Chapter 11: Analysis of variance

Learning objectives

After studying this chapter you should be able to:

1. explain basic concepts of experimental design
2. apply the one-way analysis of variance to test for differences between the means of several groups
3. construct and apply a randomised block design
4. conduct a two-way analysis of variance and interpret the interaction

11.1 (a) $df\ B = c - 1 = 4 - 1 = 3$
(b) $df\ W = n - c = 20 - 4 = 16$
(c) $df\ T = n - 1 = 20 - 1 = 19$

11.2 (a) $SSW = SST - SSB = 120 - 60 = 60$
(b) $MSB = \frac{SSB}{c-1} = \frac{60}{4-1} = 20$ new
(c) $MSW = \frac{SSW}{n-c} = \frac{60}{20-4} = 3.75$
(d) $F_{calc} = \frac{MSB}{MSW} = \frac{20}{3.75} = 5.33$ new

11.3 (a)

Source	Df	SS	MS	F
Among groups	3	60	20	5.33
Within groups	16	60	3.75	
Total	19	120		

(b) $F_{3, 16} = 3.24$
(c) Decision rule: If $F_{calc} > 3.24$, reject H_0.
(d) Decision: Since $F_{calc} = 5.33$ is greater than the critical bound 3.24, reject H_0.

11.4 (a) $df\ B = c - 1 = 6 - 1 = 5$
(b) $df\ W = n - c = 42 - 6 = 36$
(c) $df\ T = n - 1 = 42 - 1 = 41$

11.5

Source	Df	SS	MS	F
Among groups	6 − 1 = 5	(80) (5) = 400	80	80/23.33 = 3.43
Within groups	30 − 6 = 24	560	560/24 = 23.33	
Total	30 − 1 = 29	400 + 560 = 960		

11.6 (a) Decision rule: If $F_{calc} > 2.62$, reject H_0.
(b) Since $F_{calc} = 3.43$ is greater than the critical bound of $F_{5,24} > 2.62$, reject H_0.

(c) There are 5 degrees of freedom in the numerator and $n - c = 30 - 6 = 24$ degrees of freedom in the denominator. The critical value, $Q_\alpha = 4.17$.

(d) To perform the Tukey–Kramer procedure, the critical range is

$$Q_\alpha \sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)} = 4.17\sqrt{\frac{23.33}{2}\left(\frac{1}{5}+\frac{1}{5}\right)} = 9.01$$

11.7

Source of variation	**Degrees of freedom**	**Sum of squares**	**Mean square**	***F* calc**
Between groups	c − 1 = 4	400	100	8.036
Within groups	n − c = 45	560	12.444	
Total	n − 1 = 49	960	112.444	

11.8 (a) All of these tests should start with stating the hypotheses (see 11.9 as an example)

Source of variation	**Degrees of freedom**	**Sum of squares**	**Mean square**	***F* calc**
Between groups	c − 1 = 3	3,799,436	1,266.479	0.144
Within groups	n − c = 16	2,917.665	182.354	
Total	n − 1 = 19	6,717,101	1448.833	

(b) $F_{3,16} = 3.24$. Since $F_{calc} = 0.144 < 3.24$ do not reject the null. Conclude there is no significant difference in the average grades.

(c) Since there is not enough evidence of a difference in means it is inappropriate to perform the Tukey–Kramer procedure.

11.9 (a) $H_0 : \mu_1 = \mu_2 = \mu_3$

H_1: Not all the means are equal

ANOVA: Single Factor

SUMMARY

Groups	Count	Sum	Average	Variance
urban	6	25	4.166666667	2.9667
regional	6	39	6.5	5.1000
rural	6	58	9.666666667	2.2667

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	91.4444	2	45.7222	13.2742	0.0005	3.6823
Within Groups	51.6667	15	3.4444			
Total	143.1111	17				
			Level of significance			0.05

Decision rule: if $F_{calc} > 3.68$ reject H_0

Since $F_{calc} = 13.27 > 3.68$ reject the null. Conclude there is enough evidence of a difference in the mean unemployment rates

(b) The Tukey–Kramer procedure is used to establish which of the means are significantly different from one another.

$$Q_{U(c,n-c)} = \mathrm{Q}(3,15) = 3.67$$

Critical range$=Q_{U(c,n-c)}\sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)}=3.67\sqrt{\frac{3.4444}{2}\left(\frac{1}{6}+\frac{1}{6}\right)}=2.8707$

Pairwise comparisons

urban-regional	2.33
urban-rural	5.5
regional-rural	3.17

The average return is different between urban and rural, and regional and rural.

(c) $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$

H_1: not all variances are equal

Using absolute differences

urban	regional	Rural
1	1	0
2	1	1
2	2	3
2	4	1
1	1	1
1	2	0

ANOVA: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
urban	**6**	**9**	**1.5**	**0.3000**
regional	**6**	**11**	**1.833333333**	**1.3667**
rural	**6**	**6**	**1**	**1.2000**

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	**2.1111**	**2**	**1.0556**	**1.1047**	**0.3568**	**3.6823**
Within Groups	**14.3333**	**15**	**0.9556**			
Total	**16.4444**	**17**				

Decision rule: If $F_{calc} > 3.68$ reject H_0

Since $F_{calc} = 1.1047 < 3.68$ do not reject the null. Conclude there is no significant difference between three variances.

11.10 (a) $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ where 1 = Main, 2 = Satellite1, 3 = Satellite2, 4 = Satellite3

H_1: Not all μ_j are equal

Excel output:

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	6312.444	3	2104.148	6.371691	0.000859	2.769431
Within Groups	18493.09	56	330.2338			
Total	24805.54	59				

Decision Rule: If p-value < 0.05, reject H_0. Since p-value = 0.0009 < 0.05, reject the null hypothesis. There is enough evidence to conclude that there is a significant difference in the mean waiting time in the four locations

(b) PHStat output of the Tukey-Kramer procedure:

	Sample	**Sample**		Absolute	Std. Error	Critical	
Group	**Mean**	**Size**	**Comparison**	Difference	of Difference	Range	**Results**
1	69.8427	15	**Group 1 to Group 2**	28.588667	4.692077	17.548	**Means are different**
2	41.254	15	**Group 1 to Group 3**	13.281333	4.692077	17.548	**Means are not different**
3	56.5613	15	**Group 1 to Group 4**	17.870667	4.692077	17.548	**Means are different**
4	51.972	15	**Group 2 to Group 3**	15.307333	4.692077	17.548	**Means are not different**
			Group 2 to Group 4	10.718	4.692077	17.548	**Means are not different**
Other Data			**Group 3 to Group 4**	4.5893333	4.692077	17.548	**Means are not different**
Level of significance	0.05						
Numerator d.f.	4						
Denominator d.f.	56						
MSW	330.2338						
Q Statistic	3.74						

From the Tukey-Kramer procedure, there is a difference in mean waiting time between the main campus and Satellite1, and the main campus and Satellite3

(c) H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2$

H_1: At least one variance is different

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	310.979	3	103.6597	0.8201	0.4883	2.7694
Within Groups	7078.435	56	126.4006			
Total	7389.414	59				

Since the p-value = 0.4883 > 0.05, do not reject H_0. There is not enough evidence to conclude there is a significant difference in the variation in waiting time among the four locations.

Since $F_{calc} = 22.375 > 2.45$ reject the null. Conclude there is enough evidence of a difference in the variance of price of petrol between the days of the week.

11.11 (a) $H_0: \mu_1 = \mu_2 = \mu_3$

H_1: Not all the means are equal

ANOVA: Single Factor

SUMMARY

Groups	Count	Sum	Average	Variance
young	10	120	12	42.8889
mature	10	75	7.5	33.8333
old	10	163	16.3	69.3444

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	387.2667	2	193.6333	3.9770	0.0306	3.3541
Within Groups	1314.6000	27	48.6889			
Total	1701.8667	29				

Decision rule: If $F_{calc} > 3.35$ reject H_0

Since $F_{calc} = 3.977 > 3.65$ reject H_0. There is evidence that the mean number of sick days differs across the three age groups.

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(b) The Tukey–Kramer procedure is used to establish which means are significantly different from one another

$Q_{U(c,n-c)} = Q_{U(3,27)} = 3.49$

Critical range $= Q_{U(c,n-c)}\sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)} = 3.49\sqrt{\frac{48.68889}{2}\left(\frac{1}{10}+\frac{1}{10}\right)} = 7.7009$

Pairwise differences

Young–mature	4.5
Young–old	4.3
Mature–old	8.8

The means of mature and old are different.

(c) $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$
H_1: not all variances are equal

ANOVA: Single Factor

SUMMARY

Groups	Count	Sum	Average	Variance
young	10	52	5.2	13.9556
mature	10	51	5.1	4.9333
old	10	65	6.5	22.4444

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	12.2000	2	6.1000	0.4427	0.6469	3.3541
Within Groups	372.0000	27	13.7778			
Total	384.2000	29				

Decision rule: If $F_{calc} > 3.35$ reject H_0
Since $F_{calc} = 4427 < 3.35$ do not reject H_0. Conclude there is no significant difference in the variances across the 3 age groups.

(d) The results in (a) and (b) are valid as the assumption of equal variances is reasonable based on results in (c).

11.12 (a)

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F
Among groups	2	1.879	0.9395	8.7558
Within groups	297	31.865	0.1073	
Total	299	33.744		

(b) H_0: $\mu_1 = \mu_2 = \mu_3$ H_1: At least one mean is different.

Since F_{Stat} = 8.7558 > 3.00, reject H_0. There is evidence of a difference in the mean soft-skill score of the different groups

(c)

Tukey-KramerMultiple Comparisons							
	Sample	Sample		Absolute	Std. Error	Critical	
Group	Mean	Size	Comparison	Difference	of Difference	Range	Results
Nocourseworkinlea	3.29	109	**Group1toGroup2**	0.072	0.03298958	0.1092	**Means are notdifferent**
Certificate inleaders	3.362	90	**Group1toGroup3**	0.181	0.03190896	0.1056	**Means are different**
Degree inleadership	3.471	102	**Group2toGroup3**	0.109	0.03349763	0.1109	**Means are notdifferent**

Other Data	
Level of significance	0.05
Numerator d.f.	3
Denominator d.f.	297
MSW	0.1073
QStatistic	3.31

There is evidence of a difference in the mean soft-skill score between those who had no coursework in leadership and those who had a degree in leadership.

Since $F_{calc} = 1.0927 < 2.76$ do not reject the null. Conclude there is no significant difference in the variance of rating of the five advertisements.

(d)

A	B	C	D	E
15	16	8	5	12
18	17	7	6	19
17	21	10	13	18
19	16	15	11	12
19	19	14	9	17
20	17	14	10	14
18.000	17.667	11.333	9.000	15.333

The average rating of the advertisements C and D are not different and both have the lowest ratings. The advertisements C and D should not be used. Then, the advertisement E is not different compared with A, B and C. So, it should not be used.

11.13 (a)

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Front	6	36.400	6.067	2.715
Middle	6	12.400	2.067	0.427
Rear	6	22.400	3.733	2.011

ANOVA

Source of Variation	*SS*	*Df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	48.444	2	24.222	14.105	0.000	3.682
Within Groups	25.760	15	1.717			
Total	74.204	17				

Since $F_{calc} = 14.11 > 3.68$ reject the null. Conclude there is enough evidence of a difference in the mean sales volumes across three store aisle locations.

(b) The Tukey–Kramer procedure is used to establish which of the means are significantly different from one another.

$Q_{U(c,n-c)} = Q_{U(3,15)} = 3.67$

Critical range = $Q_{U(c,n-c)}\sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)} = 3.67\sqrt{\frac{1.717}{2}\left(\frac{1}{6}+\frac{1}{6}\right)} = 1.963$

Front-Middle	4.000
Front-Rear	2.333
Middle-Rear	-1.667

Because 4.000 > 1.963 and 2.333 > 1.963, conclude that there is a significant difference between the mean sales volumes between front location and middle location as well as between front location and rear location.

(c)

SUMMARY

Groups	Count	Sum	Average	Variance
abs_dif_front	6	7.6	1.267	0.875
abs_dif_middle	6	2.8	0.467	0.199
abs_dif_rear	6	6.8	1.133	0.603

ANOVA

Source of Variation	SS	Df	MS	F	P-value	F crit
Between Groups	2.204	2	1.102	1.973	0.174	3.682
Within Groups	8.380	15	0.559			
Total	10.584	17				

Since $F_{calc} = 1.973 < 3.682$ do not reject the null. Conclude there is no significant difference in the variance of sales volumes across different locations.

(d) The front aisle appears to be the best location for the sale of fluffy toys. The manager should consider switching the location of fluffy toys to the front aisle if it yields marginal profit that is greater than that of the product currently displayed at the front aisle.

11.14 (a)

SUMMARY

Groups	Count	Sum	Average	Variance
Design1	10	2365.810	236.581	110.407
Design2	10	2366.210	236.621	139.665
Design3	10	2437.200	243.720	43.163
Design4	10	2434.180	243.418	60.300

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	485.994	3	161.998	1.833	0.159	2.866
Within Groups	3181.820	36	88.384			
Total	3667.814	39				

Since $F_{calc} = 1.83 < 2.87$ do not reject the null. Conclude there is no significant difference in the mean distance travelled by the golf balls with different designs.

(b) Since there is not enough evidence of a difference in means it is inappropriate to perform the Tukey–Kramer procedure.

(c) Three assumptions needed in (a) are (i) samples are drawn randomly and independently, (ii) populations have normal distributions, and (iii) populations have equal variances.

(d)

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
abs_dif_des1	10	96.710	9.671	7.679
abs_dif_des2	10	103.030	10.303	24.026
abs_dif_des3	10	52.200	5.220	13.563
abs_dif_des4	10	64.660	6.466	13.859

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	181.487	3	60.496	4.093	0.013	2.866
Within Groups	532.142	36	14.782			
Total	713.629	39				

Since $F_{calc} = 4.09 > 2.87$ reject the null. Conclude there is enough evidence of a difference in the mean distance travelled by the golf balls with different designs.