4.8

Let $\mathbb{V}_h$ be a finite element space on a trianglation T_h of the domain $\Omega \subset \mathbb{R}^d$ satisfying

$$\|u - \pi_h u\|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}$$

Given $u \in L_2(\Omega)$, let $u_h \in \mathbb{V}_h$ be the $L_2(\Omega)$ projection of u onto $\mathbb{V}_h$ i.e.

$$(u_h, v) = (u, v) \forall v \in \mathbb{V}_h$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$. Prove the error estimate:

$$\|u - u_h\|_{L_2(\Omega)} \leq \inf_{v \in \mathbb{V}_h} \|u - v\|_{L_2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}$$

and that

$$\|u_h\|_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}$$

Proof :

Let $v \in \mathbb{V}_h$ be arbitrary, then

$$(u - u_h, v) = 0 \quad \forall v \in \mathbb{V}_h$$

Let $e = u_h - v \in \mathbb{V}_h$, then $(e, v) = 0 \ \forall v \in \mathbb{V}_h$

$$\begin{aligned} \|u - u_h\|^2 &= (u - u_h, u - u_h) = (u - v + e, u - v + e) \\ &= (u - v, u - v) + (e, e) + 2(u - v, e) \\ &= (u - v, u - v) + (e, e) \\ &= \|u - v\|^2 + \|e\|^2 \end{aligned}$$

Therefore,

$$\|u - u_h\| \leq \inf_{v \in \mathbb{V}_h} \|u - v\|$$

Now, let $v = \pi_h u$, then

$$\inf_{v \in \mathbb{V}_h} \|u - v\| \leq \|u - \pi_h u\| \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}$$

Therefore,

$$\|u - u_h\| \leq \inf_{v \in \mathbb{V}_h} \|u - v\| \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}$$

Now, we prove that $\|u_h\|_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}$.

Because $u_h \in \mathbb{V}_h$, $(u - u_h, u) = 0$, then

$$\begin{aligned} \|u_h\|^2 - \|u\|^2 &= (u_h + u, u_h - u) \\ &= (u - u_h, u_h - u) = -\|u - u_h\|^2 \leq 0 \end{aligned}$$

Therefore, $\|u_h\|_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}$