Homework 5. Week 9.

2.3. 由 $u^{(4)}(x) = f$, $u(0) = u^{(2)}(0) = u^{(1)}(1) = u^{(3)}(1) = 0$

$$\Rightarrow \int_0^1 u^{(4)} \cdot v\,dx = -\int_0^1 u^{(3)} \cdot v^{(1)} dx + u^{(3)} \cdot v\Big|_0^1$$

$$= \int_0^1 u^{(2)} \cdot v^{(2)} dx + u^{(3)} \cdot v\Big|_0^1 - u^{(2)} \cdot v^{(1)}\Big|_0^1$$

So. def: $V = \{ v \in H^2, \ v(0) = v^{(1)}(1) = 0 \}$

对 $\forall v \in V$, $\exists! u \in V$

$$a(u, v) = \int_0^1 u^{(2)} \cdot v^{(2)} dx = \int_0^1 f \cdot v\,dx = L(v)$$

检验:

双线性: 由积分线性性

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w) \quad \checkmark$$

对称性: $a(u, v) = \int_0^1 u^{(2)} v^{(2)} dx = a(v, u) \quad \checkmark$

连续性 $a(u, v) = \int_0^1 u^{(2)} \cdot v^{(2)} dx \leq \|u^{(2)}\|_2 \cdot \|v^{(2)}\|_2 \leq \|u\|_{H^2} \cdot \|v\|_{H^2}$ 有界 $\Rightarrow$ 连续

强制性: $|u'(x)| = \left| \int_x^1 u''(t) dt \right| \leq \left| \int_x^1 |u''(t)| dt \right|$

$$\leq \int_0^1 |u''(x)|\,dx$$

$$\leq \left(\int_0^1 dx \right)^{\frac{1}{2}} \left(\int_0^1 |u''|^2 dx \right)^{\frac{1}{2}} = \|u''\|_2$$

$$\Rightarrow \|u'\|_2^2 = \int_0^1 |u'(x)|^2 dx \leq \int_0^1 \|u''\|_2^2 dx = \|u''\|_2^2$$

同理: $\|u\|_2 \leq \|u'\|_2$

故 $a(u, u) = \int_0^1 (u'')^2 dx \geq \frac{1}{3} \left(\|u\|_2^2 + \|u'\|_2^2 + \|u''\|^2 \right) = \frac{1}{3} \|u\|_{H_2} \quad \checkmark$

以上: 由 Lax-Milgram 定理, 解存在唯一

2.5 由 $\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g, & \text{on } \Gamma \end{cases}$

$$\int_\Omega (-\Delta u + u) v\,d\Omega = \int_\Omega \nabla u \cdot \nabla v\,d\Omega + \int_\Omega u \cdot v\,d\Omega - \int_\Gamma \frac{\partial u}{\partial n} v\, \vec{n} \cdot ds$$

$$= \int_\Omega f \cdot v\,d\Omega$$

$\Rightarrow$ 定义: $a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,d\Omega$

$$L(v) = \int_\Omega f v \, d\Omega + \int_\Gamma g v \, \vec{n} \cdot d\vec{s}$$

$$V = H^1$$

那么，有变分问题：

find $u \in V$, for $\forall v \in V$, satisfy,

$$a(u,v) = L(v)$$

对于 $a(u,v)$：

对称性：$a(u,v) = a(v,u)$

线性 $a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w)$

$\Big\}$ 双线性.

连续性 $a(u,v) = \int_\Omega (uv + \nabla u \cdot \nabla v) \, d\Omega \le \|u\|_2 \cdot \|v\|_2 + \|\nabla u\|_2 \cdot \|\nabla v\|_2$

$$\le (\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}} \cdot (\|v\|_2^2 + \|\nabla v\|_2^2)^{\frac{1}{2}}$$

$$= \|u\|_{H_1} \cdot \|v\|_{H_1} .$$

强制性：$a(u,u) = \int_\Omega (u^2 + \nabla u^2) \, d\Omega = \|u\|_H^2$

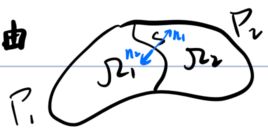
对于 f：

$$f(v) = \int_\Gamma v g \, ds + \int_\Omega f v \, ds$$

$$\le \|v\|_{L^2(\Gamma)} \|g\|_{L^2(\Gamma)} + \|v\|_{L^2(\Omega)} \cdot \|f\|_{L^2(\Omega)}$$

$$\le C \|v\|_{H_1(\Omega)} \cdot \|g\|_{L^2(\Gamma)} + \|v\|_{H_1} \|f\|_{L^2}$$

$$\le (C \|g\|_{L^2(\Gamma)} + \|f\|_{L^2(\Omega)}) \|v\|_{H_1(\Omega)}$$

2.7 由 

$$k(x) = \begin{cases} \chi_1 & x \in \Omega_1 \\ \chi_2 & x \in \Omega_2 \end{cases}$$

$$\Rightarrow -\int_\Omega k(x) \Delta u \cdot v \, d\Omega = -\int_{\Omega_1} \chi_1 v \Delta u \, d\Omega - \int_{\Omega_2} \chi_2 v \Delta u \, d\Omega$$

$$= \int_{\Omega_1} \chi_1 \nabla u \cdot \nabla v \, d\Omega + \int_{\Omega_2} \chi_2 \nabla u \cdot \nabla v \, d\Omega - \int_{\partial\Omega_1} \chi_1 v \frac{\partial u}{\partial n_1} \vec{n}_1 \cdot d\vec{s} - \int_{\partial\Omega_2} \chi_2 v \frac{\partial u}{\partial n_2} \vec{n}_2 \cdot d\vec{s}$$

$$= \int_\Omega k(x) \nabla u \cdot \nabla v - \int_{\Gamma_1} \chi_1 v \frac{\partial u}{\partial n_1} \vec{n}_1 \cdot ds - \int_S \chi_1 v \frac{\partial u}{\partial n_1} \vec{n}_1 \cdot d\vec{s} - \int_S \chi_2 v \frac{\partial u}{\partial n_2} \vec{n}_2 \cdot d\vec{s} - \int_{\Gamma_2} \chi_2 v \frac{\partial u}{\partial n_2} \vec{n}_2 \cdot d\vec{s}$$

$$= \int_\Omega k(x) \nabla u \cdot \nabla v - \int_{\Gamma_1+\Gamma_2} \left(k(x) \frac{\partial u}{\partial n}\right) v \vec{n} d\vec{s} - \int_S \left(\chi_1 \frac{\partial u_1}{\partial n} - \chi_2 \frac{\partial u_2}{\partial n}\right) v \, ds \quad [\vec{n}_1 = -\vec{n}_2]$$

考虑到：$-k(x) \Delta u = f$

所以：$\int_\Omega k(x) \nabla u \cdot \nabla v \, d\Omega - \int_S v \left(\chi_1 \frac{\partial u_1}{\partial n} - \chi_2 \frac{\partial u_1}{\partial n}\right) ds = \int_{\Gamma_2} g v \, ds + \int_\Omega f v \, d\Omega$

$\Leftrightarrow \quad a(u,v) - \int_S v(\chi_1 \frac{\partial u_1}{\partial n} - \chi_2 \frac{\partial u}{\partial n}) ds = L(v) \quad v \in V$

由条件: $\chi_1 \frac{\partial u_1}{\partial n} = \chi_2 \frac{\partial u_2}{\partial n}$ on S

$\Leftrightarrow \quad a(u,v) = L(v)$.

得变分问题 对 $\forall v \in V = H_1$, 找 $u \in V$, 得

$a(u,v) = L(v)$

$a(u,v) = \int_\Omega k(x) \nabla u \cdot \nabla v d\Omega, \quad L(v) = \int_{\Gamma_2} g v ds + \int_\Omega f v d\Omega$

2.x.12

$a(u,v) = \int_0^1 (u'v' + u'v + uv) dx, \quad V = \{v \in H^1, v(0) = u(1) = 0\}$

$\Rightarrow a(v,v) = \int_0^1 (v'^2 + v'v + v^2) dx$

考虑: $\int_0^1 v'v dx = \int_0^1 v dv = v^2 \Big|_0^1 - \int_0^1 v v' dx$

$\Rightarrow 2\int_0^1 v'v dx = 0$

故 $a(v,v) = \int_0^1 (v'^2 + v^2) dx$ 对 $v \in C^\infty$, $v(0) = v(1) = 0$ 成立.

由于 $C^\infty \cap H^1$ 在 H^1 中稠密,

故 $\forall v \in H^1$ 有, $\exists \{v_n\} \subset C^\infty$,

$\lim_{n\to\infty} v_n = v \Leftrightarrow \lim_{n\to\infty} \|v_n - v\| = 0$

由 $a(u,v)$ 连续性,

即对 $\forall v \in V$, 有 $a(v,v) = \int_0^1 (v^2 + v'^2) dx$ 成立