8.6

Proof:

(a) 因为 $f = 0$,

$$(U^n - U^{n-1}, v) + k_n a(U^n, v) = 0 \quad \forall v \in \mathbb{V}_h, n = 1, ..., N$$

$$\begin{aligned}(U^n, U^n) - (U^{n-1}, U^{n-1}) &= (U^n + U^{n-1}, U^n - U^{n-1}) \\ &= -2k_n a(U^n, U^n) + k_n a(U^n, U^n - U^{n-1}) \\ &= -2k_n a(U^n, U^n) - (U^n - U^{n-1}, U^n - U^{n-1}) \\ &\leq 0\end{aligned}$$

这说明了:

$$\|U^n\| \leq \|U^{n-1}\| \leq ... \leq \|u^0\|$$

(b) 左侧求和号内的表达式为:

$$\begin{aligned}k_n \left\| \frac{U^n - U^{n-1}}{k_n} \right\|^2 &= -a(U^n, U^n - U^{n-1}) \\ &= -\left(\frac{1}{2} a(U^n + U^{n-1}, U^n - U^{n-1}) + \frac{1}{2} a(U^n - U^{n-1}, U^n - U^{n-1})\right) \\ &\leq -\frac{1}{2} a(U^n + U^{n-1}, U^n - U^{n-1}) \\ &= -\left(\frac{1}{2} a(U^n, U^n) - \frac{1}{2} a(U^{n-1}, U^{n-1})\right)\end{aligned}$$

令 $a^n = a(U^n, U^n)$:

$$\begin{aligned}\sum_{n=1}^{N} t_n k_n \left\| \frac{U^n - U^{n-1}}{k_n} \right\|^2 &= -\sum_{n=1}^{N} t^n (a^n - a^{n-1}) \\ &= -t^n a^n + \sum_{n=1}^{N-1} a^n (t^{n+1} - t^n) \\ &= -t^n a^n + \sum_{n=1}^{N-1} a^n k_n \\ &\leq \sum_{n=1}^{N-1} k_n a(U^n, U^n) \\ &= -\sum_{n=1}^{N-1} (U^n - U^{n-1}, U^n) \\ &= \sum_{n=1}^{N-1} \frac{1}{2}\left(\|U^n\|^2 + \|U^{n-1}\|^2\right) - \|U^n\|^2 \\ &= \frac{1}{2}\|U^0\|^2 - \|U^N\|^2 \leq C\|u^0\|^2\end{aligned}$$

因此:

$$\left(\sum_{n=1}^{N} t_n k_n \left\|\frac{U^n - U^{n-1}}{k_n}\right\|^2\right)^2 \le C\|u^0\|$$

(c)

$$\begin{aligned}
\left(\sum_{n=1}^{N} \left\|\frac{U^n - U^{n-1}}{k_n}\right\|\right)^2 &\le \left(\sum_{n=1}^{N} t_n k_n \left\|\frac{U^n - U^{n-1}}{k_n}\right\|\right)\left(\sum_{n=1}^{N} \frac{k_n}{t_n}\right) \\
&\le C^2\|u_0\|^2\left(1 + \sum_{n=2}^{N} \frac{k_n}{t_n}\right) \\
&\le C^2\|u_0\|^2\left(1 + \sum_{n=2}^{N} C\frac{k_{n-1}}{t_n}\right) \\
&= C^2\|u_0\|^2\left(1 + \int_{I_{n-1}} \frac{1}{t}\,\mathrm{d}t\right) \\
&= C^2\|u_0\|^2\left(1 + C\log\frac{T}{k_1}\right) \\
&\le C\left(1 + \log\frac{T}{k_1}\right)\|u_0\|^2
\end{aligned}$$

(d) 取 $V = U$，那么

$$\int_{I_n} \left(\dot{U}, U\right) + a(U,U)\,\mathrm{d}t = -(U_+^{n-1} - U_-^{n-1}, U_+^{n-1})$$

$$\Rightarrow \int_{I_n} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2\,\mathrm{d}t + \left\|U_+^{n-1}\right\|^2 \le (U_-^{n-1}, U_+^{n-1}) \le \frac{1}{2}\left\|U_-^{n-1}\right\|^2 + \frac{1}{2}\left\|U_+^{n-1}\right\|^2$$

$$\Rightarrow \frac{1}{2}\|U_-^n\|^2 - \frac{1}{2}\left\|U_+^{n-1}\right\|^2 + \left\|U_+^{n-1}\right\|^2 \le \frac{1}{2}\left\|U_-^{n-1}\right\|^2 + \frac{1}{2}\left\|U_+^{n-1}\right\|^2$$

$$\Rightarrow \|U_-^n\|^2 \le \left\|U_-^{n-1}\right\|^2 \le \dots \le \left\|u^0\right\|^2$$