Homework 1.

1. 假设 $\exists x_0 \in (0,1)$, $w(x_0) > a > 0$

由 $w(x) \in C[0,1]$ 可知,

$\exists B(x_0, r) \subset (0,1)$, 对 $\forall x \in B(x_0, r)$ 有 $w(x) > a$

那么, 可以构造如下形式 $v(x)$:

$$v(x) = \begin{cases} \frac{x - x_0 + r}{r} & x_0 - r \le x < x_0 \\ \frac{x_0 + r - x}{r} & x_0 \le x \le x_0 + r \\ 0 & \text{else.} \end{cases} \qquad v'(x) = \begin{cases} \frac{1}{r} & x_0 - r \le x < x_0 \\ -\frac{1}{r} & x_0 \le x \le x_0 + r \\ 0 & \text{else.} \end{cases}$$

$\Rightarrow v(x) \in \mathring{V}$

而 $\int_0^1 wv\,dx > a \int_{x_0-r}^{x_0+r} v\,dx > 0$

与条件 $\int_0^1 wv\,dx = 0$ 矛盾

因此 $\forall x \in (0,1)$, $w(x) = 0$

由 w 连续性, $w(0) = w(1) = 0$

$\Rightarrow w(x) = 0$, $x \in [0,1]$

2. 令 $v \in \mathring{V} = \{ w \mid w \in C_{[0,1]}, w' \text{分段连续有界}, w(0) = w(1) = 0 \}$

那么 对两边积分

$$\int_0^1 (-u'' + u) v\,dx = \int_0^1 fv\,dx$$

$$\Leftrightarrow \int_0^1 (u'v' + uv)\,dx = \int_0^1 fv\,dx$$

令 $a(u,v) = \int_0^1 (u'v' + uv)$, $(f, v) = \int_0^1 fv\,dx$.

(W): 求 u 满足

$a(u,v) = (f,v)$, 对 $\forall v \in \mathring{V}$ 成立.

3. 令双线性型 $a(u,v) = \int_0^1 (au'v' + uv)\,dx$,

线性空间 $V = \{ w \mid w \in C_{[0,1]}, w' \text{分段连续有界}, w(0) = 0 \}$

此时 $\int_0^1 fv\,dx = \int_0^1 -v(au')' + uv\,dx$

$$= -(v(x)a(x)u'(x))\Big|_0^1 + \int_0^1 au'v'\,dx + \int_0^1 uv\,dx$$

$$= \int_0^1 au'v'\,dx + \int_0^1 uv\,dx = a(u,v)$$

(M)：求 $u\in V$, s.t. $a(u,v)=(f,v)$ ，$\forall v\in V$

4. 已知有限元方法应用在变分问题上有：

$$V_n=\mathrm{span}\{\phi_1,\phi_2,\cdots\phi_N\}$$

$$u_n=\sum_{j=1}^{N}u_n(x_j)\phi_j(x)\in V_n.$$

对 $\forall v\in V_n$

$$a(u_n,v_n)=\int_0^1 u_n'v_n'dx=\sum_{i=1}^{N}\sum_{j=1}^{N}u_n(x_j)m(\phi_i,\phi_j)u_n(x_i)=\sum_{i=1}^{N}\sum_{j=1}^{N}v_n(x_j)a(\phi_i,\phi_j)u_n(x_i)=V^TAU$$

其中 $a(\phi_i,\phi_j)=\int_0^1\phi_i'\phi_j'dx$ ，$A=\left(a(\phi_i,\phi_j)\right)$ ，$U=\begin{pmatrix}u_n(x_1)\\ \vdots\\ u_n(x_N)\end{pmatrix}$，$V=\begin{pmatrix}v_n(x_1)\\ \vdots\\ v_n(x_N)\end{pmatrix}$

$$(f,v)=\int_0^1 fv\,dx=\sum_i^N\sum_j^N\int_0^1 v_n(x_j)\phi_j(x)\phi_i(x)f_i\,dx=V^TMF$$

其中 $m(\phi_i,\phi_j)=\int_0^1\phi_i\phi_j\,dx$ ，$M=(m(\phi_i,\phi_j))$

以上：变分问题在有限元方法下写作．

求解 u_n，使得对 $\forall v_n\in V$，满足

$V^TAU=V^TMF$ 成立．

$\Leftrightarrow$ 求解 $AU=MF$．