8.2

Let η be the solution of 8.11 with $g=0$ given by 8.20:

$$\eta(t)=\sum_{j=1}^{M}(\eta^0,\xi^j)e^{-\mu_j t}\xi^j$$

(a) prove that

$$|\dot{\eta}|+\left|\bar{A}\eta(t)\right|\le\frac{C}{t}|\eta^0|$$

(b) Using (a) and the fact that $\left|\bar{A}\xi\right|\le Ch^{-2}|\xi|$ prove that:

$$\int_0^T\left(|\dot{\eta}(s)|+\left|\bar{A}\eta(s)\right|\right)ds\le C\left(1+\left|\log\frac{T}{h^2}\right|\right)|\eta^0|$$

Proof: (a) 令 $\eta_j^0=(\eta^0,\xi^j)$, 则

$$\begin{aligned}t\bar{A}\eta(t)&=\sum_{j=1}^{M}t(\eta^0,\xi^j)e^{-\mu_j t}\bar{A}\xi^j\\&=\sum_{j=1}^{M}(\eta^0,\xi^j)e^{-\mu_j t}t\mu_j\xi^j\\\Longrightarrow\left|t\bar{A}\eta(t)\right|^2&=\sum_{j=1}^{M}\left(\mu_j t\eta_j^0e^{-\mu_j t}\right)^2\end{aligned}$$

因为所有的 $\mu_j>0,t>0$，令 $x=\mu_j t$，对于所有的 $x>0$ 有$xe^{-x}\le\frac{1}{e}$，因此

$$\left|t\bar{A}\eta(t)\right|^2=\sum_{j=1}^{M}\left(\mu_j t\eta_j^0e^{-\mu_j t}\right)^2\le\frac{1}{e}\sum_j\eta_j^2=\frac{1}{e}|\eta^0|^2$$

另一方面:

$$|t\dot{\eta}|^2=\left|\sum_{j=1}^{M}(\eta^0,\xi^j)e^{-\mu_j t}t\mu_j\xi^j\right|^2\le\frac{1}{e}|\eta^0|^2$$

因此,

$$|t\dot{\eta}|^2+\left|t\bar{A}\eta(t)\right|^2\le2\frac{1}{e}|\eta^0|^2$$

利用均值不等式 $\frac{1}{2}(|a|+|b|)\le(a^2+b^2)^{\frac{1}{2}}$:

$$|\dot{\eta}|+\left|\bar{A}\eta(t)\right|\le\frac{C}{t}|\eta^0|$$

(b) 由(a)得: $|\dot{\eta}|=\left|\bar{A}\eta\right|$，那么:

$$\begin{cases}|\dot{\eta}|+\left|\bar{A}\eta(t)\right|\le\frac{C}{t}|\eta^0|\\|\dot{\eta}|+\left|\bar{A}\eta(t)\right|\le C\frac{|\eta^0|}{h^2}\end{cases}$$

当 $t<h^2$ 时，$\frac{1}{h^2}\le\frac{1}{t}$，因此:

$$
\begin{aligned}
\int_0^T \left(|\dot{\eta}(s)| + \left|\bar{A}\eta(s)\right|\right) ds &\leq C|\eta^0| \left(\int_0^{h^2} \frac{1}{h^2}\,\mathrm{d}s + \int_{h^2}^T \frac{1}{s}\,\mathrm{d}s \right) \\
&\leq C|\eta^0| \left(1 + \left| \log\left(\frac{T}{h^2}\right) \right| \right)
\end{aligned}
$$