Homework 1.

1. 假设 $\exists\, x_0 \in (0,1)$，$w(x_0) > a > 0$

由 $w(x) \in C[0,1]$ 可知，

$\exists\, B(x_0, r) \subset (0,1)$，对 $\forall x \in B(x_0, r)$ 有 $w(x) > a$

那么，可以构造如下形式 $v(x)$：

$$v(x) = \begin{cases} \dfrac{x - x_0 + r}{r} & x_0 - r \le x < x_0 \\ \dfrac{x_0 + r - x}{r} & x_0 \le x \le x_0 + r \\ 0 & \text{else.} \end{cases} \qquad v'(x) = \begin{cases} \dfrac{1}{r} & x_0 - r \le x < x_0 \\ -\dfrac{1}{r} & x_0 \le x \le x_0 + r \\ 0 & \text{else.} \end{cases}$$

$\Rightarrow v(x) \in \tilde{V}$

而 $\int_0^1 wv\,dx > a\int_{x_0-r}^{x_0+r} v\,dx > 0$

与条件 $\int_0^1 wv\,dx = 0$ 矛盾

因此 $\forall x \in (0,1)$，$w(x) = 0$

由 w 连续性，$w(0) = w(1) = 0$

$\Rightarrow w(x) = 0$，$x \in [0,1]$

2. 令 $v \in \tilde{V} = \{ w \mid w \in C_{[0,1]},\ w' \text{分段连续有界},\ w(0) = w(1) = 0 \}$

那么 对两边积分

$$\int_0^1 (-u'' + u)v\,dx = \int_0^1 fv\,dx$$

$$\Leftrightarrow \int_0^1 (u'v' + uv)\,dx = \int_0^1 fv\,dx$$

令 $a(u,v) = \int_0^1 (u'v' + uv)$，$(f, v) = \int_0^1 fv\,dx$.

(W)：求 u 满足

$a(u,v) = (f,v)$，对 $\forall v \in \tilde{V}$ 成立.

3. 令双线性型 $a(u,v) = \int_0^1 (au'v' + uv)\,dx$,

线性空间 $V = \{ w \mid w \in C_{[0,1]},\ w' \text{分段连续有界},\ w(0) = w(1) = 0 \}$

此时 $\int_0^1 fv\,dx = \int_0^1 -v(au')' + uv\,dx$

$$= -\left(v(x)a(x)u'(x)\right)\Big|_0^1 + \int_0^1 au'v'\,dx + \int_0^1 uv\,dx$$

$$= \int_0^1 au'v'\,dx + \int_0^1 uv\,dx = a(u,v)$$

(M)：求 $u \in V$, s.t. $a(u,v) = (f,v)$, $\forall v \in V$

4. 已知有限元方法应用在变分问题上有：

$$V_n = \mathrm{span}\{\phi_1, \phi_2, \cdots \phi_N\}$$

$$u_n = \sum_{j=1}^{N} u_n(x_j)\phi_j(x) \in V_n.$$

对 $\forall v \in V_n$

$$a(u_n, v_n) = \int_0^1 u_n' v_n' dx = \sum_{i=1}^{N}\sum_{j=1}^{N} v_n(x_j) m(\phi_i,\phi_j) u_n(x_i) = \sum_{i=1}^{N}\sum_{j=1}^{N} v_n(x_j)\, a(\phi_i,\phi_j)\, u_n(x_i) = V^T A U$$

其中 $a(\phi_i,\phi_j) = \int_0^1 \phi_i' \phi_j' dx$, $A = (a(\phi_i,\phi_j))$, $U = \begin{pmatrix} u_n(x_1) \\ \vdots \\ u_n(x_N) \end{pmatrix}$, $V = \begin{pmatrix} v_n(x_1) \\ \vdots \\ v_n(x_N) \end{pmatrix}$

$$(f,v) = \int_0^1 f v dx = \sum_i^N \sum_j^N \int_0^1 v_n(x_j)\phi_j(x)\phi_i(x) f_i dx = V^T M F$$

其中 $m(\phi_i,\phi_j) = \int_0^1 \phi_i \phi_j dx$, $M = (m(\phi_i,\phi_j))$

以上：变分问题在有限元方法下写作.

求解 u_n, 使得对 $\forall v_n \in V$, 满足

$V^T A U = V^T M F$ 成立.

$\Leftrightarrow$ 求解 $AU = MF$.