CS 87/187, Spring 2016

RENDERING ALGORITHMS

Monte Carlo Integration I

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Today

- Probability review
- Monte Carlo integration
- Random sampling

Monte Carlo History

- Use random numbers to solve numerical problems
- Early use during development of atomic bomb
- Von Neumann, Ulam, Metropolis
- Named after the casino in Monte Carlo




Monte Carlo vs Las Vegas

Random variation creeps into the results

Always gives the correct answer

e.g. a randomized sorting algorithm

Monte Carlo Methods

- Pros
  - Flexible
  - Easy to implement
  - Easily handles complex integrands
  - Efficient for high dimensional integrands
- Cons
  - Variance (noise)
  - Slow convergence* $O(1/\sqrt{N})$

Random Variables

- Random variable X
- Cumulative distribution function (CDF)

$$P(x) = \mathrm{Prob}\{X \leq x\} \qquad P(x) \in [0, 1]$$

- Probability density function (PDF)

$$p(x) = \frac{dP(x)}{dx} \qquad p(x) \geq 0 \qquad \int p(x) = 1$$

- Therefore

$$\mathrm{Prob}\{a \leq X \leq b\} = \int_a^b p(x)\, dx = P(b) - P(a)$$

Random Variables

- Uniform random variables:

$$p(x) = \frac{dP}{dx}(x) = \text{const}$$

- Canonical uniform random variable ξ

$$p(x) = \begin{cases} 1 & x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Expected Value & Variance

- Random variable $Y = f(X)$
- Expected Value: $E[Y] = \int_D f(x)p(x)\, dx$
- Variance: $V[Y] = E[(Y - E[Y])^2]$
- Properties:

$$E[aY] = aE[Y]$$
$$E[Y_1 + Y_2] = E[Y_1] + E[Y_2]$$
$$V[aY] = a^2 V[Y]$$

- Therefore:

$$V[Y] = E[Y^2] - E[Y]^2$$

Expected Value

- Expected value:

$$E[Y] = \int_D f(x)p(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

where the x_i are distributed according to $p(x_i)$

- Strong law of large numbers:

$$\text{Prob}\left\{E[Y] = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right\} = 1$$

Monte Carlo Integration

- We want to compute the value F of an integral

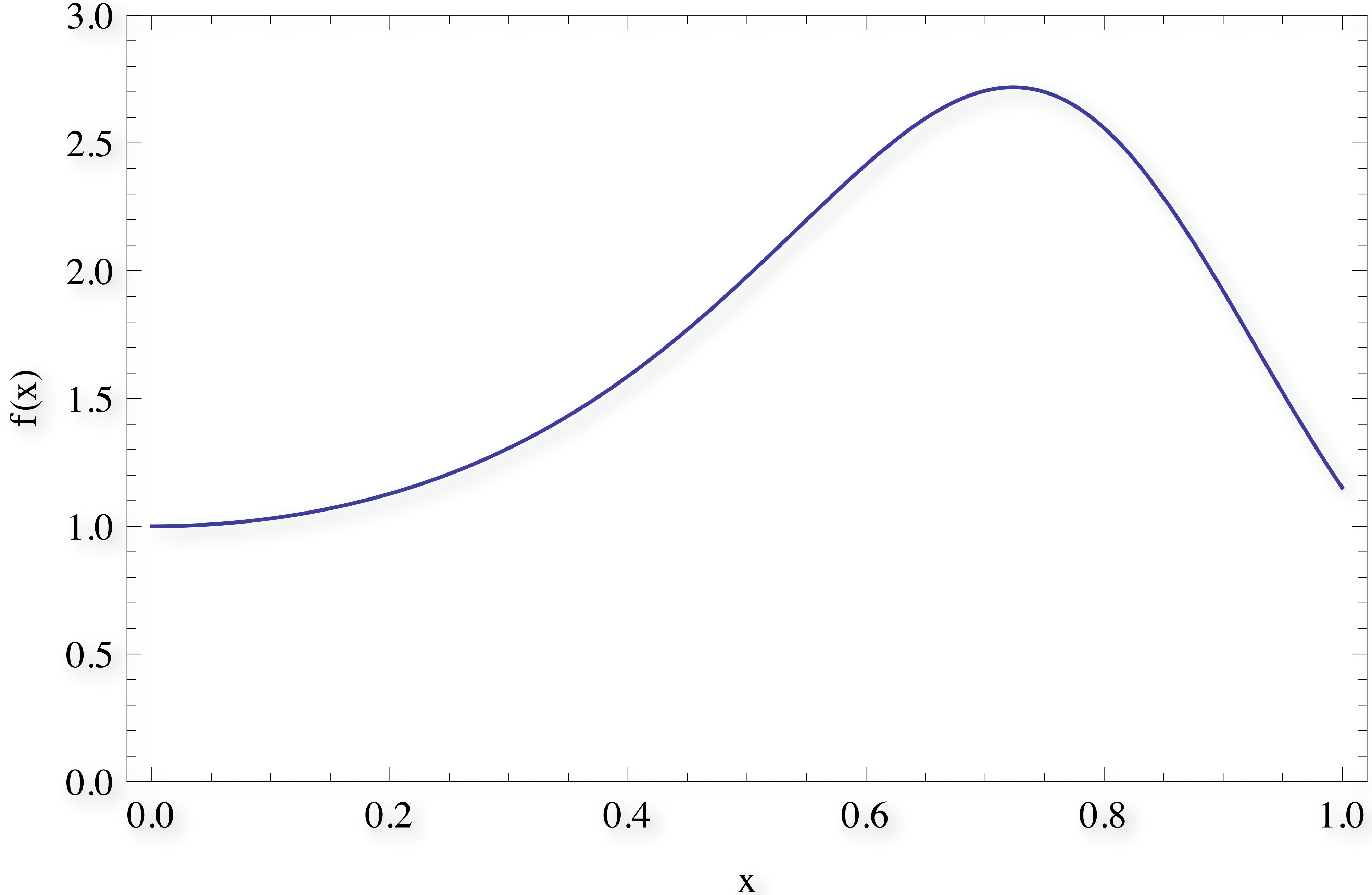
$$E\left[Y\right] = \int_D f(x)p(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

$$F = \int_0^1 f(x)dx$$

$$= \int_D \left[\frac{f(x)}{p(x)}\right] p(x)dx \approx \boxed{\frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} = F_N}$$

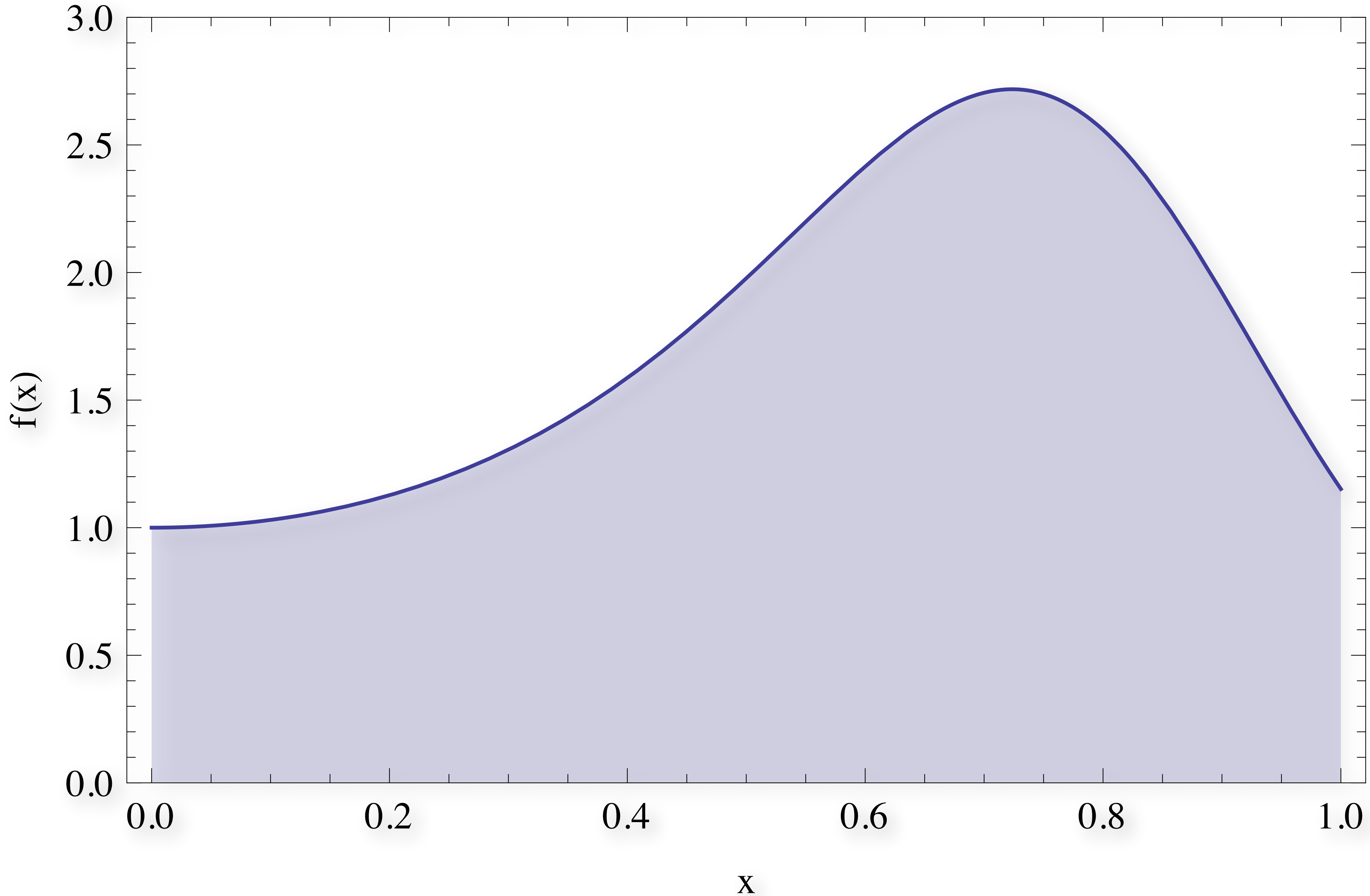
Monte Carlo Integration

$$f(x) = e^{\sin(3x^2)}$$



Monte Carlo Integration

$$F = \int_0^1 e^{\sin(3x^2)}\, dx$$



Monte Carlo Integration

$$F = \int_0^1 e^{\sin(3x^2)}\,dx \approx F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)} \Rightarrow \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

```
double integrate(int N)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = randf();
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}
```

$$p(x_i) = 1$$

Monte Carlo Integration

$$F = \int_{\boxed{a}}^{\boxed{b}} e^{\sin(3x^2)}\, dx \approx F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

```
double integrate(int N, double a, double b)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = randf();
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}
```

Monte Carlo Integration

$$F = \int_a^b e^{\sin(3x^2)}\, dx \approx F_N = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

```
double integrate(int N, double a, double b)
{
   double x, sum=0.0;
   for (int i = 0; i < N; ++i) {
      x = a + randf()*(b-a);
      sum += exp(sin(3*x*x));
   }
   return sum / double(N);
}
```

$$p(x_i) = \frac{1}{b-a}$$

Monte Carlo Integration

$$F = \int_a^b e^{\sin(3x^2)}\, dx \approx F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)} \Rightarrow \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)$$

```
double integrate(int N, double a, double b)
{
   double x, sum=0.0;
   for (int i = 0; i < N; ++i) {
      x = a + randf()*(b-a);
      sum += exp(sin(3*x*x));
   }
   return sum * (b-a) / double(N);
}
```

$$p(x_i) = \frac{1}{b-a}$$

Monte Carlo Integration

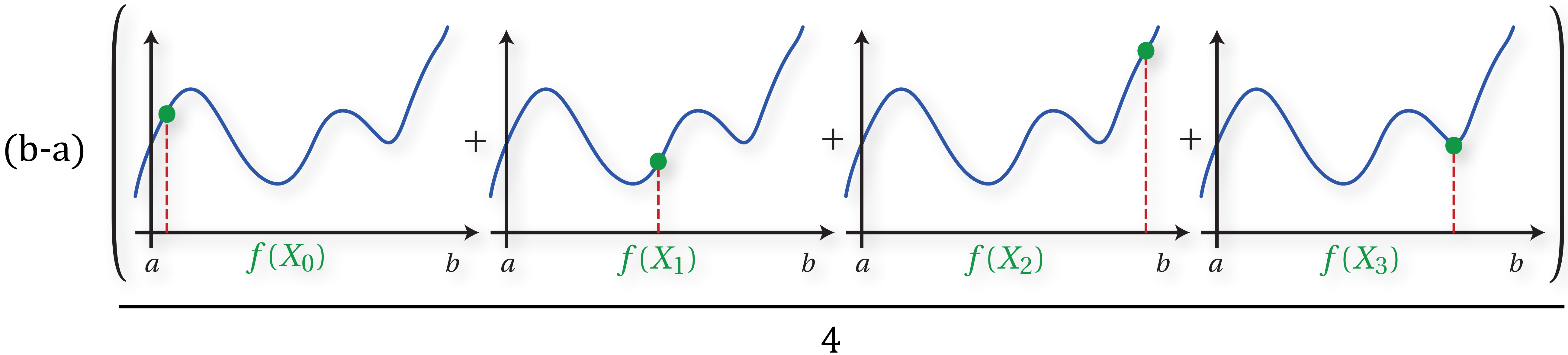
$$f(x) = e^{\sin(3x^2)}$$

N	F_N
1	2.75039
10	1.9893
100	1.79139
1000	1.75146
10000	1.77313
100000	1.77862

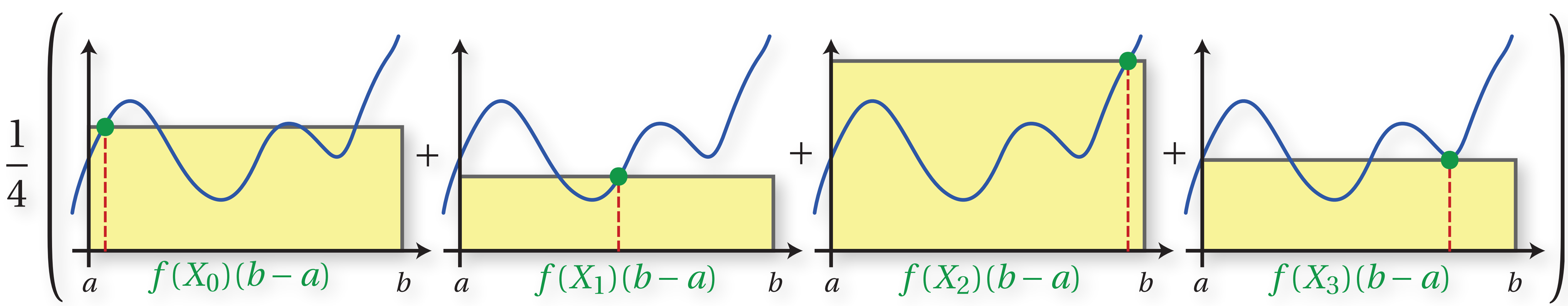
True value: 1.760977217585905…

Monte Carlo Visual Interpretation

Average function value or "height"



Area of a random box



Monte Carlo Integration Summary

- Goal: evaluate integral $\int_a^b f(x)dx$

- Random variable $X_i \sim p(x)$

- Monte Carlo Estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$

- Expectation $E[F_N] = \int_a^b f(x)dx$

Monte Carlo Estimator

- *Unbiased* estimator
  - expected value equals integral
- Extension to higher dimensions straightforward
  - number of samples independent of dimension (cf. standard quadrature)
- Convergence* $O(1/\sqrt{N})$
  - reducing the error by a factor of 2 requires 4 times as many samples!

Today's Agenda

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$

- Main practical issues:
  - How to choose $p(x)$
  - How to generate x_i according to $p(x)$
- Ambient Occlusion

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

Visual Break

Visual Break

Visual Break

Sampling Random Variables

- Sampling the function domain:
  - Uniform unit interval (0,1)
  - Uniform interval (a,b)
  - Circle?
  - Sphere?
  - Hemisphere?
  - More complex domains?

Example: uniformly sampling a disk

- Uniform probability density on a unit disk

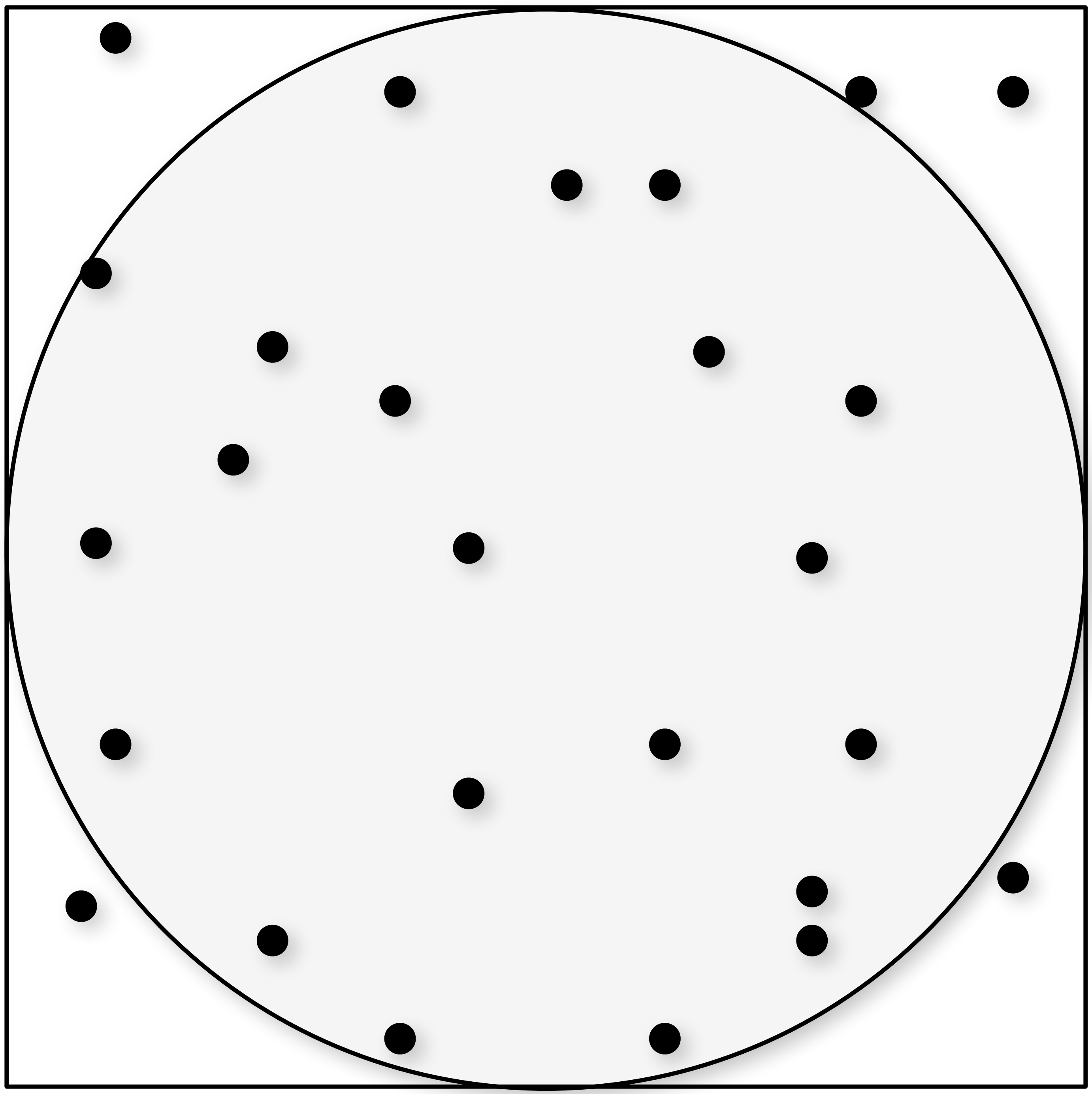
$$p(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Goal: draw samples X_i,Y_i that are distributed as:

$$(X_i, Y_i) \sim p(x, y)$$

- Problem: pseudo-random number generators only allow us to draw samples from a canonical uniform distribution

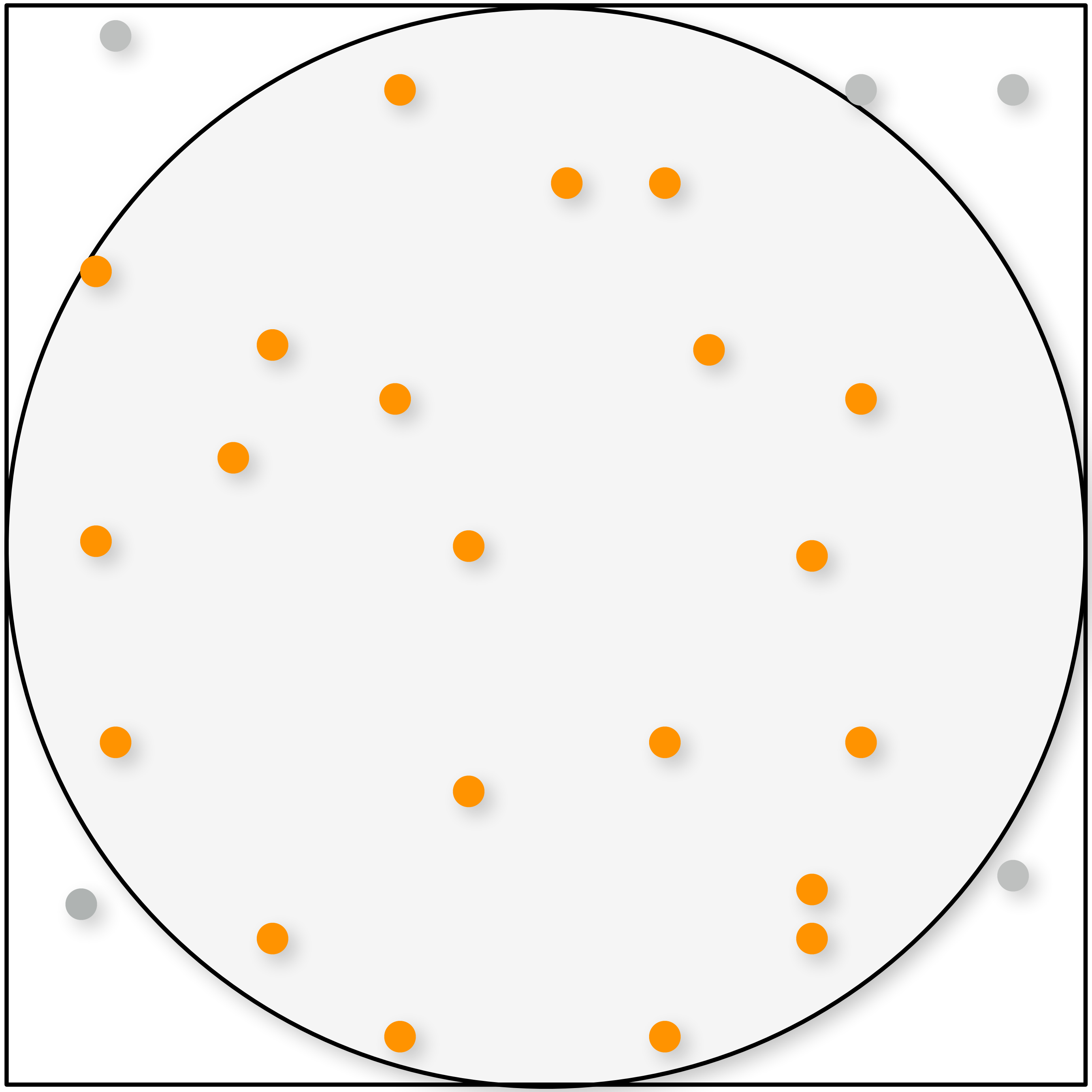
Rejection Sampling a Disk



```
Vector2 v;


    v.x = 1-2*randf();
    v.y = 1-2*randf();
```

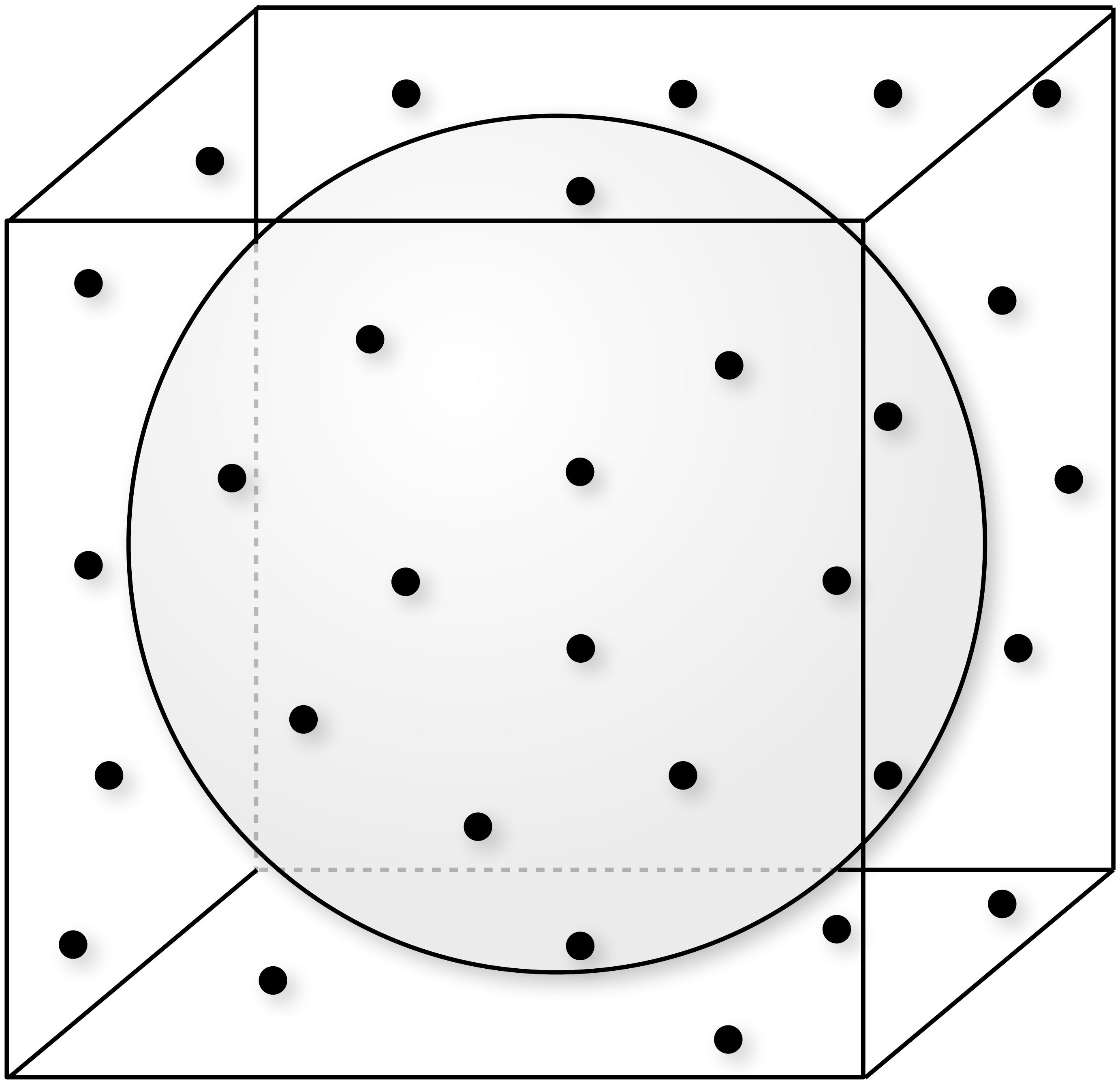
Rejection Sampling a Disk



```
Vector2 v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
} while(v.length²() > 1)
```

- Similar technique for sampling a sphere

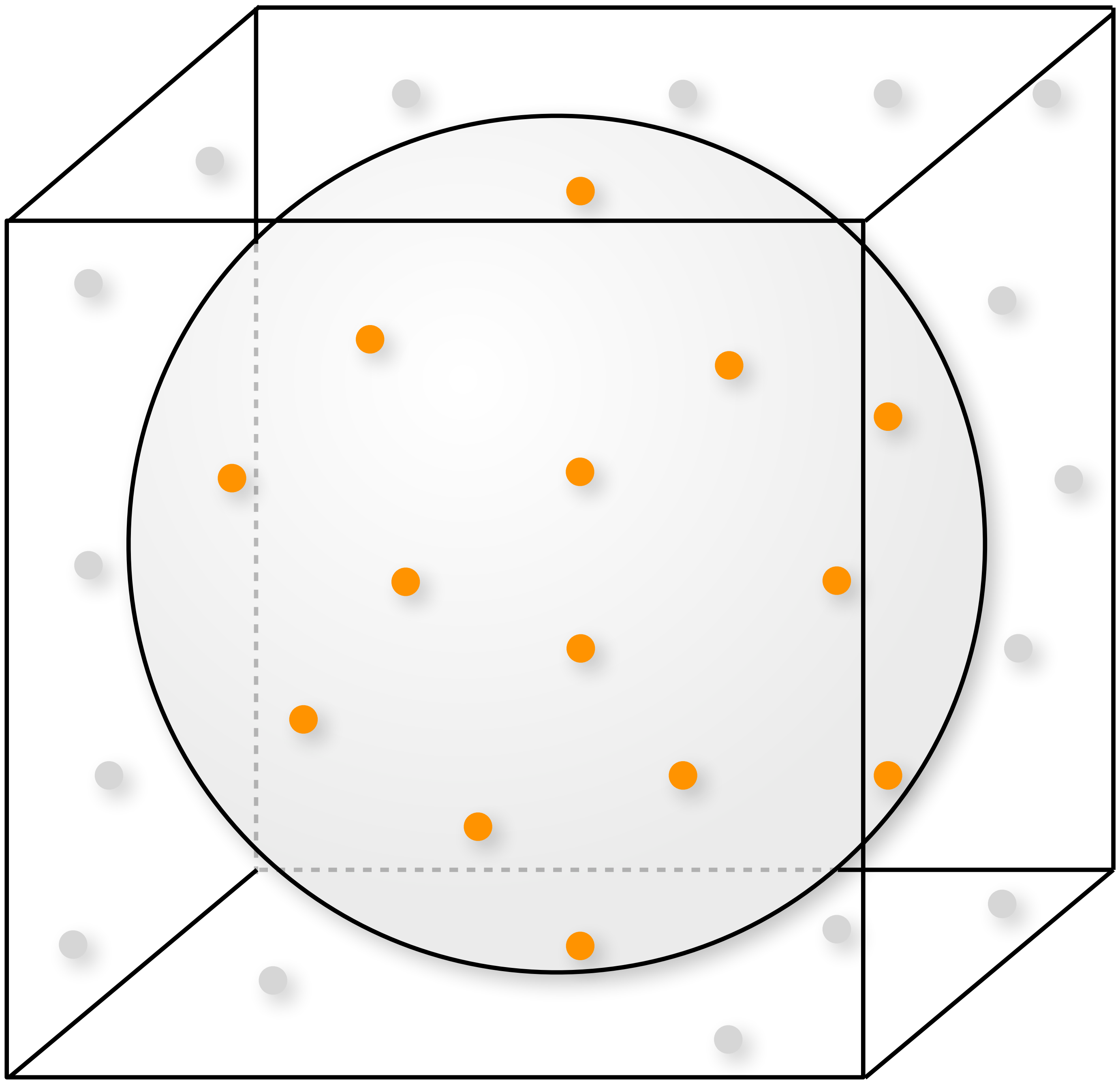
Rejection Sampling a Sphere



```
Vector3D v;

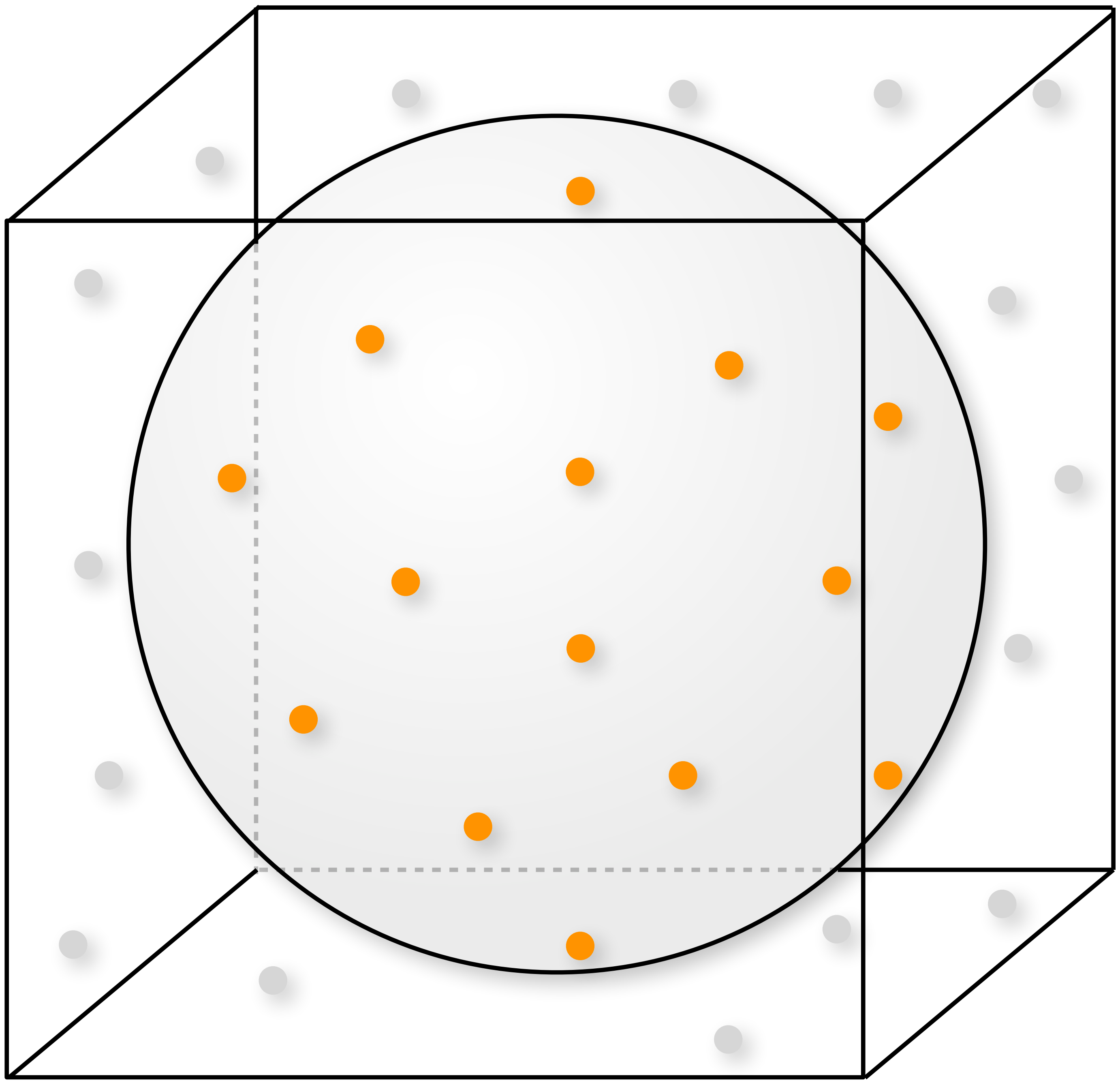
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
```

Rejection Sampling a Sphere



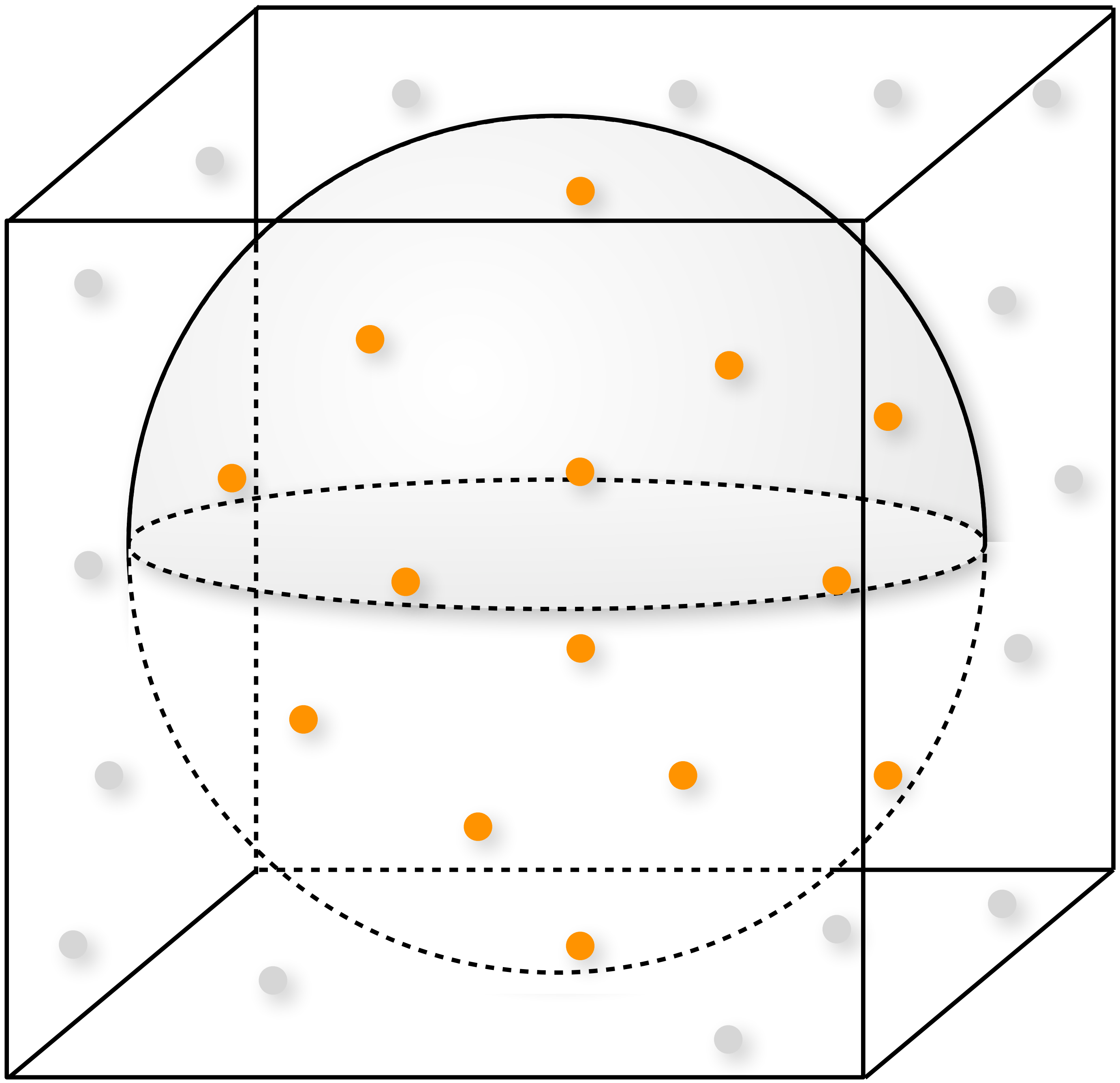
```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1)
```

Rejection Sampling a Sphere



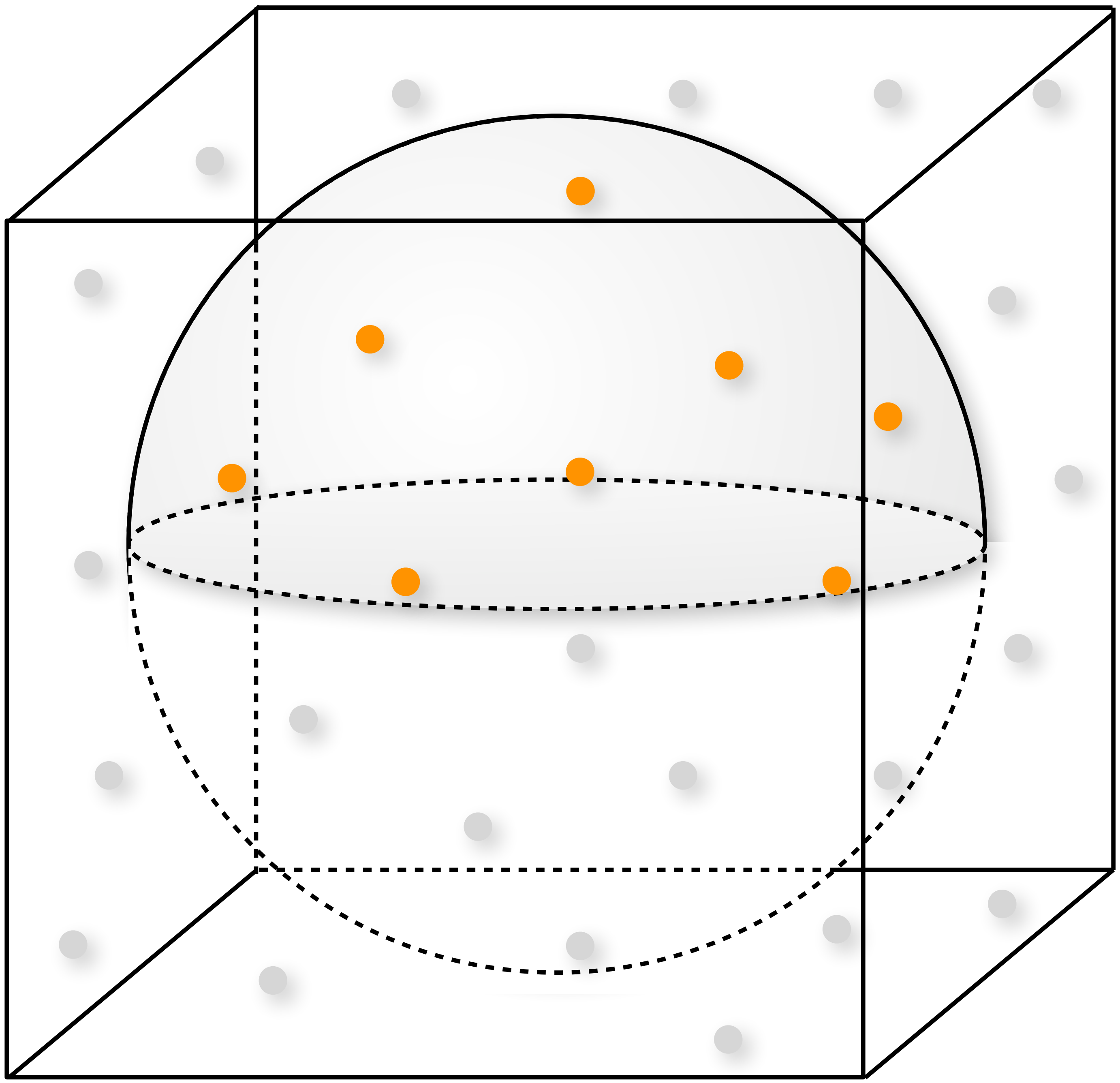
```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1)
// Project onto sphere
v /= v.length();
```

Rejection Sampling a Hemisphere



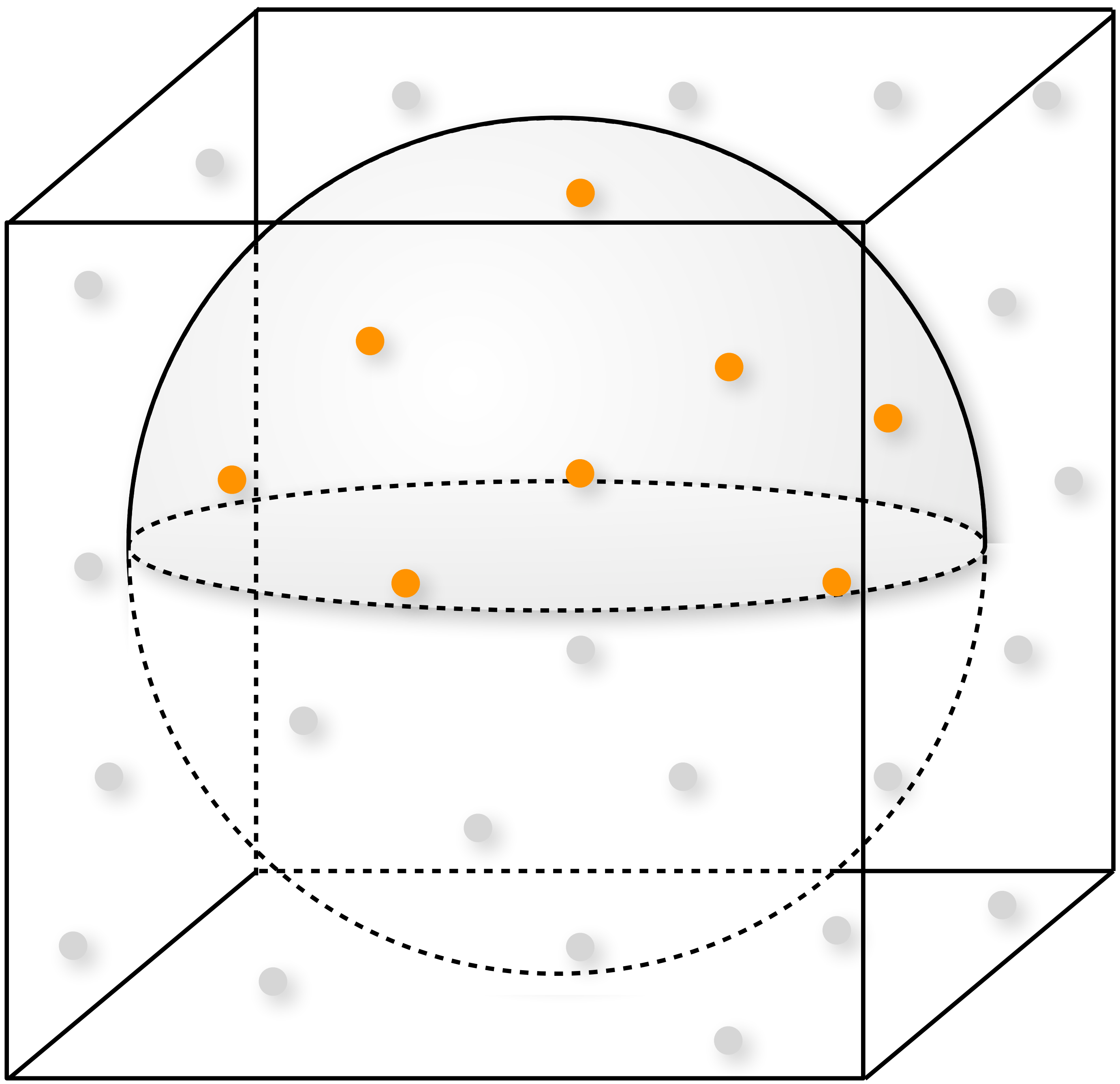
```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1)
```

Rejection Sampling a Hemisphere



```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1 ||
        v.z < 0)
```

Rejection Sampling a Hemisphere



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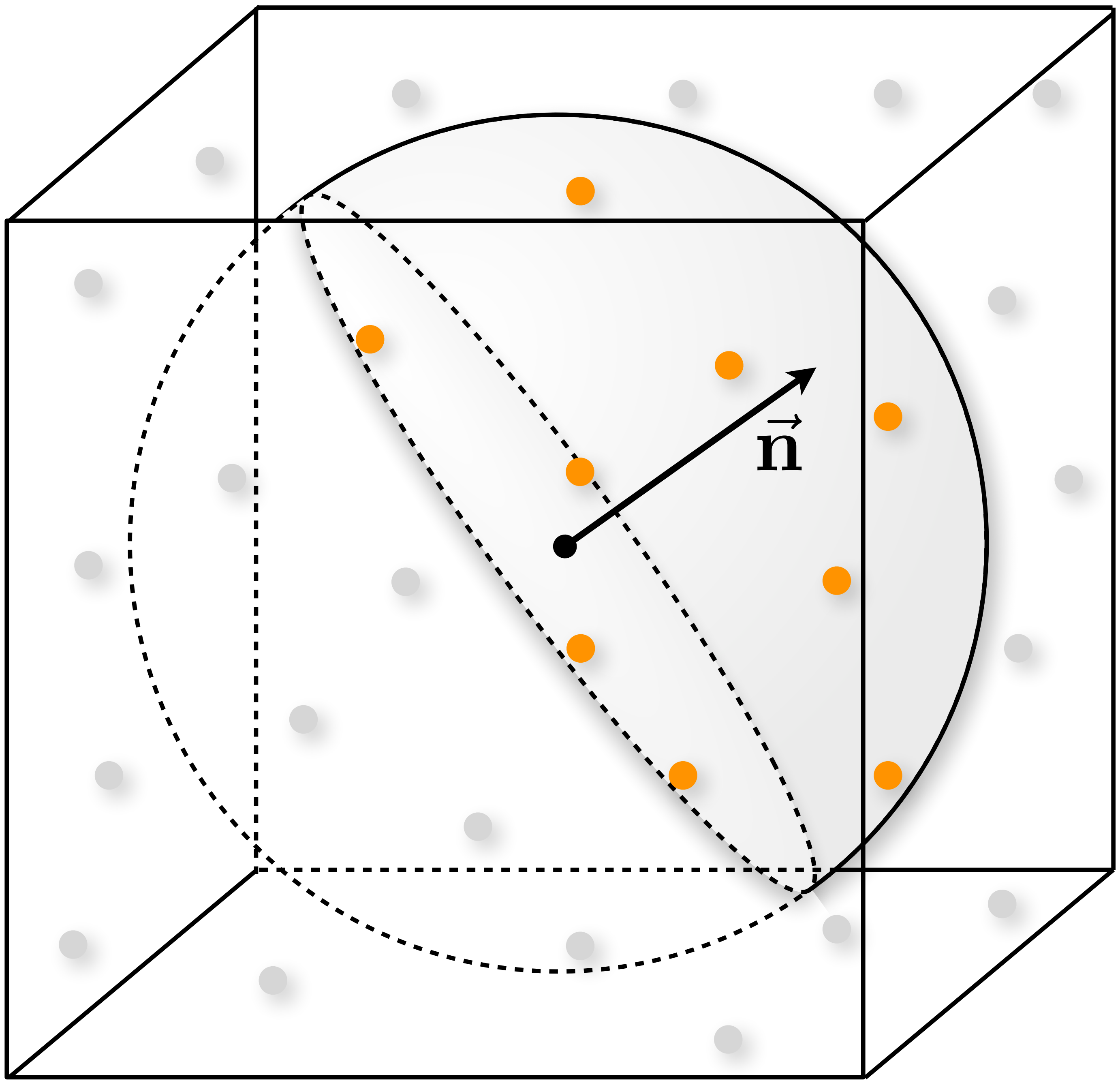
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1 ||
        v.z < 0)

// Project onto hemisphere
v /= v.length();
```

Rejection Sampling a Hemisphere




```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
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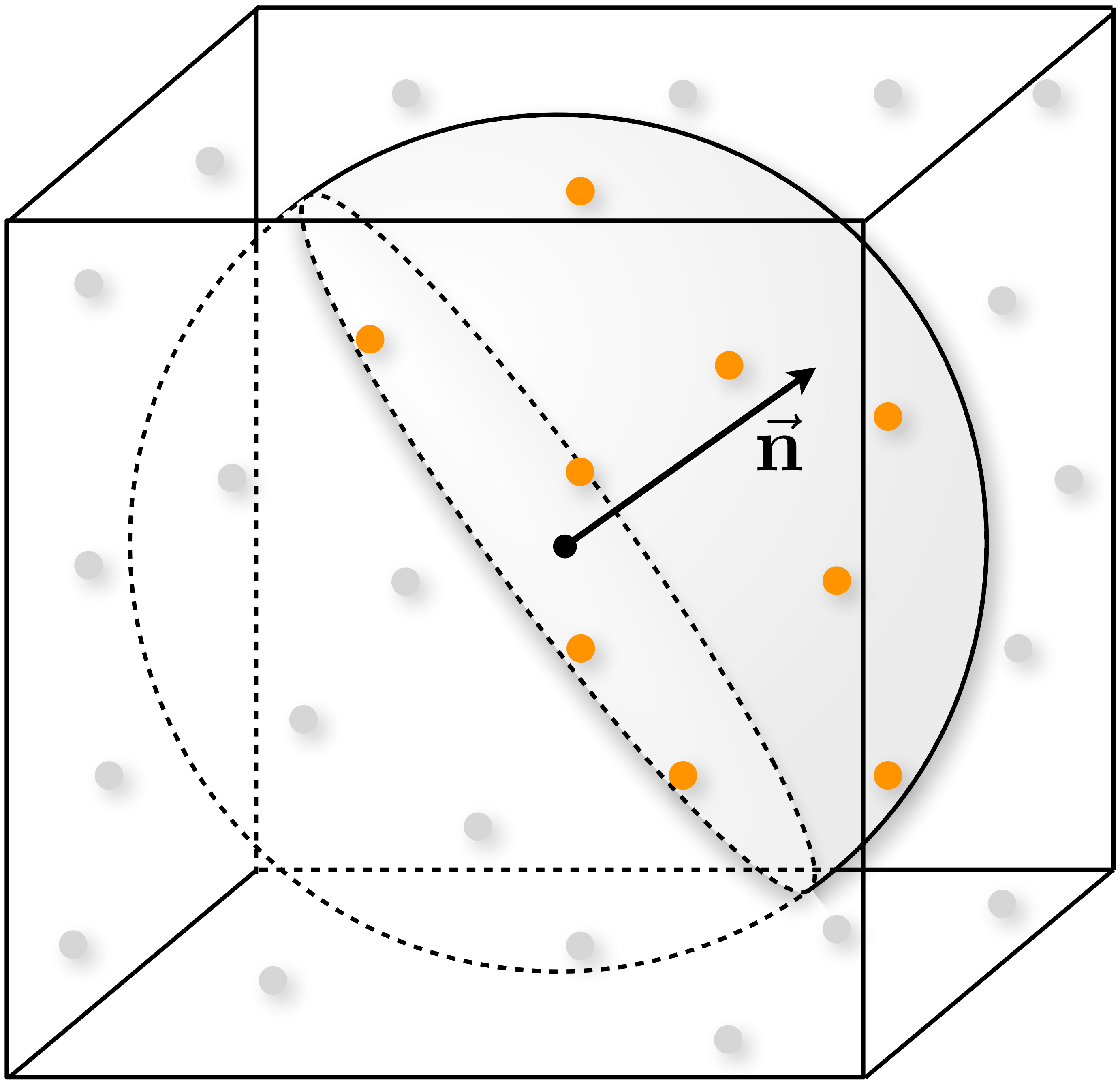
        v.z < 0)

// Project onto hemisphere
v /= v.length();
```

- Arbitrary orientation?

Rejection Sampling a Hemisphere

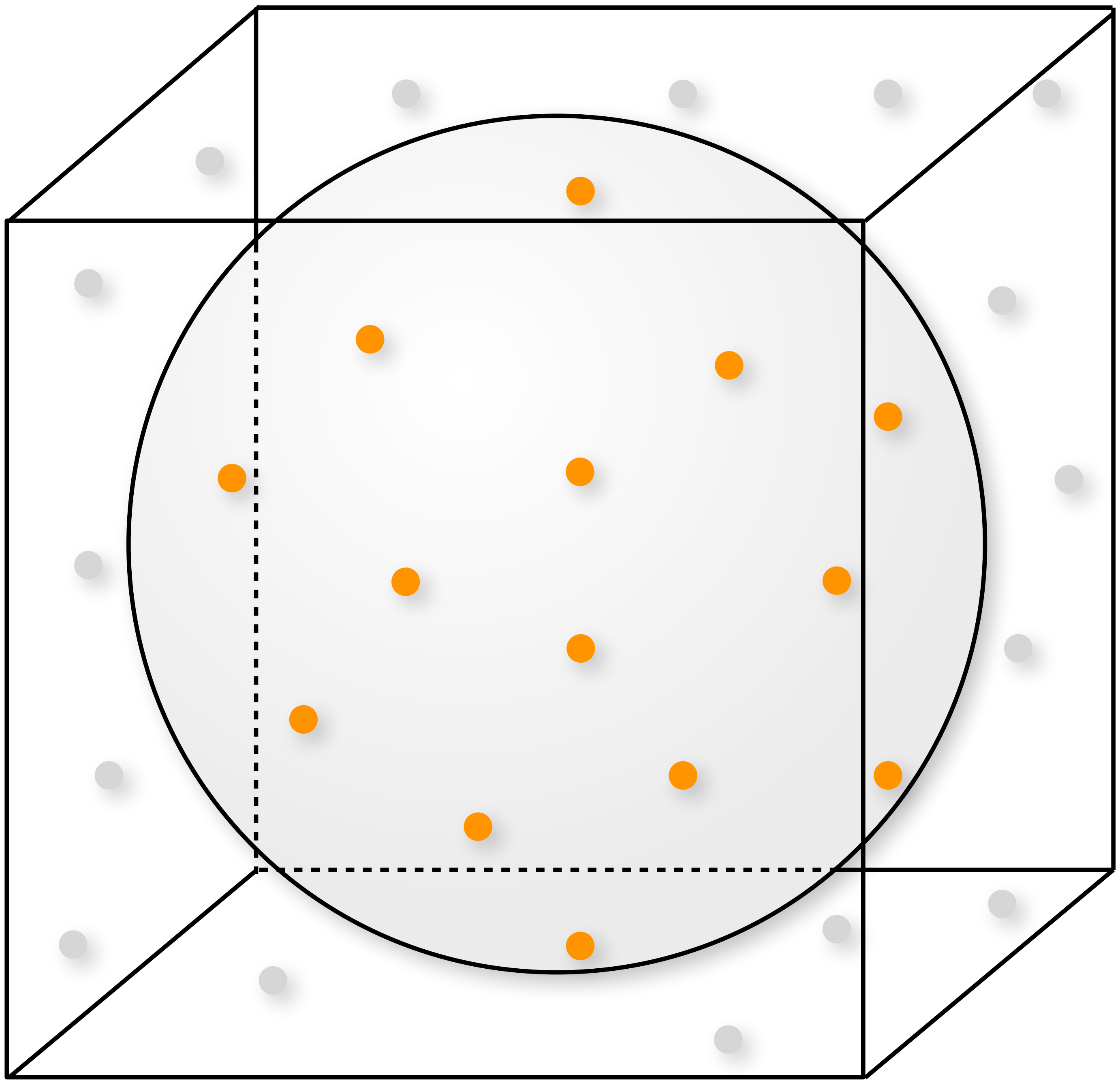



```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1 ||
         dot(v,n) < 0)

// Project onto hemisphere
v /= v.length();
```

- Arbitrary orientation?
- **Inefficient!**

Rejection Sampling a Hemisphere



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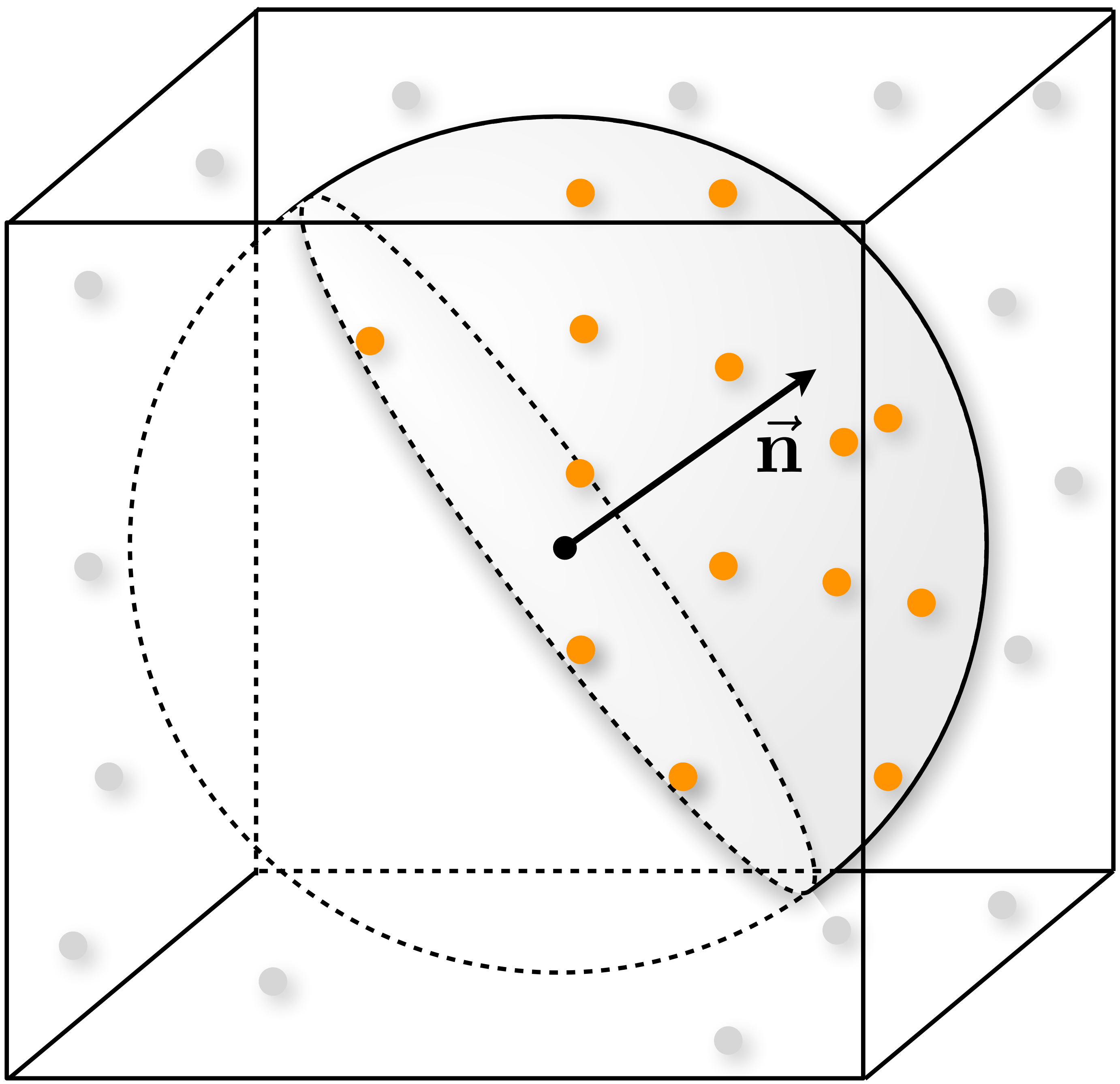
    v.z = 1-2*randf();
} while(v.length²() > 1)
```

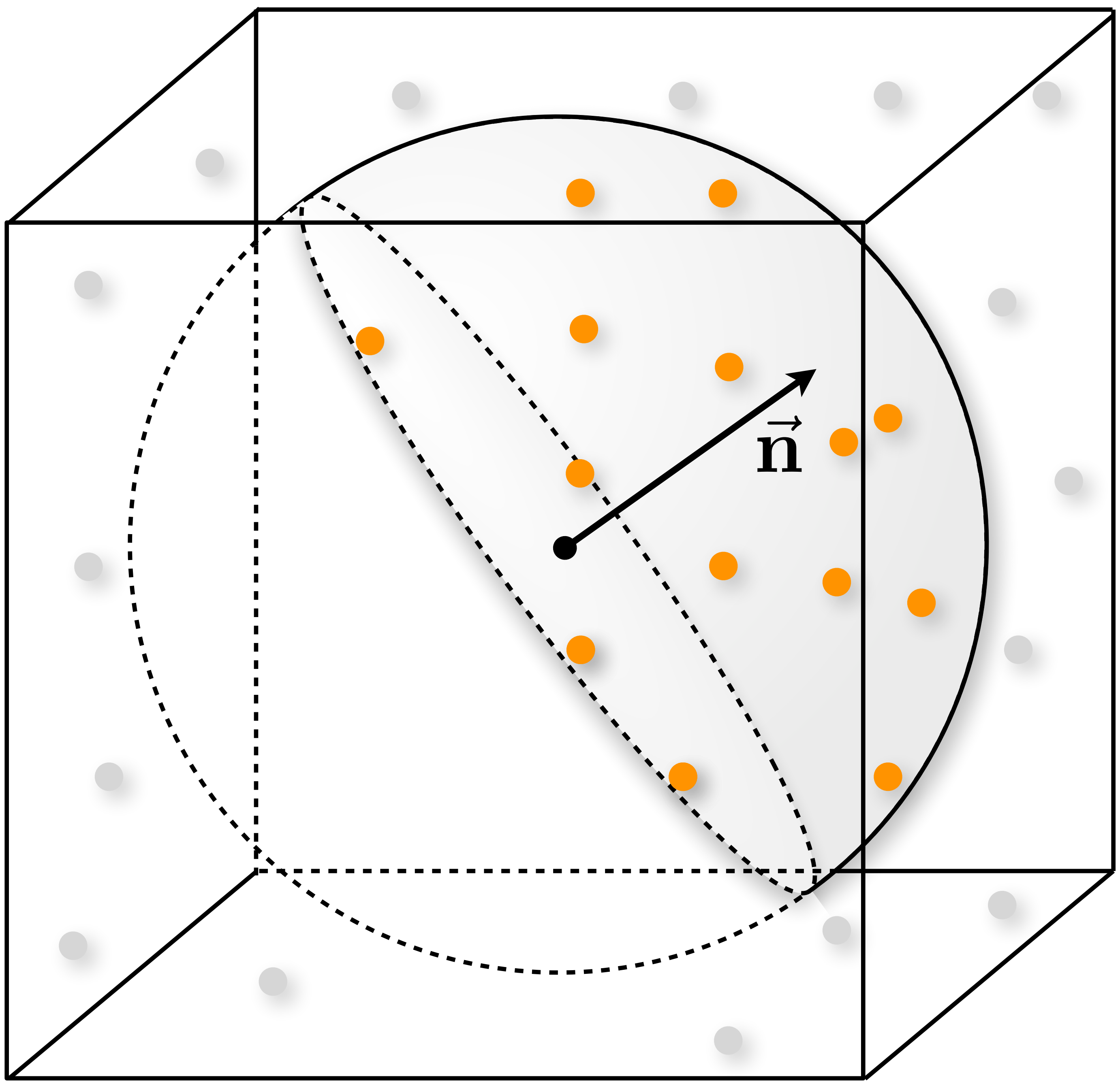
Rejection Sampling a Hemisphere




```
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1)

// flip to proper hemisphere
if (dot(v,n) < 0)
    v = -v;
```

Rejection Sampling a Hemisphere

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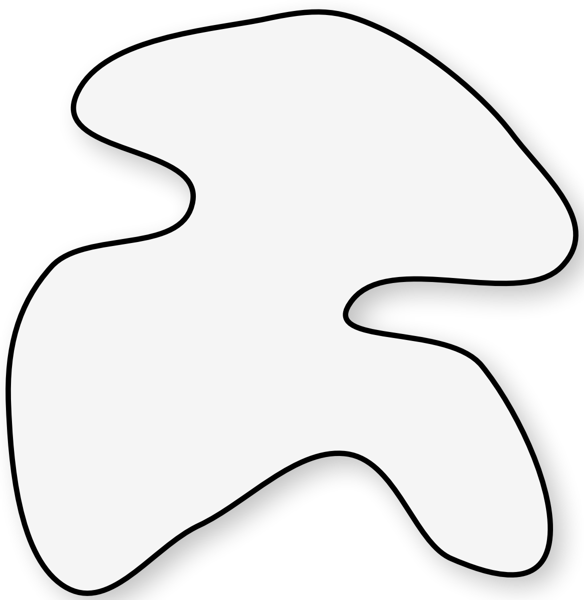
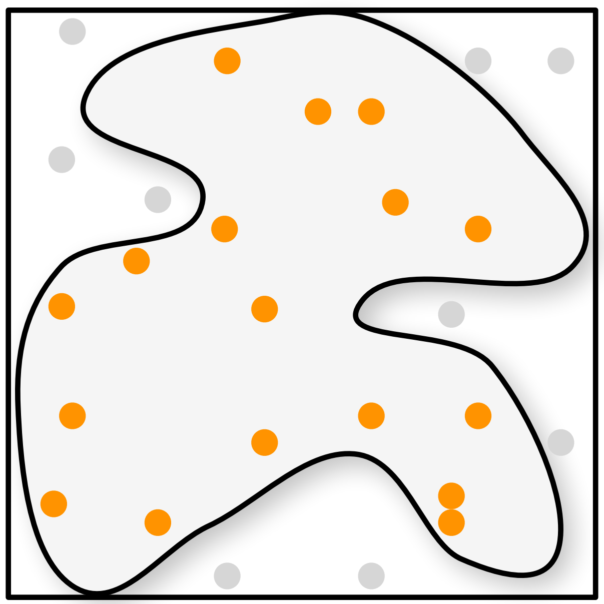
Vector3D v;
do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(v.length²() > 1)
// flip to proper hemisphere
if (dot(v,n) < 0)
    v = -v;
// Project onto hemisphere
v /= v.length();
```

- Or, generate in canonical orientation, and then rotate

Rejection Sampling

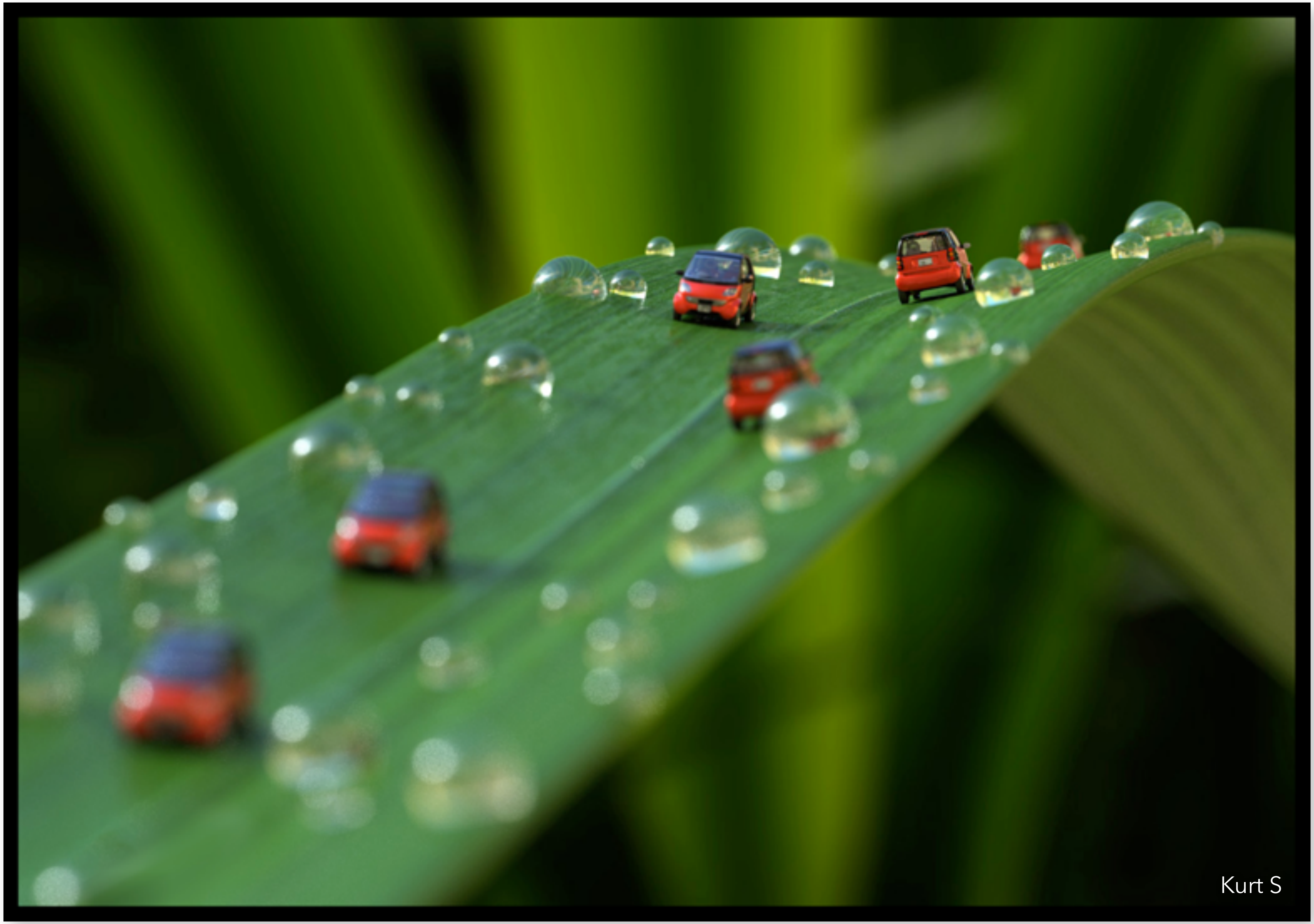
- More complex shapes





- Pros:
    + Flexible
- Cons:
    - Inefficient
    - Difficult/impossible to combine with stratification or quasi-Monte Carlo

Visual Break



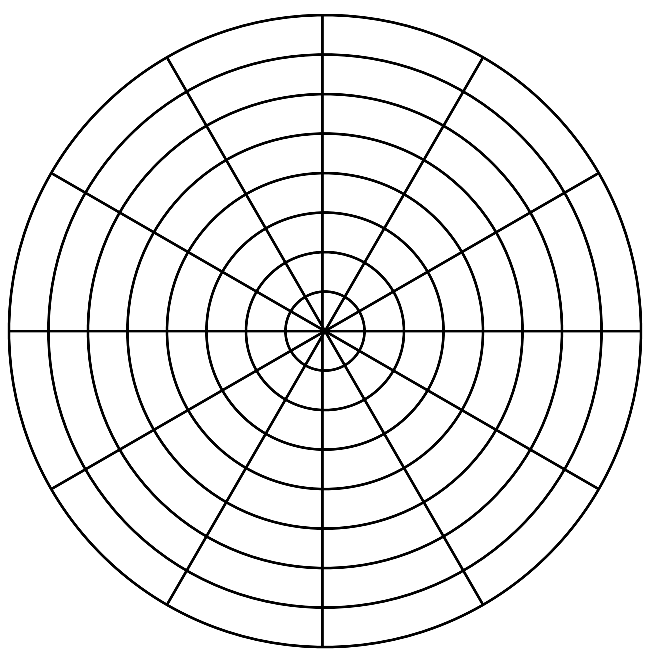
http://communities.bentley.com/blogs/kurtss_blog/default.aspx

Visual Break

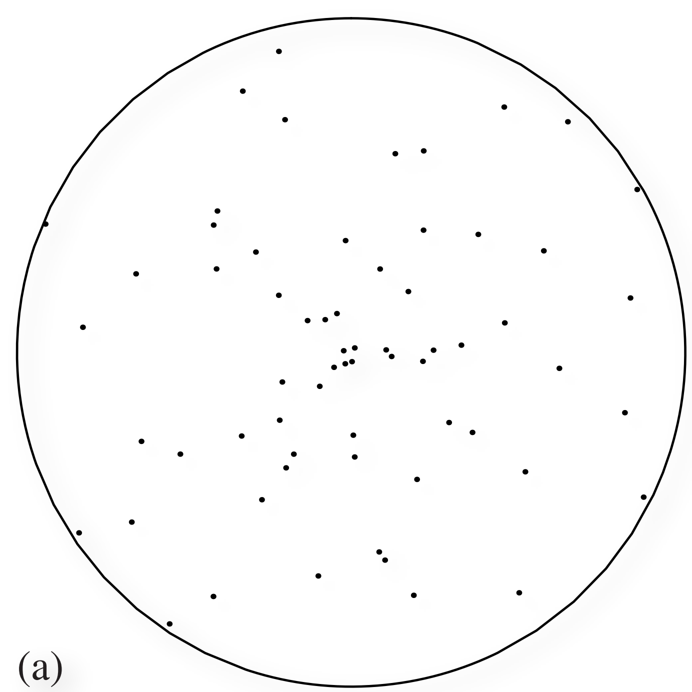
Visual Break

Directly Sampling a Disk

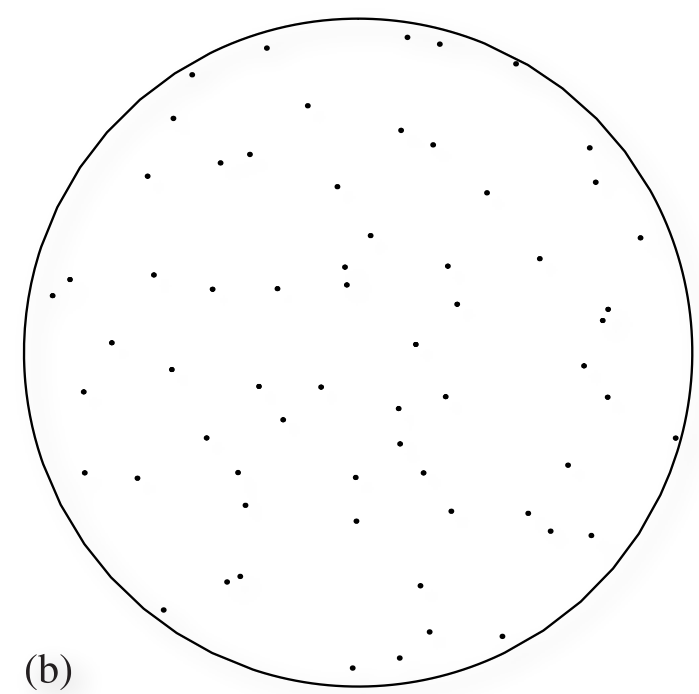
- Idea: transform samples to polar coordinates:
  - pick two uniform random variables ξ_1, ξ_2
  - select point at (r, θ) with $r = \xi_1$ and $\theta = 2\pi\xi_2$
  - problem?
    - not uniform with respect to area!



not equi-areal



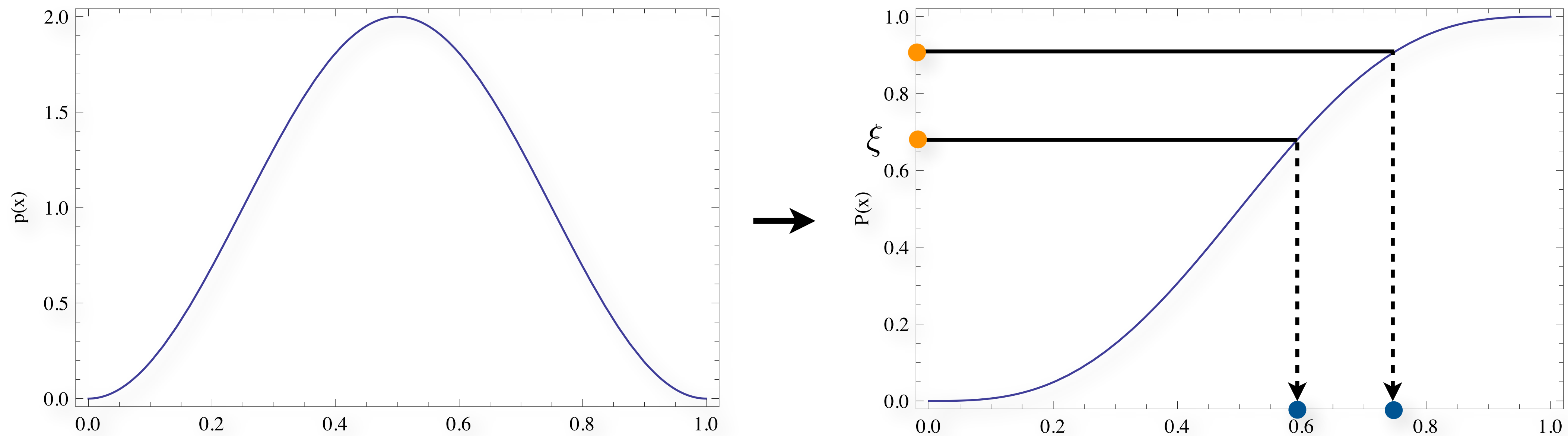
(a)



(b)

Sampling arbitrary distributions

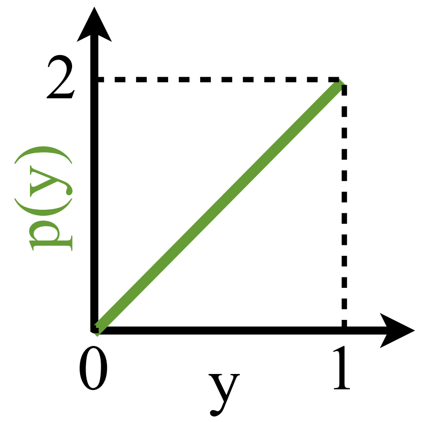
- The inversion method:
  1. Compute the CDF $P(x) = \int_0^x p(x')\,dx'$
  2. Compute its inverse $P^{-1}(x)$
  3. Obtain a uniformly distributed random number ξ
  4. Compute $X_i = P^{-1}(\xi)$



Sampling a linear ramp

- Goal: sample with PDF

$$p(y) = 2y$$



$$P(y) = y^2$$

$$P^{-1}(y) = \sqrt{y}$$

$$Y_i = \sqrt{\xi}$$

Transforming Between Distributions

- Given an n-dimensional random variable $X_i \sim p_x(x)$
- Lets say we have a one-to-one (bijective) transformation T
- What is the distribution of $Y_i = T(X_i)$?
- New density is:

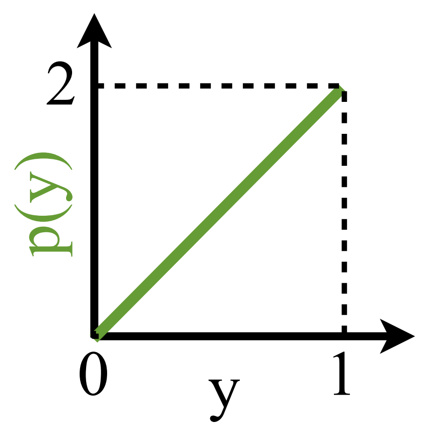
$$p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|J_T(x)|}$$

- where $|J_T(x)|$ is the absolute value of the determinant of the Jacobian of T

Validation: Sampling a linear ramp

- Goal: sample with PDF

$$p(y) = 2y$$



$$P(y) = y^2$$

$$P^{-1}(y) = \sqrt{y}$$

$$Y_i = \sqrt{\xi}$$

- Confirm PDF: $p(y) = p(T(x)) = \dfrac{p(x)}{|J_T(x)|}$

$$p(x) = 1$$

$$Y = \sqrt{X}$$

$$\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|^{-1} = 2x^{\frac{1}{2}}$$

$$\begin{aligned} p(y) &= 2x^{\frac{1}{2}} \\ &= 2(y^2)^{\frac{1}{2}} \\ &= 2y \end{aligned}$$

Sampling 2D Distributions

- Draw samples (X, Y) from a 2D distribution $p(x, y)$
- If $p(x, y)$ is separable, i.e., $p(x,y) = p_x(x)\, p_y(y)$, we can independently sample $p_x(x)$, and $p_y(y)$
- Otherwise, compute the marginal density function:

$$p(x) = \int p(x, y)\; dy$$

- and, the conditional density:

$$p(y|x) = \frac{p(x, y)}{p(x)}$$

- Procedure: first sample $X_i \sim p(x)$, then $Y_i \sim p(y|x)$

Example: **uniformly sampling a disk**

- Desired distribution

$$p(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

- We will sample polar coordinates r, θ where:

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

- Jacobian matrix:

$$J_T = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

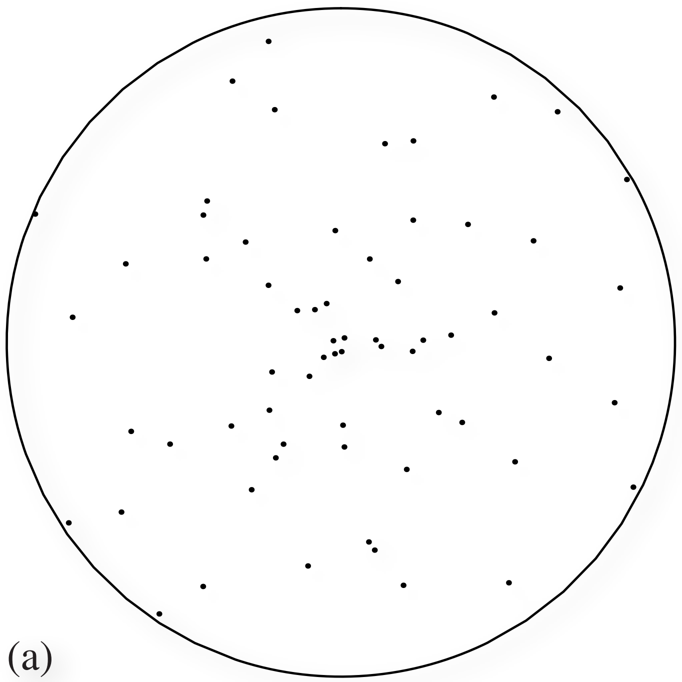
- Determinant is: $|J_T| = r(\cos^2\theta + \sin^2\theta) = r$

Naïve (incorrect) method

- pick two uniform random variables ξ_1, ξ_2
- select point at (r, θ) with $r = \xi_1$ and $\theta = 2\pi\xi_2$

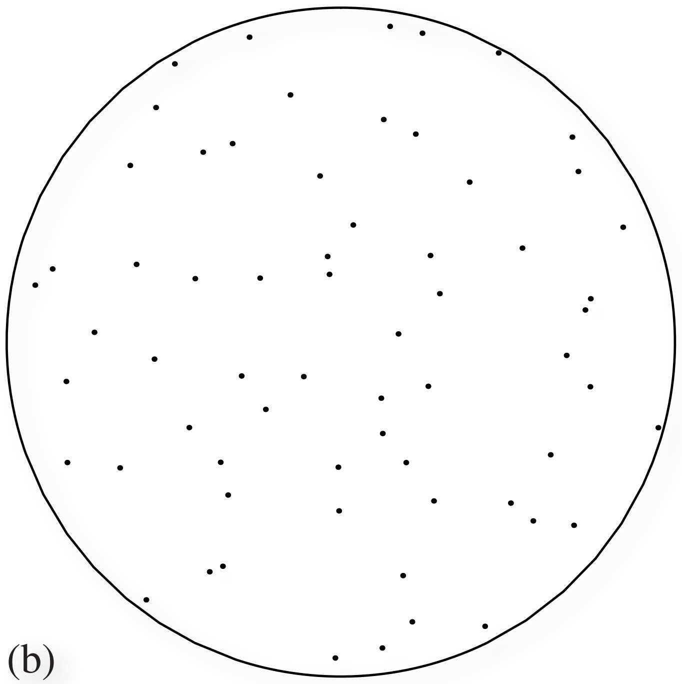
$$p(r,\theta) = p_r(r)p_\theta(\theta) = \frac{1}{2\pi}$$

$$p(x,y) = \frac{p(r,\theta)}{|J_T(r,\theta)|} = \frac{1}{2\pi r} = \frac{1}{2\pi\sqrt{x^2+y^2}}$$



(a)

$$|J_T| = r$$



(b)

Example: **uniformly sampling a disk**

- Desired distribution

$$p(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Determinant is: $|J_T| = r(\cos^2\theta + \sin^2\theta) = r$

- Therefore:

$$p(x,y) = p(r,\theta)/r \qquad\qquad p(r,\theta) = rp(x,y) = r/\pi$$

- Marginal and conditional distributions:

$$p(r) = \int_0^{2\pi} p(r,\theta)\; d\theta = 2r \qquad\qquad p(\theta|r) = \frac{p(r,\theta)}{p(r)} = \frac{1}{2\pi}$$

Example: **uniformly sampling a disk**

- Marginal and conditional distributions:

$$p(r) = \int_0^{2\pi} p(r,\theta)\ d\theta = 2r \qquad p(\theta|r) = \frac{p(r,\theta)}{p(r)} = \frac{1}{2\pi}$$

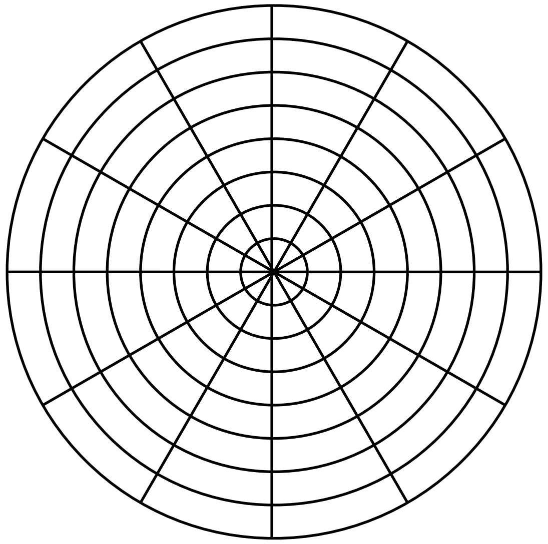
- Apply inversion method with these two PDFs
  - Compute $P(r), P^{-1}(r), P(\theta), P^{-1}(\theta)$
  - Applying inverse CDFs give:

$$r = \sqrt{\xi_1} \qquad \theta = 2\pi\xi_2$$

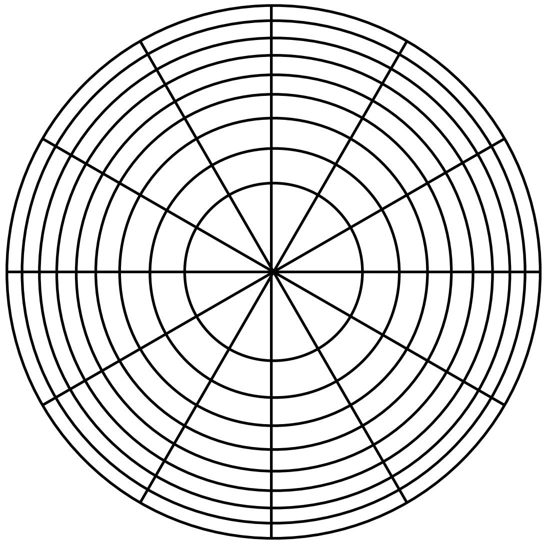
Constant PDF in θ, linearly increasing PDF in r

Directly Sampling a Disk

- Uniformly sample a unit disk
  - pick two uniform random variables ξ_1, ξ_2
  - select point at (r, θ) with $r = \xi_1$ and $\theta = 2\pi\xi_2$
  - problem: not uniform with respect to area!
  - correct solution: $r = \sqrt{\xi_1}$ and $\theta = 2\pi\xi_2$



not equi-area



equi-area

Recipe

1. Express the desired distribution in a convenient coordinate system
   - Requires computing the determinant of the Jacobian
2. Compute marginal and conditional 1D PDFs
3. Sample 1D PDFs using the inversion method

Visual Break

http://communities.bentley.com/blogs/kurtss_blog/default.aspx

Visual Break

Visual Break

Directly Sampling a Sphere

- Uniformly sample a unit sphere (3D direction)
  - pick two uniform random variables ξ_1, ξ_2
  - select point at (θ, ϕ) with $\theta = \pi\xi_1$ and $\phi = 2\pi\xi_2$
  - problem: not uniform with respect to area!
  - correct solution: $\theta = \cos^{-1}(2\xi_1 - 1)$ and $\phi = 2\pi\xi_2$

Algorithm

$$
\begin{aligned}
\theta &= \cos^{-1}(2\xi_1 - 1)\\
\phi &= 2\pi\xi_2\\
\vec{\omega}_x &= \sin\theta\cos\phi\\
\vec{\omega}_y &= \sin\theta\sin\phi\\
\vec{\omega}_z &= \cos\theta
\end{aligned}
$$

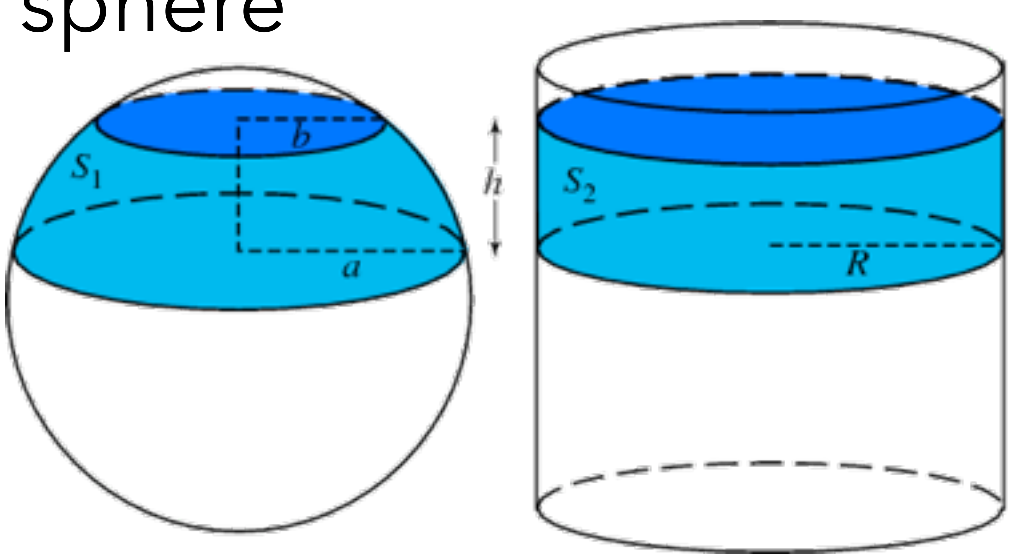
$\rightarrow$

Better

$$
\begin{aligned}
\vec{\omega}_z &= 2\xi_1 - 1\\
r &= \sqrt{1 - \vec{\omega}_z^2}\\
\phi &= 2\pi\xi_2\\
\vec{\omega}_x &= r\cos\phi\\
\vec{\omega}_y &= r\sin\phi
\end{aligned}
$$

Archimedes' Hat-Box Theorem

- The surface area of a sphere between any two horizontal planes is equal to the corresponding area on the circumscribing cylinder.
  - What is the det. of Jacobian?
  - i.e.: uniform areas on a cylinder project to uniform areas on a sphere



Weisstein, Eric W. "Archimedes' Hat-Box Theorem." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

Archimedes' Hat-Box Theorem

- The surface area of a sphere between any two horizontal planes is equal to the corresponding area on the circumscribing cylinder.
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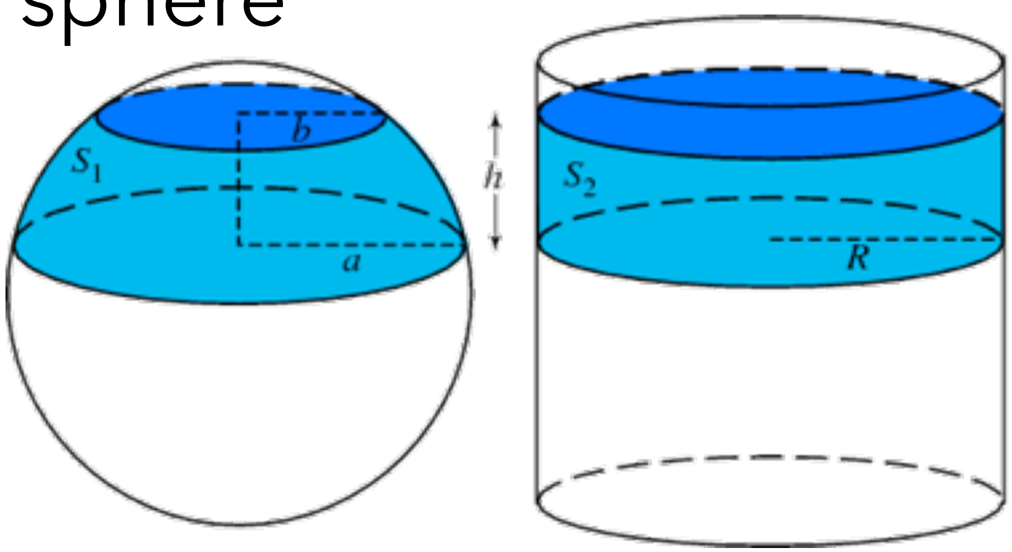
$$
\begin{aligned}
\vec{\omega}_z &= \boxed{2\xi_1 - 1} \\
r &= \boxed{\sqrt{1 - \vec{\omega}_z^2}} \\
\phi &= \boxed{2\pi\xi_2} \\
\vec{\omega}_x &= \boxed{r}\boxed{\cos\phi} \\
\vec{\omega}_y &= \boxed{r}\boxed{\sin\phi}
\end{aligned}
$$



- point on cylinder
- projection onto sphere

Weisstein, Eric W. "Archimedes' Hat-Box Theorem." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

Directly Sampling a Hemisphere

- Just like a sphere
- Use Hat-Box theorem with shorter cylinder

Ambient Occlusion

- Consider diffuse objects illuminated by an ambient overcast sky

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

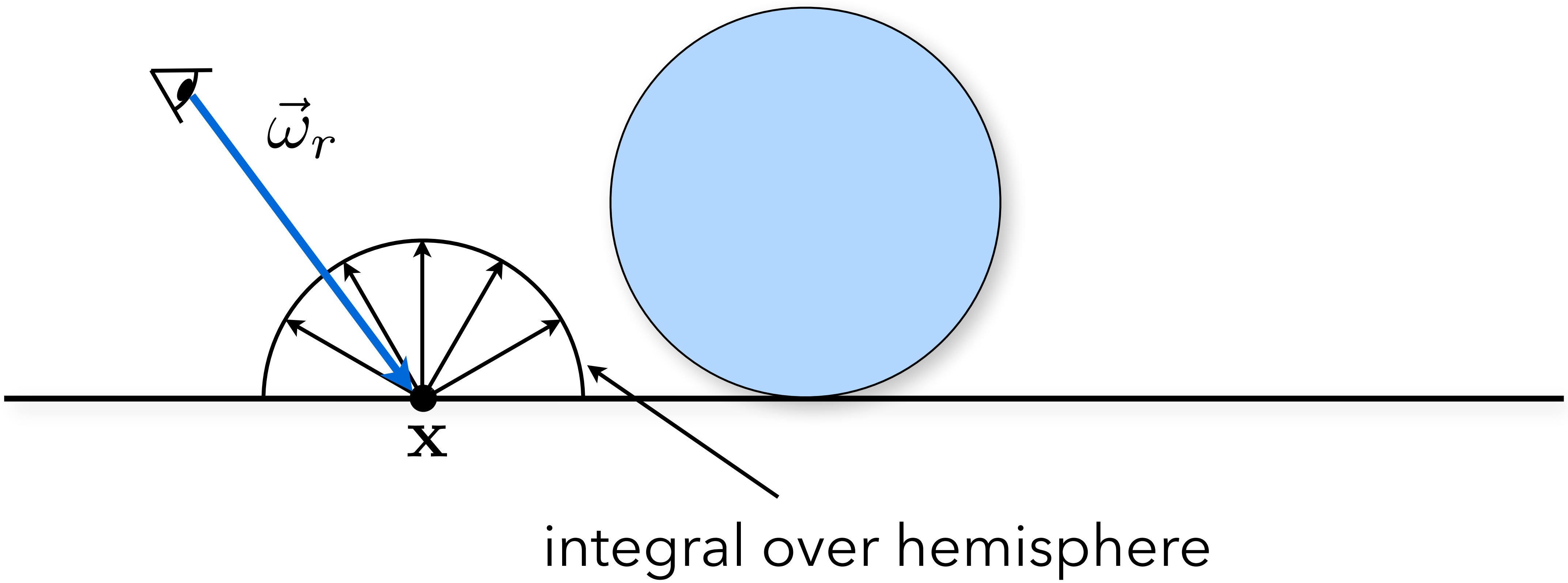
$$\approx \frac{\rho}{\pi N} \sum_{i=1}^{N} \frac{V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i}{p(\vec{\omega}_i)}$$

- Uniform hemispherical sampling, $p(\vec{\omega}_i) = 1/2\pi$:

$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

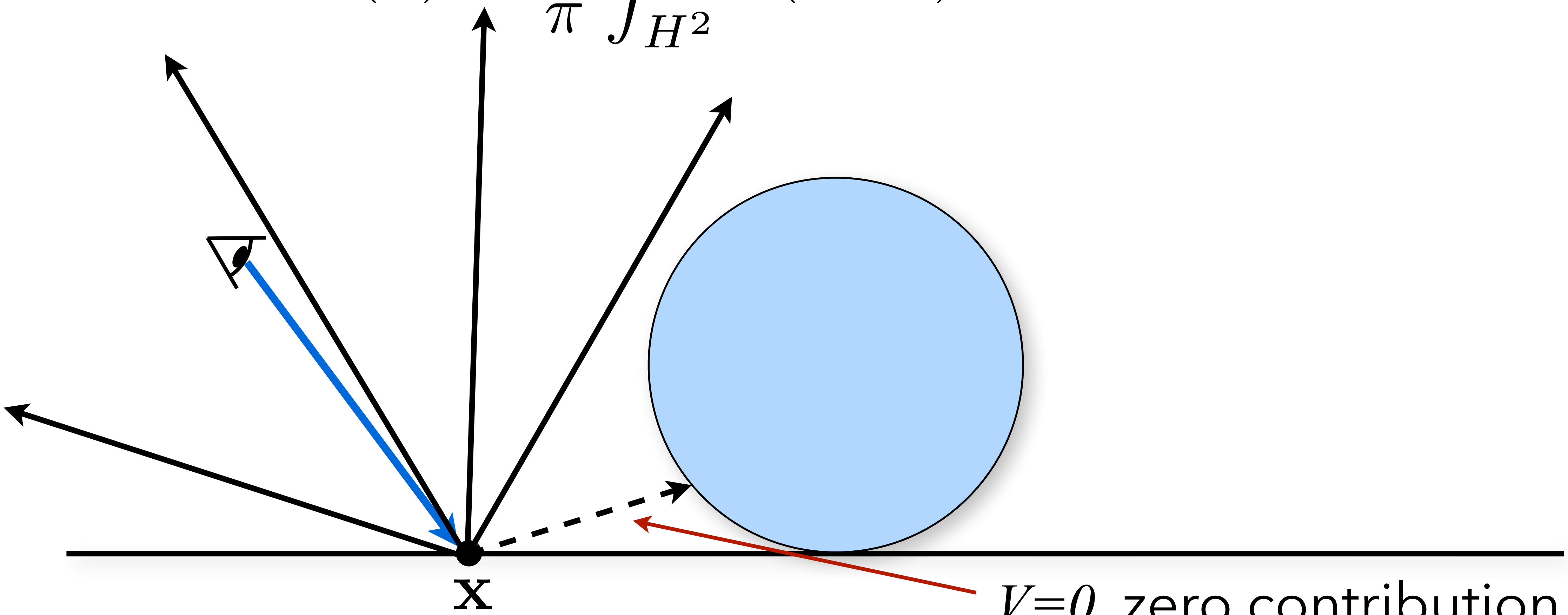
Ambient Occlusion

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$



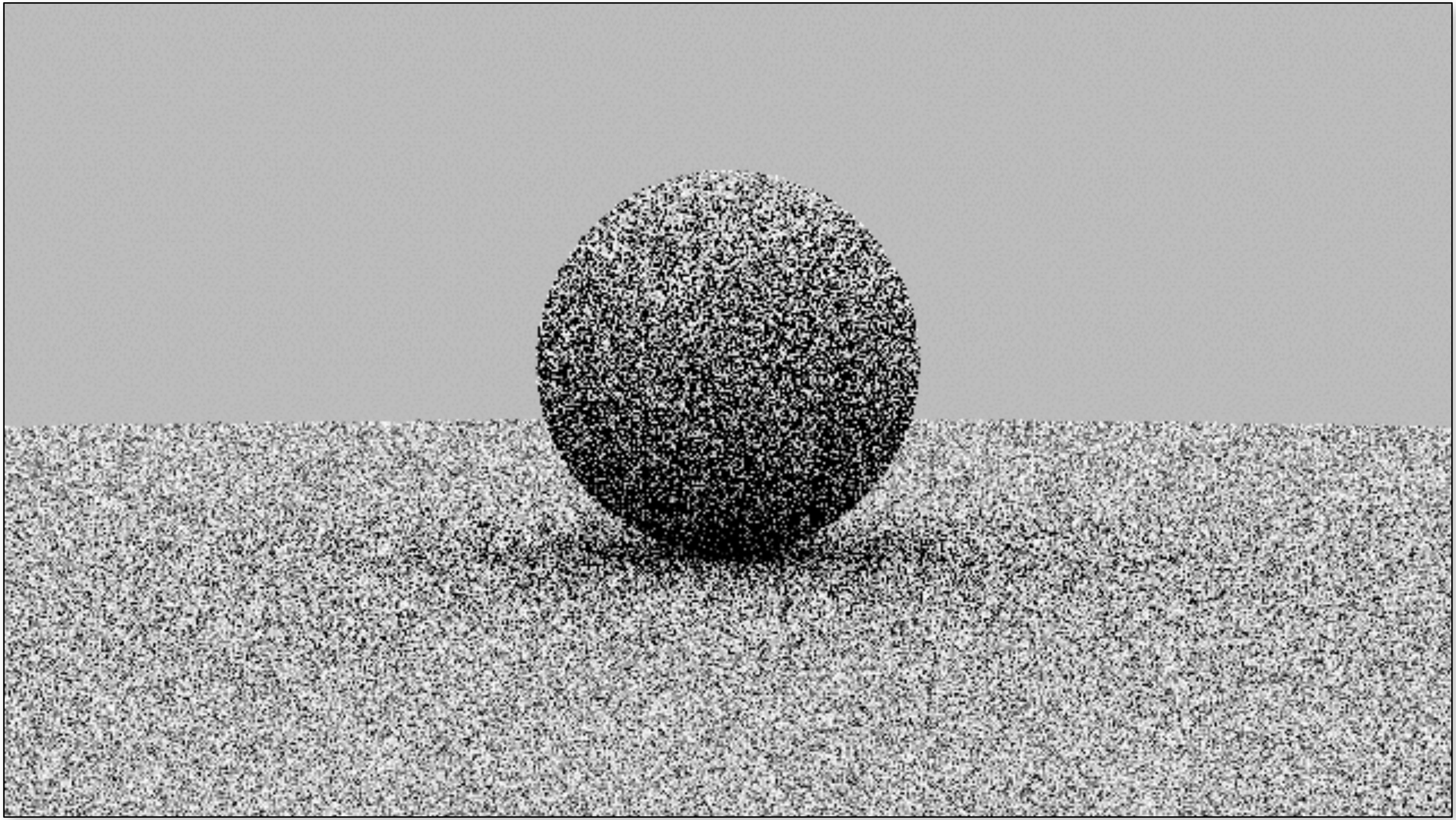
Ambient Occlusion

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

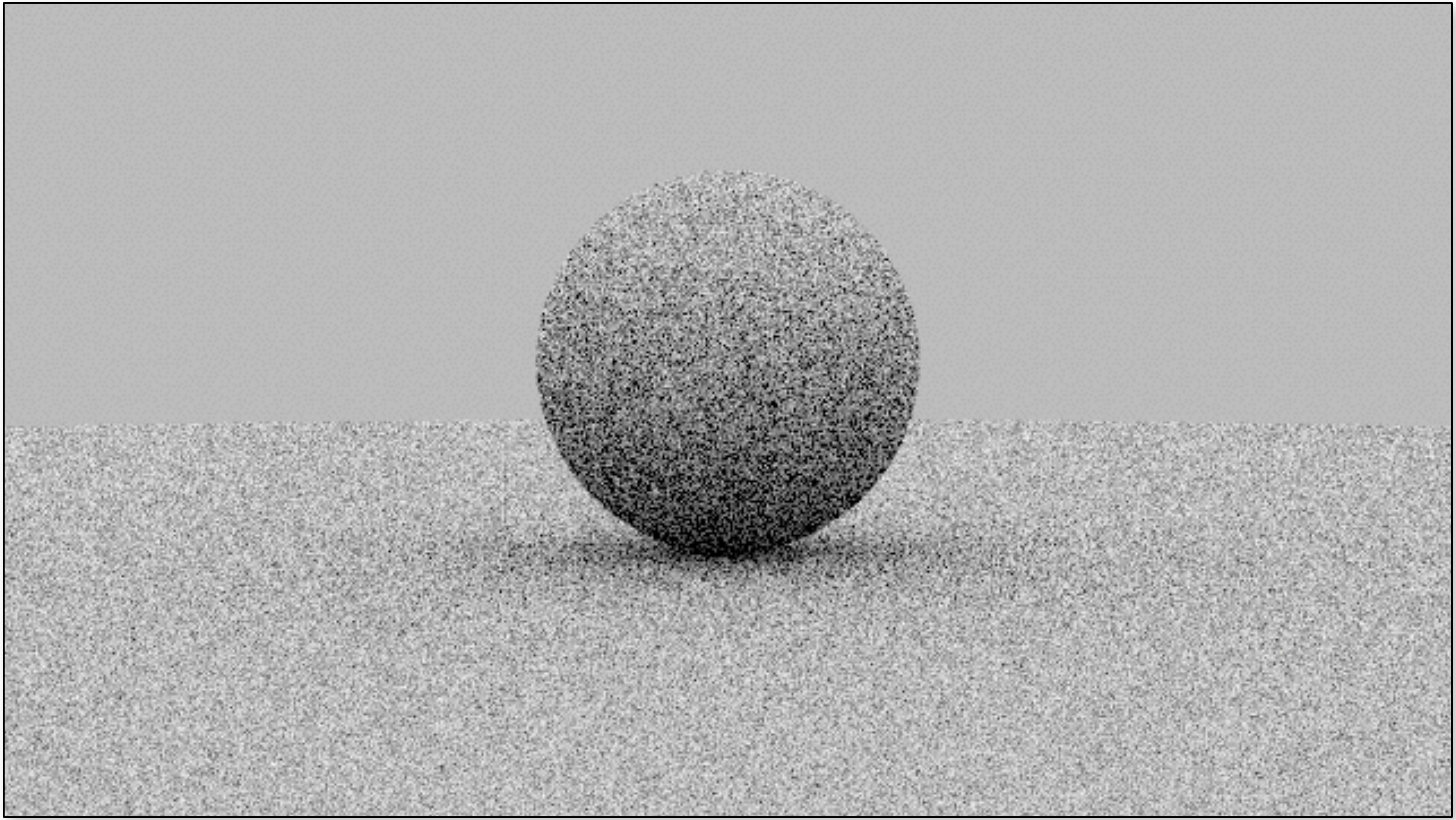


$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

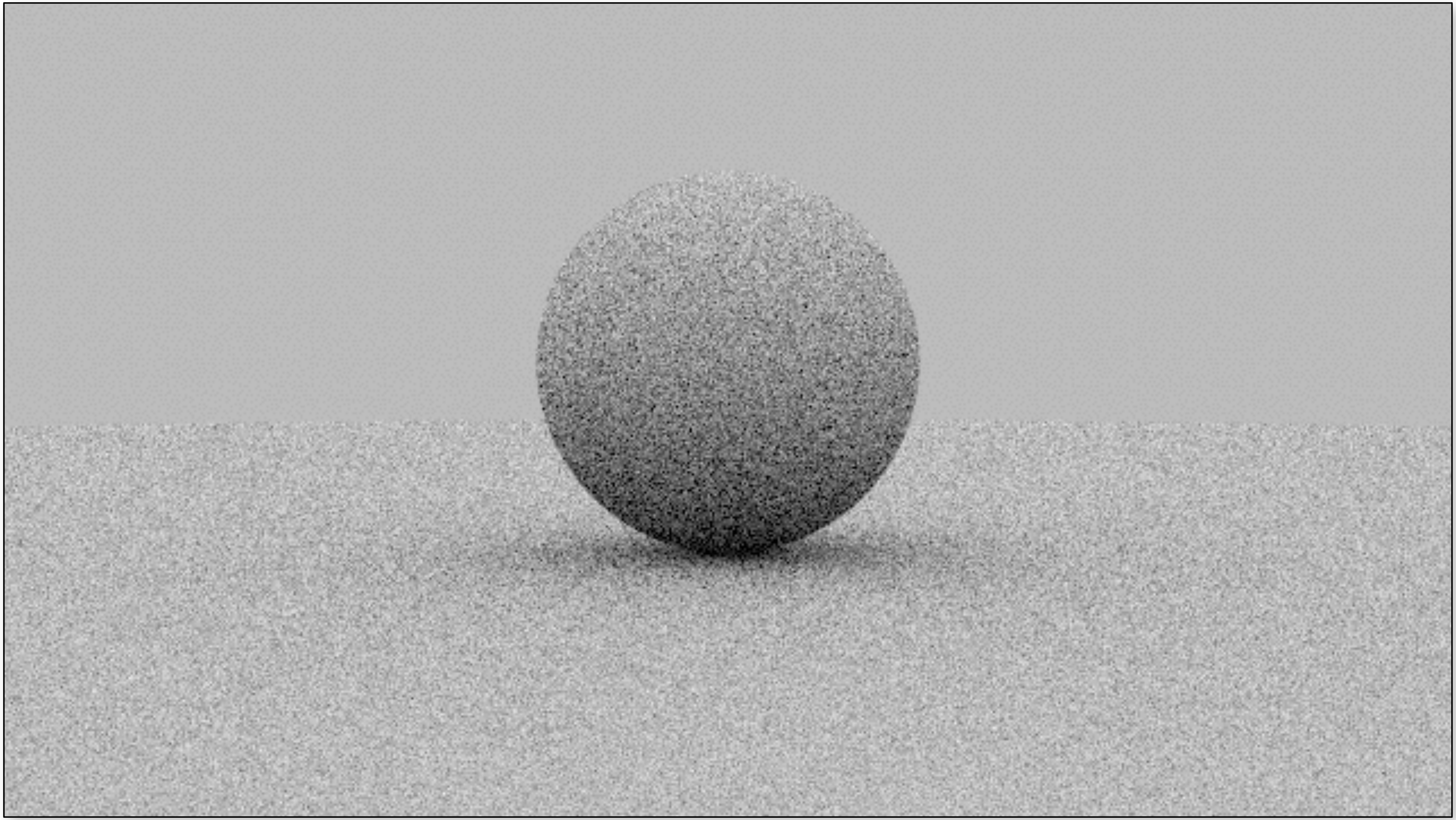
Hemispherical Sampling (1 Sample)



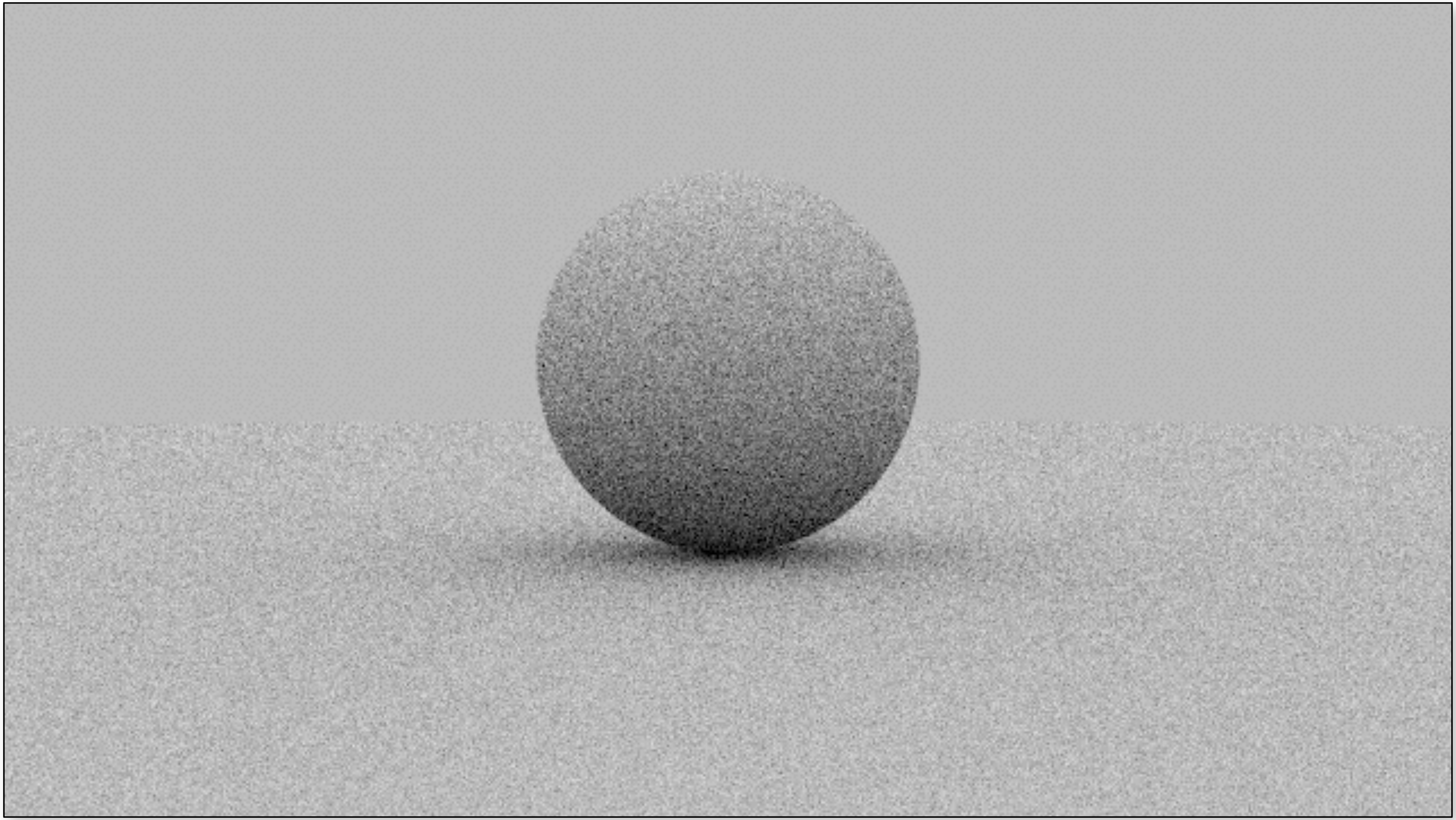
Hemispherical Sampling (4 Samples)



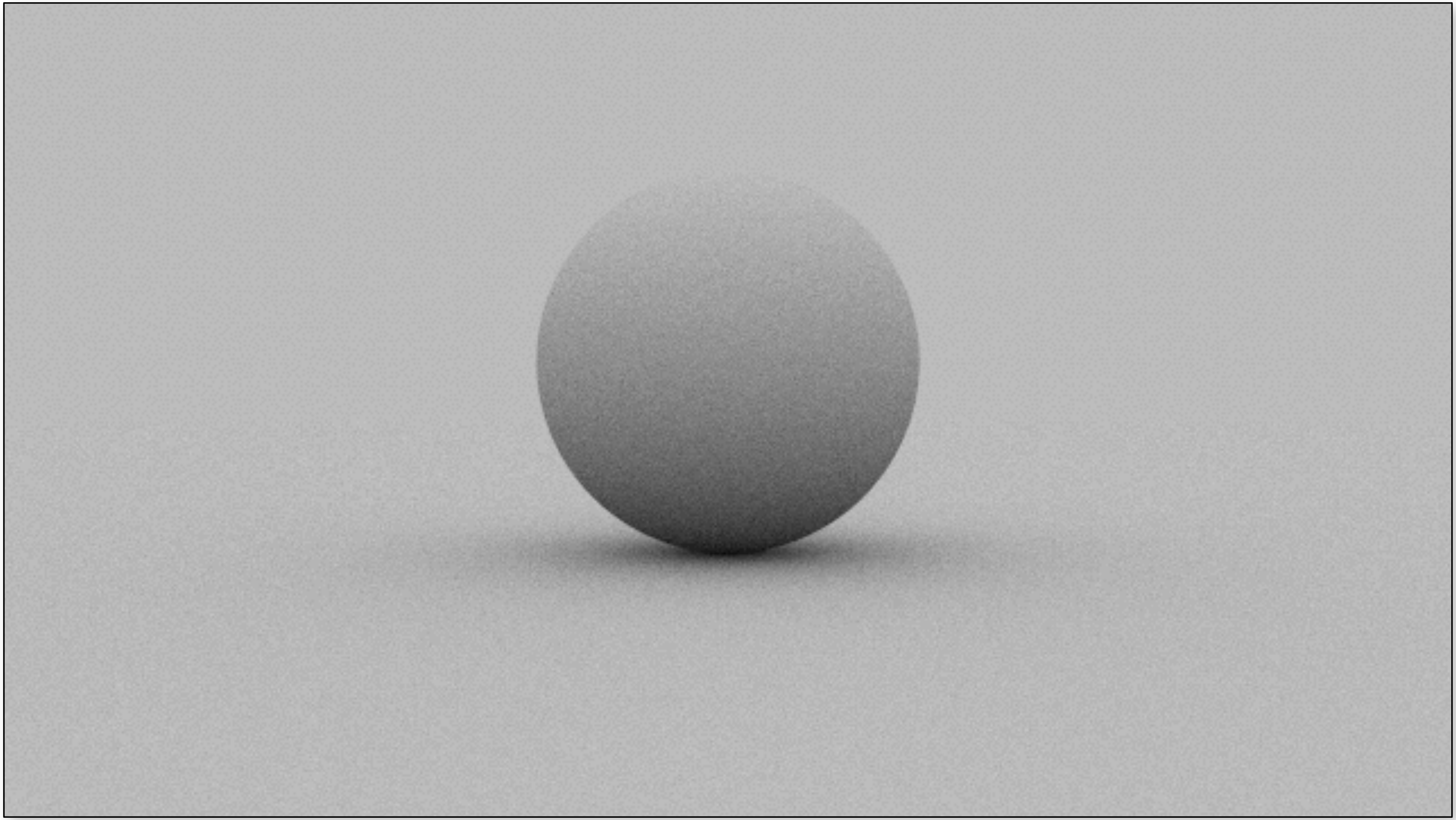
Hemispherical Sampling (9 Samples)



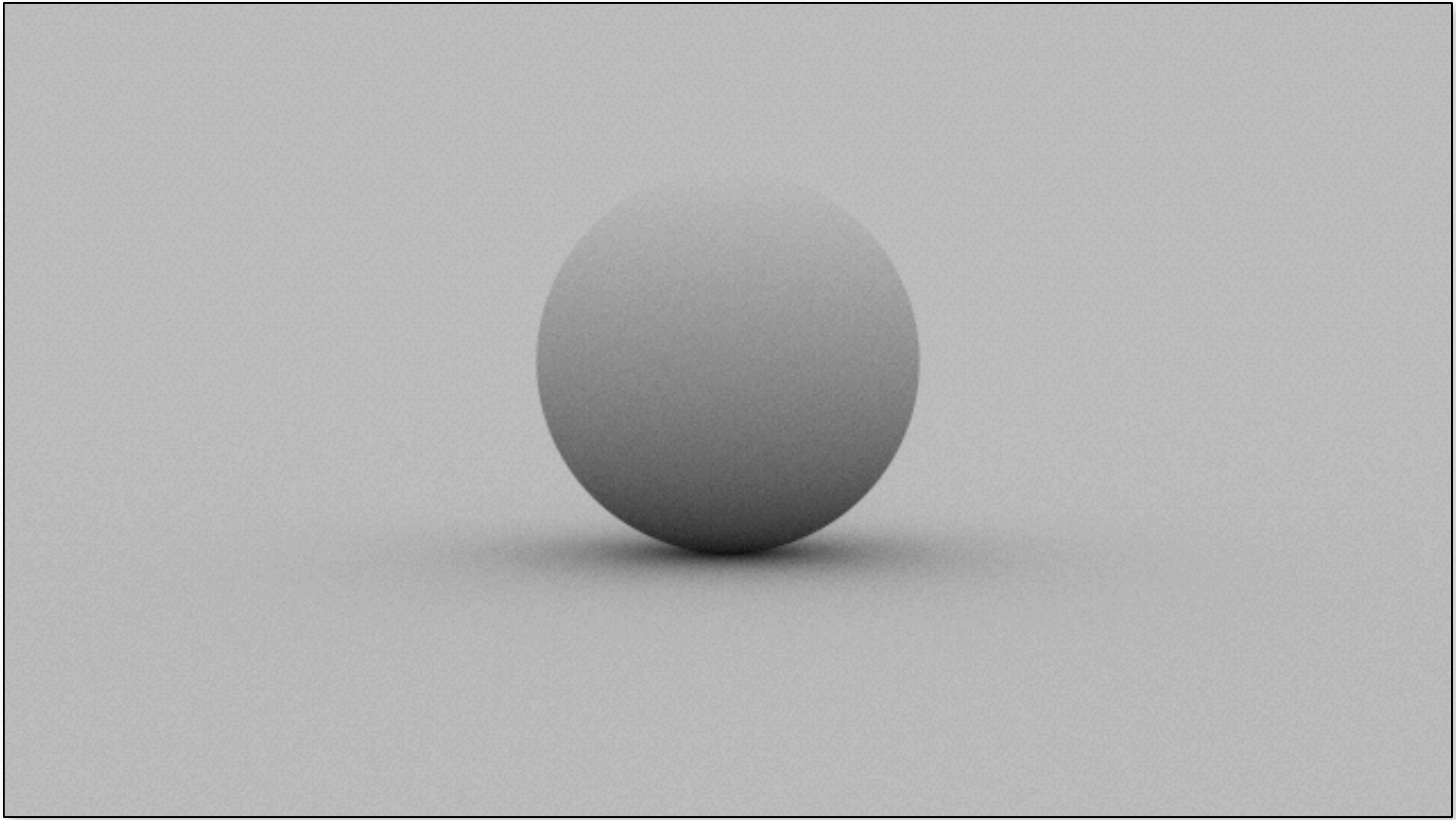
Hemispherical Sampling (16 Samples)



Hemispherical Sampling (256 Samples)



Hemispherical Sampling (1024 Samples)



Reducing Variance: Importance Sampling

- Importance sampling

$$\int f(x)dx \qquad\qquad F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

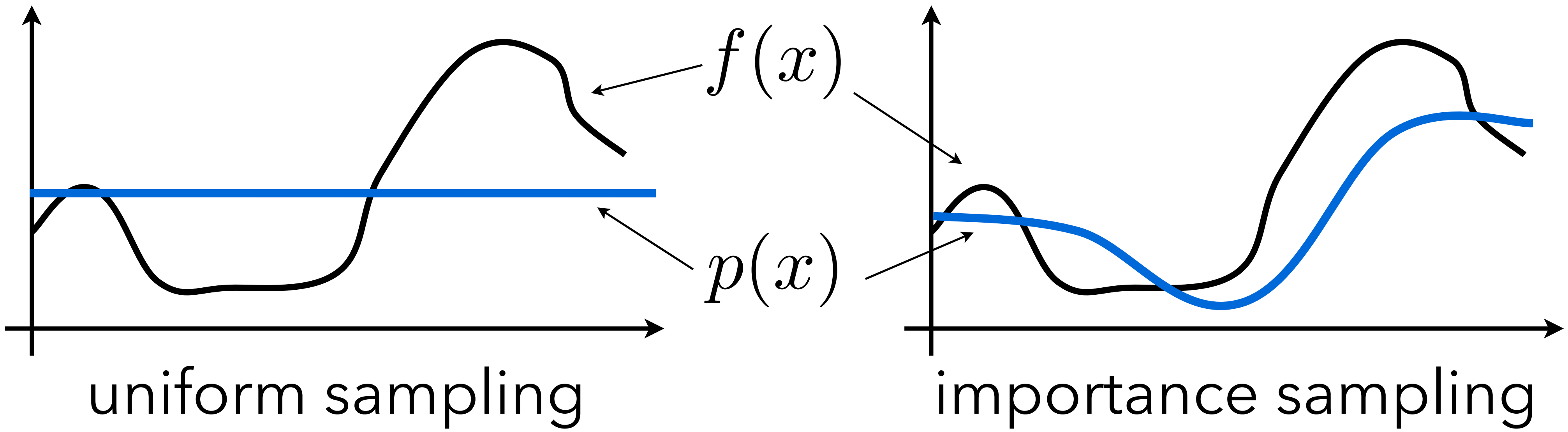
- assume $p(x) = cf(x)$

$$\int p(x)dx = 1 \quad \rightarrow \quad c = \frac{1}{\int f(x)dx}$$

- estimator $\frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x)dx$ zero variance!

Reducing Variance: Importance Sampling

- $p(x) = cf(x)$ requires knowledge of integral, which is what we are trying to solve!
- But: If PDF is similar to integrand, variance can be significantly reduced
- **Common strategy: sample according to part of the integrand**



Ambient Occlusion

- Ambient occlusion estimator:

$$L_r(\mathbf{x}) \approx \frac{\rho}{\pi N} \sum_{i=1}^{N} \frac{V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i}{p(\vec{\omega}_i)}$$

- Uniform hemispherical sampling, $p(\vec{\omega}_i) = 1/2\pi$:

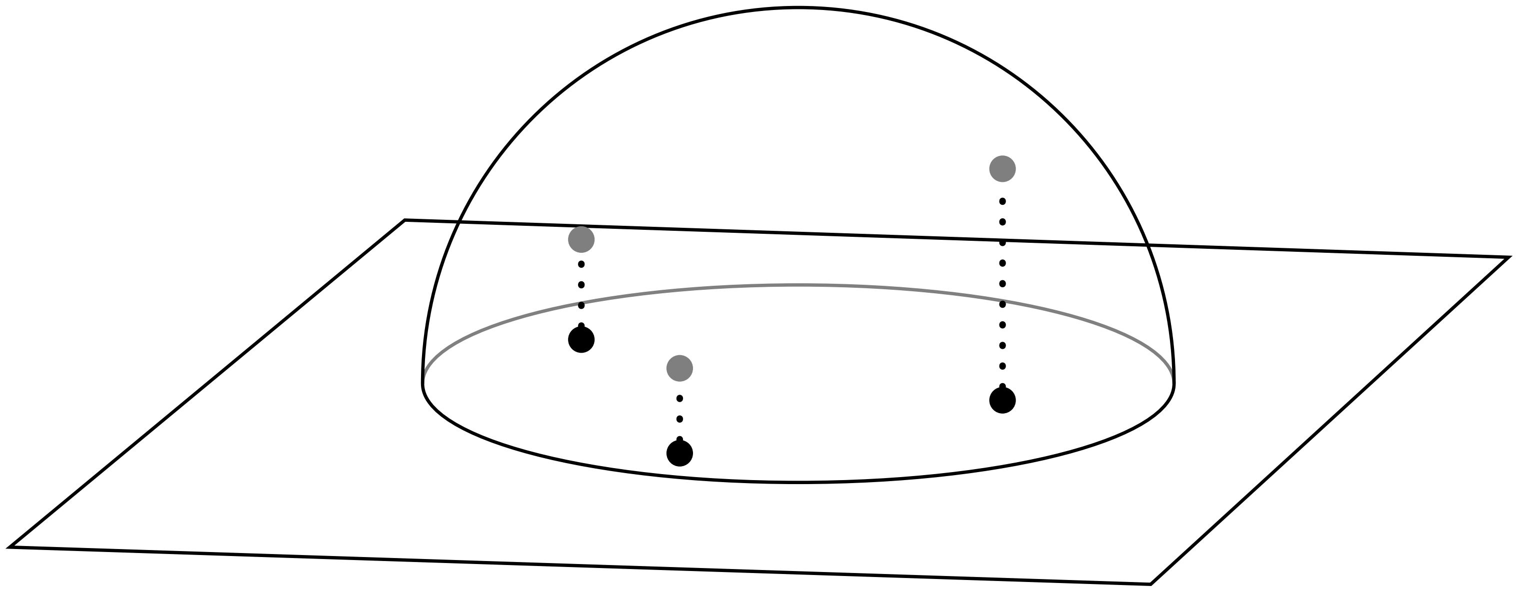
$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

- Using a PDF proportional to cosine, $p(\vec{\omega}_i) = \cos\theta_i / \pi$

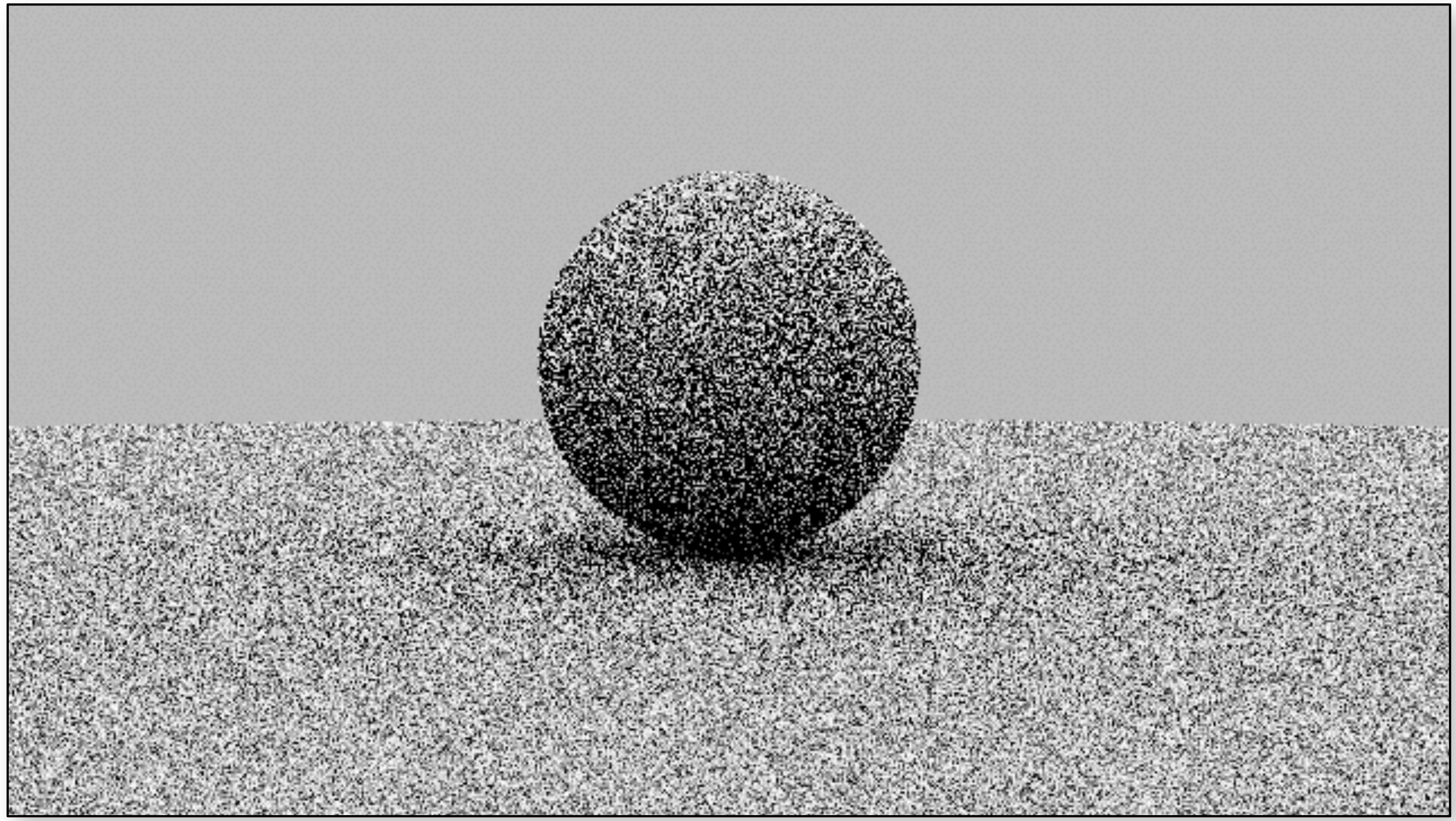
$$L_r(\mathbf{x}) \approx \frac{\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i)$$

Cosine-weighted Hemispherical Sampling

- Could proceed as before: compute marginal and conditional densities, then use inversion method.
- It turns out that:
  - Generating points uniformly on the disc, and then project these points to the surface of the hemisphere produces the desired distribution.

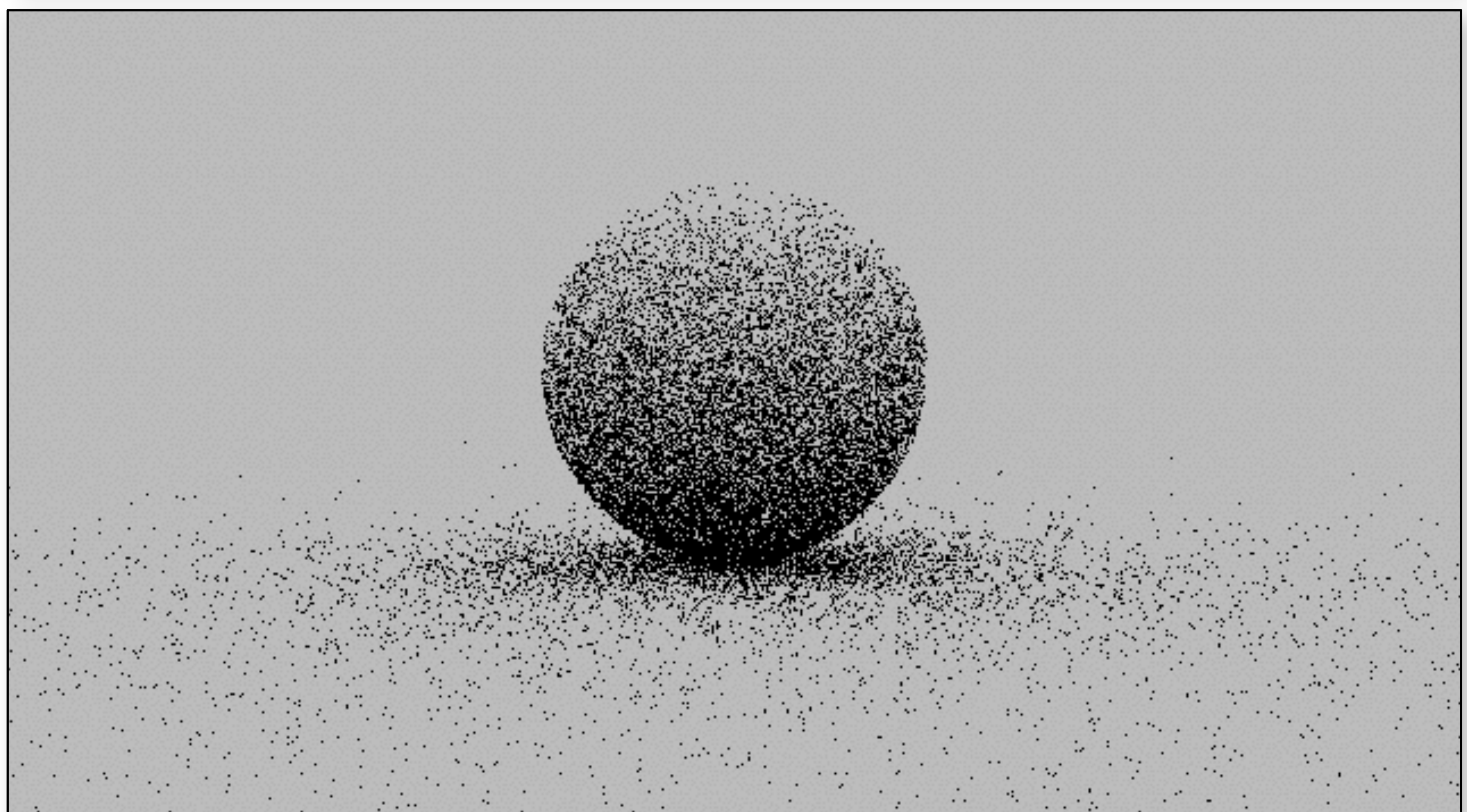


Uniform Hemispherical Sampling

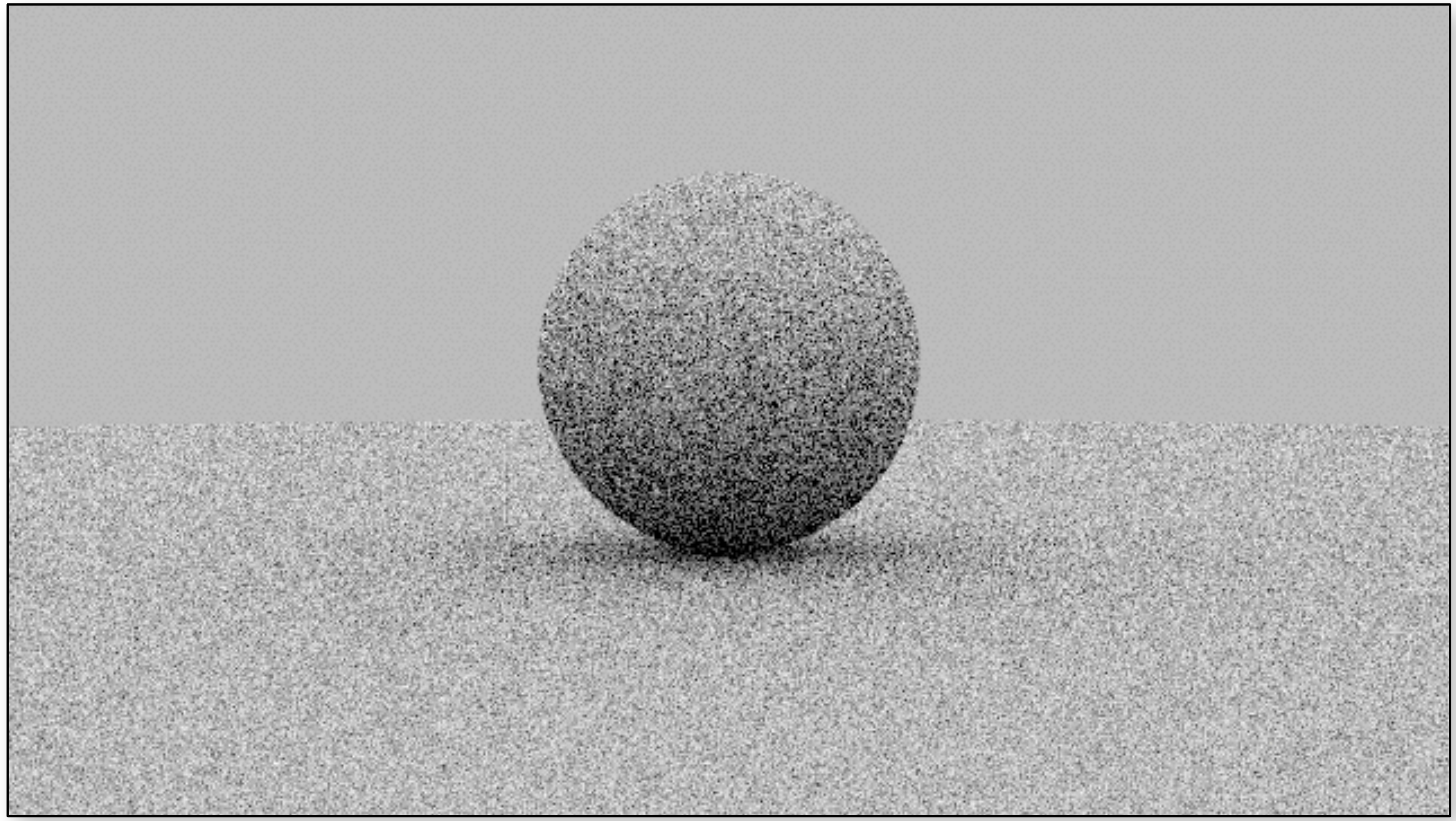


1 sample

Cosine-weighted Hemispherical Sampling

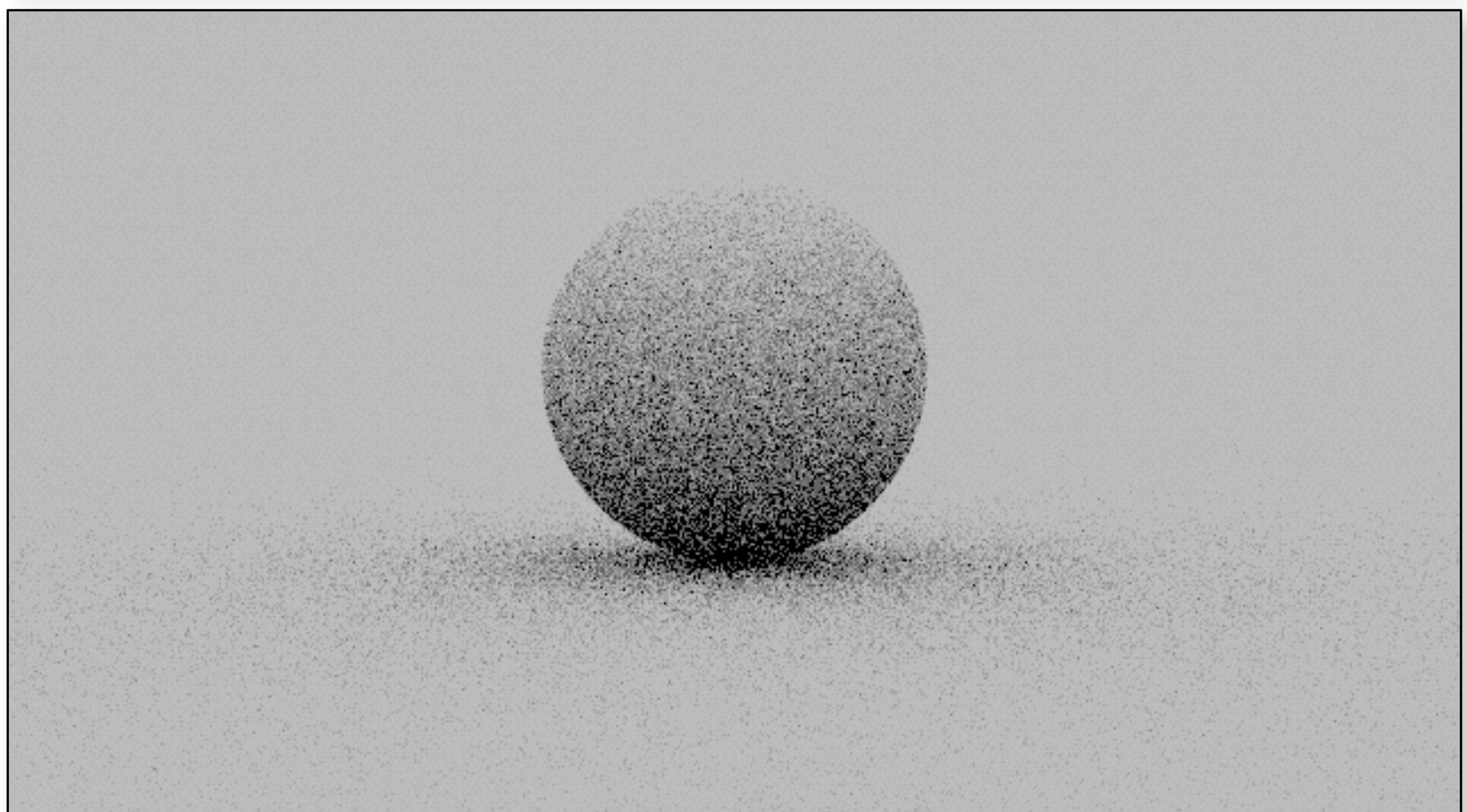


Uniform Hemispherical Sampling

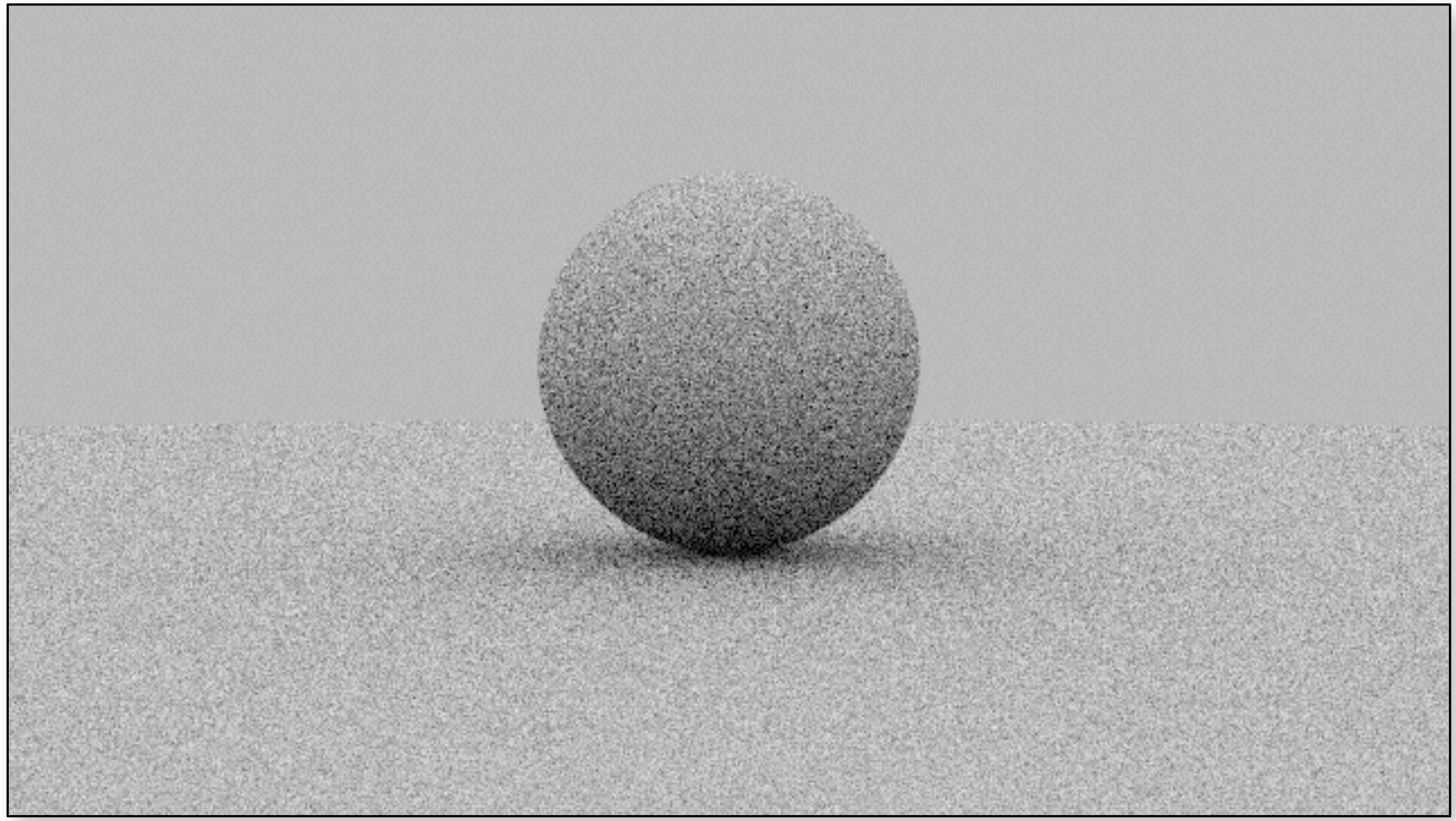


4 samples

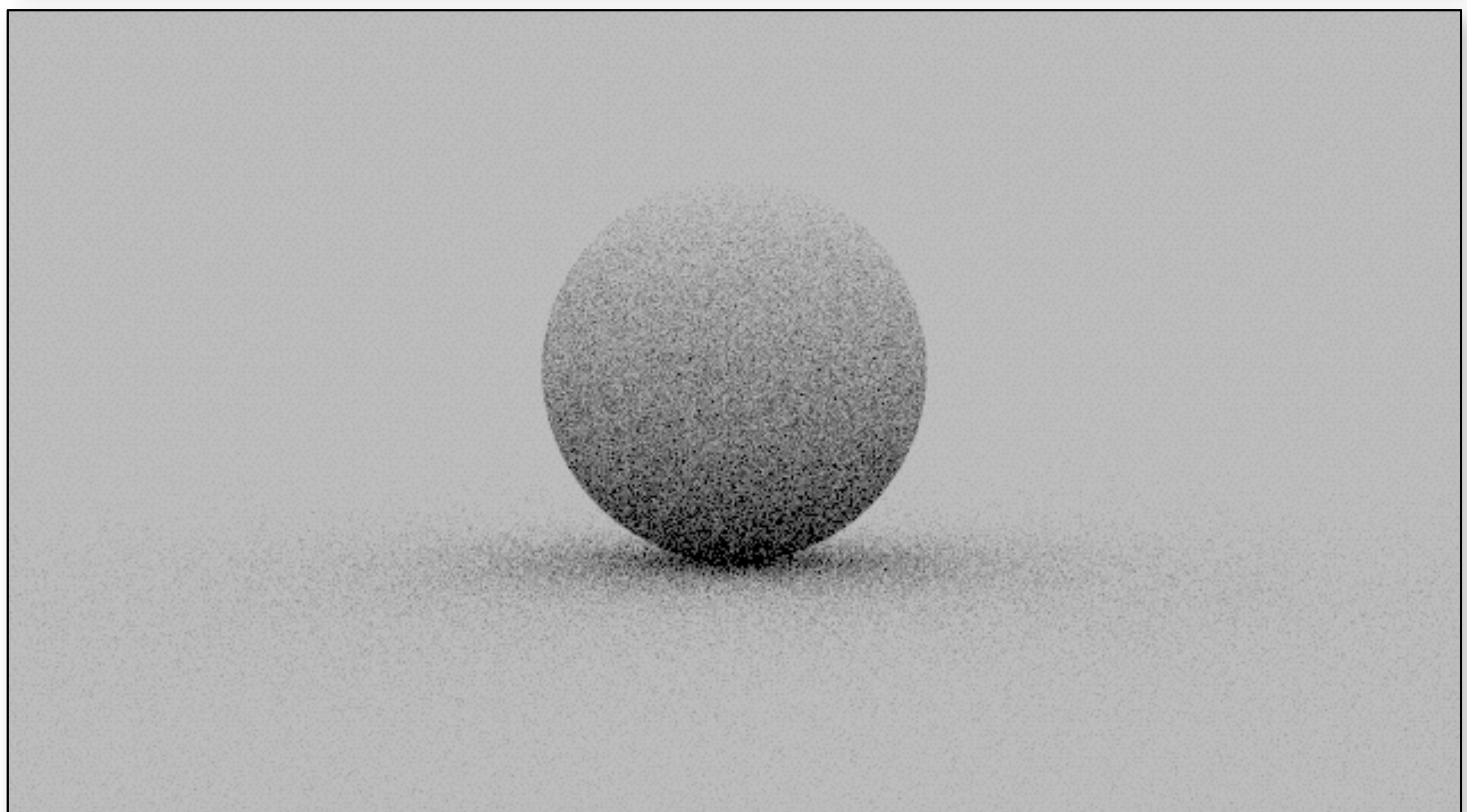
Cosine-weighted Hemispherical Sampling



Uniform Hemispherical Sampling

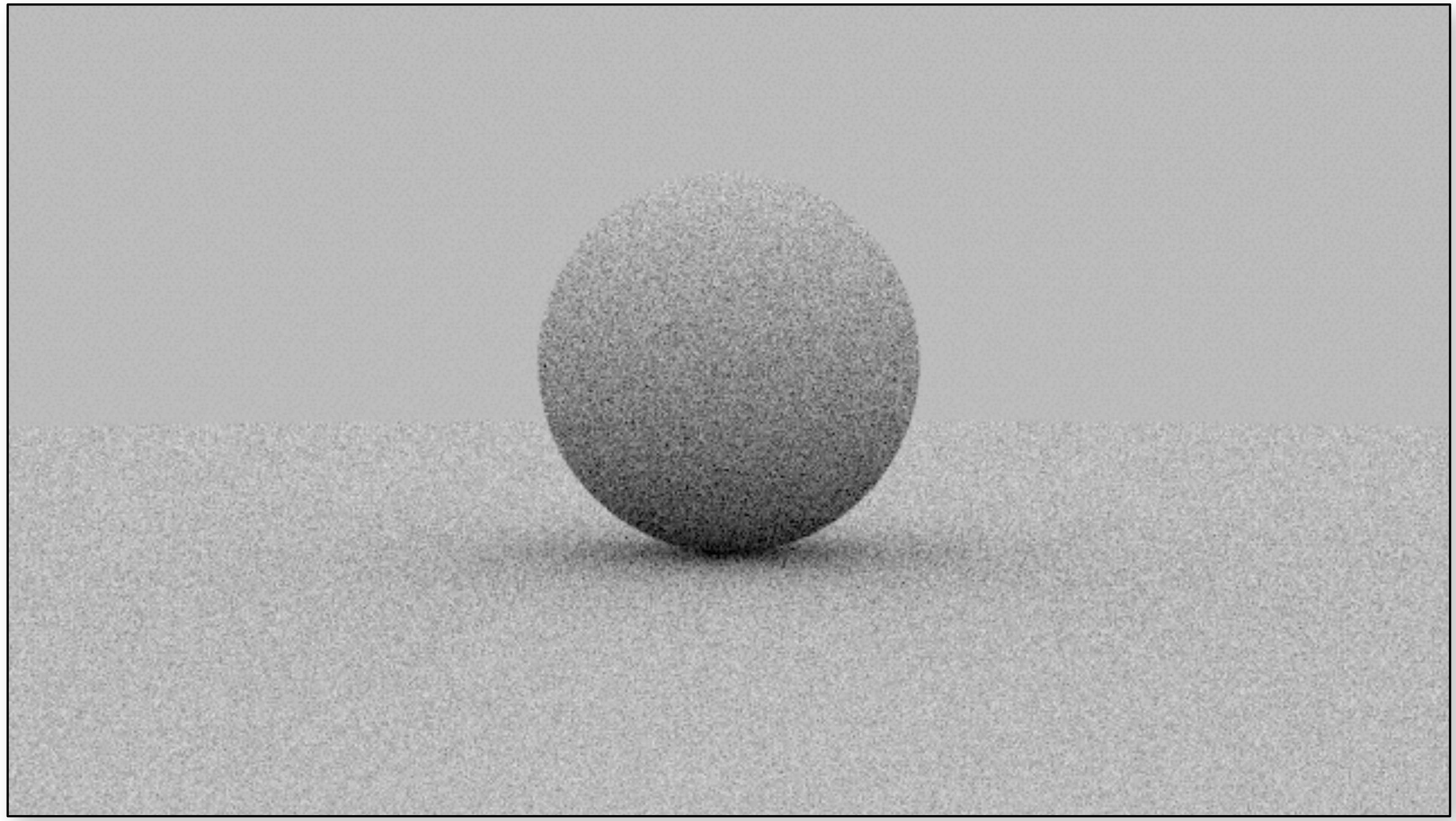


9 samples



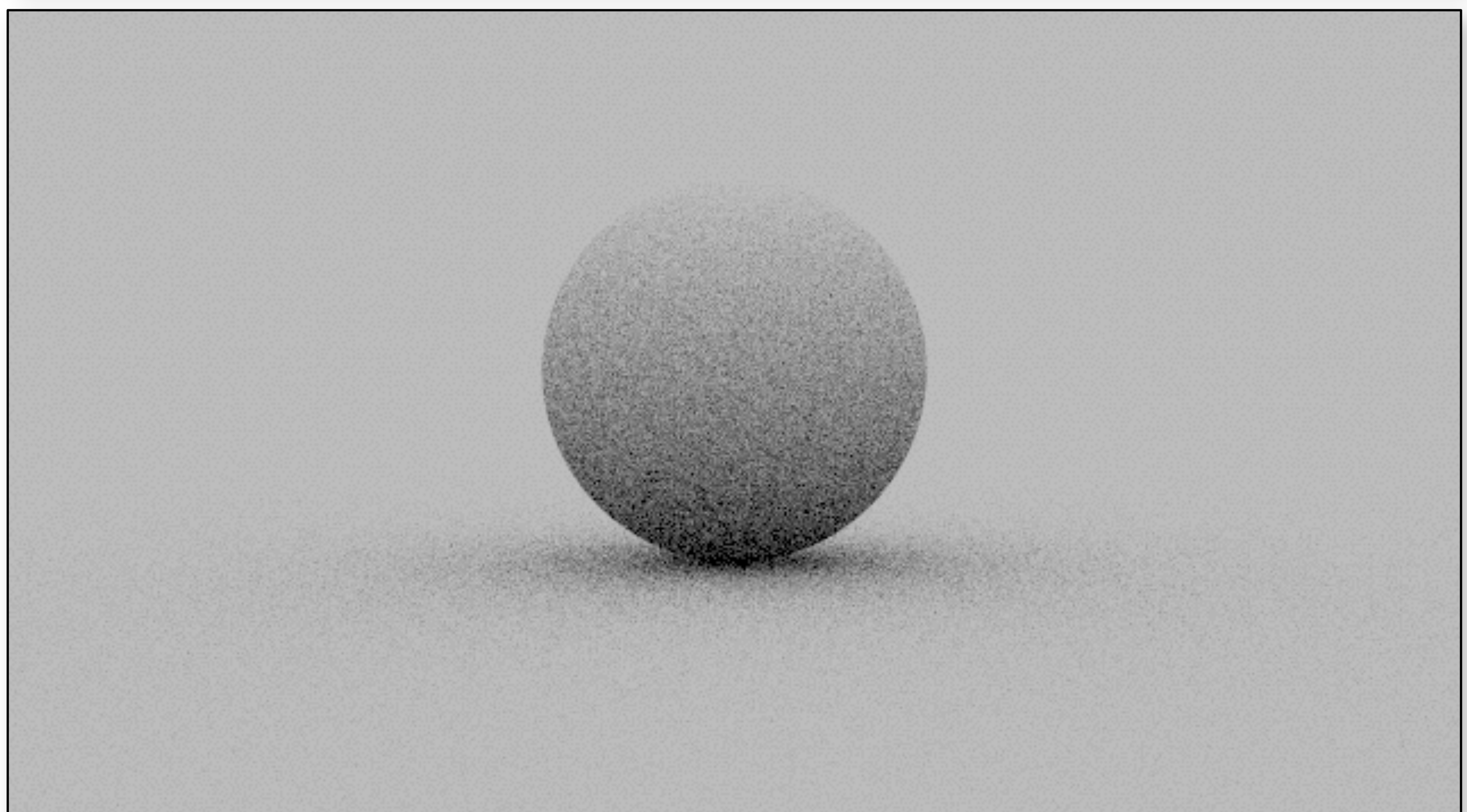
Cosine-weighted Hemispherical Sampling

Uniform Hemispherical Sampling

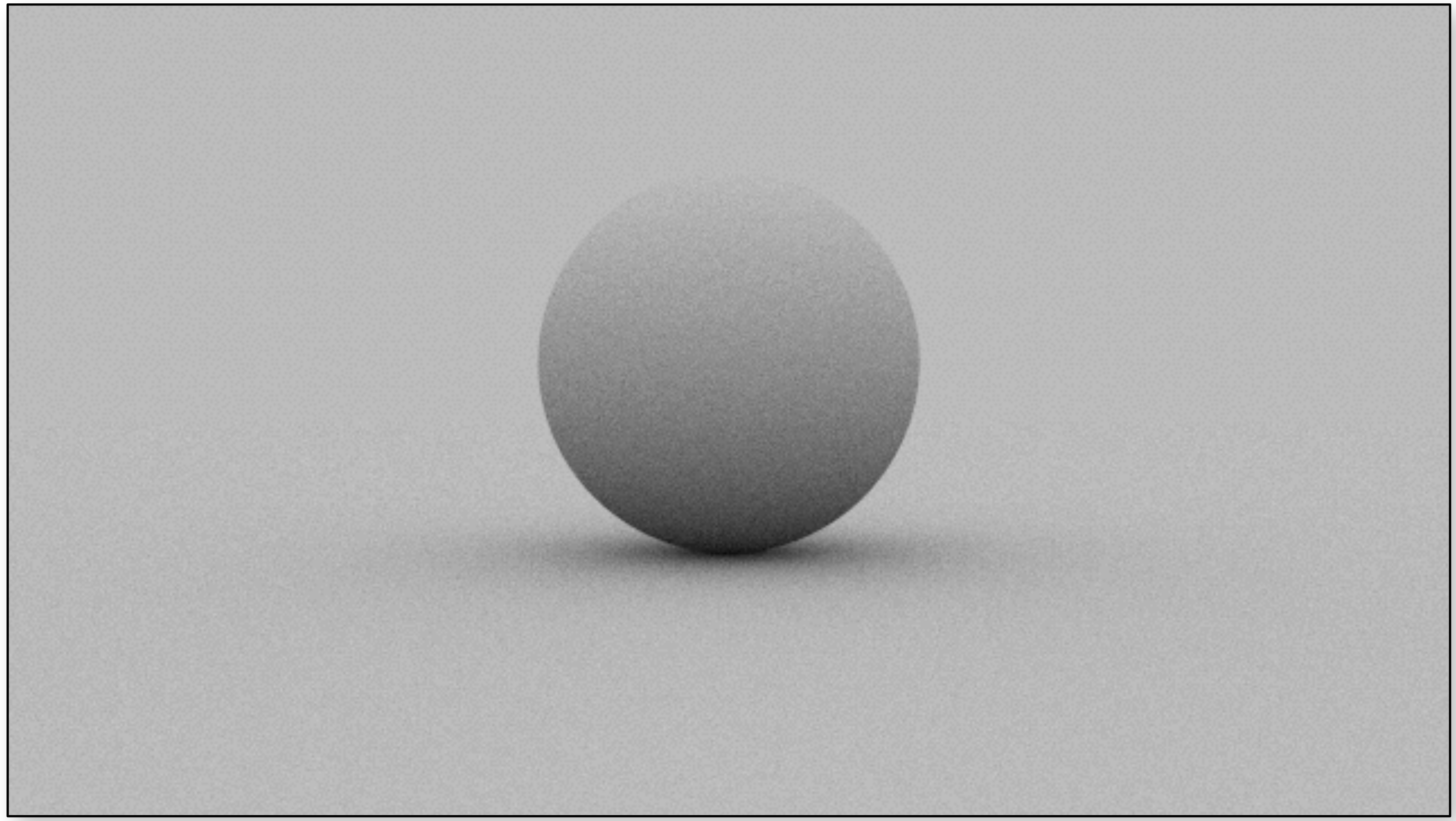


16 samples

Cosine-weighted Hemispherical Sampling

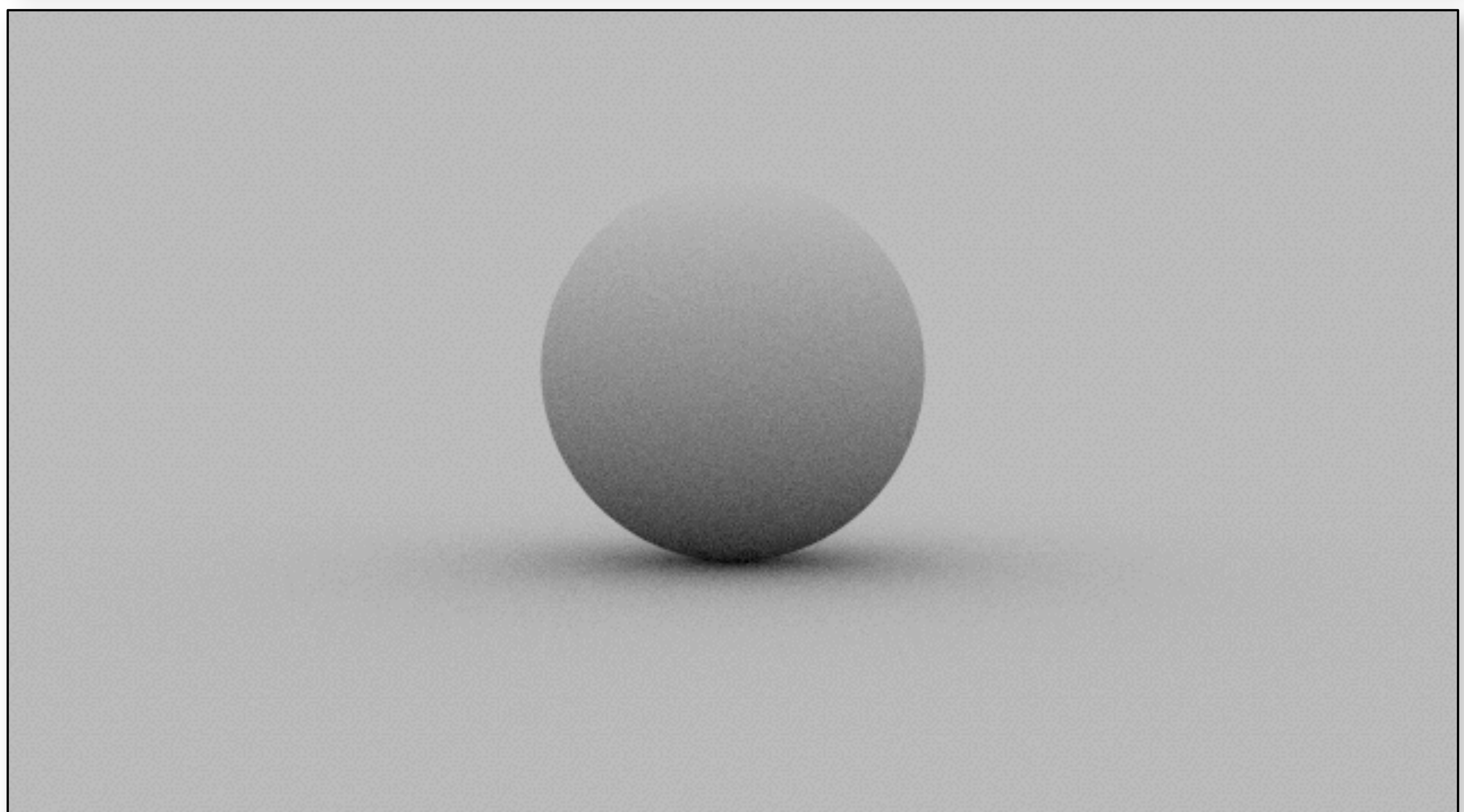


Uniform Hemispherical Sampling

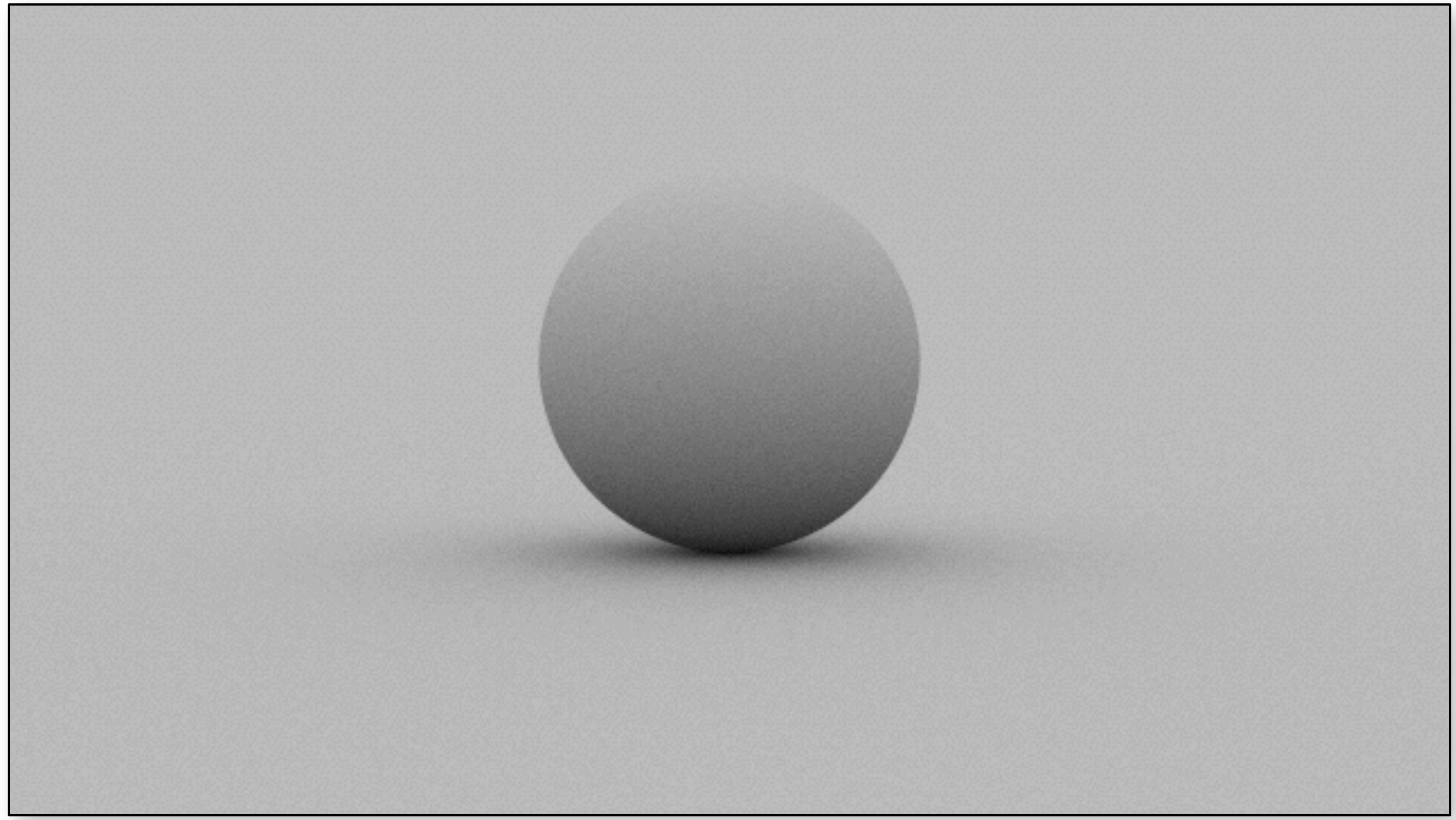


256 samples

Cosine-weighted Hemispherical Sampling

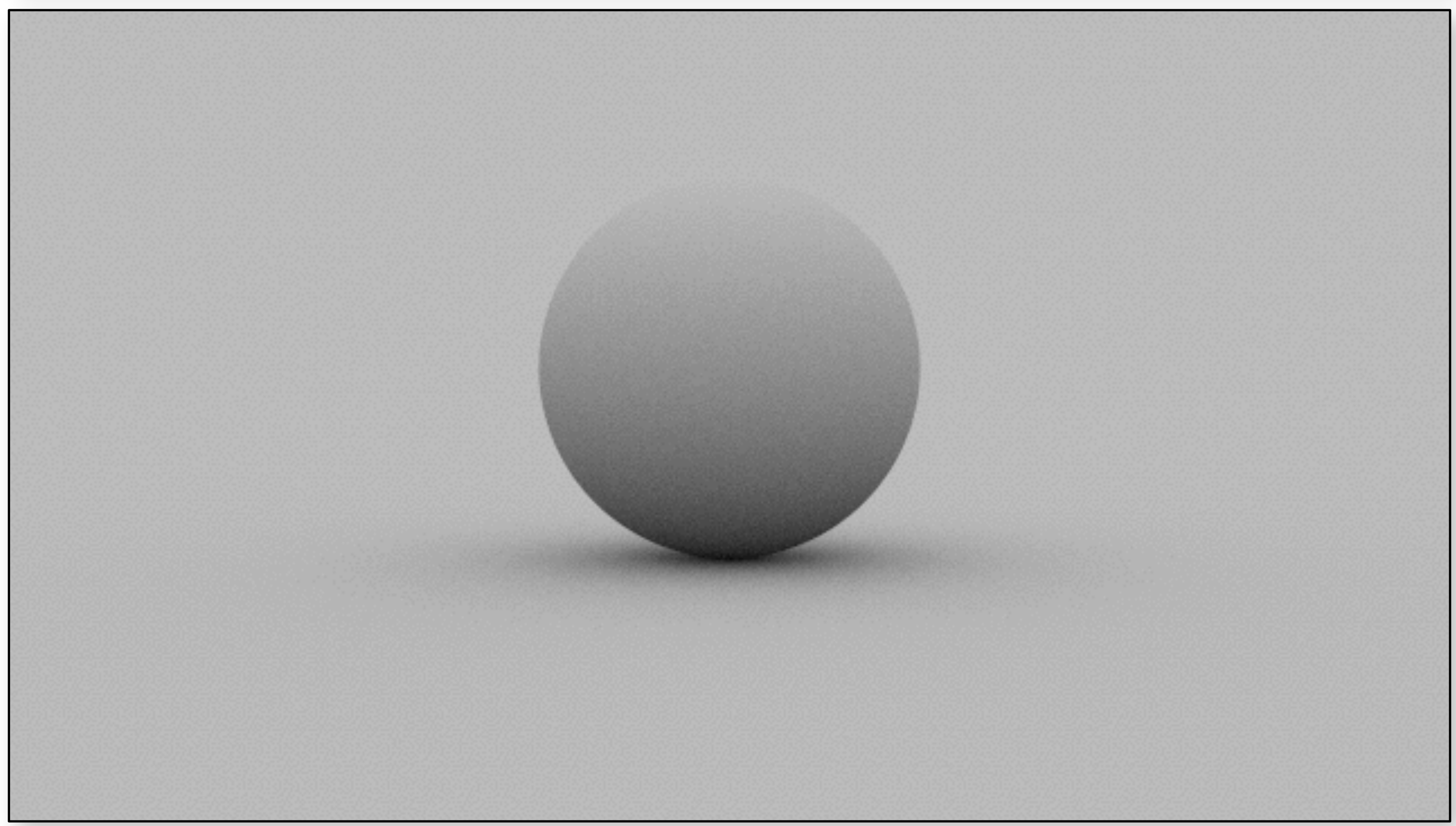


Uniform Hemispherical Sampling

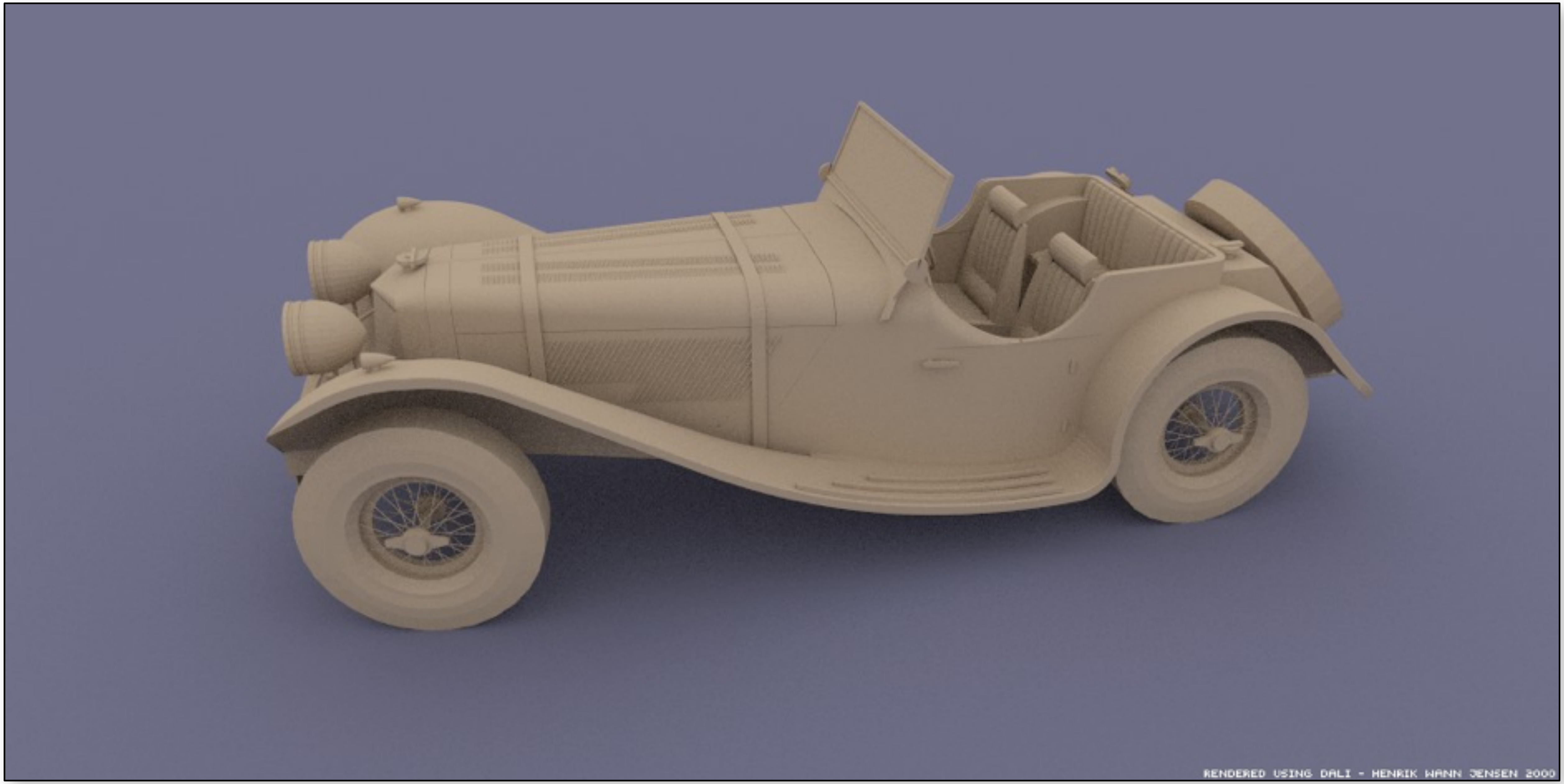


1024 samples

Cosine-weighted Hemispherical Sampling

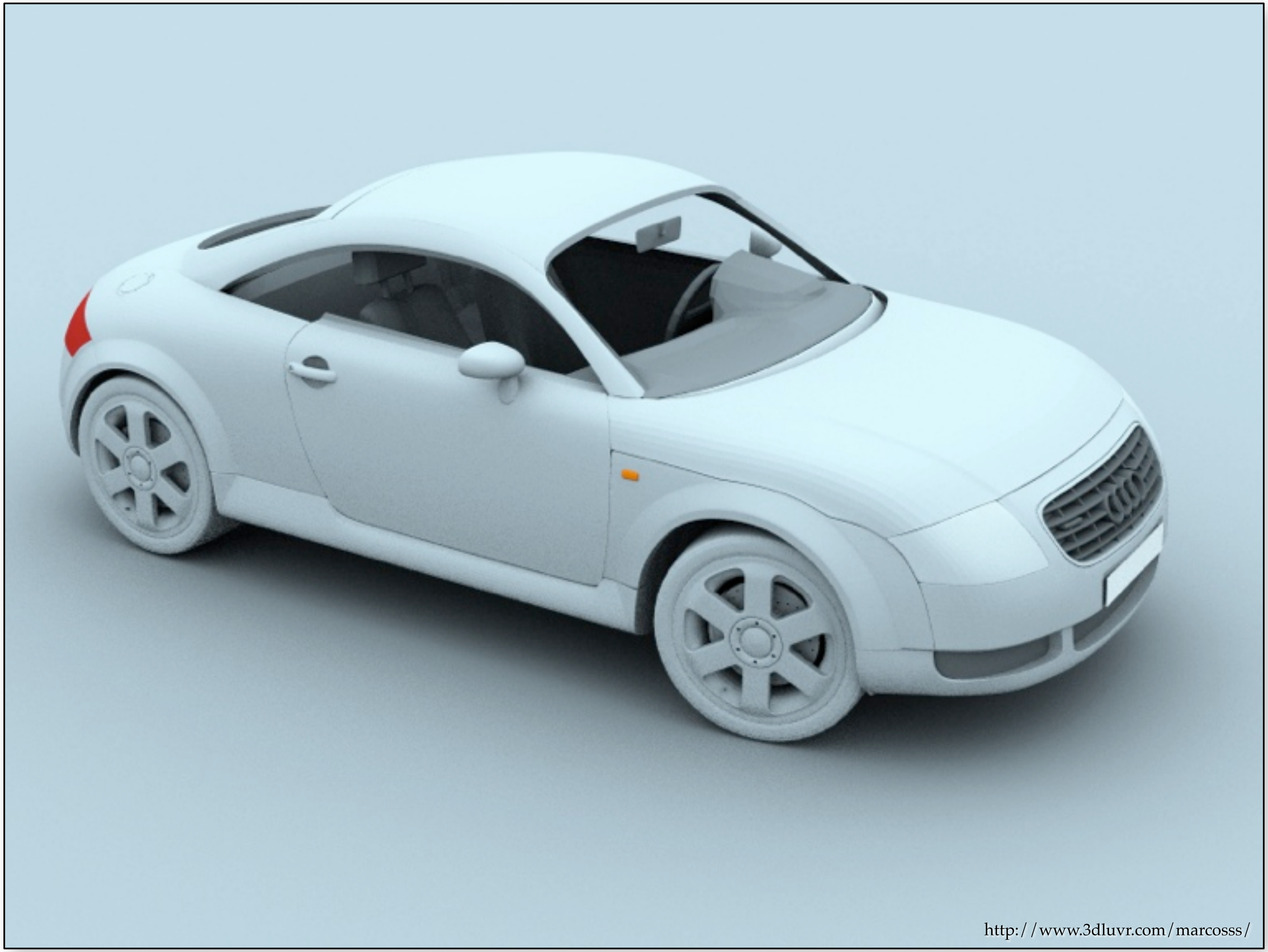


Ambient Occlusion

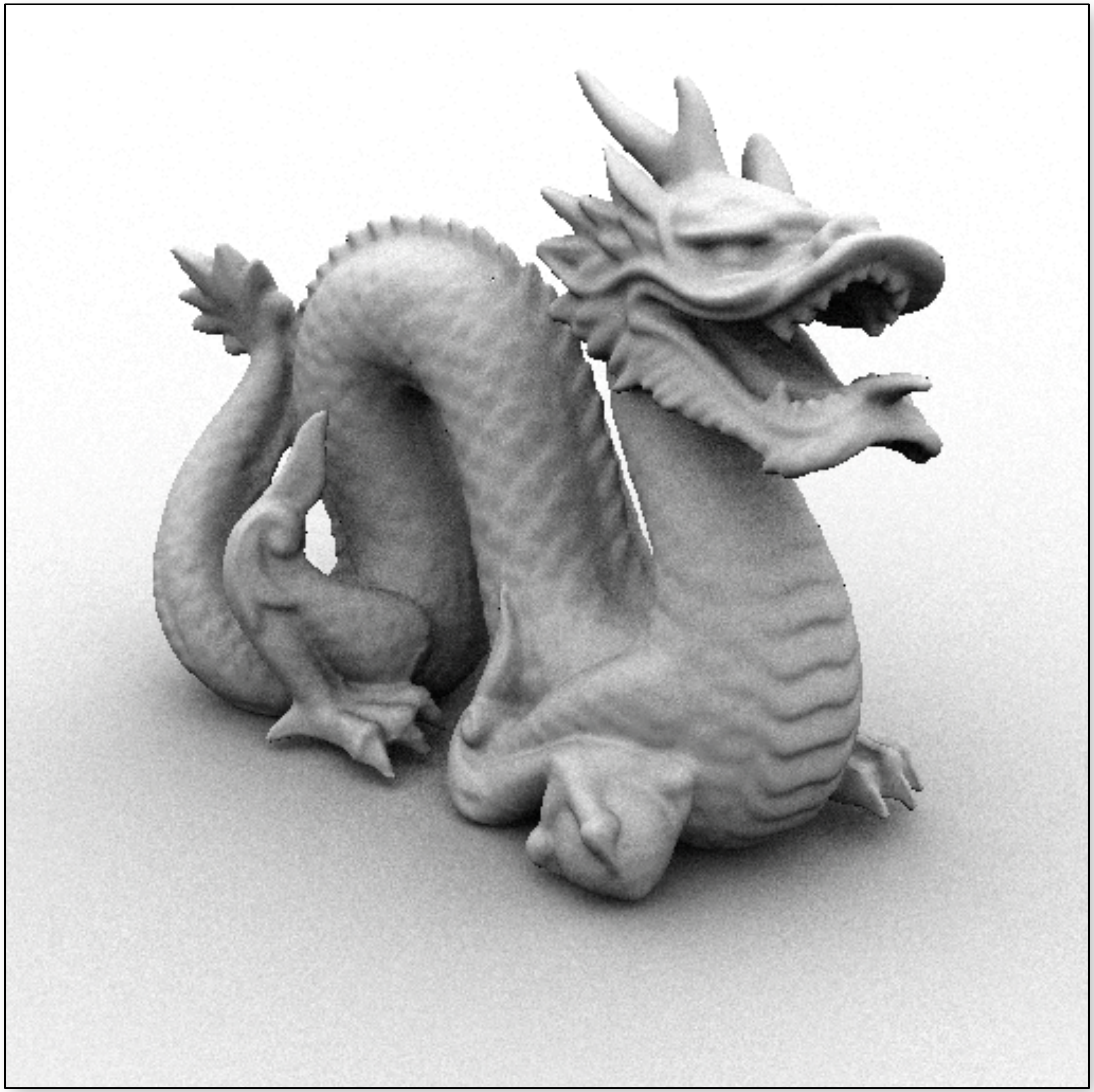


Ambient Occlusion

Ambient Occlusion


http://www.3dluvr.com/marcosss/

Ambient Occlusion



Ambient Occlusion

More Random Sampling

- Other useful sampling domains:
  - triangles
  - 1- or 2-D discrete PDFs (e.g. environment maps)
- Much more!

PBRe2: Chapters 13 & 14

Sampling Various Distributions

Target space	Density	Domain	Transformation
Radius R disk	$p(r,\theta) = \frac{1}{\pi R^2}$	$\theta \in [0, 2\pi]$ $r \in [0, R]$	$\theta = 2\pi u$ $r = R\sqrt{v}$
Sector of radius R disk	$p(r,\theta) = \frac{2}{(\theta_2 - \theta_1)(r_2^2 - r_1^2)}$	$\theta \in [\theta_1, \theta_2]$ $r \in [r_1, r_2]$	$\theta = \theta_1 + u(\theta_2 - \theta_1)$ $r = \sqrt{r_1^2 + v(r_2^2 - r_1^2)}$
Phong density exponent n	$p(\theta,\phi) = \frac{n+1}{2\pi}\cos^n\theta$	$\theta \in \left[0, \frac{\pi}{2}\right]$ $\phi \in [0, 2\pi]$	$\theta = \arccos\left((1-u)^{1/(n+1)}\right)$ $\phi = 2\pi v$
Separated triangle filter	$p(x,y)(1 - \lvert x\rvert)(1 - \lvert y\rvert)$	$x \in [-1, 1]$ $y \in [-1, 1]$	$x = \begin{cases} 1 - \sqrt{2(1-u)} & \text{if } u \geq 0.5 \\ -1 + \sqrt{2u} & \text{if } u < 0.5 \end{cases}$ $y = \begin{cases} 1 - \sqrt{2(1-v)} & \text{if } v \geq 0.5 \\ -1 + \sqrt{2v} & \text{if } v < 0.5 \end{cases}$
Triangle with vertices a_0, a_1, a_2	$p(a) = \frac{1}{\text{area}}$	$s \in [0, 1]$ $t \in [0, 1 - s]$	$s = 1 - \sqrt{1-u}$ $t = (1 - s)v$ $a = a_0 + s(a_1 - a_0) + t(a_2 - a_0)$
Surface of unit sphere	$p(\theta,\phi) = \frac{1}{4\pi}$	$\theta \in [0, \pi]$ $\phi \in [0, 2\pi]$	$\theta = \arccos(1 - 2u)$ $\phi = 2\pi v$
Sector on surface of unit sphere	$p(\theta,\phi) = \frac{1}{(\phi_2 - \phi_1)(\cos\theta_1 - \cos\theta_2)}$	$\theta \in [\theta_1, \theta_2]$ $\phi \in [\phi_1, \phi_2]$	$\theta = \arccos[\cos\theta_1 + u(\cos\theta_2 - \cos\theta_1)]$ $\phi = \phi_1 + v(\phi_2 - \phi_1)$
Interior of radius R sphere	$p = \frac{3}{4\pi R^3}$	$\theta \in [0, \pi]$ $\phi \in [0, 2\pi]$ $R \in [0, R]$	$\theta = \arccos(1 - 2u)$ $\phi = 2\pi v$ $r = w^{1/3}R$

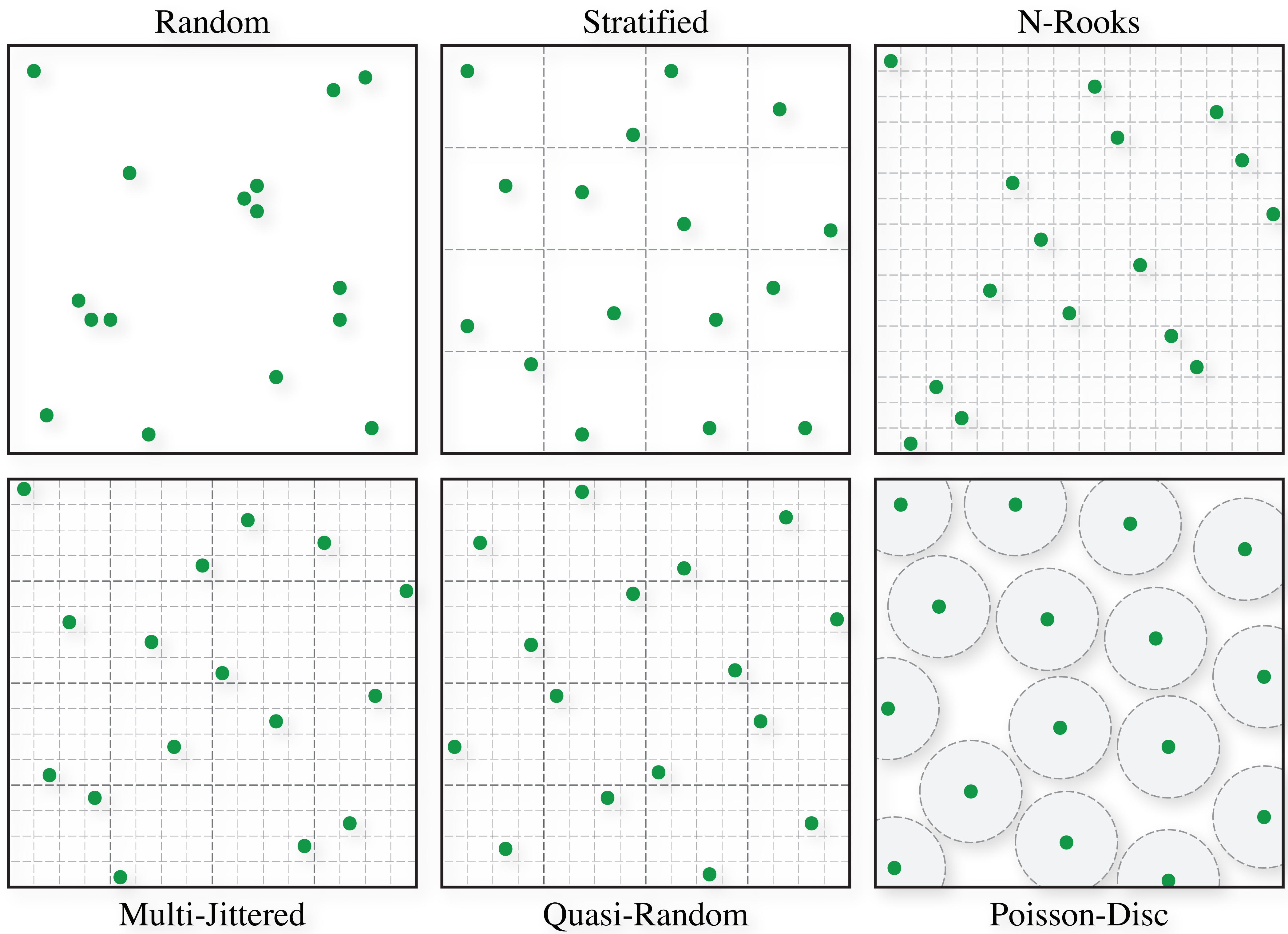
[a] The symbols u, v, and w represent instances of uniformly distributed random variables ranging over $[0, 1]$.

from: Peter Shirley. "Nonuniform random point sets via warping." Graphics Gems III, 1992.

More Integration Dimensions

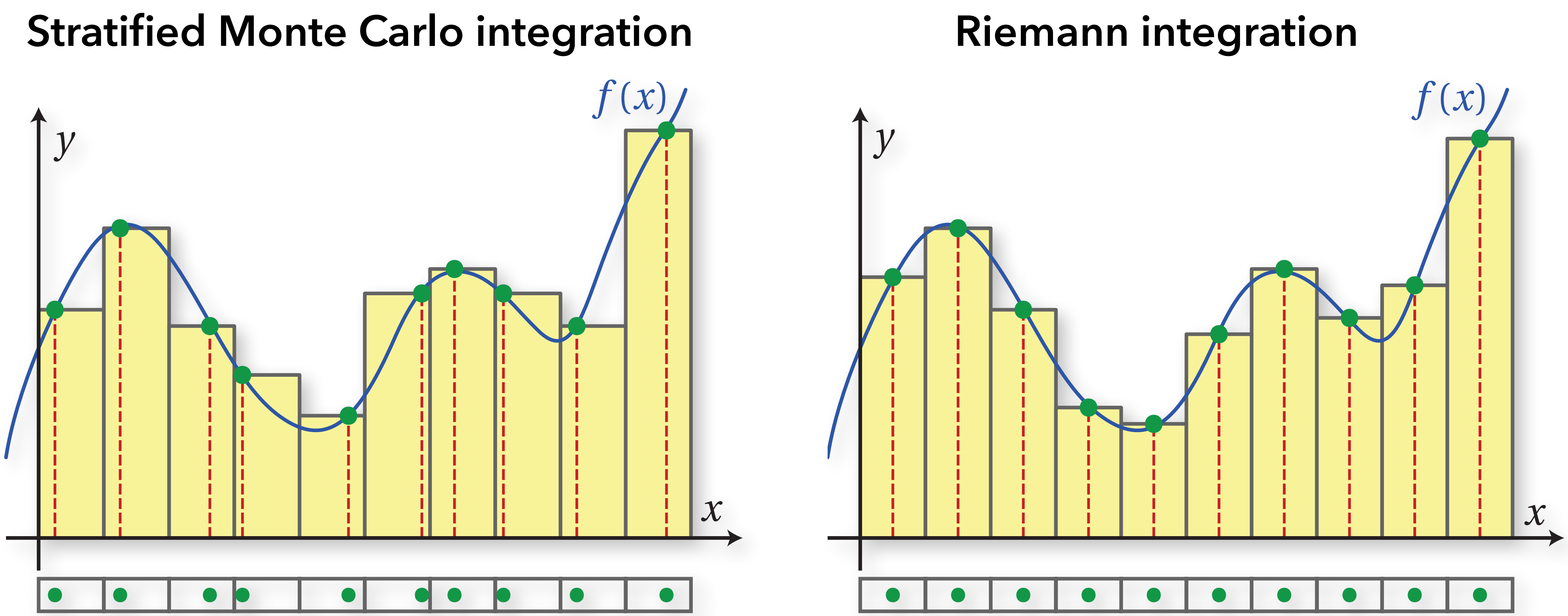
- Anti-aliasing (image space)
- Light visibility (surface of area lights)
- Depth-of-field (camera aperture)
- Motion blur (time)
- Many lights
- Multiple bounces of light
- Participating media (volume)

Careful Sample Placement



Variance Reduction: Stratification

- Subdivide integration domain into disjoint regions
- Place one random sample in each region
- Provably cannot increase variance
  - often reduces variance considerably



Next Lecture

- Better canonical random points
  - Stratification
  - Quasi Monte Carlo

Next Programming Assignment

- Random Sampling
  - warp canonical random points to various domains
  - interactive OpenGL viewer