Homework 3.

1. Prove covercivity: $\|v\|^2 + \|v'\|^2 \le C\, a(v,v)$. for $\forall\, v \in V$

证: $v \in V = \{ v \in L^2(0,1) \mid a(v,v) < +\infty\ ,\ v(0)=0 \}$

$$v(x) = \int_0^x v'(t)\,dt$$

$$\Rightarrow v^2(x) = \left(\int_0^x v'(t)dt\right)^2 \le \int_0^x 1^2 dt \cdot \int_0^x v'^2(t)dt = x\int_0^x v'^2(t)dt$$

$$\Rightarrow \int_0^1 v^2(x)dx \le \int_0^1 x\int_0^x v'^2(t)dt\,dx$$

$$= \frac{1}{2}x^2\int_0^x v'^2(t)dt\Big|_0^1 - \frac{1}{2}\int_0^1 x^2 v'^2(x)dx$$

$$= \frac{1}{2}\|v'\|^2 - \frac{1}{2}\int_0^1 x^2 v'^2(x)dx$$

在 $v(x) = x$ 时取等.

即 $\int_0^1 v^2(x)dx = \frac{1}{3} = \frac{1}{2}\int_0^1 v'^2(x)dx - \frac{1}{2}\int_0^1 x^2 v'^2 dx = \frac{1}{3}\|v'\|^2$ $\qquad n = \underline{t+1}$

故可知: $\|v\|^2 \le \frac{1}{3}\|v'\|^2$

由 $a(v,v) := \int_0^1 v'\cdot v'\,dx = \|v'\|^2$

$$\Rightarrow \|v\|^2 + \|v'\|^2 \le \frac{4}{3}\, a(v,v)\ ,\qquad C = \frac{4}{3}$$

2. 由 $v \in V$

$$\Rightarrow v^2(x) = \left(\int_0^x v'(t)dt\right)^2 \le x\int_0^x v'^2(t)\,dx$$

$$\Rightarrow \|v^2\|_{max} = \sup v^2(x) \le \sup\left(x\int_0^x v'^2(t)dt\right) \le \int_0^1 v'^2(x)dx = a(v,v$$

where. $C = 1$

3. Formulate a difference method for Poisson Equation when Ω is square.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases}$$

记: $\frac{\partial^2 u}{\partial x_1^2} \approx \frac{u_{i+N} + u_{i-N} - 2u_i}{h^2}$ $\qquad \frac{\partial^2 u}{\partial x_2^2} \approx \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}$

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_N \\ u_{N+1} & u_{N+2} & \cdots & u_{2N} \\ \vdots & & & \vdots \\ u_{N^2-1} & & & u_{N^2} \end{bmatrix}$$

故 $-\Delta u^2 = f$ 差分形式为

$$\Rightarrow -u_{i+1} + 4u_i - u_{i-1} - u_{i-N} - u_{i+N} = h^2 f(x_i)$$

假设 $f(x)$ 充分光滑，在有限元方法中，

$$\langle \varphi_i, f\rangle = \int \varphi_i f(x)dx = h^2(f(x_i)+O(h))$$

即两者相同

4. 证明：(1) $\Leftrightarrow$ (2)

$\langle u-u_h, v\rangle = 0 \quad \forall v \in V_h \qquad (1)$

$\|u-u_h\|_{H^1(\Omega)} \leq \|u-v\|_{H^1(\Omega)}, \ \forall v \in V_h \qquad (2)$

$proof_1$: (1) $\Rightarrow$ (2)

由 $u_h \in V_h$

$\Rightarrow u_h - v \in V_h$

由 $\forall v \in V_h \quad \langle u-u_h, v\rangle = 0$

$\Rightarrow \langle u-u_h, u_h-v\rangle = 0$

因此：$\|u-v\|^2_{H_1(\Omega)} = \langle u-v, u-v\rangle$

$= \langle u-u_h+u_h-v, u-u_h+u_h-v\rangle$

$= \|u-u_h\|^2 + 2\langle u-u_h, u_h-v\rangle + \|u_h-v\|^2$

$= \|u-u_h\|^2 + \|u_h-v\|^2$

$\leq \|u-u_h\|^2$

$proof_2$: (2) $\Rightarrow$ (1).

定义 $f(\varepsilon) = \|u-u_h+\varepsilon v\|_{H_1}$

由 (2) $\Rightarrow f(0) \leq f(\varepsilon)$

$\Rightarrow f'(0) = 0 \Rightarrow \langle u-u_h, v\rangle = 0.$

5. (P) 原问题：$\begin{cases} -u'' = f \\ u(0) = u'(1) = 0 \end{cases}$

(W) 变分问题：Find $u \in V = \{v \in H^1(I), v(0) = 0\}$,

satisfy：$(u', v') = (f, v), \quad \forall v \in V$

"$\Rightarrow$"：u 是 (P) 的解，那么

$$
\begin{aligned}
(u', v') &= u'(1)v(1) - u'(0)v(0) - (u'', v) \\
&= 0 - 0 + (-u'', v) \\
&= (f, v)
\end{aligned}
$$

即 u 是 (M) 的解

"$\Leftarrow$" u 是 (M) 的解

$$\Rightarrow u'(1)v(1) = (u''+f, v)$$

设 $\exists x_1, x_2$, $\forall x_1 < x < x_2$,

s.t. $(u''+f)(x) > 0$,

此时取: $v = \begin{cases} -(x-x_1)(x-x_2) & x_1 < x < x_2 \\ 0 & \text{else} \end{cases}$

$\Rightarrow 0 = u'(1)v(1) = (u''+f, v) > 0$ 矛盾.

$\therefore (u''+f, v) = u'(1)v(1)$

取 $v(x) = x \in V$, 得 $u'(1) = 0$

即 u 是 (P) 的解

"FEM", 取 $V_h = \{v \mid v$ 为分段线性函数. $v(0)=0\}$

Find $u_h \in V_h$,

Satisfy: $(u_h', v') = (f, v) \quad \forall v \in V_h$

令 $0 = x_0 < x_1 \cdots < x_{N+1} = 1$, $h_i = x_i - x_{i-1}$, $I_i = [x_{i-1}, x_i]$

$$\varphi_i = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & x \in I_i \\ \dfrac{x_{i+1} - x}{h_{i+1}} & x \in I_{i+1} \\ 0 & \text{else.} \end{cases} \qquad i = 1, 2, \cdots N$$

$$\varphi_{N+1} = \begin{cases} \dfrac{x - x_N}{h_{N+1}} & x \in I_{N+1} \\ 0 & \text{else} \end{cases}$$

令 $u_h = \sum_{i=1}^{N+1} u_h(x_i)\varphi_i$

那么：$(\varphi_i', \varphi_i') = \frac{1}{h_i} + \frac{1}{h_{i+1}}$

$(\varphi_{i-1}', \varphi_i') = -\frac{1}{h_{i-1}}$ $\quad i=1,2,\cdots N.$

$(\varphi_i', \varphi_{i-1}') = -\frac{1}{h_i}$

$(\varphi_N', \varphi_N') = \frac{1}{h_{N+1}}$

$(\varphi_N', \varphi_{N-1}') = -\frac{1}{h_{N+1}}$

$f_i = (f, \varphi_i) \approx \frac{1}{h_i} f(x_i)$

$f_{N+1} = (f, \varphi_{N+1}) \approx \frac{1}{2h_{N+1}} f(1).$

$$\Rightarrow \quad K = \begin{bmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & & & \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & & & \\ & & \ddots & & \\ & & & \frac{1}{h_N}+\frac{1}{h_{N+1}} & -\frac{1}{h_{N+1}} \\ & & & -\frac{1}{h_{N+1}} & \frac{1}{h_{N+1}} \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \\ f_{N+1} \end{bmatrix}$$

求解线性方程 $KU=F$ 即得 u_h 的解

6. (W). Find $u \in H^1(\Omega)$, s.t. $a(u,v) = (f,v) + \langle g, v\rangle$

where $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, d\Omega$

(M) Find $u \in H^1$, s.t. $F(u) \le F(v)$, $\forall v \in H^1(\Omega)$,

where. $F(v) = \frac{1}{2} a(v,v) - (f,v) - \langle g, v\rangle$

Proof: (W) $\Rightarrow$ (M).

$\forall v \in V$, satisfy: $a(u,v) = (f,v) + \langle g, v\rangle$

$u - v \in V \Rightarrow a(u, u-v) = (f, u-v) + \langle g, u-v\rangle$

$\Rightarrow F(v) = F(v-u+u) = \frac{1}{2} a(u,u) + \frac{1}{2} a(v-u, v-u) - (f,u) - \langle g, u\rangle$

$= F(u) + \frac{1}{2} a(v-u, v-u)$

$\geq F(u)$

(M) $\Rightarrow$ (W)

Define: $f(\varepsilon) = F(u+\varepsilon v)$

$\forall v \in V$, $F(u) \le F(v)$

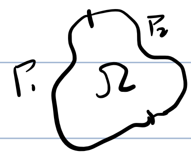
$\Rightarrow$ $\varepsilon = 0$ is minimal point.

$\Rightarrow$ $f'(\varepsilon) = 0$

$\Rightarrow$ $a(u,v) - (f,v) - \langle g, v\rangle = 0$

7. Give a variational function for problem:

$$(P)\begin{cases} \Delta u = f & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_2. \end{cases}$$



proof.

设 u 是 (p) 问题的解.

则 $\int_\Omega f v \, d\Omega = \int_\Omega (\Delta u) v \, d\Omega = \int_\Gamma v \frac{\partial u}{\partial n} ds - \int_\Omega \nabla u \cdot \nabla v \, d\Omega$

令 $V = \{ v \in H^1 : v|_{\Gamma_1} = u_0 \}$, $V_0 = \{ v \in H^1 , v|_{\Gamma_2} = 0 \}$

对 $\forall v \in V_0$

$$\int_\Omega f v \, d\Omega = \int_{\Gamma_2} v \cdot g \, dS - \int_\Omega \nabla u \cdot \nabla v \, d\Omega$$

$\Rightarrow$ (W) find $u \in V$, s.t.

$$\int \nabla u \cdot \nabla v \, d\Omega = -\int_\Omega f \cdot v \, d\Omega + \int_{\Gamma_2} v \cdot g \, ds \quad \forall v \in V_0$$

$\Rightarrow$ $a(u,v) = (-f, v) + \langle g, v\rangle$

"FEM":

考虑: $V_h = \{ v \mid v$ 为分片线性函数, $v|_{\Gamma_1} = u_0 \}$

$V_{h_0} = \{ v \mid v$ 为分片线性函数, $v|_{\Gamma_2} = 0 \}$

$\Rightarrow$ Find $u_h \in V_h$,

$$a(u_h, v) = (-f, v) + \langle g, v\rangle \quad \forall v \in V_{h_0}$$

"Physical interpretation"

线性弹性模型材质，其构型空间为 Ω，

定义 Ω 上 x 处的形变为 $u(x)$.

在每点处有外力 $-f(x)$

$\Rightarrow$ 稳定方程：$-\Delta u = f$

对于材料边界 Γ_1 处为固定边界，$u=u_0$ on Γ_1

Γ_2 处为负载(额外)外力：$\frac{\partial u}{\partial n}=g$ on Γ_2.