MODULE THREE: DETERMINING CAUSE AND MAKING RELIABLE FORECASTS

TOPIC 8: SIMPLE LINEAR REGRESSION

+ Learning Objectives

At the completion of this topic, you should be able to:

- conduct a simple regression and interpret the meaning of the regression coefficients b_0 and b_1
- use regression analysis to predict the value of a dependent variable based on an independent variable
- assess the adequacy of your estimated model
- evaluate the assumptions of regression analysis
- make inferences about the slope and correlation coefficient
- comprehend the pitfalls in regression and ethical issues

+Introduction to Regression Analysis

Regression analysis is used to:

- predict the value of a dependent variable (Y) based on the value of at least one independent variable (X)
- explain the impact of changes in an independent variable on the dependent variable

Dependent variable (Y): the variable we wish to predict or explain (response variable)

Independent variable (X): the variable used to explain the dependent variable (explanatory variable)

+Types of Regression Models

Simple Linear Regression Model

Population Y intercept

Population slope coefficient

Independent variable

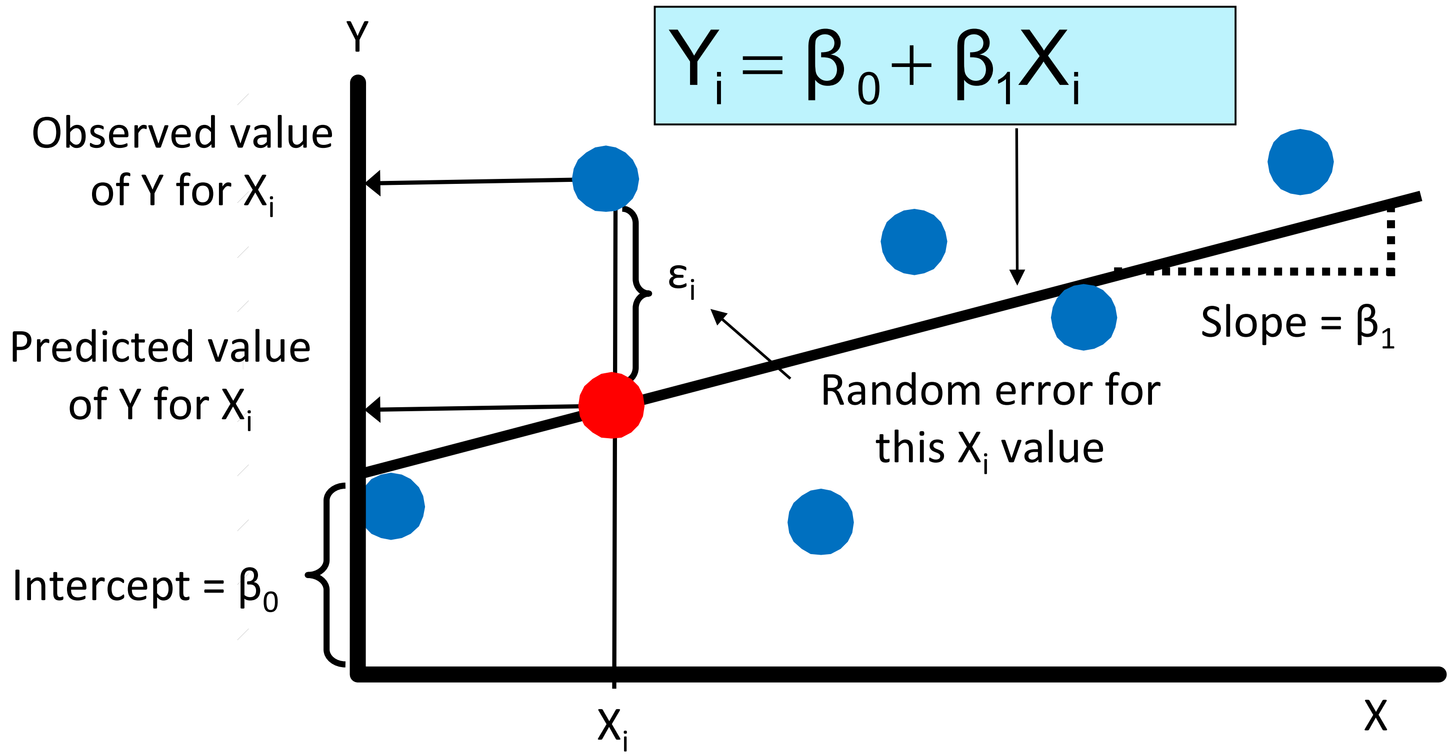
Random error term

$$Y_i = \underbrace{\beta_0 + \beta_1 X_i}_{\text{Linear component}} + \underbrace{\varepsilon_i}_{\text{Random error component}}$$

Dependent variable

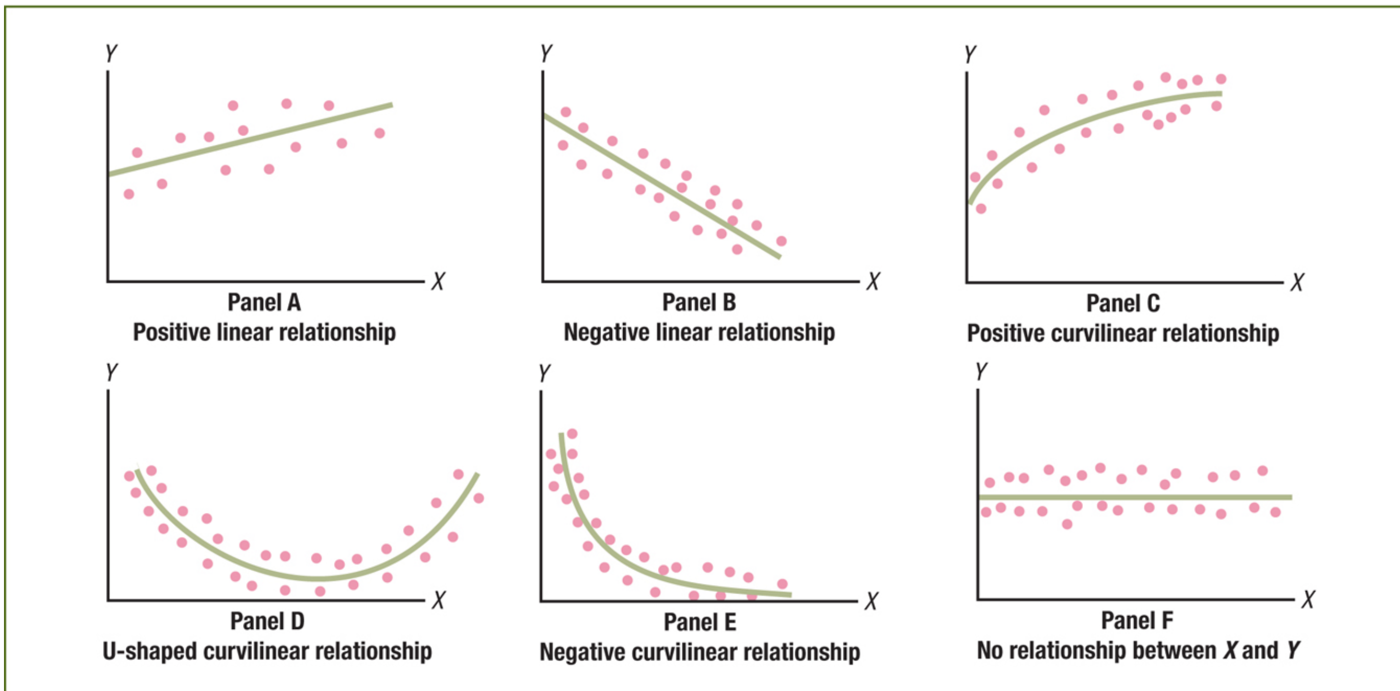
+Types of Regression Models

Simple Linear Regression Model



+Types of Regression Models (cont)

Figure 12.2
Examples of types of relationships found in scatter diagrams



+Simple Linear Regression

Simple linear regression:

- Only **one independent variable**, X
- Relationship between X and Y is described by a linear function
- Changes in Y are assumed to be caused by changes in X

+Simple Linear Regression Equation

The simple linear regression equation provides an estimate of the population regression line

Estimated (or predicted) Y value for observation i

Estimate of the regression intercept

Estimate of the regression slope

$$\hat{Y}_i = b_0 + b_1 X_i$$

Value of X for observation i

+Simple Linear Regression

Example:

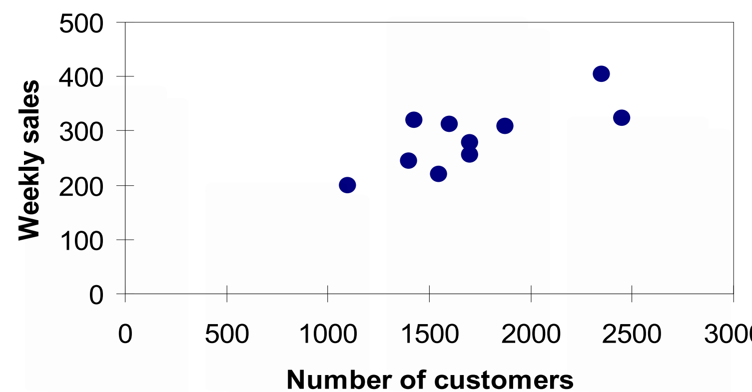
A manager of a local computer games store wishes to:

- examine the relationship between weekly sales (Y) and the number of customers making purchases (X) over a 10 week period; and
- use the results of that examination to predict future weekly sales

+Simple Linear Regression (Cont)

Weekly sales in $1,000s (Y)	Number of Customers (X)
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

Weekly sales model: scatter plot



+Simple Linear Regression (Cont)

The regression equation is:

$$\text{Weekly sales} = 98.24833 + 0.10977 \text{ (customers)}$$

Regression Statistics	
Multiple R	0.762113713
R Square	0.580817312
Adjusted R Square	0.528419476
Standard Error	41.33032365
Observations	10

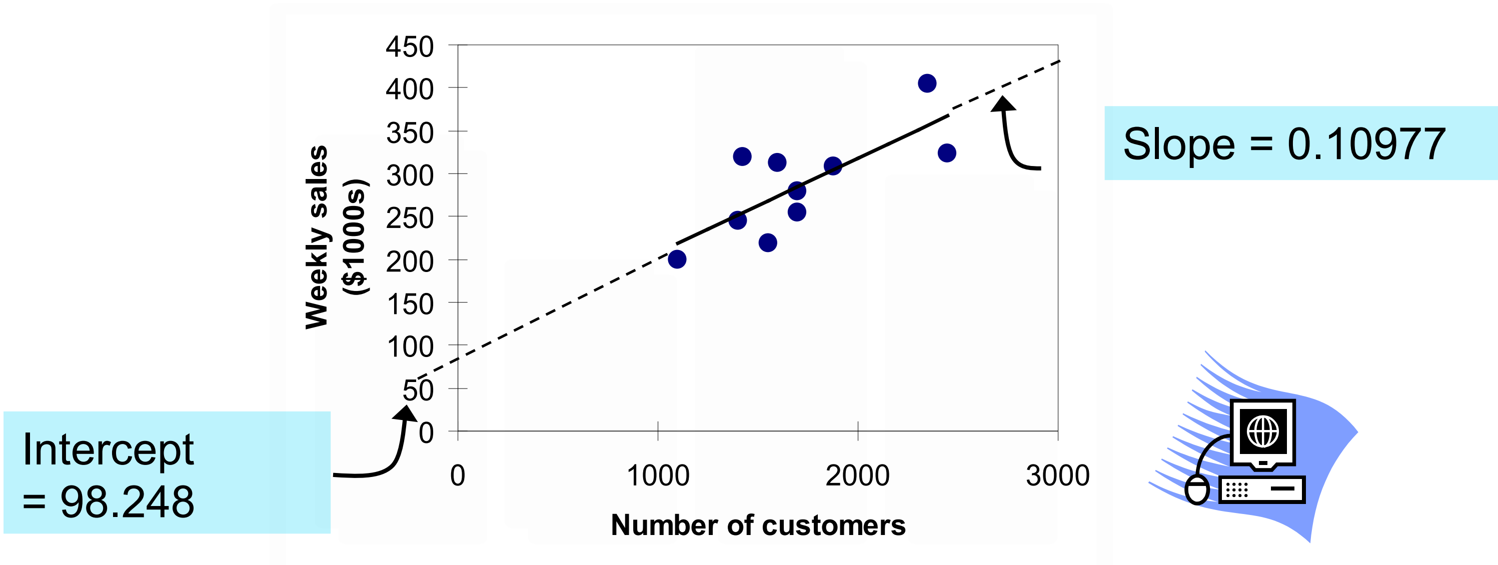
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	18934.93478	18934.93478	11.08475762	0.010394016
Residual	8	13665.56522	1708.195653		
Total	9	32600.5			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	98.24832962	58.03347858	1.692959513	0.128918812	-35.57711186	232.0737711
Number of customers	0.109767738	0.032969443	3.329377962	0.010394016	0.033740065	0.18579541

+Simple Linear Regression (Cont)

Weekly sales model: scatter plot and regression line



$$\widehat{\text{Weekly sales}} = 98.24833 + 0.10977\ (\text{customers})$$

+Simple Linear Regression (Cont)

$$\text{Weekly sales} = 98.24833 + 0.10977\,(\text{customers})$$

b_0 is the estimated average value of Y when the value of X is zero (if X = 0 is in the range of observed X values)

- Here, for no customers, b_0 = 98.2483 which appears nonsensical. However, the intercept simply indicates that over the sample size selected, the portion of weekly sales not explained by number of customers is $98,248.33. Also note that X=0 is outside the range of observed values

b_1 measures the estimated change in the average value of Y as a result of a one-unit change in X

- Here, b_1 = .10977 tells us that the average value of weekly sales increases by .10977($1,000) = $109.77, on average, for each additional customer

+Simple Linear Regression (Cont)

Predict the weekly sales for the local store for 2,000 customers:

$$\widehat{\text{Weekly sales}} = 98.25 + 0.1098\,(2000)$$
$$= 98.25 + 0.1098(2000)$$
$$= 317.85$$

The predicted weekly sales for the local computer games store for 2,000 customers is 317.85 ($1,000s) = $317,850

+The Least-Squares Method

b_0 and b_1 are obtained by finding the values of b_0 and b_1 that **minimise the sum of the squared differences** between actual values (Y) and predicted values ($\hat{Y}$)

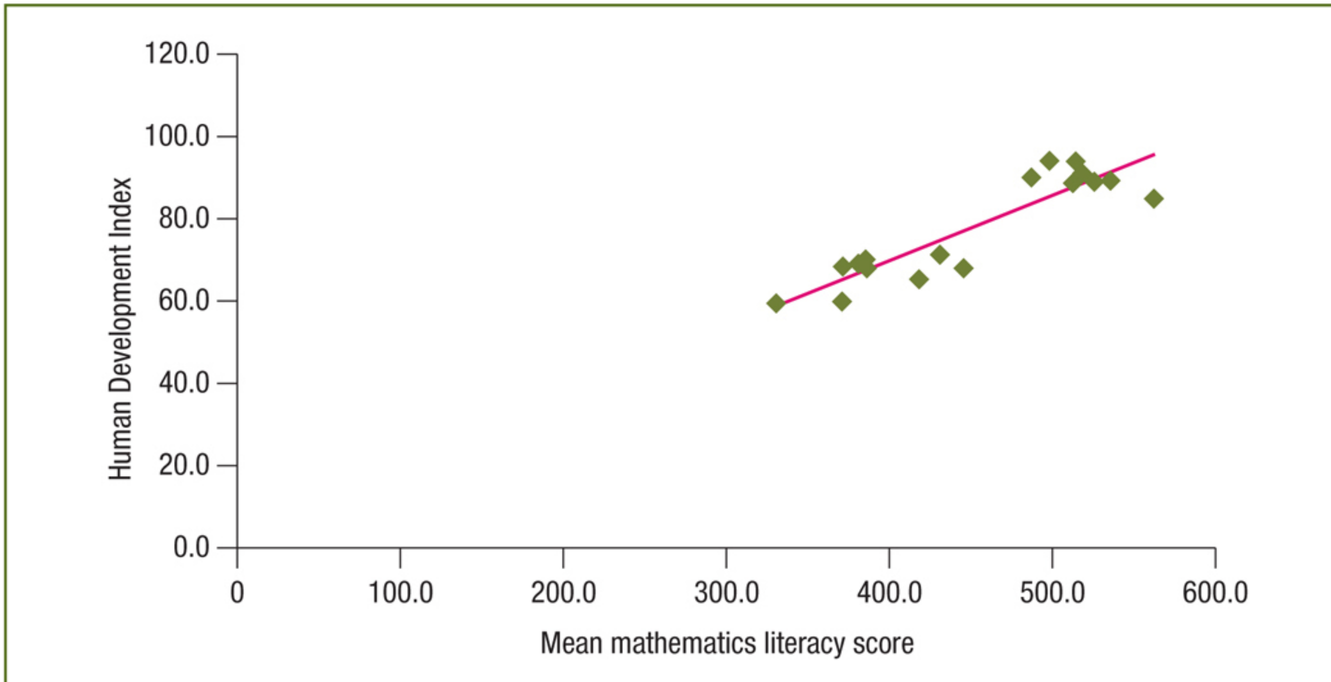
$$\boxed{\min \sum (Y_i - \hat{Y}_i)^2 = \min \sum (Y_i - (b_0 + b_1 X_i))^2}$$

b_0 is the estimated average value of Y when the value of X is zero

b_1 is the estimated change in the average value of Y as a result of a one-unit change in X

+The Least-Squares Method

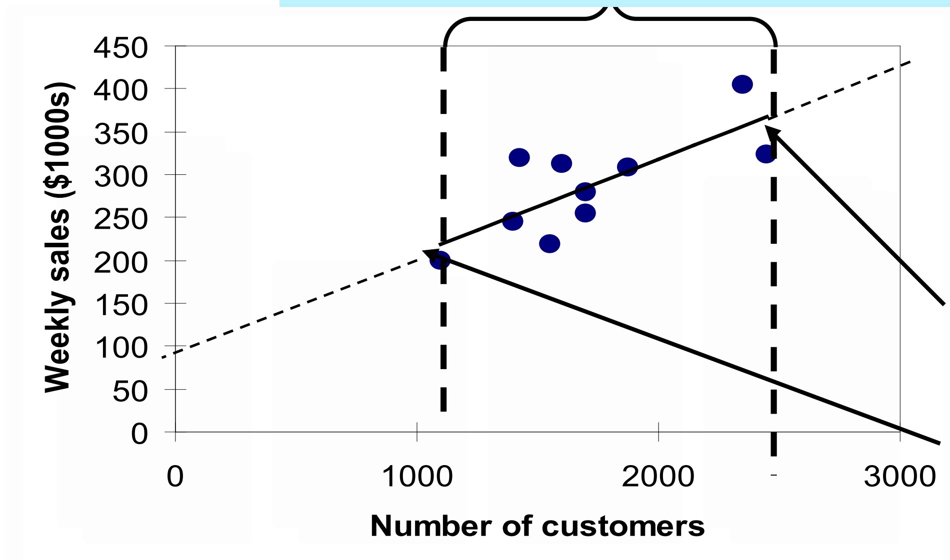
Figure 12.5
Microsoft Excel scatter diagram and prediction line for the Human Development Index data



+ Predictions in Regression Analysis: Interpolation versus Extrapolation

When using a regression model for prediction, only predict within the relevant range of data

Relevant range for interpolation



Do not try to extrapolate beyond the range of observed Xs

+Measures of Variation

Total variation is made up of two parts:

$$\text{SST} = \text{SSR} + \text{SSE}$$

Total Sum of Squares	Regression Sum of Squares	Error Sum of Squares
$\text{SST} = \sum (Y_i - \overline{Y})^2$	$SSR = \sum (\hat{Y}_i - \overline{Y})^2$	$SSE = \sum (Y_i - \hat{Y}_i)^2$
Measures the variation of the Y_i values around their mean Y	Explained variation attributable to the relationship between X and Y	Variation attributable to factors other than the relationship between X and Y

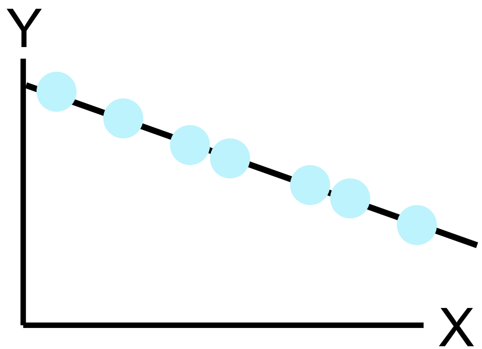
+The Coefficient of Determination, r^2

The Coefficient of Determination (r^2) is equal to the regression sum of squares (i.e. the explained variation) divided by the total sum of squares (i.e. the total variation)

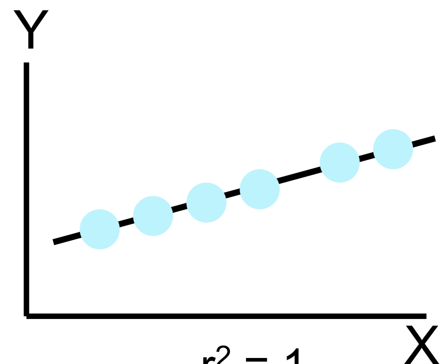
$$r^2 = \frac{\text{regression sum of squares}}{\text{total sum of squares}} = \frac{SSR}{SST}$$

It measures the proportion of the variation in Y that is explained by the Independent variable X in the regression model

+The Coefficient of Determination, r^2 (Cont)

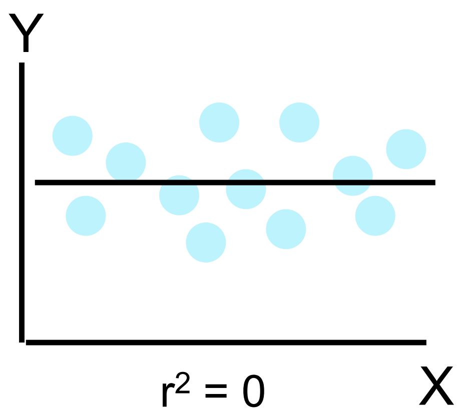


$r^2 = 1$



$r^2 = 1$

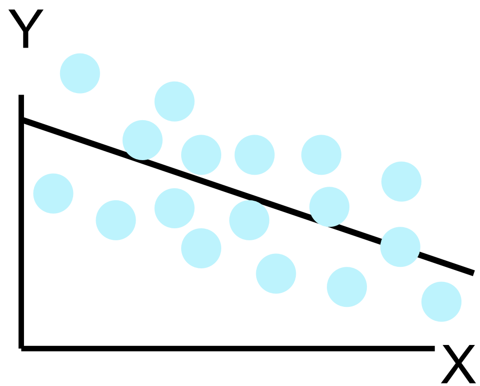
- Perfect linear relationship between X and Y
- 100% of the variation in Y is explained by variation in X

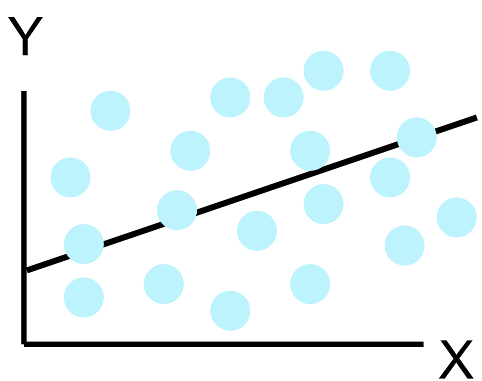


$r^2 = 0$

- No linear relationship between X and Y
- The value of Y does not depend on X (none of the variation in Y is explained by variation in X)

+The Coefficient of Determination, r^2 (Cont)





$0 < r^2 < 1$

Weaker linear relationships between X and Y:

Some, but not all, of the variation in Y is explained by variation in X

+The Coefficient of Determination, r^2 (Cont)

	A	B	C	D	E	F	G
1	*Regression Statistics*						
2	Multiple R	0.762113713					
3	R Square	0.580817312					
4	Adjusted R Square	0.528419476					
5	Standard Error	41.33032365					
6	Observations	10					
7							
8	ANOVA						
9		*df*	*SS*	*MS*	*F*	*Significance F*	
10	Regression	1	18934.93478	18934.93478	11.08475762	0.010394016	
11	Residual	8	13665.56522	1708.195653			
12	Total	9	32600.5				
13							
14		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
15	Intercept	98.24832962	58.03347858	1.692959513	0.128918812	-35.57711186	232.0737711
16	Number of customers	0.109767738	0.032969443	3.329377962	0.010394016	0.033740065	0.18579541

$$r^2 = \frac{SSR}{SST} = \frac{18934.9348}{32600.5000} = 0.58082$$

58.08% of the variation in weekly sales is explained by variation in number of customers

+Standard Error of the Estimate

The standard deviation of the variation of observations around the regression line is estimated by:

$$S_{YX} = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-2}}$$

Where:

SSE = error sum of squares

n = sample size

+Standard Error of the Estimate (Cont)

Excel Output:

	A	B	C	D	E	F	G
1	*Regression Statistics*						
2	Multiple R	0.762113713					
3	R Square	0.580817312					
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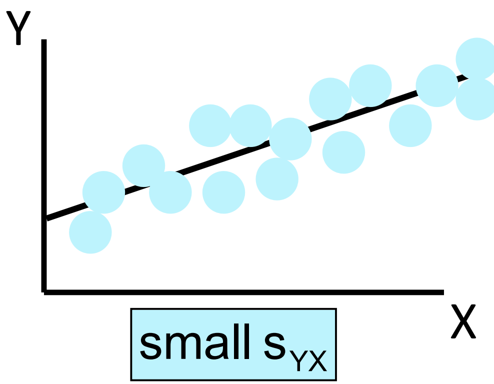
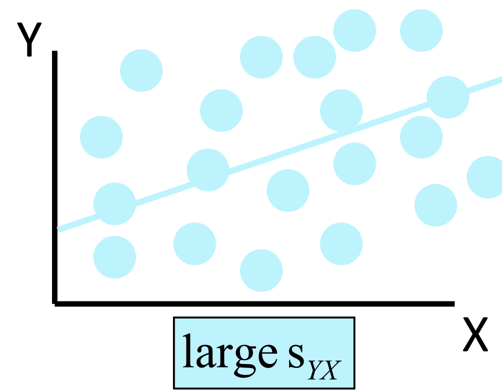
$$S_{YX} = 41.33032$$

+ Standard Error of the Estimate - Comparing Standard Errors

S_{YX} is a measure of the variation of observed Y values from the regression line





The magnitude of S_{YX} should always be judged relative to the size of the Y values in the sample data

i.e. S_{YX} = \$41.33K is moderately small relative to weekly sales in the \$200 - \$300K range

+Assumptions

Use the acronym LINE:

Linearity

- The underlying relationship between X and Y is linear

Independence of errors

- Error values are statistically independent

Normality of error

- Error values (ε) are normally distributed for any given value of X

Equal variance (homoscedasticity)

- The probability distribution of the errors has constant variance

+Residual Analysis

The residual for observation i, e_i, is the difference between its observed and predicted value

$$e_i = Y_i - \hat{Y}_i$$

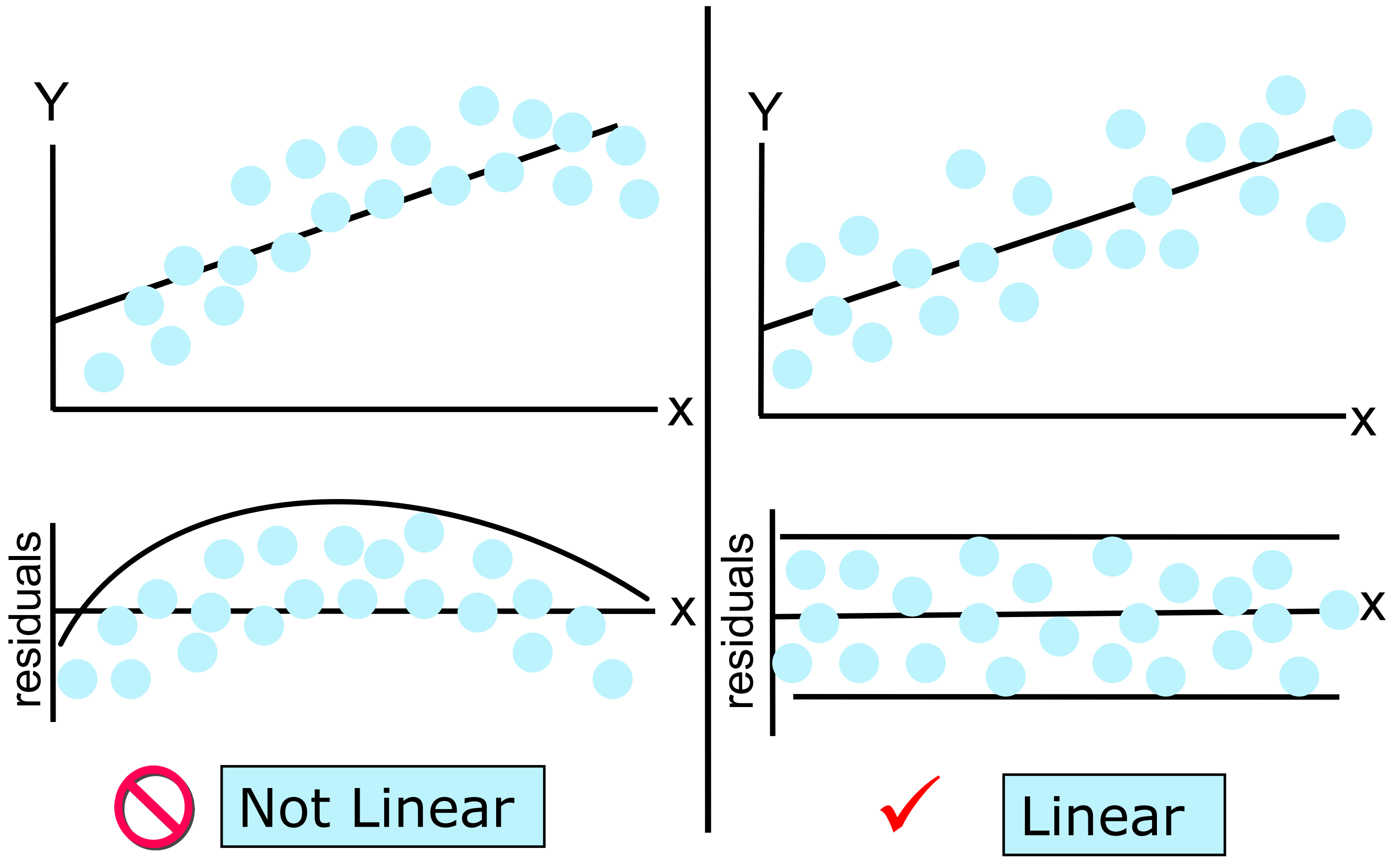
Check the assumptions of regression by examining the residuals:

- Examine for linearity assumption
- Evaluate independence assumption
- Evaluate normal distribution assumption
- Examine for constant variance for all levels of X (homoscedasticity)

Graphical Analysis of Residuals

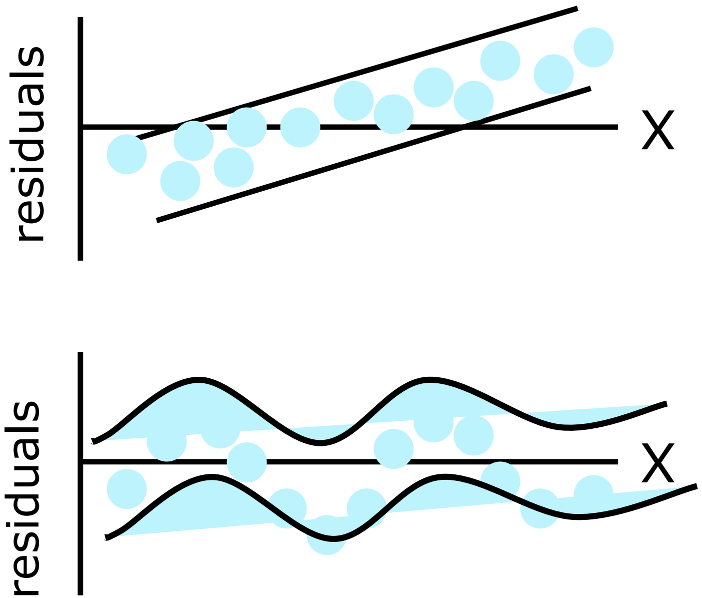
Can plot residuals vs. X

+Residual Analysis for Linearity

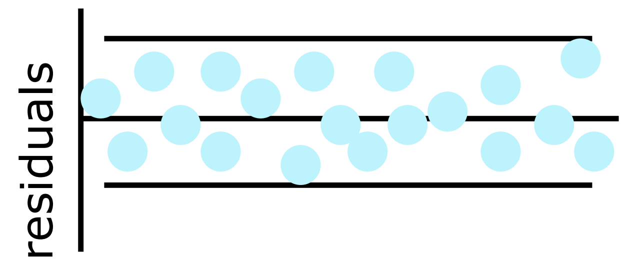


+Residual Analysis for Independence

Not Independent

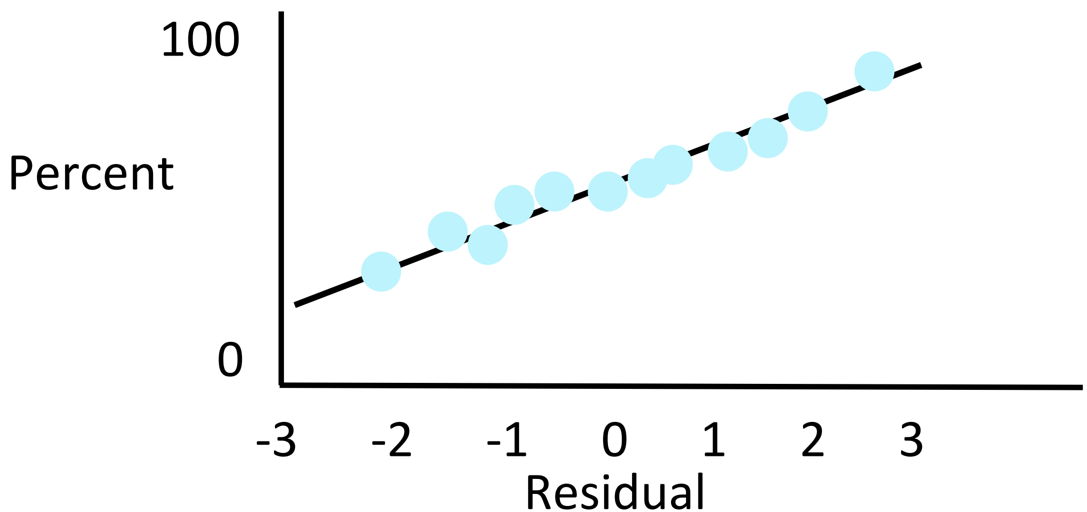


Independent

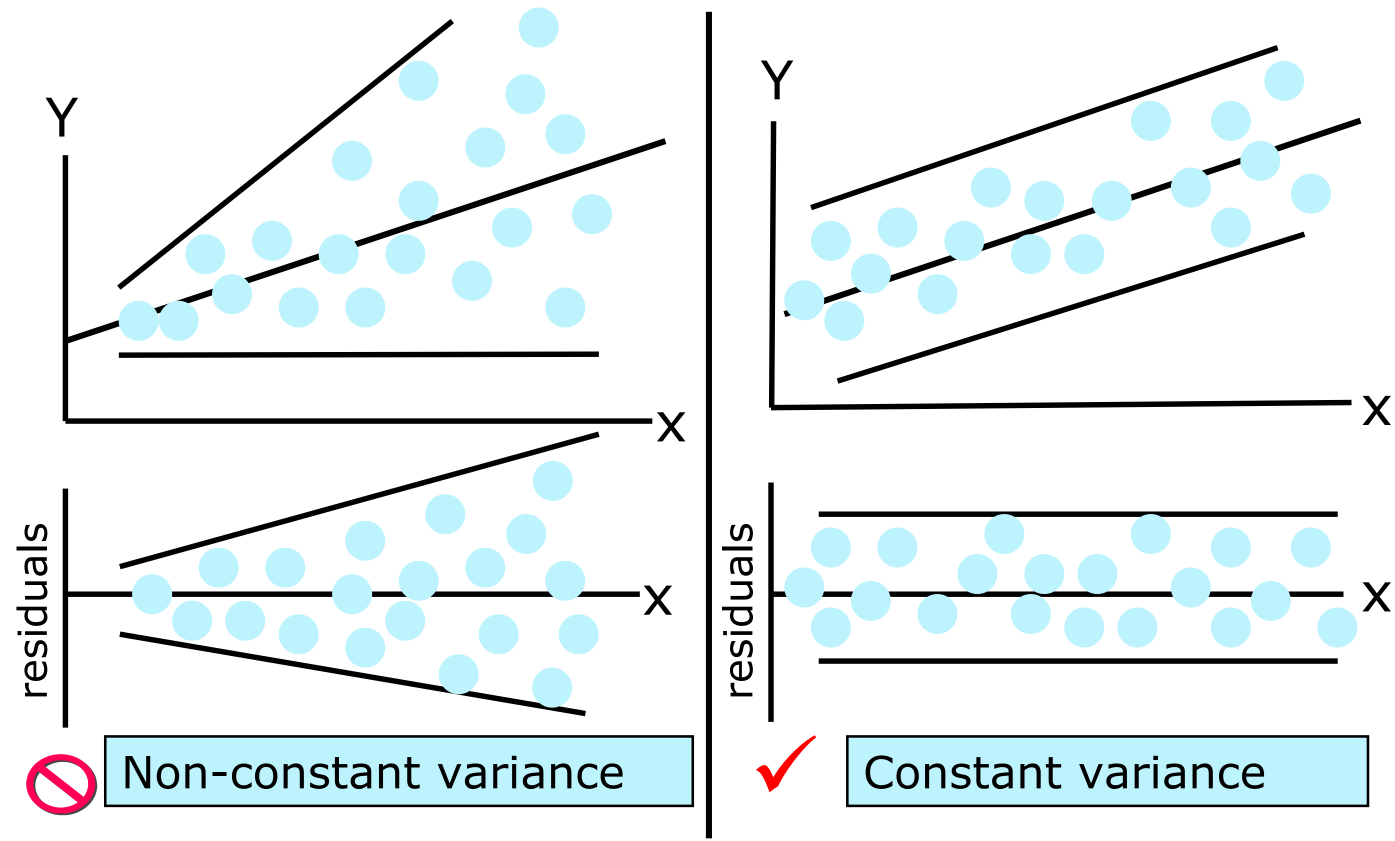


+Residual Analysis for Normality

A normal probability plot of the residuals can be used to check for normality:

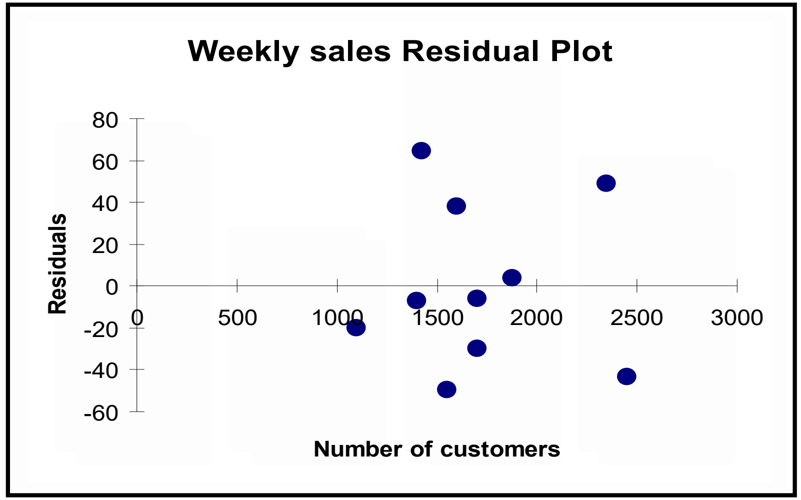


+ Residual Analysis for Equal Variance (Homoscedasticity)



+Residual Analysis – Excel Residual Output

RESIDUAL OUTPUT		
	Predicted Weekly Sales	*Residuals*
1	251.92316	-6.923162
2	273.87671	38.12329
3	284.85348	-5.853484
4	304.06284	3.937162
5	218.99284	-19.99284
6	268.38832	-49.38832
7	356.20251	48.79749
8	367.17929	-43.17929
9	254.6674	64.33264
10	284.85348	-29.85348



Does not appear to violate any regression assumptions

+Inferences About the Slope

The standard error of the regression slope coefficient (b_1) is estimated by:

$$S_{b_1} = \frac{S_{YX}}{\sqrt{SSX}} = \frac{S_{YX}}{\sqrt{\sum (X_i - \overline{X})^2}}$$

where:

S_{b_1} = Estimate of the standard error of the least squares slope

$S_{YX} = \sqrt{\frac{SSE}{n-2}}$ = Standard error of the estimate

+Inferences About the Slope – Excel Output

$$S_{b_1} = 0.03297$$

	A	B	C	D	E	F	G
1	*Regression Statistics*						
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16	Number of customers	0.109767738	0.032969443	3.329377962	0.010394016	0.033740065	0.18579541

+*t* Test for the Slope

t test for a population slope

- Is there a linear relationship between X and Y?

Null and alternative hypotheses:

H_0: $\beta_1 = 0$ (no linear relationship)

H_1: $\beta_1 \neq 0$ (linear relationship does exist)

Test statistic with d.f. = n-2

$$\boxed{t = \frac{b_1 - \beta_1}{S_{b_1}}}$$

Where: b_1 = regression slope coefficient

β_1 = hypothesised slope

S_b = standard error of the slope

+ t Test for the Slope

$$\widehat{\text{Weekly sales}} = 98.25 + 0.1098\,(\text{customers})$$

The slope of this model is 0.1098
Does number of customers affect weekly sales?

H_0: $\beta_1 = 0$

H_1: $\beta_1 \neq 0$

b_1 S_{b_1}

	Coefficients	*Standard Error*	*t Stat*	*P-value*
Intercept	**98.24833**	**58.03348**	**1.69296**	**0.12892**
Number of customers	**0.10977**	**0.03297**	**3.32938**	**0.01039**

$$t = \frac{b_1 - \beta_1}{S_{b_1}} = \frac{0.10977 - 0}{0.03297} = 3.32938$$

+ *t* Test for the Slope

H_0: $\beta_1 = 0$

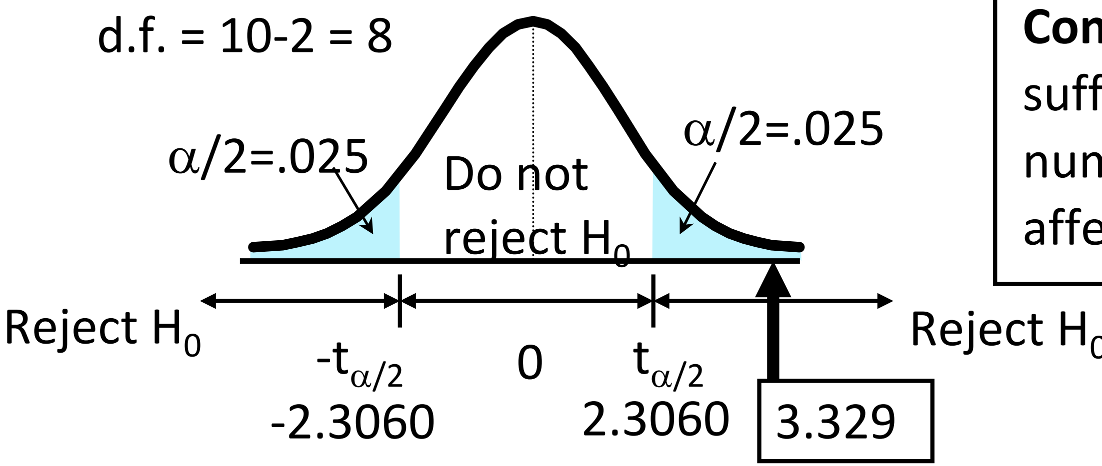
H_1: $\beta_1 \neq 0$

Test Statistic: t = 3.329

T critical = +/- 2.3060 (from t tables)

Decision: Reject H_0

Conclusion: There is sufficient evidence that number of customers affects weekly sales



+ *F* Test for Significance

F Test statistic

$$\boxed{F = \frac{MSR}{MSE}}$$

where:

$$\boxed{\begin{aligned} \text{MSR} &= \frac{\text{SSR}}{\text{k}} \\ \text{MSE} &= \frac{\text{SSE}}{\text{n} - \text{k} - 1} \end{aligned}}$$

F follows an F distribution with k numerator and (n – k - 1) denominator degrees of freedom

k = the number of independent (explanatory) variables in the regression model

+ *F* Test for Significance – Excel Output

	A	B	C	D	E	F	G
2	Multiple R	0.762113713					
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$$F = \frac{MSR}{MSE} = \frac{18934.9348}{1708.1957} = 11.0848$$

With 1 and 8 degrees of freedom

P-value for the F Test

+*F* Test for Significance - Example

H_0: $\beta_1 = 0$

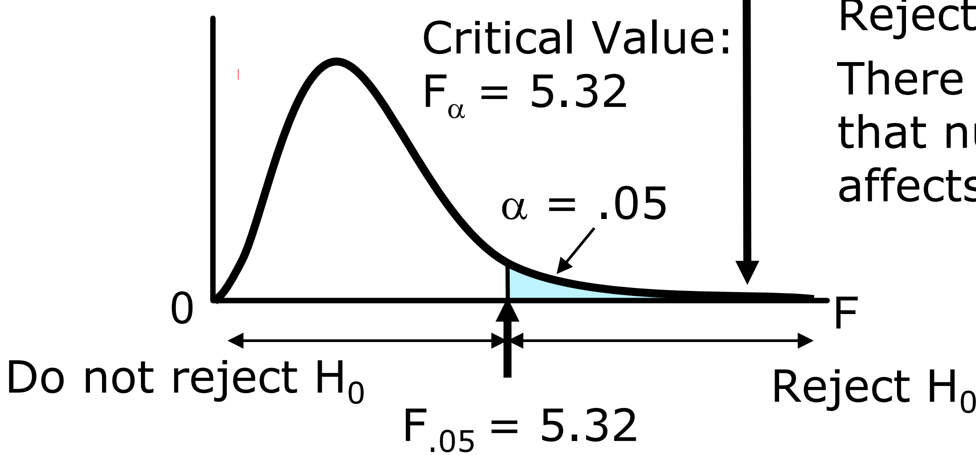
H_1: $\beta_1 \neq 0$

$\alpha = .05$

$\underbrace{df_1 = 1 \qquad df_2 = 8}$

Test Statistic:

$$F = \frac{MSR}{MSE} = 11.08$$



Conclusion:

Reject H_0 at $\alpha = 0.05$

There is sufficient evidence that number of customers affects weekly sales

+Confidence Interval Estimation for the Slope (β_1)

$$b_1 \pm t_{n-2} S_{b_1}$$

d.f. = n - 2 Excel Printout for Weekly sales:

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	**98.24833**	**58.03348**	**1.69296**	**0.12892**	**-35.57720**	**232.07386**
Customers	**0.10977**	**0.03297**	**3.32938**	**0.01039**	**0.03374**	**0.18580**

At 95% level of confidence, the confidence interval for the slope is (0.03374, 0.18580); i.e. we are 95% confident that the average impact on weekly sales is between $33.74 and $185.80 per customer

This 95% confidence interval does not include 0.
Conclusion: There is a significant relationship between weekly sales and number of customers at the .05 level of significance

+ *t* Test for the Correlation Coefficient

Hypotheses

$H_0: \rho = 0$	no association (correlation) between X and Y
$H_1: \rho \neq 0$	statistically significant association (correlation) exists

Test statistic

$$t = \frac{r - \rho}{\sqrt{\frac{1 - r^2}{n - 2}}}$$

where

$$r = +\sqrt{r^2} \quad \text{if } b_1 > 0$$

$$r = -\sqrt{r^2} \quad \text{if } b_1 < 0$$

(with n – 2 degrees of freedom)

+ *t* Test for the Correlation Coefficient - Example

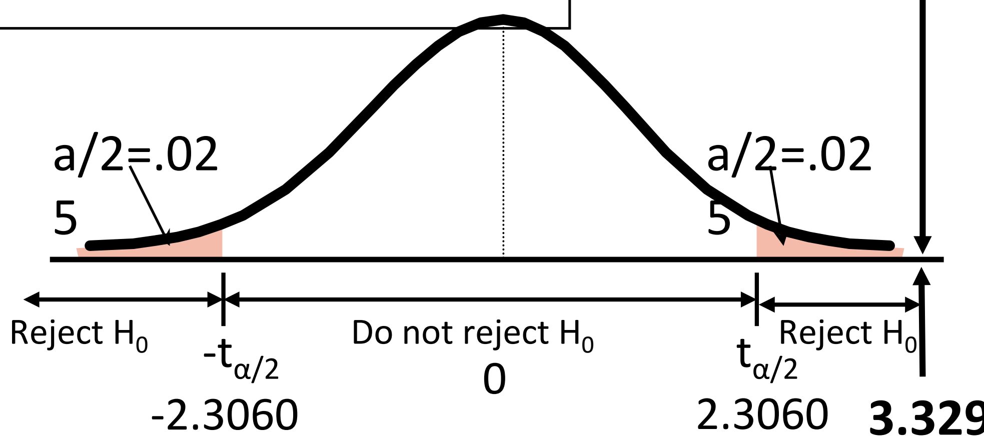
Is there evidence of a significant linear relationship between weekly sales and number of customers at the .05 level of significance?

H_0: ρ = 0 (No correlation)

H_1: ρ ≠ 0 (correlation exists)

α =.05 , df = 10 - 2 = 8

$$t = \frac{r-\rho}{\sqrt{\frac{1-r^2}{n-2}}} = \frac{.762-0}{\sqrt{\frac{1-.762^2}{10-2}}} = 3.329$$



Decision: Reject H_o

Conclusion: There is evidence of a significant linear association at the 5% level of significance

+Pitfalls in Regression and Ethical Issues

- Lacking an awareness of the assumptions underlying least-squares regression
- Not knowing how to evaluate the assumptions
- Not knowing the alternatives to least-squares regression if a particular assumption is violated
- Using a regression model without knowledge of the subject matter
- Extrapolating outside the relevant range
- Concluding that a significant relationship in observational study is due to a cause and effect relationship