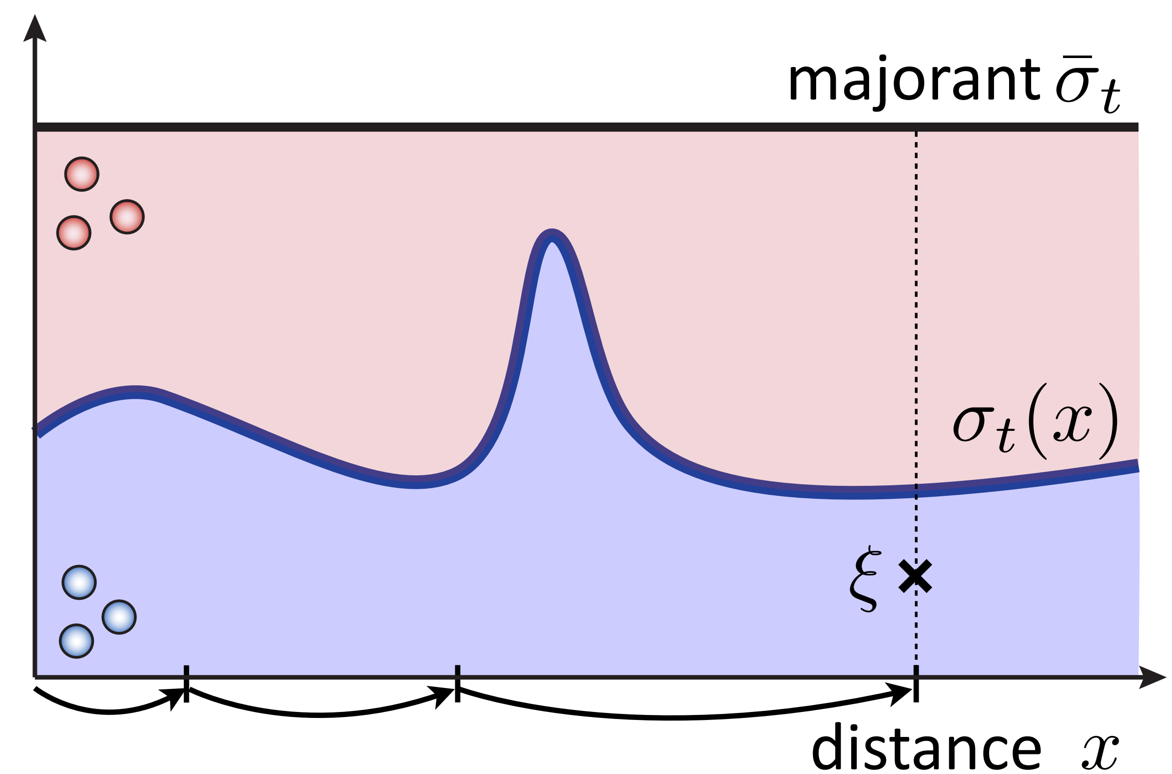


Woodcock Tracking

- Add fictitious particles that do not impact transport
- Majorant extinction $\bar{\sigma}_t$ = real + fictitious extinction

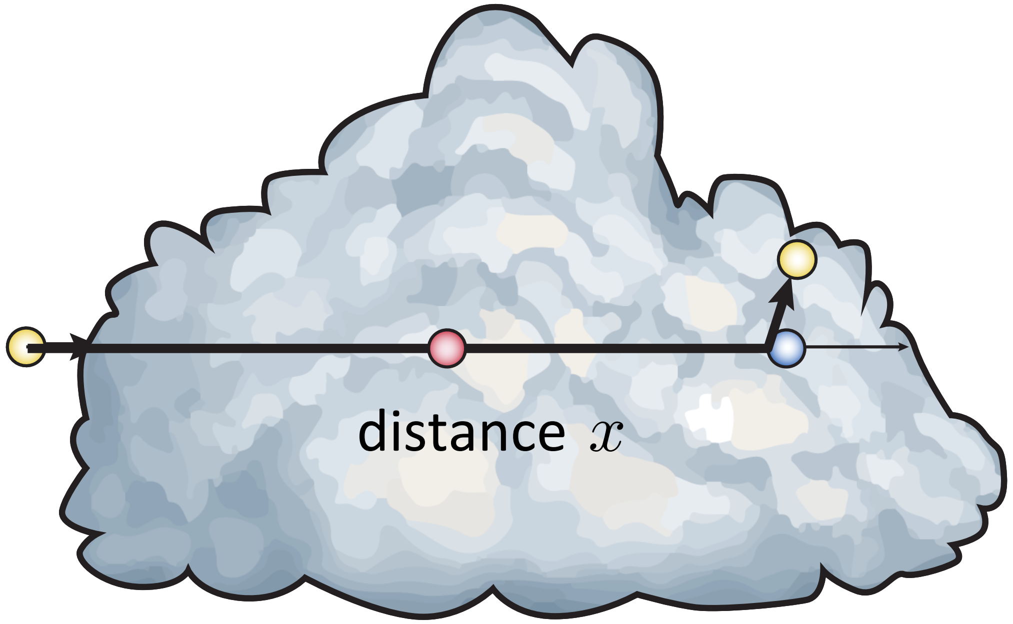


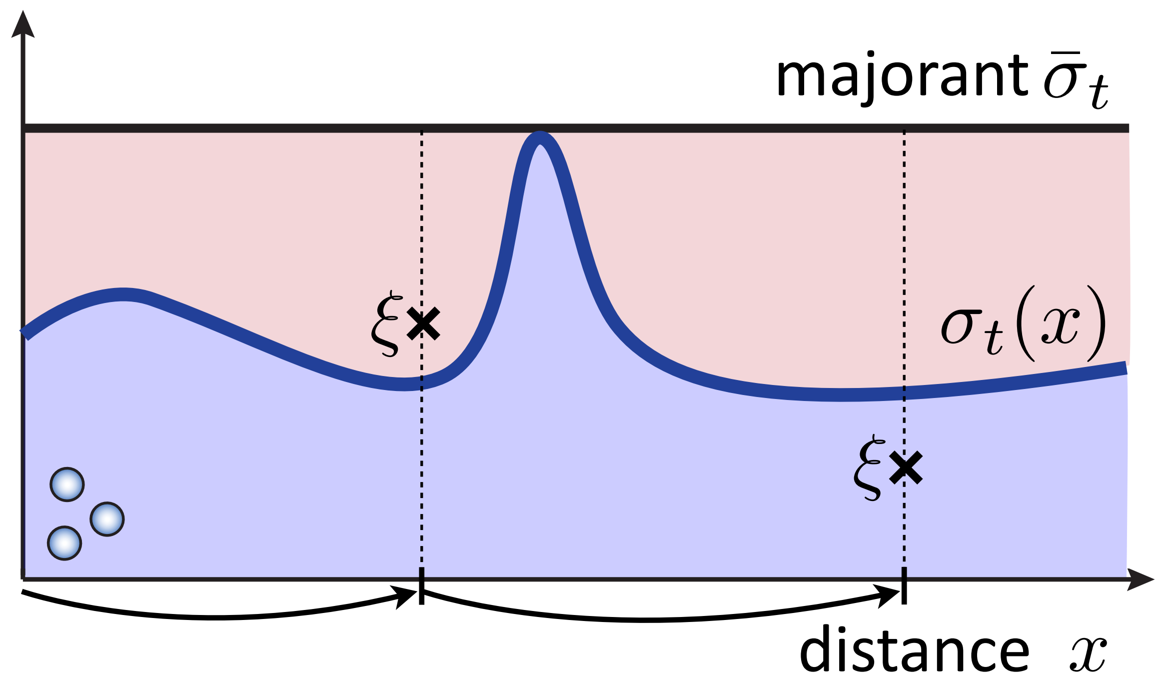
$$P_r = \frac{\sigma_t(x)}{\bar{\sigma}_t} \qquad P_f = 1 - P_r$$



Woodcock Tracking

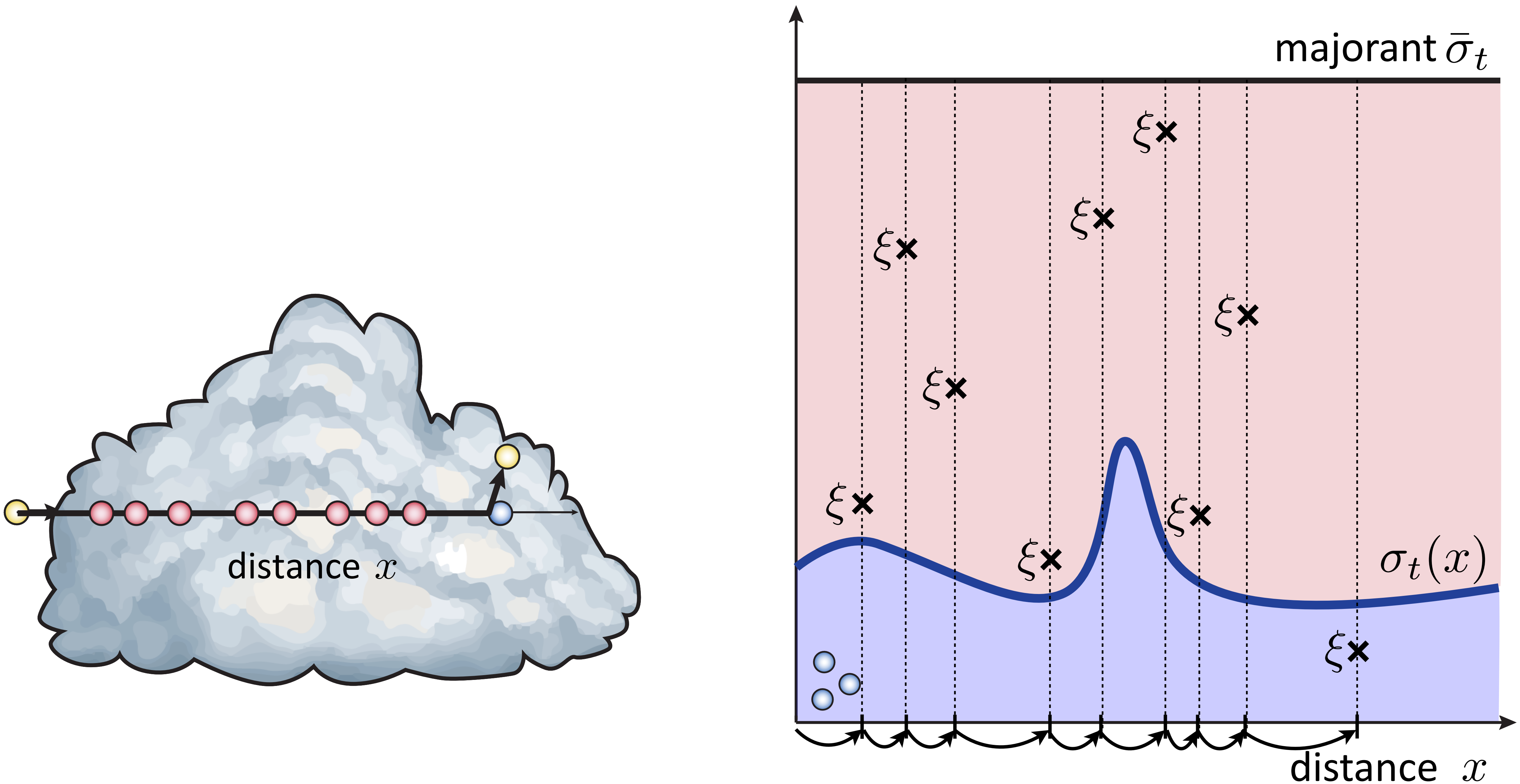
- Tight majorant = high efficiency





Woodcock Tracking

- Loose majorant = many fictitious collisions, high cost



Woodcock Tracking

```
void preprocess()
    majorant = findMaximumExtinction()

void sampleFreePath(x, ω)
    t = 0
    do:
        // Sample distance to next tentative collision
        t += -log(1 - rand()) / majorant
        // Compute probability of a real collision
        Pr = getExtinction(x + t*ω) / majorant
    while Pr < rand()
    return t
```

Woodcock Tracking Summary

- Unbiased, see [Coleman 86] for a proof

Mathematical Verification of a Certain Monte Carlo Sampling Technique and Applications of the Technique to Radiation Transport Problems

W. A. Coleman

Woodcock Tracking Summary

- Unbiased, see [Coleman 86] for a proof
- Majorant extinction
  - defines the combined homogeneous volume
  - must bound the real extinction
  - loose majorants lead to many fictitious collisions

Transmittance via Woodcock Tracking

- Can be used to estimate transmittance $T_r(\mathbf{x}, \mathbf{y})$
- Recipe:
  - Sample free path from $\mathbf{x}$ towards $\mathbf{y}$
  - If a real collision occurs before $\mathbf{y}$, then $T_r(\mathbf{x}, \mathbf{y}) = 0$
  - else $T_r(\mathbf{x}, \mathbf{y}) = 1$
- To get a finer estimate, sample multiple free paths and count the fraction that made it beyond $\mathbf{y}$

Next Time

- Fast(er) multiple scattering
  - volumetric photon mapping
  - photon beams
  - virtual ray lights…

Questions?

Errata