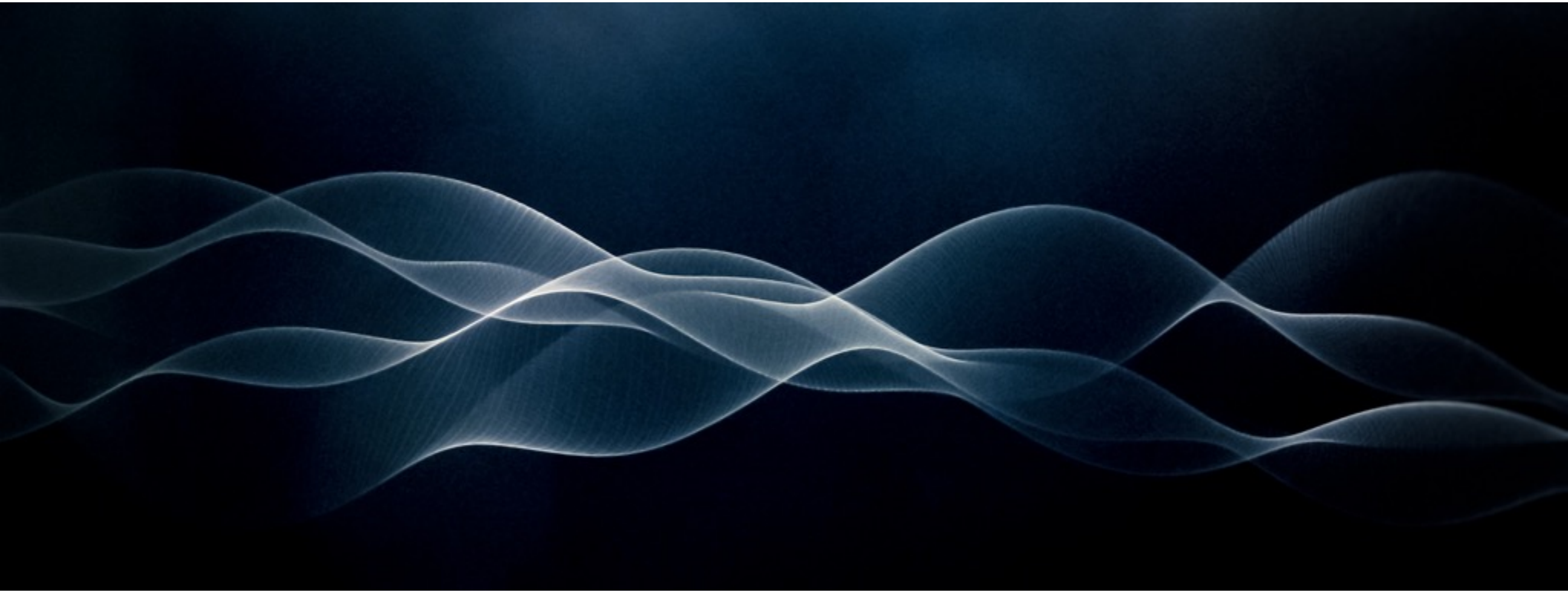
CS 87/187, Spring 2016

RENDERING ALGORITHMS

Swimming in Harmonic Waves



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Dartmouth

À la Carte

Starters

Main Course

Desserts

À la Carte

- Fourier Transform, Fourier Series
- Frequency Spectrum

Main Course

Desserts

À la Carte

- Fourier Transform, Fourier Series
- Frequency Spectrum
- Monte Carlo integration
- Harmonic Analysis of Point Samples

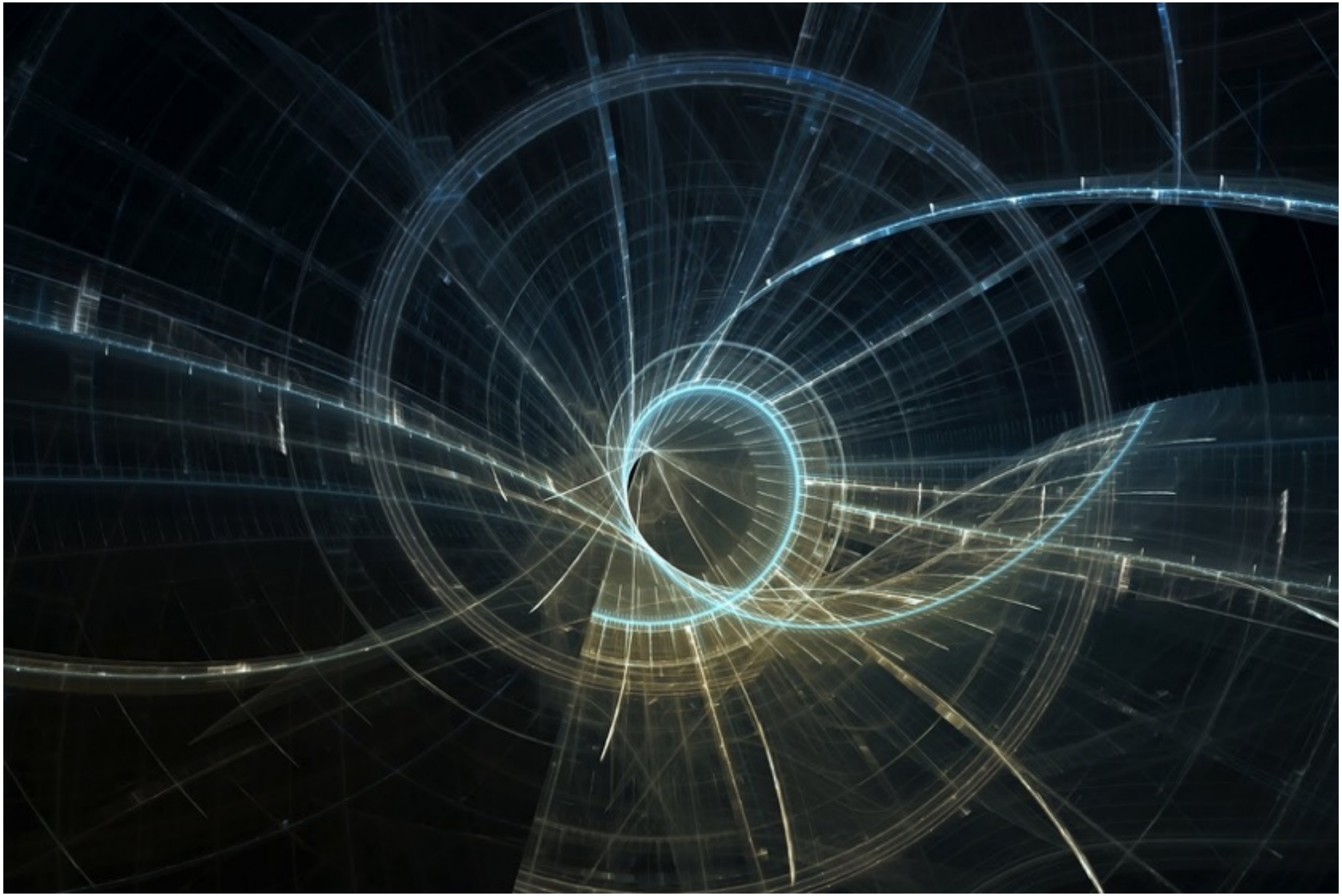
Desserts

À la Carte

- Fourier Transform, Fourier Series
- Frequency Spectrum
- Monte Carlo integration
- Harmonic Analysis of Point Samples
- Variance Convergence Analysis

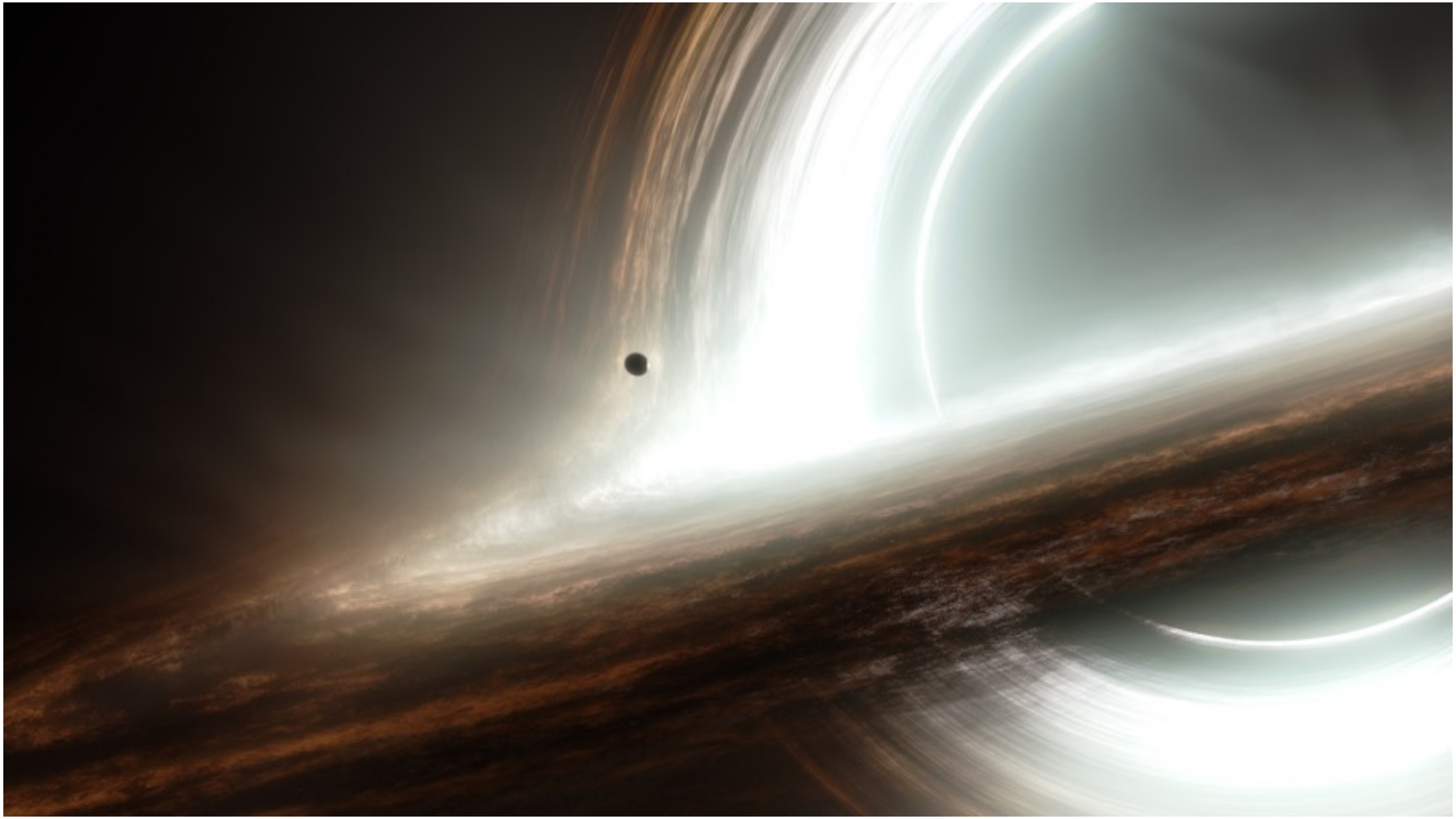
World Through the “Fourier” Looking Glass

Applicability: Quantum Mechanics



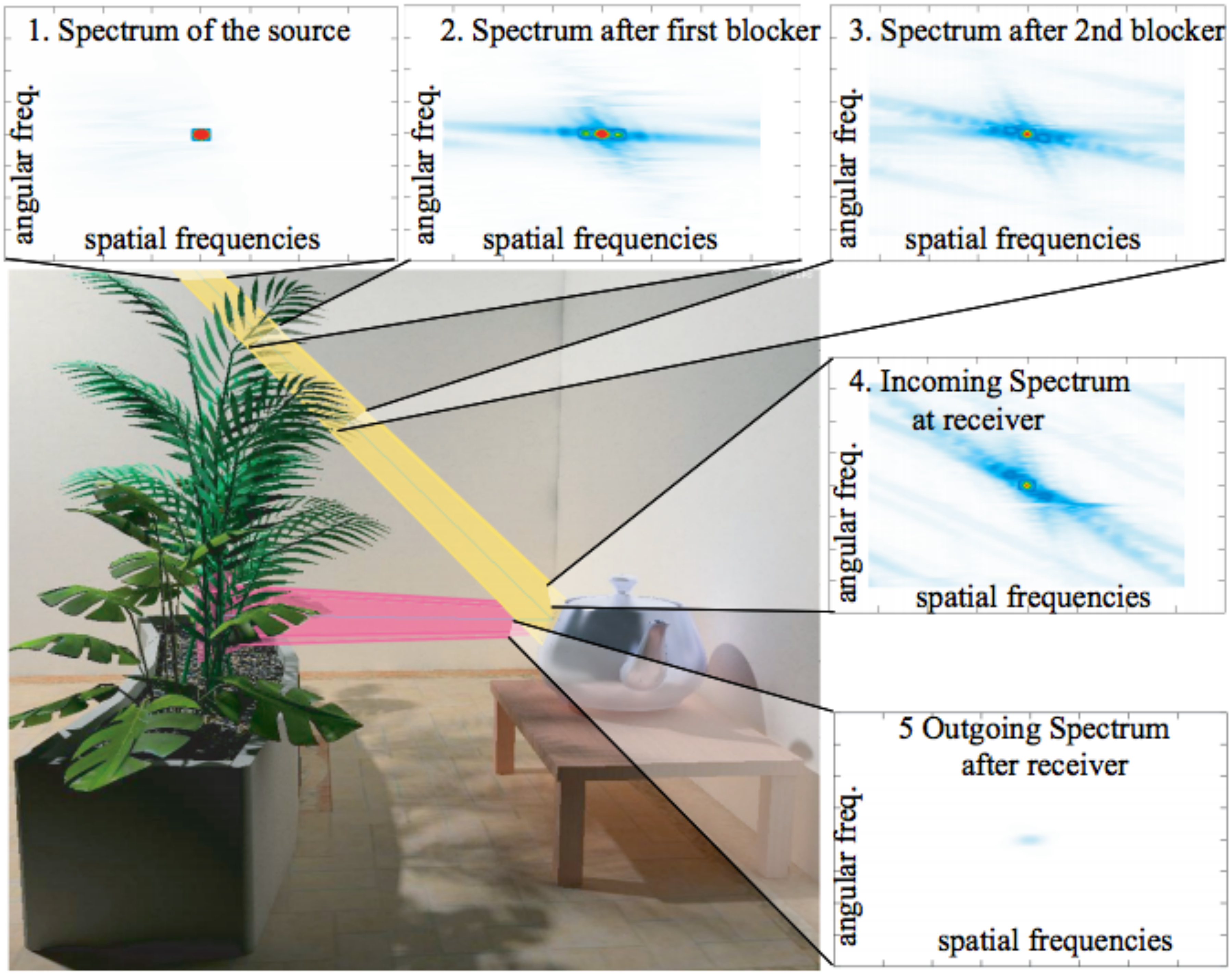
Courtesy, Documentary: Quantum Mechanics

Applicability: Black Holes



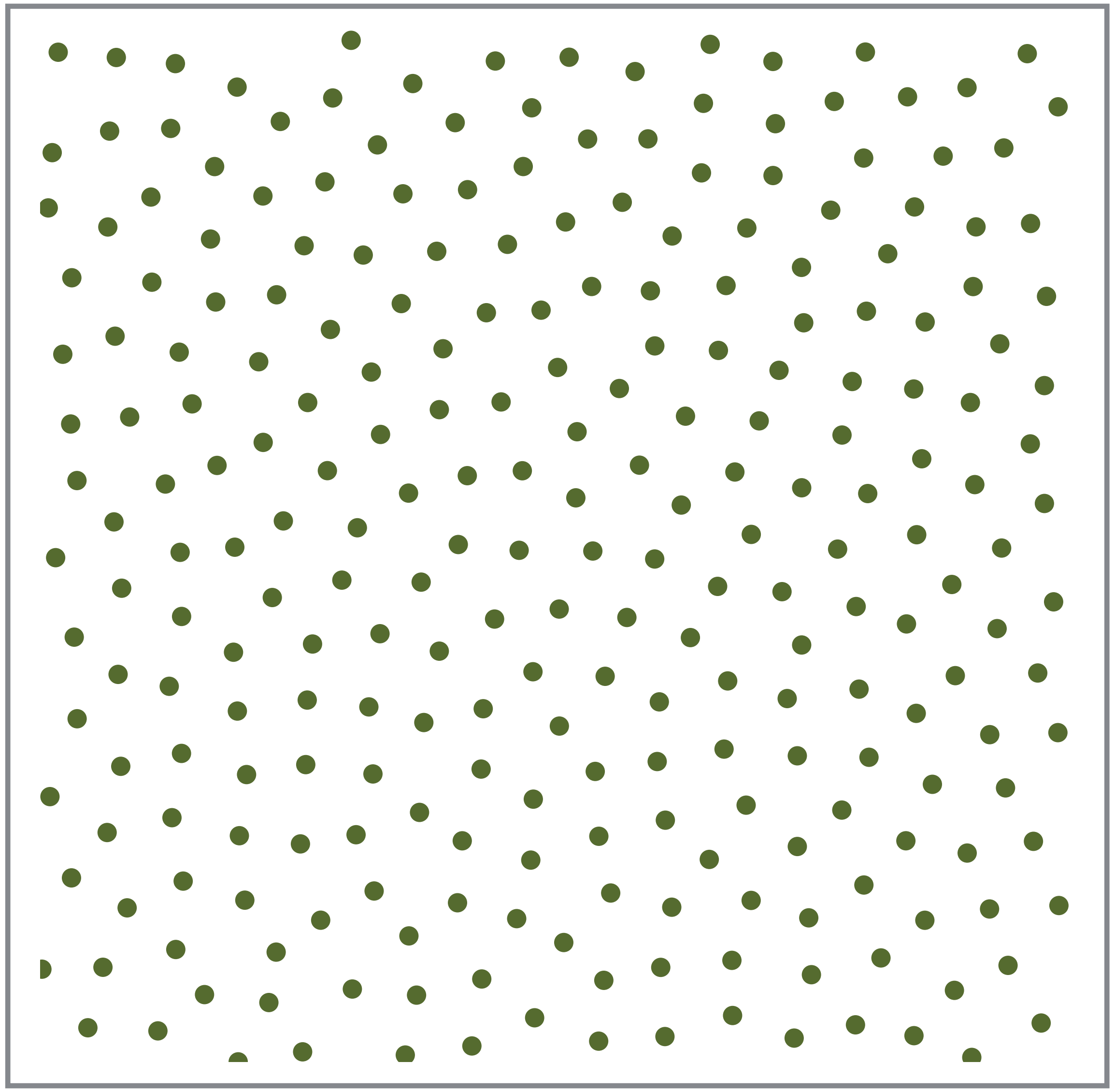
Warner Bros. Interstellar 2014

Applicability: Light Transport

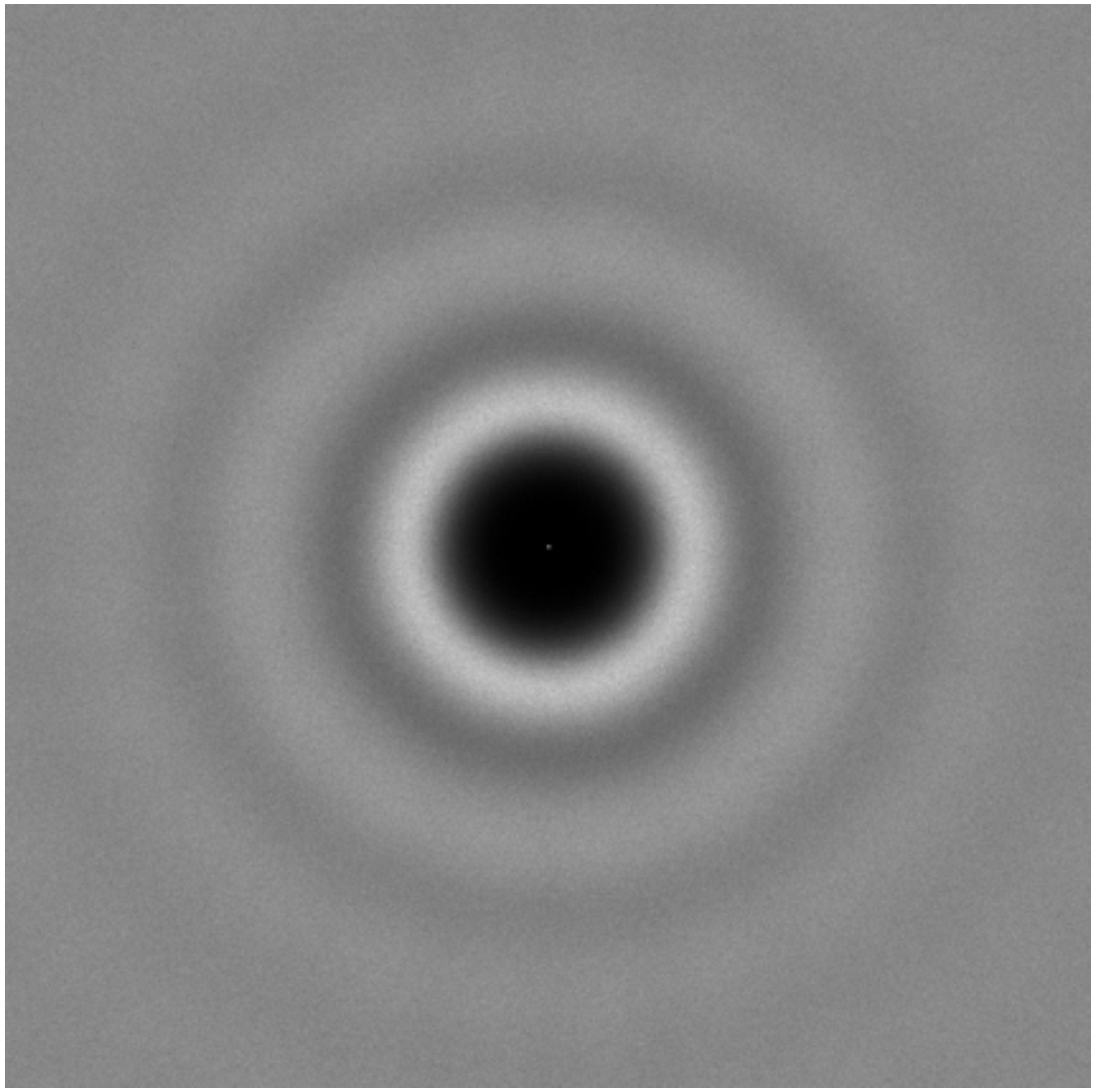


Durand et al. [2005]

Applicability: Point Samples



Point Samples



Frequency Spectrum

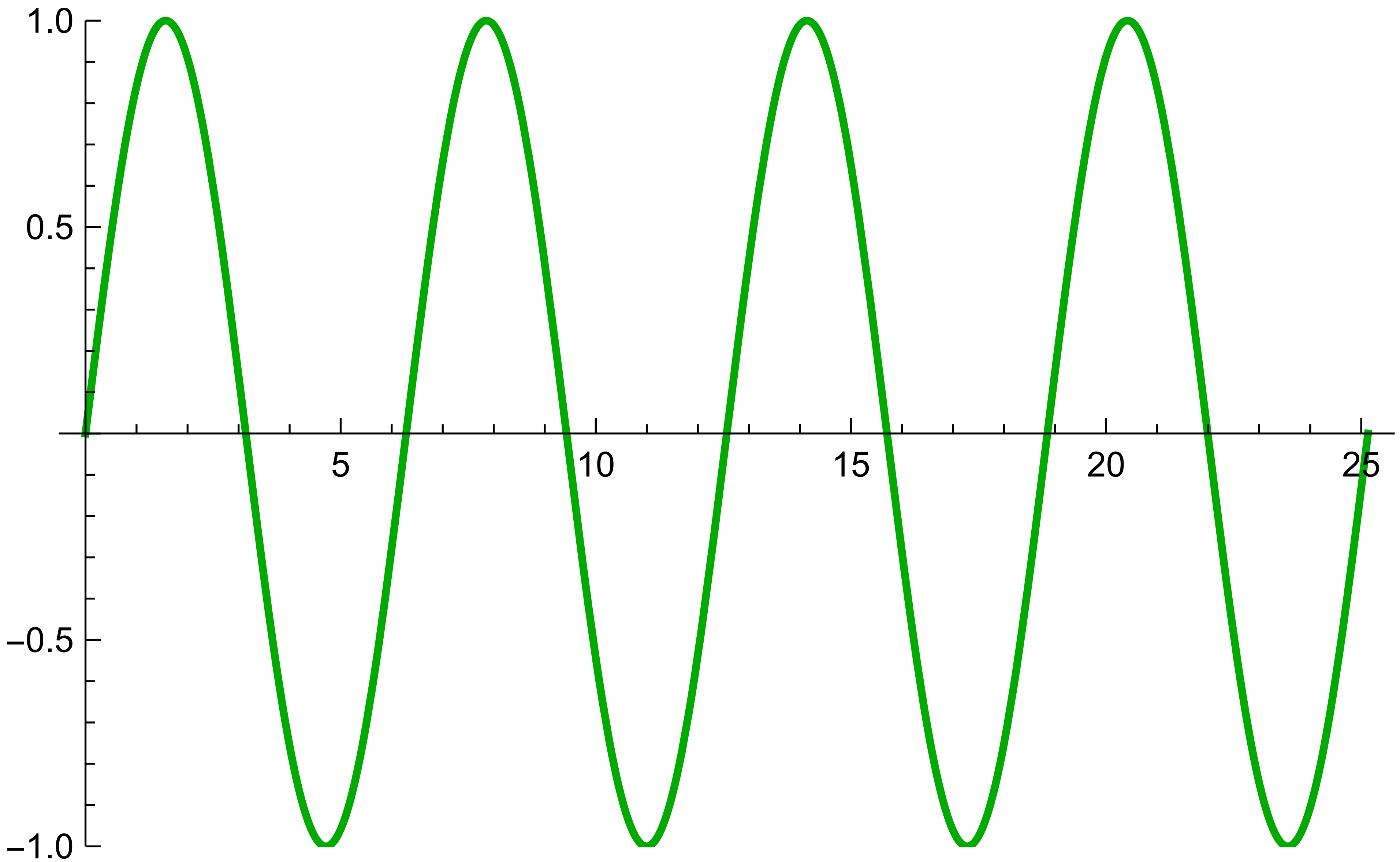
Introduction

Jean Baptiste Joseph Fourier (1768-1830)

Fourier claimed that:
“*Any function*
of a variable, whether continuous
or discontinuous, can be
expanded in a series of sines
of multiples of the variable.”

1822-Théorie analytique de la chaleur

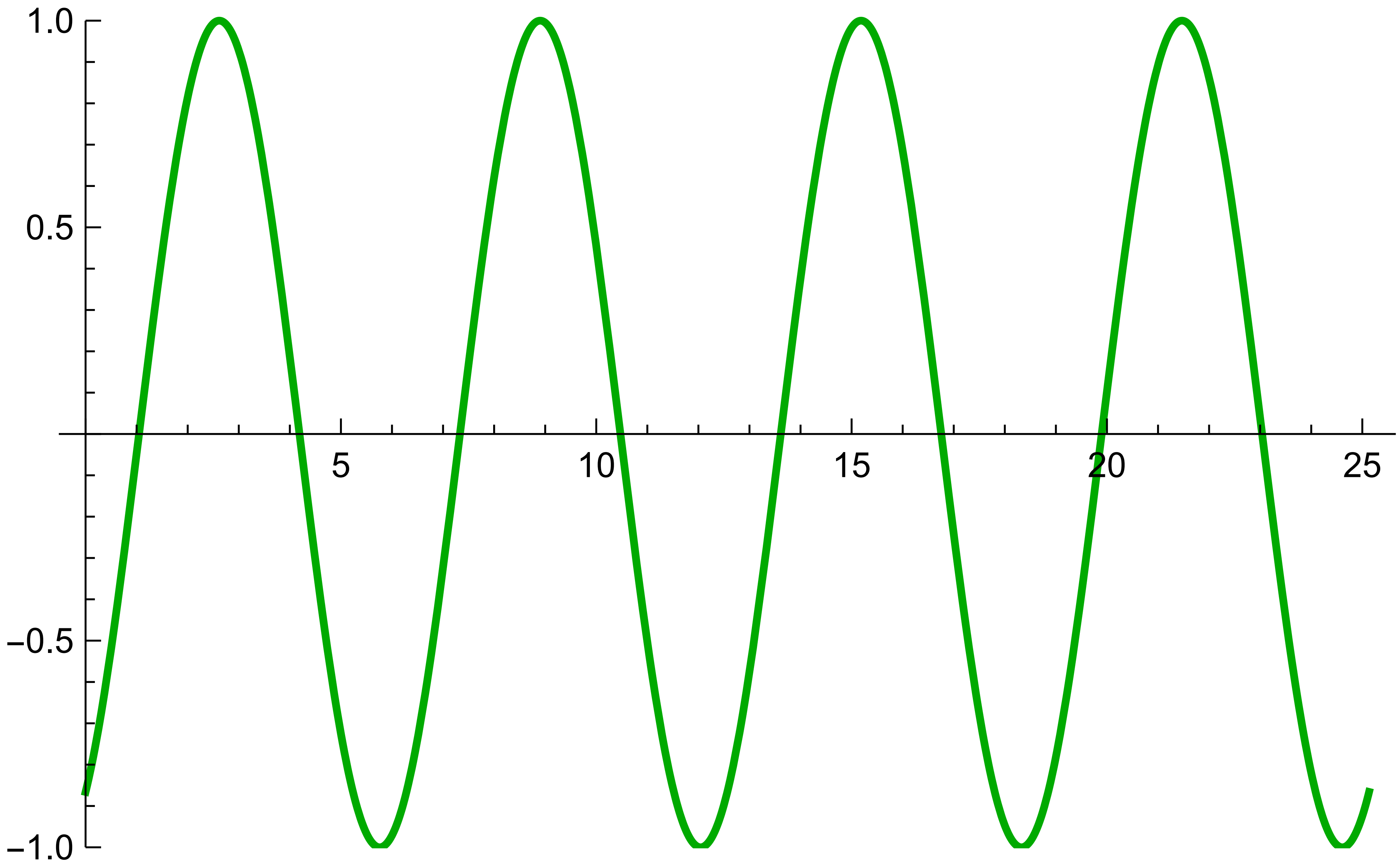
Sine Waves



$Sin(x)$

Sine Waves

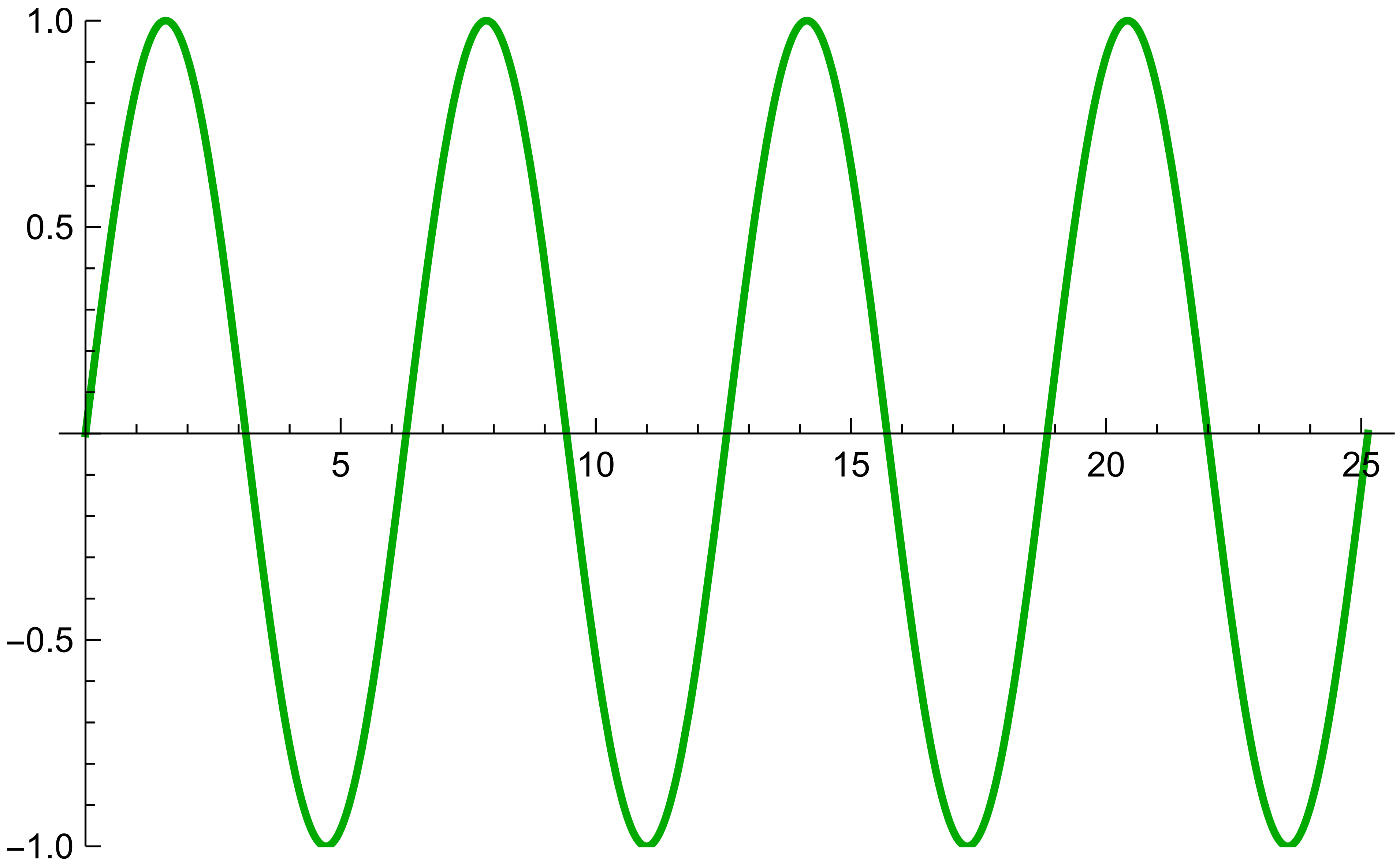
- Phase Shift



$$Sin(x - \frac{\pi}{3})$$

Sine Waves

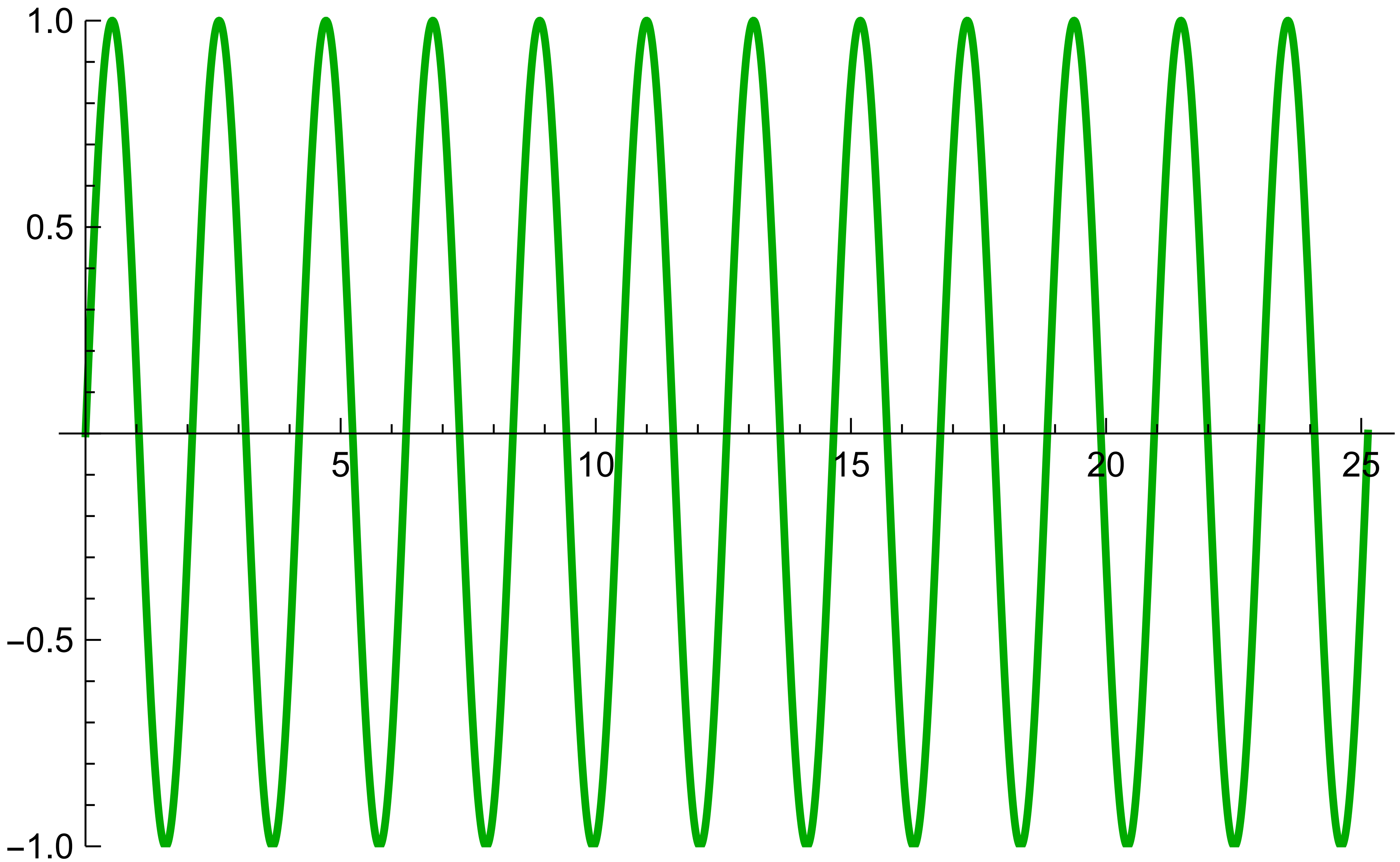
- Phase Shift



$$Sin(x - 2\pi)$$

Sine Waves

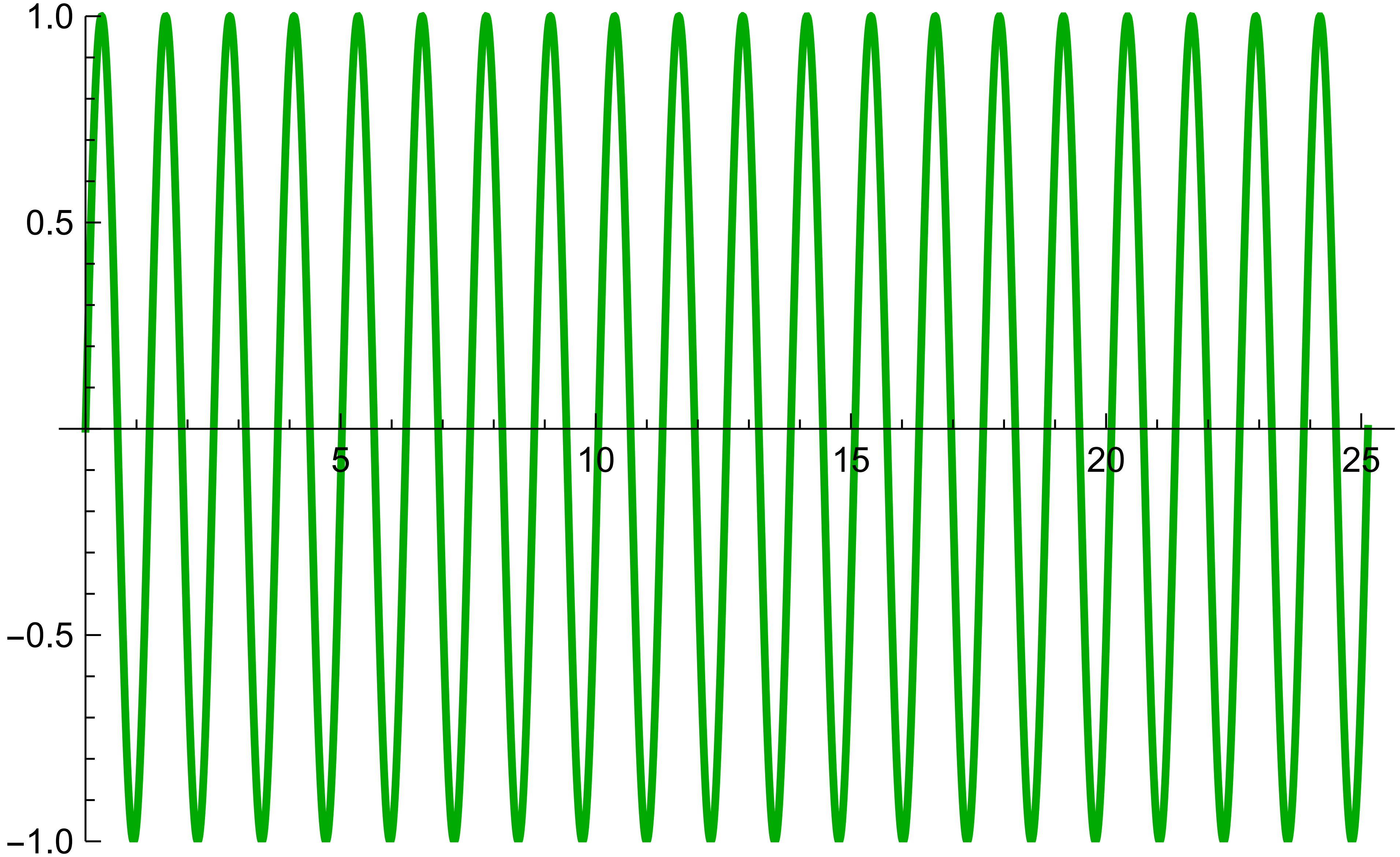
- Frequency: 3x



$$Sin(3x)$$

Sine Waves

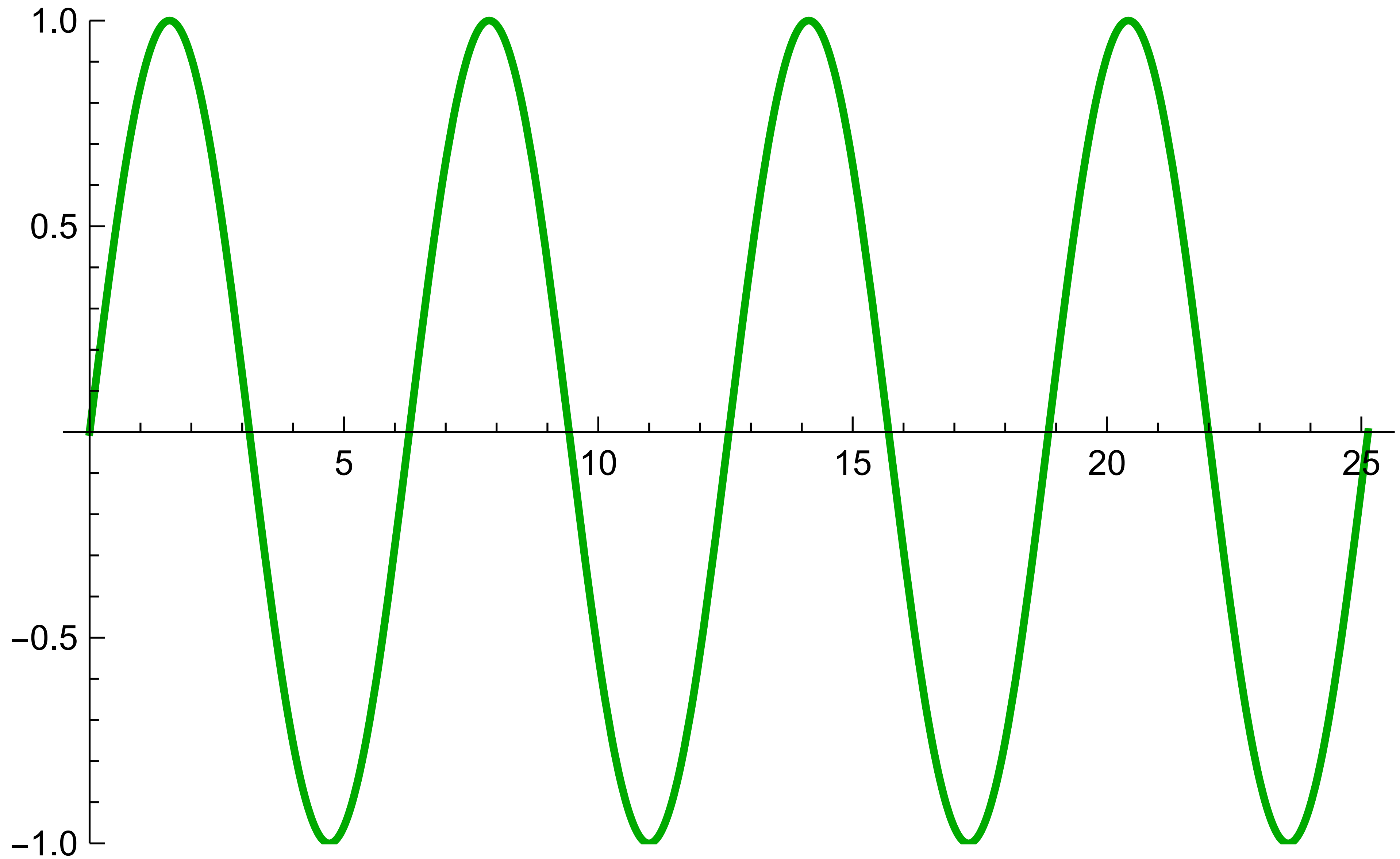
- Frequency: 5x



$$Sin(5x)$$

Sine Waves

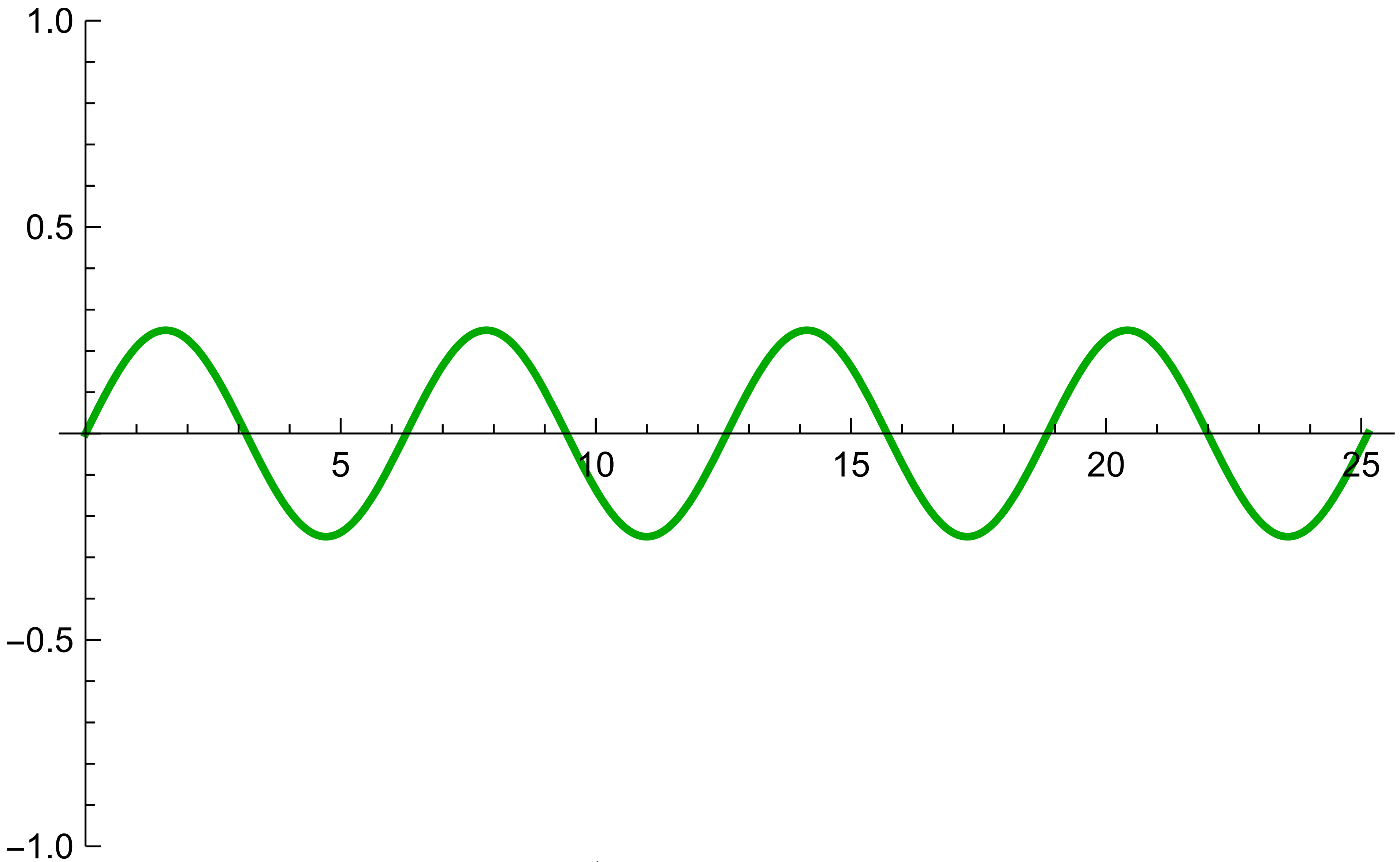
- Frequency: 1x



$$Sin(x)$$

Sine Waves

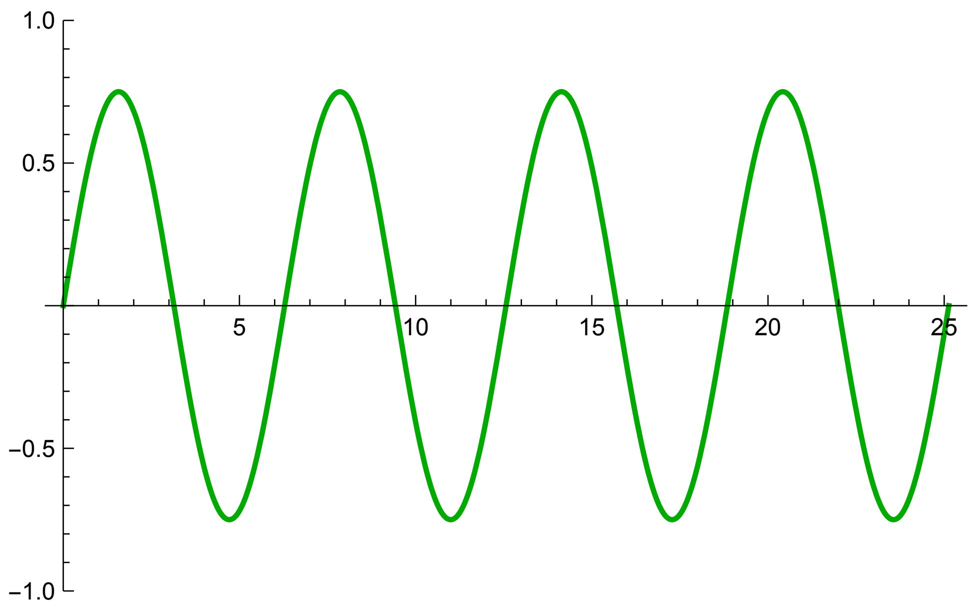
- Amplitude



$$\frac{1}{4} Sin(x)$$

Addition of Sinusoids

- Addition of two sinusoids is another sine wave



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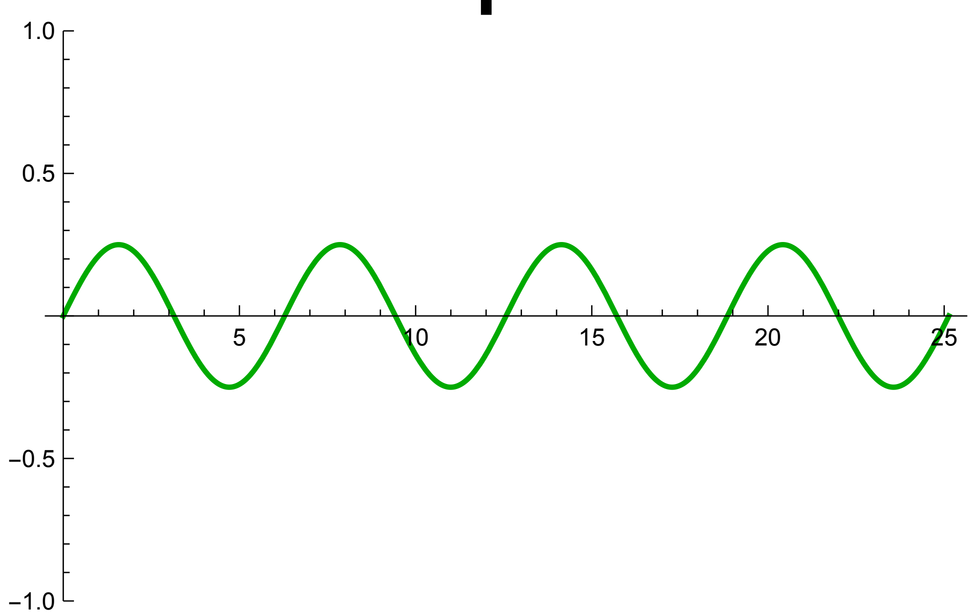

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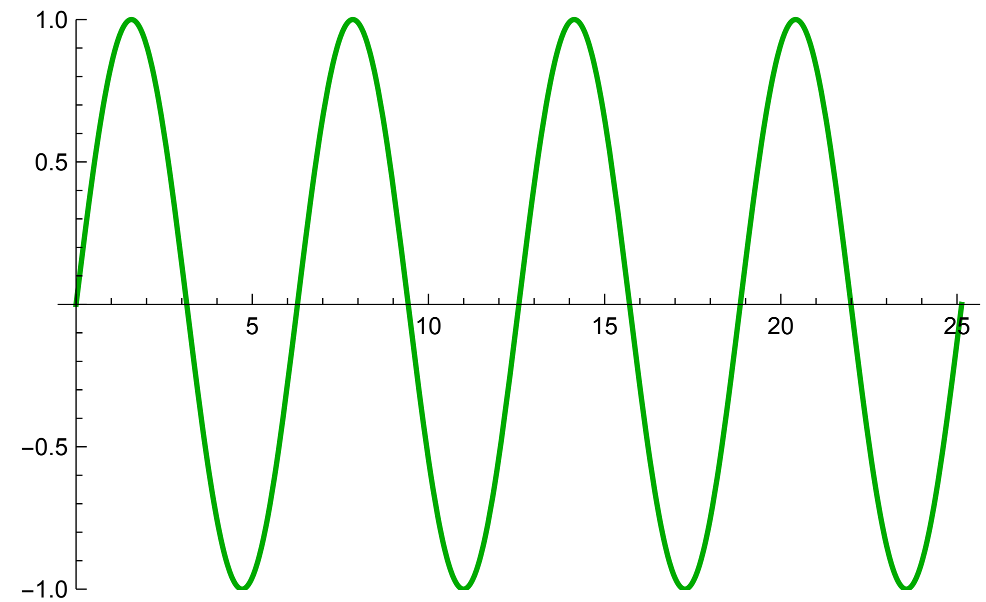
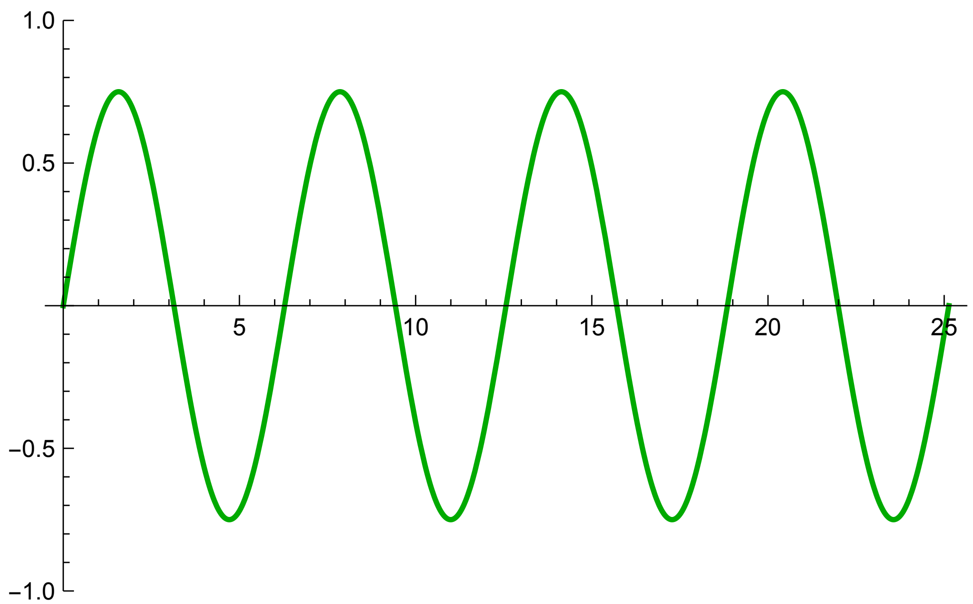


$$\frac{3}{4}Sin(x) + \frac{1}{4}Sin(x) = Sin(x)$$

Addition of Sinusoids

- Phase shifted sinusoids is another sine wave



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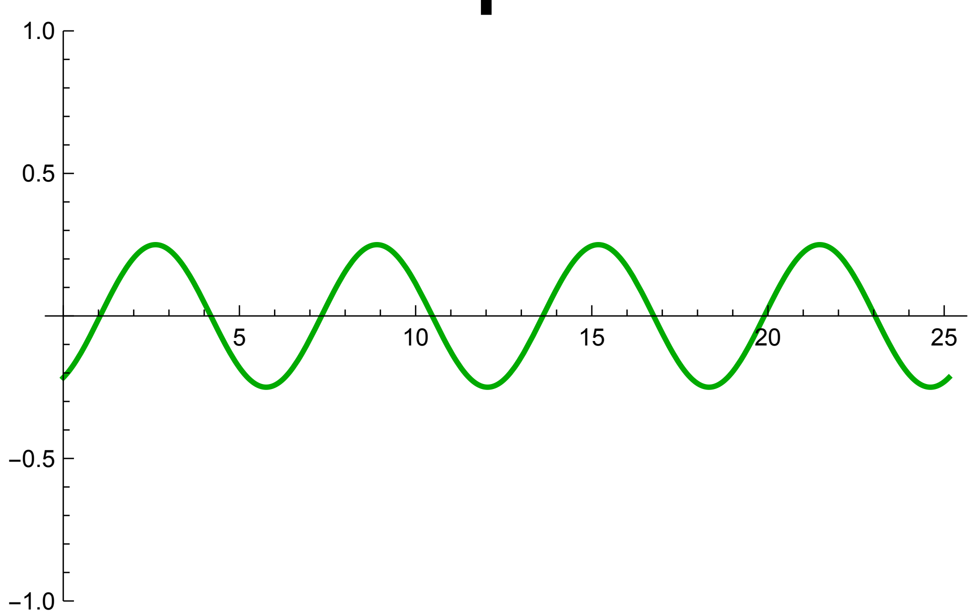

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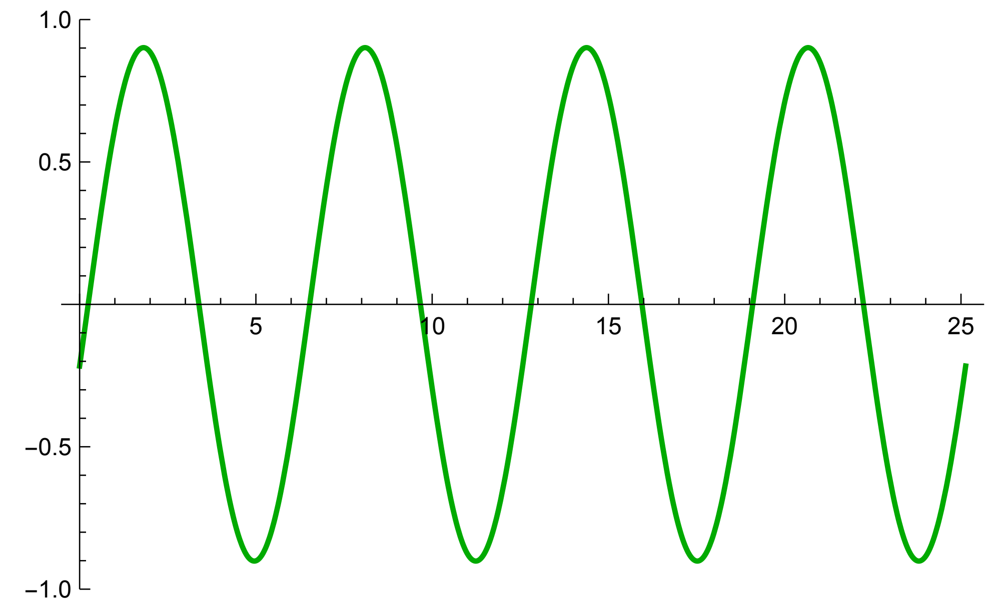
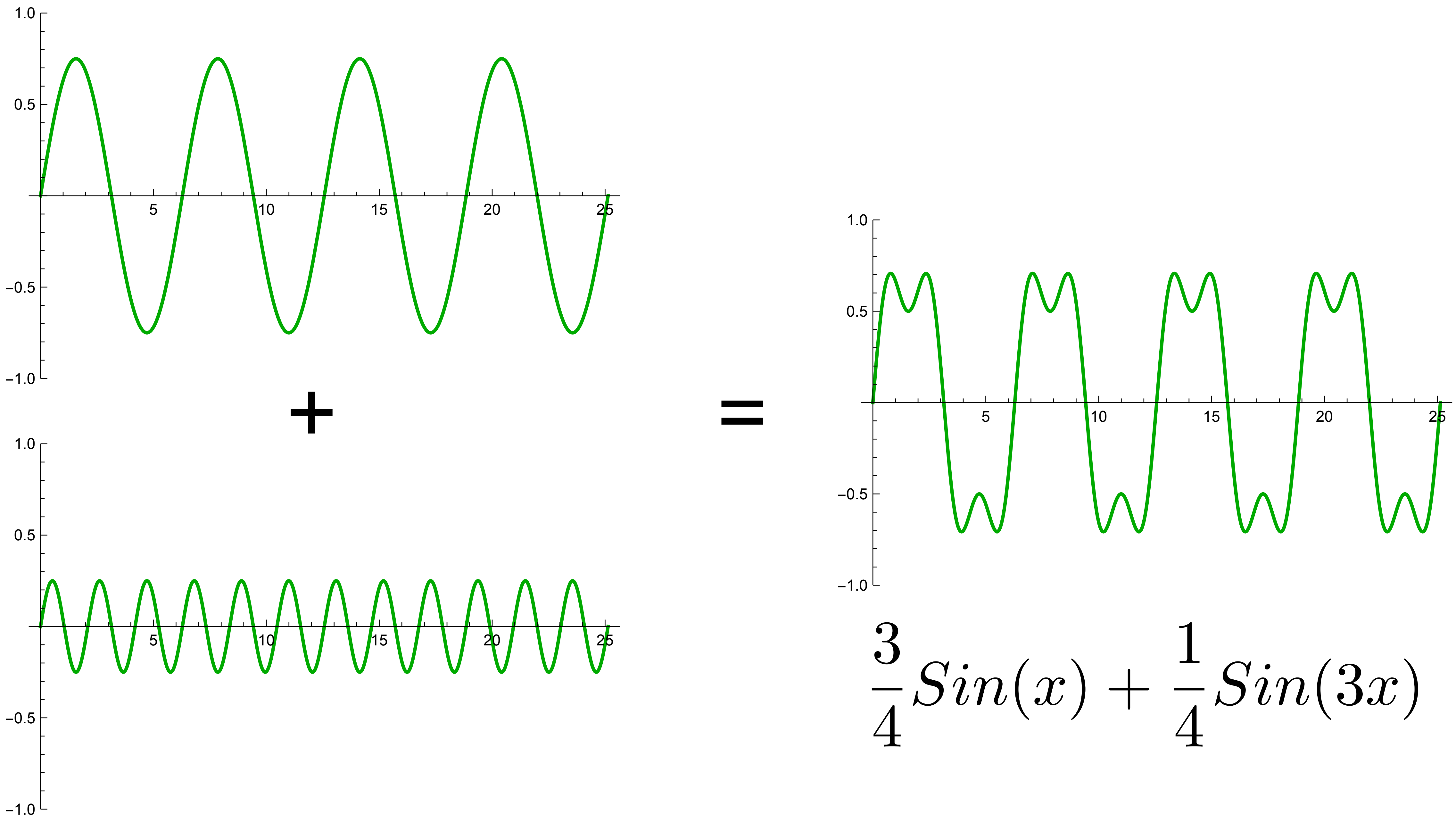


$$\frac{3}{4}Sin(x) + \frac{1}{4}Sin(x - \frac{\pi}{3})$$

Addition of Sinusoids

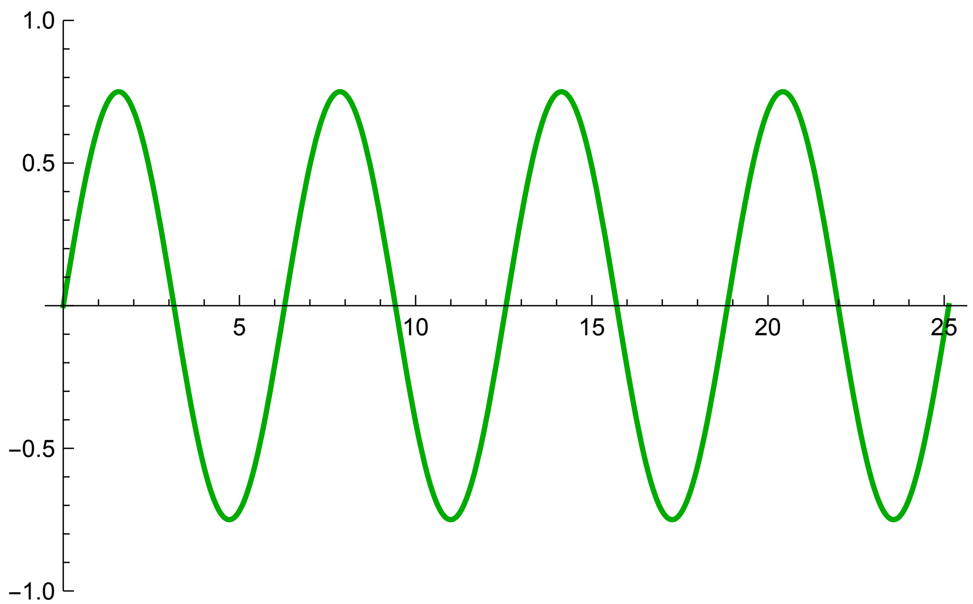
- As long as the two sines have the same frequency, their sum will always be another sine wave of the exact same frequency, but with a different amplitude and phase.

Addition of Sinusoids



$$\frac{3}{4}Sin(x) + \frac{1}{4}Sin(3x)$$

Addition of Sinusoids



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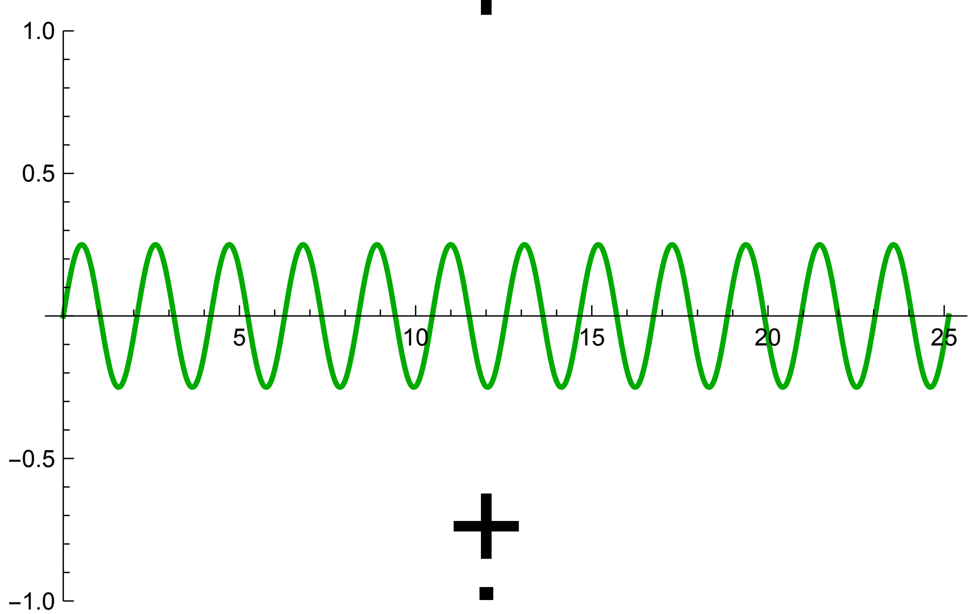

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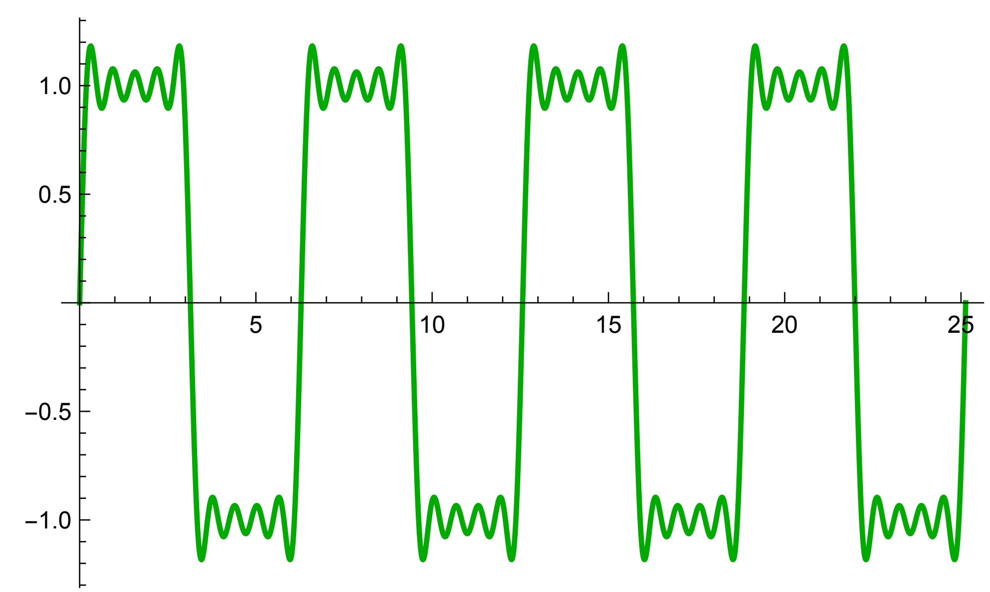
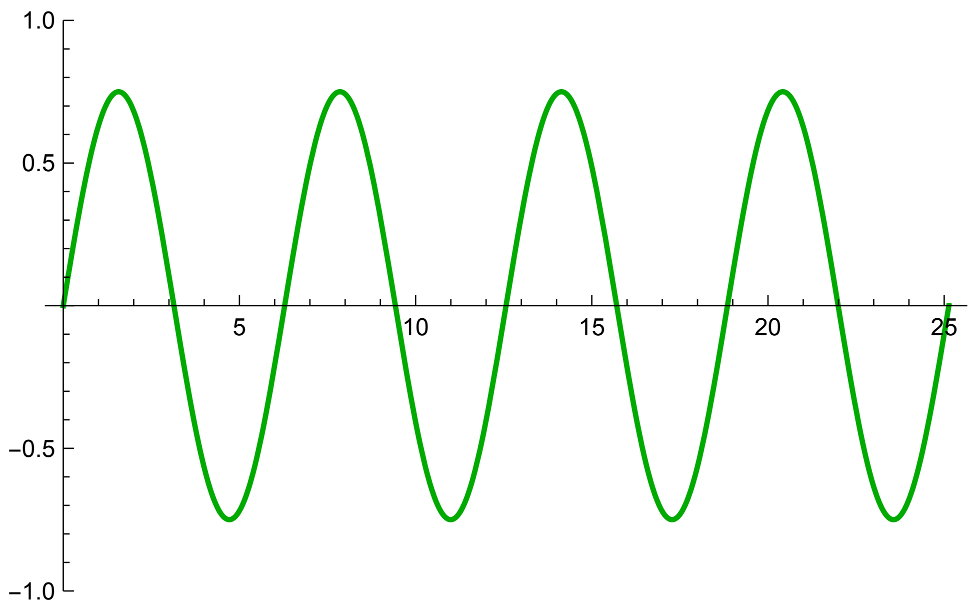


$$\sum_{n=}^{10} \frac{4Sin((2n-1)x)}{(2n-1)\pi}$$

Addition of Sinusoids



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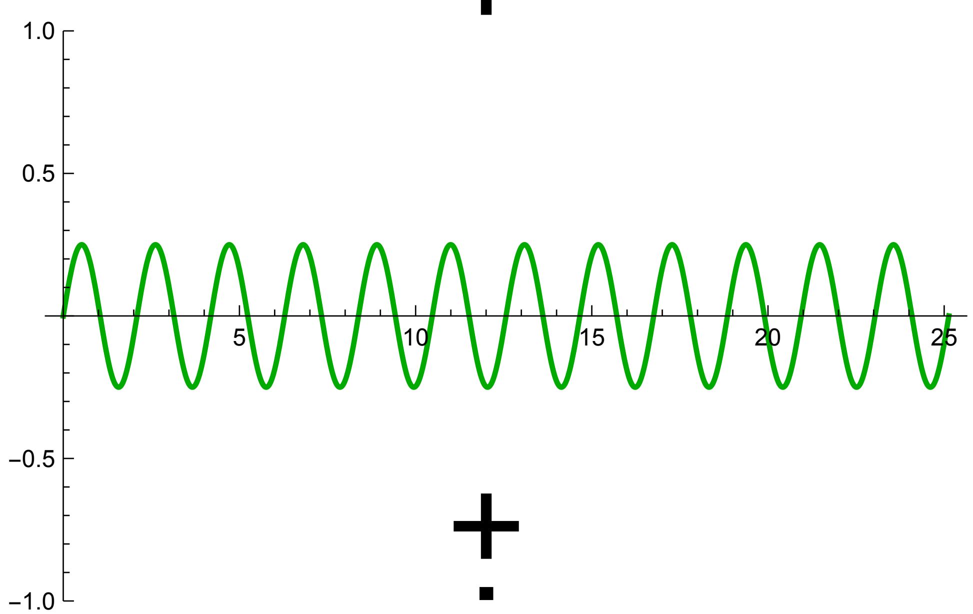

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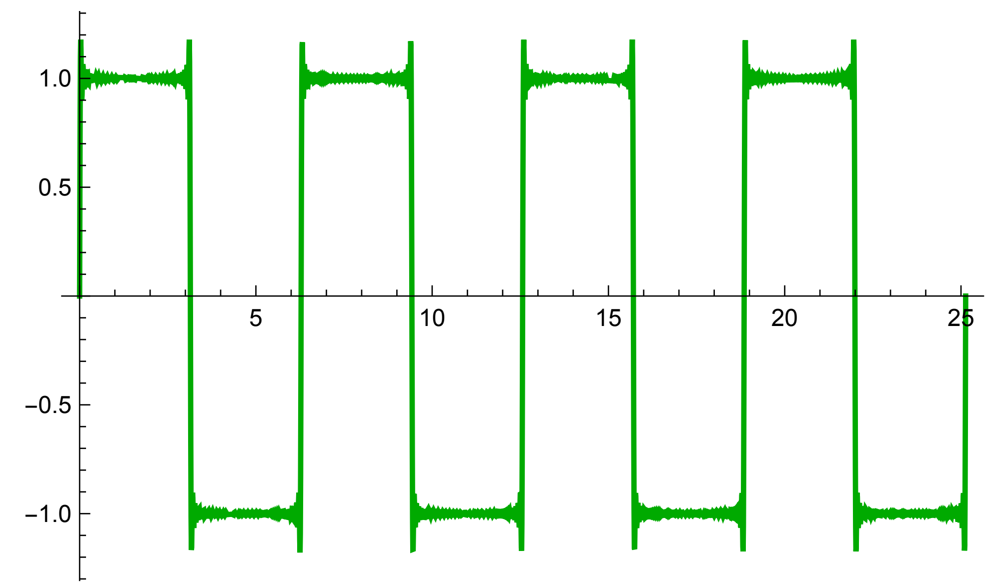
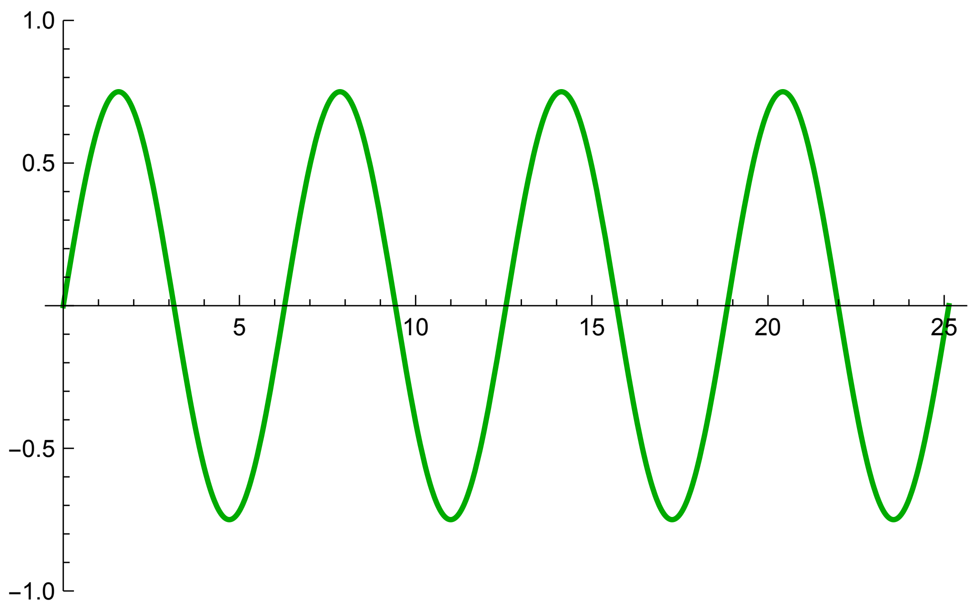


$$\sum_{n=}^{100} \frac{4Sin((2n-1)x)}{(2n-1)\pi}$$

Addition of Sinusoids



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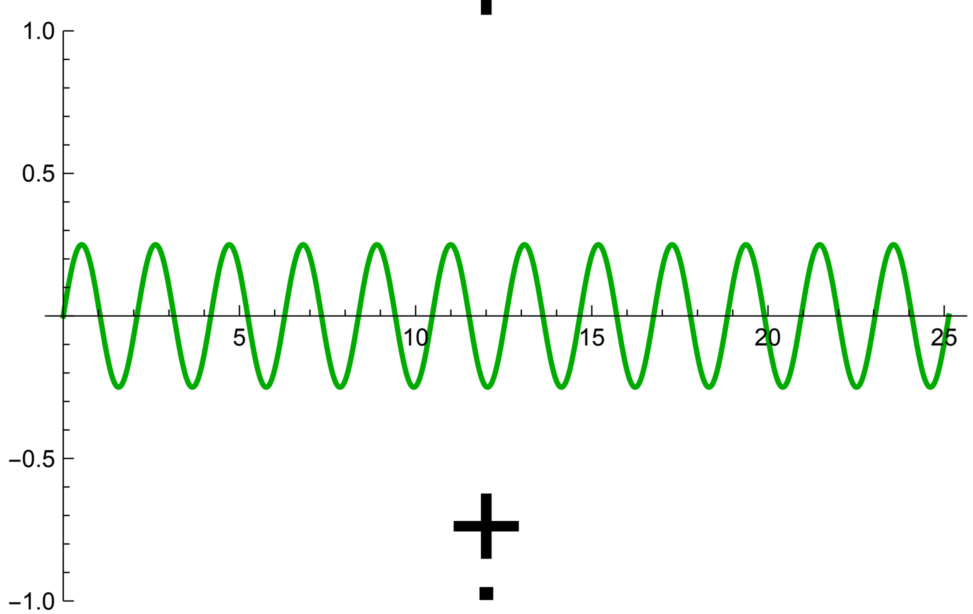

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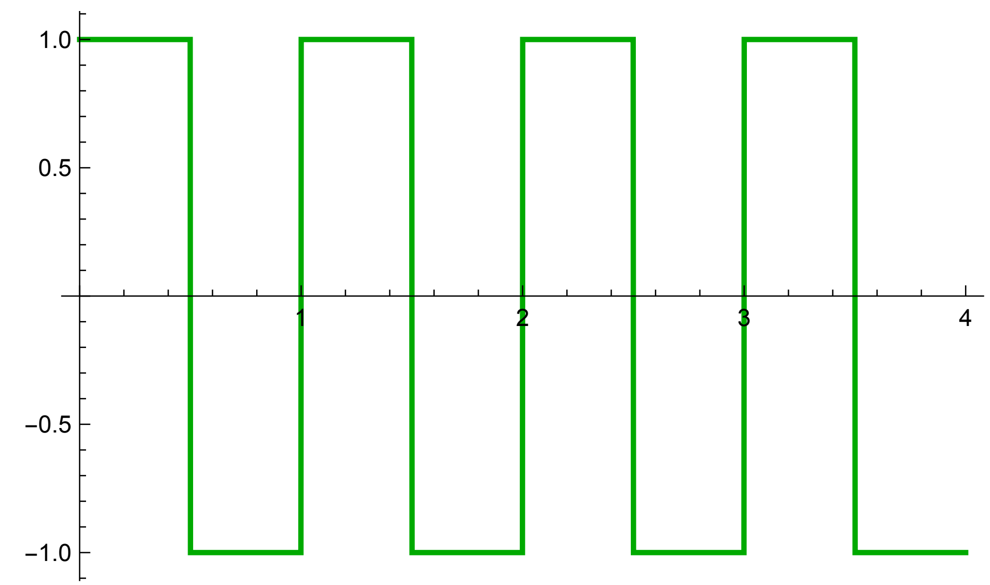
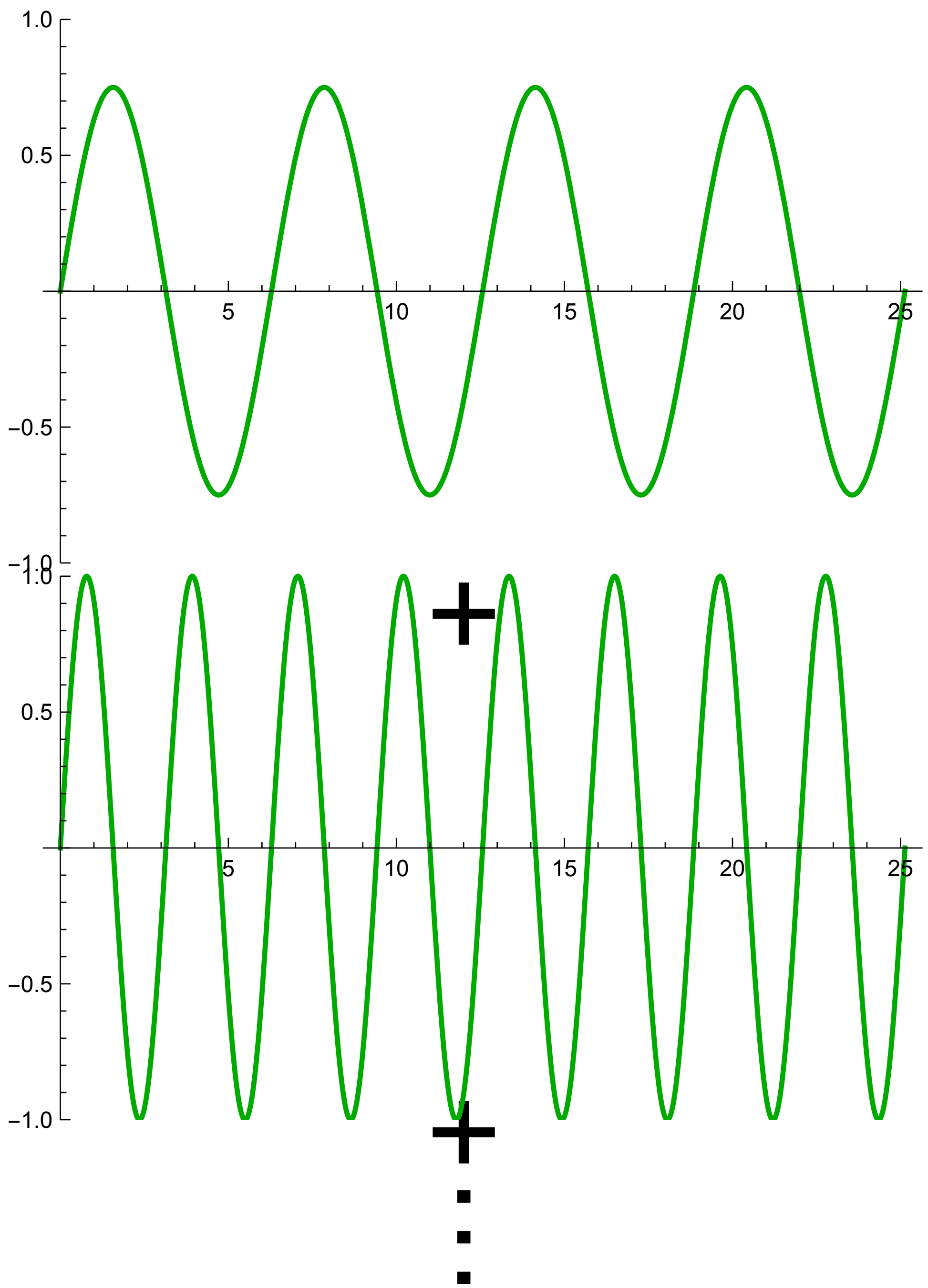


$$\sum_{n=1}^{\infty} \frac{4Sin((2n-1)x)}{(2n-1)\pi}$$

Addition of Sinusoids



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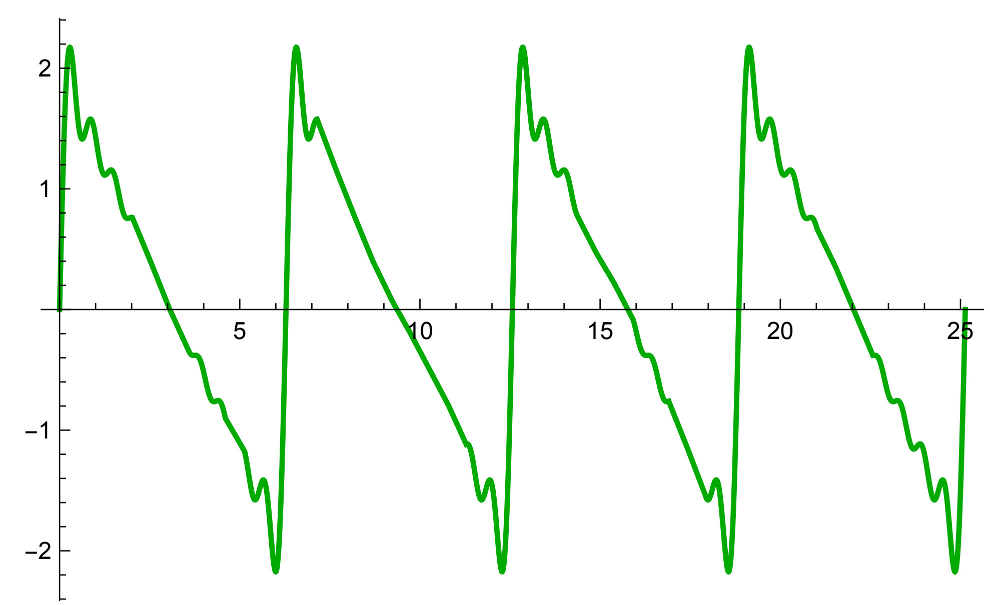
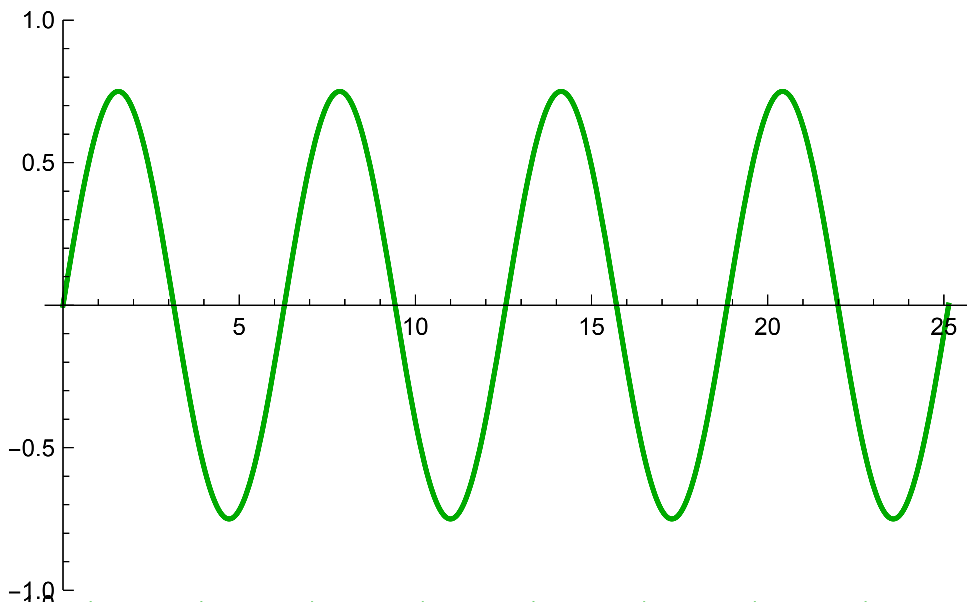


$$\sum_{n=1}^{10} \frac{(-1)^n 2Sin(nx)}{n\pi}$$

Addition of Sinusoids



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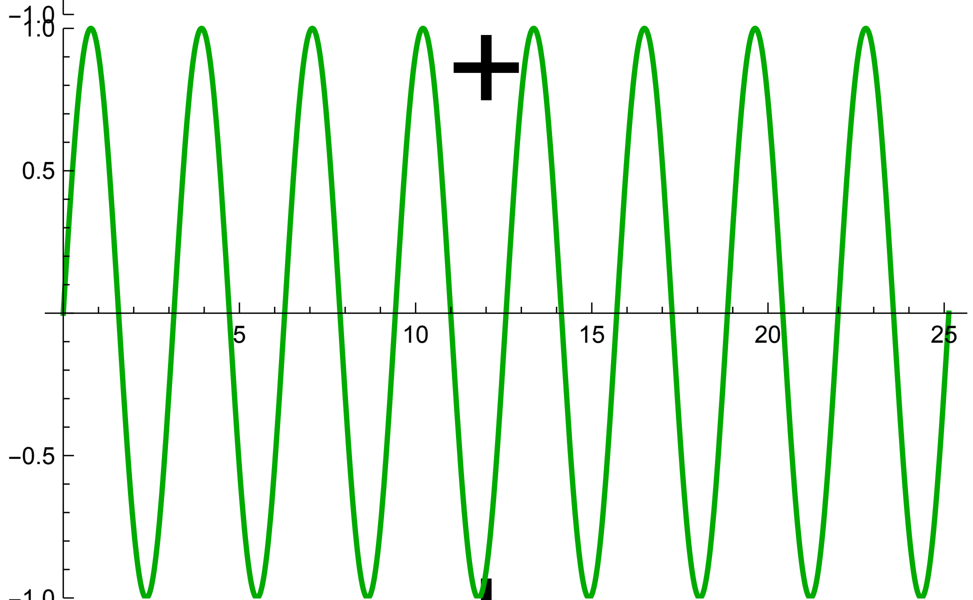

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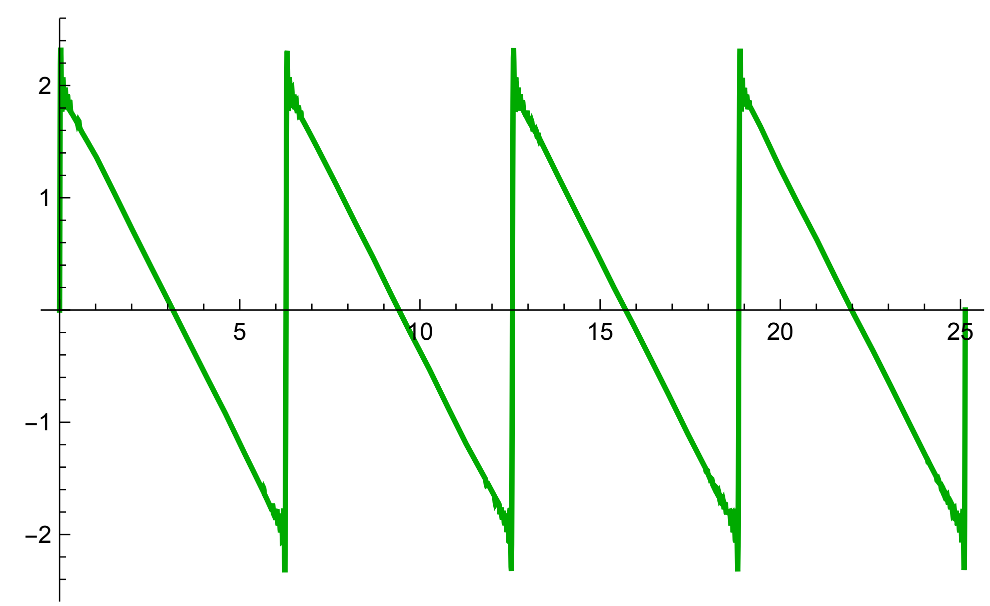


$$\sum_{n=1}^{100} \frac{(-1)^n 2Sin(nx)}{n\pi}$$

Addition of Sinusoids

- If we add together two different frequency sine waves, the resultant waveform is not sine wave.
- In the two examples, the two examples happened to be repeating waveforms.
  - in these cases, only certain frequencies were needed
  - and each of these sine waves had a measurable amplitude.

Fourier Series

- The Fourier series of a function f(x) is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Fourier Series

- The Fourier series of a function f(x) can also be written in the complex plane:

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nx}$$

where

$$e^{-i2\pi nx} = \cos(2\pi nx) + isin(2\pi nx)$$

Euler formula [1760]

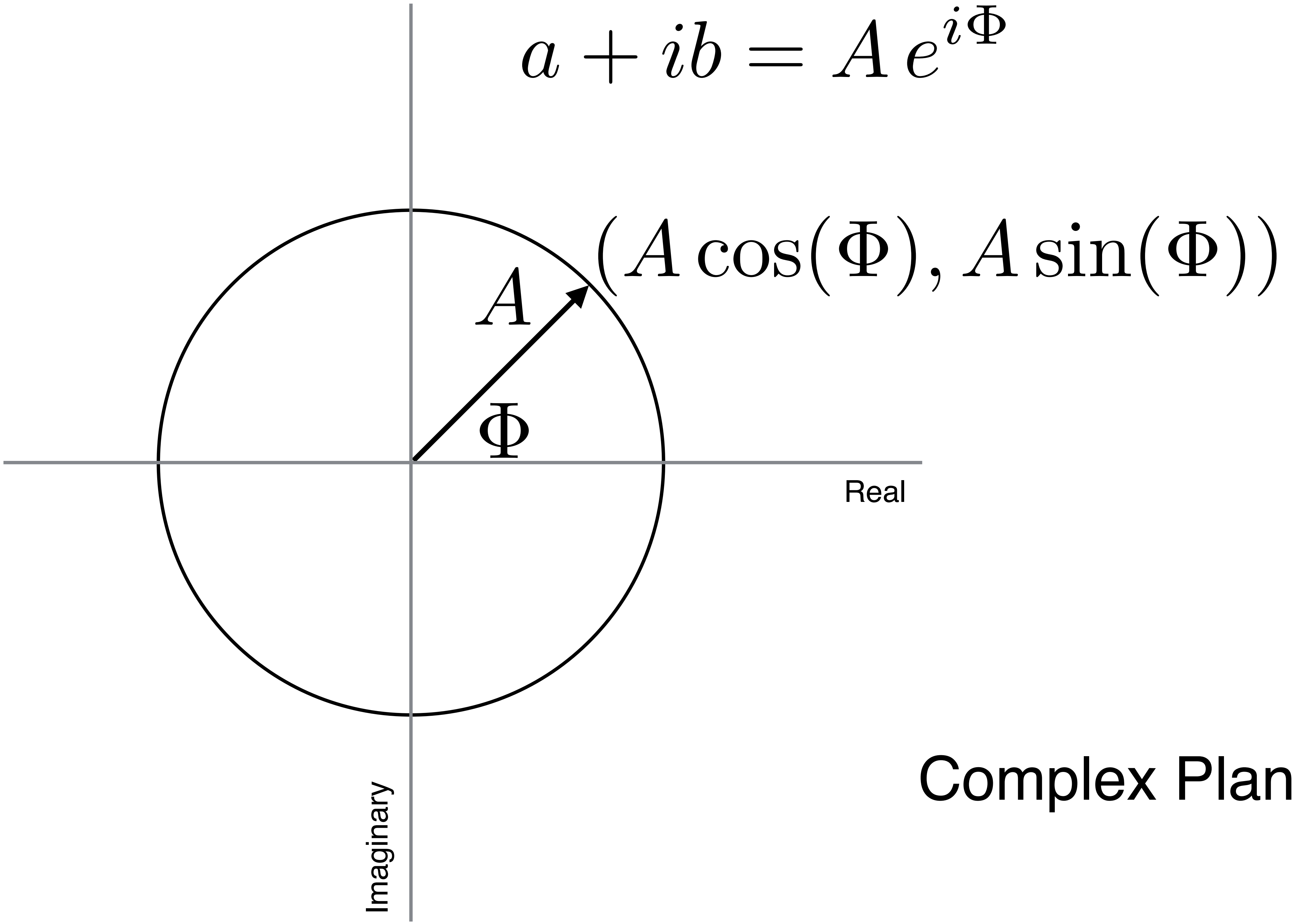
Why Complex Numbers ?

Why Complex Numbers ?

- Spatial Domain → Frequency Domain
- In the Frequency domain, which is represented by Sine waves, we have both Phase and Amplitude
- To represent both these informations simultaneously we need something more than real numbers

Complex Numbers

$$a + ib = A\,e^{i\Phi}$$



Complex Plane

Complex Numbers

- Complex Numbers: $a + ib$

$$a + ib = A\, e^{i\Phi}$$

$$A = (\sqrt{a * a + b * b})$$

$$e^{i\Phi} = \sin(\Phi) + i\cos(\Phi)$$

Here,

Φ is Phase and A is the amplitude

Video

- Fourier Animating Circles

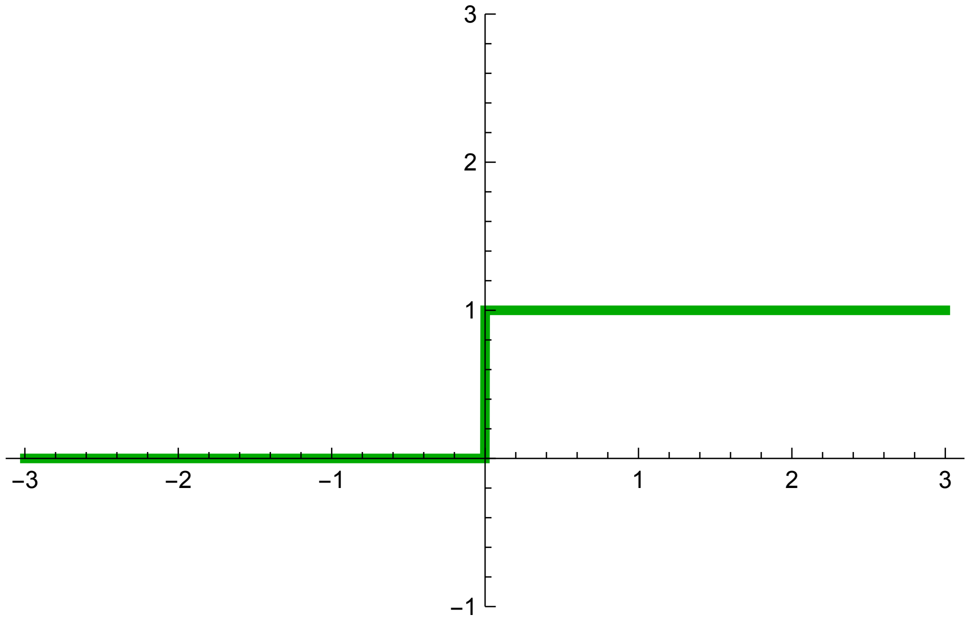
Visual Break

Internal Chaos by Gaétan Weltzer

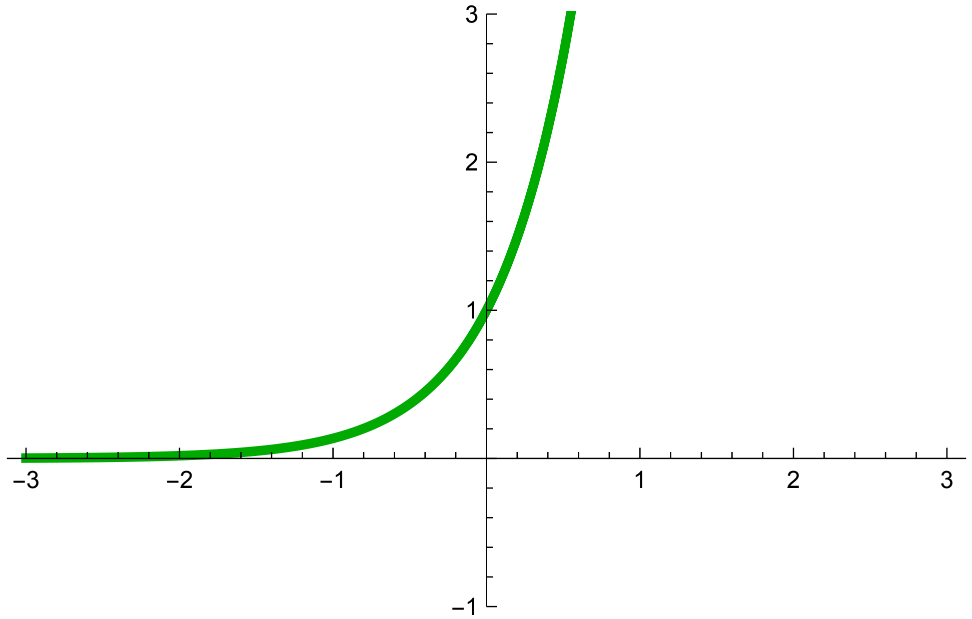
What about Non-Periodic Signals ?

What about non-periodic functions ?

- Non-repeating waveforms can also be generated by adding sine waves together



Step



Exponential

but, in these cases, sine waves of every possible frequency may be needed

What about non-periodic functions ?

- The signals and waveforms that we see in real life have a start and an end.
- However, each of these signals and waveforms can be thought of as having infinite number of sine waves, with each sine wave having no beginning or end.
- It is just that all these sine waves exactly cancel each other out at all locations except during the extent in which the signal is present.

Fourier Transform

- Fourier transform of a real-valued function f(x) is given by:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\omega x} dx$$

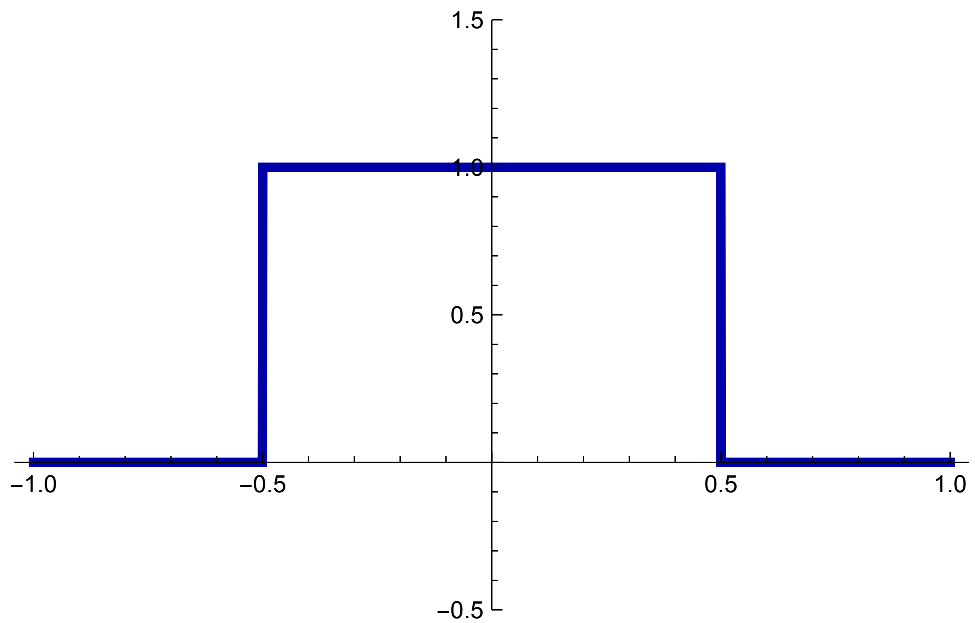
where

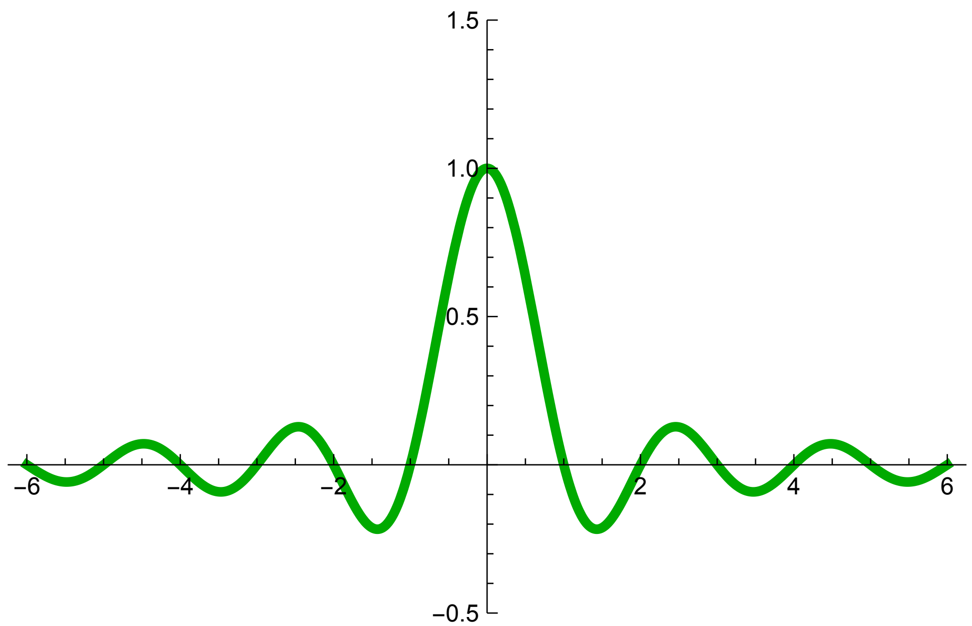
$$e^{-i2\pi\omega x} = \cos(2\pi\omega x) + isin(2\pi\omega x)$$

1D Fourier Transform Pairs

Fourier Transform Pairs

- Box function

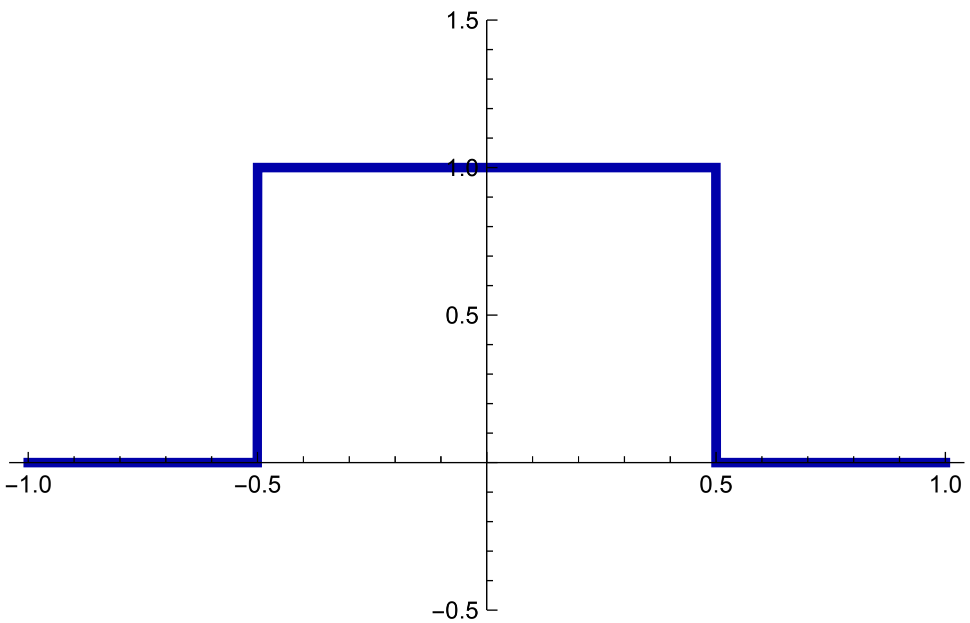


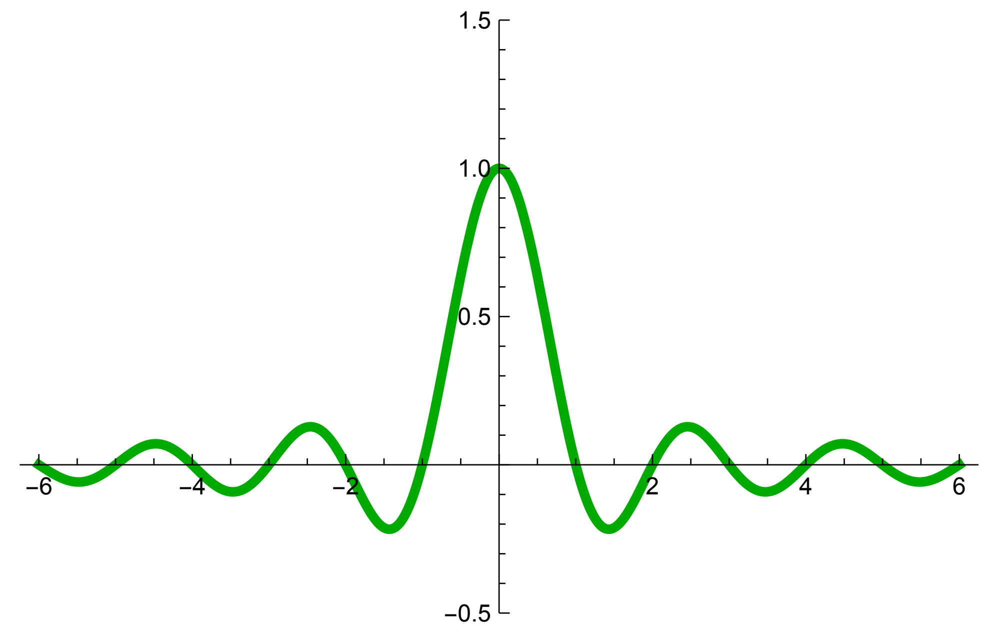


$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{otherwise} \end{cases} \qquad F(\omega) = a\frac{\sin(a\pi\omega)}{a\pi\omega}$$

Fourier Transform Pairs

- Box function

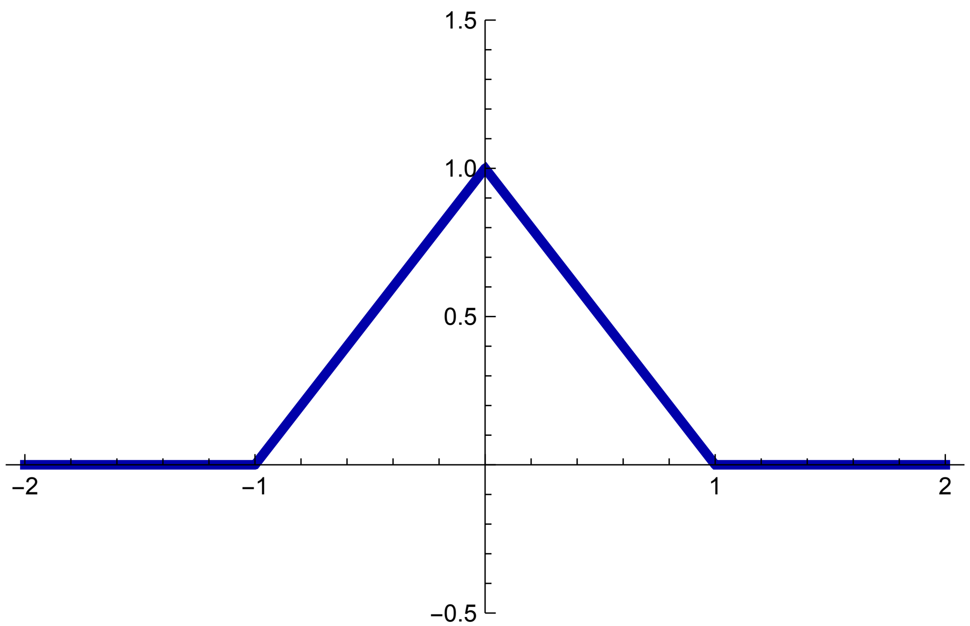


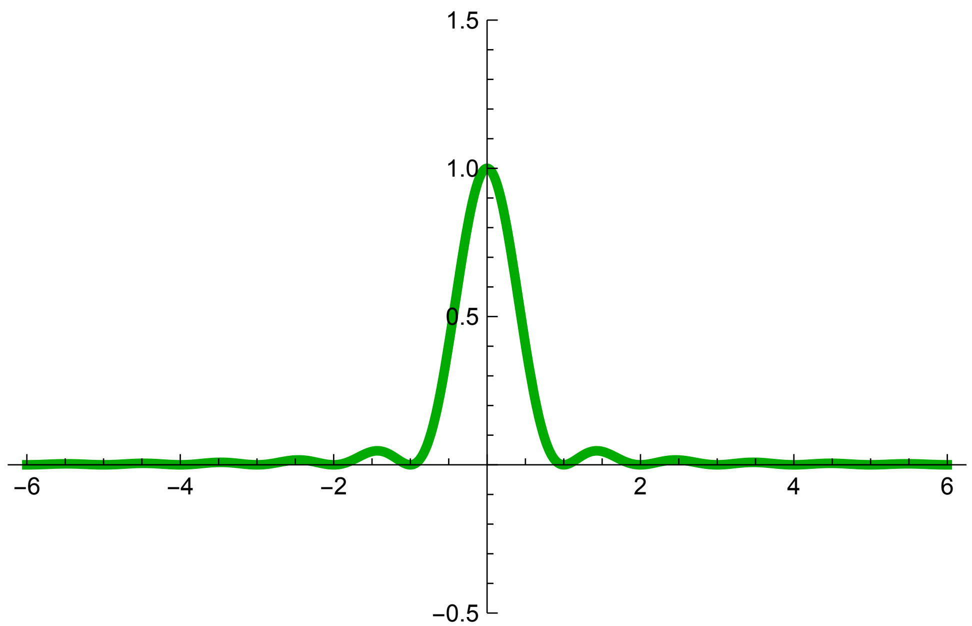


$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{otherwise} \end{cases} \qquad F(\omega) = a\,Sinc(a\pi\omega)$$

Fourier Transform Pairs

- Triangle function

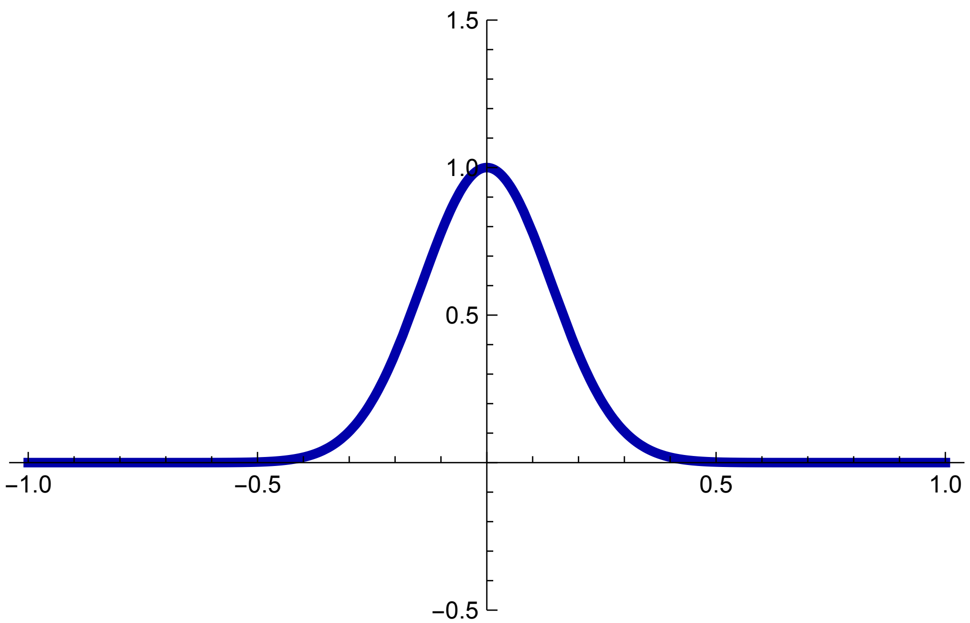


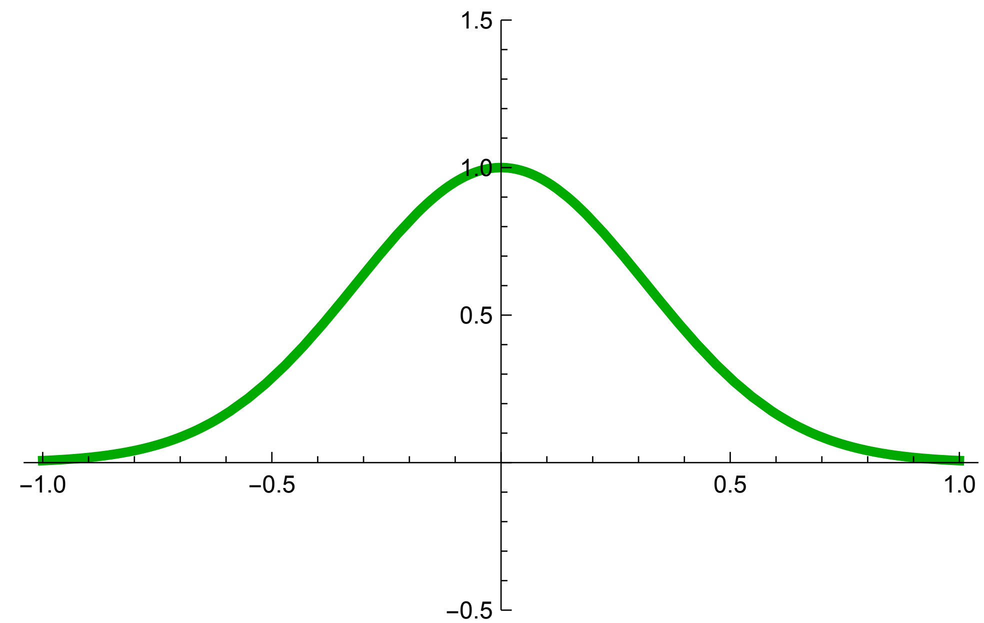


$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{otherwise} \end{cases} \qquad F(\omega) = a^2\, Sinc(a\pi\omega)^2$$

Fourier Transform Pairs

- Gaussian function



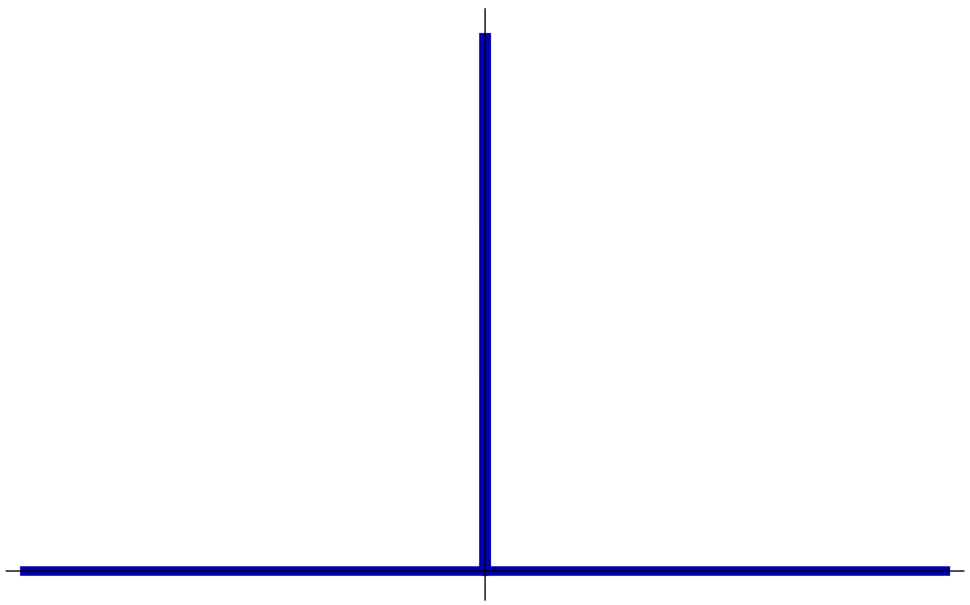


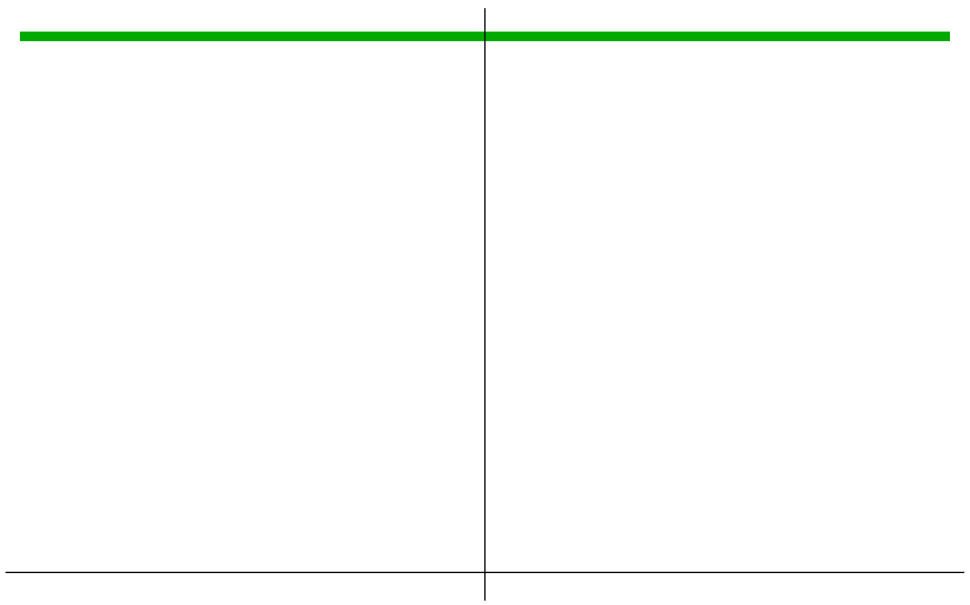
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$F(\omega) = \frac{1}{\sigma'\sqrt{2\pi}} e^{-\frac{\omega^2}{2\sigma'^2}}$$

Fourier Transform Pairs

- DiracDelta function (Impulse function)



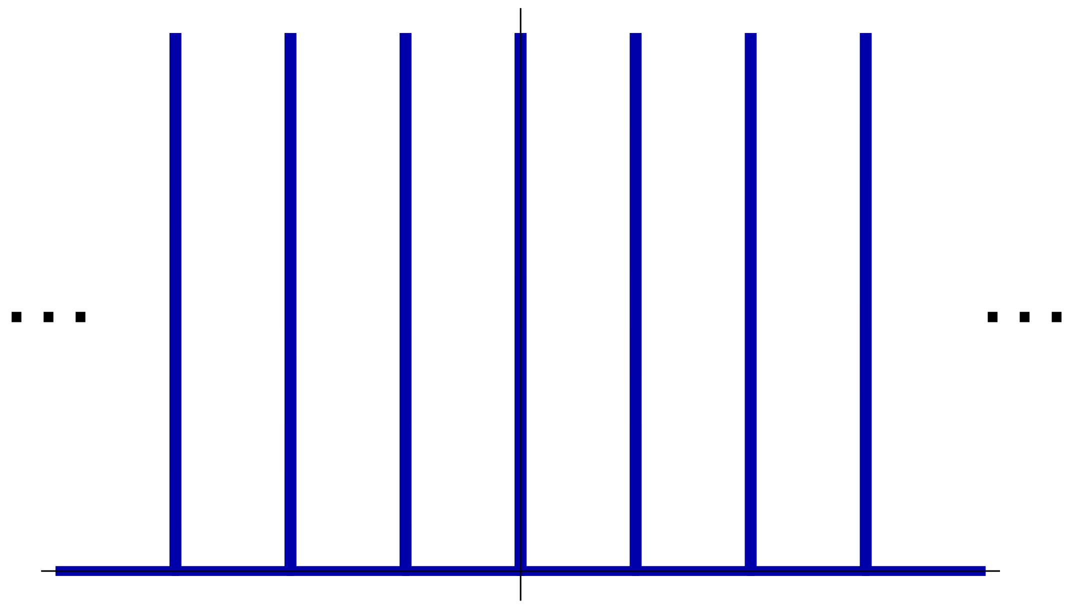


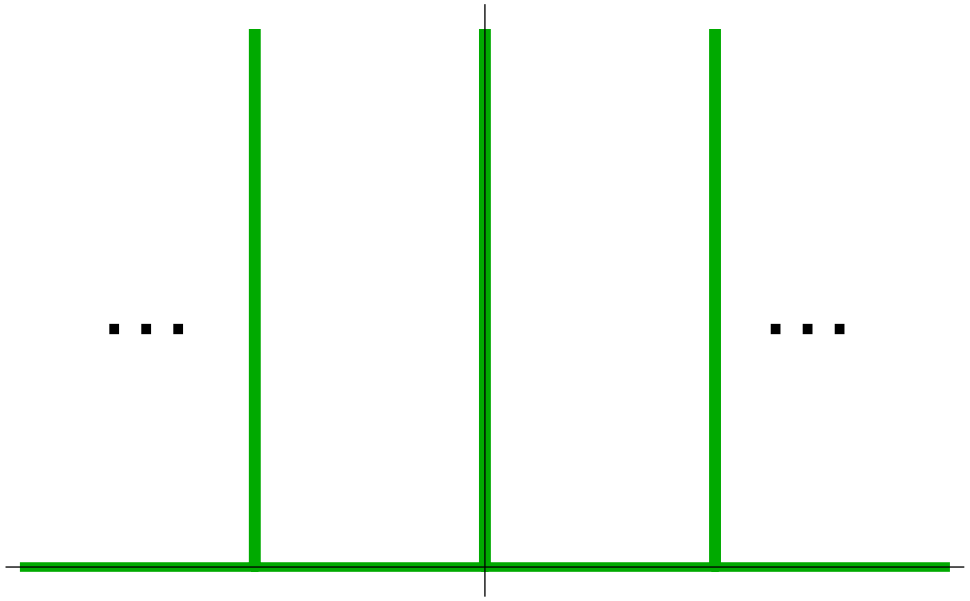
$$\delta(x) = \begin{cases} a & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\omega) = a$$

Fourier Transform Pairs

- DiracComb function





$$f(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

$$F(\omega) = f(a\omega)$$

2D Fourier Transforms

Fourier Transform in 2D

- 1D

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\omega x} dx$$

- 2D

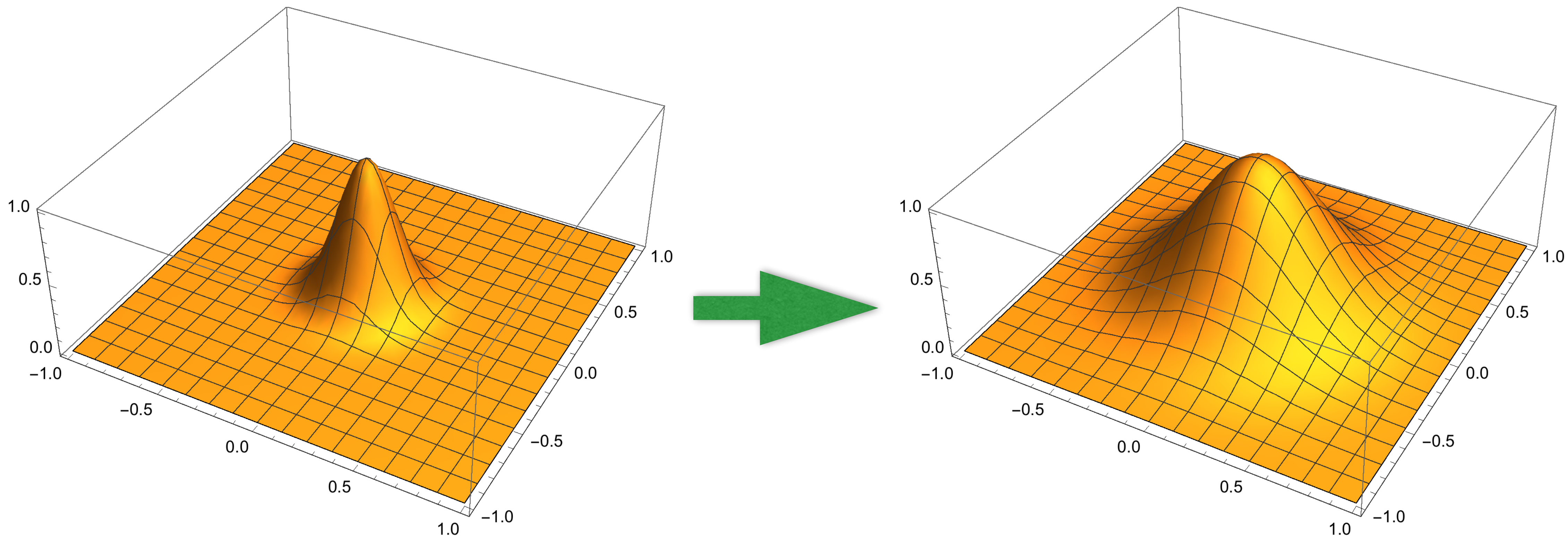
$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(x\omega_x + y\omega_y)} dx dy$$

$$F(\omega_x, \omega_y) = F_R(\omega_x, \omega_y) + iF_I(\omega_x, \omega_y)$$

$$||F(\omega_x, \omega_y)||^2 = (F_R(\omega_x, \omega_y) F_R(\omega_x, \omega_y) + F_I(\omega_x, \omega_y) F_I(\omega_x, \omega_y))$$

2D Fourier Transform Pairs

- Gaussian 2D function

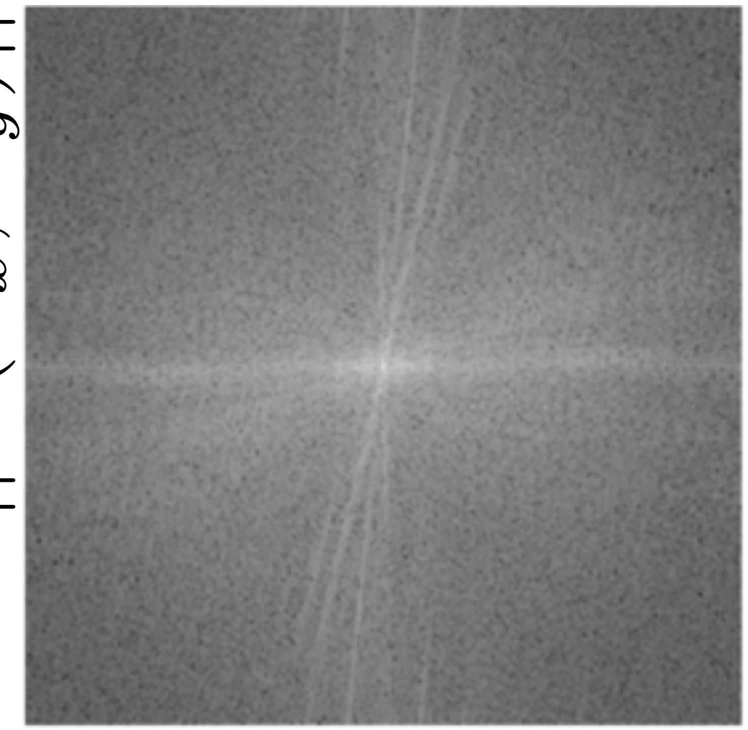


$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \qquad F(\omega_x,\omega_y) = \frac{1}{2\pi\sigma'^2} e^{-\frac{{\omega_x}^2+{\omega_y}^2}{2\sigma'^2}}$$

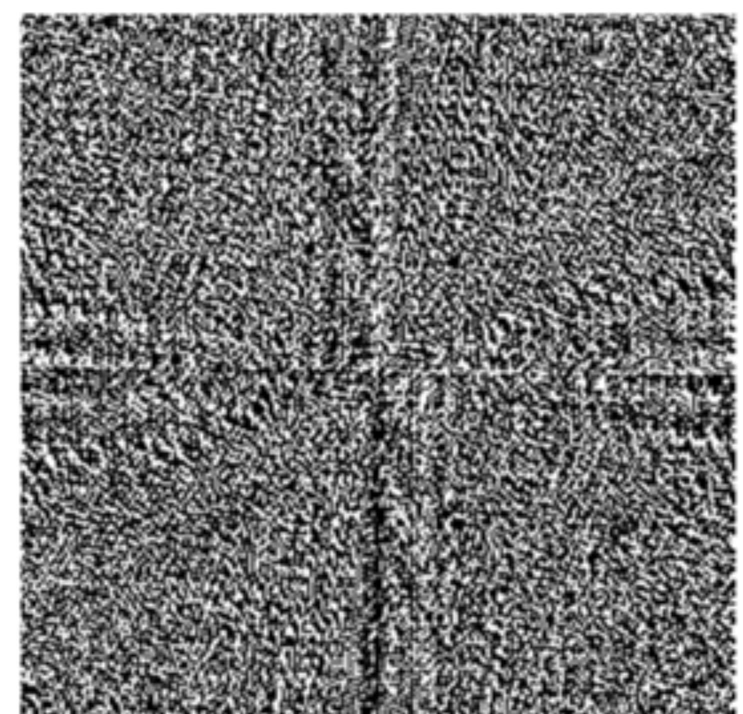
Fourier Transform in 2D

$f(x, y)$

$A = ||F(\omega_x, \omega_y)||$



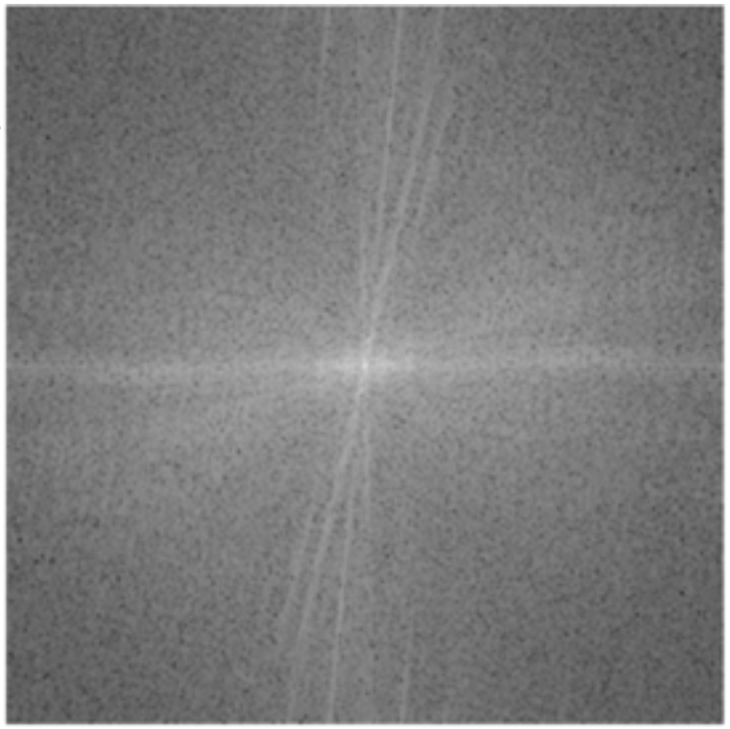
Phase



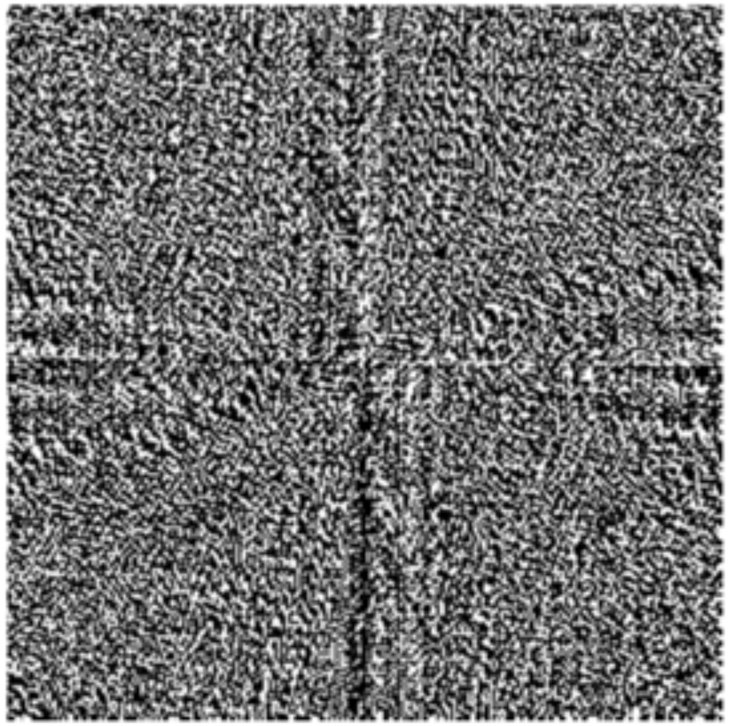
Slide after Andrew Zisserman

Inverse Fourier Transform in 2D

$A = ||F(\omega_x, \omega_y)||$



Phase



$f(x, y)$

Slide after Andrew Zisserman

Inverse Fourier Transform in 2D

- Inverse Fourier Transform

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x,\omega_y)e^{i2\pi(x\omega_x+y\omega_y)}d\omega_x d\omega_y$$

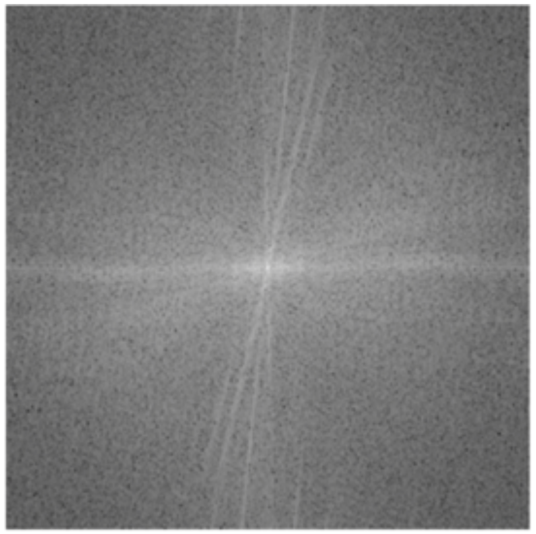
- Forward Fourier Transform

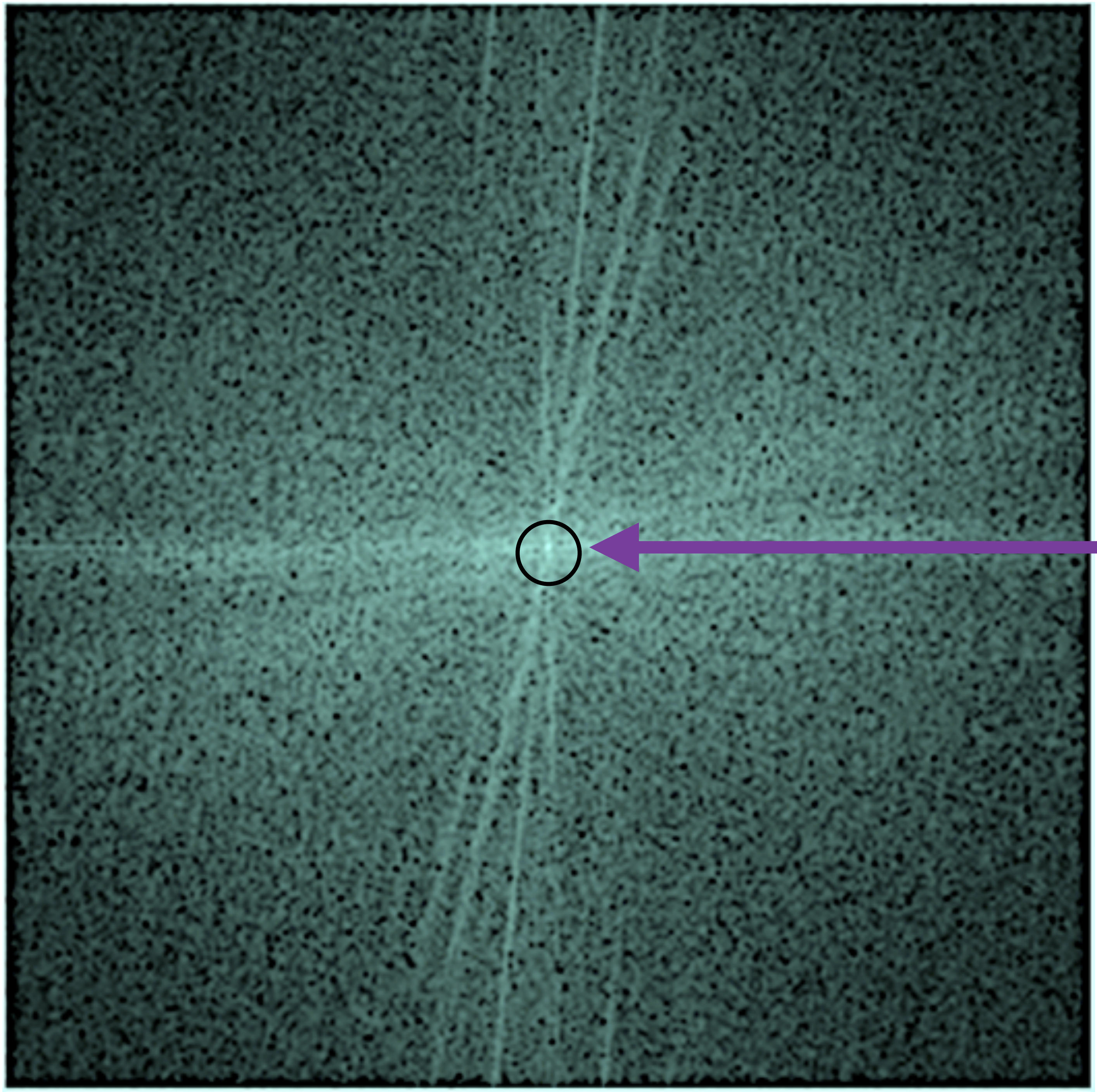
$$F(\omega_x,\omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(x\omega_x+y\omega_y)}dx dy$$

Amplitude and Phase

Fourier Transform: Amplitude

$$A = ||F(\omega_x, \omega_y)||$$

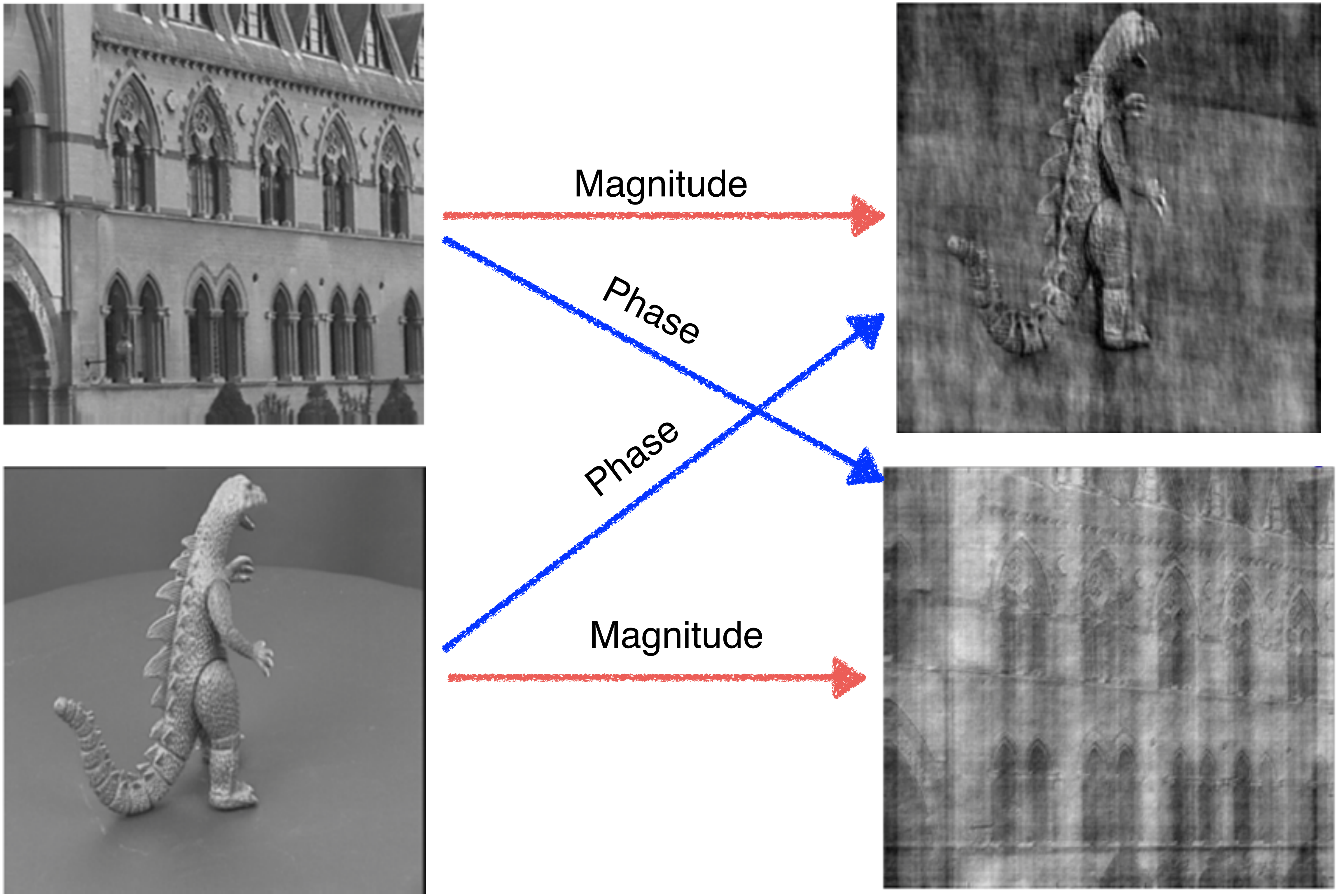




DC peak

$(\omega_x = 0, \omega_y = 0)$

What about the Phase ?



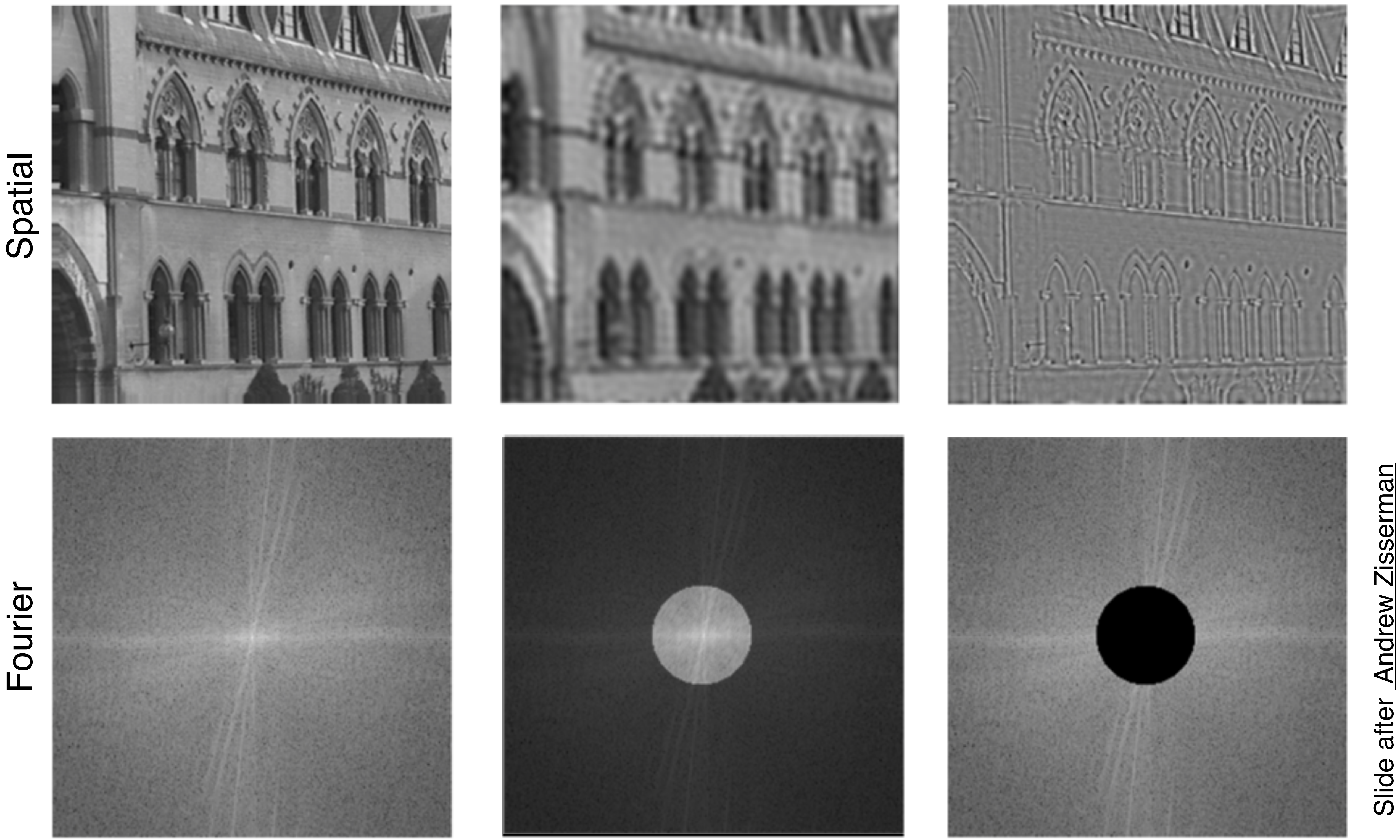
Slide after Andrew Zisserman

Purpose: Monte Carlo Integration

- We are interested in Amplitude or/and
- Power = Amplitude*Amplitude
- Phase is not important for random samples

Understanding Low and High Frequency Fourier Regions

Spatial operators in Fourier Domain



The Convolution Theorem

Convolution

- Filtering an image

$f(x)$									
	100	200	100	200	80	100	80	100	100

$g(x)$			
	1/4	1/2	1/4

	150	200	145	115	90	90	95	

$$(f * g)(x)$$

Slide after Andrew Zissermann

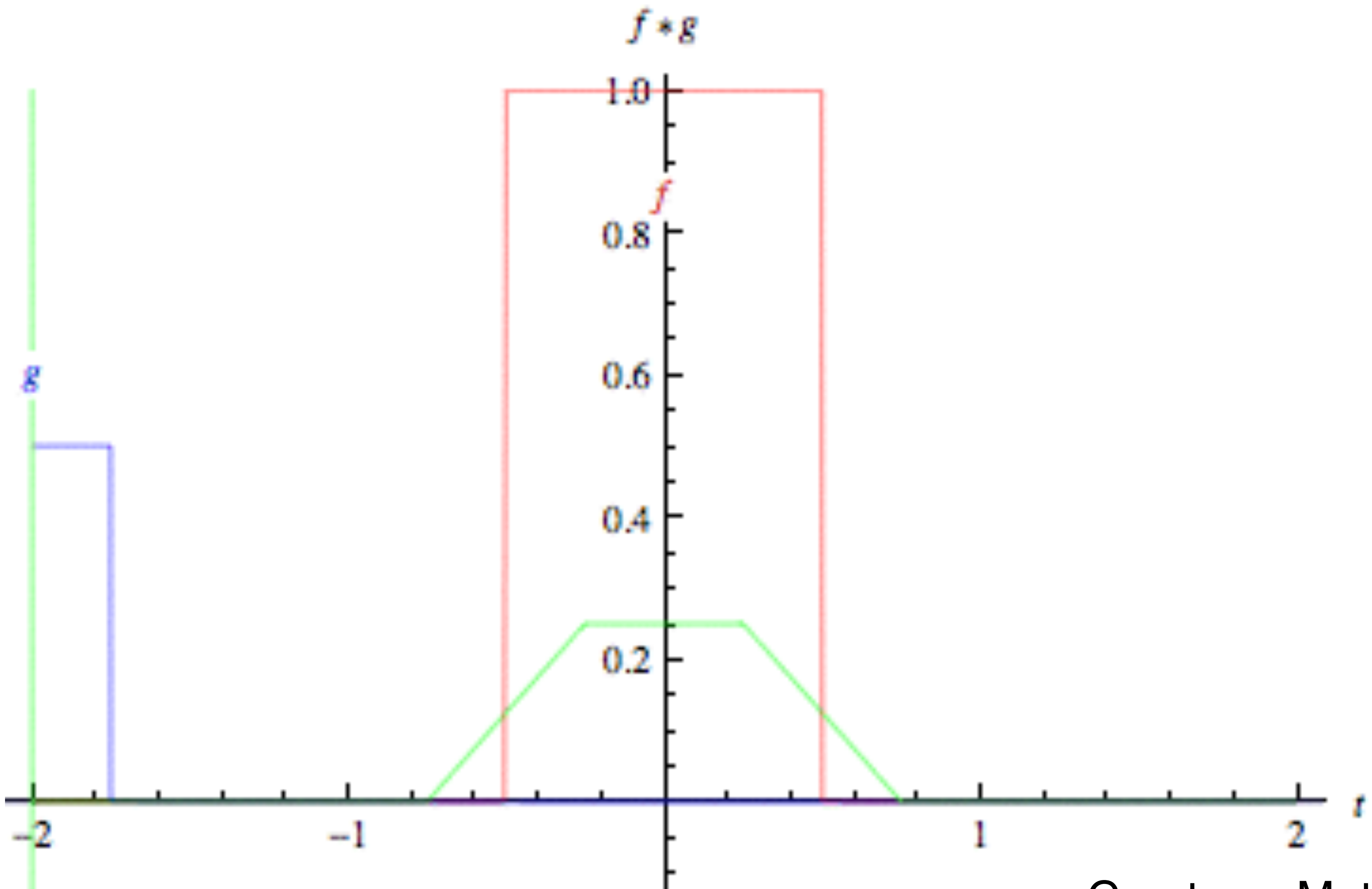
Convolution in 1D

$$(f * g)(\tau) = \int_{-\infty}^{\infty} f(x)g(\tau - x)dx$$

Convolution operator

Convolution in 1D

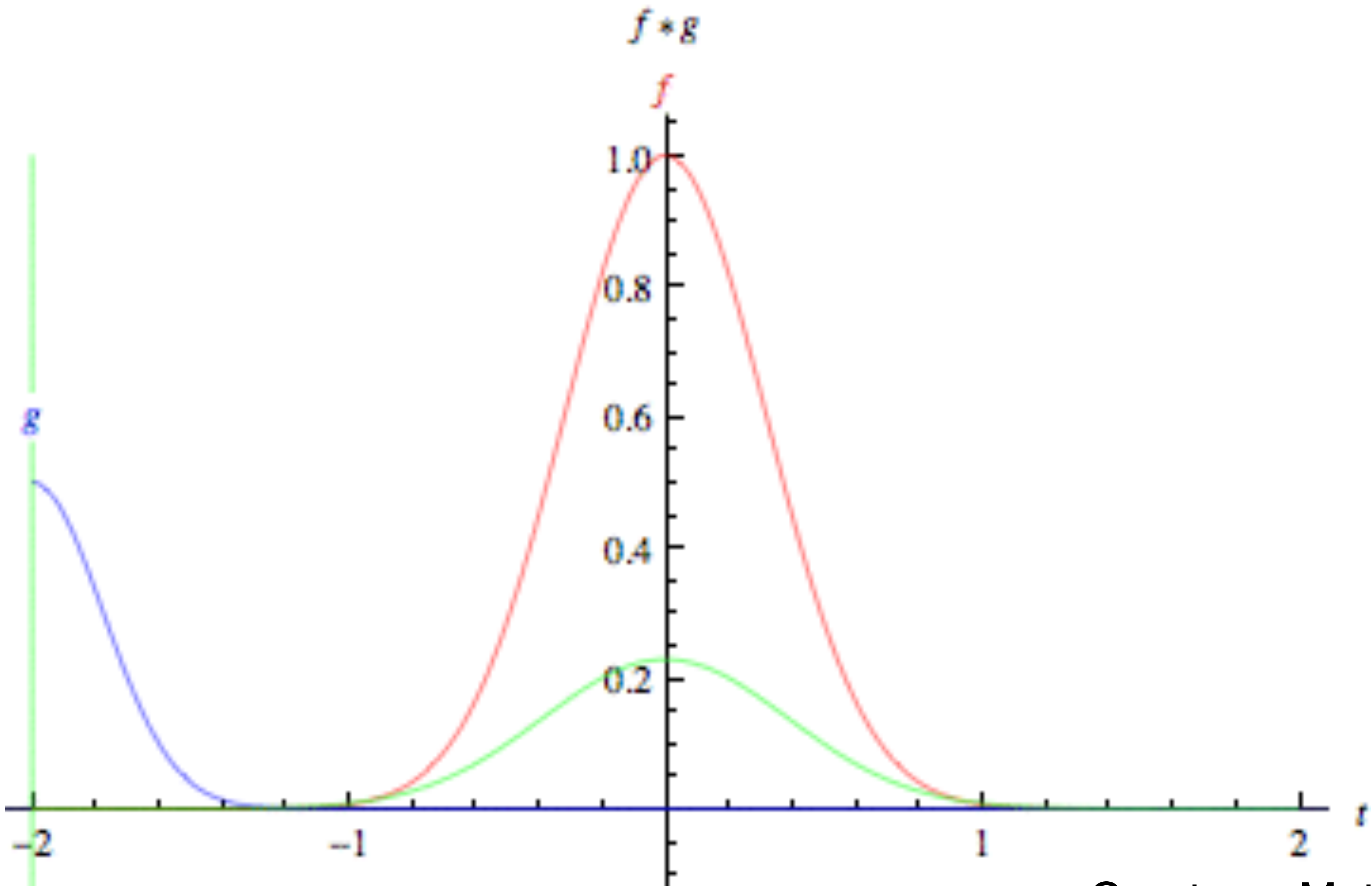
- Box Convolved with a Box



Courtesy: Math Wolfram

Convolution in 1D

- Gaussian Convolved with a Gaussian



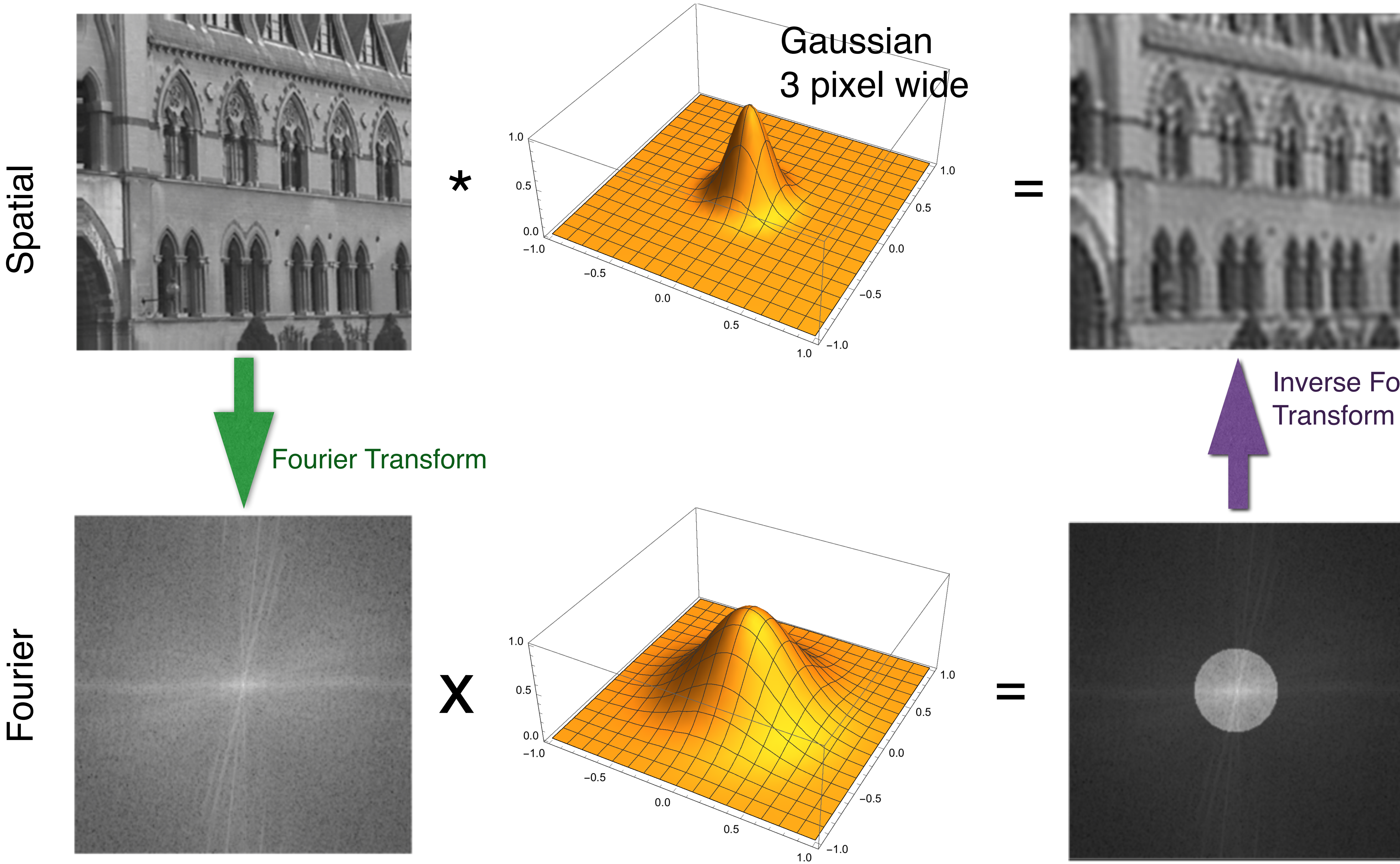
Courtesy: Math Wolfram

Convolution

- Multiplication vs Convolution

Spatial Domain	*Fourier Domain*
Multiplication $f(x)\,g(x)$	Convolution $F(\omega) * G(\omega)$
Convolution $f(x) * g(x)$	Multiplication $F(\omega)G(\omega)$

Convolution in 2D



Slide after Andrew Zisserman

Visual Break

The Alchemist's Letter

Monte Carlo Integration

Monte Carlo Integration

$$I = \int_a^b f(x)dx$$

$$I_N = \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

$$I_N = \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$

Monte Carlo Integration

$$I = \int_0^1 f(x)dx$$

$$I_N = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

$$I_N = \frac{1}{N}\int_0^1 f(x)\sum_{i=1}^{N}\delta(x - x_i)dx$$

Monte Carlo Integration

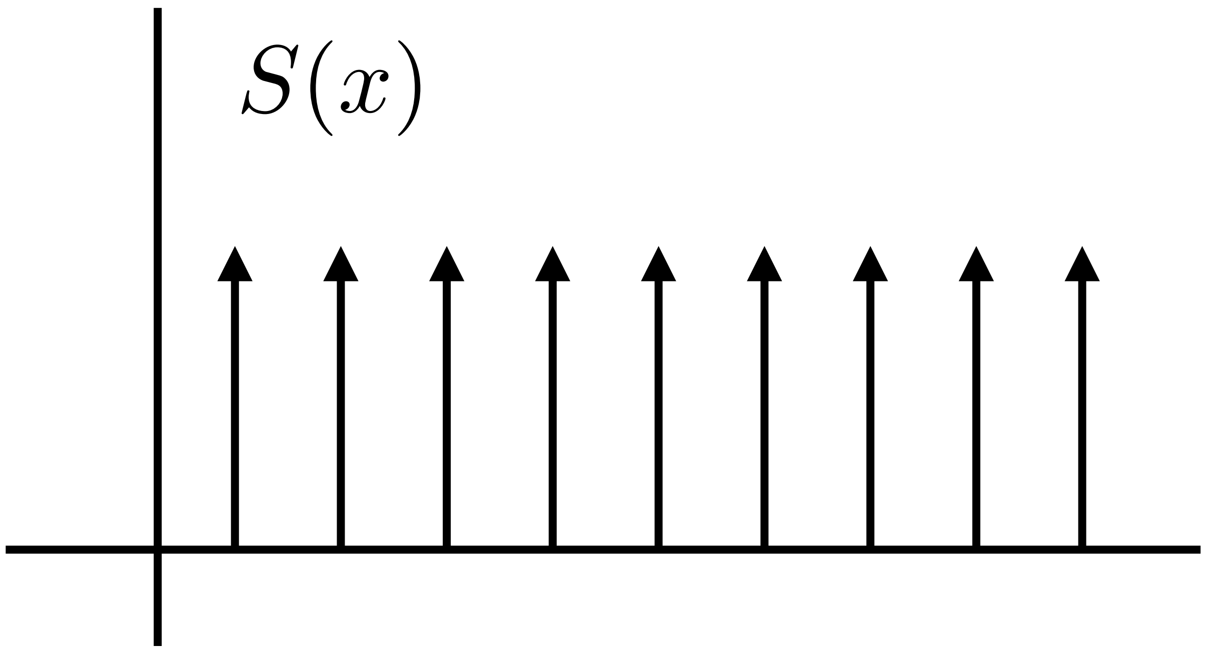
$$I_N = \int_0^1 f(x) \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) dx$$

$$I_N = \int_0^1 f(x) S(x) dx$$

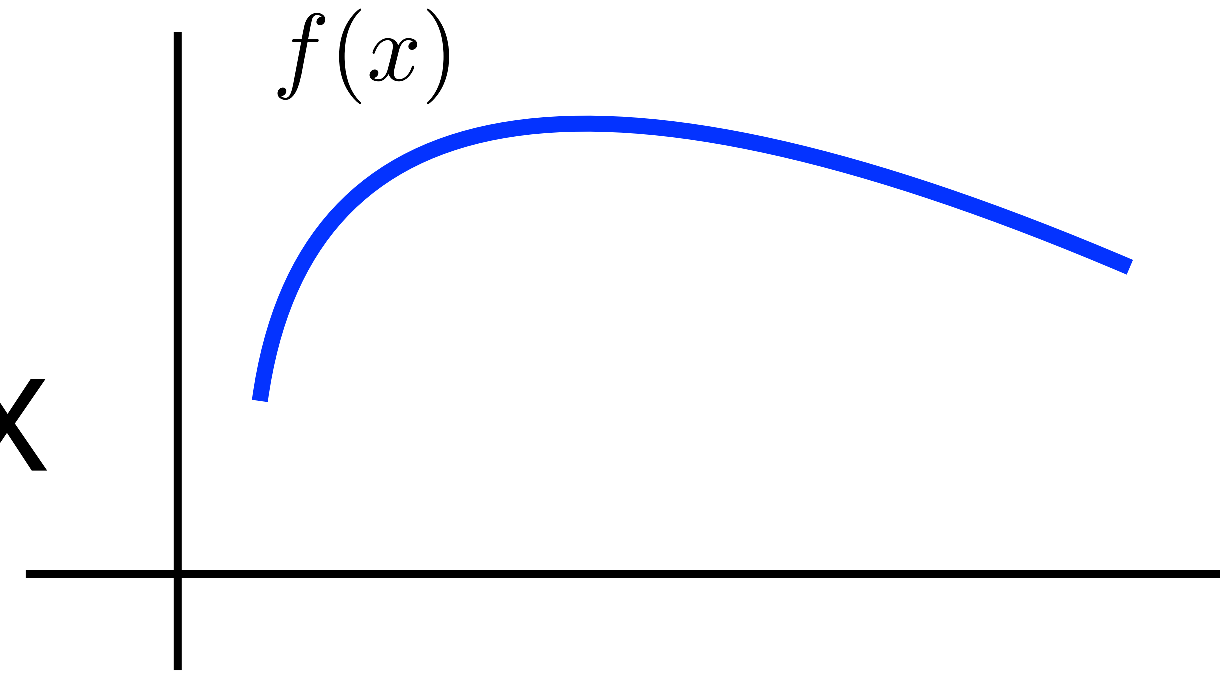
$$S(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$$

Frequency Analysis of Monte Carlo Integration

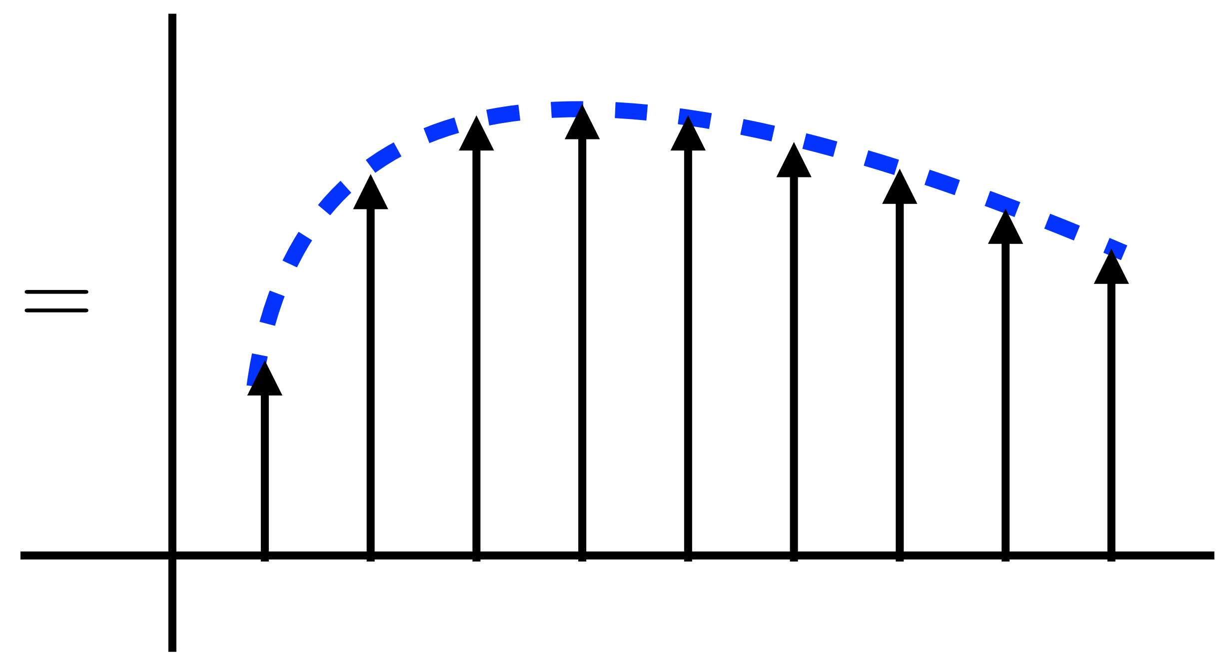
Spatial: Monte Carlo Integration



X

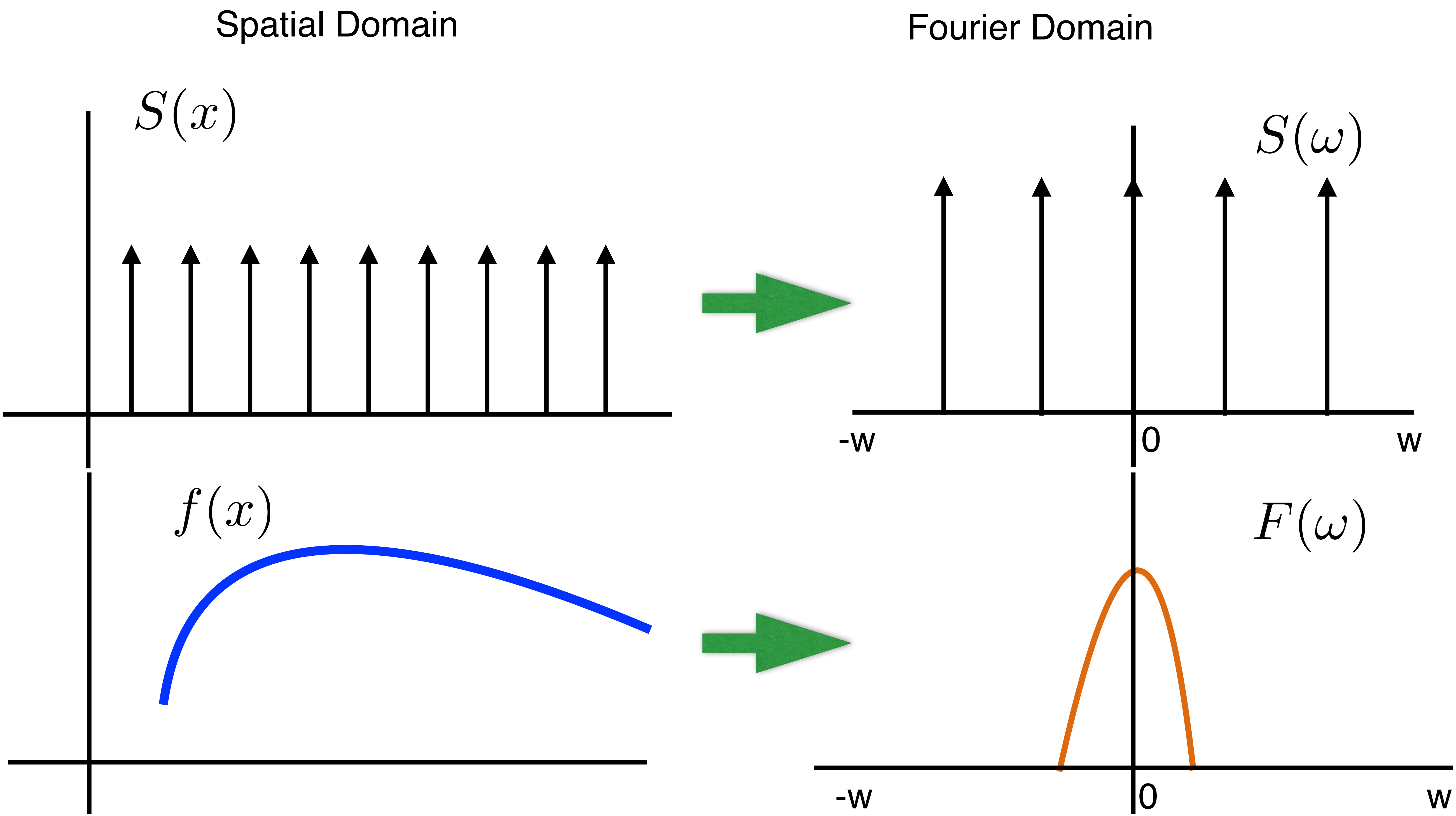


$f(x)S(x) =$



$$I_N = \int_0^1 f(x)S(x)\,dx$$

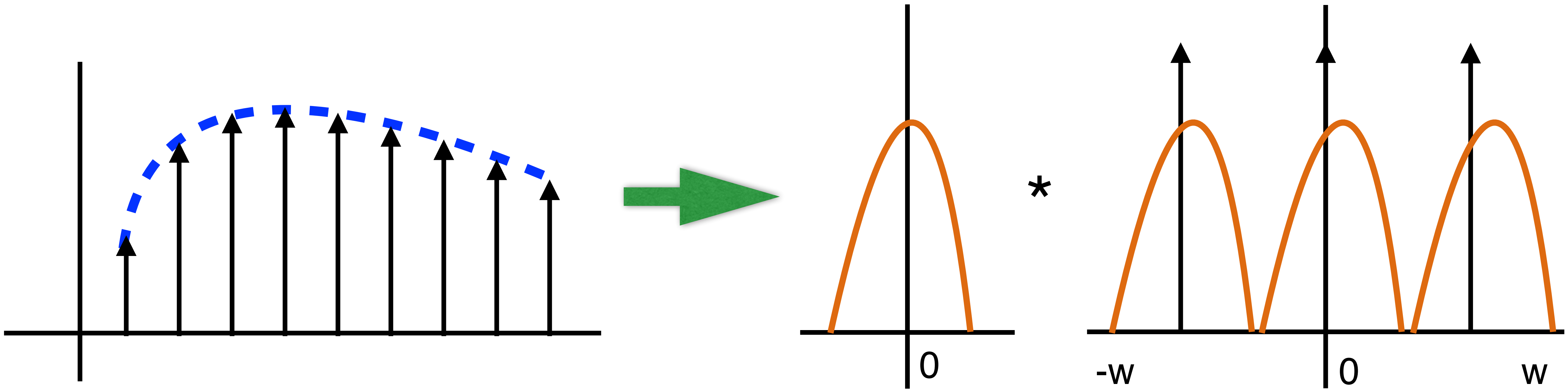
Fourier: Monte Carlo Integration



Fourier: Monte Carlo Integration

Spatial Domain

Fourier Domain



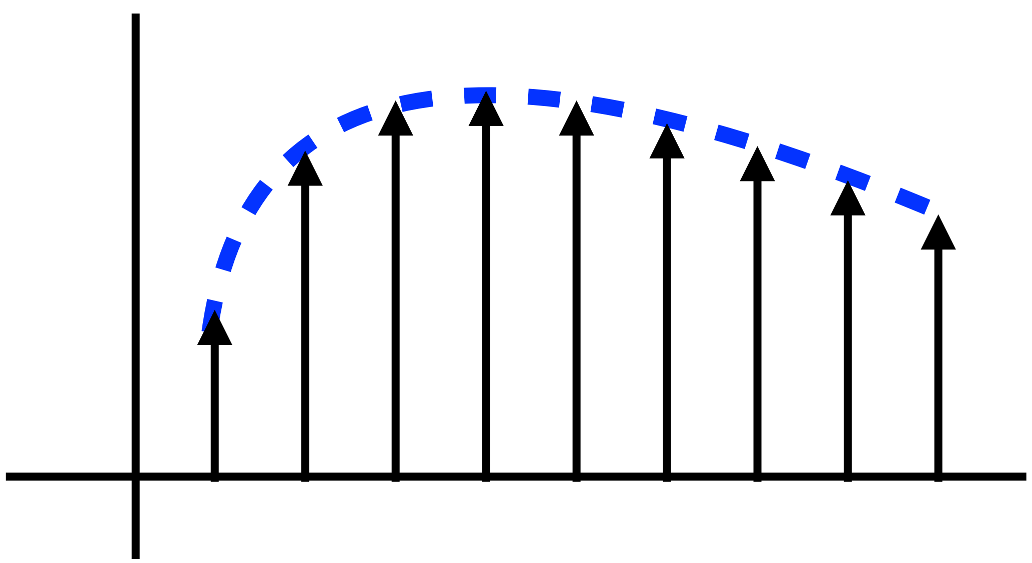
$$f(x)S(x)$$

$$F(\omega) * S(\omega)$$

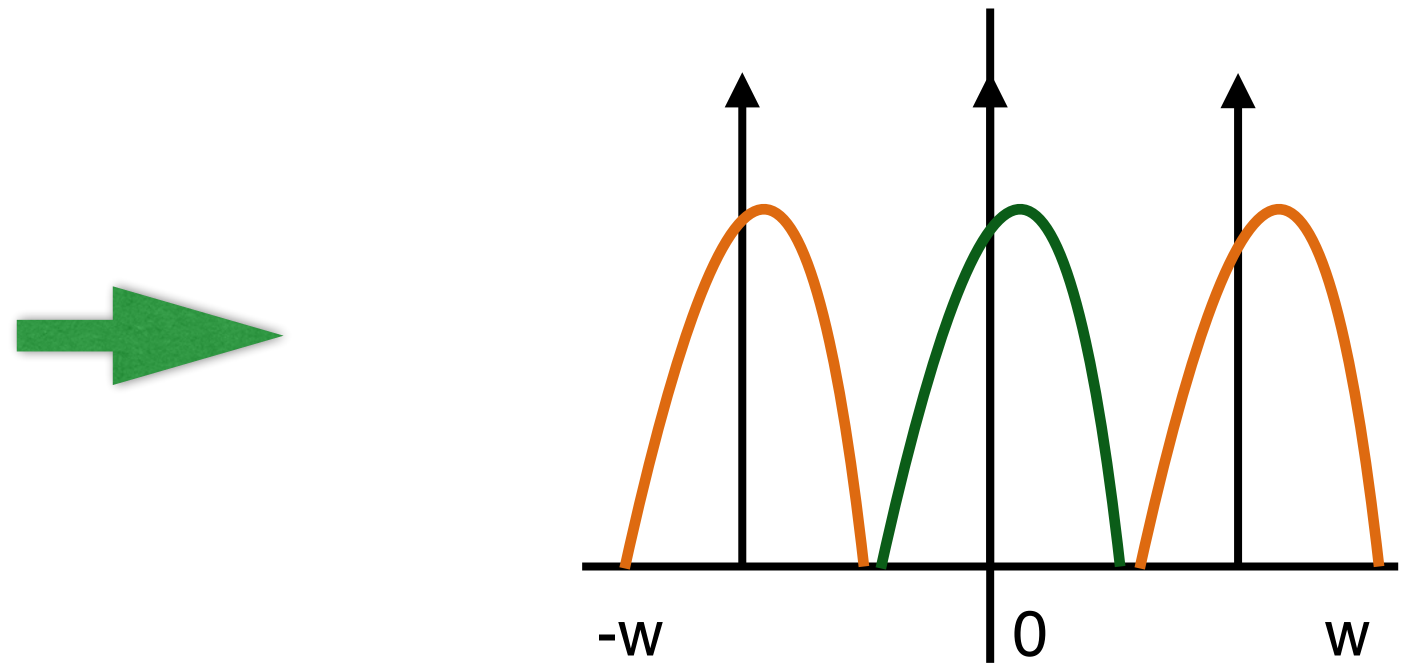
Fourier: Monte Carlo Integration

Spatial Domain

Fourier Domain



$$f(x)S(x)$$



$$F(\omega) * S(\omega)$$

Monte Carlo Integration in the Fourier Domain

$$I_N = \int \overline{F(-\omega)} S(\omega) d\omega$$

Fourier: Monte Carlo Integration

Monte Carlo Integration in the Spatial Domain

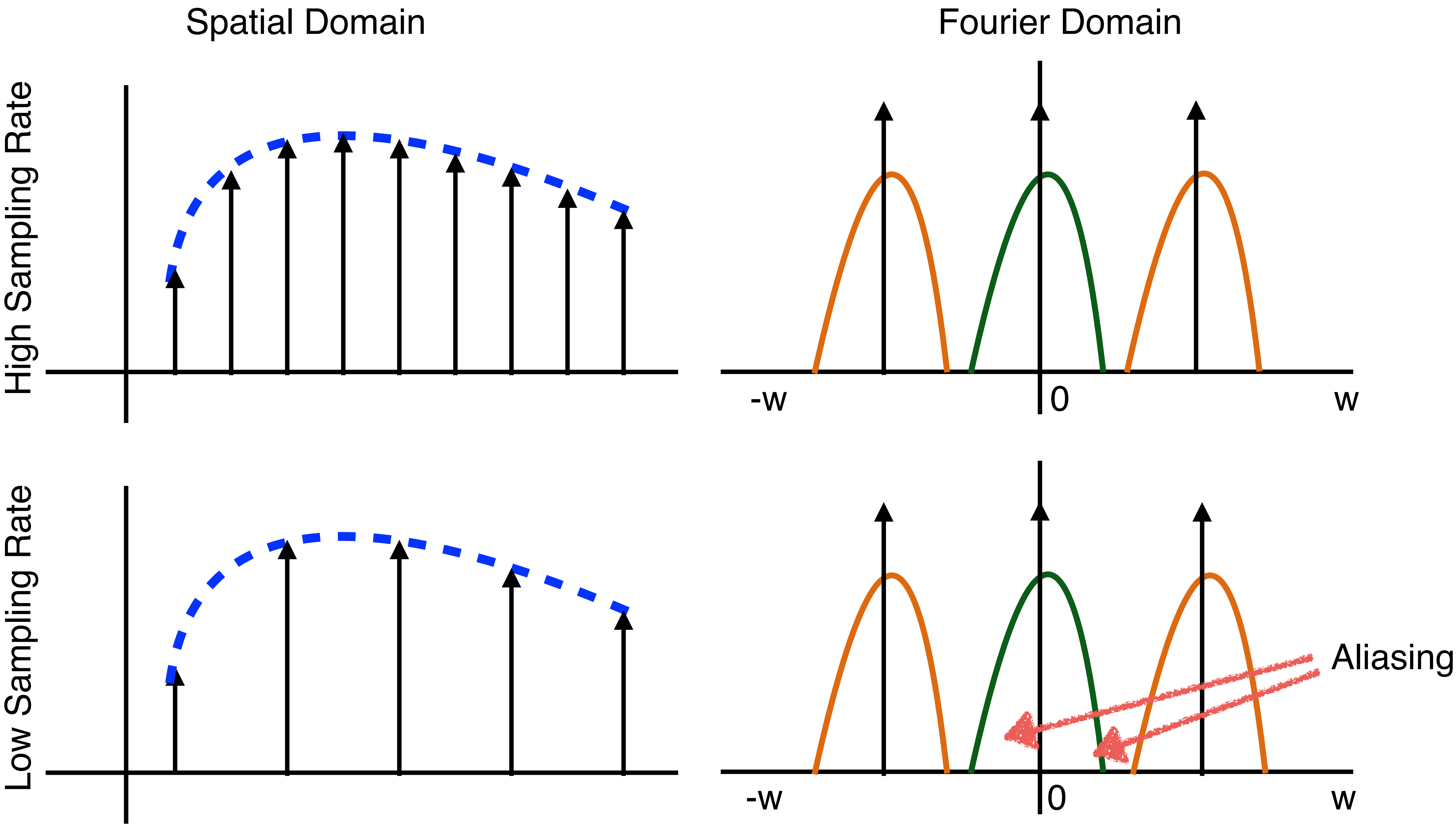
$$I_N = \int_0^1 f(x)S(x)dx$$

Monte Carlo Integration in the Fourier Domain

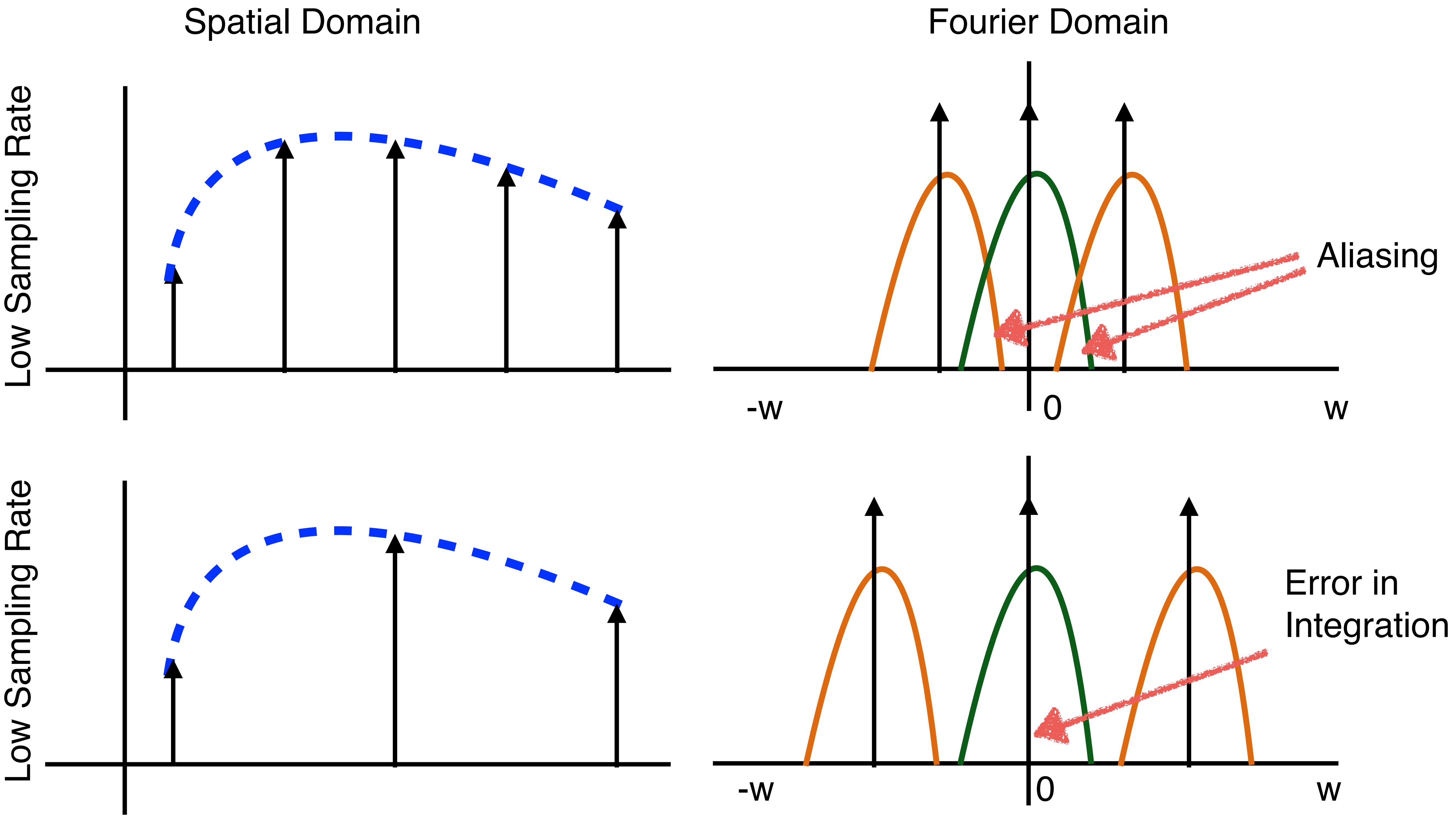
$$I_N = \int_{-\infty}^{\infty} \overline{F(-\omega)} S(\omega) d\omega$$

Fredo Durand [2011]

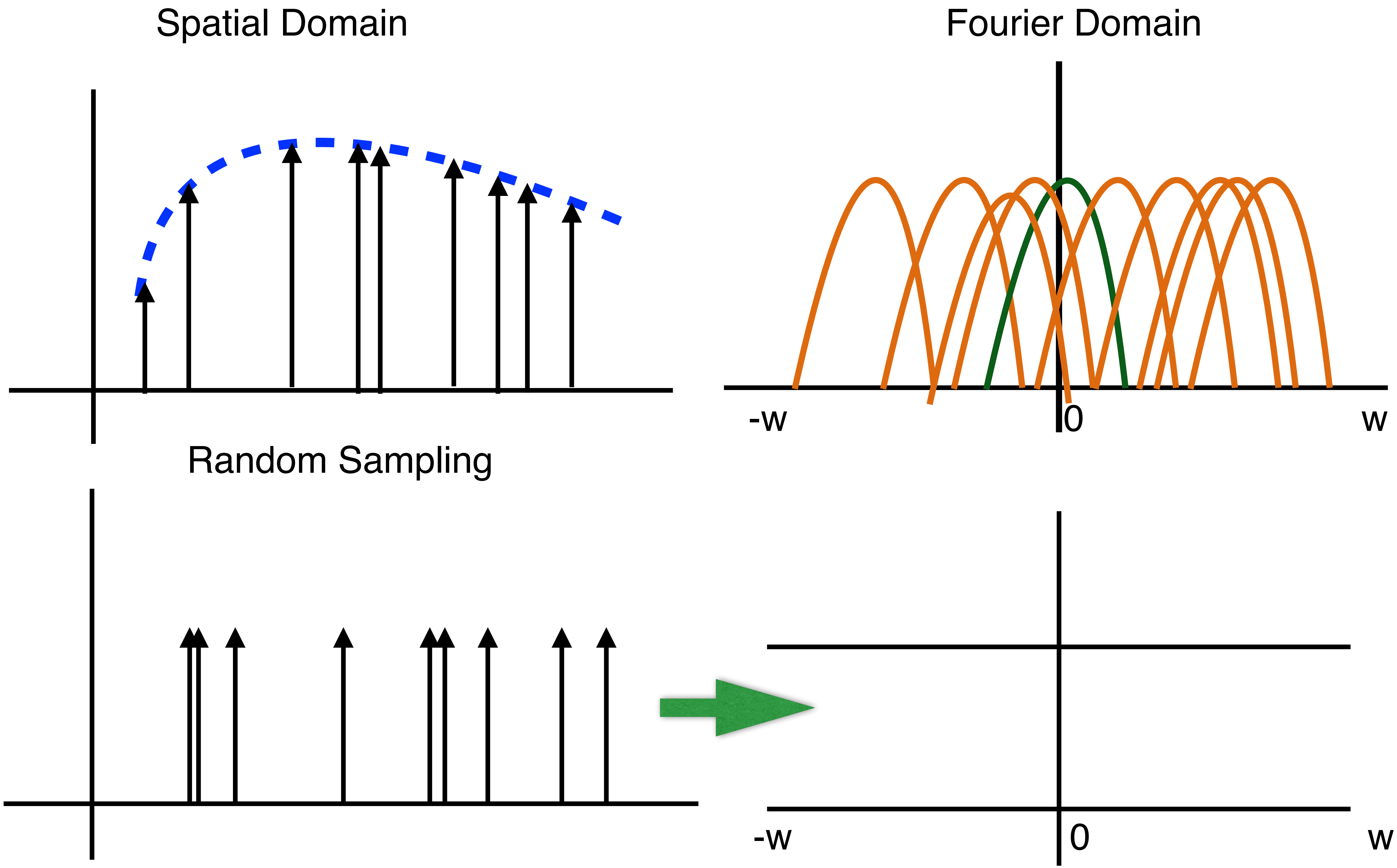
Fourier: Monte Carlo Integration



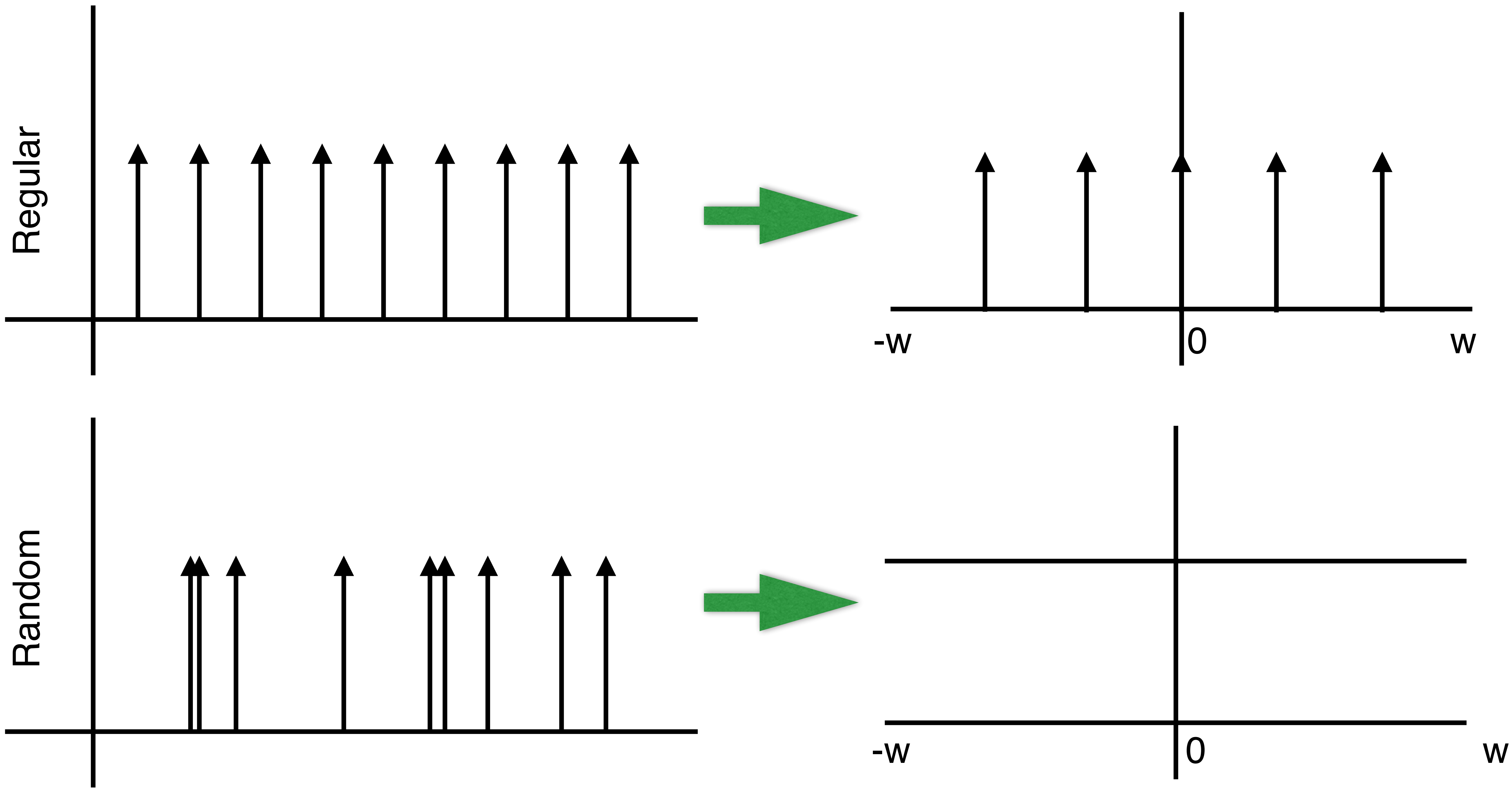
Fourier: Monte Carlo Integration



Fourier: Monte Carlo Integration

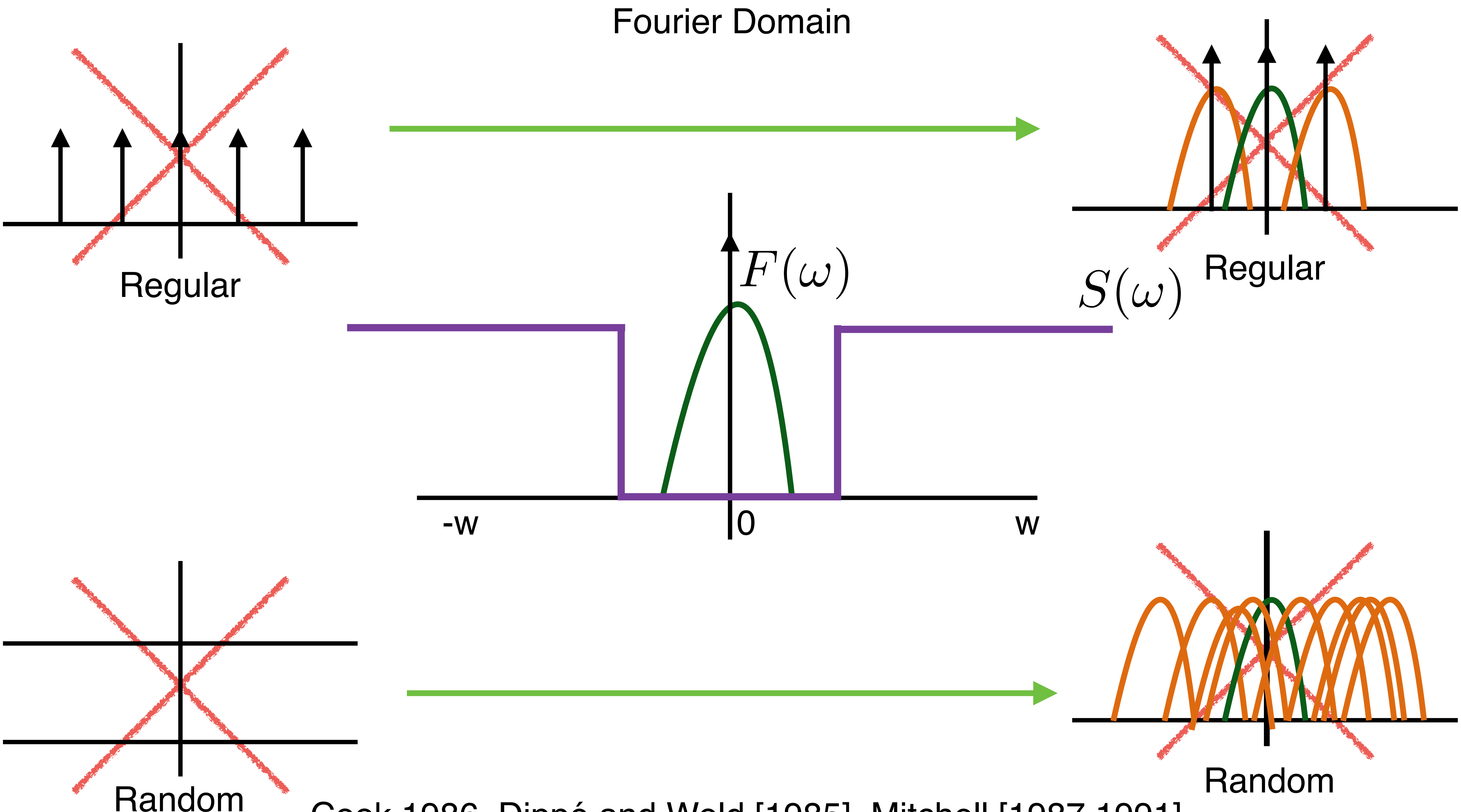


Fourier: Monte Carlo Integration



What is the desired Sampling Power Spectrum?

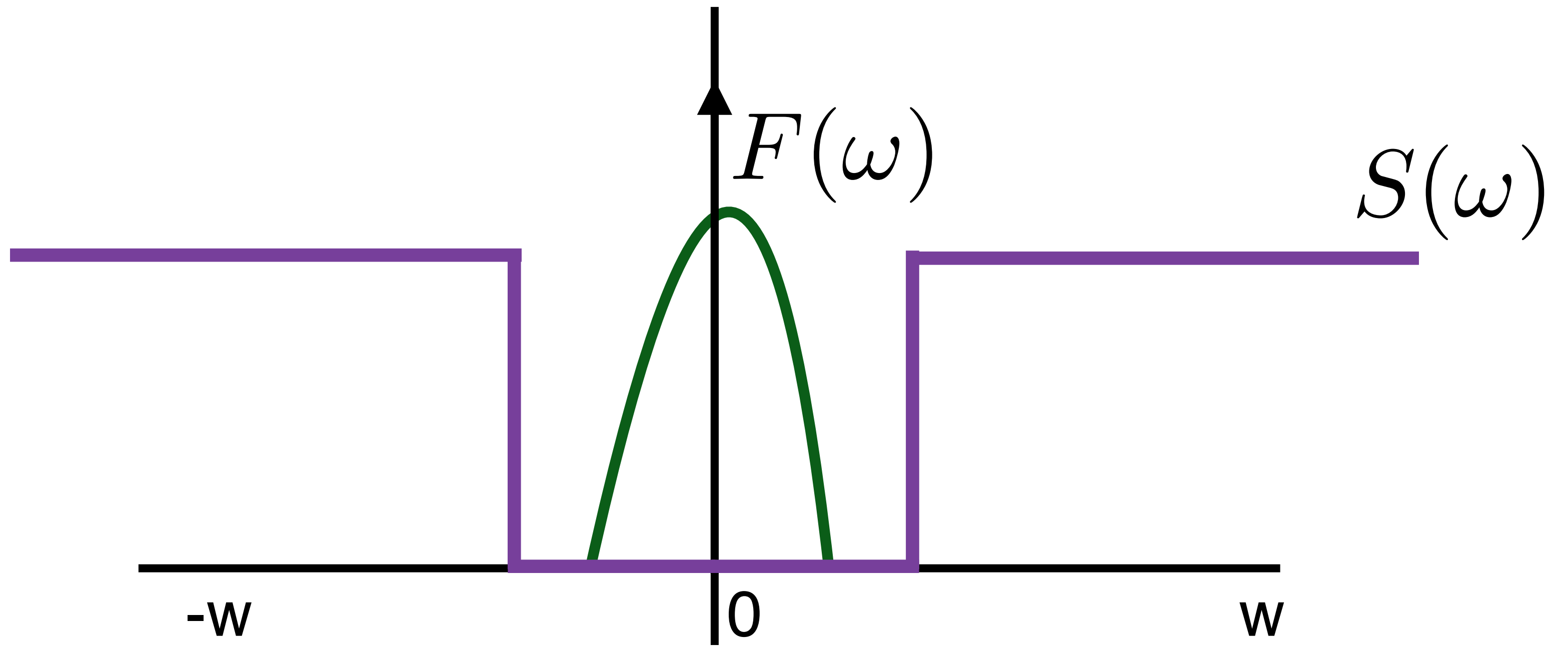
Desired Sampling Frequency Spectrum



Cook 1986, Dippé and Wold [1985], Mitchell [1987,1991]

Desired Sampling Frequency Spectrum

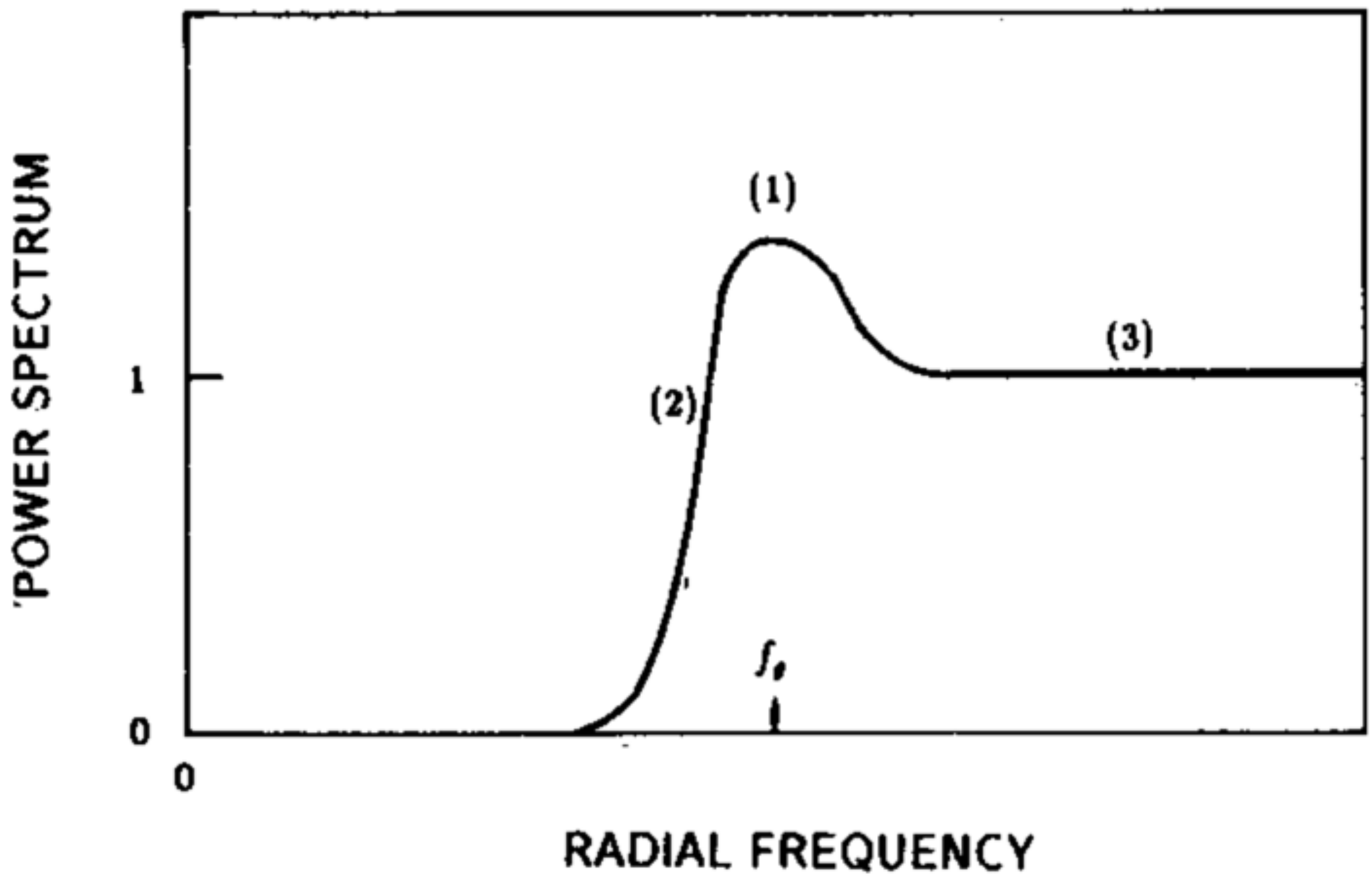
Fourier Domain



Cook 1986, Dippé and Wold [1985], Mitchell [1987,1991]

Blue Noise Power Spectrum

- Robert Ulichney [1987]



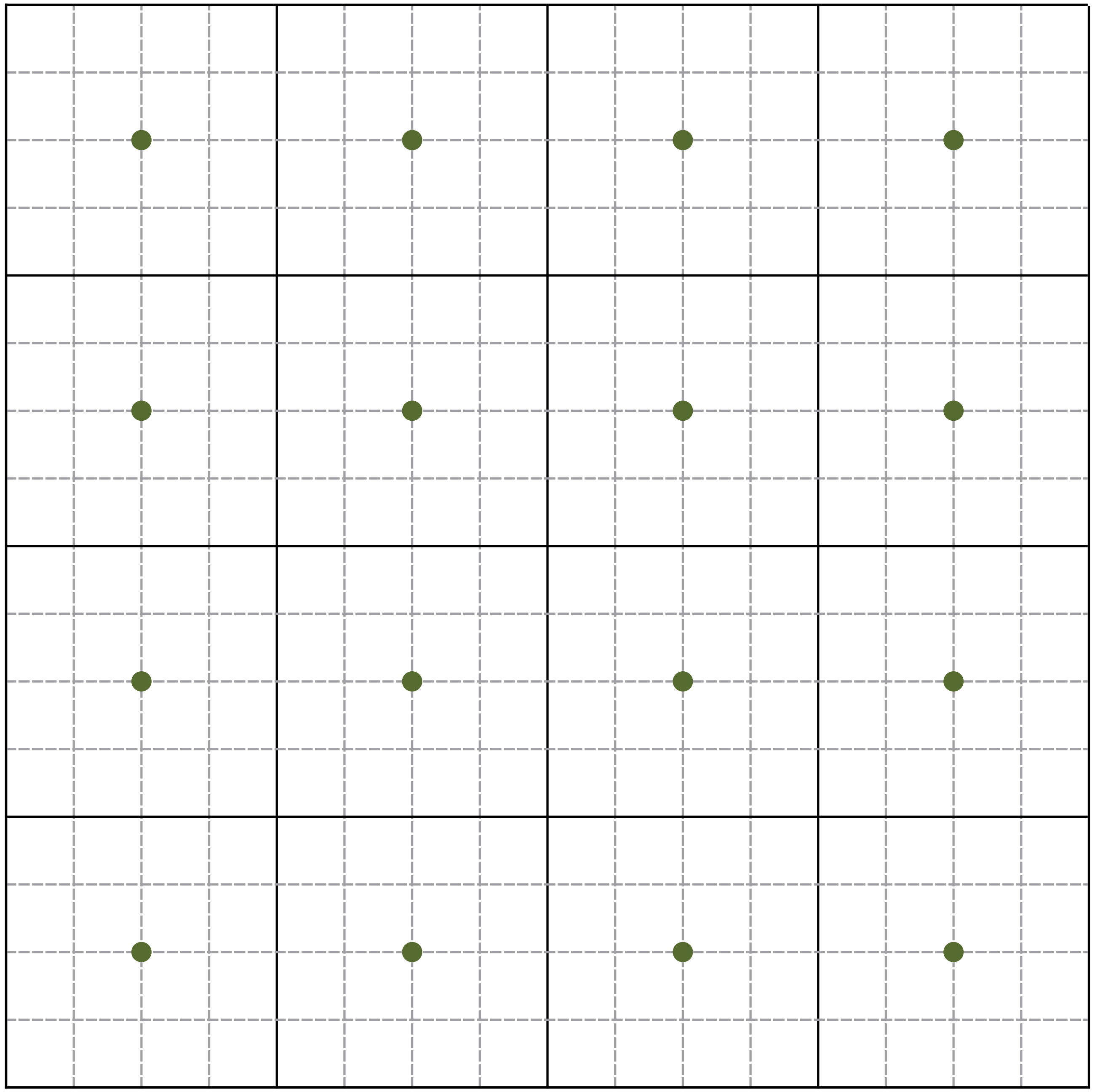
Visual Break

Dragon Hunter by CG Society

Frequency Analysis of Sampling Patterns

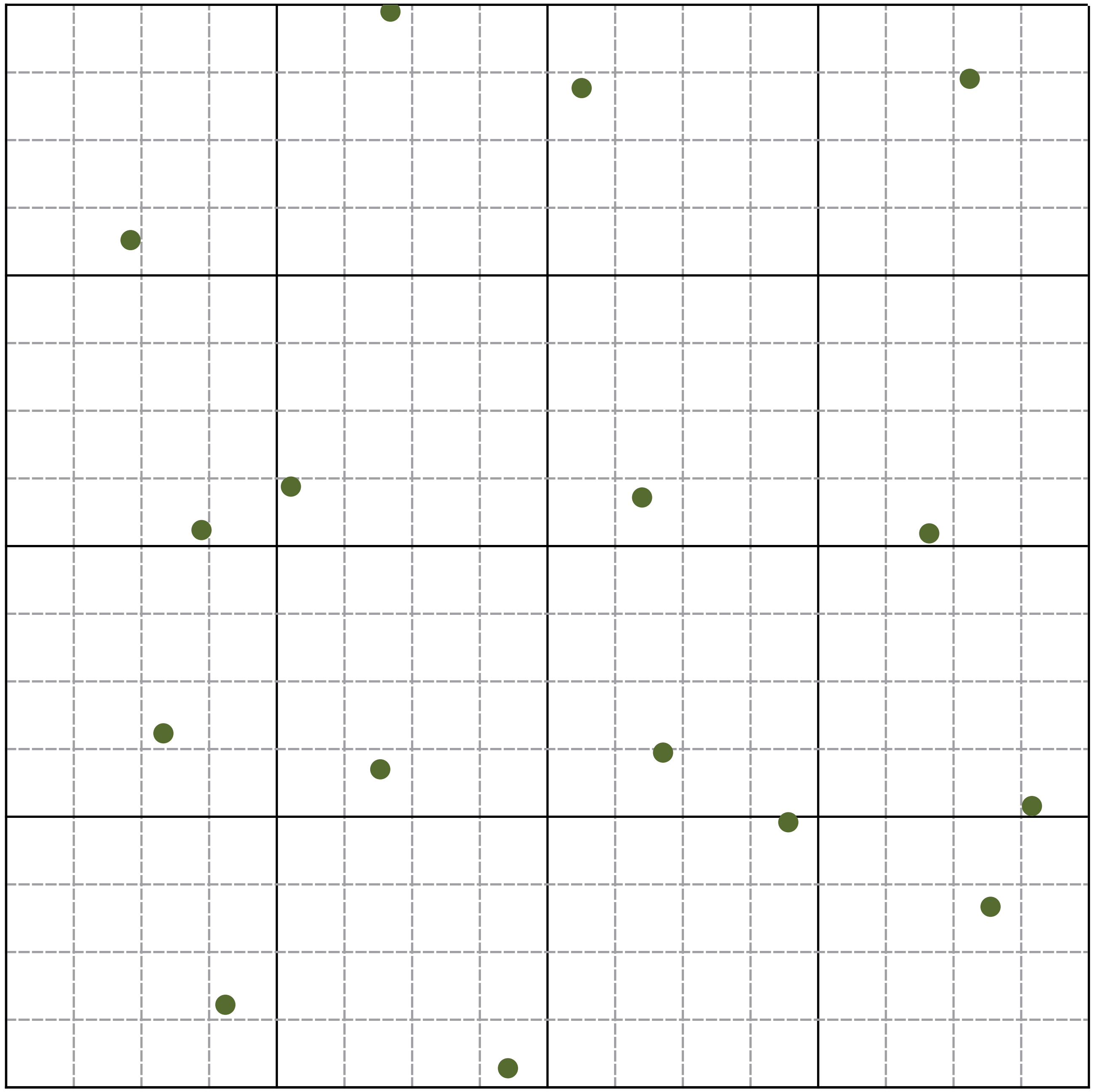
Sampling Patterns

- Regular



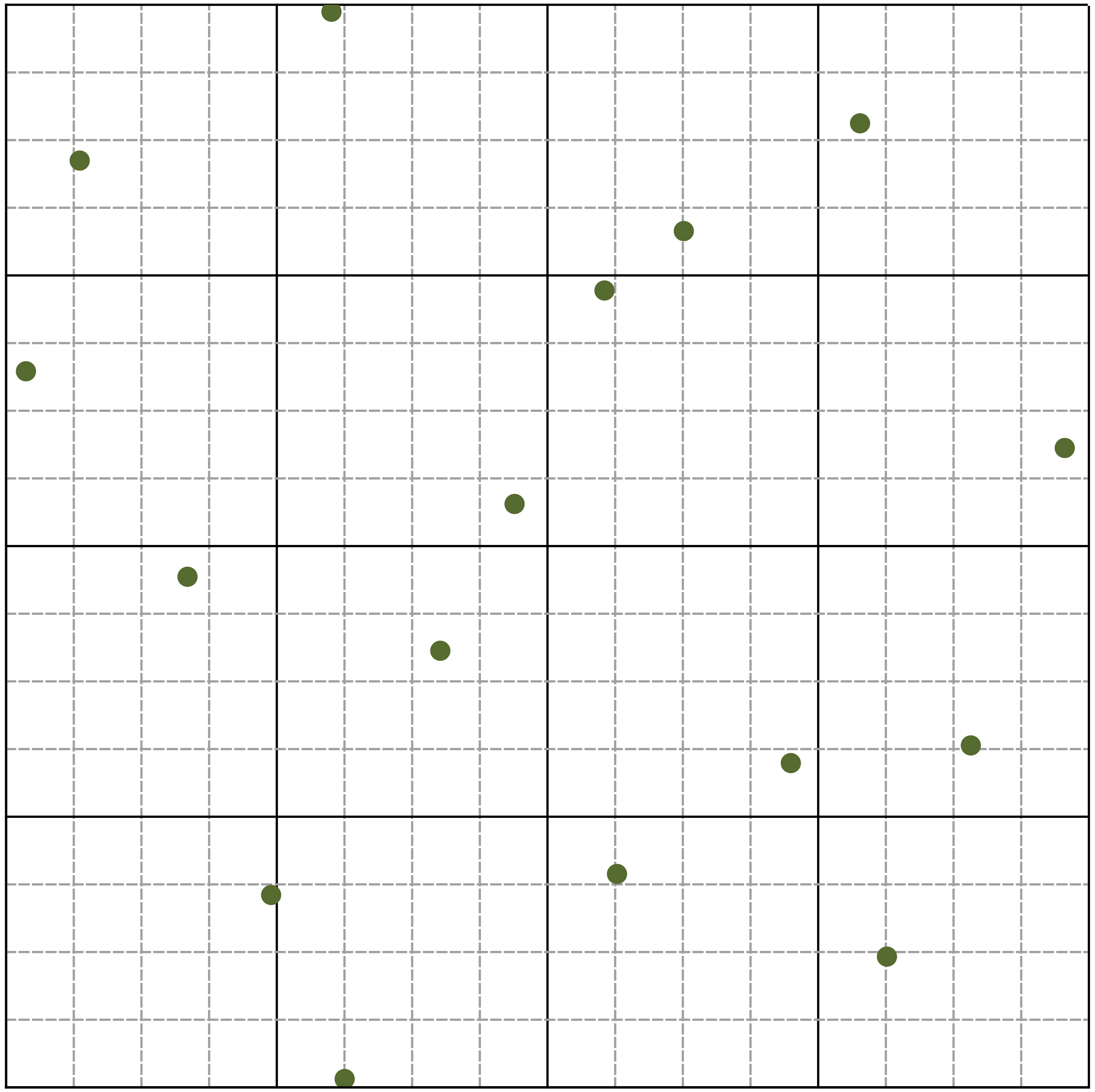
Sampling Patterns

- Jitter



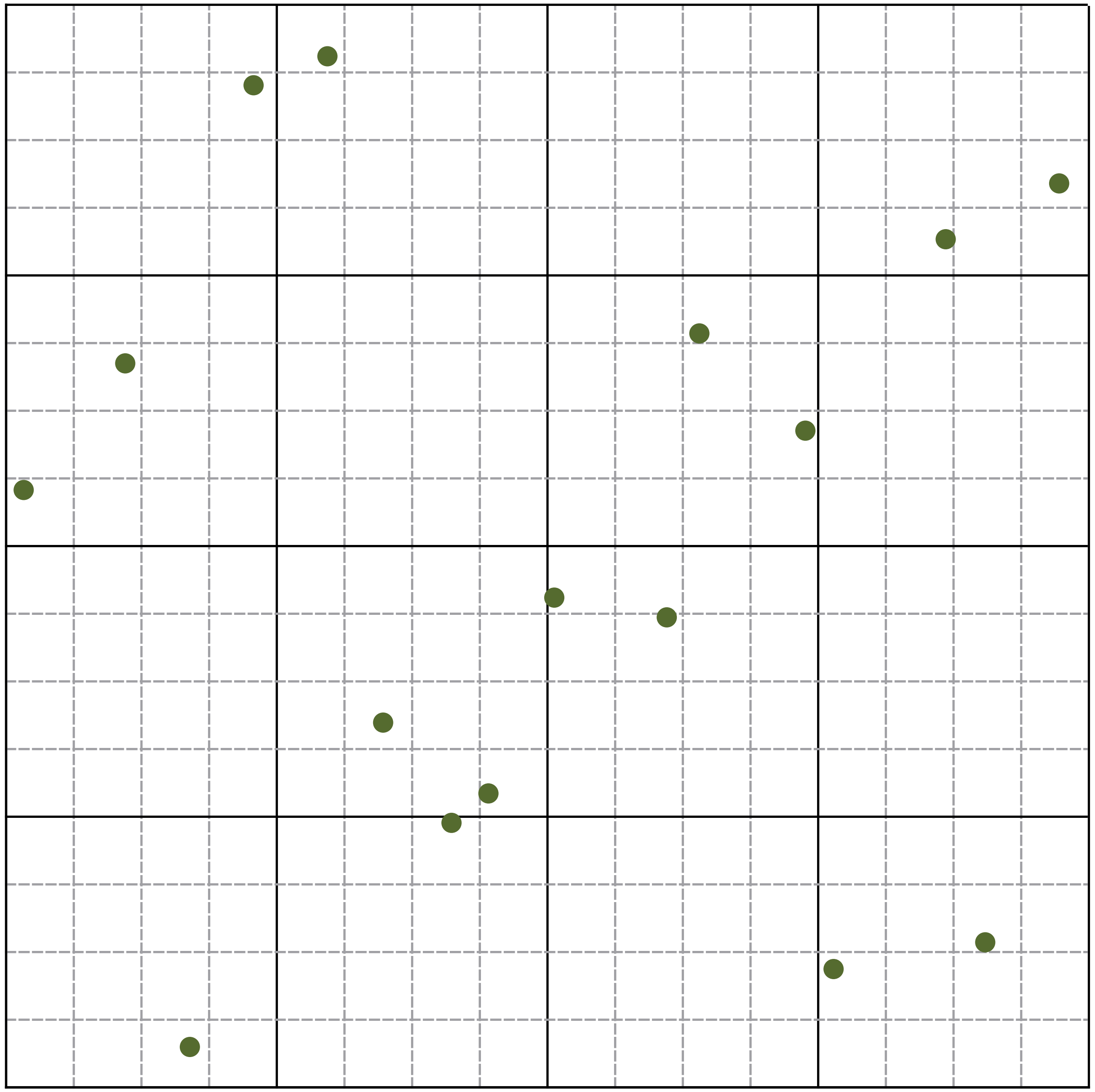
Sampling Patterns

- Multi jitter



Sampling Patterns

- N-rooks



Frequency Spectrum

- Fourier Transform of Point Samples (2D)
- Fourier coefficient at (ω_x, ω_y)

```
std::complex<double> fourierCoeff(0.,0.);

double fx = 0.0, fy = 0.0;

for(int i=0; i < N; i++){
    double exp = -2π * (ωx * points[2*i+0] + ωy * points[2*i+1]);
    fx += cos(exp);
    fy += sin(exp);
}

fourierCoeff.real(fx);
fourierCoeff.imag(fy);
```

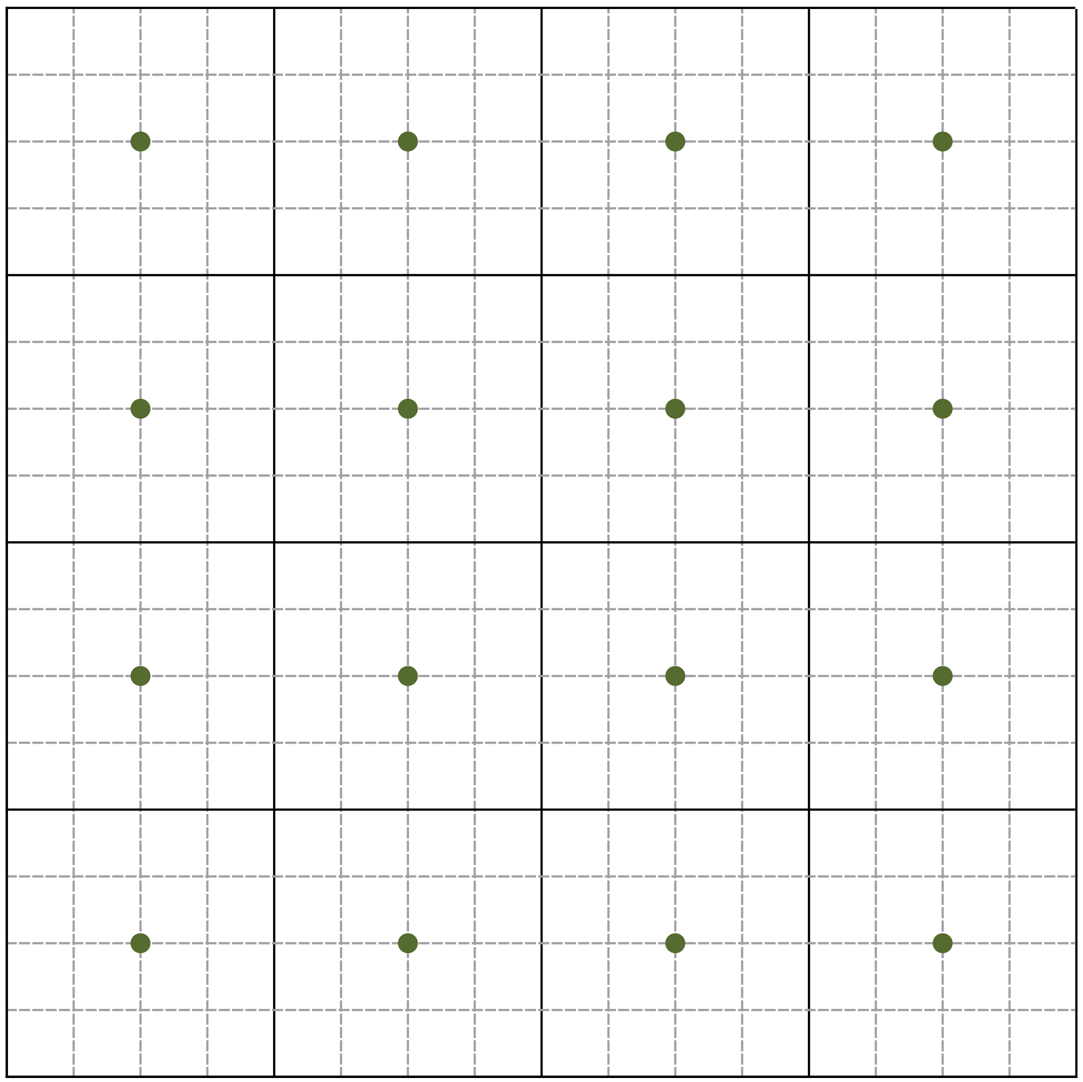
Power Spectrum

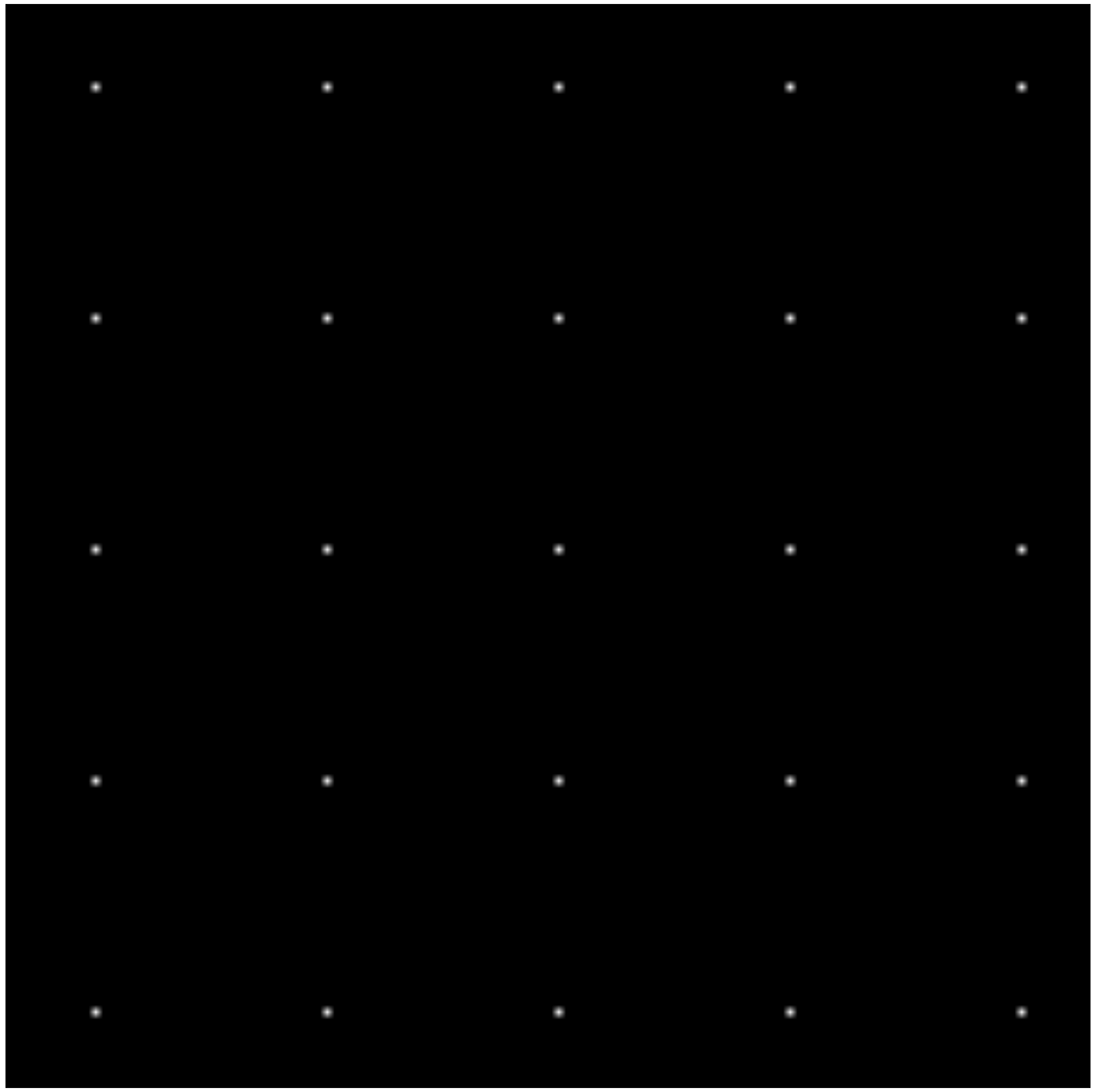
- Fourier Transform of Point Samples (2D)
- Fourier coefficient at (ω_x, ω_y)

```
double Power(ωx,ωy) = fx*fx+fy*fy;
```

Point Samples' Power Spectrum

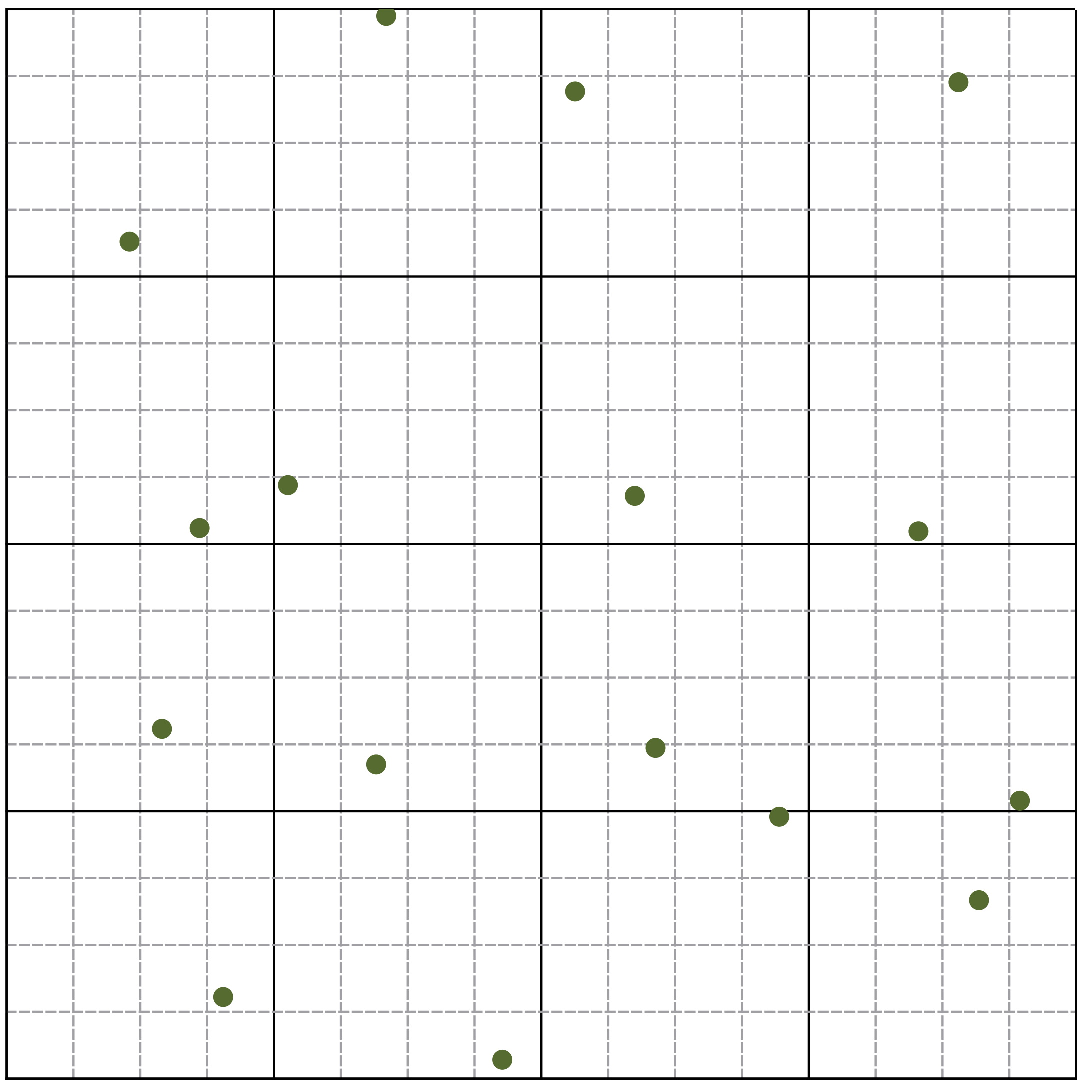
- Fourier Transform of Regular Pattern

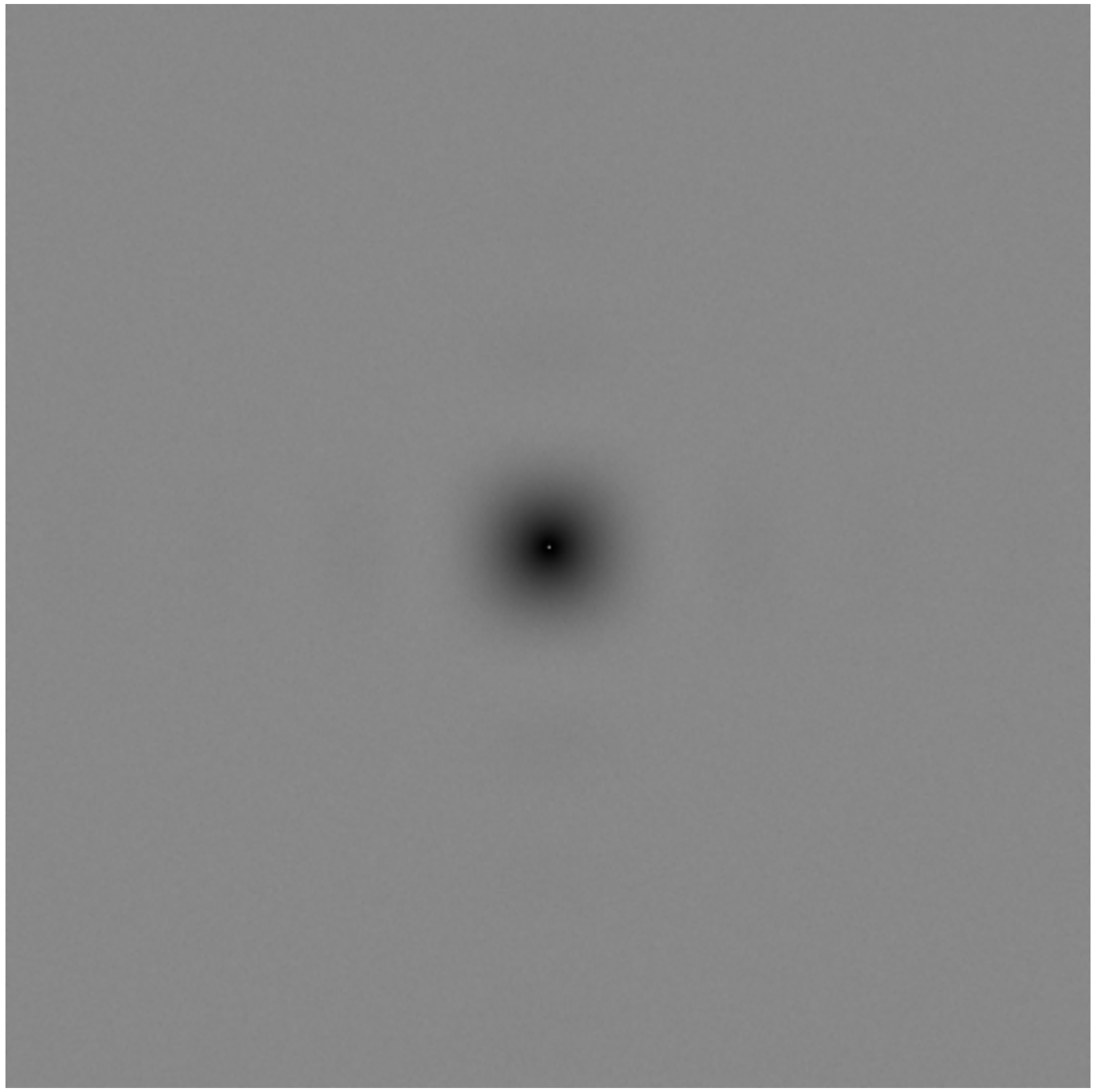




Point Samples' Power Spectrum

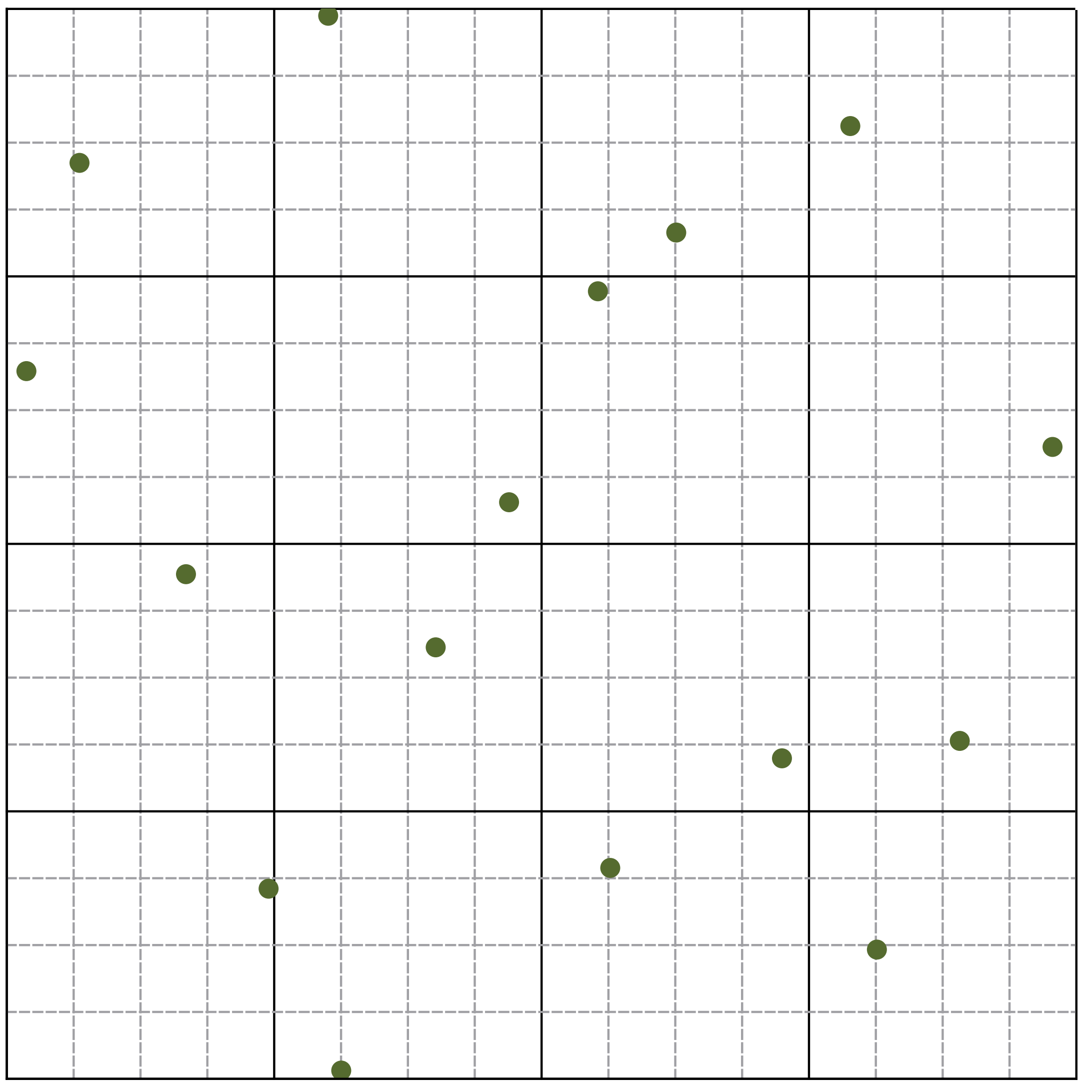
- Fourier Transform of Jitter Pattern

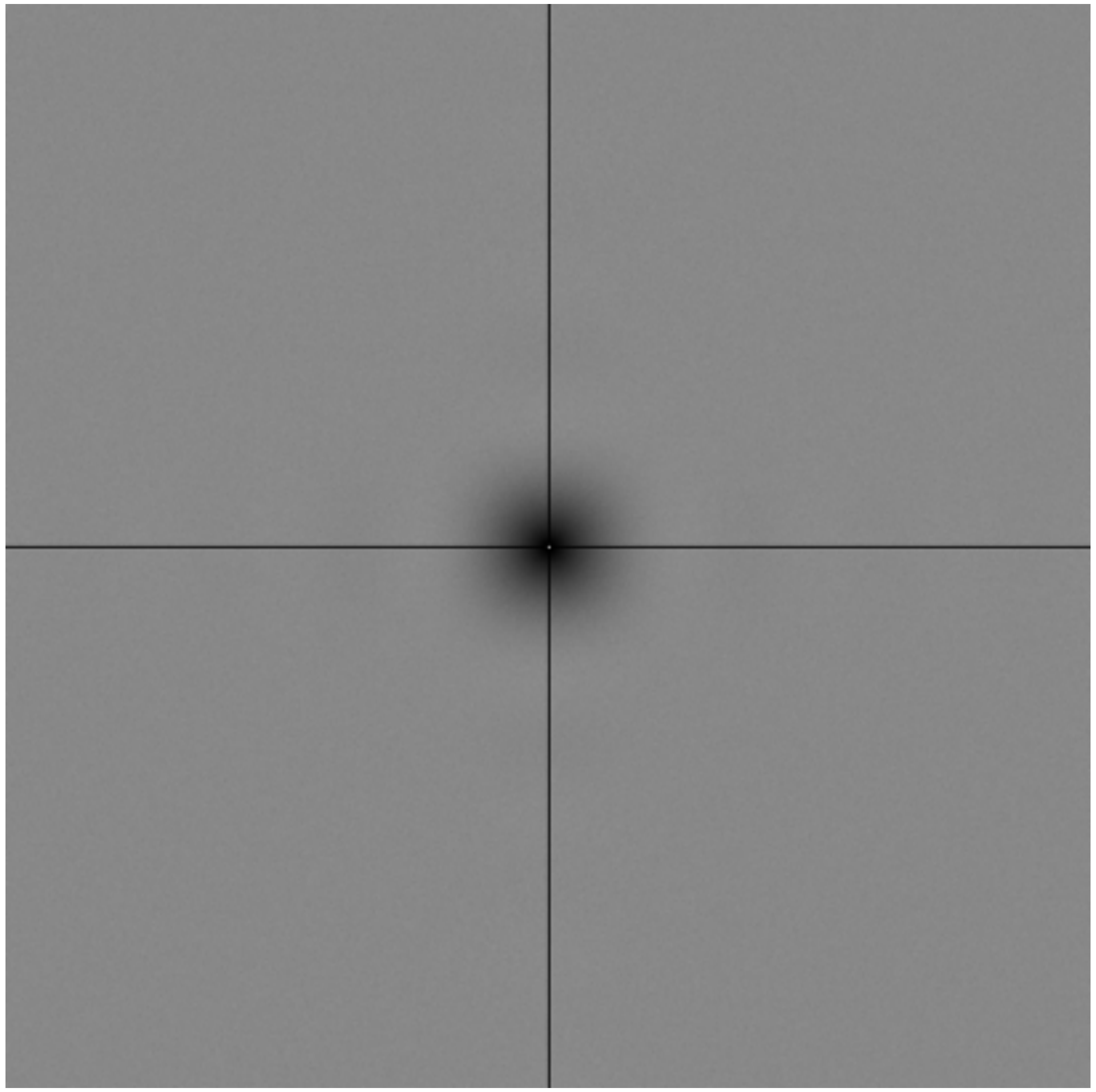




Point Samples' Power Spectrum

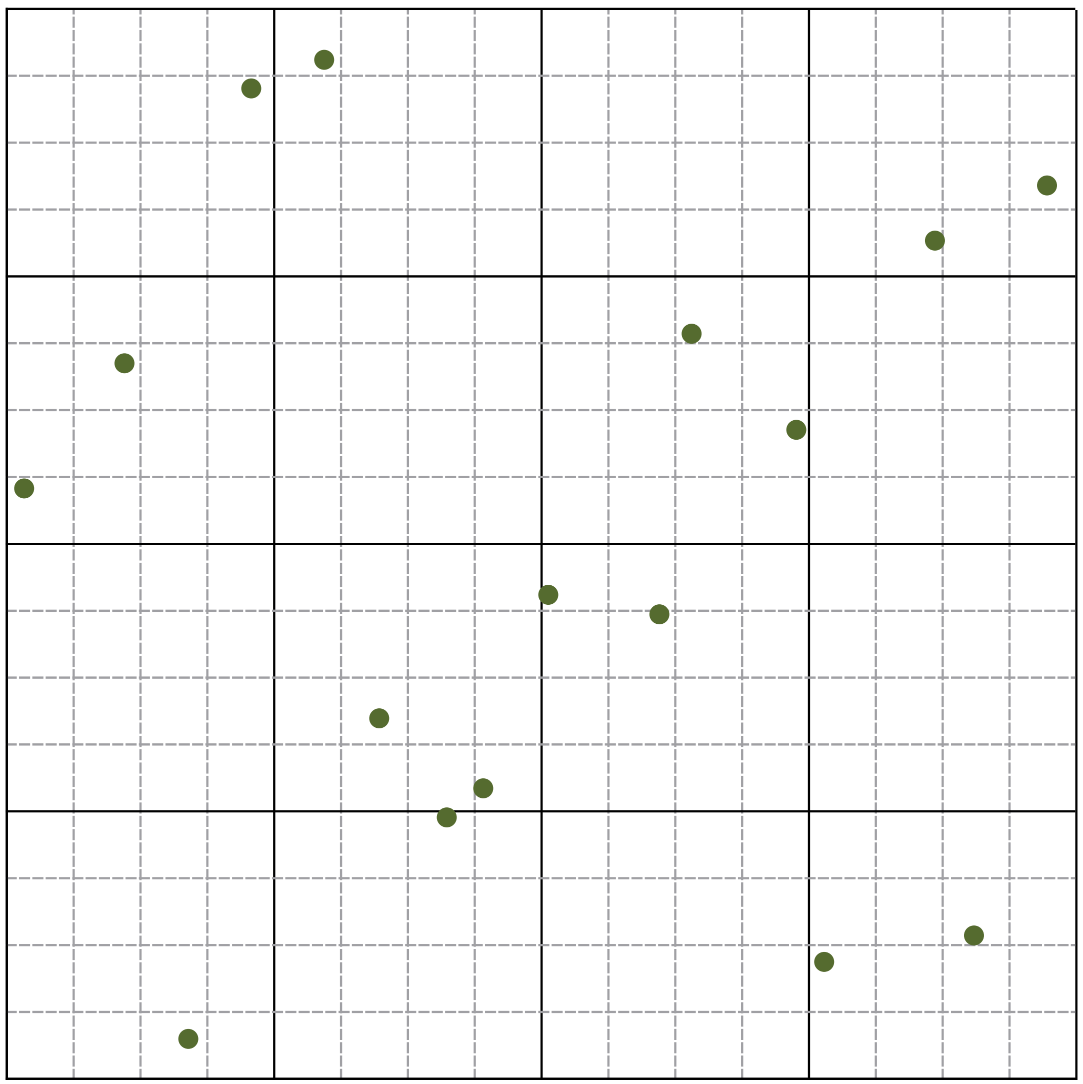
- Fourier Transform of Multi-jitter Pattern

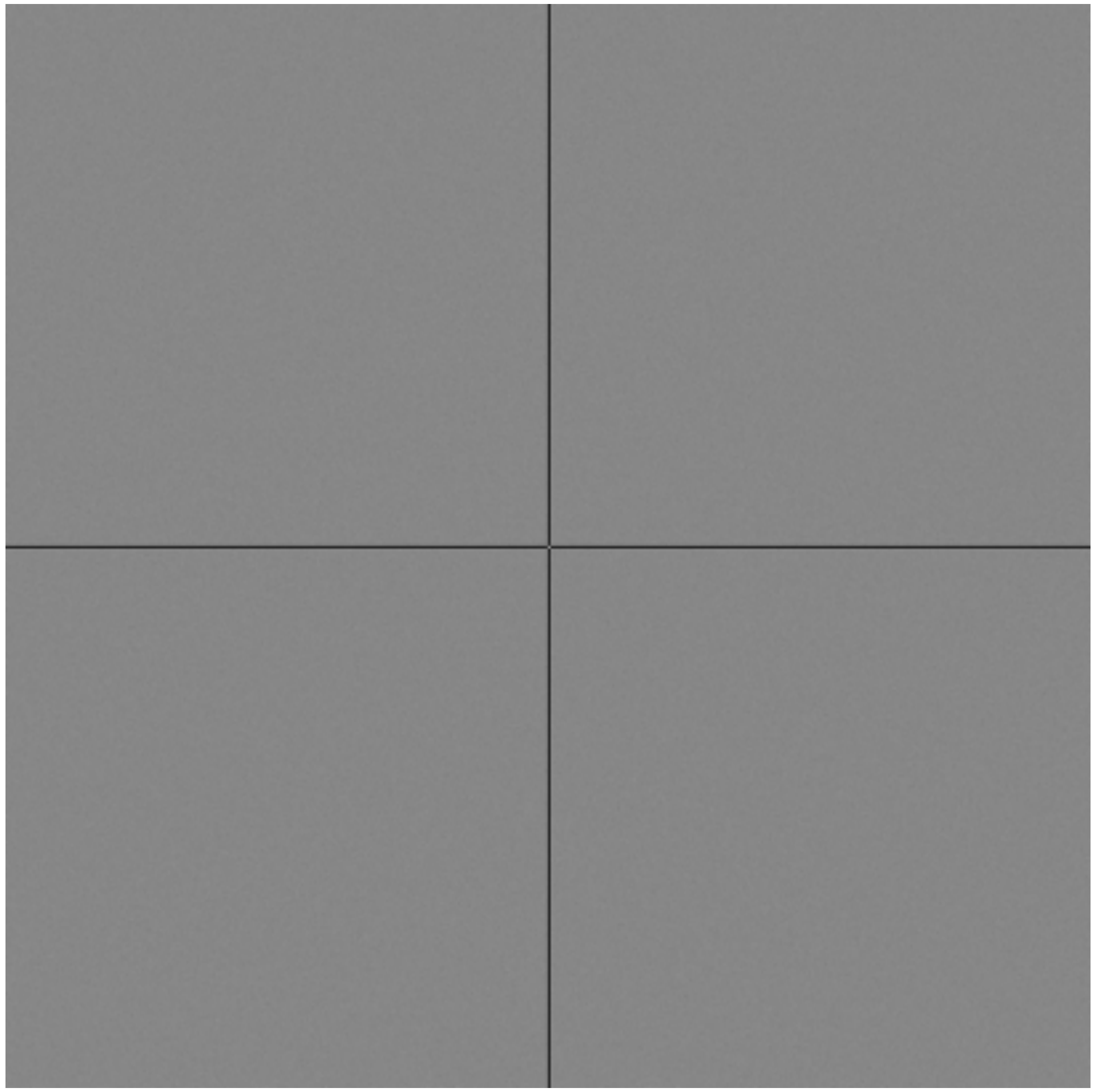




Point Samples' Power Spectrum

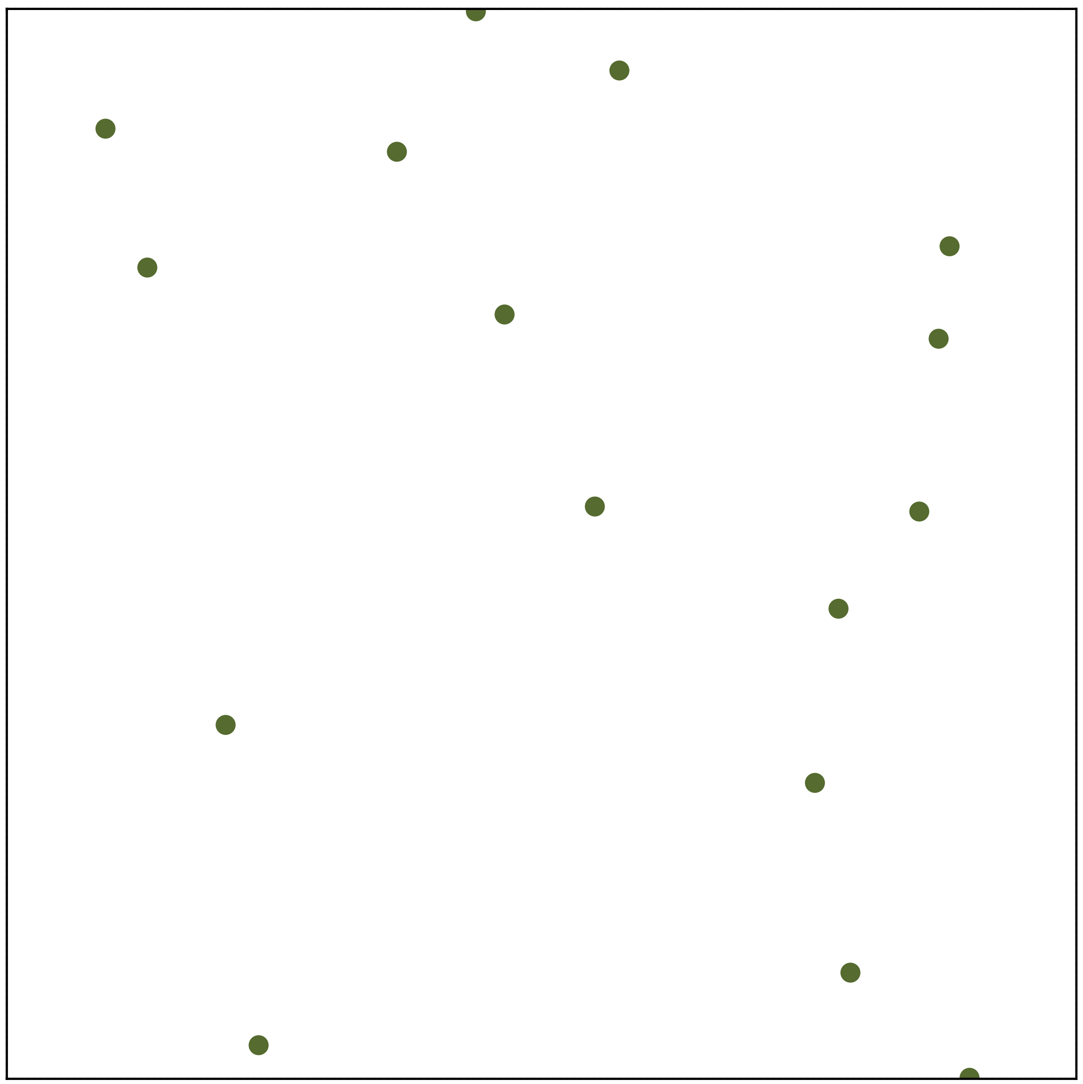
- Fourier Transform of N-rooks Pattern

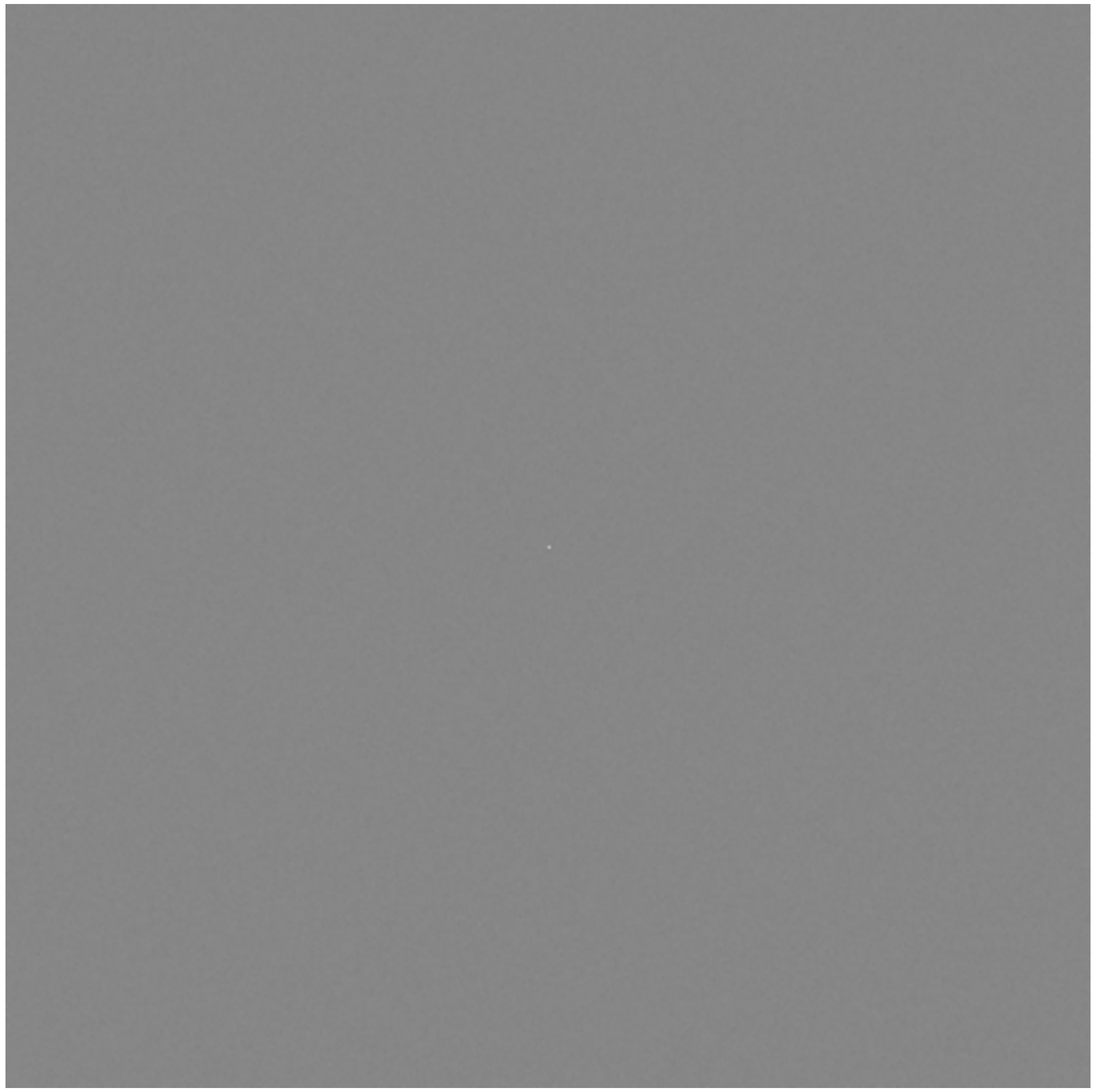




Point Samples' Power Spectrum

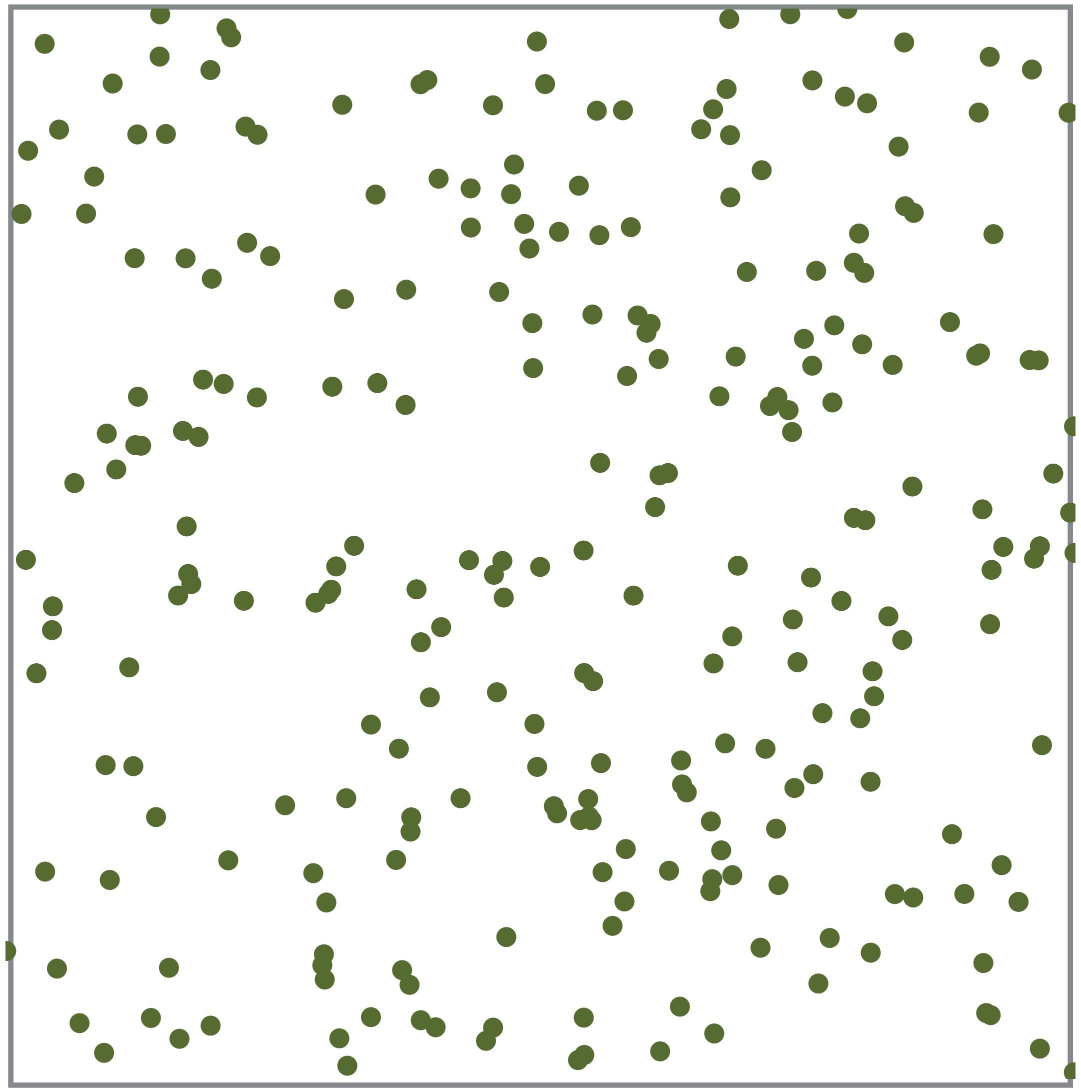
- Fourier Transform of Random Pattern

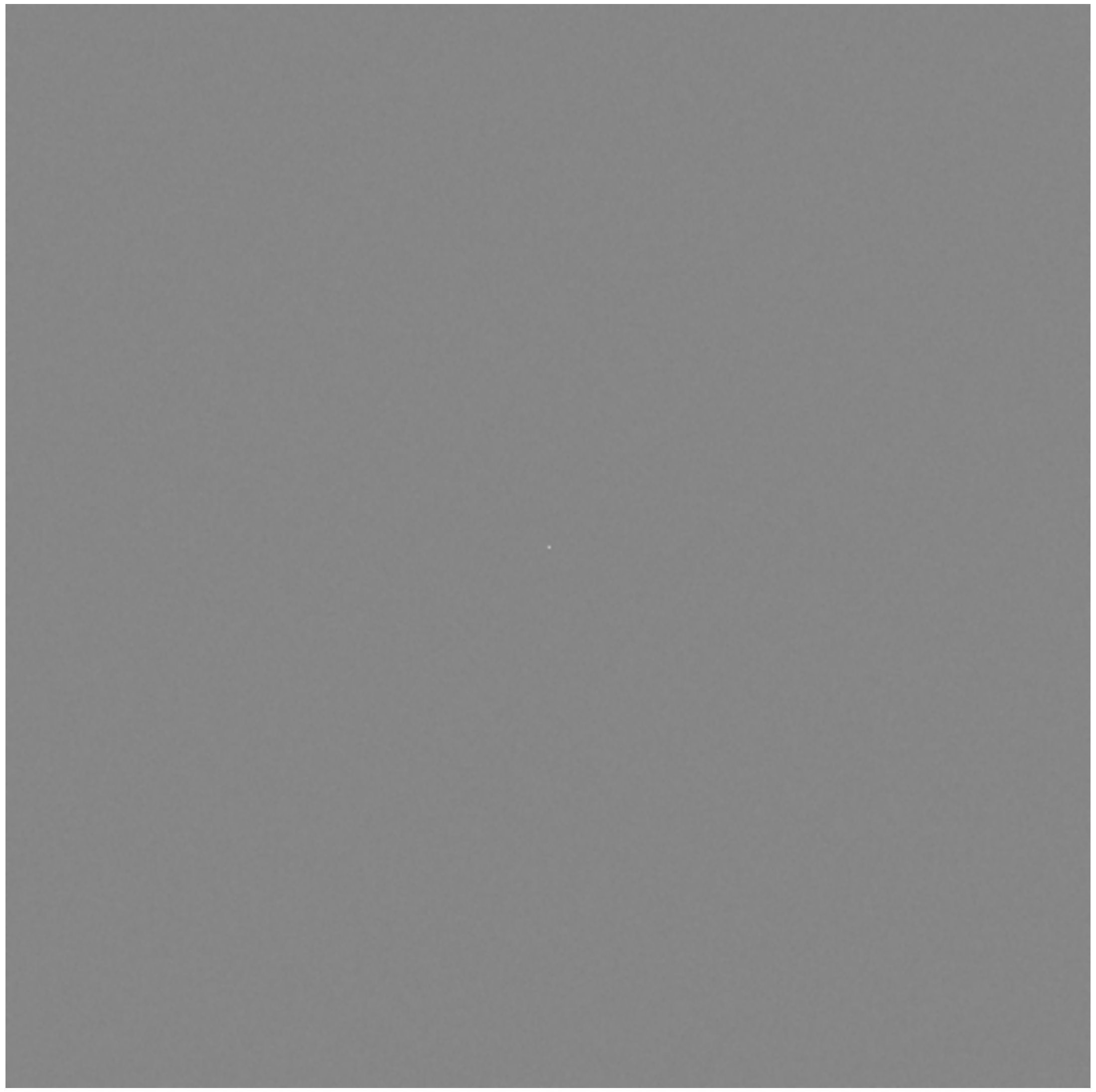




Point Samples' Power Spectrum

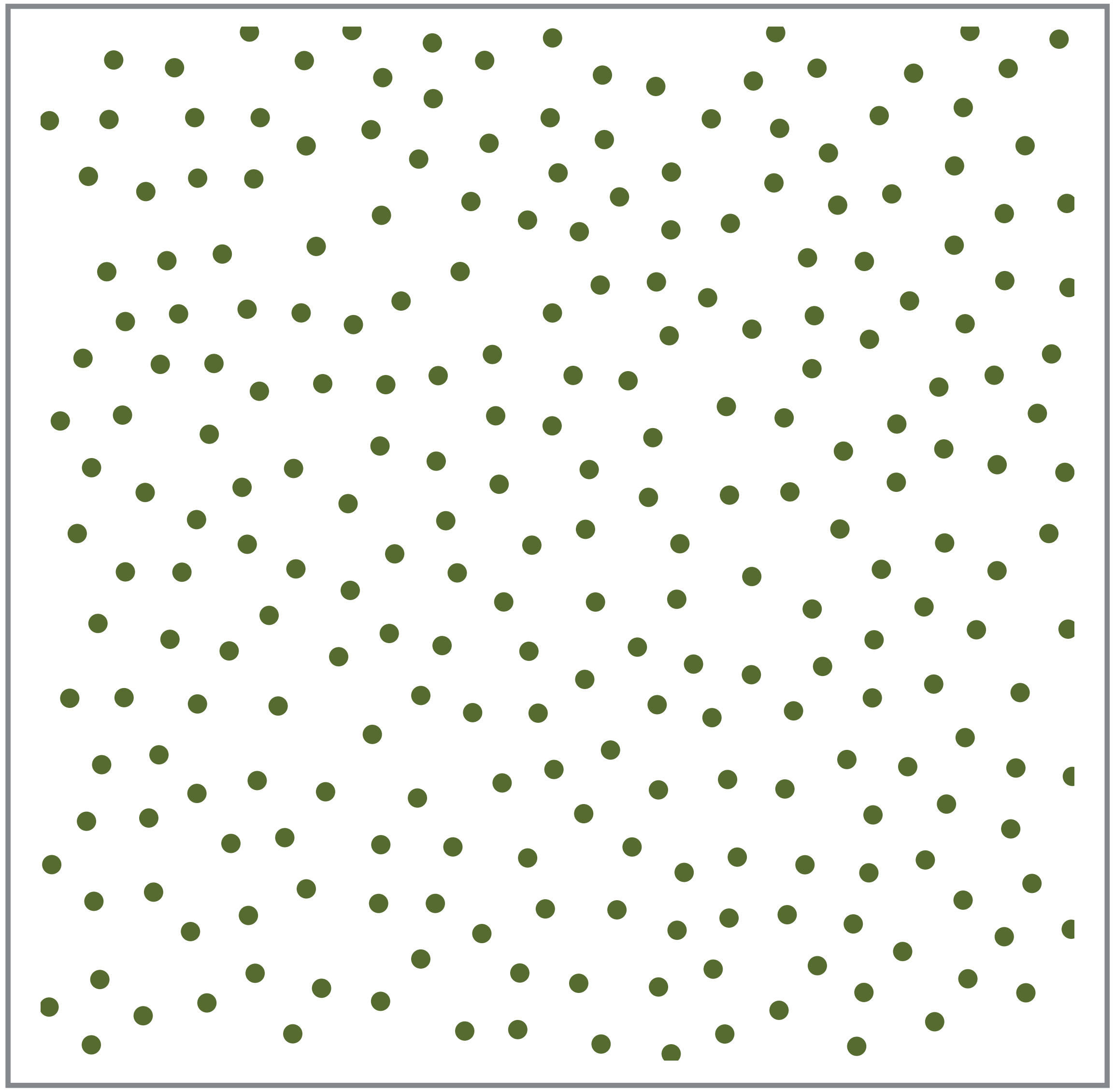
- Fourier Transform of Random Pattern

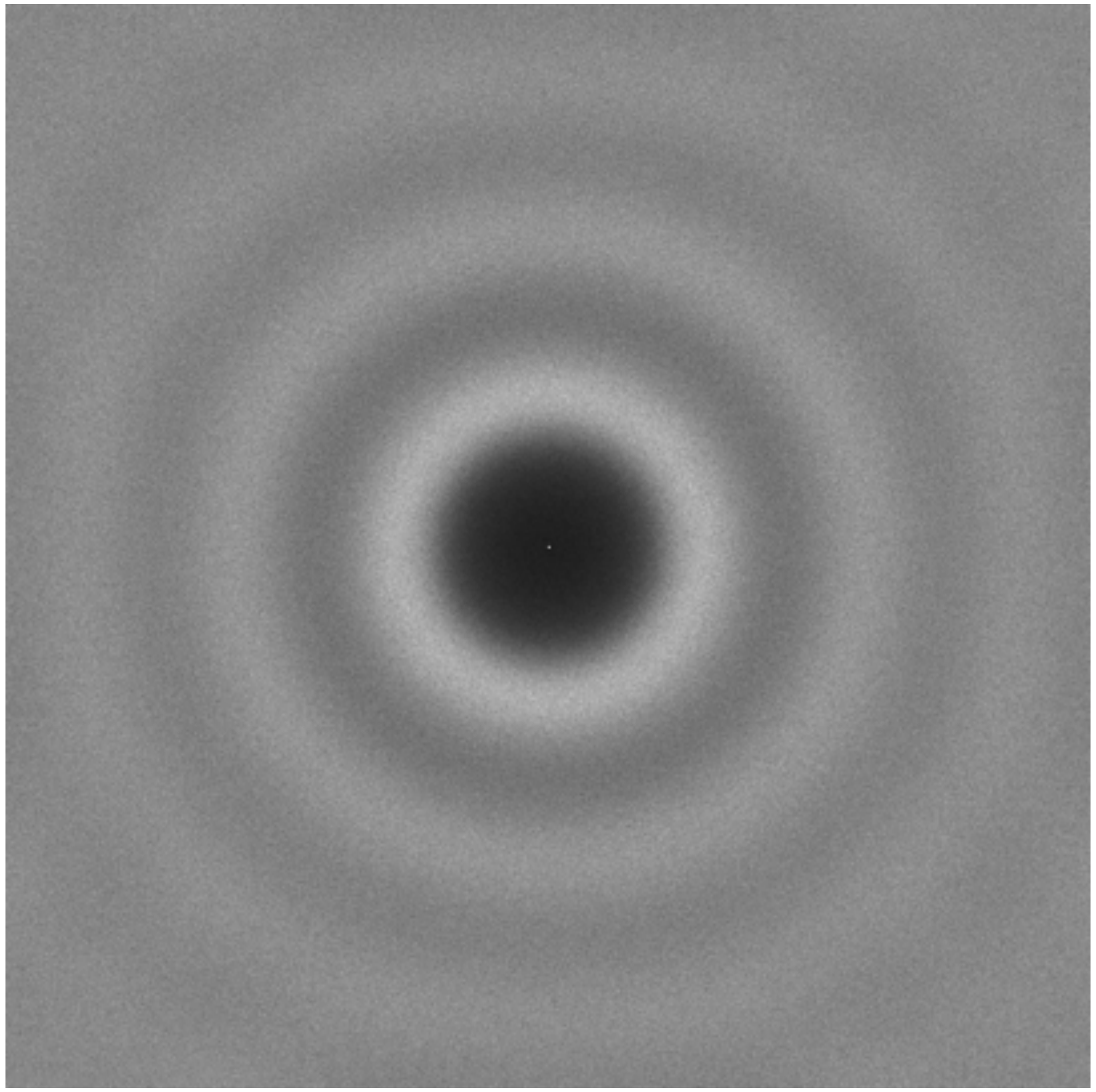




Point Samples' Power Spectrum

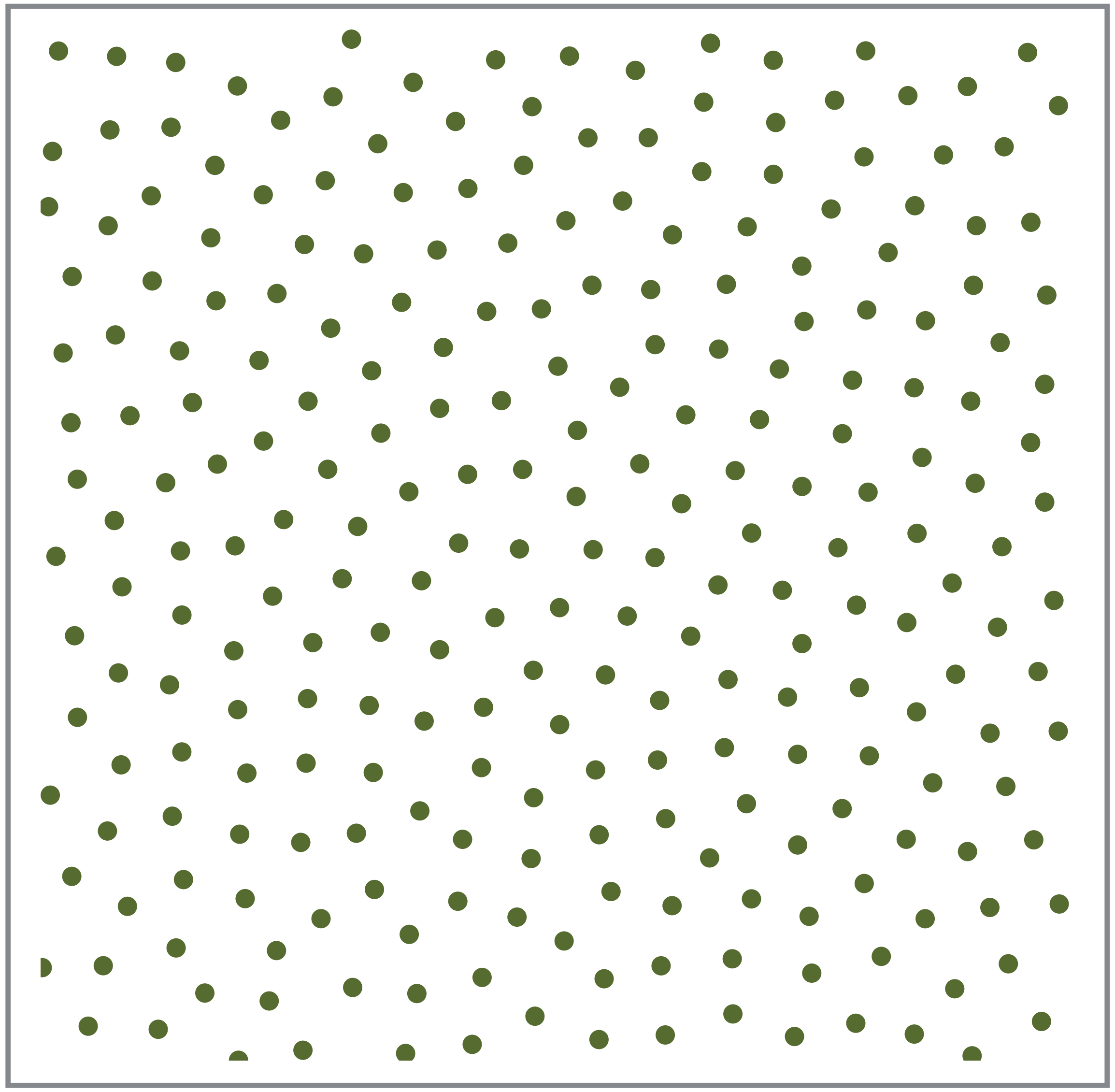
- Fourier Transform of Poisson disk Pattern



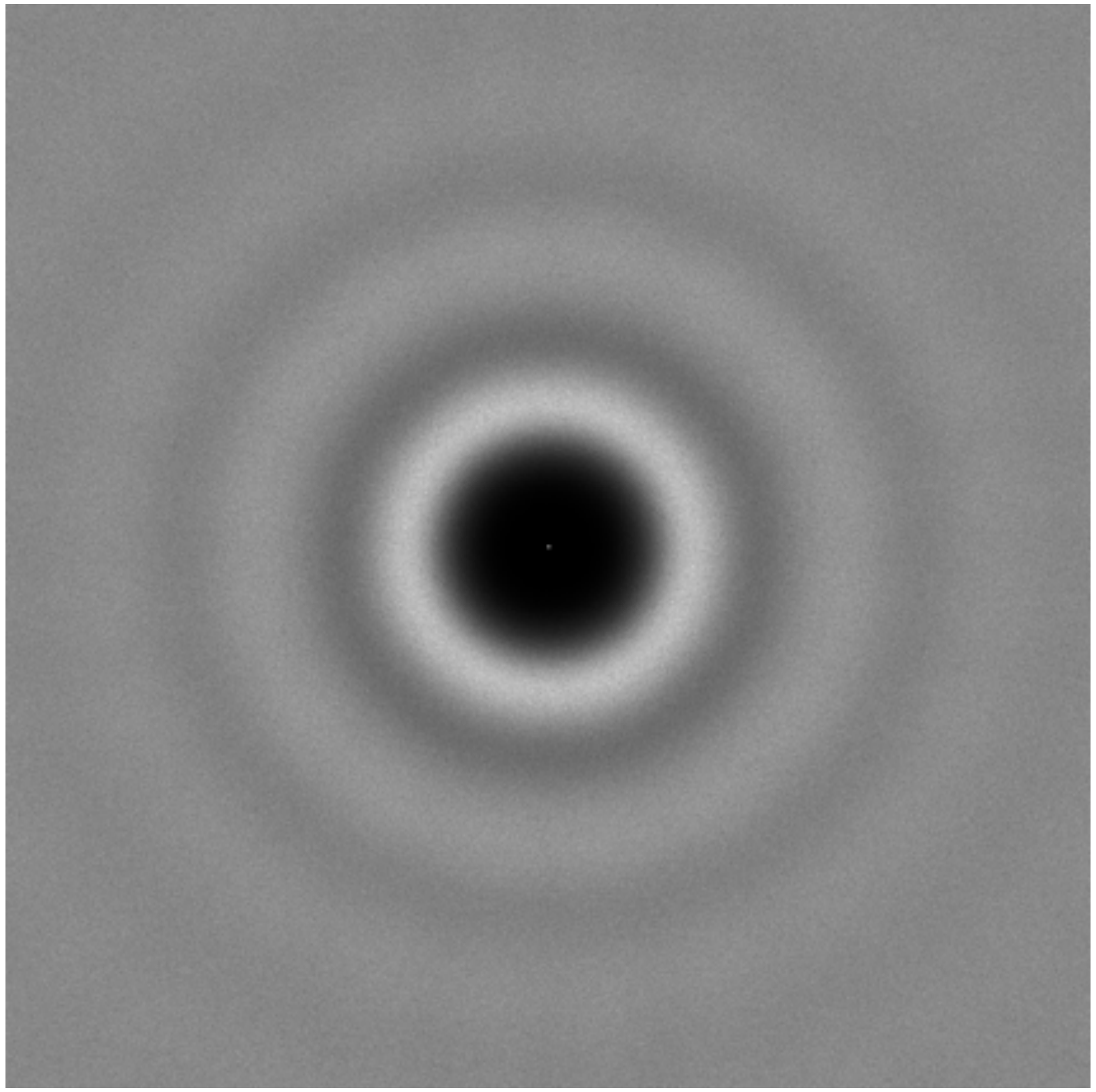


Point Samples' Power Spectrum

- Fourier Transform of CCVT Pattern

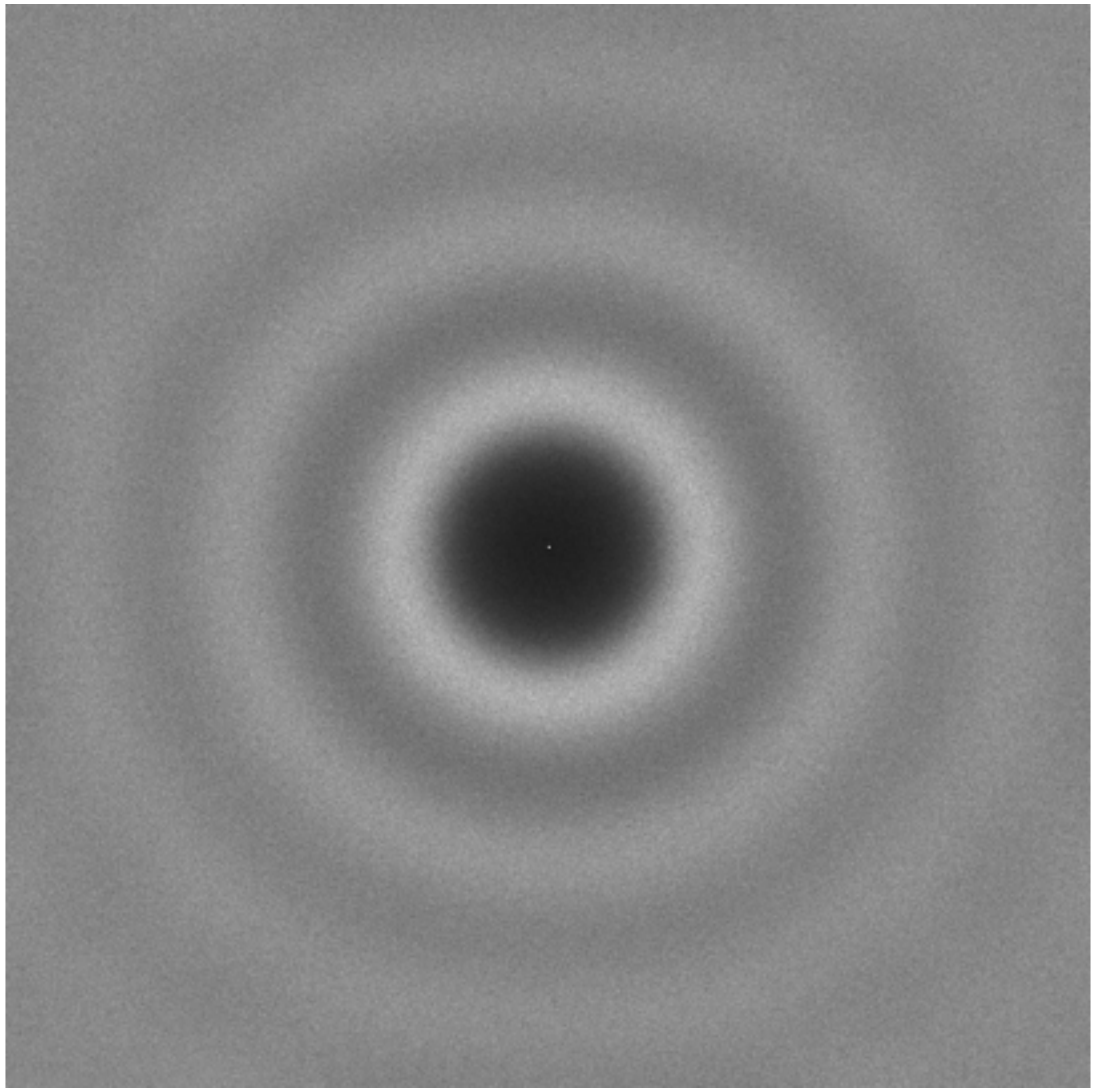


Balzer et al [2009]

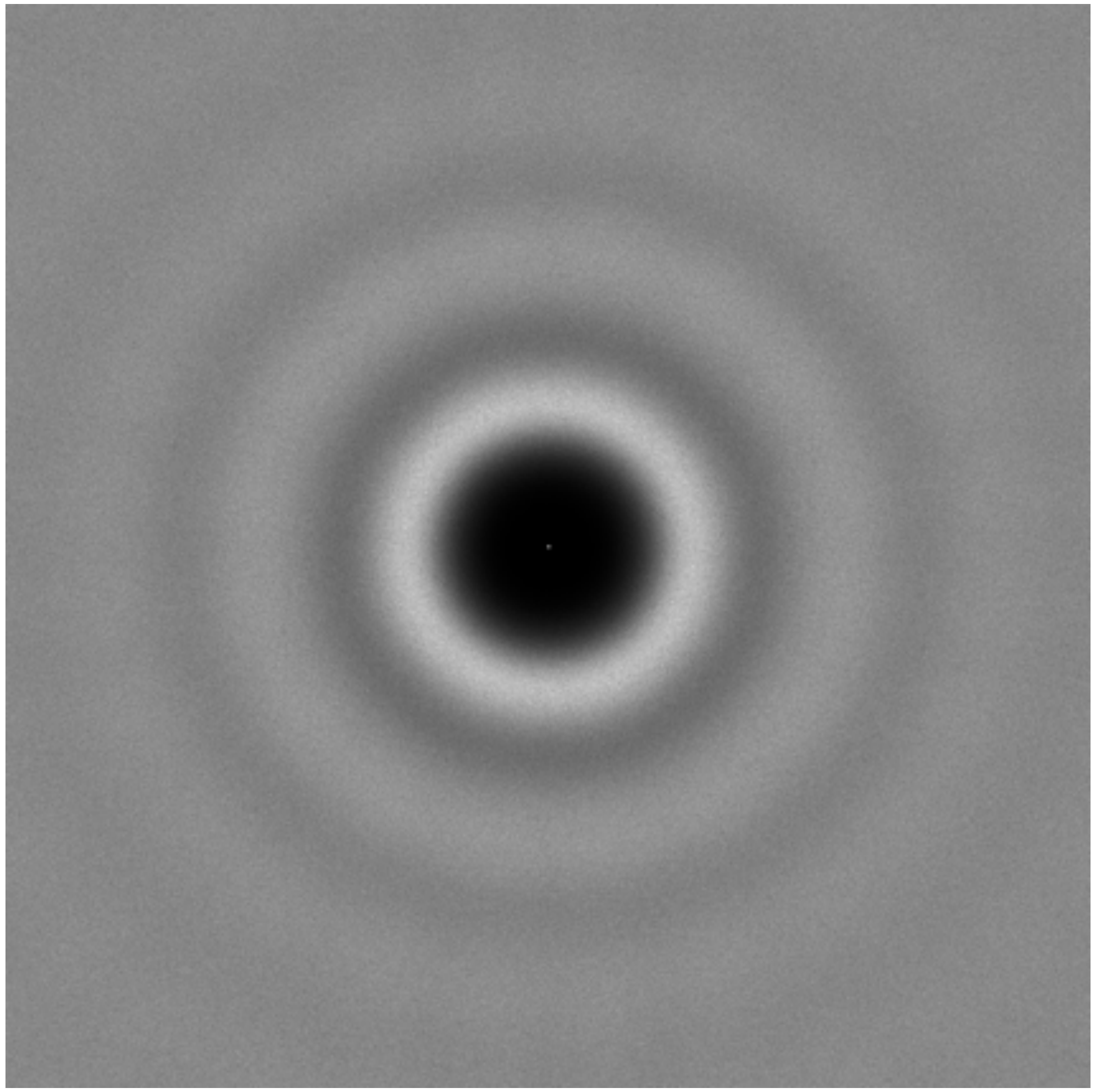


Point Samples' Power Spectrum

- Poisson Disk vs CCVT



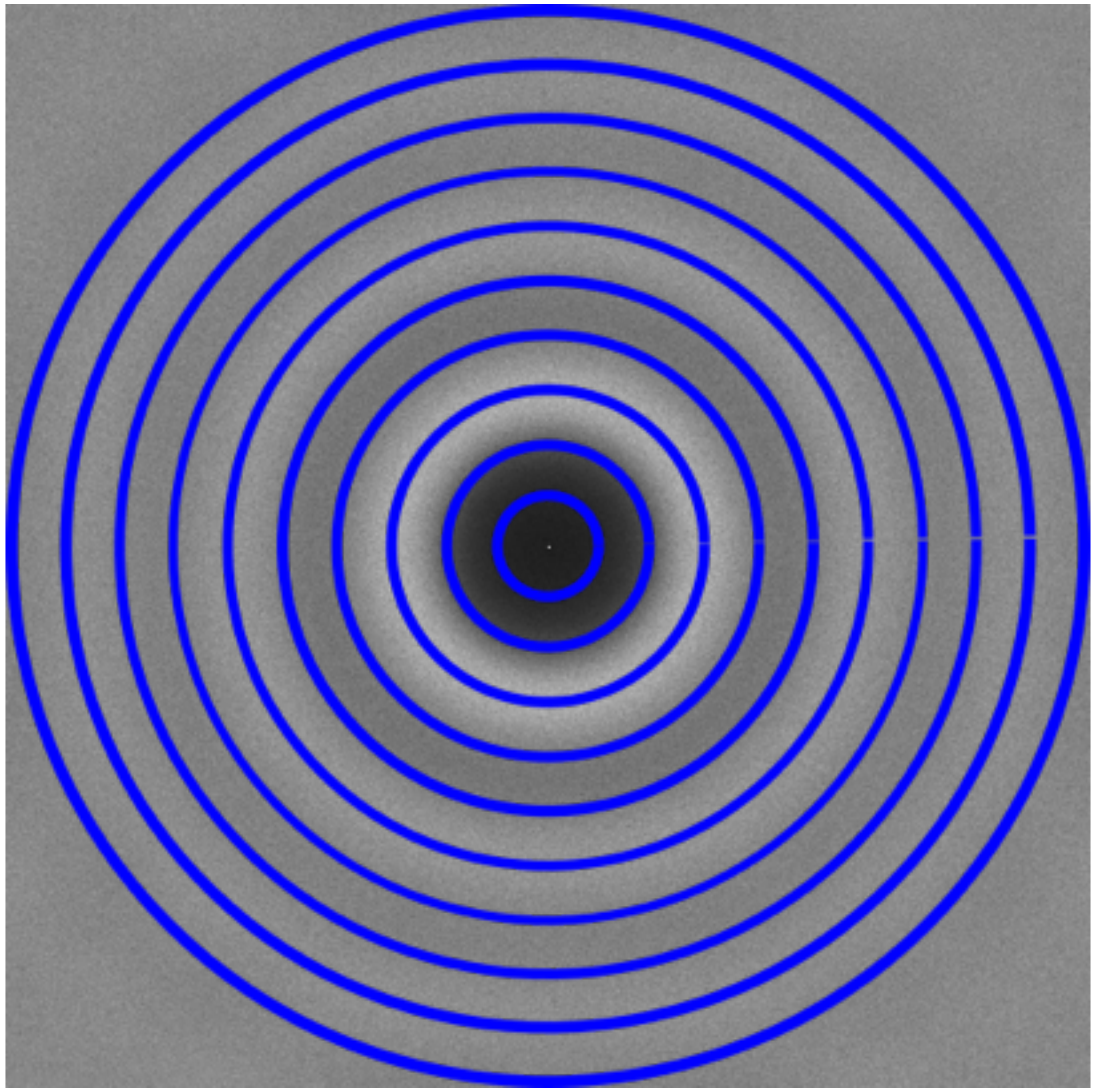
Poisson Disk

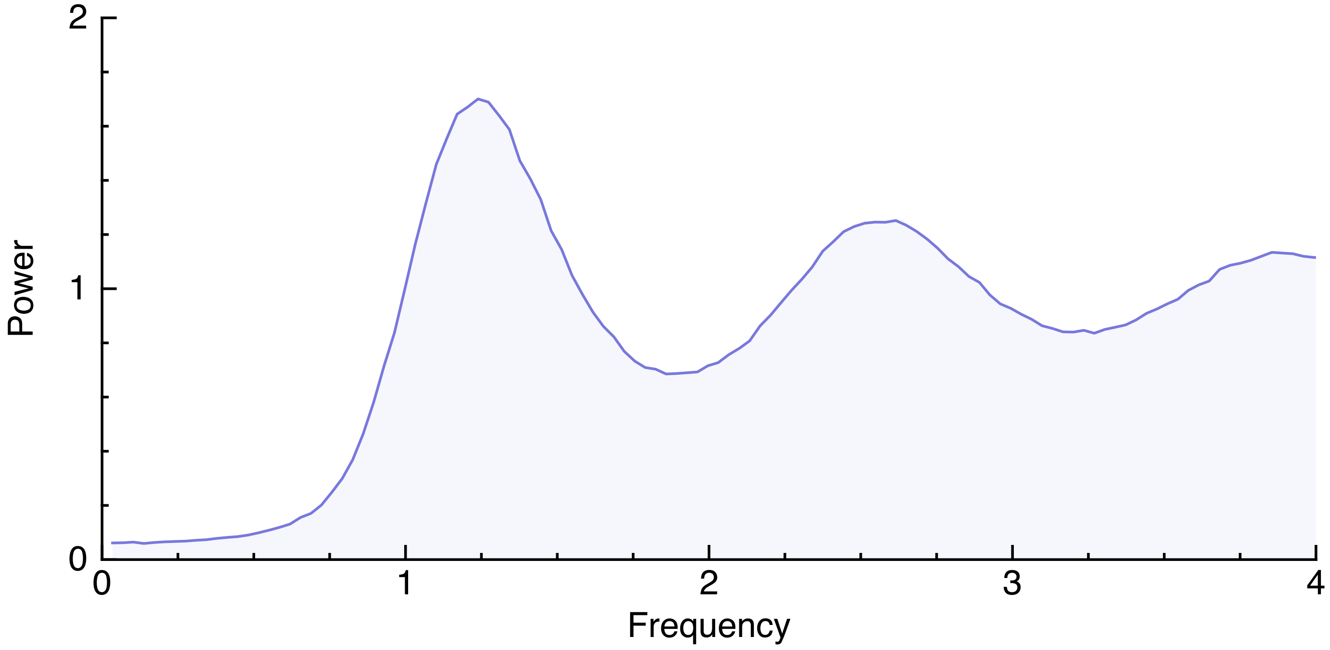


CCVT

Fourier Radial Power Spectrum

- Radial Spectrum of Poisson Disk

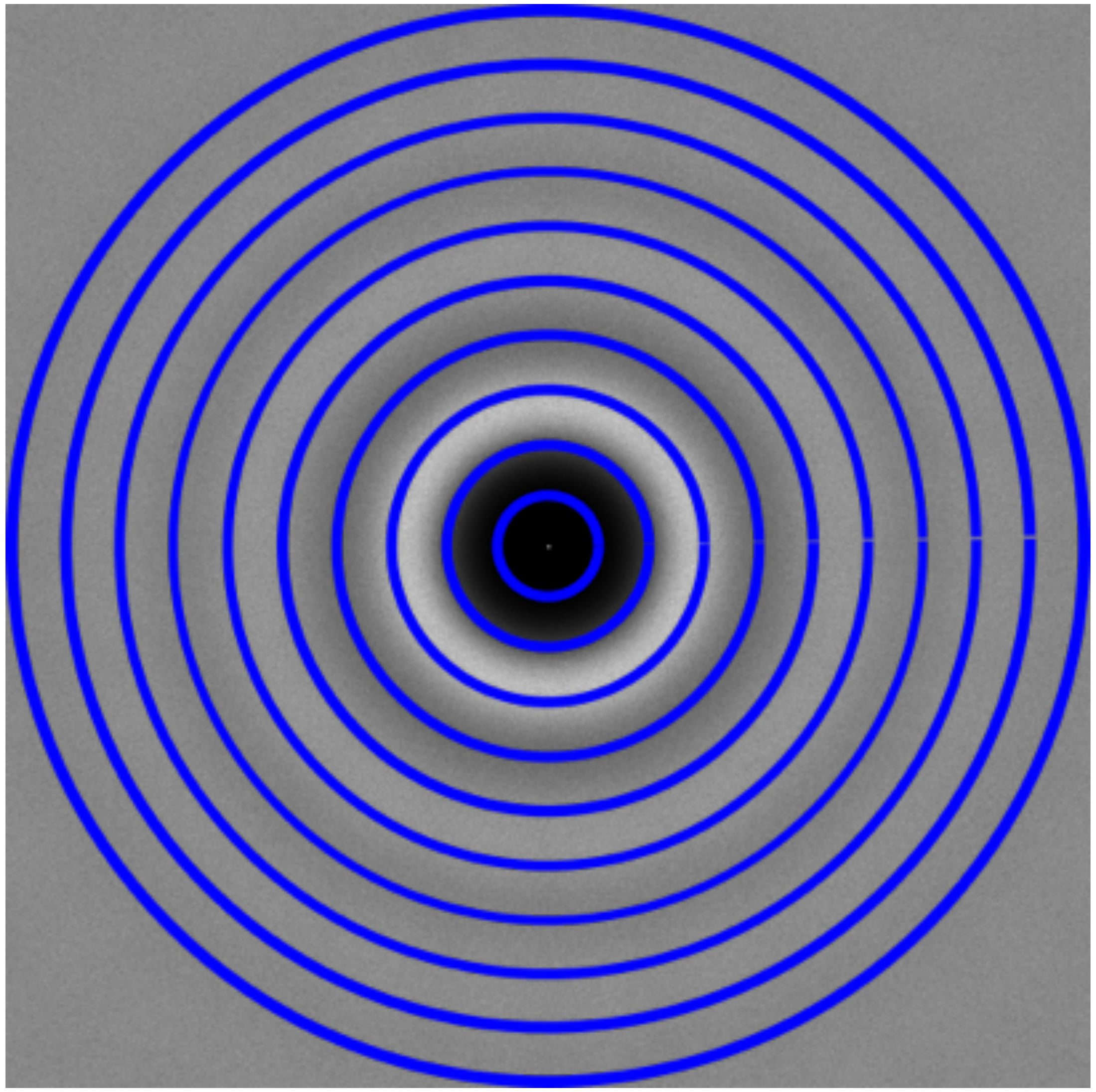


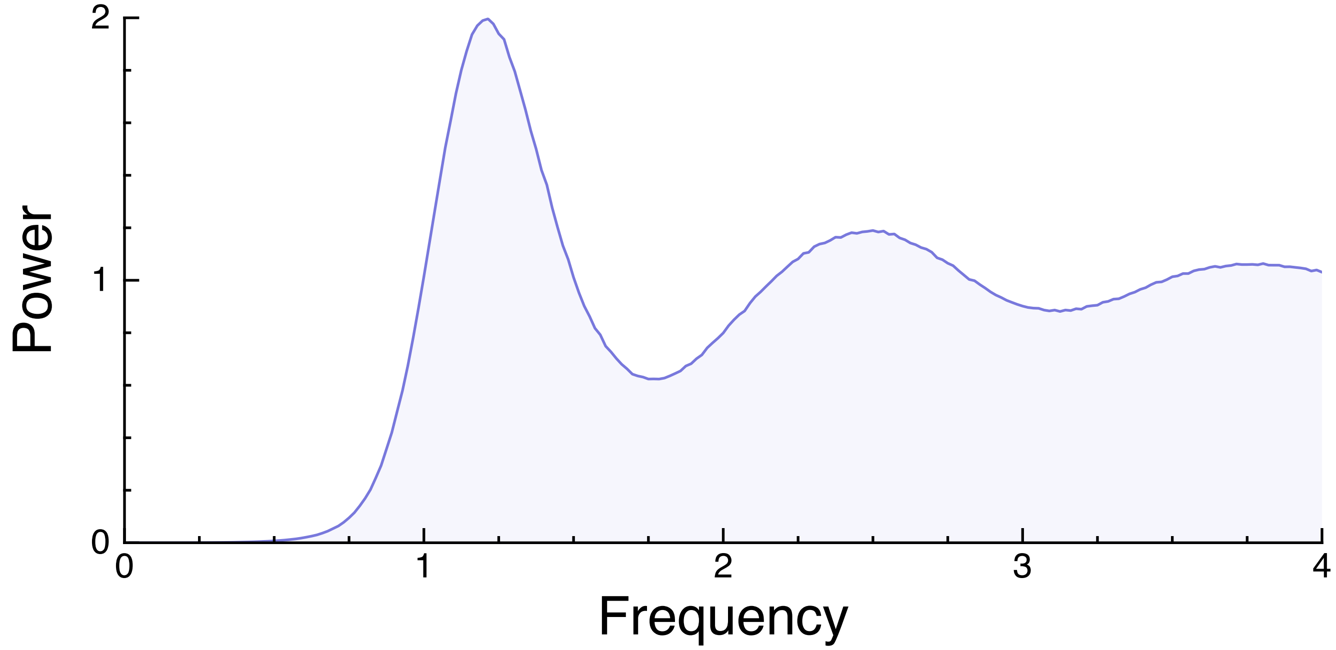


Poisson Disk

Fourier Radial Power Spectrum

- Radial Spectrum of CCVT

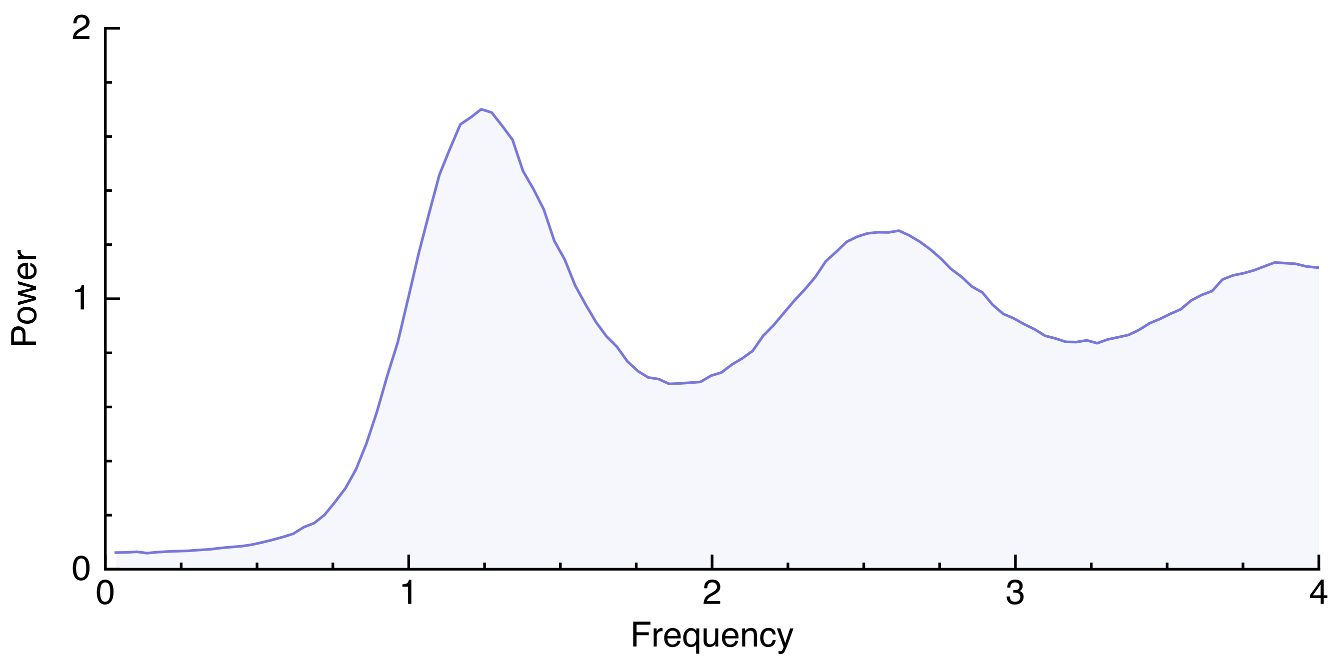




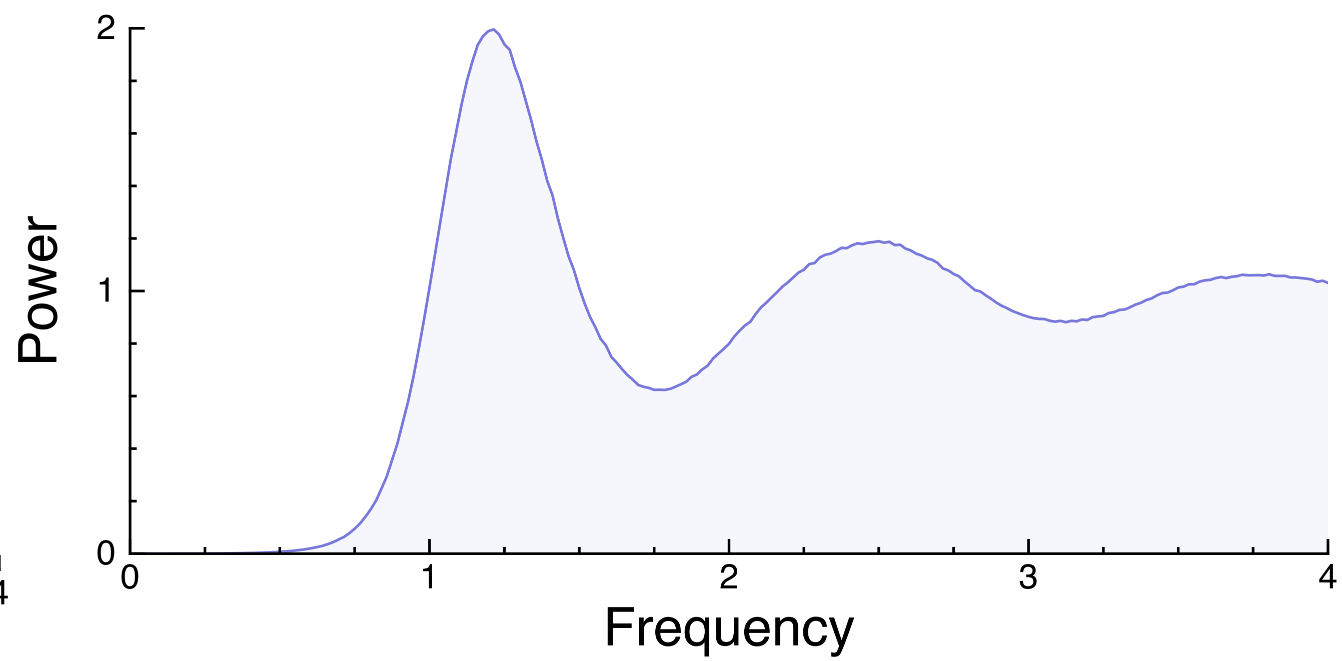
Poisson Disk

Fourier Radial Power Spectrum

- Radial Power Spectrum for Blue Noise



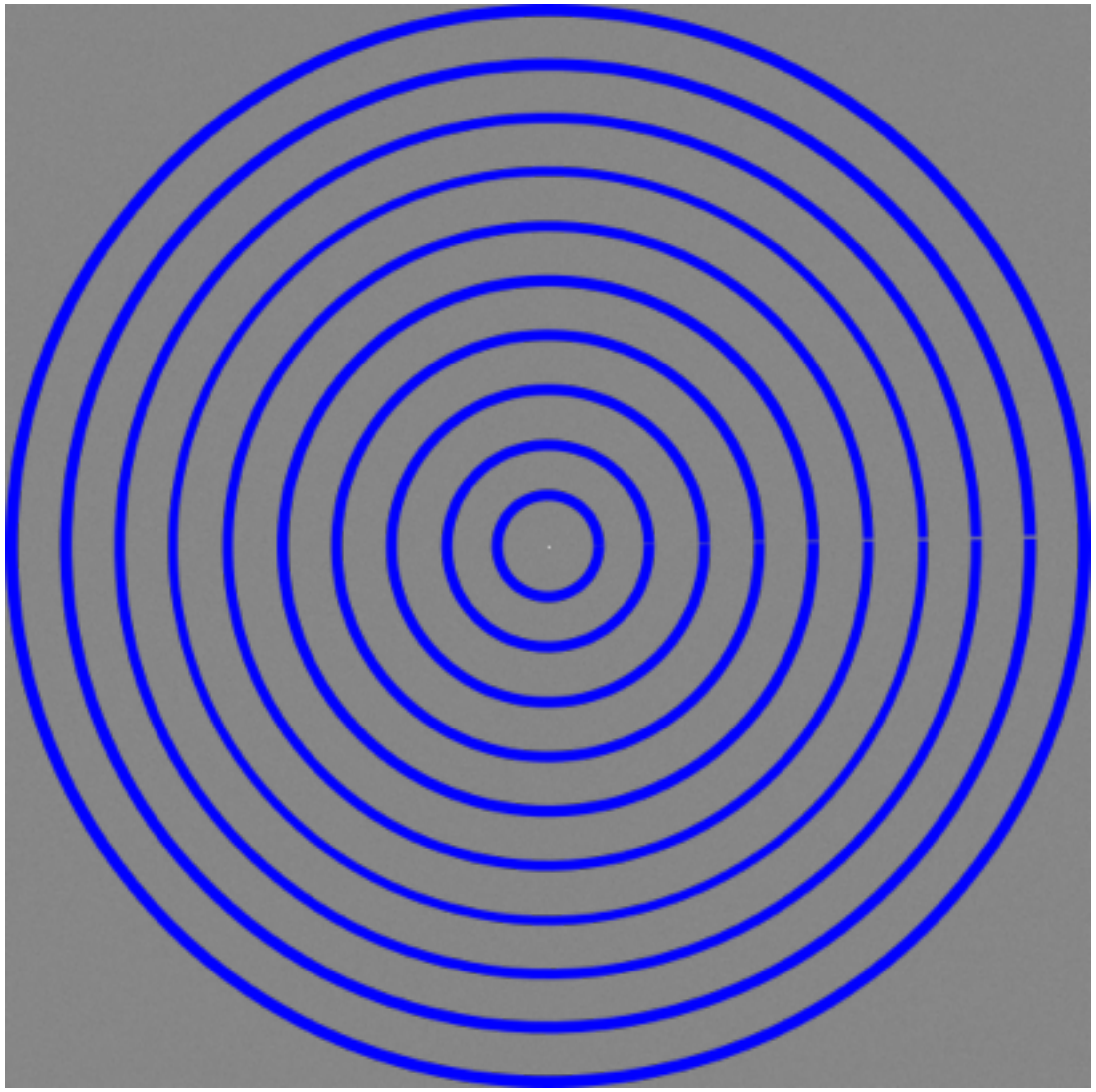
Poisson Disk

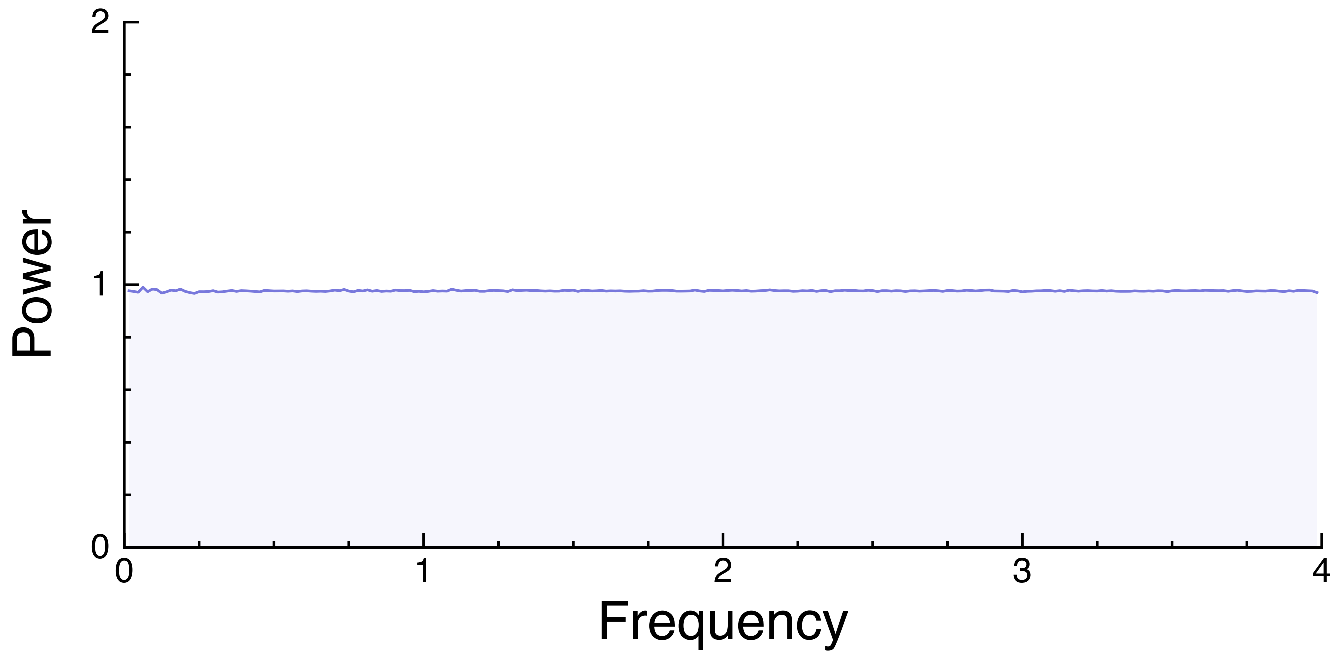


CCVT

Fourier Radial Power Spectrum

- Radial Spectrum of Random Samples

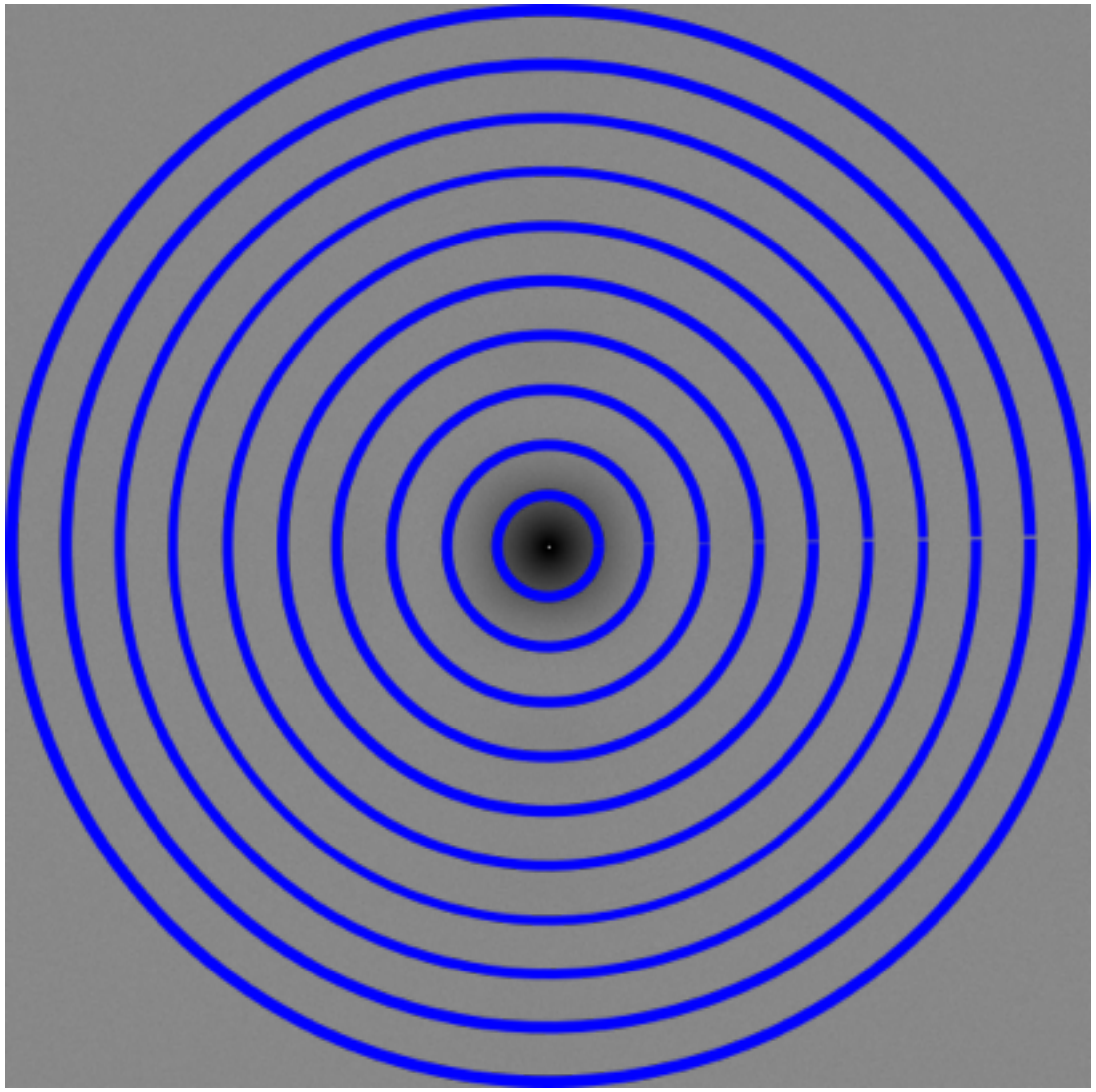


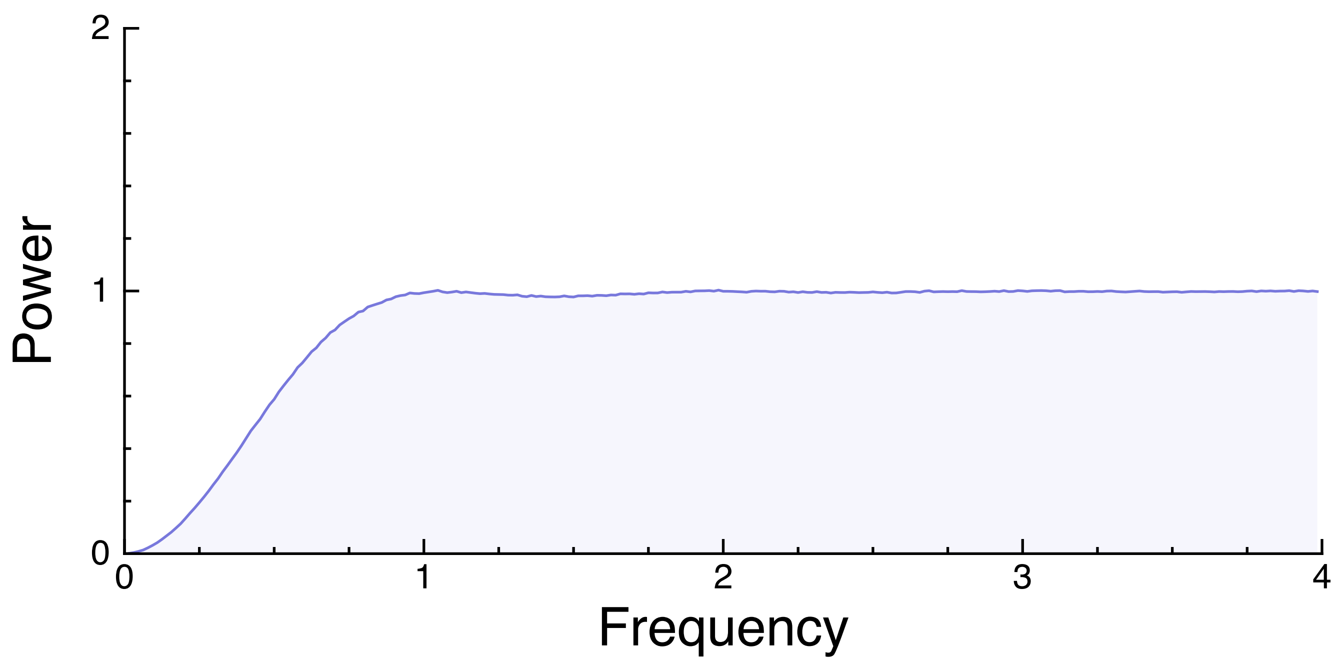


Random

Fourier Radial Power Spectrum

- Radial Spectrum of Jitter





Poisson Disk

Variance in Fourier Domain

Variance in Fourier Domain

$$I_N = \int \overline{F(-\omega)} S(\omega) d\omega$$

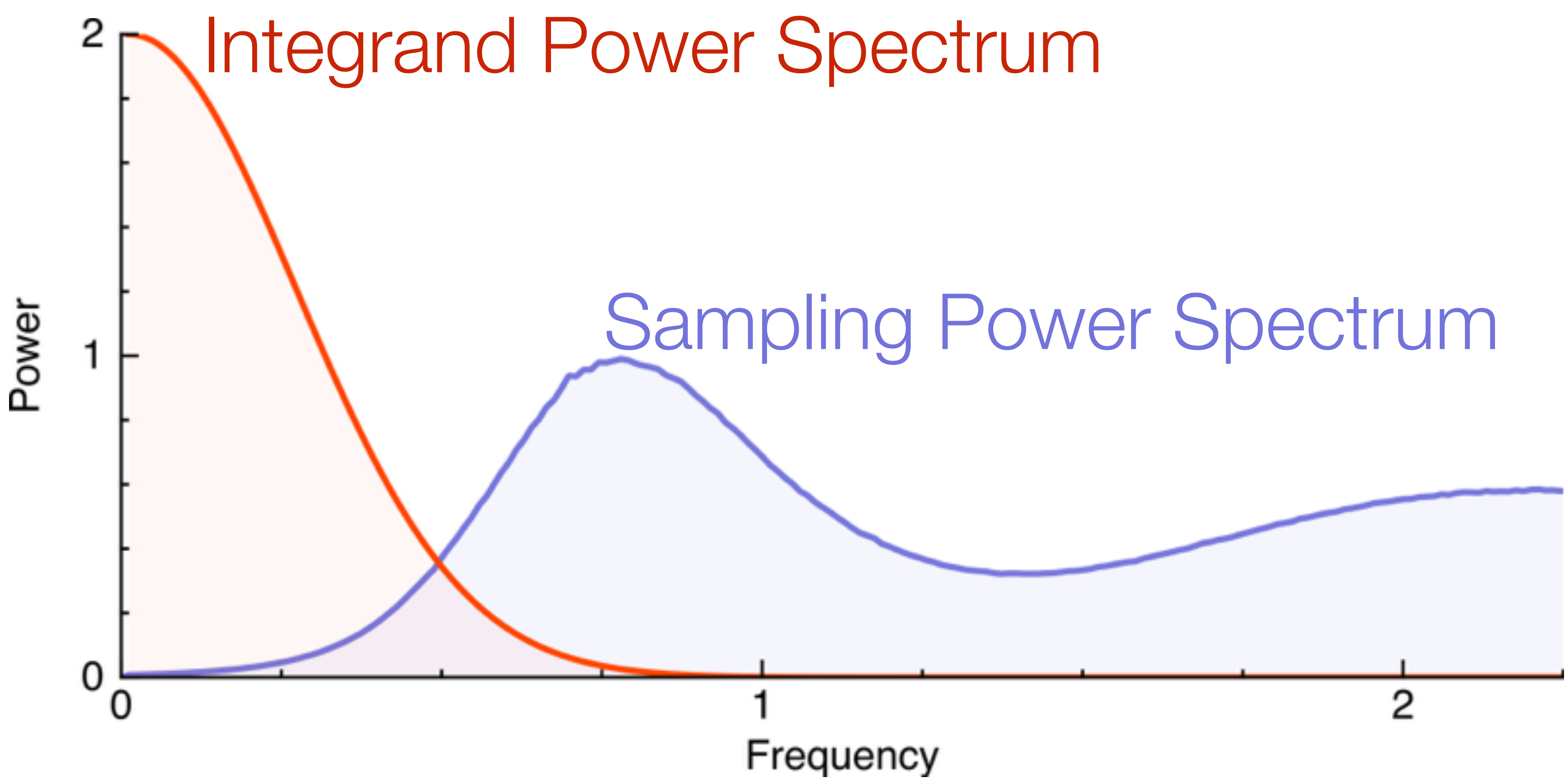
Fredo Durand [2011]

$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega) \langle \mathcal{P}_S(\omega) \rangle d\omega$$

Pilleboue et al. [2015]

Variance in Fourier Domain

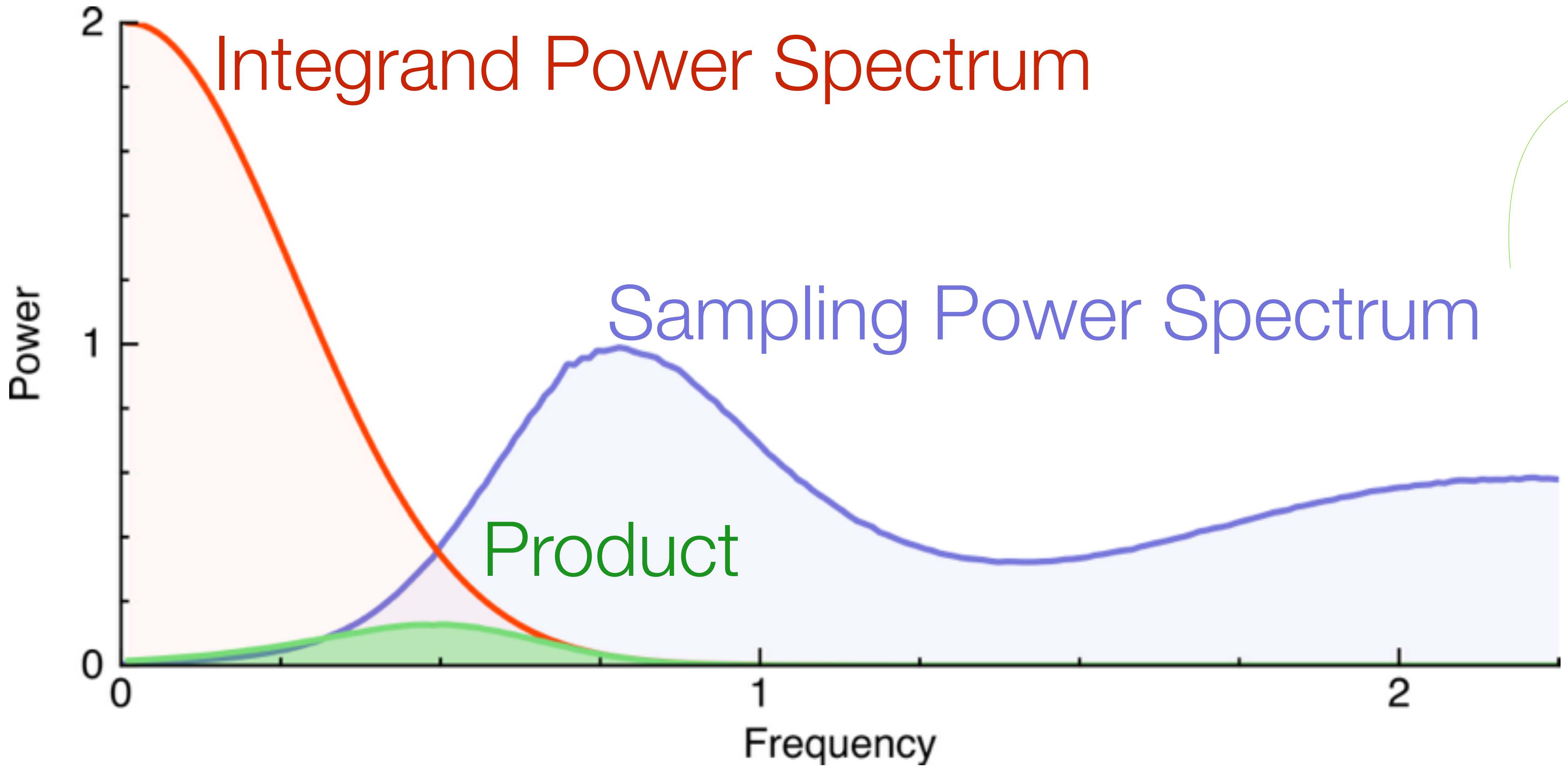
$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega) \langle \mathcal{P}_S(\omega) \rangle d\omega$$



For given number of Samples

Variance in Fourier Domain

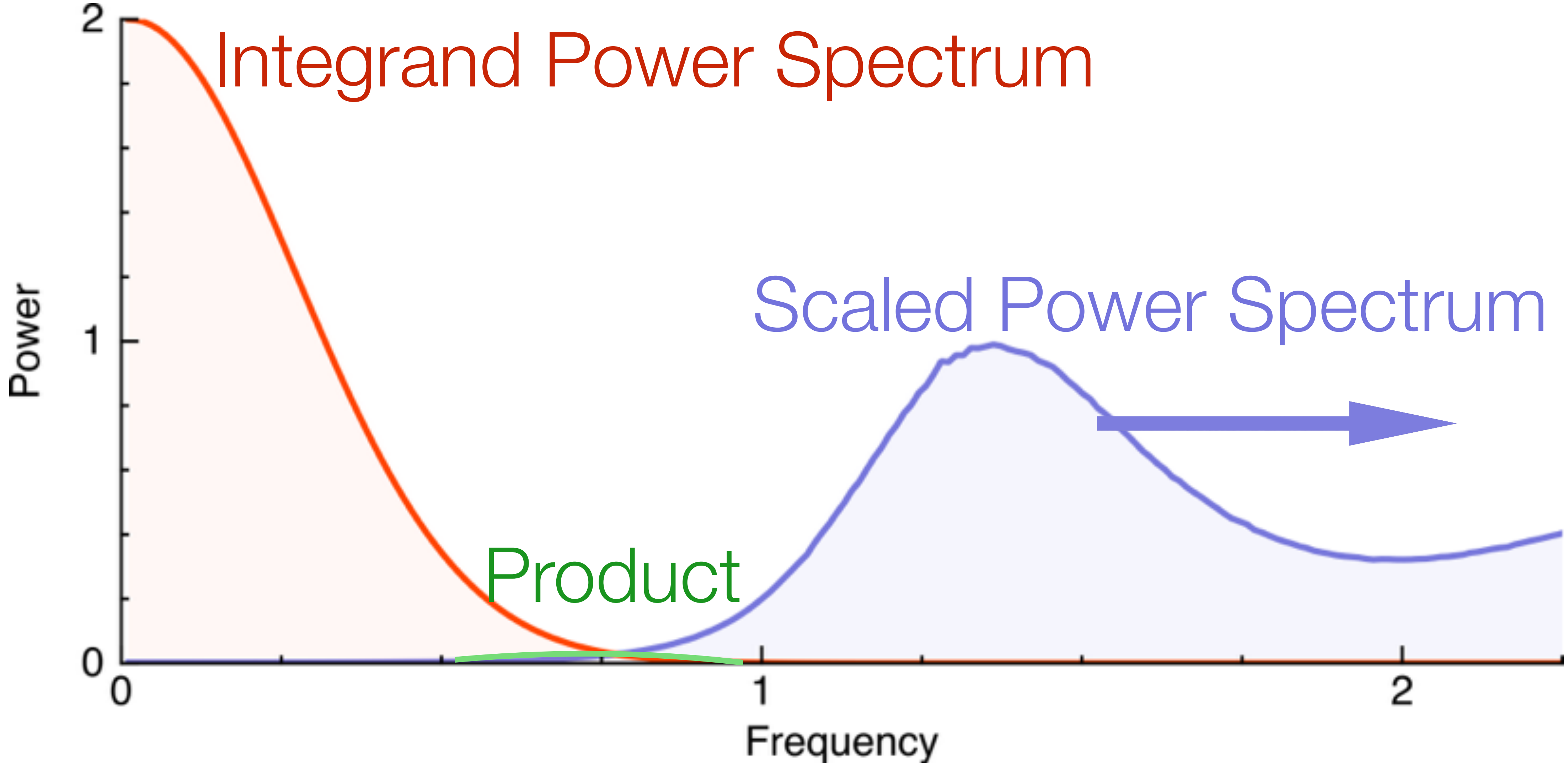
$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega)\langle \mathcal{P}_S(\omega)\rangle d\omega$$



For given number of Samples

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega) \langle \mathcal{P}_S(\omega) \rangle d\omega$$



As we increase the number of samples

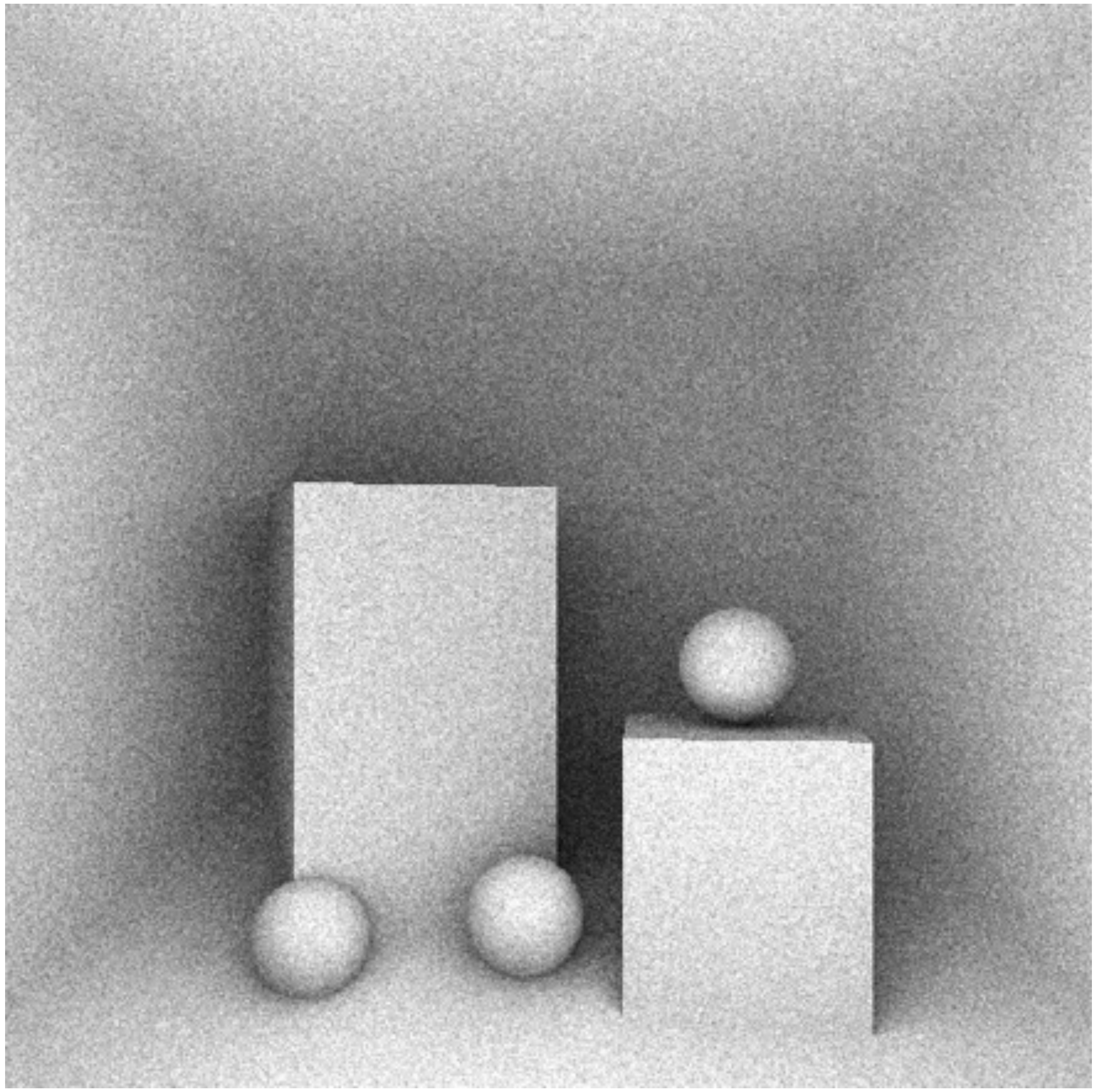
Visual Break

The Alchemist's Letter

Variance Convergence Analysis

Variance Comparison

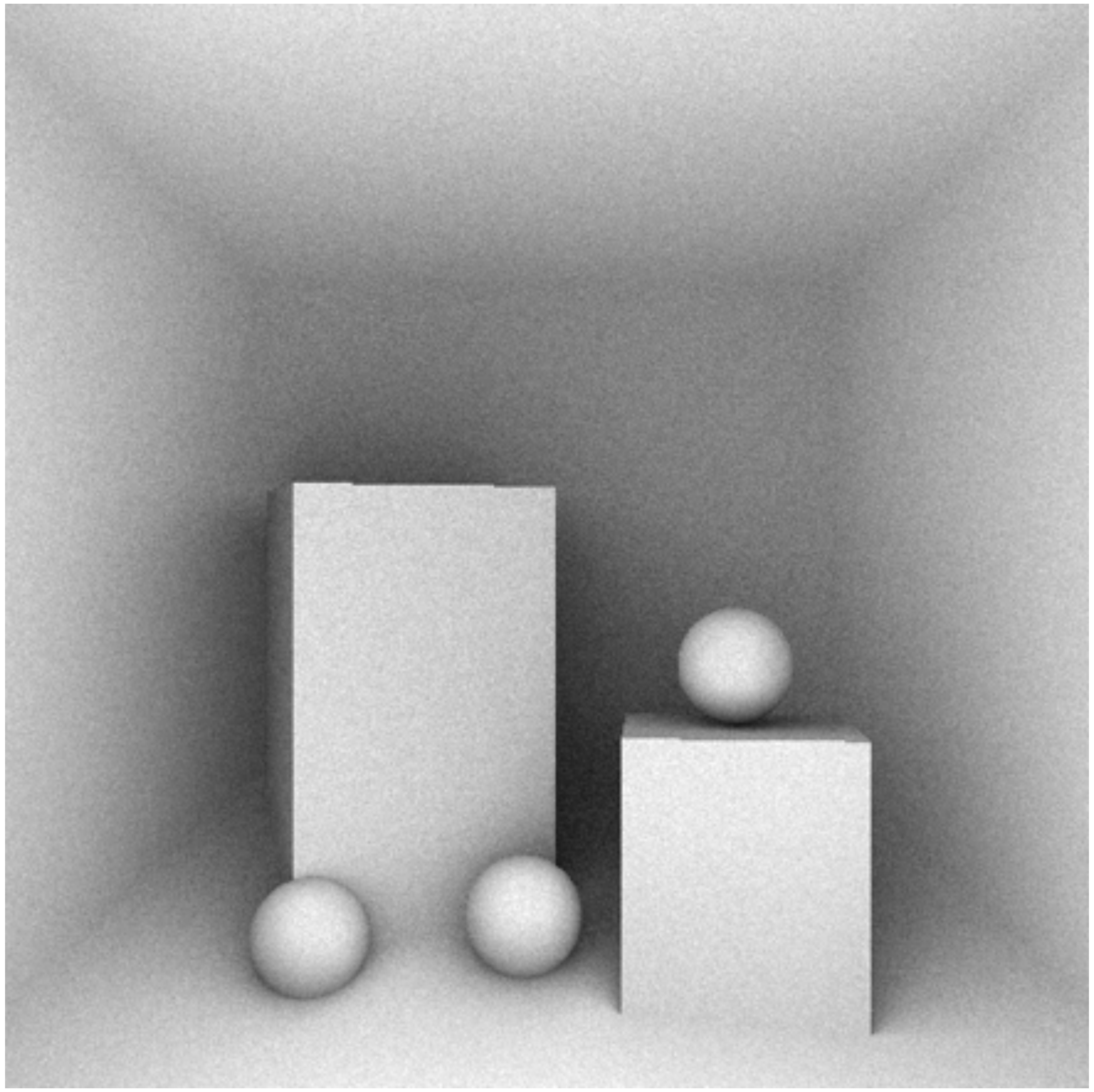
- Random Samples (96 shadow rays)



MSE: 4.74 x 10e-3

Variance Comparison

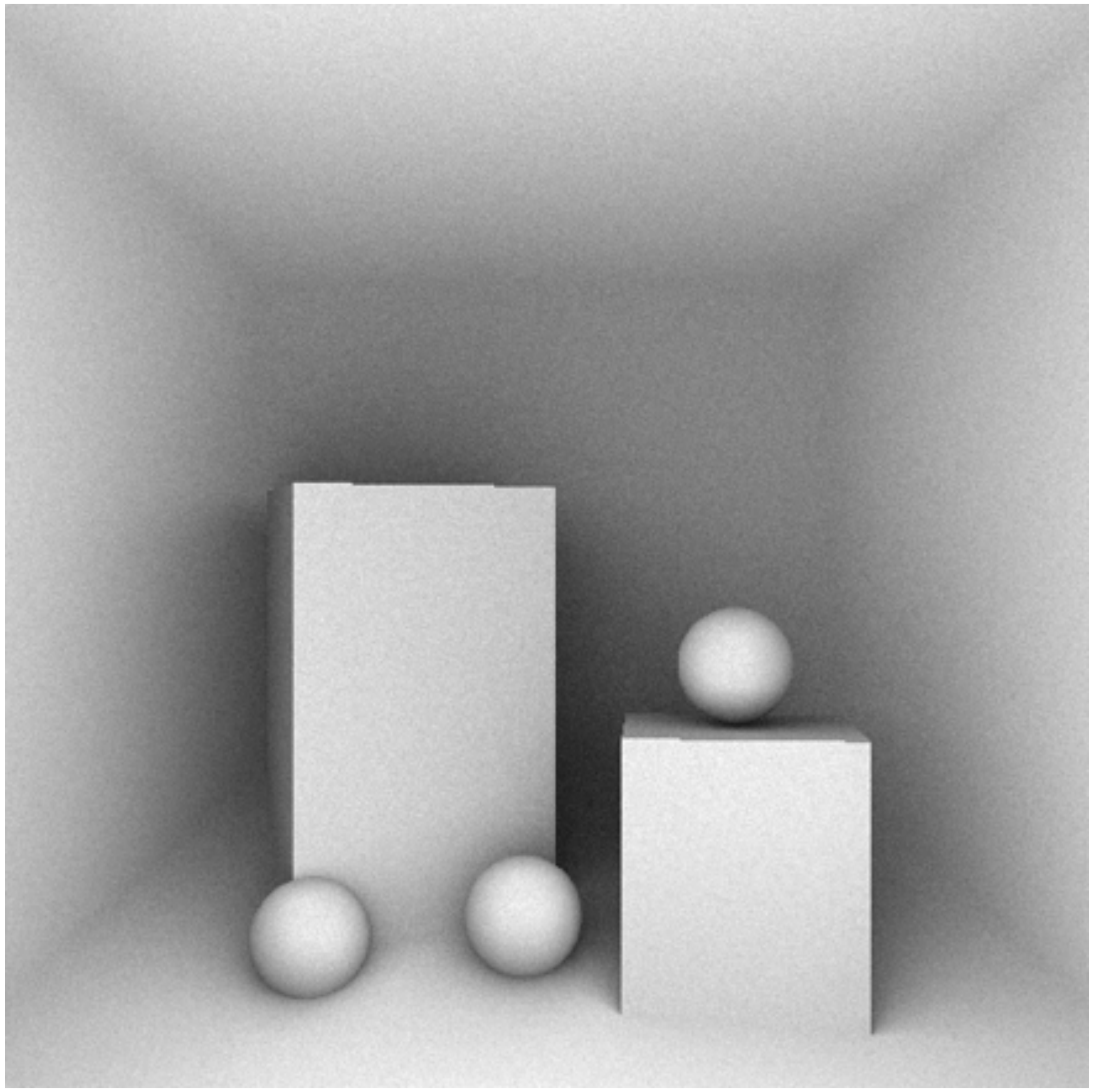
- Jitter Samples (96 shadow rays)



MSE: 8.56 x 10e-4

Variance Comparison

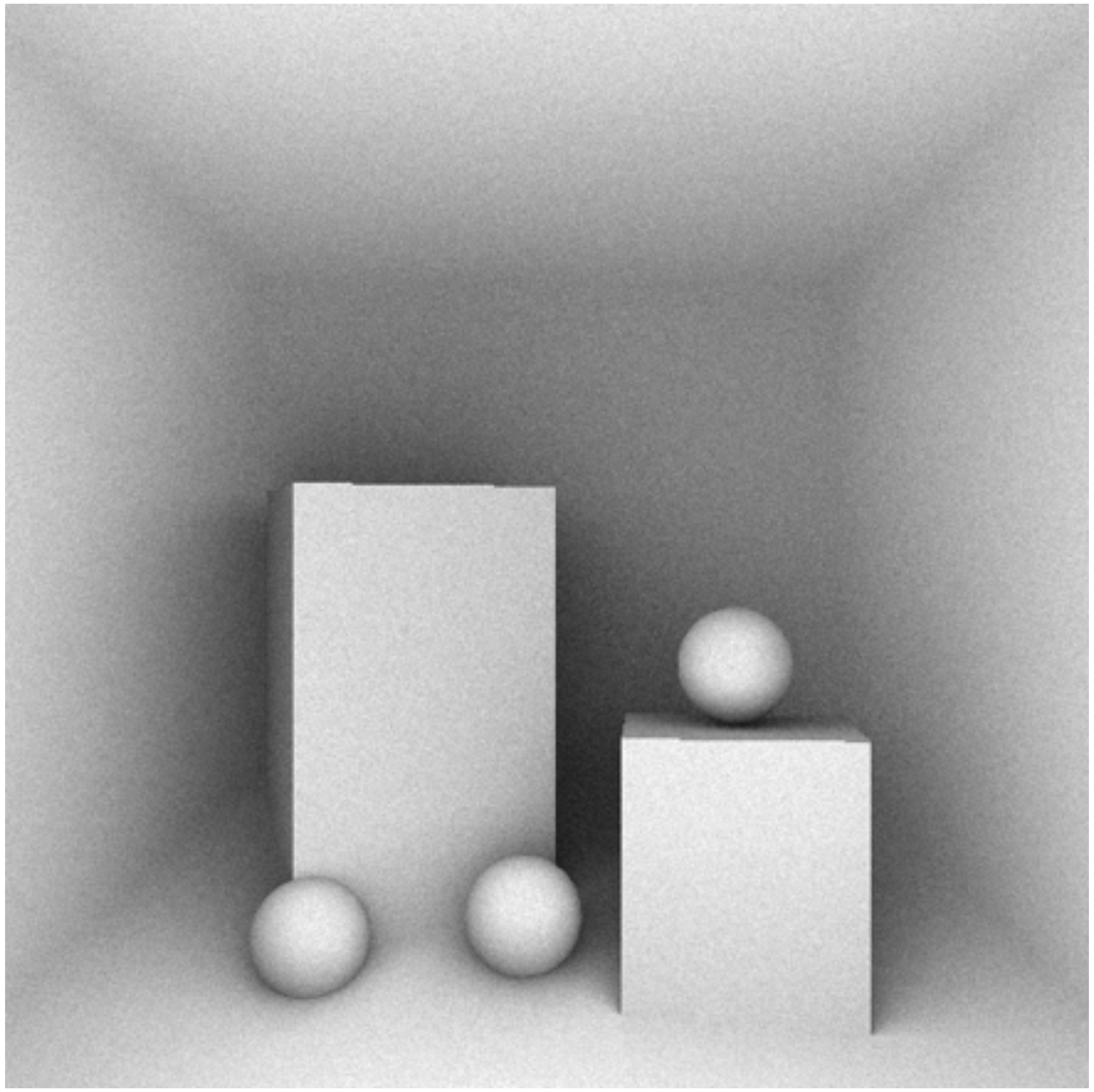
- CCVT (Blue noise) Samples (96 shadow rays)



MSE: 4.24 x 10e-4

Variance Comparison

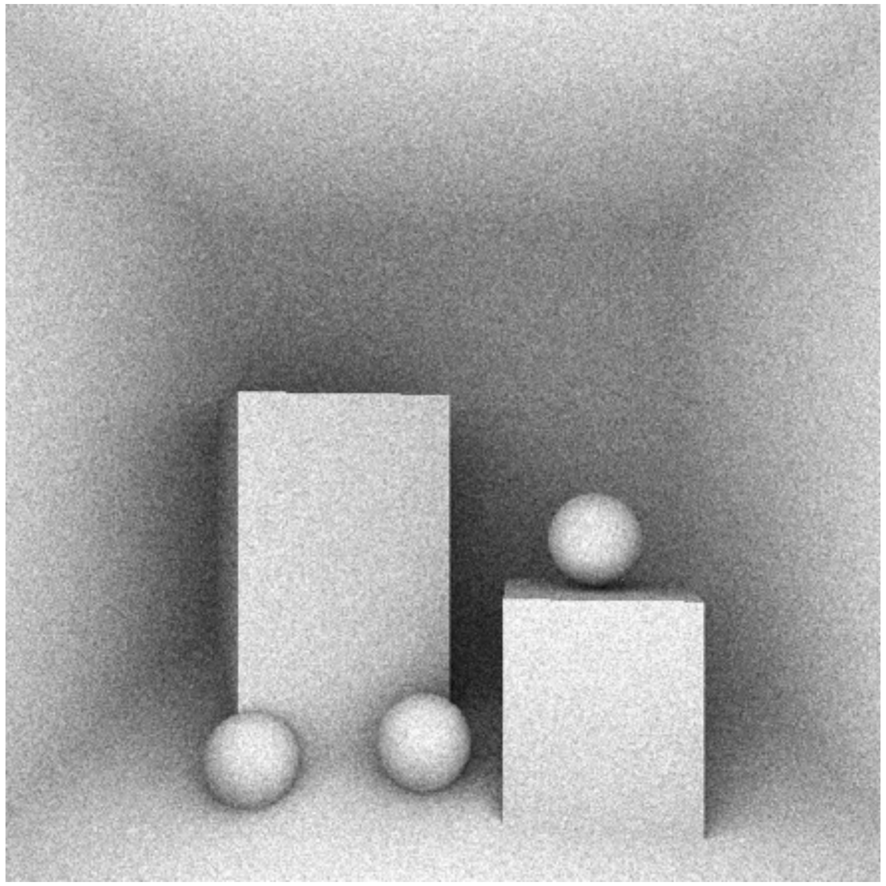
- Poisson Disk (Blue noise) Samples (96 shadow rays)



MSE: 6.95 x 10e-4

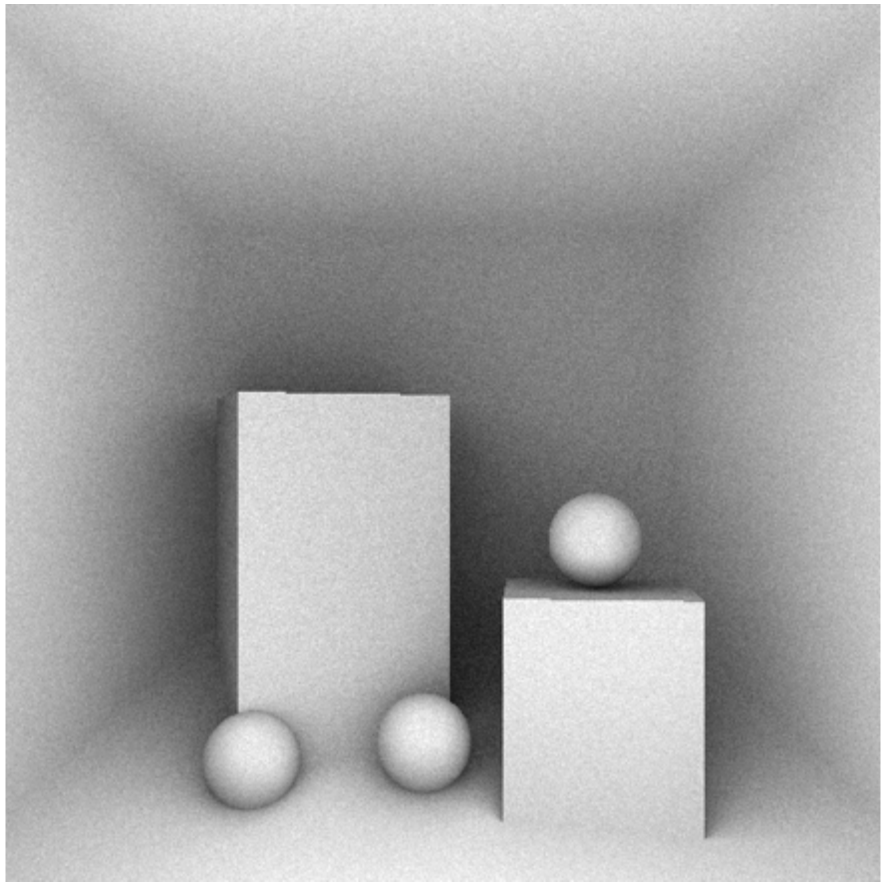
Variance Comparison

Random



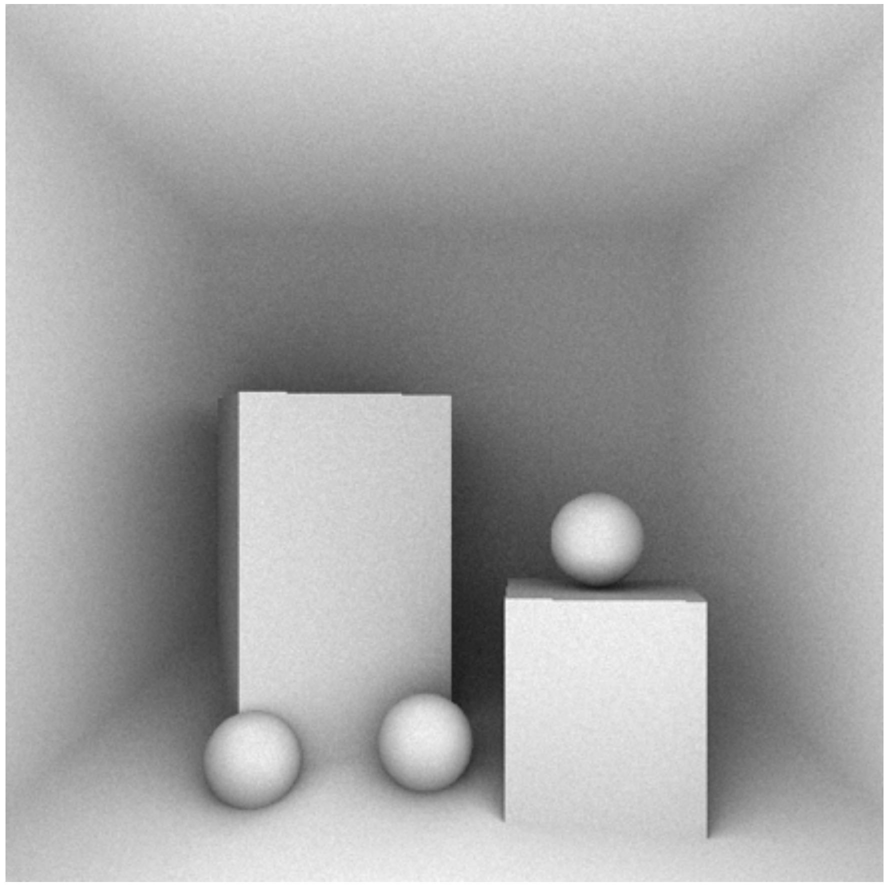
MSE: 4.74 x 10e-3

Jitter



MSE: 8.56 x 10e-4

CCVT

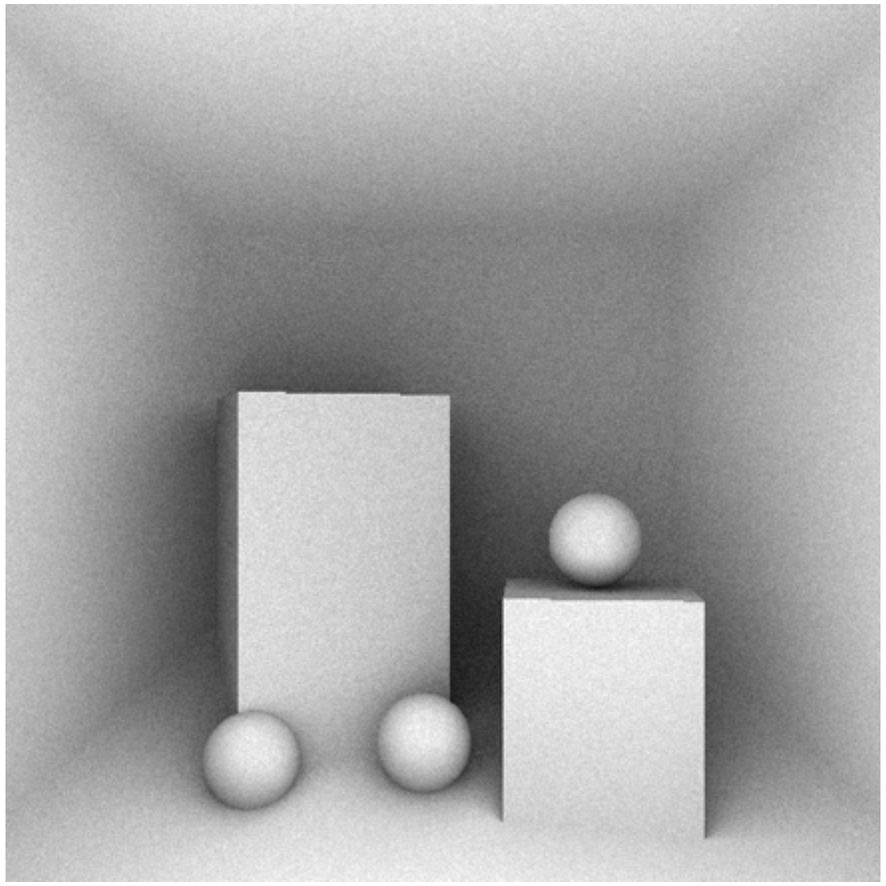


MSE: 4.24 x 10e-4

96 Shadow rays

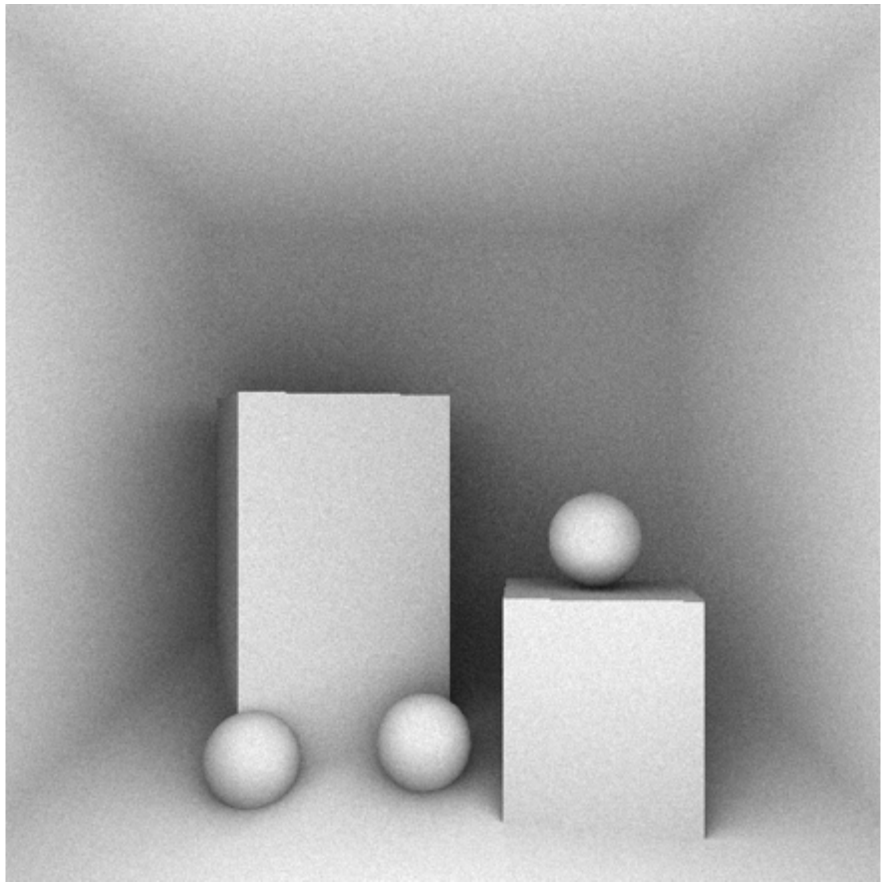
Variance Comparison

Jitter



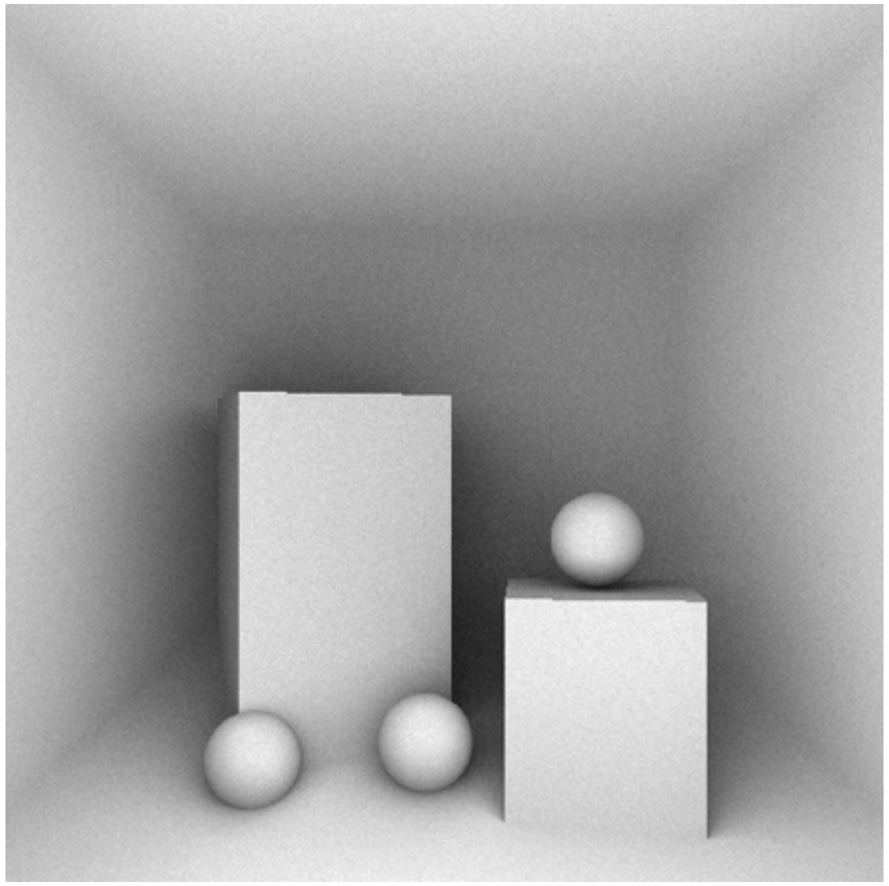
MSE: 8.56 x 10e-4

Poisson Disk



MSE: 6.95 x 10e-4

CCVT



MSE: 4.24 x 10e-4

96 Shadow rays

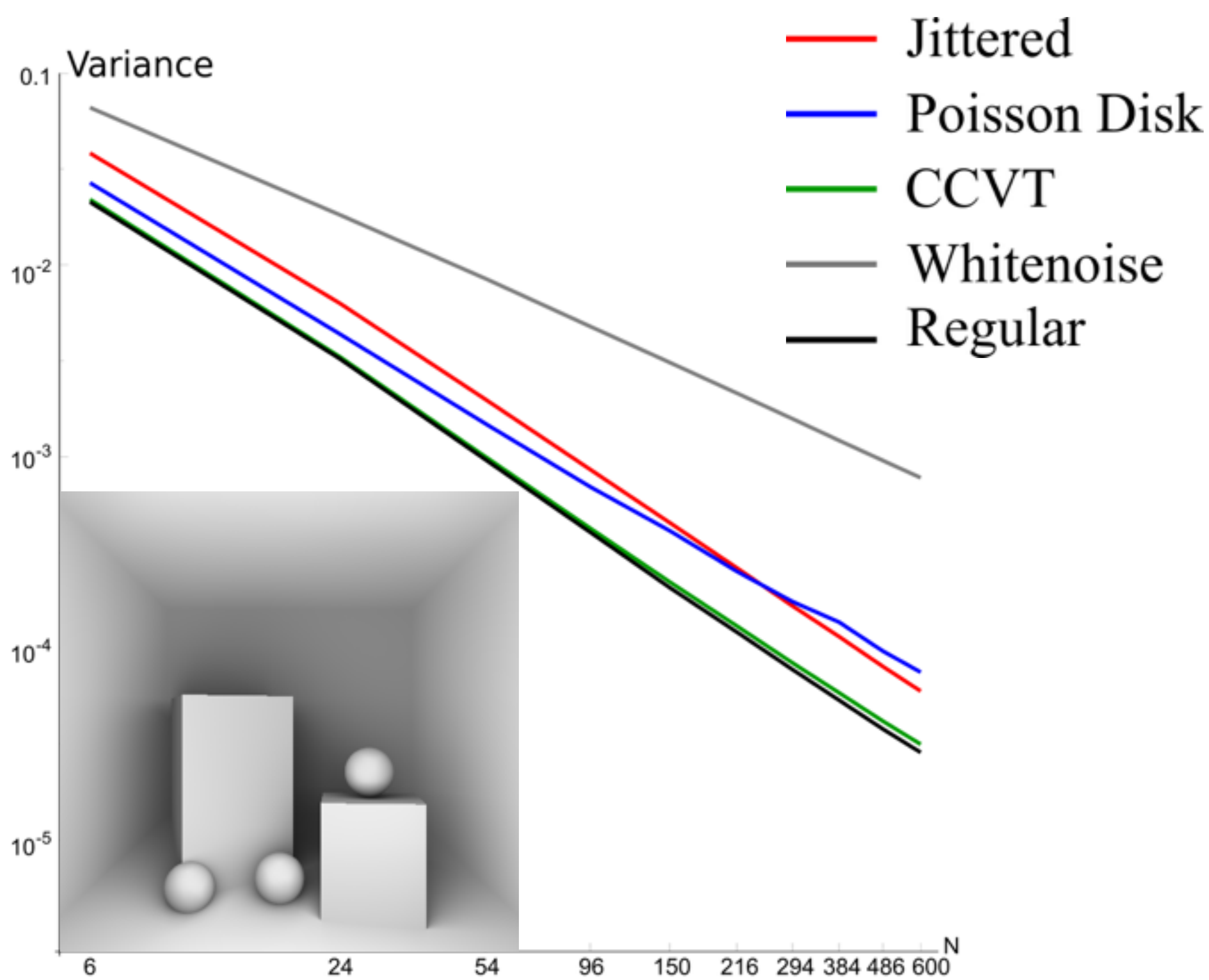
Convergence Rate Comparison

Convergence rate

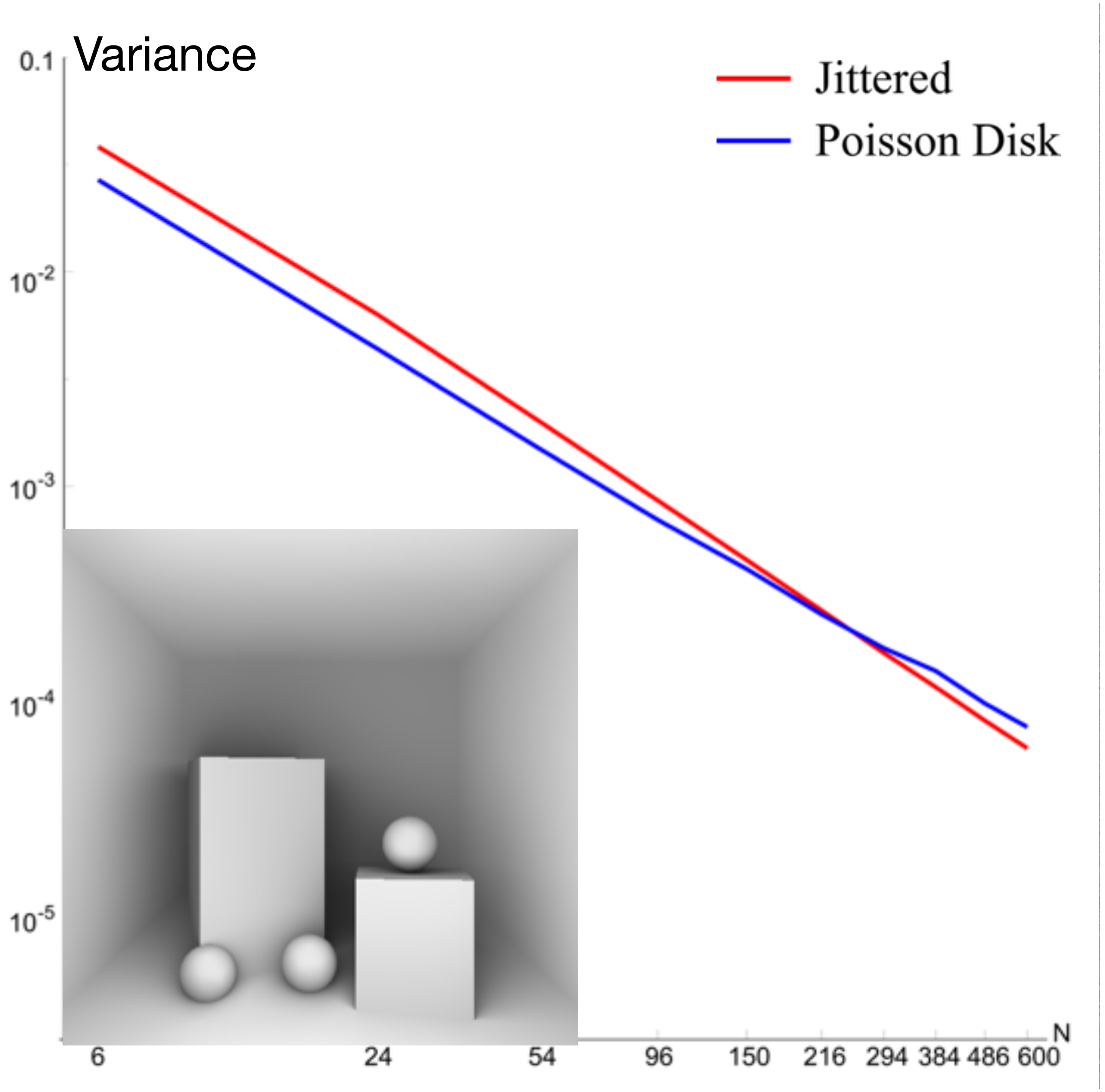
- How fast the variance (noise) in the image is vanishing as we increase the number of samples ?

Variance vs Number of Samples

Convergence analysis



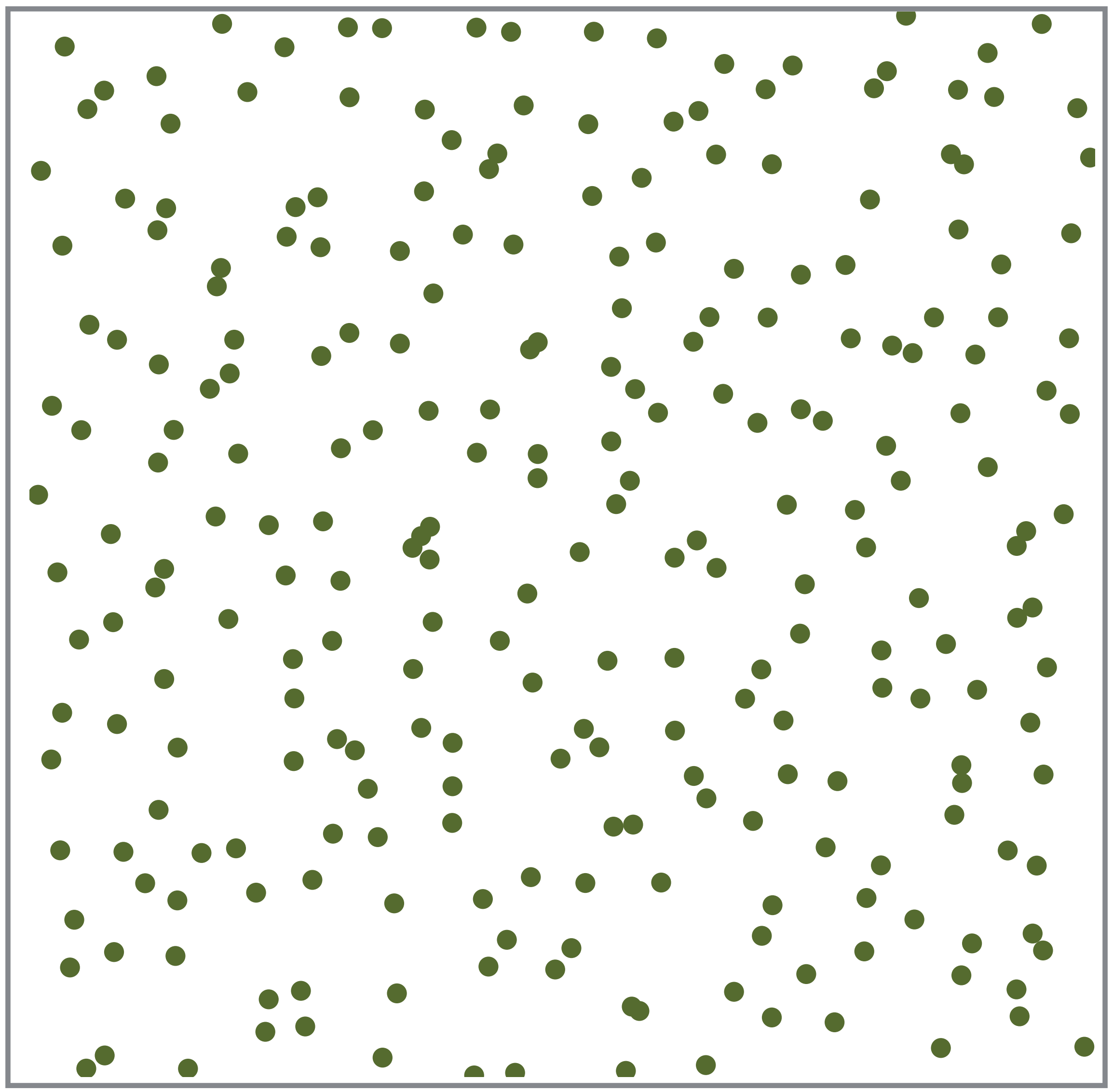
Convergence analysis



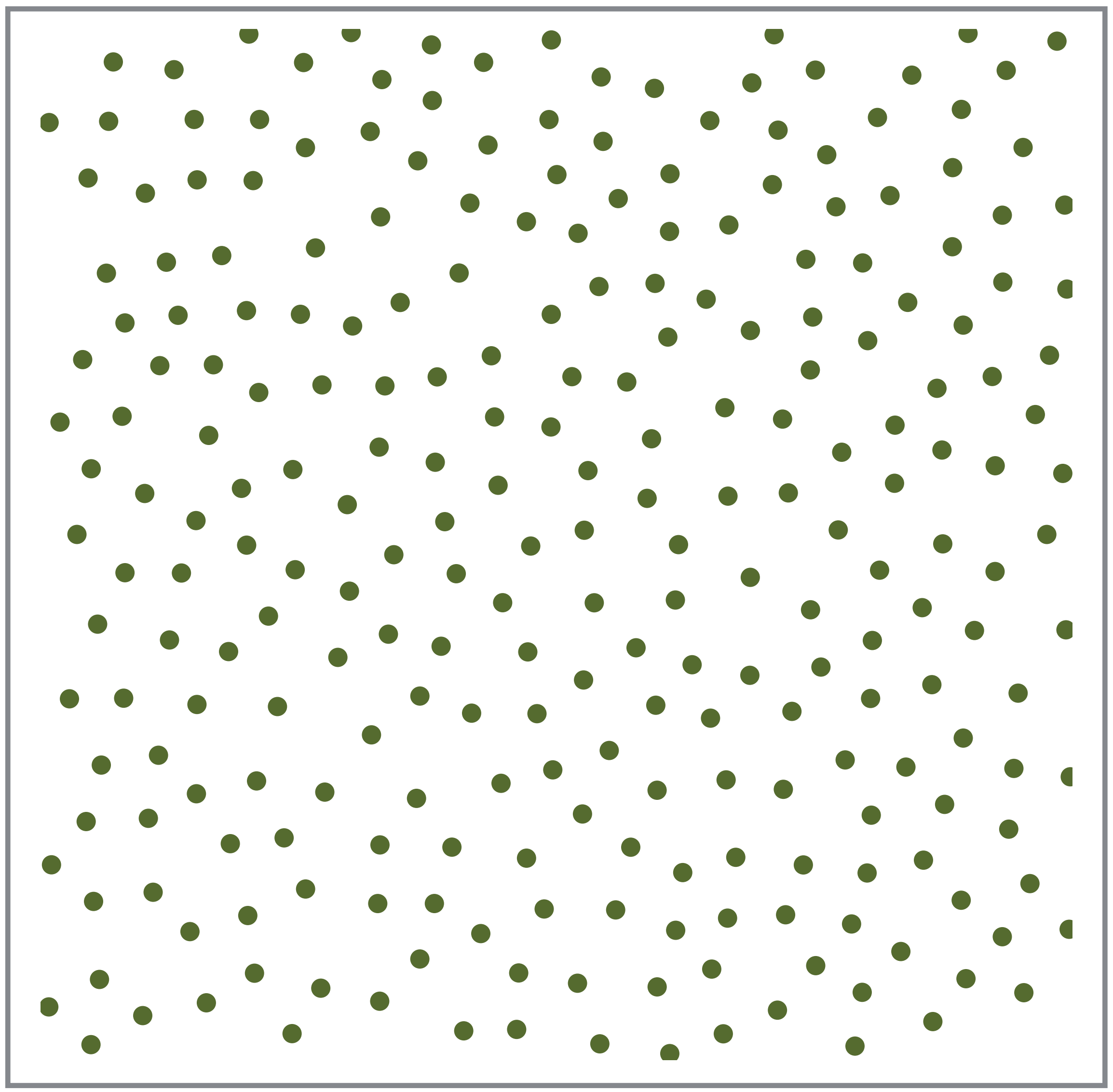
Spatial Distribution
vs
Fourier Power Spectrum

Sample distribution

Jitter

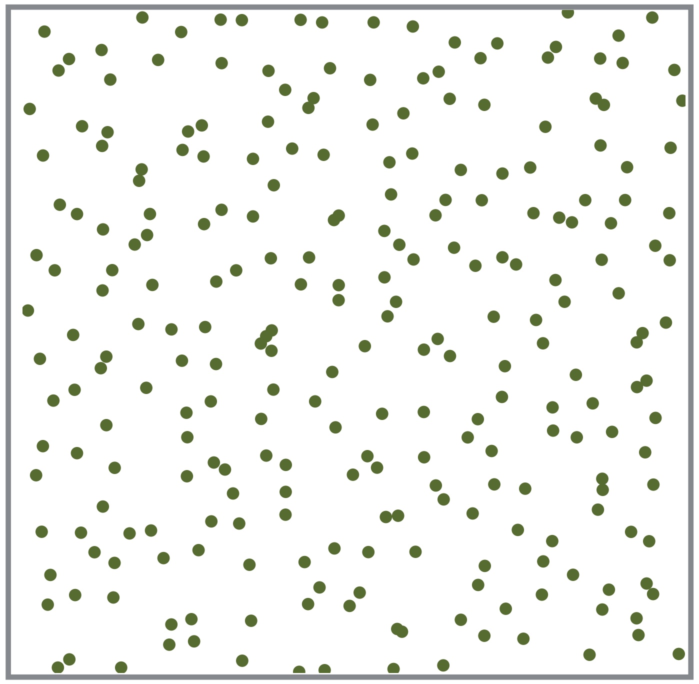


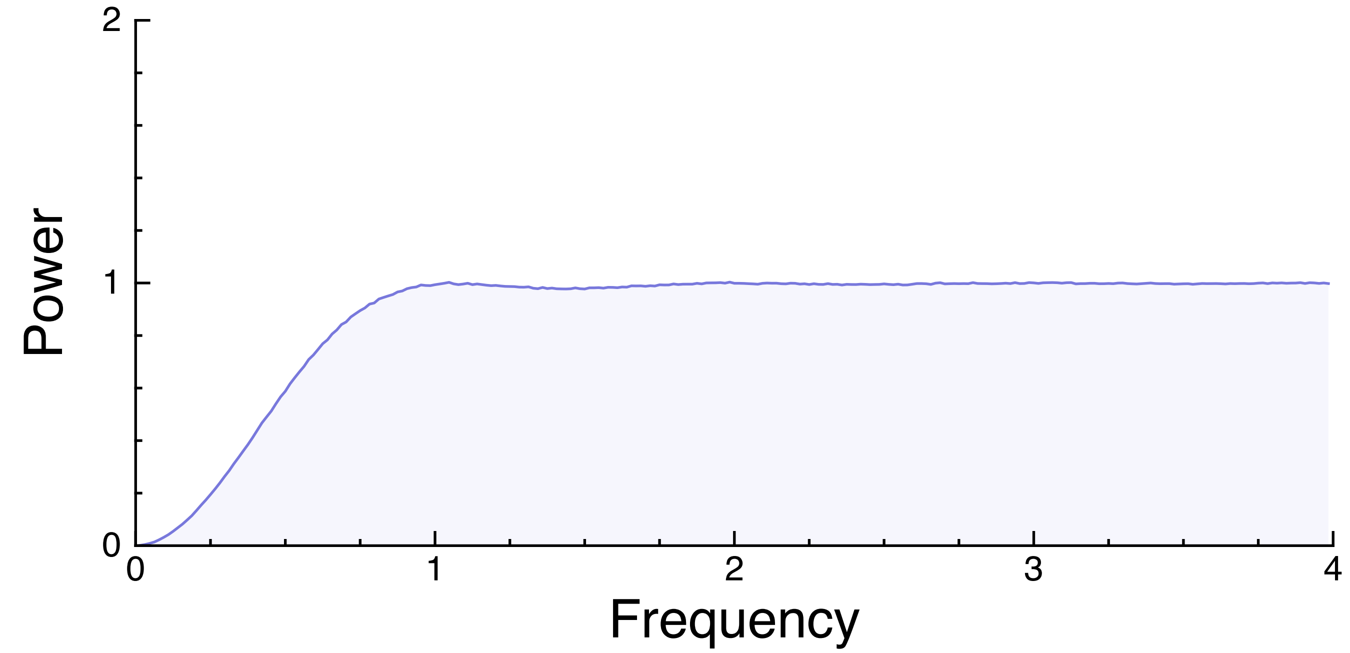
Poisson Disk



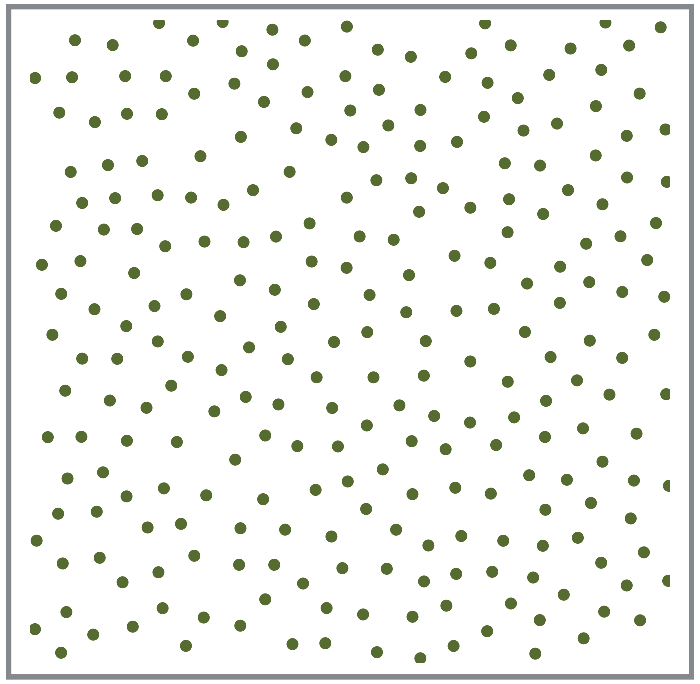
Fourier Radial Power Spectrum

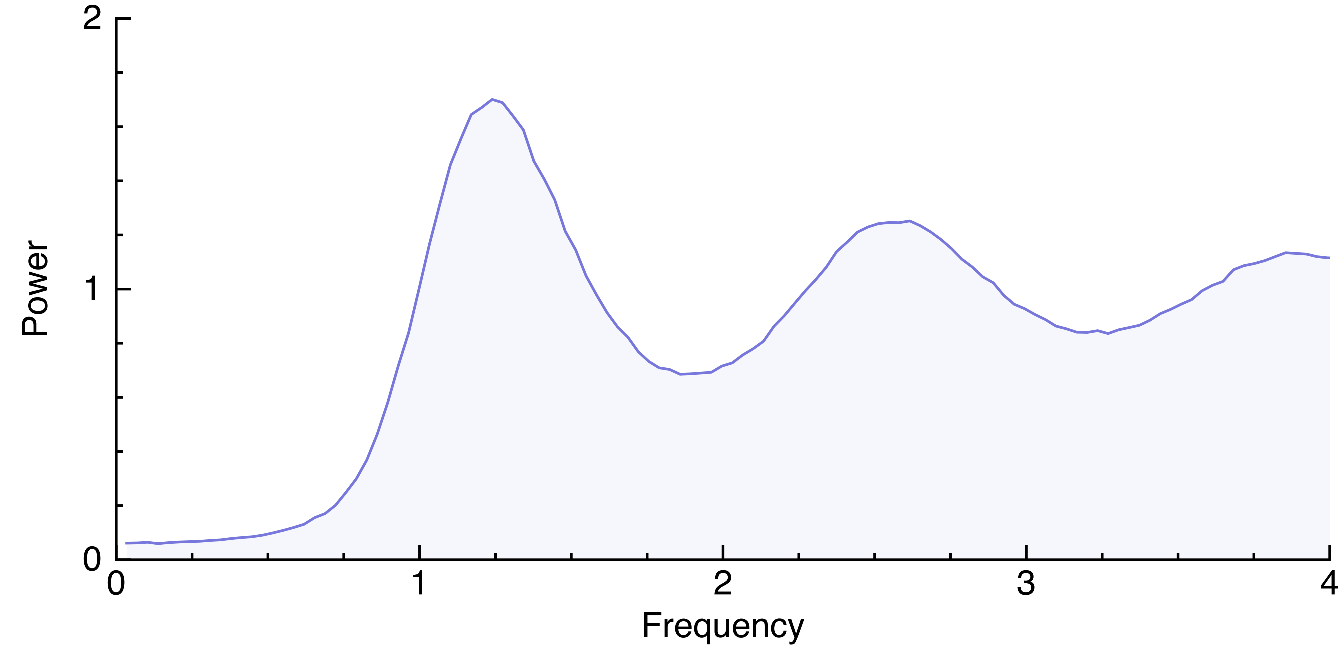
Jitter



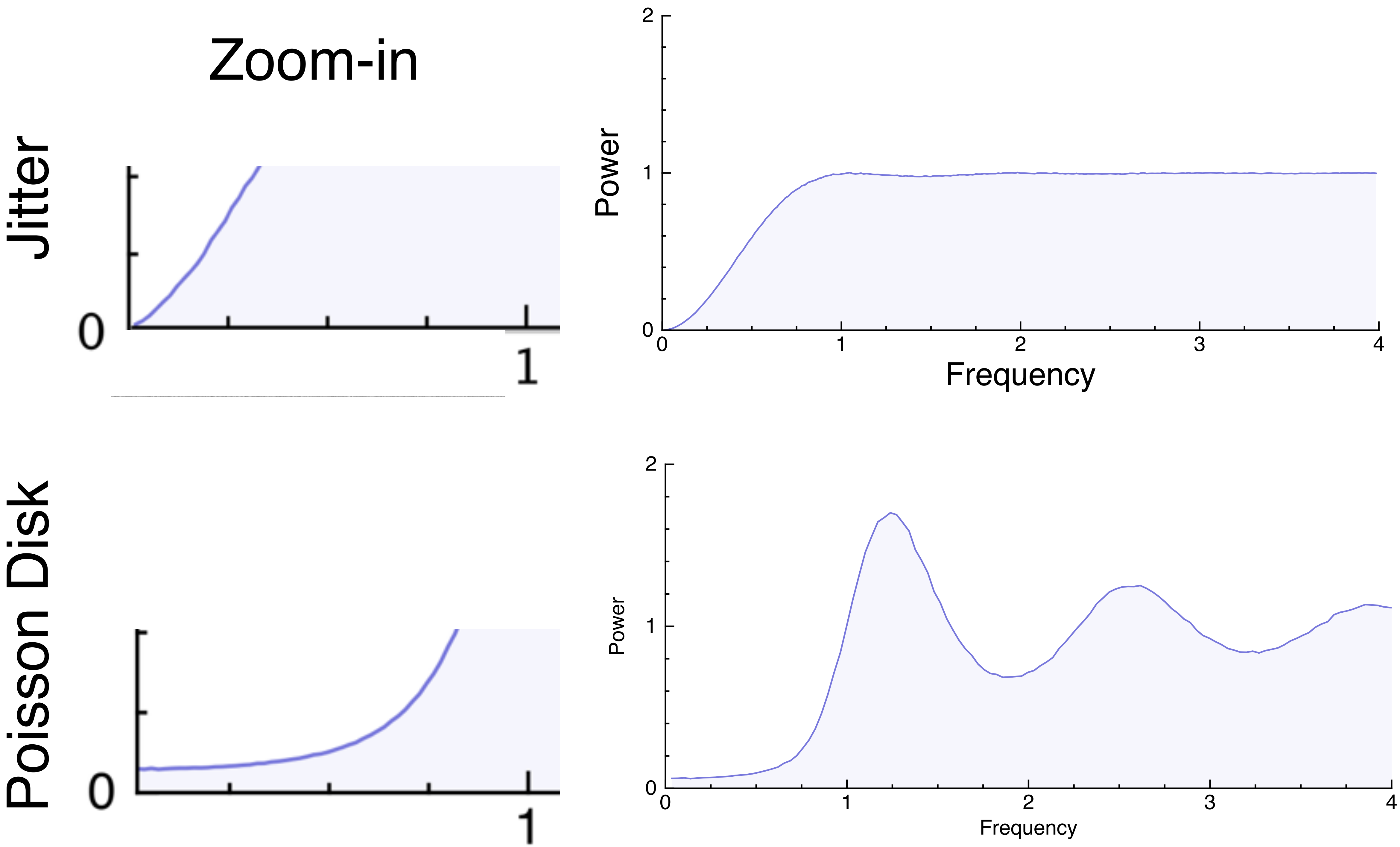


Poisson Disk





Fourier Radial Power Spectrum



Pilleboue et al. [2015]

Visual Break

Godfire from CG Society

Deriving Convergence Rates

Variance Formulation

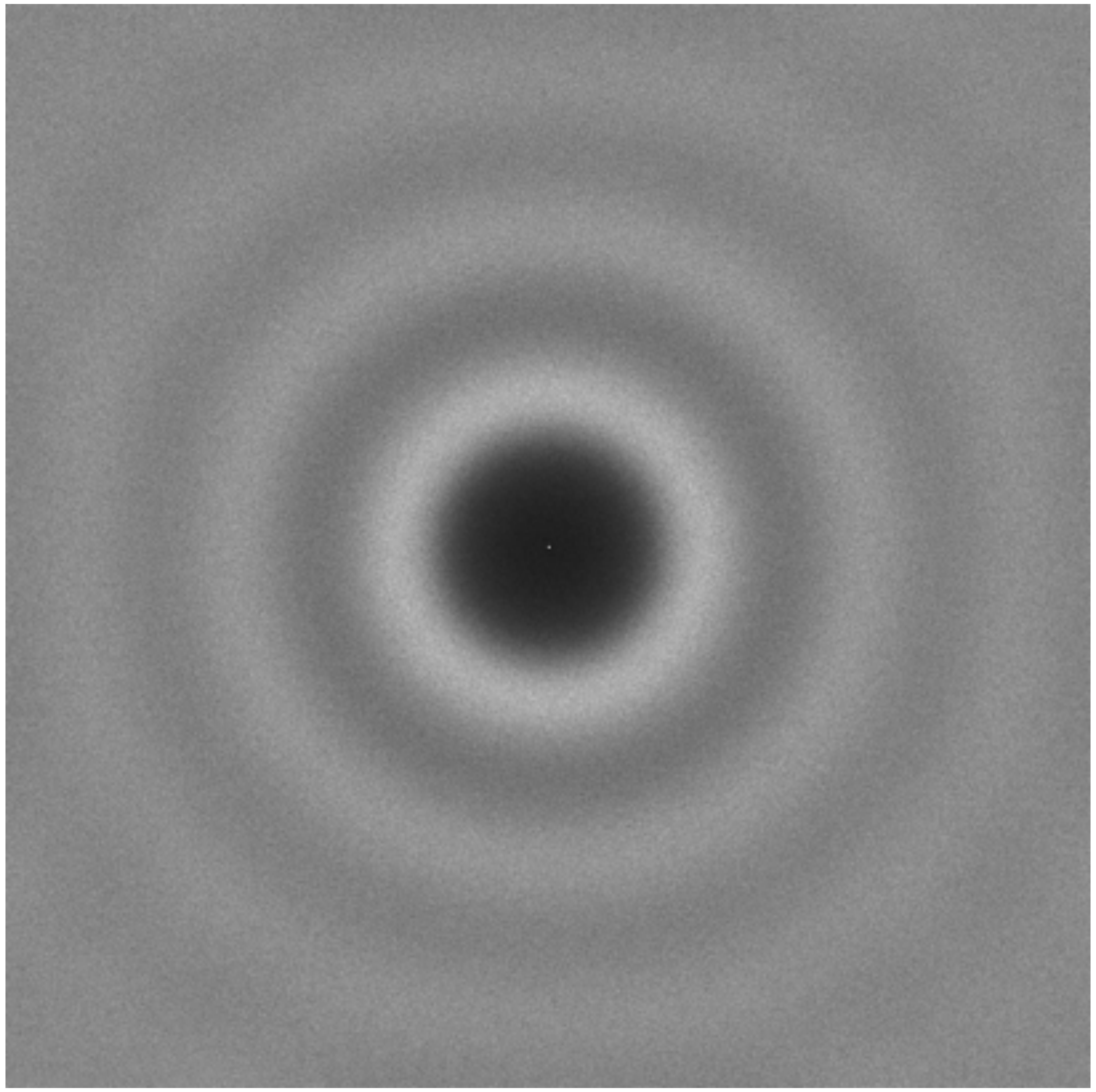
$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega) \langle \mathcal{P}_S(\omega) \rangle d\omega$$

Can we simplify this ?

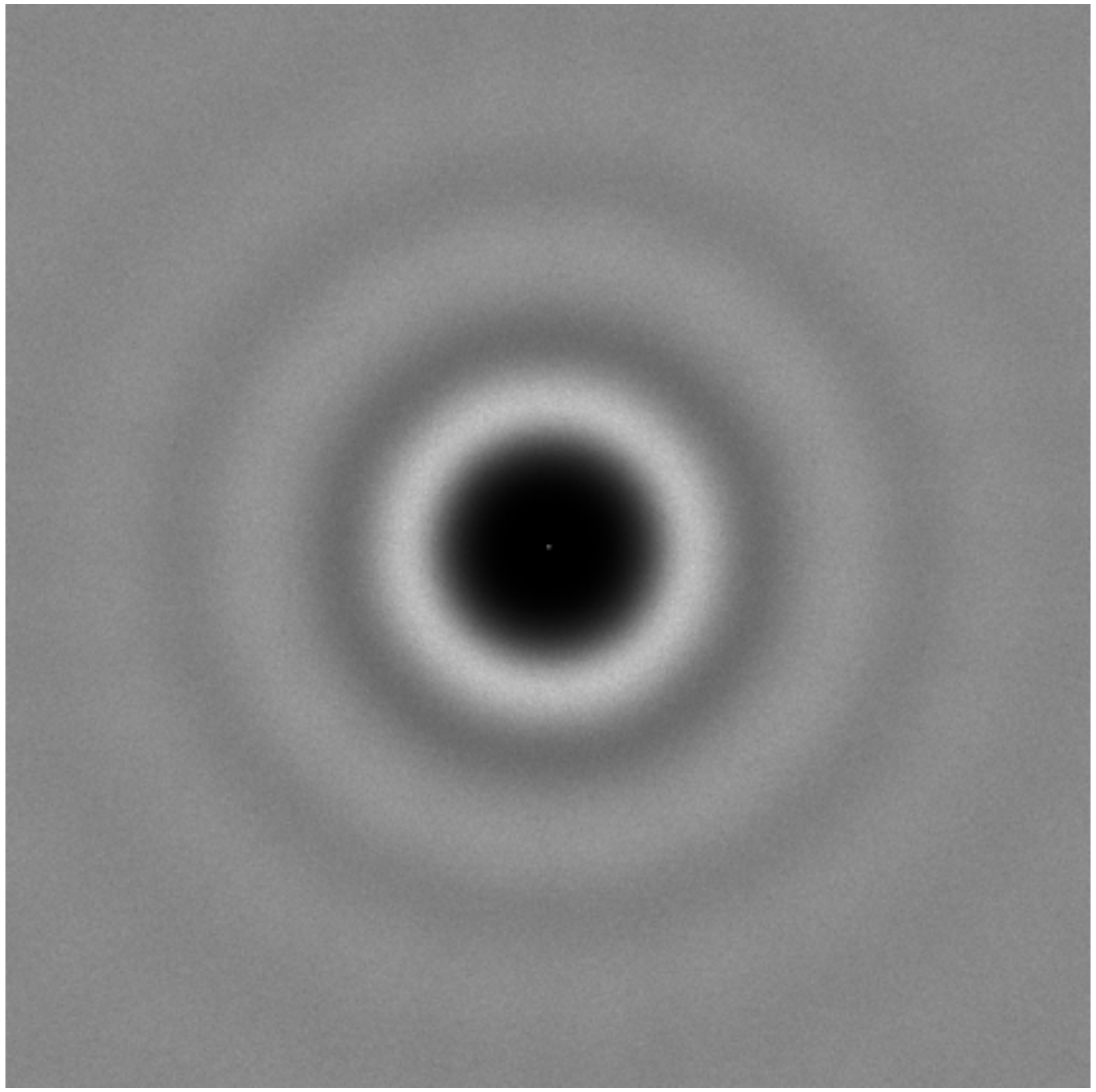
Pilleboue et al. [2015]

Point Samples' Power Spectrum

- Isotropic Sampling Power Spectra



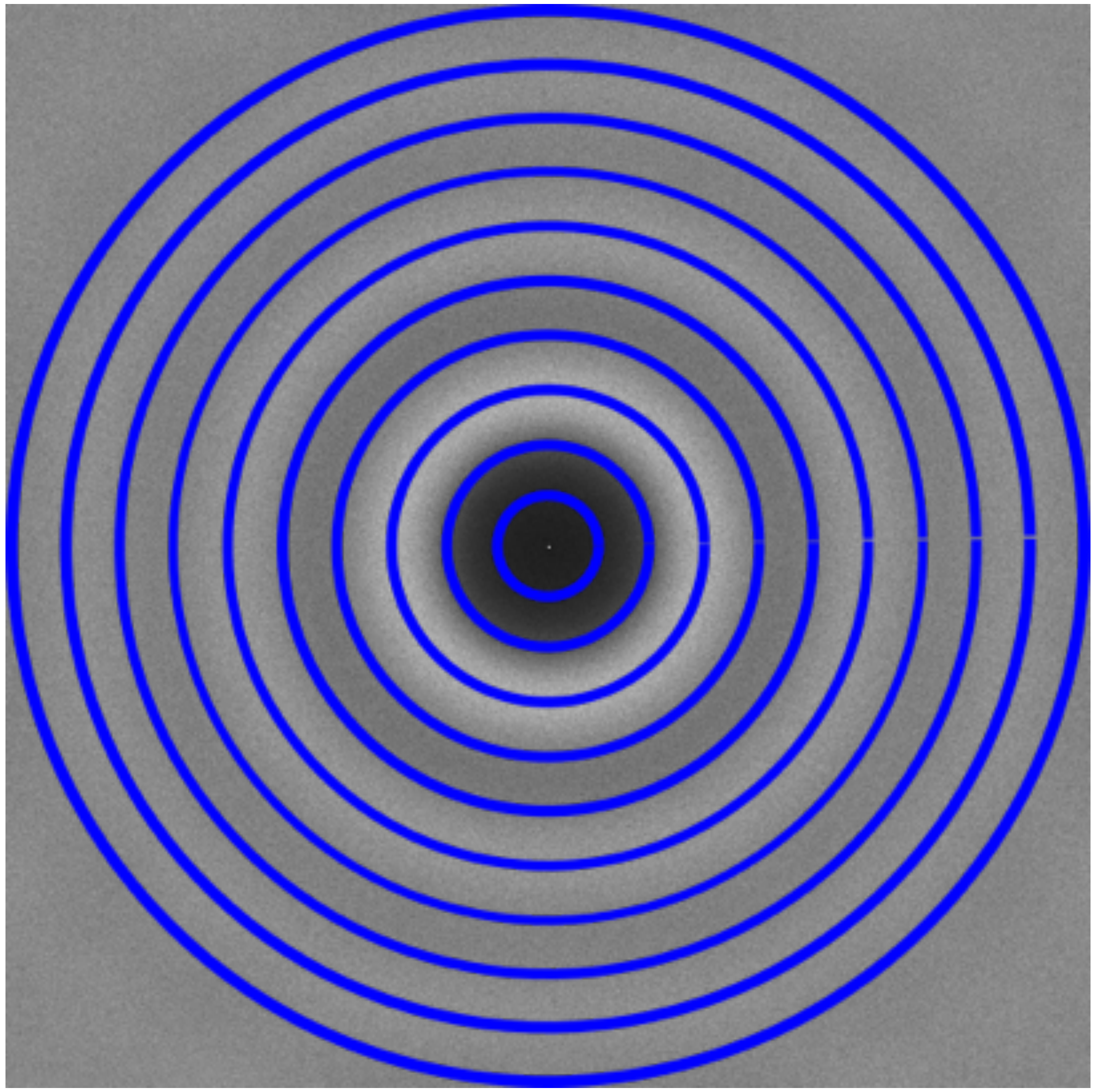
Poisson Disk



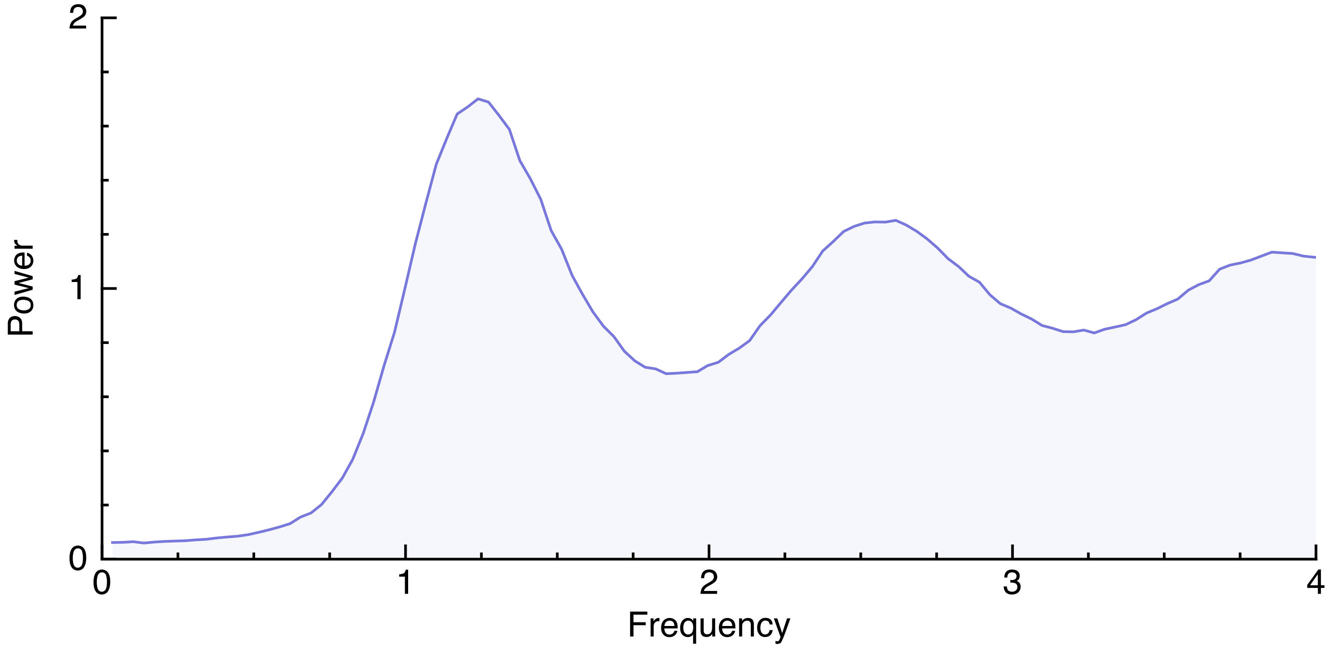
CCVT

Fourier Radial Power Spectrum

- Radial Spectrum



Poisson Disk



Variance Formulation

$$Var(I_N) = \int_{-\infty}^{\infty} \mathcal{P}_F(\omega) \langle \mathcal{P}_S(\omega) \rangle d\omega$$

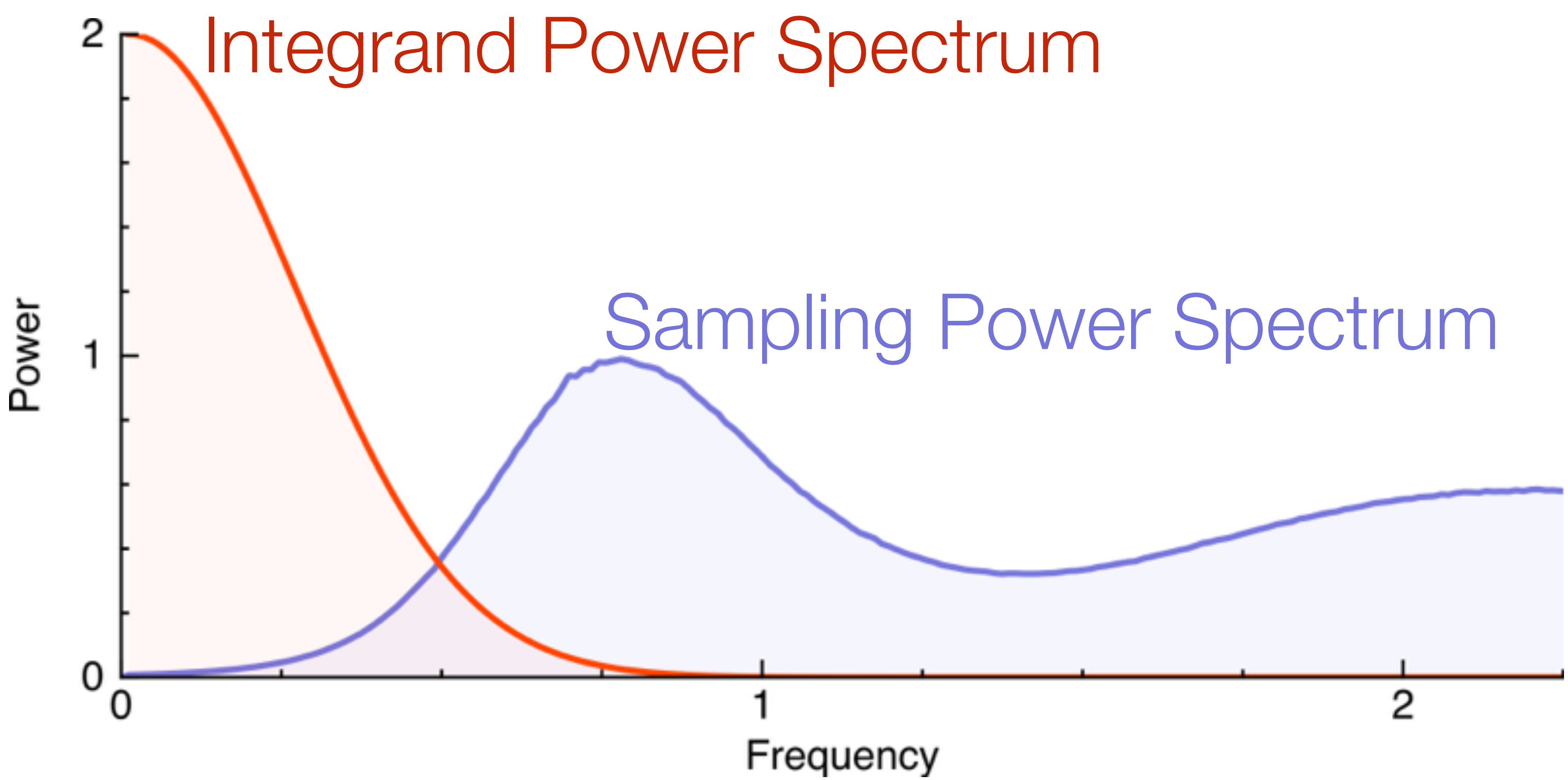
For Isotropic power spectrum, we can rewrite in radial form:

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \mathcal{P}_F(\rho) \langle \mathcal{P}_S(\rho) \rangle d\rho$$

Pilleboue et al. [2015]

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \mathcal{P}_F(\rho) \langle \mathcal{P}_S(\rho) \rangle d\rho$$

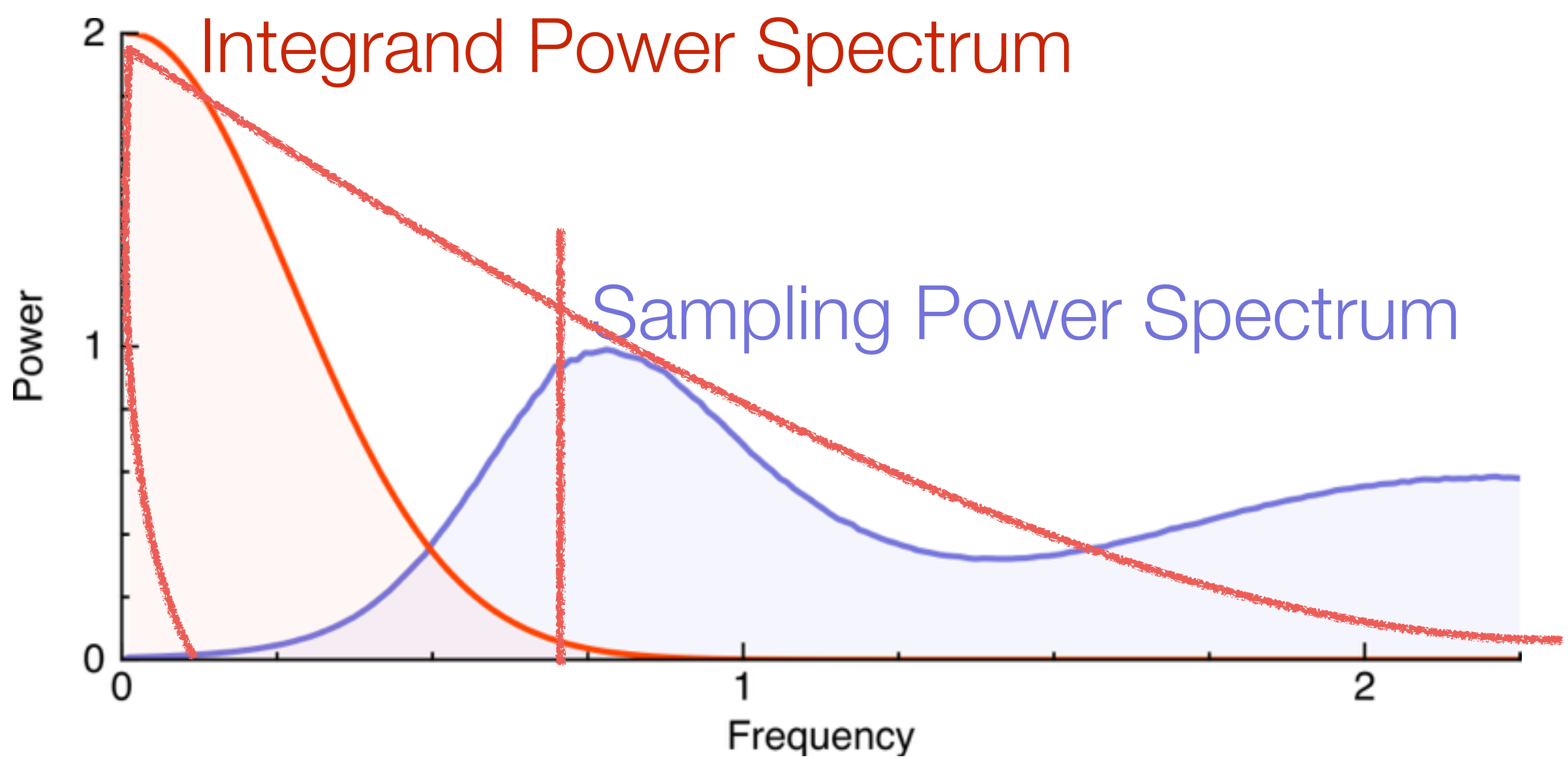


For given number of Samples

Pilleboue et al. [2015]

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \boxed{\mathcal{P}_F(\rho)} \langle \boxed{\mathcal{P}_S(\rho)} \rangle d\rho$$



For given number of Samples

Pilleboue et al. [2015]

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \mathcal{P}_F(\rho) \langle \mathcal{P}_S(\rho) \rangle d\rho$$

Best possible Integrand Power spectrum

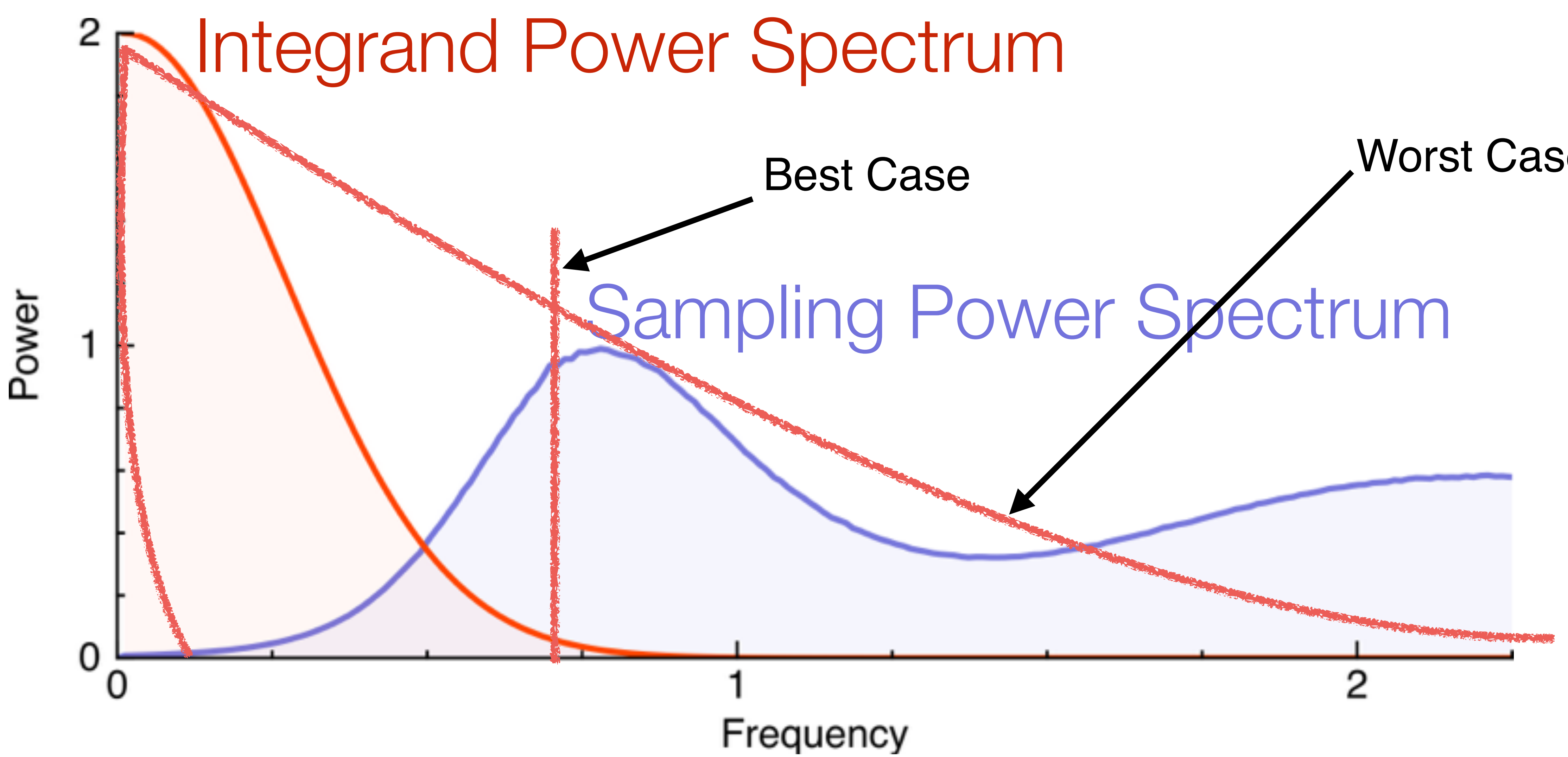
⋮

Worst possible Integrand Power spectrum

Pilleboue et al. [2015]

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \mathcal{P}_F(\rho) \langle \mathcal{P}_S(\rho) \rangle d\rho$$



For given number of Samples

Pilleboue et al. [2015]

Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \boxed{\mathcal{P}_F(\rho)} \langle \boxed{\mathcal{P}_S(\rho)} \rangle d\rho$$

???

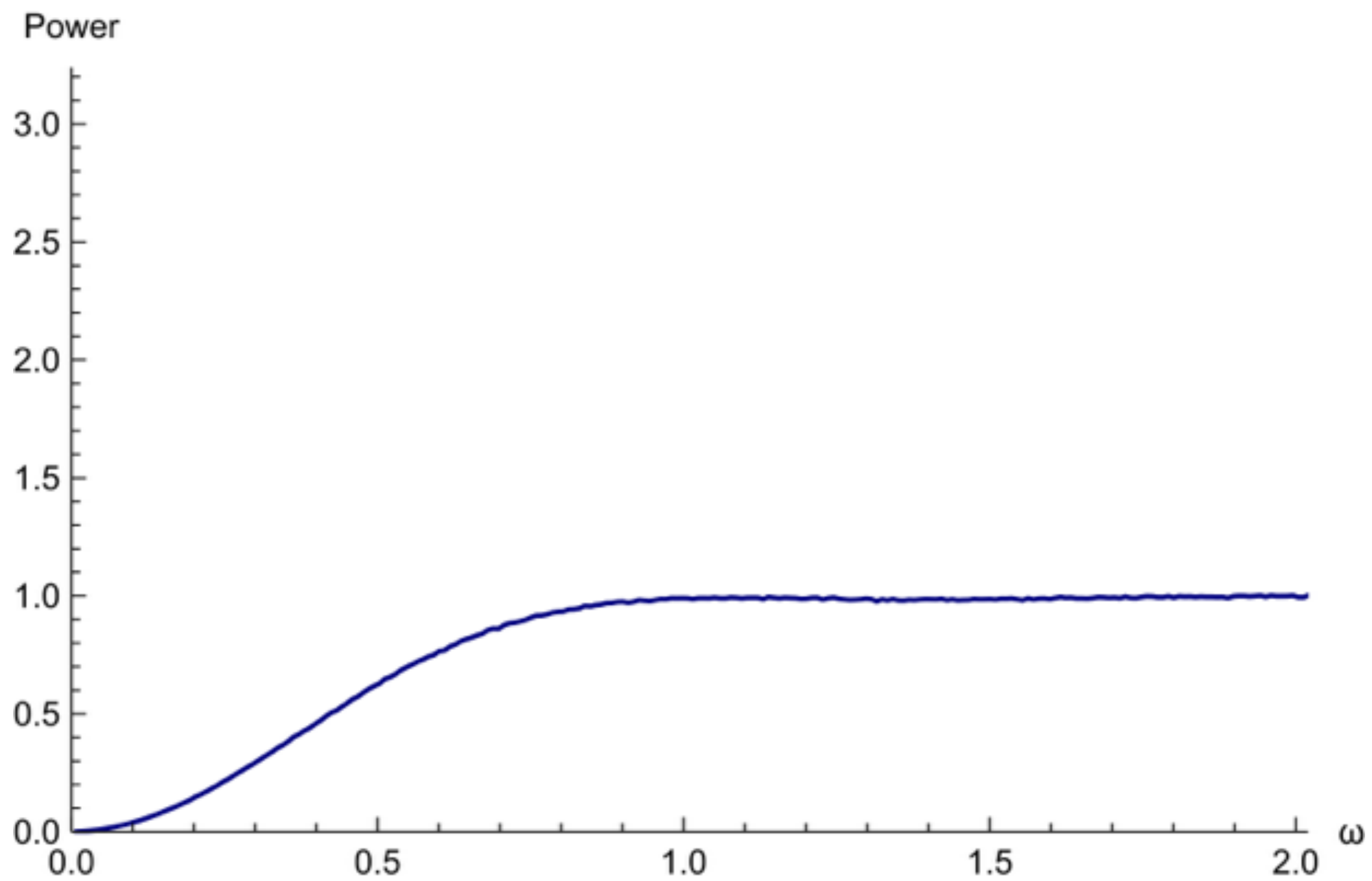
Best possible Integrand Power spectrum

⋮

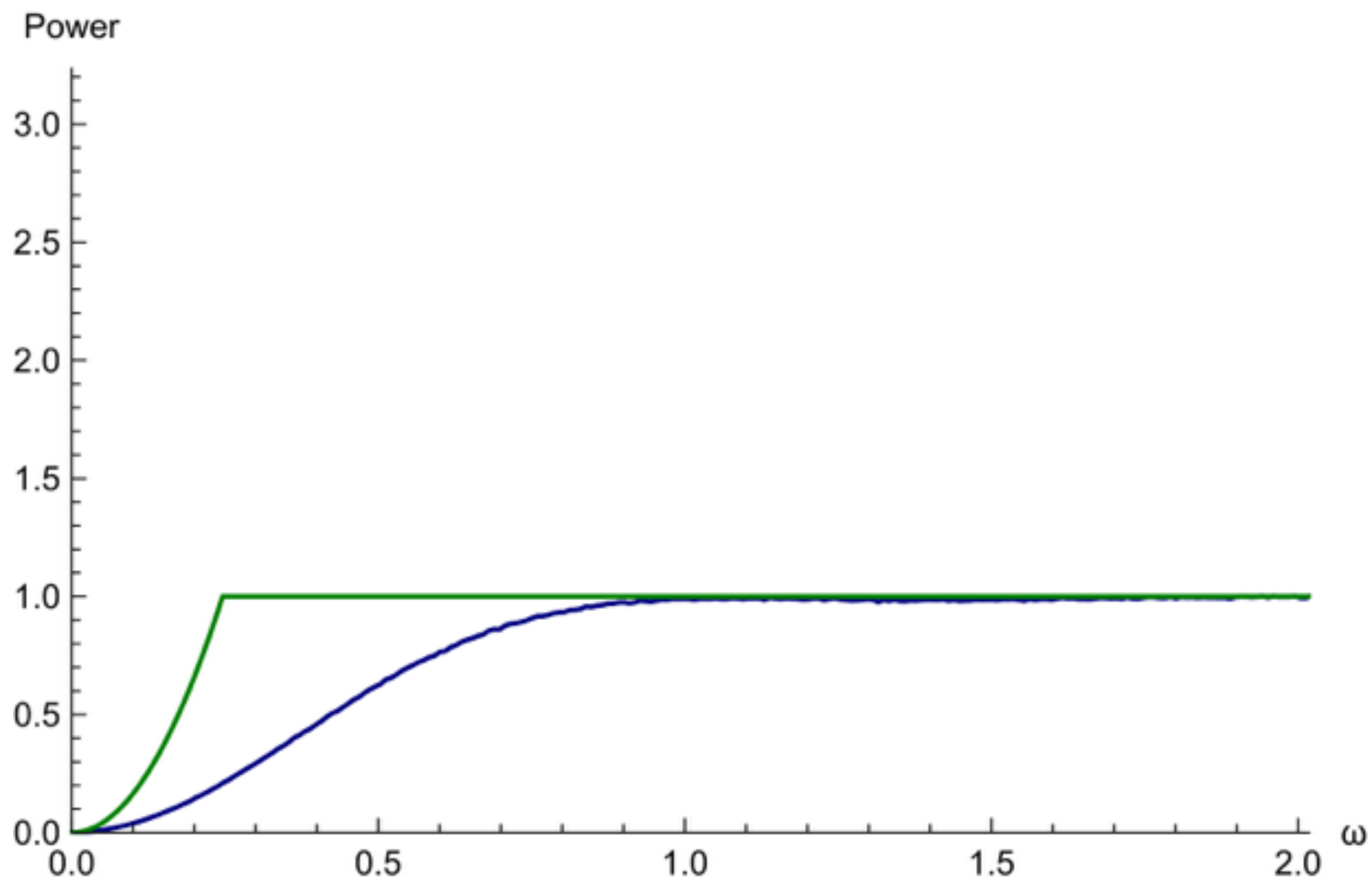
Worst possible Integrand Power spectrum

Pilleboue et al. [2015]

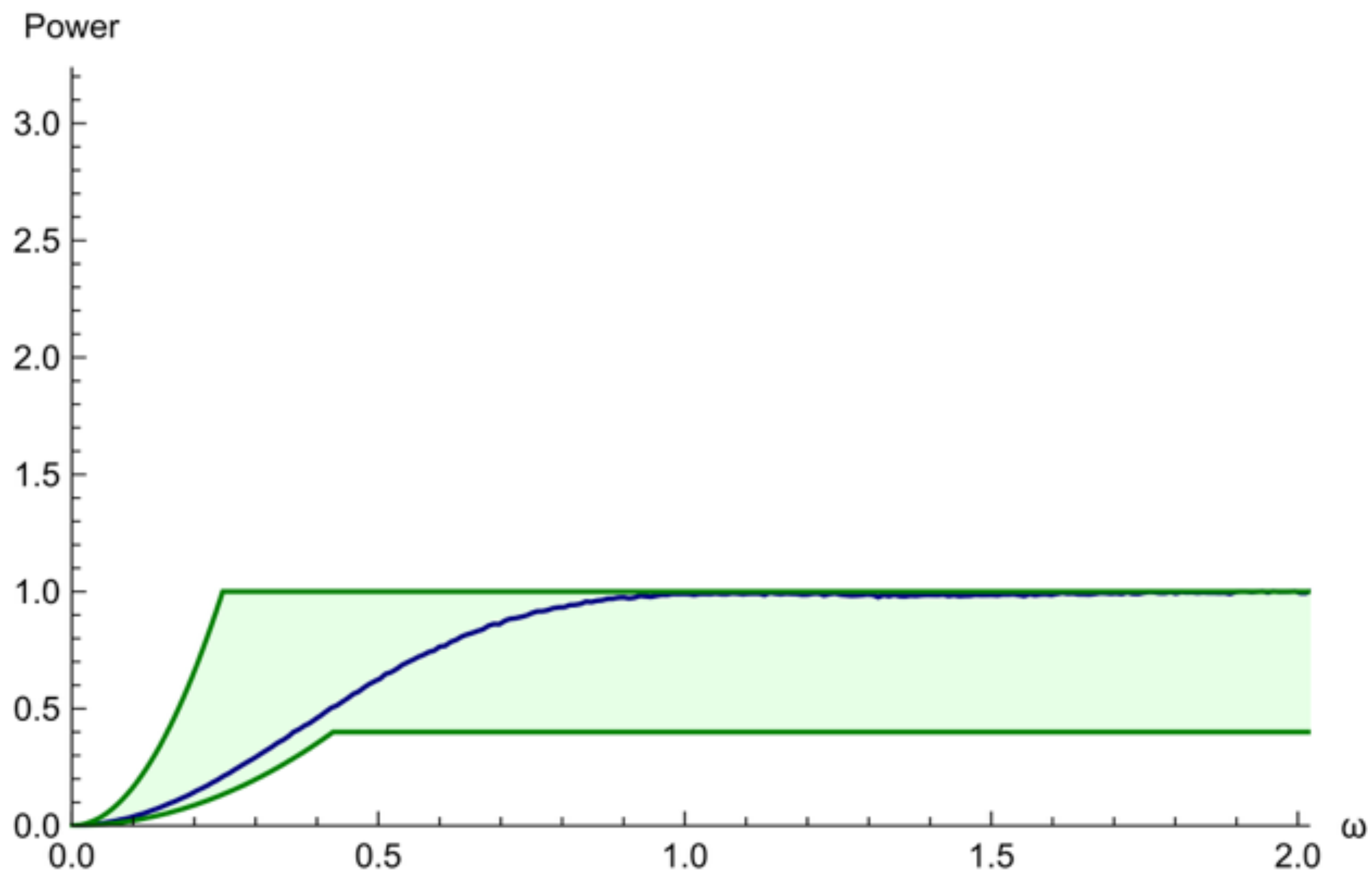
Sampling Radial Spectrum



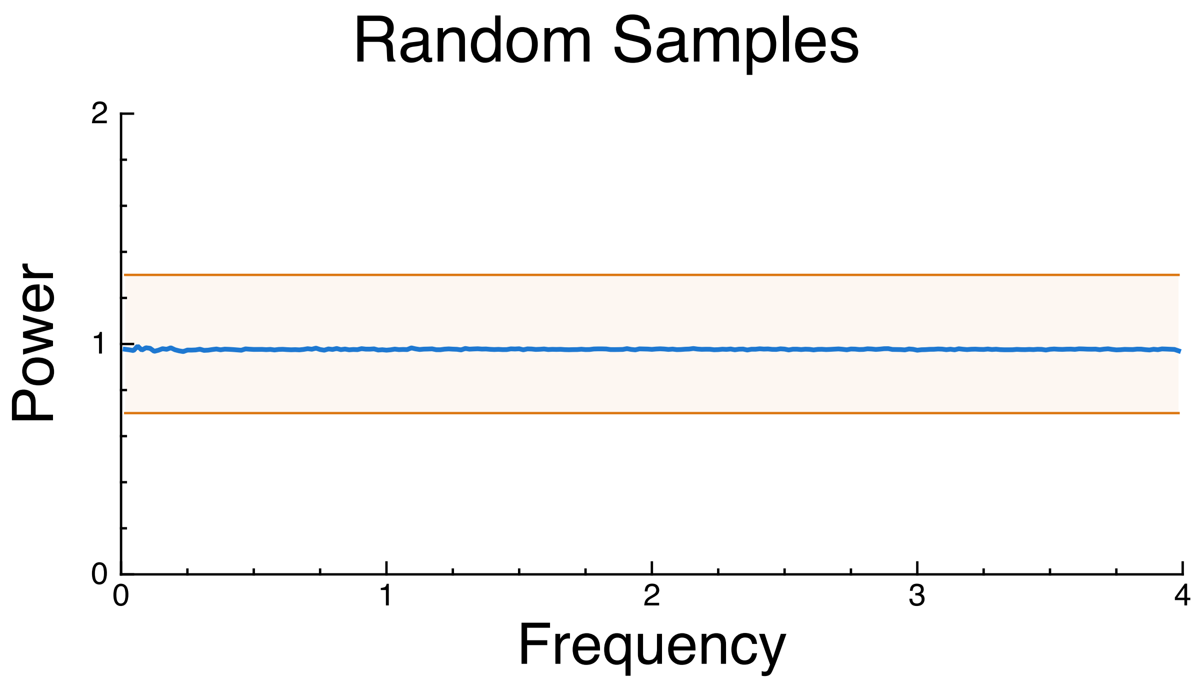
Sampling Radial Spectrum

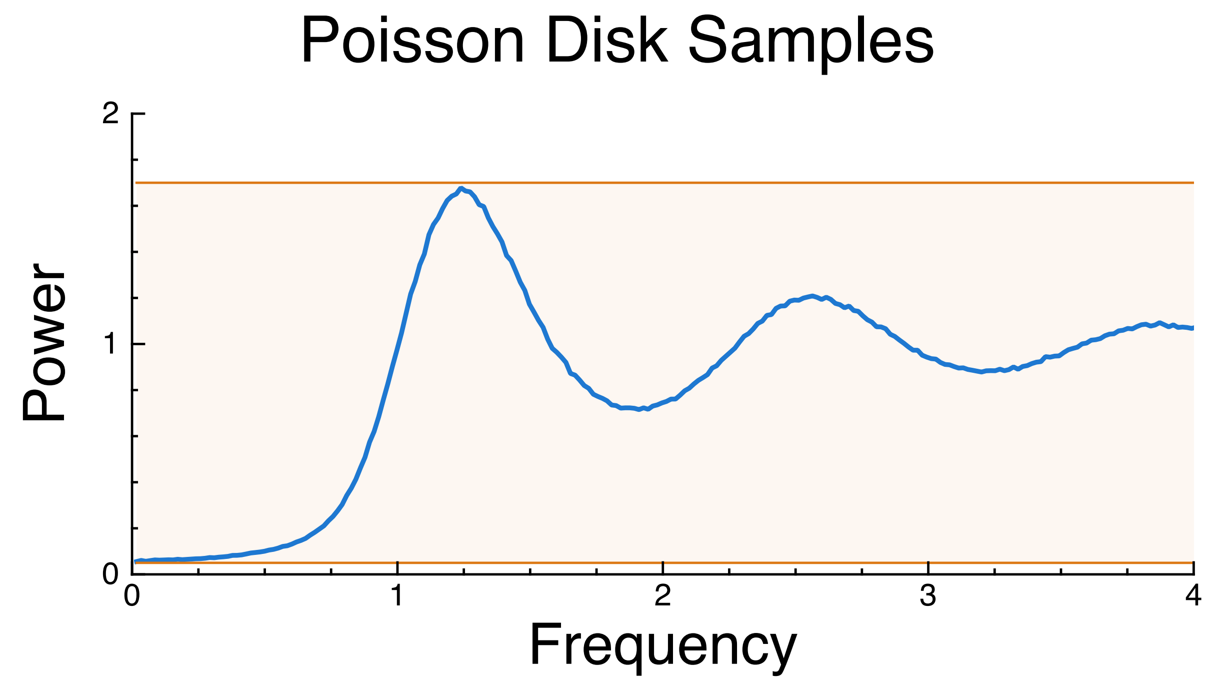


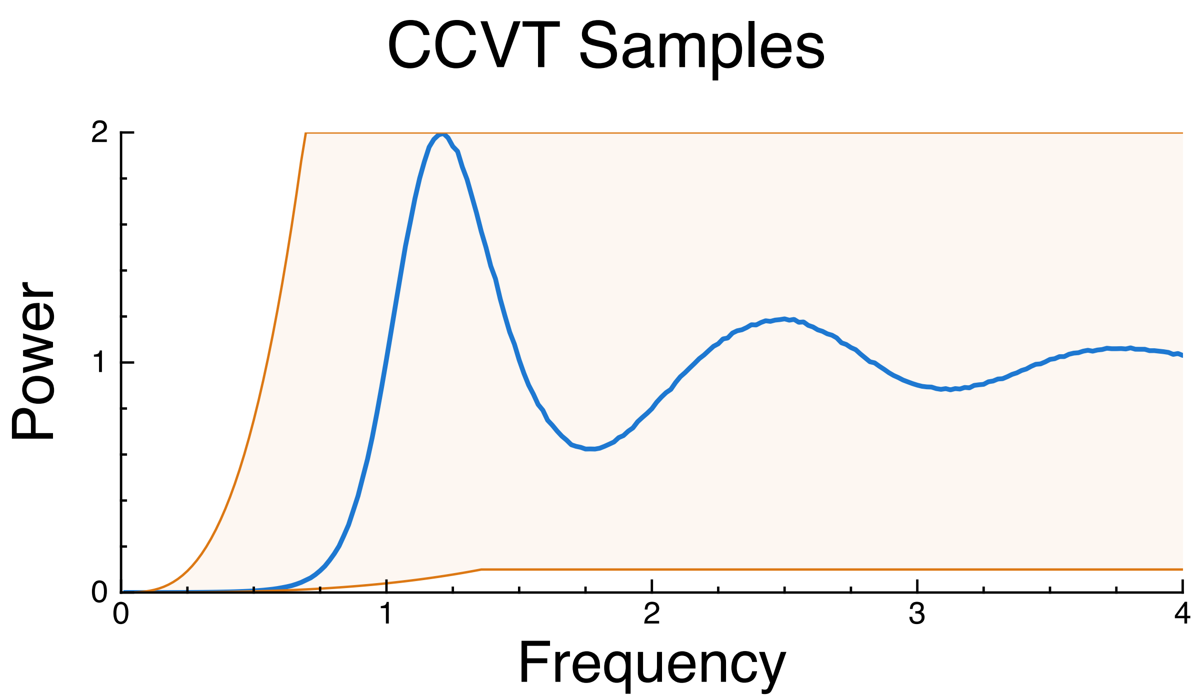
Sampling Radial Spectrum

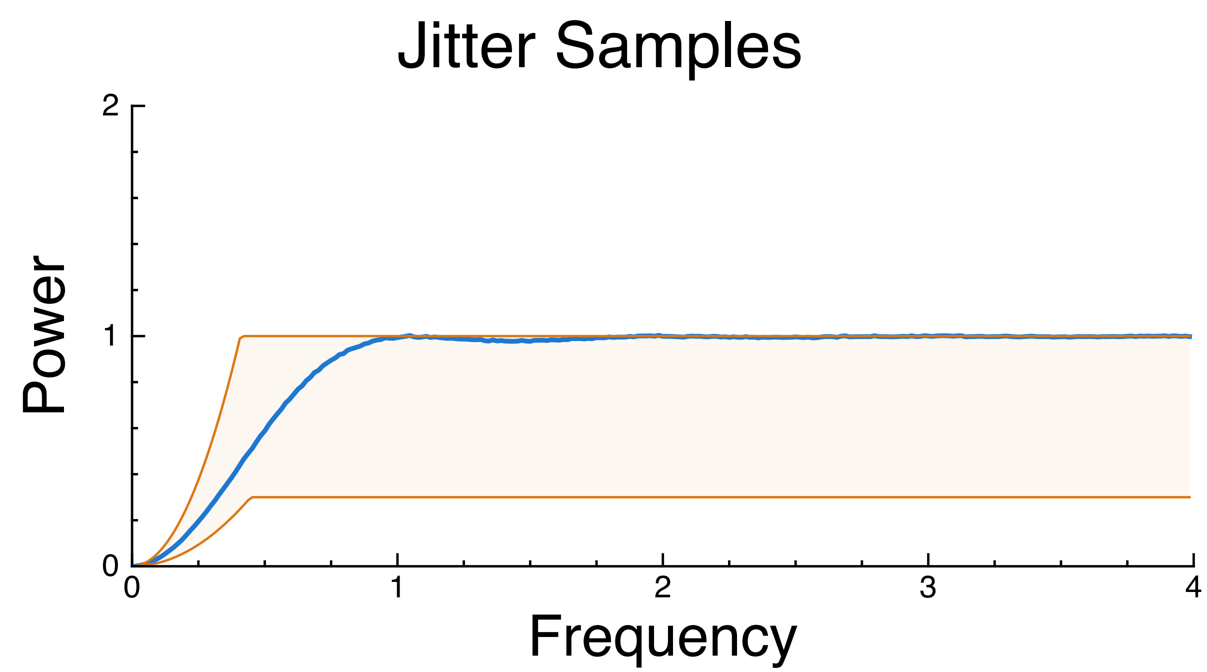


Sampling Radial Spectra









Variance in Fourier Domain

$$Var(I_N) = \int_{-\infty}^{\infty} \rho^{d-1} \boxed{\mathcal{P}_F(\rho)} \langle \boxed{\mathcal{P}_S(\rho)} \rangle d\rho$$

Best possible Integrand Power spectrum

⋮

Worst possible Integrand Power spectrum

Constant monomials

Quadratic monomials

Other monomials

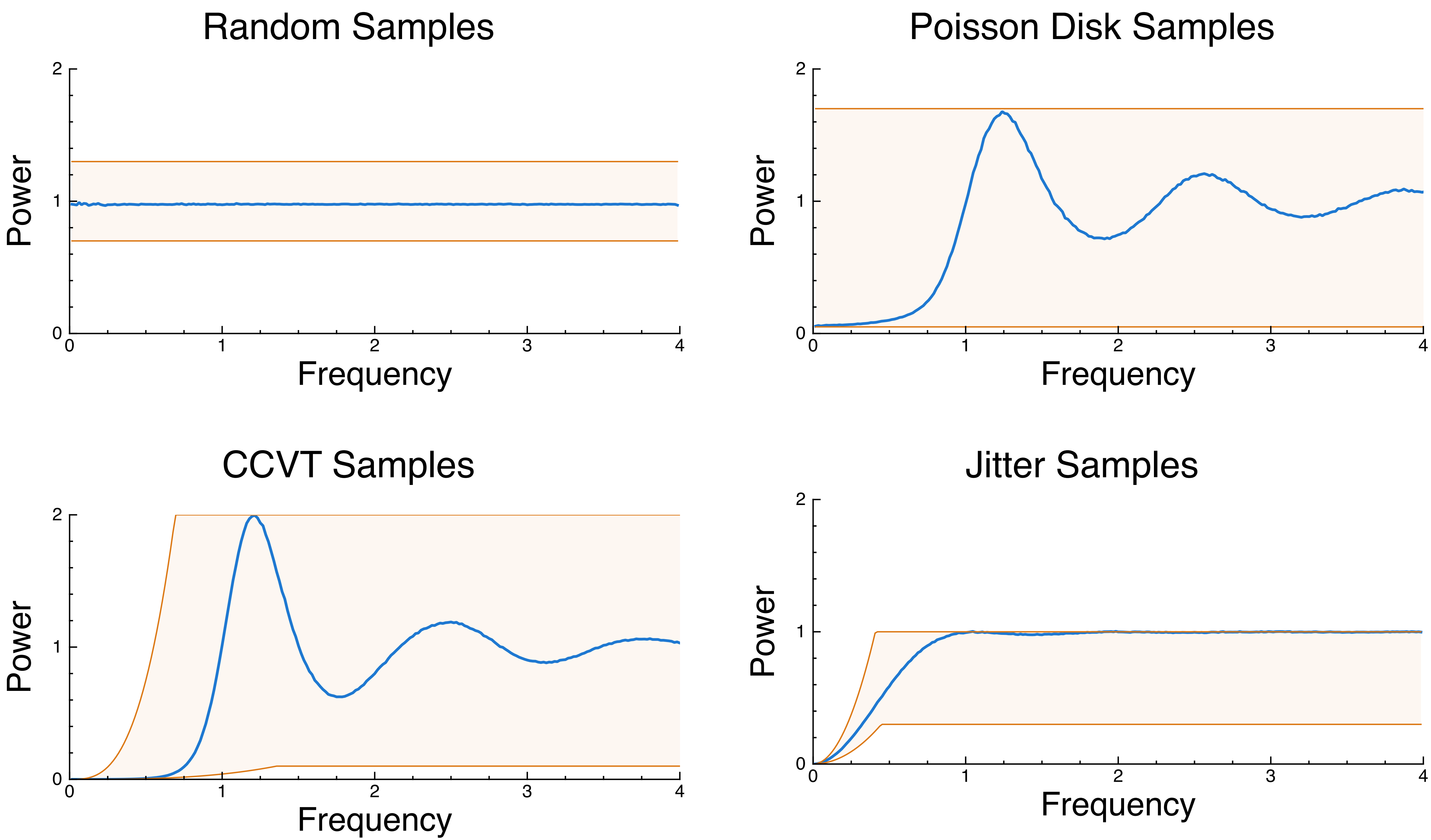
Pilleboue et al. [2015]

Theoretical Convergence Rates in 2D

Samplers	*Worst Case*	*Best Case*
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$

Pilleboue et al. [2015]

Sampling Radial Spectra



Visual Break

Glacial by Courtney Trowbridge