MODULE TWO: MEASURING UNCERTAINTY; AND DRAWING CONCLUSIONS ABOUT POPULATIONS BASED ON SAMPLE DATA

TOPIC 4: PROBABILITY AND DISCRETE DISTRIBUTIONS

Learning Objectives

At the completion of this topic, you should be able to:

- recognise basic probability concepts
- calculate probabilities of simple, marginal and joint events
- calculate conditional probabilities and determine whether events are independent or not
- revise probabilities using Bayes' theorem
- use counting rules to calculate the number of possible outcomes
- recognise and use the properties of a probability distribution
- calculate the expected value and variance of a probability distribution
- identify situations that can be modelled by Binomial and Poisson distributions and calculate their probabilities

+4.1 Basic Probability Concepts

A **probability** is a numerical value that represents the *chance*, *likelihood* or *possibility* that a particular event will occur (always between 0 and 1)

There are 3 approaches to assigning a probability to an event:

1. a priori classical probability
   - based on prior knowledge
2. empirical classical probability
   - based on observed data
3. subjective probability
   - based on individual judgment or opinion about the probability of occurrence

+Events and Sample Spaces

Events

Simple event (denoted A)

- An outcome from a sample space with one characteristic

  e.g. planned to purchase TV

Complement of an event A (denoted A’)

- All outcomes that are not part of event A

  e.g. did not plan to purchase TV

Joint event (denoted $A \cap B$)

- Involves two or more characteristics simultaneously

  e.g. planned to purchase a TV and did actually purchase TV

+Events and Sample Spaces (cont)

Space

- The sample space is the collection of ALL possible events

  e.g. all 6 faces of a die

  all 52 playing cards

+Joint Probability

Mutually Exclusive Events

- Events that *cannot occur* together

e.g. Event A = Male

Event B = Event A’= Other

- Events A and B are mutually exclusive

Collectively Exhaustive Events

- One of the events *must occur*
- The set of events covers the entire sample space

e.g. member of loyalty program or not member of loyalty program

+Visualising Events

Contingency Tables

- Event A = Order > $50
- Event B = Member loyalty program

		Member loyalty program		
		Yes (B)	No (B')	Total
Order > $50				
Yes (A)		210	70	280
No (A')		110	110	220
Total		320	180	500

Joint Event

Sample space

+Visualising Events

Venn Diagrams

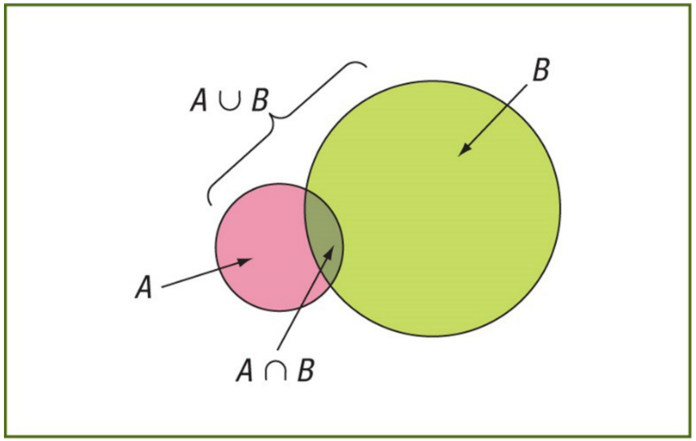


Figure 4.1 Venn diagram for events A and B

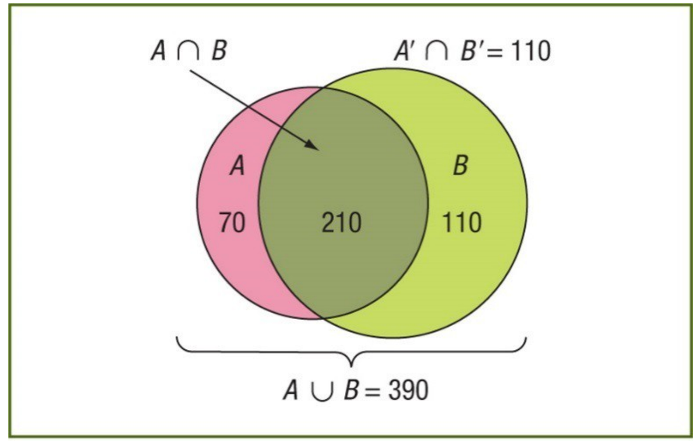


Figure 4.2 Venn diagram for Gaia Cruises scenario

+ Probability and Events

The probability of any event must be between 0 and 1, inclusively

$$\boxed{0 \le P(A) \le 1 \quad \text{For any event A}}$$

The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1

If A and B are mutually exclusive and collectively exhaustive

$$\boxed{P(A) + P(B) = 1}$$

+Computing Joint and Marginal Probabilities

The probability of a joint event, A and B:

$$P(\text{A and B}) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

+ Computing Joint Probability

Example:

P (Order >$50 AND member of loyalty program)

$$= \frac{210}{500}$$

		Member loyalty program		
		Yes (B)	**No (B')**	**Total**
Order > $50				
Yes (A)		210	70	280
No (A')		110	110	220
Total		320	180	500

+ Computing Marginal (or simple) Probability

$$P(A) = P(A \text{ and } B_1) + P(A \text{ and } B_2) + \cdots + P(A \text{ and } B_k)$$

where: B_1, B_2, …, B_k are k *mutually exclusive* and *collectively exhaustive* events

Example:

$$P\ (\text{Order} > \$50) = \frac{280}{500}$$

		Member loyalty program		
		Yes (B)	**No (B')**	**Total**
Order > $50				
Yes (A)		210	70	280
No (A')		110	110	220
Total		320	180	500

+Joint and Marginal Probabilities Using Contingency Tables

Event	Event: B	Event: B'	Total
A	P(A and B)	P(A and B')	P(A)
A'	P(A' and B)	P(A' and B')	P(A')
Total	P(B)	P(B')	1

Joint Probabilities

Marginal (Simple) Probabilities

+General Addition Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are mutually exclusive, then

P(A and B) = 0, so the rule can be simplified

$$P(A \text{ or } B) = P(A) + P(B)$$

+General Addition Rule (cont)

Example:

P(Order > $50 OR member of loyalty program)

$P(A) + P(B) - P(A \cap B)$

$= 280 / 500 + 320 / 500 - 210 / 500$

$= 390 / 500$

		Member loyalty program		
		Yes (B)	No (B')	Total
Order > $50				
Yes (A)		210	70	280
No (A')		110	110	220
Total		320	180	500

Note: $P(A \cap B)$ is (double) counted in both P(A) AND P(B) so we must subtract it

+Conditional Probability

A conditional probability is the probability of one event, *given that* another event has occurred

- The conditional probability of A *given that* B has occurred

$$\boxed{P(A|B) = \frac{P(A \text{ and } B)}{P(B)}}$$

- The conditional probability of B *given that* A has occurred

$$\boxed{P(B|A) = \frac{P(A \text{ and } B)}{P(A)}}$$

where P(A and B) = joint probability of A and B
P(A) = marginal probability of A
P(B) = marginal probability of B

+Calculating Conditional Probabilities

Example:

P (member of loyalty program | order > $50

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{210/500}{280/500} = 0.75$$

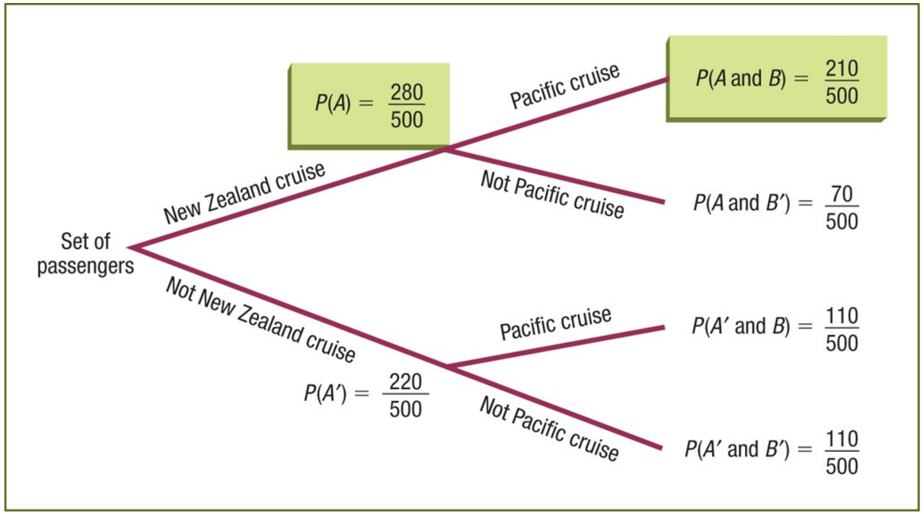
		Member loyalty program		
		Yes (B)	No (B')	Total
Order > $50				
Yes (A)		210	70	280
No (A')		110	110	220
Total		320	180	500

+Decision Trees

A Decision Tree:

- is an alternative to contingency tables
- allows sequential events to be graphed
- allows calculation of joint probabilities by multiplying respective branch probabilities

Figure 4.3 Decision tree for Gaia Cruises scenario



+Statistical Independence

Two events are independent if *and only if*:

$$\boxed{P(A \mid B) = P(A)}$$

$$\boxed{\text{or } \; P(B \mid A) = P(B)}$$

Events A and B are independent when the probability of one event is not affected by the other event

+Multiplication Rules

Multiplication rule for two events A and B:

$$\boxed{P(A \text{ and } B) = P(A \mid B)\,P(B)}$$

Note: If A and B are independent then:

$$\boxed{P(A \mid B) = P(A)}$$

and the multiplication rule simplifies to:

$$\boxed{P(A \text{ and } B) = P(A)P(B)}$$

Marginal Probability Using the General Multiplication Rule

Marginal probability for event A:

$$\boxed{P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k)}$$

where: B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

+Bayes' Theorem

A technique used to revise previously calculated probabilities with the addition of new information

Need to identify:

- Prior probabilities $P(S_i)$ where $i = 1, \ldots, k$
- Conditional probabilities $P(F|S_i)$ where $i = 1, \ldots, k$

Then we can calculate:

- Joint probabilities $P(F \cap S)$
- Revised probabilities $P(S_i|F)$ where $i = 1, \ldots, k$

+Bayes' Theorem (cont)

Example:

Suppose a Consumer Electronics Company is considering marketing a new model of television. In the past, 40% of the televisions introduced by the company have been successful and 60% have been unsuccessful.

Before introducing a television to the marketplace, the marketing research department always conducts an extensive study and releases a report, either favourable or unfavourable. In the past, 80% of the successful televisions had received a *favourable* market research report and 30% of the unsuccessful televisions had received a *favourable* report.

For the new model of television under consideration, the marketing research department has issued a *favourable* report. What is the probability that the television will be successful, given this favourable report?

+Bayes' Theorem (cont)

Event S_i	Prior probability $P(S_i)$	Conditional probability $P(F \mid S_i)$	Joint probability $P(F \mid S_i)P(S_i)$	Revised probability $P(S_i \mid F)$
S = successful television set	0.40	0.80	0.32	$0.32/0.50 = 0.64 = P(S \mid F)$
S' = unsuccessful television set	0.60	0.30	0.18	$0.18/0.50 = 0.36 = P(S' \mid F)$
			0.50	

Table 4.3 Bayes' theorem calculations for the television-marketing example

where: S = successful television

S' = unsuccessful television (i.e. the complement of S)

F = favourable report

F' = unfavourable report (i.e. the complement of F)

+Counting Rules

Counting Rule 1:

- If any one of *k* different mutually exclusive and collectively exhaustive events can occur on each of *n* trials, the number of possible outcomes is equal to:

$$\boxed{k^n}$$

Example:

- Suppose you toss a coin 5 times. What is the number of different possible outcomes (i.e. the sequence of heads and tails)

Answer:

- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ or $(2)(2)(2)(2)(2) = 32$ possible outcomes

+Counting Rules

Counting Rule 2:

- If there are k_1 events on the first trial, k_2 events on the second trial, … and k_n events on the n^{th} trial, the number of possible outcomes is:

$$\boxed{(k_1)(k_2)\ldots(k_n)}$$

Example:

- Standard New South Wales vehicle registration plates previously consisted of 3 letters followed by 3 digits. How many possible combinations were there?

Answer:

- 26 x 26 x 26 x 10 x 10 x 10 = 26^3 x 10^3 = 17,576,000 possible outcomes

+Counting Rules

Counting Rule 3:

- The number of ways that n items can be arranged in order is:

$$\boxed{n! = (n)(n-1)\ldots(1)}$$

Example:

- If a set of 6 textbooks are to be placed on a shelf, in how many ways can the 6 books be arranged?

Answer:

- $6! = (6)(5)(4)(3)(2)(1) = 720$ possible outcomes

+Counting Rules

Counting Rule 4 - Permutations:

- The number of ways of arranging *X* objects selected from *n* objects in order is:

$$ {}_nP_x = \frac{n!}{(n-X)!} $$

Example:

- If there are 6 textbooks but room for only 4 books on a shelf, in how many ways can these books be arranged on the shelf?

$$ {}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360 $$

Answer:

- 360 different permutations

+Counting Rules

Counting Rule 5 - Combinations:

- The number of ways of arranging *X* objects selected from *n* objects irrespective of order is:

$$\boxed{{}_{n}C_{x} = \frac{n!}{X!(n-X)!}}$$

Example:

- How many ways can you choose 4 textbooks out of the 6 to place on a shelf?

$$\boxed{{}_{6}C_{4} = \frac{6!}{4!(6-4)!} = \frac{720}{(24)(2)} = 15}$$

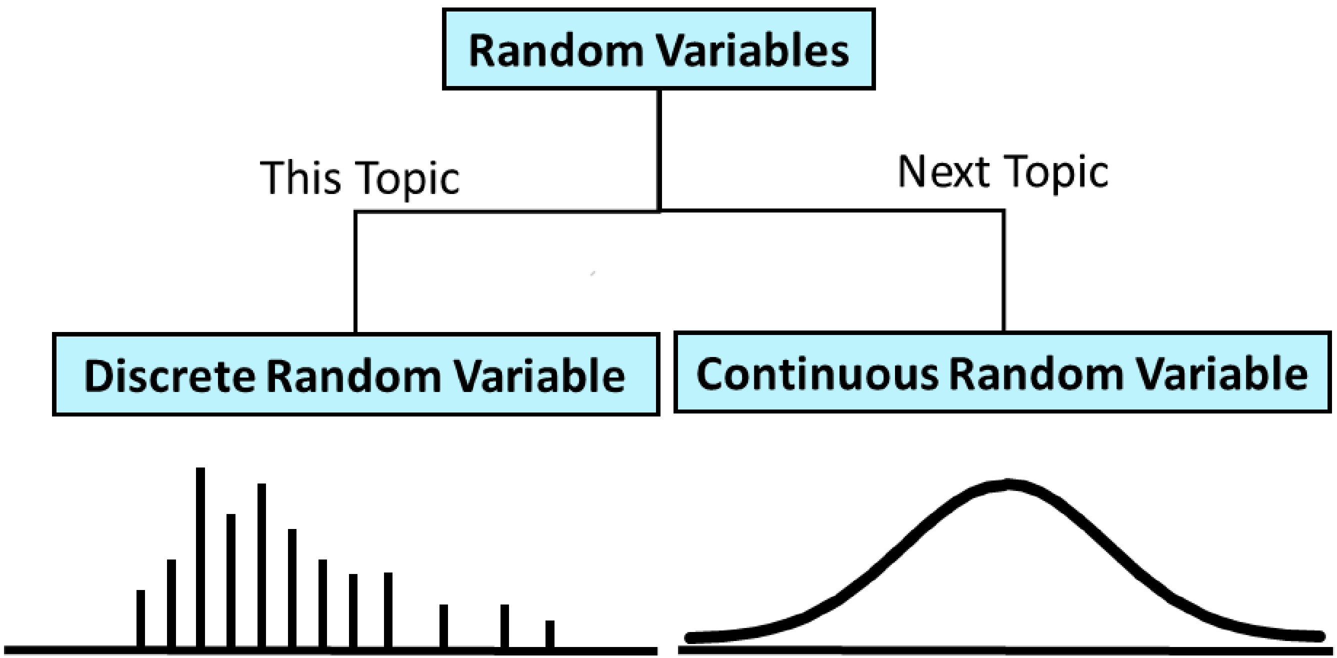
Answer:

- 15 different combinations

+Introduction to Probability Distributions

Random variable

- Represents a possible numerical value from an uncertain event



+Probability Distribution for a Discrete Random Variable

A probability distribution for a **discrete random variable** is a *mutually exclusive* list of all possible numerical outcomes of the random variable with the *probability of occurrence* associated with each outcome

Home mortgages approved per week	Probability
0	0.10
1	0.10
2	0.20
3	0.30
4	0.15
5	0.10
6	0.05

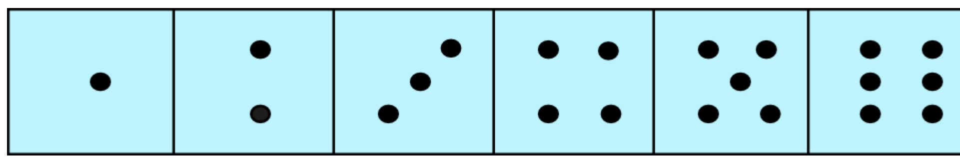
Table 5.1
Probability distribution of the number of home mortgages approved per week

+Discrete Random Variable

Can only assume a countable number of values

Examples:

- Roll a dice twice. Let X be the number of times 4 comes up; thus X could be 0, 1, or 2 times

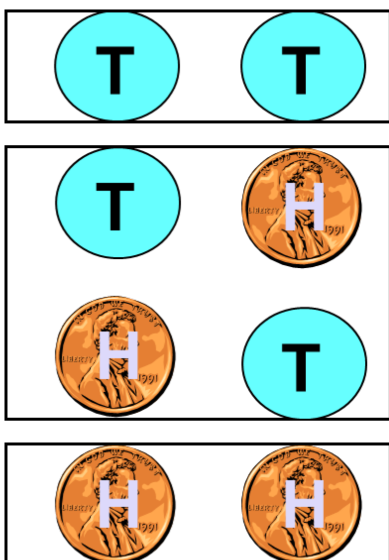


- Toss a coin five times. Let X be the number of heads; thus X could = 0, 1, 2, 3, 4, or 5

+Discrete Probability Distribution

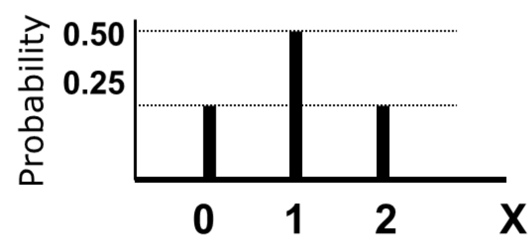
Experiment: Toss 2 Coins. Let X = # heads

4 possible outcomes



Probability Distribution

X	Probability
0	1/4 = 0.25
1	2/4 = 0.50
2	1/4 = 0.25



+Expected Value of a Discrete Random Variable

Expected value (or mean) of a discrete random variable (weighted average)

$$\mu = E(X) = \sum_{i=1}^{N} X_i P(X_i)$$

Toss 2 coins, X = # of heads, calculate expected value of X:

E(X) = (0 x 0.25) + (1 x 0.50) + (2 x 0.25) = 1.0

X	P(X)
0	0.25
1	0.50
2	0.25

+Variance and Standard Deviation of a Discrete Random Variable

Example:

- Toss two coins, X = # heads, calculate the variance, σ^2, and standard deviation, σ (from previous slide, E(X) = 1)

$$\sigma^2 = \sum_{i=1}^{N} X_i^2 P(X_i) - E(X)^2$$

$$\sigma^2 = \left(0^2 * 0.25 + 1^2 * 0.5 + 2^2 * 0.25\right) - \left(1^2\right) = 0.5$$

$$\text{Standard deviation } \sigma = \sqrt{\sigma^2} = \sqrt{0.5} = 0.707$$

where: E(X) = expected value of the discrete random variable X

X_i = the i^{th} outcome of the discrete random variable X

$P(X_i)$ = probability of the i^{th} occurrence of X

+Variance and Standard Deviation of a Discrete Random Variable

Home mortgages approved per week X_i	$P(X_i)$	$X_iP(X_i)$	$X_i^2P(X_i)$
0	0.10	0.0	0.0
1	0.10	0.1	0.1
2	0.20	0.4	0.8
3	0.30	0.9	2.7
4	0.15	0.6	2.4
5	0.10	0.5	2.5
6	0.05	0.3	1.8
	1.00	$\mu = E(X) = 2.8$	10.3

Table 5.2 Calculating the mean and variance of the number of home mortgages approved per week

+Binomial Distribution

A binomial distribution can be thought of as simply the probability of a SUCCESS or FAILURE outcome in an experiment or survey that is repeated multiple times

The binominal distribution is a mathematical model

Possible binominal scenarios:

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of ‘yes, I will buy’ or ‘no, I will not’
- A new job applicant either accepts the offer or rejects it

+Binomial Distribution

There are 4 essential properties of the binominal distribution:

A fixed number of observations, or trials, *n*

- e.g. 15 tosses of a coin; 10 light bulbs taken from a warehouse

Two mutually exclusive and collectively exhaustive categories

- e.g. head or tail in each toss of a coin; defective or not defective light bulb
- generally called ‘success’ and ‘failure’
- probability of success is p, probability of failure is 1–p

+Binomial Distribution (cont)

Constant probability for each observation

- e.g. probability of getting a tail is the same each time we toss the coin

Observations are independent

- the outcome of one observation does not affect the outcome of the other
- two sampling methods can be used to ensure independence; either:

  - selected from infinite population without replacement; or

  - selected from finite population with replacement

+Binomial Distribution

The Binomial Distribution Formula

$$P(X) = \frac{n!}{X!(n-X)!} p^X (1-p)^{n-X}$$

where:

P(X) = probability of X successes in n trials, with the probability of success p on each trial

X = number of 'successes' in sample, (X = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

p = probability of 'success'

1-p = probability of failure

+Binomial Distribution

Example:

A customer has a *35%* probability of making a purchase. *Ten* customers enter the shop. What is the probability of *three* customer making a purchase?

Let x = # customer purchases:
where:
$n = 10$
$p = 0.35$
$1-p = (1-0.35) = 0.65$
$X = 3$

$$
\begin{aligned}
P(X=3) &= \frac{n!}{X!(n-X)!} p^X (1-p)^{n-X} \\
&= \frac{10!}{3!(10-3)!} (0.35)^3 (1-0.35)^{10-3} \\
&= (120)(0.35)^3 (0.65)^7 \\
&= (120)(0.042875)(0.04902227890625) \\
&= 0.2522
\end{aligned}
$$

+Binomial Distribution

n = 10									
x	…	p=.20	p=.25	p=.30	**p=.35**	p=.40	p=.45	p=.50	
0	…	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010	10
1	…	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098	9
2	…	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439	8
3	…	0.2013	0.2503	0.2668	**0.2522**	0.2150	0.1665	0.1172	7
4	…	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051	6
5	…	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461	5
6	…	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051	4
7	…	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172	3
8	…	0.0001	**0.0004**	0.0014	0.0043	0.0106	0.0229	0.0439	**2**
9	…	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098	1
10	…	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0
	…	p=.80	**p=.75**	p=.70	p=.65	p=.60	p=.55	p=.50	**x**

n = 10, p = 0.35, x = 3: $P(x = 3 | n = 10, p = 0.35) = 0.2522$

n = 10, p = 0.75, x = 2: $P(x = 2 | n = 10, p = 0.75) = 0.0004$

+Binomial Distribution

Table 5.4 Finding a binomial probability for $n = 4$, $X = 2$ and $p = 0.1$ (extracted from Table E.6)

			p		
n	X	**0.01**	**0.02**	....	$\boxed{\textbf{0.10}}$
4	0	0.9606	0.9224	....	0.6561
	1	0.0388	0.0753	....	0.2916
	$\boxed{2}$	0.0006	0.0023	....	$\boxed{0.0486}$
	3	0.0000	0.0000	....	0.0036
	4	0.0000	0.0000	....	0.0001

+Binomial Distribution

Characteristics of the Binomial Distribution

Mean

$$\boxed{\mu = E(x) = np}$$

Variance and standard deviation

$$\boxed{\sigma^2 = np(1-p)}$$

$$\boxed{\sigma = \sqrt{np(1-p)}}$$

Where: n = sample size
p = probability of success
(1 – p) = probability of failure

+Binomial Distribution

Figure 5.2
Microsoft Excel worksheet for calculating binomial probabilities

	A	B	C
3	**Data**		
4	Sample size	4	
5	Probability of success	0.1	
6			
7	**Statistics**		
8	Mean	0.4	=B4*B5
9	Variance	0.36	=B8*(1-B5)
10	Standard deviation	0.6	=SQRT(B9)
11			
12	Binomial probabilities table		
13	*X*	*P*(*X*)	
14	0	0.6561	=BINOM.DIST(A14,B4,B5,FALSE)
15	1	0.2916	=BINOM.DIST(A15,B4,B5,FALSE)
16	2	0.0486	=BINOM.DIST(A16,B4,B5,FALSE)
17	3	0.0036	=BINOM.DIST(A17,B4,B5,FALSE)
18	4	0.0001	=BINOM.DIST(A18,B4,B5,FALSE)

+Poisson Distribution

We can apply the Poisson distribution to calculate probabilities when counting the number of times a particular event occurs in an interval of time or space if:

- the probability an event occurs in any interval is the same for all intervals of the same size
- the number of occurrences of the event in one interval is independent of the number in any other interval
- the probability that two or more occurrences of the event in an interval approaches zero as the interval becomes smaller

+Poisson Distribution (cont)

Mean

$$\boxed{\mu = \lambda}$$

Variance and Standard Deviation

$$\boxed{\sigma^2 = \lambda}$$

$$\boxed{\sigma = \sqrt{\lambda}}$$

where: λ = expected number of events

+Poisson Distribution (cont)

The Poisson distribution has one parameter λ (lambda) which is the mean or expected number of events per interval

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{X!}$$

where:

P(X)	= the probability of X events in a given interval
λ	= expected number of events in the given interval
e	= base of the natural logarithm system (2.71828...)

+Poisson Distribution (cont)

X	λ								
	0.10	0.20	0.30	0.40	**0.50**	0.60	0.70	0.80	0.90
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	**0.0758**	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example: Find $P(X = 2)$ if $\lambda = 0.50$

$$P(X=2) = \frac{e^{-\lambda}\lambda^{X}}{X!} = \frac{e^{-0.50}(0.50)^{2}}{2!} = 0.0758$$

+Poisson Distribution (cont)

X	9.1	9.2	λ	10
0	0.0001	0.0001	....	0.0000
1	0.0010	0.0009	....	0.0005
2	0.0046	0.0043	....	0.0023
3	0.0140	0.0131	....	0.0076
4	0.0319	0.0302	....	0.0189
5	0.0581	0.0555	....	0.0378
6	0.0881	0.0851	....	0.0631
7	0.1145	0.1118	....	0.0901

Table 5.5
Calculating a Poisson probability for $\lambda = 10$ (extracted from Table E.7 in Appendix E of this book)

+Poisson Distribution (cont)

	A	B	C	D	E
3	**Data**				
4	Average/expected number of successes:				10
5					
6	Poisson probabilities table				
7	*X*	*P*(*X*)			
8	0	0.000045	=POISSON.DIST($A8,$E$4,FALSE)		
9	1	0.000454	=POISSON.DIST($A9,$E$4,FALSE)		
10	2	0.002270	=POISSON.DIST($A10,$E$4,FALSE)		
11	3	0.007567	=POISSON.DIST($A11,$E$4,FALSE)		
12	4	0.018917	=POISSON.DIST($A12,$E$4,FALSE)		
13	5	0.037833	=POISSON.DIST($A13,$E$4,FALSE)		
14	6	0.063055	=POISSON.DIST($A14,$E$4,FALSE)		
15	7	0.090079	=POISSON.DIST($A15,$E$4,FALSE)		
16	8	0.112599	=POISSON.DIST($A16,$E$4,FALSE)		
17	9	0.125110	=POISSON.DIST($A17,$E$4,FALSE)		
18	10	0.125110	=POISSON.DIST($A18,$E$4,FALSE)		
19	11	0.113736	=POISSON.DIST($A19,$E$4,FALSE)		
20	12	0.094780	=POISSON.DIST($A20,$E$4,FALSE)		
21	13	0.072908	=POISSON.DIST($A21,$E$4,FALSE)		
22	14	0.052077	=POISSON.DIST($A22,$E$4,FALSE)		
23	15	0.034718	=POISSON.DIST($A23,$E$4,FALSE)		
24	16	0.021699	=POISSON.DIST($A24,$E$4,FALSE)		
25	17	0.012764	=POISSON.DIST($A25,$E$4,FALSE)		
26	18	0.007091	=POISSON.DIST($A26,$E$4,FALSE)		
27	19	0.003732	=POISSON.DIST($A27,$E$4,FALSE)		
28	20	0.001866	=POISSON.DIST($A28,$E$4,FALSE)		

Figure 5.4 Microsoft Excel worksheet for calculating Poisson probabilities