CAR-FOLLOWING THEORY OF STEADY-STATE TRAFFIC FLOW

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(Received February 13, 1959)

The steady-state flow is examined for a car-following model in which the acceleration at time t of a car attempting to follow a lead car is proportional to the relative velocity at a time $t-\Delta$ and in which the sensitivity λ is no longer taken constant as in previous work but is inversely proportional to the car spacing The characteristics of the steady-state flow for this model are described and compared with experimental data

IN TWO recent papers[1 2] a detailed analysis has been given of the manner in which vehicles can follow each other on a highway without passing It was shown experimentally that an approximate description of the way in which a car follows a leader is given by relative velocity control in which the acceleration at time t of the following car is proportional to the relative velocity of the two cars at a time $t-\Delta$, the time lag Δ being about 1 5 sec Conditions were derived for the local and asymptotic stability of a chain of cars when a fluctuation in the motion of the lead car is introduced It is interesting to note that similar considerations were made and results obtained independently by KOMETANI AND SASAKI [3]

It is the purpose of the present paper to investigate some of the properties of the steady-state flow of traffic based on a follow-the-leader theory of vehicle interaction One of the characteristics of traffic flow which has been measured experimentally is that there exists an optimum density for which the traffic flow is a maximum Figure 1 illustrates this feature for results obtained by GREENBERG[4] for the flow of traffic through the Lincoln Tunnel LIGHTHILL AND WHITHAM[5] have given a phenomenological description of this type of flow on the basis of the equation of continuity of a compressible fluid Greenberg proposed a more specific fluid theory and derived the formula

$$u=c\ln(\rho_j/\rho) \qquad (1)$$

for the velocity u and the density ρ of the stream of traffic, c and ρ_j are parameters which depend on the road and vehicle characteristics, c being

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interpreted as the optimum speed, i e, the speed for maximum flow, and ρ_j the density of the traffic when a traffic jam occurs In Fig 2 the stream velocity u is plotted against the density ρ for the results obtained by Greenberg, the solid curve represents Greenberg's relation (1) which has been fitted to the results by choosing $c = 17.2$ miles/hr and $\rho_j = 228$ vehicles per mile For the flow of traffic, equation (1) gives

$$q = \rho u = c\rho \ln(\rho_j/\rho), \tag{2}$$

and with the above values of c and ρ_j this relation is plotted in Fig 1

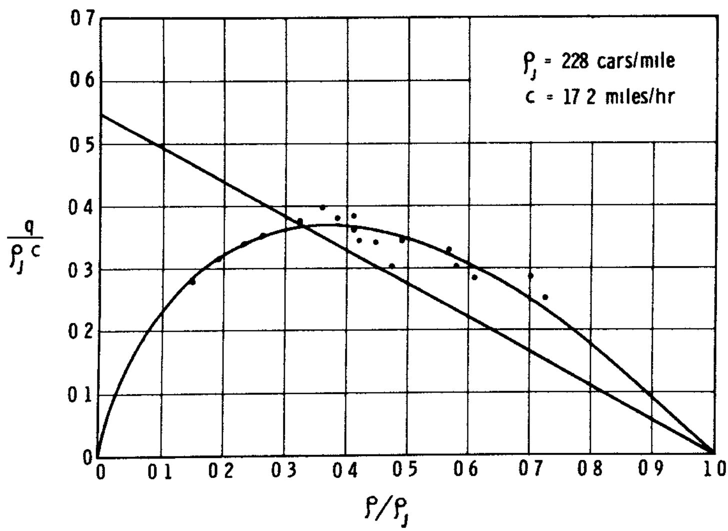

Fig 1 Normalized flow $q/\rho_j c$ *versus* normalized density ρ/ρ_j The data are those of Greenberg (reference 4) from the Lincoln Tunnel The curve corresponds to the steady-state theory for variable sensitivity and the straight line for the constant sensitivity case

Although the agreement between Greenberg's theory and the experimental results is satisfactory, no justification has been given for the assumed dynamic equation of fluid motion on which the theory is based Indeed it is difficult to see why there should be any connection between this equation and the actual dynamic laws of motion of the vehicles

It is the purpose of the present paper to show that a follow-the-leader theory with a sensitivity inversely proportional to the distance of separation gives a law of vehicle interaction which leads to equation (2) with the desired traffic flow characteristics The fact that Greenberg's result is obtained is not to be interpreted as justifying his assumptions since these assumptions are not related to those used in the follow-the-leader theory

FOLLOW-THE-LEADER THEORY

THE EQUATIONS of motion of a line of identical vehicles each of which follows the car ahead by velocity control are

$$M\, x_{n+1}(t) = \lambda\, (x_n - x_{n+1})_{t-\Delta}, \qquad (n=1, 2, 3, \quad) \quad (3)$$

where $x_n(t)$ is the position of the nth car in the line, M the mass of each car, λ the sensitivity (assumed constant), and Δ the time lag of the driver-car system Suppose that at time t_0 the cars are each travelling with speed u_0, and that then the lead car accelerates or decelerates in any manner

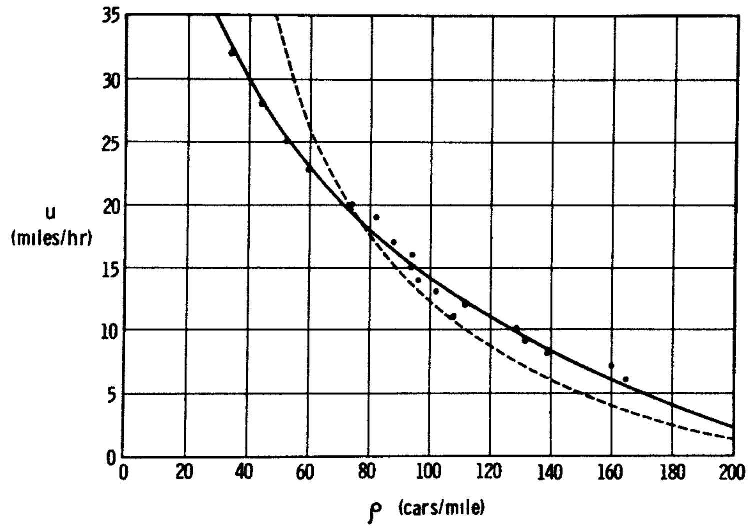

Fig 2 Speed (miles per hour) *versus* density (cars per mile) The data are those of Greenberg (reference 4) as in Fig 1 The solid curve corresponds to the steady-state flow theory for variable sensitivity and the dashed curve for constant sensitivity

until its speed is u Because of the velocity control between the following cars, a steady state is eventually reached in which each car moves with speed u. The separation distance $y_n = x_n - x_{n+1}$ between cars (measured from front bumper to front bumper, for example) alters in the process by the amount

$$y - y_0 = M\, (u - u_0)/\lambda, \qquad (4)$$

the same for all pairs of cars, as proved in reference 2, equation (23) Notice that this expression is independent of Δ, but, of course, it has been tacitly assumed that the stability conditions, which do involve Δ, have been satisfied Since the car separation is the inverse of the density

$$\rho^{-1} - \rho_0^{-1} = M\, (u - u_0)/\lambda \qquad (5)$$

or $$u=(\lambda/M)(\rho^{-1}-\rho_j^{-1}), \tag{6}$$

where the constant ρ_j is the jam density when $u=0$ and is equal to $1/L$ where L is the effective length of each car It is evident that equation (5) is the integral of equation (3) provided that Δ is neglected, a condition which is valid for steady-state flow In Fig 2, the dashed curve represents equation (6) for $\rho_j=228$ vehicles per mile (corresponding to $L\sim 23\,2$ feet) and $\lambda/M=0\,60\ \text{sec}^{-1}$ which is in the range of values obtained experimentally [1] Although the agreement between this curve and the experimental results appears reasonable, equation (6) leads to the result

$$q=(\lambda/M)[1-(\rho/\rho_j)], \tag{7}$$

for the traffic flow, and this does not exhibit the required qualitative behavior of a maximum flow at an optimum density (*see* Fig 1)

In the following it will be shown that the velocity control equation (3) can be altered to give equation (2) by supposing that the sensitivity λ is not constant but is inversely proportional to the separation distance This is, of course, a reasonable assumption, but one which has been avoided in the previous theory in order to retain the linearity of the equations of motion Even in the nonlinear case, however, it is possible to calculate the steady-state velocity and traffic flow as functions of the density In fact equation (3) becomes

$$M\, x_{n+1}(t)=\lambda_1\, [(x_n - x_{n+1})/(x_n-x_{n+1})]_{t-\Delta}, \tag{8}$$

where λ_1 is a new constant which shall be referred to as the sensitivity coefficient This equation, like equation (3), can be integrated once immediately giving

$$M\, x_{n+1}(t)=\lambda_1 \ln[L^{-1}\,(x_n-x_{n+1})]_{t-\Delta}, \tag{9}$$

where L is the length of each car For the steady state, the time lag Δ can be neglected as justified above, yielding

$$u=c\ln(\rho_j/\rho), \tag{10}$$

and $$q=c\rho\ln(\rho_j/\rho), \tag{11}$$

with $c=\lambda_1/M$ and $\rho_j=L^{-1}$ Thus the follow-the-leader theory for velocity control with a sensitivity inversely proportional to the separation distance leads to equations (10) and (11), which are the same as equations (1) and (2) obtained from the 'fluid dynamic equations of motion.' Conversely differentiation of equation (1) with respect to t with the use of the relationship $\rho=(x_n-x_{n+1})^{-1}$ leads immediately to equation (8) with, of course, $\Delta=0$

It is interesting to note that the steady-state flow equations for traffic can be derived in a general sort of way from the close-up distance relation given in equation (4). First we assume that equation (4) can be written in differential form for a small change in velocity as*

$$dy=(M/\lambda)\,du \tag{12}$$

If we suppose that

$$\lambda/M=c/y, \tag{13}$$

we obtain the differential equation

$$dy/y=du/c,$$

so that

$$y=y_j\,e^{u/c}, \tag{14}$$

and

$$u=c\ln(y/y_j) \tag{15}$$

Finally using the relation $y=\rho^{-1}$ we obtain the results already stated in equations (10) and (11).

It is instructive to compare the experimental data obtained for the Lincoln Tunnel with the steady-state results given above for constant sensitivity and a sensitivity inversely proportional to the car spacing. For this purpose a least-squares fit has been obtained using the flow data analyzed by Greenberg. In Figs. 1 and 2 we have plotted the experimental data and the least-squares fits for the variable and constant sensitivity cases. In the variable sensitivity case, $c=17.2$ miles/hr and $\rho_j=228$ cars/mile as obtained by Greenberg. For comparison purposes, the least squares fit for the constant sensitivity case was made assuming $\rho_j=228$ cars/mile with a resulting value of $\lambda/M\approx0.60$ sec^{-1}.

It is seen that the variable sensitivity theory fits the experimental data very much better than the theory for constant sensitivity. This is more evident in Fig. 1 where $q/(\rho_j c)$ is plotted versus ρ/ρ_j, inasmuch as the constant sensitivity theory does not exhibit the maximum flow characteristic at an optimum density and does not yield $q=0$ when $\rho=0$. It should be pointed out that in the region of low traffic density, where there are no experimental data, there is little or no interaction between vehicles. This means that the cars will travel with essentially maximum desired speed so that the flow will be proportional to the traffic density.

The above considerations stimulated us to re-examine the car-following experiments reported in reference 1. In Table 1 of that paper the sensitivity is given for eight drivers who participated in those experiments. In Fig. 3 we have plotted the values obtained for the proportionality coefficient λ/M, versus the reciprocal of car spacing in order to determine whether a correlation exists. The car spacing was taken to be the average

* It is assumed that $d\lambda/dy$ is finite for the entire range of y, which is true in the sequel.

distance between the pair of cars involved in the car-following experiment even though no special effort had been made at that time to study distance effects It may be seen from Fig 3 that there is a definite trend of sensitivity decreasing with increasing distance Even though the data are scanty, we performed a least-squares fit through the origin and obtained the following result

$$\lambda/M = (40\ 2/y)\ \sec^{-1}, \tag{16}$$

where y is measured in feet This result corresponds to a value of $c = 27\ 4$

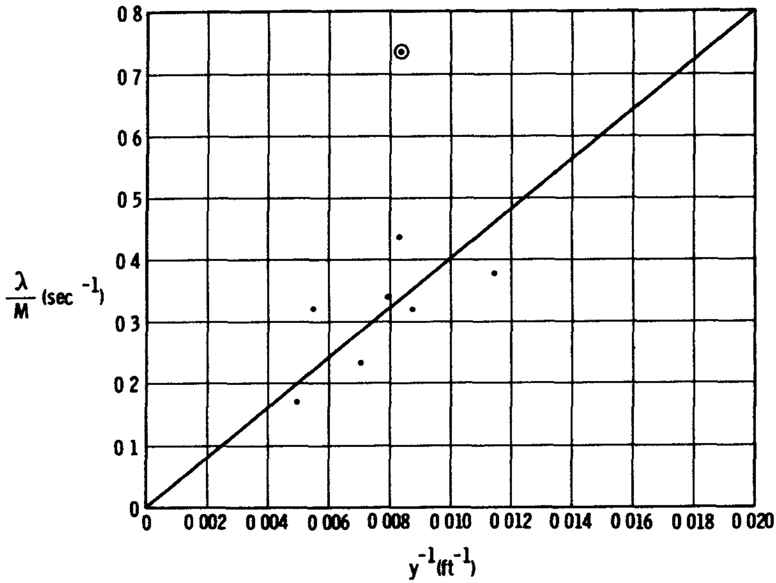

Fig 3 Proportionality coefficient, λ/M ($\sec^{-1}$), *versus* the reciprocal of the car spacing, $1/y$ (ft^{-1}) The least-squares straight line is represented by $\lambda/M = (40\ 2/y)\ \sec^{-1}$ The data were obtained from reference 1 The encircled point was not included in the fit

mi/hr for the optimum speed in the car-following experiment, which is to be compared with Greenberg's value of 17 2 mi/hr for the optimum speed of traffic flow through the Lincoln Tunnel These speeds are surprisingly low and it would be very interesting to determine the values for traffic on streets, expressways, and throughways in different cities and different countries since driving habits undoubtedly vary all over the world The value of the sensitivity coefficient will clearly depend on the driver-car-highway complex

The present study emphasizes once again the great importance of intelligent data taking in the field, as well as the constant need to devise and conduct sımple controlled experıments whıch hand-ın-hand wıth theoretıcal studıes wıll eventually lead to greater understandıng of the very complex traffic problem

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