NONLINEAR FOLLOW-THE-LEADER MODELS OF TRAFFIC FLOW

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A variety of nonlinear follow-the-leader models of traffic flow are discussed in the light of available observational and experimental data Emphasis is placed on steady-state flow equations Some trends regarding the advantages of certain follow-the-leader functionals over others are established However, it is found from extensive correlation studies that more data are needed before one can establish the unequivocal superiority of one particular model A discussion is given of some ideas concerning the possible reasons for the existence of a bimodal flow versus concentration curve especially for multilane highways

THE DEVELOPMENT of various theories concerning traffic phenomena has received considerable attention during the last few years An increasing number of investigators with different backgrounds and points of view have considered various aspects of traffic phenomena with very gratifying results Virtually all of the theories are in various stages of development and, of course, still far from perfect However, some of the groundwork has been done and the methodology developed for the treatment of this highly complex socio-economic problem of traffic

One of the above mentioned theories has come to be known as the follow-the-leader theory of traffic Such a theory pertains to single-lane dense traffic with no passing, and is based on the assumption that each driver reacts in some specific fashion to a stimulus from the car (or cars) ahead of and/or behind him. The list of contributions to this theory is now a long one and rapidly becoming longer at an increasing rate [1-15] The purpose of this paper is to present a reappraisal of the state of the theory and contribute several additional results, both theoretical and experimental, pertaining to nonlinear effects with an emphasis on the equations of steady-state flow

A point worth mentioning here is that the follow-the-leader theory applies mainly to fairly dense traffic The results of the theory in its simple form should not necessarily be extrapolated to the range of very low concentrations Interactions between cars disappear statistically at low concentrations, even though there may exist such interactions within some isolated clusters of vehicles A different approach may be called for at low concentrations For example, PRIGOGINE[16, 17] has used the methods of statistical mechanics to discuss traffic of low to moderate concentrations including the added complication of passing

The basic differential difference equations of the follow-the-leader theory express the idea that each driver of a vehicle responds to a given stimulus according to a relation such as

$$\text{response} = \text{sensitivity} \times \text{stimulus} \tag{1}$$

The stimulus could be a functional of the positions of a number of cars and their time derivatives, and perhaps also some other parameters The response has been taken as the acceleration of the vehicle, since a driver actually has a direct control of this quantity through the gas and brake pedals

In equation (1) one may assume various functionals for their mathematical advantages, and proceed to develop a theory with or without recourse to experimental evidence It is, of course, preferable if one can check these assumptions against experimental evidence Unfortunately, carefully controlled experiments are extremely difficult to perform and very demanding in time and manpower There does not seem to be, however, any easier way for arriving at a realistic theory

The follow-the-leader theory has proceeded as follows First it has been established that there is a high correlation between the response of a driver and the relative speed of his car and the one ahead The stimulus has therefore been taken as this relative speed The sensitivity was initially taken as constant[3] and later as inversely proportional to the spacing of the lead and following car [10] Other functionals for the sensitivity have been given elsewhere,[11, 14] whereas NEWELL[15] has lumped the sensitivity and stimulus into one functional The various modifications of the sensitivity functional have, in general, been made in an attempt to account for experimental and observational data These have been obtained from both phenomenological observations on single-lane traffic flow in tunnels and from follow-the-leader experiments In what follows we shall discuss the extent to which one can use these data to justify any one particular functional

EQUATIONS OF THE FOLLOW-THE-LEADER MODELS

THE stimulus-response equation, which describes the motion of the $(n+1)$th car following the nth car, in a single lane of traffic has been taken as

$$\ddot{x}_{n+1}(t+T)=\lambda[\dot{x}_n(t)-\dot{x}_{n+1}(t)], \tag{2}$$

where x_n is the position of the nth car, T is the time lag of response to the stimulus, λ is the sensitivity* and the dots as usual denote differentiation with respect to time t The following functions have been assumed

* It should be noted that the sensitivity, λ, has been referred to as λ/M in reference 3 and as proportionality coefficient, α, in reference 11

in the past for the sensitivity

1. constant [3] $\lambda = a,$ (3)
2. step function [11] $\lambda = \begin{Bmatrix} a \text{ for } s \leqq s \text{ critical} \\ b \text{ for } s > s \text{ critical} \end{Bmatrix},$ (4)
3. reciprocal spacing [10] $\lambda = c/s,$ (5)

where s is the spacing, i e, $(x_n - x_{n+1})$, and a, b, c are constants In addition to the above functions EDIE[14] has assumed the sensitivity function

$$\lambda = c\, x_{n+1}/s^2 \tag{6}$$

Furthermore, the sensitivity in a follow-the-leader model that can yield GREENSHIELDS'[18] equation of steady-state flow of traffic is

$$\lambda = c/s^2, \tag{7}$$

as will become apparent later All three functionals given by equations (5), (6), and (7), may be considered as special cases of a more general expression for the sensitivity, namely

$$\lambda = a\, x^m_{n+1}\, (t+T)/[x_n(t) - x_{n+1}(t)]^l \tag{8}$$

The choice of one equation over another of the type given in equation (8) depends on how well they describe actual traffic phenomena There are two major classes of these phenomena that deserve primary consideration, namely, stability and steady-state flow characteristics Unfortunately the investigation of stability is mathematically difficult for any equations other than the linear system obtained by assuming a constant sensitivity The stability conditions have been established for this linear system[5, 6, 7] and can be assumed as providing a 'bound' of stability even for the stable nonlinear cases This is so in the sense that linearization of a nonlinear system for small perturbations is possible so long as the linearized system is stable and therefore any perturbation introduced in a stream of traffic remains small Some numerical evaluations on the IBM 704 of certain simulated traffic problems indicate that this stability bound for a follow-the-leader model of the type given in equations (5) is very close to that of the linear model [11] Considerably more work remains to be done for a more complete investigation of the stability of a nonlinear system

For the second class of phenomena, namely, that of steady-state flow characteristics it is possible to obtain a fairly complete description corresponding to any sensitivity given by an equation such as equation (8) The stimulus response equation for the $(n+1)$th vehicle is, according to equations (2) and (8)

$$x_{n+1}\,(t+T) = \frac{a\, x^m_{n+1}(t+T)\,[x_n(t) - x_{n+1}(t)]}{[x_n(t) - x_{n+1}(t)]^l}, \tag{9}$$

or

$$\frac{x_{n+1}(t+T)}{x^m_{n+1}(t+T)} = \frac{a[x_n(t) - x_{n+1}(t)]}{[x_n(t) - x_{n+1}(t)]^l} \tag{10}$$

Equation (10) when integrated yields the steady-state flow equation

$$f_m(u)=cf_l(s)+c', \tag{11}$$

where u is the steady-state speed of a stream of traffic, s the constant average spacing, c and c' some appropriate constants consistent with physical restrictions, and the meaning of the function f_p ($p=m$ or l) is

$$f_p(x)=x^{1-p}, \qquad (p\neq 1) \tag{12}$$

and

$$f_p(x)=\ln x \qquad (p=1) \tag{13}$$

As mentioned elsewhere[10] the integration disposes of the time lag T insofar as the steady-state equations are concerned The integration constant c' is related to a 'free speed,'* u_f, or a 'jam spacing,' s_j, depending on the values of m and l It may be ascertained that

$$c'=f_m(u_f) \tag{14}$$

for $m>1$, $l\neq 1$, or $m=1$, $l>1$, and

$$c'=cf_l(s_j) \tag{15}$$

for all other combinations of m and l, except $m=1$, $l<1$ In the latter case it is not possible to satisfy either boundary condition, $k=k_j$ or $k=0$ Hence the integration constant can be assigned arbitrarily but only a portion of the flow curve may be used

Using equations (11), (14), and (15), and the definition of flow, namely,

$$q=uk, \qquad k=1/s, \tag{16}$$

one can obtain relations between the speed u, the concentration k, and the flow q Examples of such steady-state equations are those obtained for a sensitivity inversely proportional to the spacing,[10] and also for the sensitivity function of equation (6) examined by Edie [14] Figures 1 to 3 contain other examples of flow versus density curves obtained on the basis of equations (11) to (16) for various values of m and l It will be noted in Fig 1 that when $m=0$ and $l=2$, i e , the case for the sensitivity function given in equation (7), one obtains Greenshield's[18] flow relation The flow curves have been normalized to give maximum normalized flow q_n equal to unity The 'jam density,' where applicable, has been taken the same for all values of m and l

Perhaps a word might be said here regarding some apparent inconsistencies in the derivation of the steady-state flow equations It will be observed that if one assumes a stream of traffic moving at an essentially

* The boundary condition of free speed is the only one possible for these models However, in view of the remarks made in the introduction one might introduce the boundary condition of free speed at some 'critical' low concentration, rather than at zero concentration

constant speed, the basic relations given in equations (2) are satisfied regardless of the spacing between cars It appears then that no unique equation of steady state is implied by a stimulus-response equation such as equation (2) However, it may be seen that one can obtain a transition from one steady state to another after a perturbation of the lead car through a process of relaxation governed by (2) This relaxation amounts exactly to an integration of both members of (2), a procedure which eliminates the time lag and yields a relation between speed and spacing of cars in a chain of steady-state traffic It is then asked that this transition from one

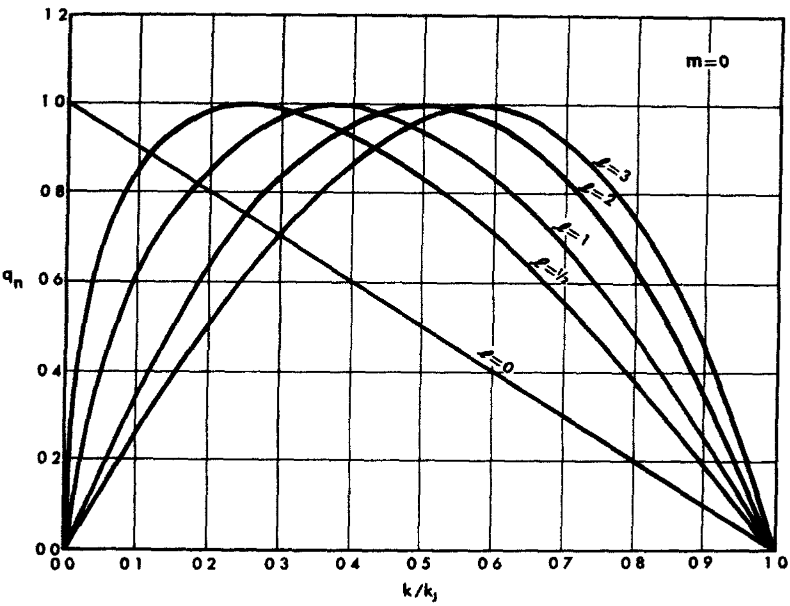

Fig. 1. Normalized flow, q_n, versus normalized concentration, k/k_j, corresponding to the steady-state solution of equations (9) for $m=0$ and various values of l

steady state to another leads to the proper 'boundary condition,' namely, either zero flow at jam density or 'free speed' at zero concentration Now, any steady-state condition is actually obtained through such transitions that may in practice include the boundary conditions, e g , traffic starting from jam density at a traffic light or a bottleneck Consequently, it is reasonable to accept the steady-state flow relation obtained in equations (11) to (16) as one implied by the basic stimulus-response equation (2) This does not preclude the possibility that a steady-state stream of traffic governed by (2) may deviate from the unique flow law associated with this equation We suggest that the deviations have some reasonable statistical distribution with our flow equation as the average

Before discussing the relative merits of any one of these equations of steady-state flow we might indicate that one can derive such equations starting from different considerations Let us assume that while driving at a speed u every driver tries to maintain a distance between his car and

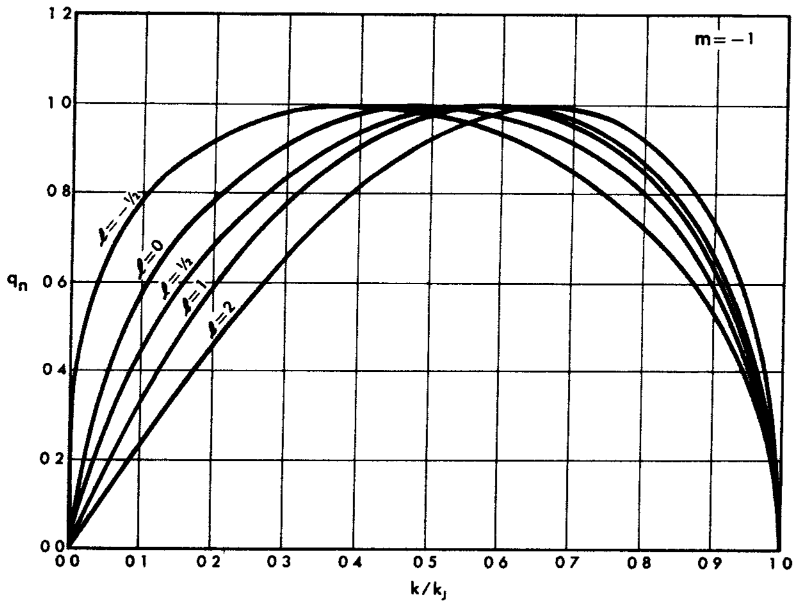

Fig. 2 Normalized flow, q_n, versus normalized concentration, k/k_j, corresponding to the steady-state solution of equations (9) for $m=-1$ and various values of l

the one ahead equal to his stopping distance minus, perhaps, some assumed stopping distance of the lead car *[19] The spacing, s, will then be given by the equation

$$s=s_0+Au+Bu^2, \tag{17}$$

with some appropriate constants s_0, A, and B As an example s_0 may be the length of every car (if spacing is defined as the distance from front bumper to front bumper), A, a time lag of reaction to sudden braking of the lead car, and

$$B=(1/2d_{n+1}-1/2d_n), \tag{18}$$

where d_{n+1} is the desired maximum deceleration of the $(n+1)$th car and d_n the maximum deceleration of the lead (nth) car, which probably has to be assumed as the maximum physically possible A consideration of this kind has led Kometani and Sasaki[12] to propose and investigate a quadratic relation between spacing and speed

* *See*, for example, reference 19 for a number of such driving formulas

Equations (16) and (17) yield a system of equations of steady state which also deserves scrutiny A relation between flow and density can be obtained by solving equation (17) for u and entering this value in equation

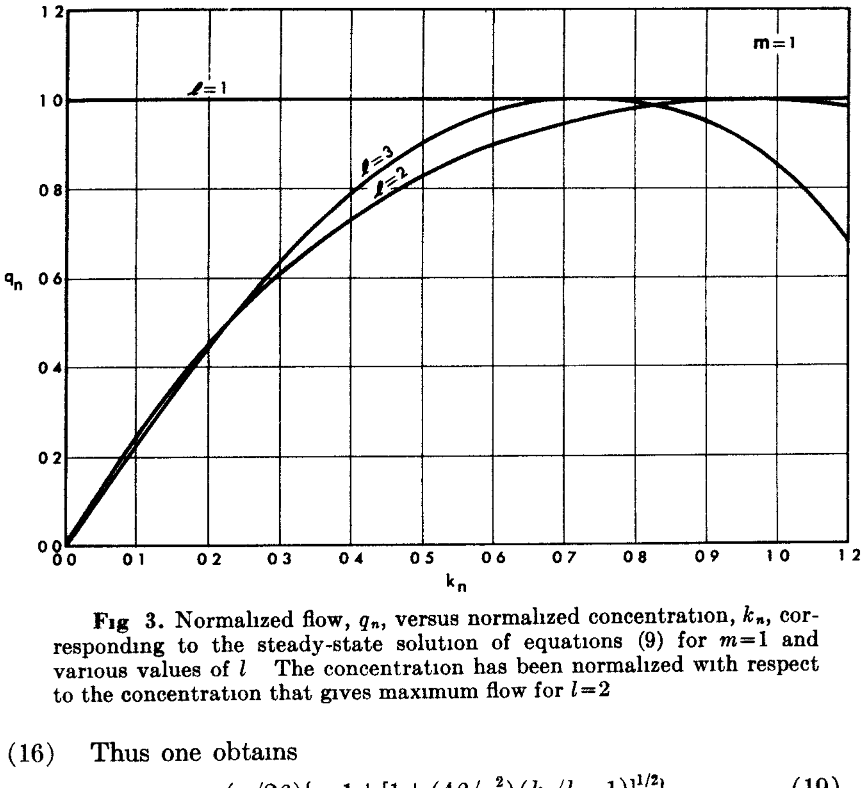

Fig 3. Normalized flow, q_n, versus normalized concentration, k_n, corresponding to the steady-state solution of equations (9) for $m=1$ and various values of l The concentration has been normalized with respect to the concentration that gives maximum flow for $l=2$

(16) Thus one obtains

$$u=(\alpha/2\beta)\{-1+[1+(4\beta/\alpha^2)(k_j/k-1)]^{1/2}\}, \tag{19}$$

where $k=1/s, \qquad k_j=1/s_0, \qquad \alpha=A/s_0, \qquad \beta=B/s_0,$ (20)

and hence $q=q_0\{-k/k_j+[k^2/k_j^2+\gamma\ (k/k_j)(1-k/k_j)]^{1/2}\},$ (21)

where $q_0=\alpha k_j/2\beta, \qquad \gamma=4\beta/\alpha^2$ (22)

The maximum flow q_{max}, is obtained for

$$k/k_j=[\sqrt{\gamma}-\gamma]/[2(1-\gamma)], \tag{23}$$

and is given by

$$q_{max}=q_0\gamma/[2(1-\sqrt{\gamma})]=k_j/(\alpha+2\sqrt{\beta}) \tag{24}$$

A plot of q/q_{max} versus k/k_j for various values of γ is given in Fig 4 In this figure the curve for $\gamma=0$ is identical with the curve obtained on the basis of the simple linear model (*see* Fig 1) Also note that for $\gamma\rightarrow\infty$ the result is the same as for the case $m=-1$ and $l=0$ (*see* Fig 2) It may be ascertained that the tangent dq/dk is infinite at $k=0$ This is not to be considered as a disadvantage of the model since, as mentioned already, the interactions between cars are statistically weak at low densities and a follow-the-leader theory is not applicable Therefore, a straight-line law such as the one shown in Fig 4 would probably describe the flow versus density law near zero density

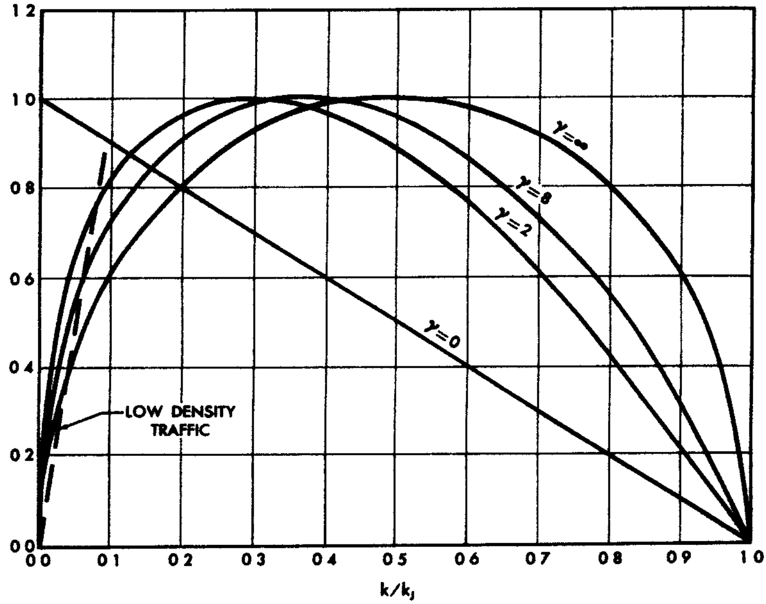

Fig. 4. Normalized flow, q_n, versus normalized concentration, k/k_j, corresponding to the steady-state flow equation (21), for various values of the parameter γ A straight line such as the dashed line near the origin would describe the case of light traffic when interactions between cars are essentially absent

An additional result may be obtained from the steady-state equation (17) By differentiating both sides with respect to time one obtains

$$s=(A+2Bu)\,u, \tag{25}$$

or

$$u=c_1\, s/(1+c_2 u), \tag{26}$$

which may be considered as a stimulus-response equation, perhaps after an introduction of a time-lag of response to stimulus, with a sensitivity given by

$$\lambda=c_1/(1+c_2\, x_{n+1}) \tag{27}$$

This procedure demonstrates yet another way of arriving at both equations of steady-state and stimulus-response laws from elementary considerations

The following section contains a discussion of the source and dependability of the available data that have been used for the purpose of determining which one, if any, of the above models agrees best with experiment and observation

EXPERIMENTAL AND OBSERVATIONAL DATA

Two TYPES of data have been used in the present study, namely, phenomenological flow and density measurements and records of follow-the-leader experiments A brief description of the source and methods of acquisition of these data is given here in order to avoid numerous references to publications

The phenomenological data have been obtained in the tunnels of the New York City area by the Port of New York Authority * These tunnels are particularly suitable for a study of single-lane traffic inasmuch as passing is actually not permitted inside them In addition, these tunnels are blessed with the dubious asset of being fairly congested for long periods every day, thus frequently displaying virtually all of the phenomena concomitant with congested traffic

The second type of data are those obtained from follow-the-leader experiments The first such experiments were conducted on the General Motors Technical Center test track[3] and led to the adoption of the linear stimulus-response law, with constant sensitivity Later experiments were conducted for the purpose of establishing the pattern of variation of the sensitivity with variation of spacing and/or speed of the vehicles In addition, follow-the-leader experiments have been conducted within the last two years both on the General Motors Technical Center test track and in the tunnels of New York, the latter for the purpose of comparing the 'macroscopic' phenomenological measurements with the 'microscopic' results of the follow-the-leader theory Some preliminary results of these experiments have already been reported elsewhere [11] The experimental set-up has been the same as that used in the first car-following experiments which are described in reference 3

The records of the follow-the-leader experiments have been used in two ways, namely, in a phenomenological study of the time averages of spacing and speed of the two vehicles participating in the experiment, and in a timewise correlation study of the response of the following car with various functionals assumed for the product of stimulus times sensitivity The phenomenological study assumed that the system of the two cars is a representative sample of a relatively homogeneous stream of traffic Hence, the average headway and speed of the two cars during the experiment is assumed to be about equal to the corresponding time and space averages of these quantities for the entire stream of traffic On the basis

* *See*, for example, reference 20

of these speeds, $\bar{u}_i$, and concentrations, $\bar{k}_i$, obtained for the various individual runs, i, we have computed the functions $f_m(\bar{u}_i)$ and $f_l(\bar{s}_i)$ entering into equation (11), where $\bar{s}_i = 1/\bar{k}_i$, for different values of m and l Setting

$$y_i = f_m(\bar{u}_i), \qquad x_i = f_l(\bar{s}_i), \tag{28}$$

we have computed the correlation coefficient as a function of m and l, for linear correlation between y_i and x_i Equation (17) has also been considered in this regard with

$$y = \bar{u} + (\beta/\alpha)\,\bar{u}^2, \qquad x = \bar{s}, \tag{29}$$

with the ratio (β/α) assumed as a parameter in the analysis We have also computed a normalized standard deviation of the constants of proportionality, c, of equation (11), or α of equation (17), which have been obtained for individual drivers Unfortunately, there are two major difficulties in this correlation study First, the number of runs, i, is very limited For example, for the Lincoln Tunnel we have only 18 runs, which, incidentally, have required several days of difficult experimentation It is obvious that the reliability of a correlation study in a population of 18 is rather poor The second difficulty is that all the drivers are not necessarily associated with exactly the same constants of proportionality of equation (11), c and c', even if such an equation describes exactly the way in which they are driving It would be desirable to have records of the same driver in situations of different concentrations and speeds However the correlation study on the records of different drivers was undertaken in the hope that it might still show a trend favoring a certain stimulus-response equation It was not necessarily intended to find a set of noninteger exponents m and l for equation (8) which would yield maximum correlation of the phenomenological flow data Psychologists might object to the assertion that an average, or even gifted, human being is able to discern and react in proportion with some unusual noninteger power of his speed and/or his spacing relative to the car in front * A correlation study of the phenomenological aspects of the follow-the-leader experiments may at best be expected to establish trends, such as, whether the exponents m and l of equation (8) are likely to be positive or negative, and also perhaps the relative magnitude of the constants s_0, A, and B of equation (17)

The time-wise correlation study of the follow-the-leader approach is the more dependable approach, since it is based on a larger population of variables, namely, the values of acceleration, spacing, absolute and relative speeds taken at many discrete intervals of time

The original analysis of this data for each driver consists of calculating

* It should be remarked that noninteger exponents m and l might arise in actual driving because of the fact that the response of the driver-car complex to a given stimulus does not depend only on the intentions of the driver but on the over-all performance characteristics of the complex

a correlation coefficient, r, for the linear correlation between $\ddot{x}_{n+1}(t+T)$ and $x_n(t)-x_{n+1}(t)$ for different values of the time lag T taken every 0 2 second Figure 5 illustrates the results for a typical data set for the correlation coefficient as a function of T The results reported by CHANDLER, HERMAN, AND MONTROLL[3] are based on this kind of a correlation study for constant sensitivity These results are shown in Table I together with the average spacing, s, and the average relative speed, u The constants

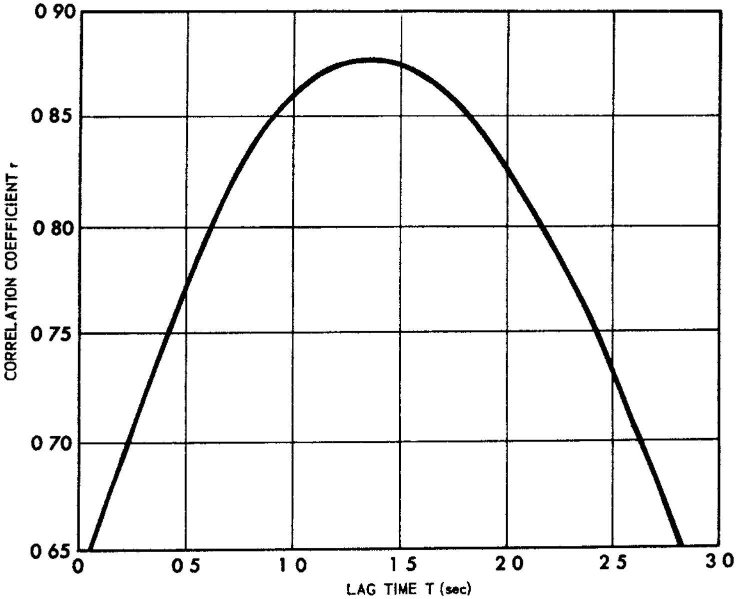

Fig. 5 Correlation coefficient, r, plotted against time lag, T, for one run of the car-following experiments in the Lincoln Tunnel The correlation was made assuming constant sensitivity

λ and T associated with a given driver are those for which r is a maximum These results can suggest a functional dependence of the sensitivity on the relative spacing and/or speed by correlating the λ's one obtains from a number of different data sets For example, GAZIS, HERMAN, AND POTTS[10] re-examined the car-following experiments reported in Table I in the light of the reciprocal spacing model, i e, c/s [*see* equation (5)] Figure 6, taken from their paper, gives a plot of λ against the reciprocal of the average spacing, s The straight line is a least-squares fit (excluding the encircled point) for a straight line passing through the origin It can be seen from this figure that there is a trend for the sensitivity to decrease with increasing distance thus suggesting a nonlinear model such as in equation (5) However, it should be emphasized that this type of analysis compares the sensitivity of different drivers in different situations An alternate and more

TABLE I

RESULTS OF CORRELATION STUDY OF REFERENCE 3

Driver	T (r = max)	λ	r	u	s
1	1 4 sec	0 74 sec^{-1}	0 87	66 1 ft/sec	120 0 ft
2	1 0	0 44	0 90	53 4	122 4
3	1 5	0 34	0 86	68 3	126 9
4	1 5	0 32	0 49	74 1	116 1
5	1 7	0 38	0 74	56 1	89 0
6	1 1	0 17	0 86	60 2	203 7
7	2 2	0 32	0 82	60 3	185 8
8	2 0	0 23	0 85	62 3	143 6

direct method of analysis consists of calculating a correlation coefficient for the correlation between $x_{n+1}(t+T)$ and a given functional for the right-hand side of the stimulus-response equation, for different values of T, for each driver, for the functional dependence of the sensitivity in the light of a given model Figure 7 illustrates the results for a typical data set for this time-wise correlation coefficient as a function of T for the constant λ, the nonlinear model of equation (5), and Edie's model of equation (6)

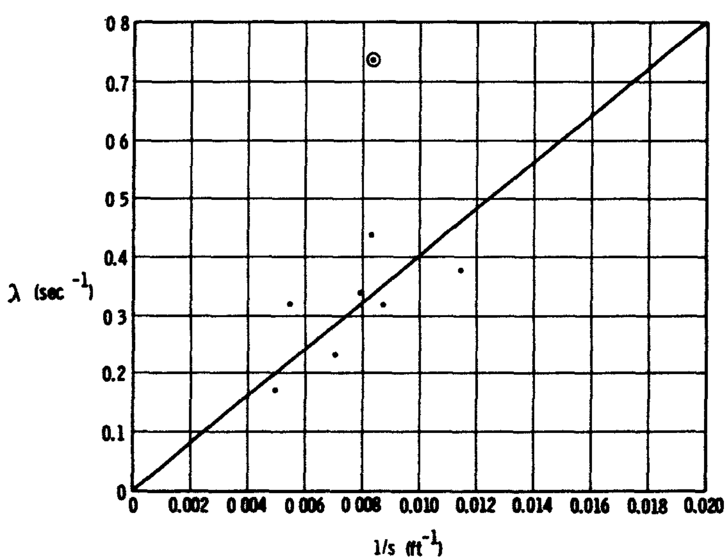

Fig. 6. Sensitivity, $\lambda(\text{sec}^{-1})$, versus the reciprocal of the car spacing, $1/s(\text{ft}^{-1})$ The least-squares straight line is represented by $\lambda = (40\ 2/s)\text{sec}^{-1}$ The data were obtained from reference 3 The encircled point was not included in the fit

COMPARISON OF THEORY WITH EXPERIMENTS AND OBSERVATIONS

Phenomenological Flow Data

Flow and density measurements at the Lincoln Tunnel of New York City were used by GREENBERG[20] for comparison with his steady-state theory The agreement appeared to be quite good A steady-state flow equation similar to Greenberg's was obtained from the follow-the-leader equations by assuming a sensitivity inversely proportional to spacing [10]

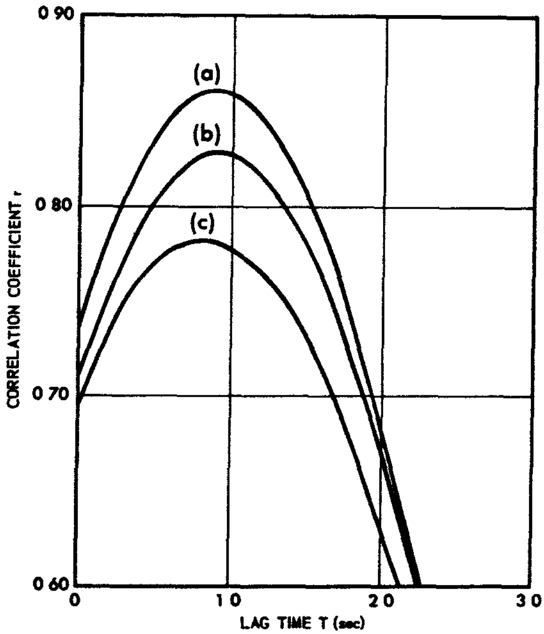

Fig. 7. Correlation coefficient, r, versus time lag, T, for one run of the car-following experiments in the Lincoln Tunnel Curve (a) corresponds to Edie's model, equation (6), curve (b) to the reciprocal spacing model of equation (5), and curve (c) to constant sensitivity

This assumption was further substantiated by a re-examination of the first follow-the-leader experiments,[3] which showed a trend for the sensitivity to increase with decreasing average spacing

Recently, Edie[14] has attempted to improve the follow-the-leader equations by introducing the sensitivity function given by equation (6), for the range of small densities Three main reasons were given by Edie for his choice of functional, namely, a finite speed corresponding to zero density, a better fit of the flow data after they are divided into low and high density regions, and a better fit of his sensitivity function with the measurements of reference 3 Although Edie's contribution is very illuminating, we have a number of remarks regarding his arguments and conclusions First, the applicability of his equation for very low densities including zero is not a very important asset, since it is not to be expected that a follow-the-leader theory can describe adequately this situation, as mentioned in the introduction Second, Edie separates the flow data in two regions, of low and high density and assumes that the region of low density is described by his model, while the region of high density is described by equations with sensitivity inversely proportional to spacing It is true that he obtains a better

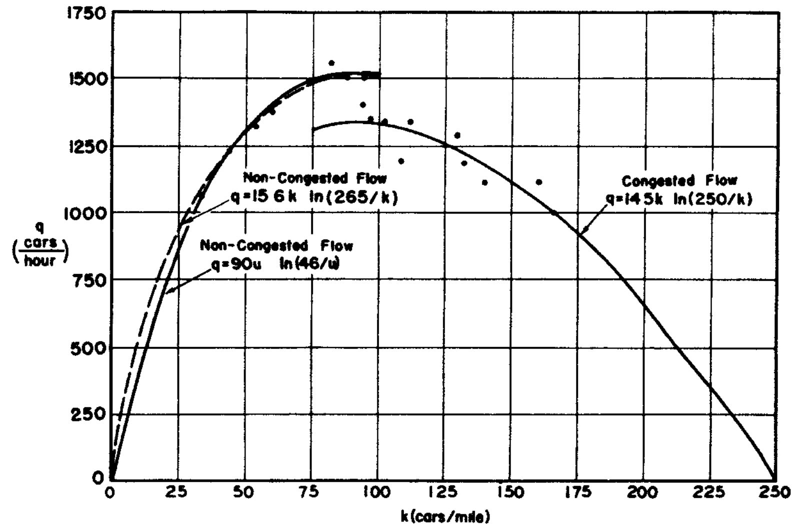

Fig 8 Flow, q(cars/hour), versus concentration, k(cars/mile), for congested and noncongested flow, steady-state models fitted to Lincoln Tunnel data A discontinuity is indicated at a density of about 90 cars/mile The solid branches of the curve are Edie's, reference 14 The dashed line in the range of noncongested flow, drawn for comparison, corresponds to the reciprocal spacing model of equation (5)

fit with the experimental data by assuming his two discontinuous branches of the theoretical flow curve However, one can obtain about just as good a fit by assuming two discontinuous flow curves governed by equation (5) with different 'critical speeds,' c, and effective jam densities, k_j This is shown in Fig 8, which is exactly the same as Edie's Fig 6 with an additional theoretical flow curve added for low densities This additional curve, plotted with dashed line, corresponds to a sensitivity given by equation (5) with

$$c = 15.6 \text{ mph}, \qquad k_j = 265 \text{ cars/mile}$$

It is seen that there is hardly any difference between Edie's curve and the new one, in the range of the available data

However, Edie's observation of the apparent discontinuity of the flow near the peak of the flow versus concentration curve is significant It is interesting to discuss a possible interpretation of the apparent discontinuity of the flow near the peak of the flow versus concentration curve This discontinuity may reflect a possible bimodal character of the flow curve The flow in the single lane no-passing case is, of course, controlled by the specific interactions between vehicles at various concentrations The initial linear portion of a realistic flow curve at low concentrations clearly arises from the fact that there are very few vehicles and that they do not interact As the concentration increases the flow increases, but because of the increase of the vehicle interactions the flow reaches a maximum and then decreases with increasing concentration As the concentration becomes rather high, large relative speeds become improbable so that a cooperative lattice-like flow may take hold and the flow pattern may become more ordered As a consequence the flow curve may exhibit some kind of bimodal character In this connection we refer to a study made by HANKIN AND WRIGHT[21] in which flow versus concentration curves have been determined for pedestrians moving through a passageway In that case the flow versus concentration curve shows a marked bimodal character that arises, we believe, for reasons similar to those mentioned above Clearly the speed of the pedestrians decreases with increasing concentration because of interactions However, when a certain high concentration is reached a relatively ordered shuffling forward motion ensues which maintains the speed at a nearly constant level almost until the jam concentration is reached It is very likely that this effect is not so marked with vehicles, and is probably qualitatively different This is so because close packing of vehicles involves considerably greater risks, at all speeds, than the risks involved in a jostling group of pedestrians

The possibility of the existence of a bimodal character of the flow versus concentration curve for a multilane highway may be greater than that for a tunnel where no passing is allowed In the multilane case the flow rise in the low to moderate concentration region is again decreased with increasing interactions, but in a manner strongly influenced by the fact that passing is possible It may be that when the flow first starts decreasing with increasing concentration the flow pattern is highly irregular, but may become more ordered with increasing concentration by the elimination of high relative speeds and passing This more ordered flow may lead to different flow versus concentration relations It would be very interesting indeed to see if one could find corroborative evidence of such phenomena from traffic measurements of expressways, thruways, and other multilane highways along stretches where there are no entrances or exits to disturb the flow of traffic

The third argument of Edie regarding the sensitivity values of reference 3 is actually based on an inaccuracy which as he suspected is the result of a computational smoothing process Edie used the values of Table I, reference 3, which unfortunately did not include the values of the average speeds of the vehicles during the various experiments He also used the average spacings given in Fig 3 of reference 10 Then he computed the speeds corresponding to these densities using equation (15) of reference 3, thus smoothing out the data This gave him the excellent fit of his Fig 2, which is reproduced here in Fig 9 The circles of Fig 9 are the values of 'sensitivity' computed by Edie Actually the average speeds in the ex-

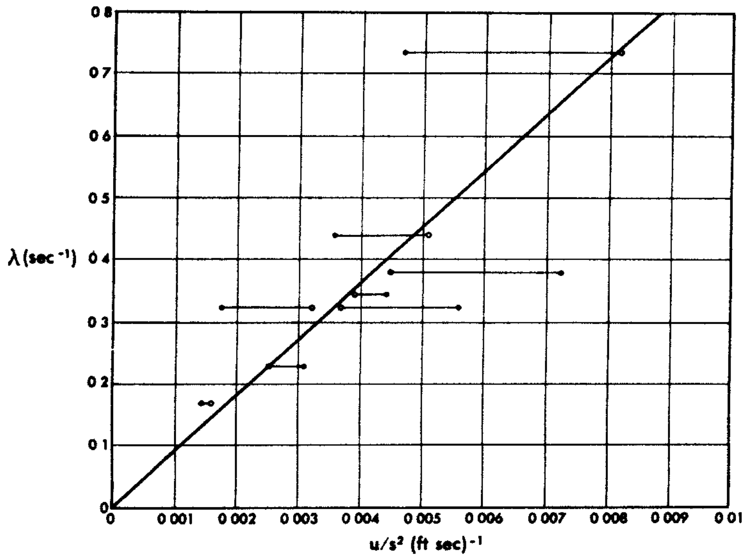

Fig 9 Sensitivity, $\lambda(\sec^{-1})$, versus the product $us^{-2}(\text{ft}^{-1}\sec^{-1})$ for the data of reference 3 The circles were computed by Edie in reference 14 assuming speeds derived using the values of concentrations and equation (15) of reference 3 The solid dots are the correct points computed on the basis of the values given here in Table I

periments in question are quite different from those computed by Edie and are given in Table I On the basis of these values the sensitivity functions computed using Edie's formula, equation (6), are those plotted as solid dots in Fig 9 The agreement is not as good as that of the circles computed by Edie

It is probably obvious by now that there are a considerable number of difficulties regarding the comparison of observations with the various models discussed in the preceding sections First of all, even if there is a unique average flow versus concentration relation associated with a particular tunnel, statistical deviations will produce some scatter of the measurements It is also possible that the flow versus concentration law is affected by a number of special conditions, such as the condition of the pavement, the consistency of traffic, and even the number of policemen present, to name a few. Another difficulty is that no flow measurements are available in the regions of very low and very high concentrations. For all these reasons no strong arguments can be given in favor of any single equation of steady state on the basis of the available data. A very limited number of characteristics may be established, such as, for example, the skewness of the actual flow versus concentration curve with a maximum to the left of $k/k_j=0.5$. If, on the other hand, one tries to break the set of data available into sub-sets for different ranges of concentration, then almost every one of the steady-state equations of the section "Equations of the Follow-the-Leader Models," can be made applicable in some subregion.

In any case, a comparison of the available data, Fig. 8, and the theoretical curves of Figs. 1 to 4 may lead to a limited number of conclusions. If one assumes an equation such as equation (8), the exponent of the spacing, l, should probably be positive and not greater than 2. Virtually nothing can be said about the exponent of the speed, m, particularly if one is to consider a nonzero m for some subrange of concentration. The steady-state flow description of equation (21) also appears to be feasible for some appropriate value of the constant γ, with the exclusion, of course, of the range of concentrations close to zero.

Follow-the-Leader Experiments

The 18 experiments performed in the Lincoln Tunnel of New York were analyzed as explained in the preceding section in order to decide whether or not any one of the models discussed has distinct advantages over the others. As mentioned there, both a correlation of the phenomenological quantities associated with each driver and a timewise correlation study of response to different functionals for the product of stimulus times sensitivity were performed.

Phenomenological aspects. The correlation coefficient for linear correlation of $y(\bar{u}_i)$ and $x(\bar{k}_i)$ ($i=1$ to 18) was computed for the various functions $y(u)$ and $x(k)$ corresponding to different models. For each model the correlation study establishes a best slope, c, and a cutoff, c', entering the equation

$$y=cx+c' \tag{30}$$

The individual coefficients of proportionality, c_i, were then computed for each driver according to the equation

$$c_i=(y_i-c')/x_i \tag{31}$$

Finally these c_i were normalized with respect to their average and a standard deviation was computed for each model, given by

$$\sigma_c = [\langle c_i^2 \rangle / \langle c_i \rangle^2 - 1]^{1/2} \qquad (32)$$

The comparison of the various models is shown in Figs 10 and 11 Figure 10 shows the variation of the correlation coefficient, r, with l for constant m, and Fig 11 shows the variation of σ_c with m for constant l As seen in Fig 10, the maximum r is obtained for quite high values of l and m However, if the values of y_i and x_i corresponding to such high l

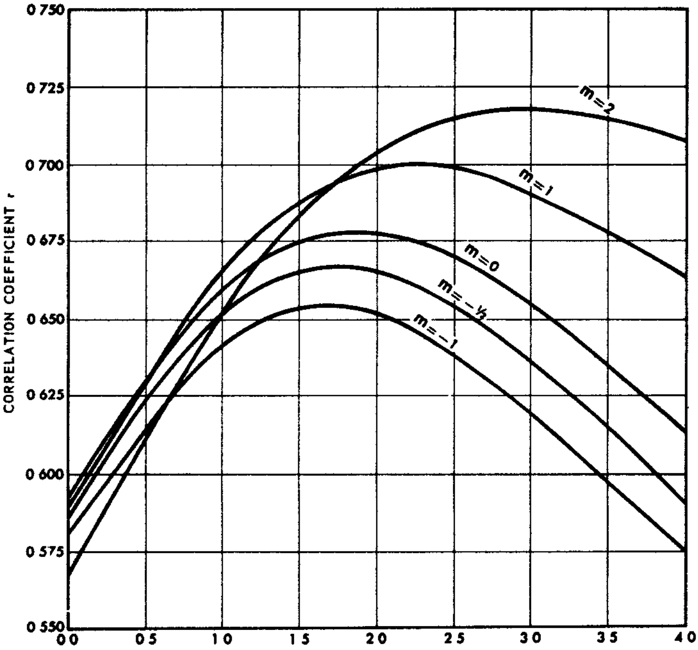

Fig 10 Phenomenological study of the 18 car-following experiments in the Lincoln Tunnel Correlation coefficient, r, versus the exponent l entering equation (9) for various values of the exponent m

and m are plotted it is seen that there is a clustering of most of the data near the origin with a few points far away Although this phenomenon yields high correlation coefficients for large values of l and m, such results do not appear to be physically reasonable, nor are they statistically reliable If we limit ourselves to values of m between -1 and 2, it is seen that the best correlation corresponds to values of l between 1 and 3 However, very little can be said about the value of m, not even whether it should be taken positive or negative This is so because the maximum correlation coefficient for the various theories ranges from $\sim$0 65 to $\sim$0 72 for $-1 \leqq m \leqq 2$

Comparison of the various models on the basis of the standard deviation σ_c associated with them, Fig 11, yields analogous conclusions, although somewhat more restrictive regarding the optimum value of l The minimum σ_c was obtained near $l=1$ for all theories However, the variation of the minimum normalized σ_c was from $\sim$0 102 to $\sim$0 093 for $-1 \leqq m \leqq 3$

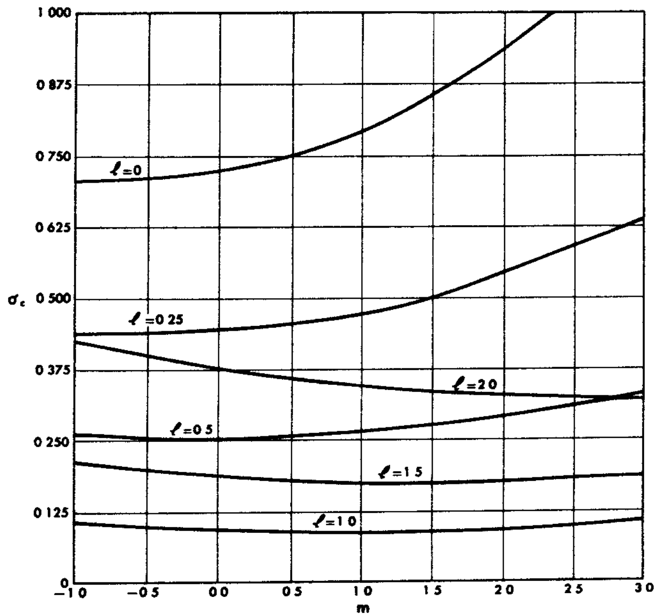

Fig 11. Phenomenological study of the 18 car-following experiments in the Lincoln Tunnel Normalized standard deviation, σ_c, of the proportionality coefficients, c, of the individual drivers according to equation (31), versus the exponent m entering equations (9) for various values of the exponent l

In the case of the model based on a stopping distance criterion, equation (17), the maximum correlation coefficient, $r=0$ 592, is obtained when $B=0$, in which case this model degenerates to the linear one of constant sensitivity The normalized standard deviation for $B=0$ is $\sigma_c=0$ 726 and decreases very slowly with increasing B.

Although some trend seems to be established we again point out that the reliability of this phase of our study is relatively poor because of the limited population We now proceed to the discussion of a potentially more dependable correlation study

Time-wise correlation The time-wise correlation analysis has been carried out for ten different functions representing the sensitivity, over a limited number of integral values of l and m, in an effort to ascertain whether this approach might suggest the best function of the form given by equation (8) Table II lists the maximum correlation coefficient for the optimum time lag, T, for the fourteen data sets that were used in this phase of the analysis for each of the ten different functionals assumed

TABLE II

CORRELATION COEFFICIENT $r(l, m)$ FOR THE OPTIMUM TIME LAG

Driver	$r(0, 0)$	$r(1,-1)$	$r(1, 0)$	$r(1, 1)$	$r(1, 2)$	$r(2,-1)$	$r(2, 0)$	$r(2, 1)$	$r(2, 2)$	$r^{(a)}$
1	0 686	0 408	0 459	0 693	0 721	0 310	0 693	0 584	0 690	0 718
2	0 878	0 776	0 843	0 847	0 746	0 719	0 847	0 827	0 766	0 847
3	0 770	0 757	0 778	0 786	0 784	0 726	0 786	0 784	0 797	0 787
4	0 793	0 730	0 748	0 803	0 801	0 685	0 801	0 786	0 808	0 803
5	0 831	0 826	0 862	0 727	0 577	0 805	0 728	0 784	0 624	0 749
6	0 720	0 665	0 709	0 721	0 709	0 660	0 720	0 713	0 712	0 721
7	0 640	0 470	0 678	0 742	0 691	0 455	0 745	0 774	0 718	0 773
8	0 865	0 845	0 881	0 890	0 862	0 818	0 890	0 903	0 907	0 896
9	0 728	0 642	0 734	0 773	0 752	0 641	0 773	0 769	0 750	0 773
10	0 898	0 890	0 898	0 893	0 866	0 881	0 892	0 889	0 865	0 893
11	0 890	0 952	0 966	0 921	0 854	0 883	0 921	0 971	0 940	0 944
12	0 846	0 823	0 835	0 835	0 823	0 793	0 835	0 821	0 821	0 836
13	0 909	0 906	0 928	0 935	0 927	0 860	0 935	0 928	0 936	0 937
14	0 761	0 790	0 790	0 771	0 731	0 737	0 772	0 783	0 775	0 775

[a] This is the maximum correlation coefficient obtained when the sensitivity is assumed to be of the form given by equation (27), with an appropriate c_2 for each driver

for the sensitivity We denote the correlation coefficient that results from the analysis for a given l and m as $r(l,m)$

It can be seen from this Table that there is little difference in the correlation coefficient for the different sensitivity coefficients for any one driver On the basis of these small differences, it would be difficult to pick out any one of the ten functions for the sensitivity as the best fit to the data with any assurance However, these results do seem to indicate that the exponent of x is not negative, since in these cases the correlation coefficient is generally the smallest for these models, i e , for $m=-1$ and $l=1$ or 2 These results also suggest that m is probably less than 2

One of the main difficulties is that in order to be able to differentiate between the various sensitivities the dependent variables, u and s should not have too small a dispersion around their mean values for any given data set For example, it would be difficult to determine the dependence of the sensitivity of the vehicle speed if the variation in this speed were small One could, of course, look for those cases where the variation was large For example, in run number 8 where the speed varies from 20 mph to 45 mph and where the standard deviation for the speed of the following vehicle is the largest of all fourteen data sets (i e , 22 ft/sec), the largest correlation coefficient is that which corresponds to the u^2/s^2 model, i e , $l=m=2$

A correlation study was also made between the sensitivities for individual drivers that result from the correlation analysis of the linear model and different functions for the sensitivity This correlation coefficient is a measure of how well the variation of the individual sensitivities, λ, of equation (2) can be accounted for, according to a specific model, by the differences in the average speed, u, and the average spacing, s, in each experimental run The results of this analysis are essentially similar to the preceding ones Three functions for the sensitivity reflect a relative superiority The correlation coefficients that result from the linear correlation between λ and the quantities u/s^2, $1/s$, and $1/s^2$ are 0 87, 0 78, and 0 72, respectively The correlation coefficients resulting from using the other models were all in the range from 0 2 to 0 5

DISCUSSION

WE HAVE discussed a number of follow-the-leader models involving different functionals of speed and spacing for the sensitivity Some of these functionals have been suggested earlier and together with others they fall under a more general category of the form given in equation (8) Another form of functionals considered is that derived on the basis of a braking-distance criterion and is given in equation (27)

All these models were considered in a study of observational and experimental data, in an attempt to establish which model, if any, best describes traffic situations It was found that the now available data are not quite sufficient to establish definite superiority of one particular model Nevertheless it has again been ascertained that a nonlinear model is necessary to account for observed flow versus concentration data However, it is not clear, on the basis of presently available data, that a somewhat more complicated nonlinear model has any distinct advantages over the simple nonlinear model [10]

In this connection, a few comments might be made regarding another follow-the-leader model, that suggested by NEWELL,[15] which lumps the sensitivity and stimulus into one functional Newell chose this model mainly because of its advantage of integrability and made no claim of a better fit with any available data other than the fact that it gives a finite speed for zero concentration He did, however, try to establish the range of applicability of the linear theory assuming that his nonlinear model is the correct one It might be pointed out that different results for this range may be obtained from different nonlinear models all of which can be linearized in the limit

We have recently obtained additional data on car-following from a series of experiments conducted in the Holland Tunnel in New York City These data are presently being analyzed and it is hoped that the results of this analysis will allow us to make some better choice of one of the nonlinear models discussed in this paper

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REFERENCES

1 A REUSHEL, "Fahrzeugbewegungen in der Kolonne," *Osterr Ing Archiv* **4,** 193–215 (1950), "Fahrzeugbewegungen in der Kolonne bei gleichformig beschleunigtem der verzogertem Leitfahrzeug," *Zeit d Osterr Ing u Architekt Vereines* **95,** 59–62, 73–77 (1950)
2 L A PIPES, "An Operational Analysis of Traffic Dynamics," *J Applied Phys* **24,** 274–281 (1953)
3 R E CHANDLER, R HERMAN, AND E W MONTROLL, "Traffic Dynamics Studies in Car Following," *Opns Res* **6,** 165–184 (1958)
4 T S CHOW, "Operational Analysis of a Traffic Dynamics Problem," *Opns Res* **6,** 827–834 (1958)
5 E KOMETANI AND T SASAKI, "On the Stability of Traffic Flow," *J Operations Research–Japan* **2,** 11–26 (1958)
6 R HERMAN, E W MONTROLL, R B POTTS, AND R W ROTHERY, "Traffic Dynamics Analysis of Stability in Car Following," *Opns Res* **7,** 86–106 (1959)
7 T SASAKI, "On the Stability of Traffic Flow (Report II)," *J Operations Research–Japan* **2,** 60–79 (1959)
8 W HELLY, "Dynamics of Single-Lane Vehicular Traffic Flow," Research Report No 2, Center for Operations Research, Massachusetts Institute of Technology, 1959
9 E KOMETANI AND T SASAKI, "A Safety Index for Traffic with Linear Spacing," *Opns Res* **7,** 704–720 (1959)
10 D C GAZIS, R HERMAN, AND R B POTTS, "Car-Following Theory of Steady-State Traffic Flow," *Opns Res* **7,** 499–505 (1959)

11 R HERMAN AND R B POTTS, "Single Lane Traffic Theory and Experiment," Proceedings—Symposium on Theory of Traffic Flow, Research Laboratories, General Motors Corporation, 1959, Elsevier Publishing Co, 1961, p 120
12 E KOMETANI AND T SASAKI, "Dynamic Behavior of Traffic with a Nonlinear Spacing-Speed Relationship," Proceedings—Symposium on Theory of Traffic Flow, Research Laboratories, General Motors Corporation, 1959, Elsevier Publishing Co, 1961, p 105
13 T KISHI, "Traffic Dynamics Analysis as Sampled-Data Control Systems," *J Operations Research-Japan* **2**, 114–123 (1960)
14 L C EDIE, "Car-Following and Steady-State Theory for Noncongested Traffic," *Opns Res* **9**, 66–76 (1961)
15 G F NEWELL, "Nonlinear Effects in the Dynamics of Car Following," *Opns Res* **9**, 209–229 (1961)
16 I PRIGOGINE, "A Boltzmann-like Approach to the Statistical Theory of Traffic Flow," Proceedings—Symposium on Theory of Traffic Flow, Research Laboratories, General Motors Corporation, 1959, Elsevier Publishing Co, 1961, p, 158
17 I PRIGOGINE AND F C ANDREWS, "A Boltzman-like Approach for Traffic Flow," *Opns Res* **8**, 789–797 (1960)
18 B D GREENSHIELDS, "A Study of Traffic Capacity", *Proc Highway Research Board* **14**, 468 (1935)
19 Highway Capacity Manual, U S Government Printing Office, Washington D C, p 3 (1950)
20 H GREENBERG, "An Analysis of Traffic Flow," *Opns Res* **7**, 79–85 (1959)
21 B D HANKIN AND R A WRIGHT, "Passenger Flow in Subways," *Operational Research Quart* **9**, 81–88 (1958)