Homework 9. Week 12

4.1 . 设 $I=[0,h]$, $\pi v\in P_1(I)$ 是 $v\in C^0(I)$ 端点的线性插值, 试估计 $\|v-\pi v\|_{L_\infty(I)}$, $\|v'-\pi v'\|_{L_\infty(I)}$

proof. 设 $v\in W^2_\infty(I)$, v 有2阶弱导数, 对 v 作 Taylor 展开

$$v(y) = v(x) + v'(x)(y-x) + R(x,y)$$

$$R(x,y) = \frac{1}{2}v''(\xi)(y-x)^2$$, ξ 与 y, x 有关.

此时. $v(0) = v(x) + v'(x)(-x) + R(x,0)$

$v(h) = v(x) + v'(x)(h-x) + R(x,h)$

对 $\pi v \doteq v(0)\frac{h-x}{h} + v(h)\frac{x}{h}$

$$= v(x) + v'(x)\left(\frac{h-x}{h}(-x) + (h-x)\frac{x}{h}\right) + \frac{h-x}{h}R(x,0) + \frac{x}{h}R(x,1)$$

$$= v(x) + \frac{h-x}{h}R(x,0) + \frac{x}{h}R(x,1)$$

有 $v(x)-\pi v(x) = -\frac{1}{2}\left(\frac{h-x}{h}x^2|v''(\xi)| + \frac{x}{h}(h-x)^2|v''(\eta)|\right)$

$$\Rightarrow \|v-\pi v\|_\infty = \sup_x\left|\frac{1}{2}\left(\frac{h-x}{h}x^2|v''(\xi_x)| + \frac{x}{h}(h-x)^2|v''(\eta_x)|\right)\right|$$

$$\le \frac{1}{2}\left(\frac{h-x}{h}x^2 + \frac{x}{h}(h-x)^2\right)|D^2v|_\infty$$

$$\le \frac{1}{8}h^2|D^2v|_\infty$$

同理 $\Rightarrow \|v'-\pi v'\|_\infty = \frac{1}{h}\sup_x|R(x,1)-R(x,0)|$

$$\le h|D^2v|_\infty$$

4.3. 估计变分问题的误差. $\|u-u_h\|_{H^2(I)}$

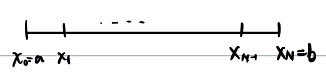
变分问题: find $u\in H^2$, $u(0)=u'(0)=u(1)=u'(1)=0$,

对 $\forall v\in H^2$, $v(0)=v'(0)=v(1)=v'(1)=0$ 满足.

$$a(u,v) = \int_I u''\cdot v''dx = L(v) = \int_I f\cdot v\,dx$$

proof. 假设 $u\in H^3$, πu 是对 u 的分片二次 Hermite 插值. $\pi u\in H^2(I)$

对区间 I 分段



其中 $I_i=[x_{i-1},x_i]$, $h_i=x_i-x_{i-1}$, $h=\max_i h_i$

那么在 I_i 上, $\pi u\in H^3(I_i)$

考虑单位单元 $[0,1]$, 有线性变换 $\lambda=\frac{x-x_{i-1}}{x_i-x_{i-1}}$

那么 $\hat{e}(\lambda) = \hat{u}(\lambda) - \hat{\pi u}(\lambda)$. 且 $\hat{e}(\lambda) \in H^3(I_i) \hookrightarrow C^2(I_i)$

且由插值定义: $\hat{e}(0) = \hat{e}'(0) = \hat{e}(1) = \hat{e}'(1) = 0$

$\Rightarrow$ 根据Rolle中值定理, $\exists \lambda_0 \in (0,1)$, $\hat{e}''(\lambda_0) = 0$

$\Rightarrow |\hat{e}''(\lambda)| = \left|\int_{\lambda_0}^{\lambda} \hat{e}^{(3)}(\tau)d\tau\right| \leq \int_{\lambda_0}^{\lambda} |\hat{e}^{(3)}(\tau)|d\tau \leq \int_0^1 |\hat{e}^{(3)}(\lambda)|d\lambda \leq 1\cdot\left(\int_0^1 (\hat{e}^{(3)}(\lambda))^2 d\lambda\right)^{\frac{1}{2}} = \|e^{(3)}\|_{L^2(0,1)}$

$\Rightarrow \|\hat{e}''(\lambda)\|_{L^2_{(0,1)}} = \int_0^1 |\hat{e}''(\lambda)|^2 d\lambda \leq \|\hat{e}^{(3)}(\lambda)\|_{L^2(0,1)}$

同理: $\|\hat{e}\|_{L^2[0,1]} \leq \|\hat{e}'\|_{L^2[0,1]} \leq \|\hat{e}''(\lambda)\|_{L^2[0,1]} \leq \|\hat{e}^{(3)}(\lambda)\|_{L^2[0,1]}$ ①

变换回 x 空间 有 $\frac{de}{dx} = \frac{d\hat{e}}{d\lambda}\frac{d\lambda}{dx} = \frac{1}{h_i}\hat{e}'(\lambda)$, $dx = \frac{dx}{d\lambda}\cdot d\lambda = h_i d\lambda$

那么 $\int_{I_i} e^2(x)dx = h_i \int_0^1 \hat{e}^2(\lambda)d\lambda \leq h_i \|\hat{e}^{(3)}(\lambda)\|^2_{L^2[0,1]}$

$\int_{I_i} e'(x)^2 dx = h_i^{-1} \int_0^1 \hat{e}'(\lambda)^2 d\lambda \leq h_i^{-1} \|\hat{e}^{(3)}(\lambda)\|_{L^2[0,1]}$

$\int_{I_i} e''(x)^2 dx = h_i^{-3} \int_0^1 \hat{e}''(\lambda)^2 d\lambda \leq h_i^{-3} \|\hat{e}^{(3)}(\lambda)\|_{L^2[0,1]}$ ②

且由 $\|\hat{e}^{(3)}(\lambda)\|^2_{L^2[0,1]} = \int_0^1 \hat{e}^{(3)}(\lambda)^2 d\lambda = h_i^{5} \int_{I_i} e^{(3)}(x)dx$

$$\Rightarrow \int_{I_i} e^2(x) + e'(x)^2 + e''(x)^2 dx \leq (h_i + h_i^{-1} + h_i^{-3})h_i^5 \int_{I_i} e^{(3)}(x)dx$$
$$\leq h^2(1+h^2+h^4)\int_{I_i} e^{(3)}(x)dx$$

$\Rightarrow \|e\|_{H^2[I]} = \sum_{i=1}^{N} \|e\|_{H^2[I_i]} \leq C\cdot h^2 |e|_{H^3[I]}$ (这里设 $h<1$)

由于 πu 是二次的 $\Rightarrow e^{(3)} = u^{(3)}$

$\Rightarrow \|u-u_h\|_{H_2} \leq \|u-\pi u\|_{H_2[I]} \leq C|u|_{H^3[0,1]}$

4.8. 设 V_h 是空间 $\Omega \subset \mathbb{R}^d$ 上三角化 T_h 的有限元空间, 满足

$$\|u-\pi_h u\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)}$$

给定 $u \in L^2(\Omega)$, $u_h \in V_h$ 是 u 在 V_h 上投影, 即

$$(u_h, v) = (u, v) \quad \forall v \in V_h$$

证明 误差估计

$$\|u-u_h\|_{L_2(\Omega)} \leq \inf_{v\in V_h}\|u-v\|_{L_2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)}$$

以及

$$\|u_h\|_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}$$

proof: 对∀ $v \in V_h$, 有

$$(u-u_h, v) = 0$$

令 $e = v - u_h \in V_h$, 易得 $(u-u_h, e) = 0$

$$\begin{aligned}
\|u-u_h\|_{L^2}^2 &= (u-u_h, u-u_h) \\
&= (u-v+e, \ u-v+e) \\
&= (u-v, u-v) + 2(u-v, e) + (e, e) \\
&= (u-v, u-v) + \cancel{2(u-u_h, e)} + 2(u_h - v, e) + (e, e) \\
&= \|u-v\|_{L^2}^2 - \|e\|_{L^2}^2
\end{aligned}$$

所以: $\|u-u_h\|_{L^2}^2 \leq \inf\limits_{v \in V_h} \|u-v\|_{L^2}^2$

令 $v = \pi u \in V_h$, 那么

$$\inf_{v \in V_h} \|u-v\|_{L^2}^2 \leq \|u-\pi u\|_{L^2}^2 \leq C \cdot h^{r+1} |u|_{H^{r+1}(\Omega)}$$

以上: $\|u-u_h\|_{L^2}^2 \leq \inf\limits_{v \in V_h} \|u-v\|_{L^2}^2 \leq C \cdot h^{r+1} |u|_{H^{r+1}(\Omega)}$

proof: 由 $u_h \in V_h$, $\Rightarrow$ $(u-u_h, u_h) = 0$

$$\begin{aligned}
\Rightarrow \|u\|_{L^2}^2 - \|u_h\|_{L^2}^2 &= (u-u_h, u+u_h) \\
&= (u-u_h, u+u_h-2u_h) \\
&= \|u-u_h\|_{L^2}^2 \geq 0
\end{aligned}$$

$$\Rightarrow \|u_h\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(\Omega)}^2$$