Homework2

1. 证明：$(f,\phi_i)=\frac{1}{2}(h_i+h_{i+1})(f(x_i)+O(h))$

proof： $\phi_i=\begin{cases}\frac{x-x_{i-1}}{x_i-x_{i-1}} & x\in[x_{i-1},x_i]\\ \frac{x_{i+1}-x}{x_{i+1}-x} & x\in[x_i,x_{i+1})\\ 0\end{cases}$

$\therefore (f,\phi_i)=\int_{x_{i-1}}^{x_i} f\cdot\frac{x-x_{i-1}}{x_i-x_{i-1}}dx+\int_{x_i}^{x_{i+1}} f\cdot\frac{x_{i+1}-x}{x_{i+1}-x_i}dx$

$=\int_0^{h_i} f(\xi+x_{i-1})\cdot\frac{\xi}{h_i}d\xi+\int_0^{h_{i+1}} f(x_{i+1}-\xi)\cdot\frac{\xi}{h_{i+1}}d\xi$

$=\int_0^{h_i} f(\xi+x_i-h_i)\frac{\xi}{h_i}d\xi+\int_0^{h_{i+1}} f(x_i+h_{i+1}-\xi)\cdot\frac{\xi}{h_{i+1}}d\xi$

$=\int_{-h_i}^{0} f(z+x_i)(\frac{z}{h_i}+1)dz+\int_0^{h_{i+1}} f(z+x_i)(1-\frac{z}{h_{i+1}})dz.$

$=\int_{-h_i}^{0} f(z+x_i)\frac{z}{h_i}dz-\int_0^{h_{i+1}} f(z+x_i)\frac{z}{h_{i+1}}dz+\int_{-h_i}^{h_{i+1}} f(z+x_i)dz.$

$=\int_{-h_i}^{0}[f(x_i)+O(z)]\frac{z}{h_i}dz-\int_0^{h_{i+1}}[f(x_i)+O(z)]\frac{z}{h_{i+1}}dz+\int_{-h_i}^{h_{i+1}}[f(x_i)+O(z)]dz.$

$\because z\in[-h_i,h_{i+1}]$ ， $h_i\leq h$

$\therefore$ 可认为 $O(z)=O(h).$

原式 $=[f(x_i)+O(h)](-\frac{h_i}{2}-\frac{h_{i+1}}{2}+h_i+h_{i+1})$

$=\frac{1}{2}(h_i+h_{i+1})[f(x_i)+O(h)]$

2. 设 $h=\max\{h_i\}$，证明

$\|u-u_I\|\leq ch^2\|u''\|$

proof： 引理： $w(0)=w(1)=0$， $\int_0^1 w^2(x)dx\leq\bar{c}\int_0^1 w'^2(x)dx.$ $\bar{c}=\frac{1}{8}$

引理证明： $w^2(t)=\left(\int_0^t 1\cdot w'(\xi)d\xi\right)^2\leq\left(\int_0^t 1^2d\xi\right)\left(\int_0^t w'^2(\xi)d\xi\right)=t\cdot\int_0^t w'^2(\xi)d\xi$

$\Rightarrow\int_0^\alpha w^2(t)dt\leq\int_0^\alpha t\cdot\int_0^t w'^2(\xi)d\xi dt$

令 $F(t)=\int_0^t w'^2(\xi)d\xi$

$\int_0^\alpha w^2(t)dt\leq\int_0^\alpha t F(t)dt=\frac{1}{2}\alpha^2F(\alpha)-\frac{1}{2}\int_0^\alpha t^2w'^2(t)dt.$

$\leq\frac{1}{2}\alpha^2F(\alpha).$ ①

同理 $w^2(t)=\left(-\int_t^1 1\cdot w'(\xi)d\xi\right)^2\leq(1-t)\int_t^1 w'^2(\xi)d\xi$

$\Rightarrow\int_\alpha^1 w^2(t)dt\leq\int_\alpha^1(1-t)\int_t^1 w'^2(\xi)d\xi dt=\int_\alpha^1(1-t)(F(1)-F(t))dt.$

$=\frac{1}{2}(1-\alpha)^2(F(1)-F(\alpha))-\frac{1}{2}\int_\alpha^1(1-t)^2w'^2(t)dt$

$\leq\frac{1}{2}(1-\alpha)^2(F(1)-F(\alpha)).$ ②

① + ② $\int_0^1 w^2(t)\,dt \leq \frac{1}{2}\alpha^2 F(\alpha) + \frac{1}{2}(1-\alpha)^2 (F(1) - F(\alpha))$.

取 $\alpha = \frac{1}{2}$ $\int_0^1 w^2(t) \leq \frac{1}{8} F(1) = \frac{1}{8}\int_0^1 w'^2 dx$ ③

引理得证，

若仅有 $w(0)=0$ 条件，由①可知 $\int_0^1 w^2(t) \leq \frac{1}{4}\int_0^1 w'^2 dx$. ④.

对原问题，考虑 $x \in [x_{i-1}, x_i]$ 时，$(u-u_I)(x_i) = (u-u_I)(x_{i-1}) = 0$ ⑤

那么 $\exists\, \xi_i \in (x_{i-1}, x_i)$，$(u-u_I)'(\xi_i) = 0$ ⑥

③,⑤ $\Rightarrow$ $\int_{x_{i-1}}^{x_i} (u-u_I)^2(x)\,dx \leq \frac{1}{8} h_i^2 \int_{x_{i-1}}^{x_i} (u-u_I)'^2(x)\,dx$

④,⑥ $\Rightarrow$ $\int_{\xi}^{x_i} (u-u_I)'^2(x)\,dx \leq \frac{1}{4}(x_i-\xi_i)^2 \int_{\xi_i}^{x_i} (u-u_I)''^2(x)\,dx$

$\leq \frac{1}{4} h_i^2 \int_{x_{i-1}}^{x_i} (u-u_I)''^2(x)\,dx = \frac{1}{4} h_i^2 \int_{x_{i-1}}^{x_i} u''^2 dx$ $[u_I''=0]$.

$\Rightarrow$ $\int_{x_{i-1}}^{x_i} (u-u_I)^2(x)\,dx \leq C h_i^4 \int_{x_{i-1}}^{x_i} u''^2(x)\,dx$

$\Rightarrow$ $\int_0^1 (u-u_I)^2 dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u-u_I)^2(x)\,dx \leq C h^4 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} u''^2 dx = C h^4 \int_0^1 u''^2 dx$

$\Rightarrow$ $\|u-u_I\| \leq C h^2 \|u''\|$