AN ANALYSIS OF TRAFFIC FLOW

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A model for traffic flow is developed by treating the traffic stream as a continuous fluid Fluid dynamic principles are then used to derive relations between speed, density, and flow

THE VOLUME of vehicular traffic in the past several years has rapidly outstripped the capacities of the nation's highways It has become increasingly necessary to understand the dynamics of traffic flow and obtain a mathematical description of the process This is especially true for extremely heavy traffic when the roadway must perform at its peak

The majority of the theories that describe traffic flow are derived from the statistical study of the flow Greenshields[1] made an experimental study of traffic flow by measuring actual flows (vehicles per hour) and velocities of observed vehicles He plotted the velocity against density (vehicles per mile) for one-lane traffic, he then fitted the points by a straight line Greenshields' results have been used by several authors[2,3] in determining traffic phenomena However, they are based on limited data and further study is warranted

LIGHTHILL AND WHITHAM,[4] instead of using a statistical approach, applied fluid dynamic principles to various highway occurrences They treated traffic as a fluid flow and arrived at a qualitative description of the flow-density curve Flow increased with increasing vehicle density until a maximum was reached, the flow decreased to zero as the density increased further In the theory presented below, the fluid dynamic analogy is developed further, resulting in functional relations for the basic interactions between vehicles

THEORY

IN THIS analysis, traffic is assumed to behave like a continuous fluid The methods of fluid dynamics may then be used except for the lowest densities of traffic

The equation of motion of a one-dimensional fluid is

$$\frac{du}{dt} = -\frac{c^2}{k}\frac{\partial k}{\partial x}, \tag{1}$$

where u = fluid velocity (traffic velocity), miles per hour,
k = density of traffic, vehicles per mile,
x = distance along road,
t = time,
c = parameter that is determined from the state of the fluid.

Since
$$u=u(x,t), \tag{2}$$
equation (1) becomes
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{c^2}{k}\frac{\partial k}{\partial x}=0 \tag{3}$$

An additional equation describing fluid flow is the equation of continuity (conservation of flow equation)

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=0, \tag{4}$$

where q is the traffic flow, vehicles per hour. As

$$q=ku, \tag{5}$$

equation (4) becomes
$$\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+k\frac{\partial u}{\partial x}=0. \tag{6}$$

Let the velocity be a function of density only*; $u=u(k)$ Then

$$\frac{\partial u}{\partial t}=\frac{du}{dk}\frac{\partial k}{\partial t}, \tag{7}$$

$$\frac{\partial u}{\partial x}=\frac{du}{dk}\frac{\partial k}{\partial x}. \tag{8}$$

After substituting (7) and (8) into (3) and (6),

$$\frac{\partial k}{\partial t}+\left[u+\frac{c^2}{u'k}\right]\frac{\partial k}{\partial x}=0, \tag{9}$$

$$\frac{\partial k}{\partial t}+[u+ku']\frac{\partial k}{\partial x}=4, \tag{10}$$

where $u'=du/dk$. Equations (9) and (10) describe the fluid flow To have a nontrivial solution† the determinant of the coefficients to the partial derivatives must equal zero Thus

$$(k\,du/dk)^2=c^2, \tag{11}$$

and
$$du/dk=-(c/k), \tag{12}$$

* This is similar to an assumption made by HESS AND BALINSKI, in an unpublished Massachusetts Institute of Technology report, on studying the effect of a traffic light on traffic flow

† In the trivial solution $\partial k/\partial x=\partial k/\partial t=0$, traffic flows with the same u and k values as the initial ones

where only the negative sign applies on taking the square root from (11). Equation (12) can be solved to yield the density dependence of the traffic velocity Thus

$$u = c \ln(k_j/k), \tag{13}$$

where c is taken as constant and k_j is the density for a traffic jam ($u=0$) It is convenient to write the headway (ft) between vehicles in terms of the velocity

$$h = h_j\, e^{u/c}, \tag{14}$$

where $h_j = 5{,}280/k_j$ The headway represents the distance between the front of one vehicle and the front of the next vehicle

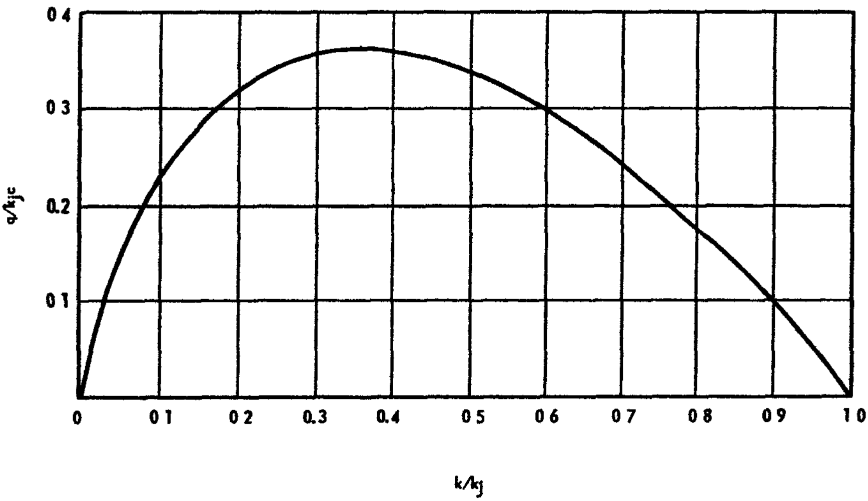

Fig. 1. Normalized traffic flow versus density.

To find the flow q in terms of the density k, substitute for u in (13) from (5), thus

$$q = ck \ln(k_j/k) \tag{15}$$

The shape of the flow-density curve may be obtained from (15) and is shown in Fig 1 As can be seen from the graph there is a maximum value of flow This occurs when $k_j/k = e$, which gives the optimum speed as $u = c$ This optimum speed depends on the parameter c, which must be obtained for a particular roadway

If equation (12) is inserted into (9) a partial differential equation results which gives the wave motion of the traffic stream This effect will not be studied here, since the primary interest is in obtaining the relations between speed, density, and flow

Equations (13), (14), and (15) are the relations for the basic interactions between vehicles They depend on two parameters* One is given by the traffic jam characteristics, the other by the stream velocity that results in maximum flow If the values of u and k, in a physical situation, do not satisfy (13), then the velocity is not strictly a function of density However, when a particular driver's behavior causes the stream to vary from the condition from (13), the tacit assumption is made that the flow achieved by the traffic stream will tend to a stabilized condition given by (13) In a sense, equation (13) represents the average situation

TABLE I

TRAFFIC DATA—LINCOLN TUNNEL

Velocity u	Headway h	Density k	Flow q
32	155	34	1088
28	120	44	1232
25	99 6	53	1325
23	88 0	60	1380
20	71 4	74	1480
19	64 4	82	1558
17	60 0	88	1496
16	56 2	94	1504
15	56 2	94	1410
14	55 0	96	1344
13	51 3	103	1339
12	47 1	112	1344
11	48 9	108	1188
10	40 9	129	1290
9	40 0	132	1188
8	38 0	139	1112
7	33 0	160	1120
6	32 0	165	990

TABLE II

TRAFFIC DATA—MERRITT PARKWAY

Velocity u	Headway h	Density k	Flow q
38 8	258 9	20 4	792
31 5	192 5	27 4	864
10 6	49 7	106 2	1128
16 1	65 7	80 4	1296
7 7	37 4	141 3	1092
8 3	40 3	130 9	1092
8 5	43 4	121 7	1032
11 1	49 5	106 5	1176
8 6	40 4	130 5	1128
11 1	52 2	101 1	1116
9 8	42 6	123 9	1212
7 8	36 6	144 2	1128
31 8	179 3	29 5	936
31 6	171 5	30 8	972
34 0	199 2	26 5	900
28 9	148 0	35 7	1032
28 8	176 0	30 0	864
10 5	49 7	106 2	1116
12 3	54 4	97 0	1188
13 2	58 6	90 1	1188
11 4	49 5	106 7	1212
11 2	53 2	99 3	1116
10 3	49 2	107 2	1104
11 4	48 3	109 1	1248

EXPERIMENTAL VERIFICATION

IN ORDER to verify the theory it is necessary to fit data using the equations presented above This can be accomplished by fitting a straight line to u versus log h data using the method of least squares

There are several methods for summarizing speed and headway data For example, Table I presents a summary of data taken with a Simplex Productograph machine in the north tube of the Lincoln Tunnel The machine was set up alongside a short length of roadway Two observers were used, one at the entrance and one at the exit to the zone The machine was actuated to record the time when a vehicle passed each observer In this manner the velocity of each vehicle and the headway between successive vehicles was obtained quite accurately The data was then separated into speed classes and the average headway was calculated for each speed class

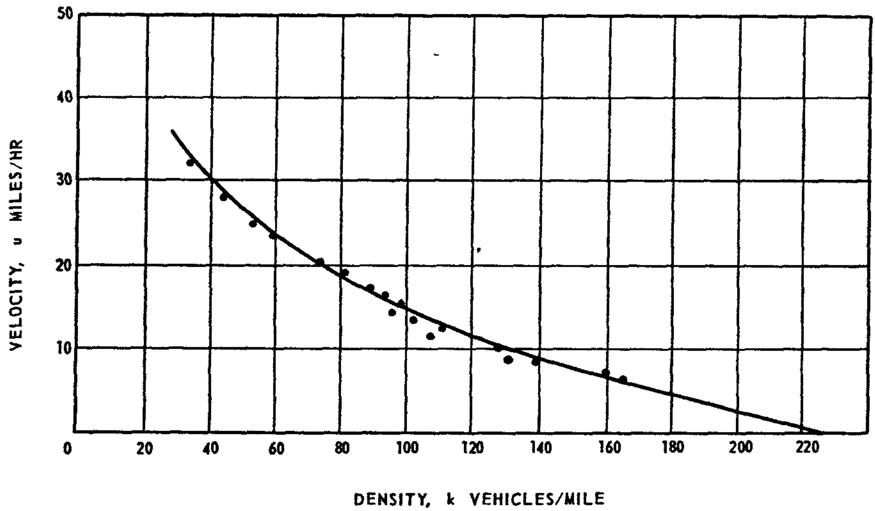

Fig. 2. Traffic velocity versus density—Lincoln Tunnel

Making a least-squares fit to the data resulted in the headway relation,

$$h=23\ 2\, e^{u/17\ 2}, \tag{16}$$

or for the density,

$$k=228\, e^{-u/17\ 2} \tag{17}$$

Figure 2 is a plot of velocity and density resulting from (17) The points shown are the empirical data indicating the high degree of correlation

Another method consists in calculating average velocity (space mean speed) and average headway for all vehicles in equal time profiles This has the effect of smoothing data taken by the first method Table II presents published data[5] taken at the Merritt Parkway using five-minute time profiles Making a least-squares fit results in

$$h=24.6\, e^{u/16\ 1}, \tag{18}$$

or

$$k=215\, e^{-u/16\ 1}. \tag{19}$$

Figure 3 again shows the excellent correlation obtained between the theory and the empirical results

In both cases the shape of the curve given in Fig. 1 remains the same where the constants for the coordinate values are obtained from (17) and (19).

BOTTLENECK BEHAVIOR

EDIE AND FOOTE[6] have concluded that bottlenecks are controlling factors in determining the flow behavior of the roadway adjacent to the bottleneck, the flow is limited before and after the bottleneck

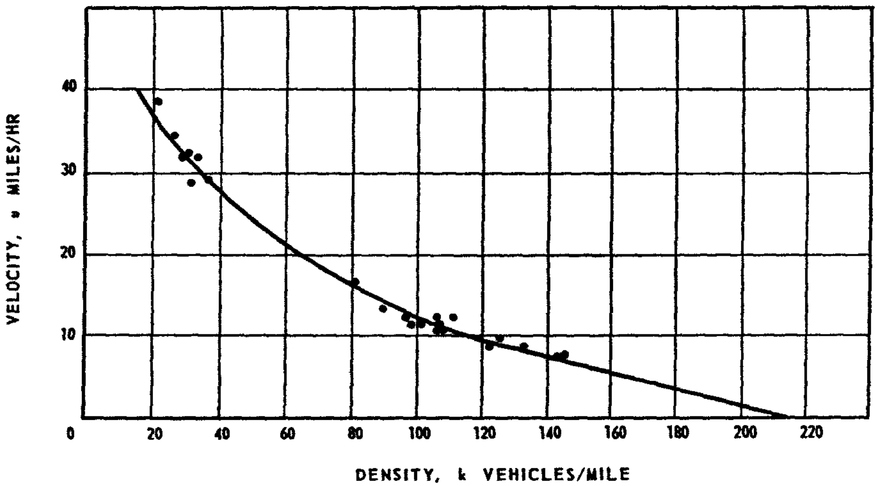

Fig. 3. Traffic velocity versus density—Merritt Parkway

The traffic flow for a bottleneck section is fluid flow and can be analyzed as above

Before the bottleneck, dual behavior occurs For traffic volumes below the bottleneck level, the flow is characteristic of the roadway section For traffic volumes above the bottleneck level, a lengthening queue develops at the bottleneck The flow is then controlled by the bottleneck. Care must be taken when applying the analysis to a bottleneck-controlled situation, the dual phenomena are handled separately

Beyond the bottleneck, the number of vehicles that enter the section is limited by the bottleneck The stream is often nonfluid and an alternate type of analysis is necessary.

CONCLUSION

THE FLUID dynamic analogy to traffic flow applies when there is continuous wave interaction between vehicles This effect takes place when a driver's speed and headway are controlled to a major extent by the other vehicles on the roadway The actual characteristics of the roadway determine the optimum velocity for maximum flow

The flow density curve as presented in Fig 1 turns out to be an excellent verification for the theory The shape of the curve, together with the coverage as given by the normalized description of the coordinates, agrees with the physical requirements of flow and density

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