SIT718 Real World Analytics

Unit Chair: Dr. Ye Zhu

School of Information Technology
Deakin University

Week 8: Modelling and Solving Linear Programming Problems

The Bank Loan Problem

[Modified from **Taha**]
Haus Bank is working out a loan policy that involves up to AUD 12 million. The type of loans, their respective interest rates, and bad-debt ratio are provided as follows.

Type of loan	Interest rate	Bad-debt ratio
Personal	18.33%	10%
Car	8.76%	7%
Home	4.62%	3%
Farm	12.5%	5%
Commercial	10%	4.5%

The Bank Loan Problem

We assume that:

1. bad-debts are unrecoverable and produce no interest income;
2. ratio of bad-debts over all loans cannot be more than 4%;
3. at least 40% of the funds must go to farm and commercial loans; and
4. home loans must equal to at least 50% of the sum of personal, home, and car loans.

The Bank Loan Problem–An LP model

Let:

- x_1 be the amount of personal loans (in million dollars)
- x_2 be the amount of car loans (in million dollars)
- x_3 be the amount of home loans (in million dollars)
- x_4 be the amount of farm loans (in million dollars)
- x_5 be the amount of commercial loans (in million dollars)

Assumption: All loans are released at the same time.

We would like to maximize the net return:

Net return = Total interest - Bad debt

The Bank Loan Problem–The Objective Function

Total Interest =

$$0.1833(1-10\%)x_1+0.0876(1-7\%)x_2+0.0462(1-3\%)x_3$$
$$+0.125(1-5\%)x_4+0.1(1-4.5\%)x_5$$

Bad Debt =

$$0.1x_1+0.07x_2+0.03x_3+0.05x_4+0.045x_5$$

The objective function:

$$\begin{aligned}\max\ z = \ & 0.16497x_1+0.081468x_2+0.044814x_3+0.11875x_4+0.0955x_5\\ & -(0.1x_1+0.07x_2+0.03x_3+0.05x_4+0.045x_5)\\ = \ & 0.06497x_1+0.011468x_2+0.014814x_3+0.06875x_4+0.0505x_5\end{aligned}$$

The Bank Loan Problem–The Constraints

1. Total funds should not exceed AUD 12 M.

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

2. Farm and commercial loans make up at least 40% of loans.

$$x_4 + x_5 \geq 0.4(x_1 + x_2 + x_3 + x_4 + x_5)$$

3. Home loan make up at least 50% of personal, car, and home loans.

$$x_3 \geq 0.5(x_1 + x_2 + x_3)$$

The Bank Loan Problem–The Constraints

4 Bad debts should not exceed 4% of all loans.

$$0.1x_1+0.07x_2+0.03x_3+0.05x_4+0.045x_5 \leq 0.04(x_1+x_2+x_3+x_4+x_5)$$

5 Non-negativity constraints.

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,$$

An optimal solution is given by:

$$z = 0.501; \quad x_1 = x_2 = x_5 = 0; \quad x_3 = 6; \quad x_4 = 6$$

The Blending Problem–Blending Crude Oils into Gasolines

The Haus' Refinery is blending crude oils into gasolines. Profit and cost details are as follows.

	Sales price per barrel		Purchase price per barrel
Gas 1	$70	Crude 1	$45
Gas 2	$60	Crude 2	$35
Gas 3	$50	Crude 3	$25

- $4 to transform one barrel of oil into one barrel of gas.
- Purchase limit: 5000 barrels of each type of crude oil.
- Maximum production: 14000 barrels per day.

The Blending Problem–Blending Crude Oils into Gasolines

Content restrictions

Crude Oil	Octane Rating	Sulphur Content	Gasoline	Octane Rating	Sulphur Content
Crude 1	12	0.5%	Gas 1	≥ 10	$\leq 1.0\%$
Crude 2	6	2.0%	Gas 2	≥ 8	$\leq 2.0\%$
Crude 3	8	3.0%	Gas 3	≥ 6	$\leq 1.0\%$

- Customer demands
  - 3000 barrels of Gas 1;
  - 2000 barrels of Gas 2;
  - 1000 barrels of Gas 3.
- Each dollar spent on advertising increases demand by 10 barrels.

Formulate an LP that will enable the company to maximize the daily profits (profits = revenues - costs)

The Blending Problem–Blending Crude Oils into Gasolines

- Decisions to make:
  - How much to spend on advertising each type of Gas
  - How to blend each type of gasoline from the 3 types of crude oil.
    - How many barrels of Crude 1 to be used for producing Gas1
    - How many barrels of Crude 1 to be used for producing Gas 2, and so on
- Decision variables:
  - Let a_i be amount of dollars spent daily on advertising Gas i, $i = 1, 2, 3$.
  - Let x_{ij} be number of barrels of Crude i used daily to produce Gas j, for $i = 1, 2, 3; j = 1, 2, 3$.

The Blending Problem–Blending Crude Oils into Gasolines

- Number of barrels of each type of crude oil used daily
  - Crude 1: $x_{11} + x_{12} + x_{13}$
  - Crude 2: $x_{21} + x_{22} + x_{23}$
  - Crude 3: $x_{31} + x_{32} + x_{33}$
- Number of barrels of each type of gas produced daily
  - Gas 1: $x_{11} + x_{21} + x_{31}$
  - Gas 2: $x_{12} + x_{22} + x_{32}$
  - Gas 3: $x_{13} + x_{23} + x_{33}$

The Blending Problem–The objective function

- Daily revenue from gas sales

$$70(x_{11}+x_{21}+x_{31})+60(x_{12}+x_{22}+x_{32})+50(x_{13}+x_{23}+x_{33})$$

- Daily cost of purchasing crude oil

$$45(x_{11}+x_{12}+x_{13})+35(x_{21}+x_{22}+x_{23})+25(x_{31}+x_{32}+x_{33})$$

- Daily advertising costs

$$a_1+a_2+a_3$$

- Daily production costs

$$4(x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23}+x_{31}+x_{32}+x_{33})$$

- Daily profit = revenue − purchasing cost − advertising costs − production costs

$$z=21x_{11}+11x_{12}+x_{13}+31x_{21}+21x_{22}+11x_{23}$$

$$+\,41x_{31}+31x_{32}+21x_{33}-a_1-a_2-a_3$$

The Blending Problem–The constraints

- Demand constraints
  - Daily demand + demand brought by advertising
- Crude oil purchase limit constraints
  - A limit of 5000 barrels for each crude oil
- Production limit constraints
  - At most 14000 barrels of gasoline in total
- Octane level constraints
  - Gas 1 ≥ 10; Gas 2 ≥ 8; Gas 3 ≥ 6
- Sulphur content constraints
  - Gas 1 $\leq 1\%$; Gas 2 $\leq 2\%$; Gas 3 $\leq 1\%$

The Blending Problem–Demand constraints

- Gas 1
  - Daily demand: 3000 barrels
  - Advertising: each dollar spent will bring up by 10

$$x_{11} + x_{21} + x_{31} = 3000 + 10a_1$$

- Gas 2

$$x_{12} + x_{22} + x_{32} = 2000 + 10a_2$$

- Gas 3

$$x_{13} + x_{23} + x_{33} = 1000 + 10a_3$$

The Blending Problem–Other constraints

- Crude oil purchase limit constraints
  - A limit of 5000 barrels for each type of crude oil

    $$x_{11} + x_{12} + x_{13} \leq 5000$$
    $$x_{21} + x_{22} + x_{23} \leq 5000$$
    $$x_{31} + x_{32} + x_{33} \leq 5000$$

- Production limit constraint
  - At most 14000 barrels of gasoline in total

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 14000$$

The Blending Problem–Octane constraints

- Octane level in Gas 1 must be at least 10
- Calculating "average" octane level. Gas 1:

$$\frac{12x_{11} + 6x_{21} + 8x_{31}}{x_{11} + x_{21} + x_{31}} \geq 10 \implies 2x_{11} - 4x_{21} - 2x_{31} \geq 0$$

- Gas 2:

$$4x_{12} - 2x_{22} \geq 0$$

- Gas 3:

$$6x_{13} + 2x_{33} \geq 0$$

The Blending Problem–Sulphur constraints

- Sulphur level in Gas 1 must be at most 1%
- Calculating "average" sulphur level. Gas 1:

$$\frac{0.005x_{11} + 0.02x_{21} + 0.03x_{31}}{x_{11} + x_{21} + x_{31}} \leq 0.01$$

$$\implies -0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0$$

- Gas 2: $-0.015x_{12} + 0.01x_{32} \leq 0$
- Gas 3: $-0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0$

The Blending Problem–The Optimal Solution

$z = 287,750;$

$$
\begin{array}{lll}
x_{11} = 2088.9; & x_{12} = 2111.1; & x_{13} = 800 \\
x_{21} = 777.78; & x_{22} = 4222.2; & x_{23} = 0 \\
x_{31} = 133.33; & x_{32} = 3166.7; & x_{33} = 200
\end{array}
$$

$a_1 = 0; \quad a_2 = 750; \quad a_3 = 0$

We should spend $750 on advertising Gas 2, and no money on the other two gases.
We are to produce:
3000 barrels of Gas 1,
9500 barrels of Gas 2, and
1000 barrels of Gas 3 respectively.

Transportation Models

The transportation problem model:

- Concern: distributing any commodity from sources to destinations.

The requirements assumption:

- Each source has a fixed supply.
- Entire supply must be distributed to the destinations.
- Each destination has a fixed demand.
- Entire demand must be received from the sources.

Transportation Models

The feasible solutions property.

- A transportation problem will have feasible solutions if and only if:

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

The cost assumption:

- Cost is directly proportional to number of units distributed.

Transportation Models

Format of a transportation simplex tableau

		Destination				Supply	u_i
		1	2	⋯	n		
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1	
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2	
	⋮	⋯	⋯	⋯	⋯	⋮	
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m	
Demand		d_1	d_2	⋯	d_n	$Z =$	
	v_j						

let x_{ij} be the number of millions of kwh produced at plant i and sent to city j.

$$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \sum_{j=1}^{n} x_{ij} \leq s_i \quad \forall i = 1, \ldots, m \text{ (Supply Constraints)}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_i \quad \forall i = 1, \ldots, m \text{ (Demand Constraints)}$$

$$x_{ij} \geq 0 \quad \forall i = 1, \ldots, m; j = 1, \ldots, m$$

Electric Power Plants

The cost matrix of sending each million kwh of electricity from a plant to a city is given in the table below.

	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	72
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	78
Demand (million kwh)	45	70	30	55	

let x_{ij} be the number of millions of kwh produced at plant i and sent to city j.

$$\begin{aligned}\min z = {} & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} \\ & + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \\ & + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}\end{aligned}$$

Electric Power Plants (cont.)

The supply constraints would be:

- Plant 1:

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 72$$

- Plant 2:

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$$

- Plant 3:

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 78$$

$$x_{ij} \geq 0 \quad \forall i = 1, 2, 3; j = 1, 2, 3, 4$$

For each city, the demand constraints are:

- City 1:

$$x_{11} + x_{21} + x_{31} \geq 45$$

- City 2:

$$x_{12} + x_{22} + x_{32} \geq 70$$

- City 3:

$$x_{13} + x_{23} + x_{33} \geq 30$$

- City 4:

$$x_{14} + x_{24} + x_{34} \geq 55$$

The optimal solution for this problem is
$x_{12} = 47, x_{13} = 25, x_{21} = 45, x_{23} = 5, x_{32} = 23, x_{34} = 55$ with the total cost being $\$1484$.

Inventory Problem

Scientific inventory management.

- Mathematical model describes system behavior.
- Goal: optimal inventory policy with respect to the model.
- Computerized information processing system maintains inventory level records.
- Apply the inventory policy to replenish inventory.

Demand.

- Number of units that will need to be withdrawn from inventory.

Deterministic inventory model.

- Used when demand is known.

Stochastic inventory model.

- Used when demand cannot be predicted well.

Inventory Problem (cont.)

Haus & Co. produces toy cars, and the demand for the next 5 quarters is 200, 150, 300, 250, and 400 units. Haus & Co. are able to supply 180, 230, 430, 300 and 300 units respectively in these five quarters. The production costs are \$100, \$96, \$116, \$102 and \$106 respectively.

Overtime can be employed, but back ordering is not allowed. The overtime production is 50% more expensive, but the overtime capacity is half that of regular time.

If a toy car is produced now but sold in a later quarter, there is a storage cost of \$4 per car per quarter.

Inventory Problem (cont.)

Cost Matrix

		Q1	Q2	Q3	Q4	Q5	Dummy	Supply
Q1	RT	100	104	108	112	116	0	180
	OT	150	154	158	162	166	0	90
Q2	RT	M	96	100	104	108	0	230
	OT	M	144	148	152	156	0	115
Q3	RT	M	M	116	120	124	0	430
	OT	M	M	174	178	182	0	215
Q4	RT	M	M	M	102	106	0	300
	OT	M	M	M	153	157	0	150
Q5	RT	M	M	M	M	106	0	300
	OT	M	M	M	M	159	0	150
	Demand	200	150	300	250	400	860	

Here we use M as a very large positive number e.g. 99999 to represent a very high cost value ensure that no cellos are used to meet demand during a quarter prior to their production.

The Assignment Problem

Special type of linear programming problem.

- Assignees are being asked to perform tasks.
- Assignees could be people, machines, plants, or time slots.

Requirements to fit assignment problem definition.

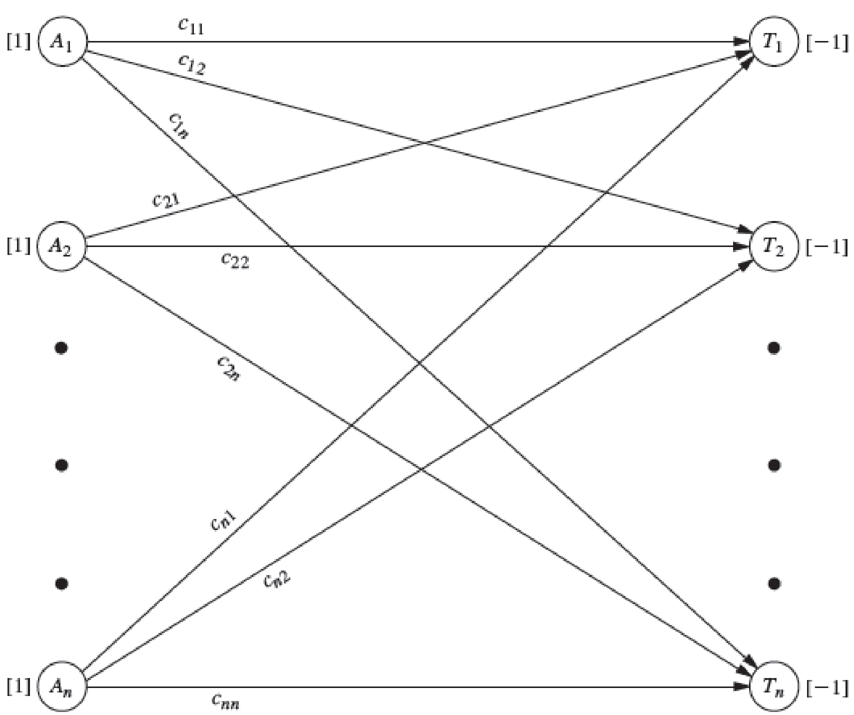
- The number of assignees and tasks are the same.

  Designated by n
- Each assignee is assigned to exactly one task.
- Each task is to be performed by exactly one assignee.
- Cost c_{ij} is associated with each assignee i performing task j.
- Objective: determine how n assignments should be made to minimize the total cost.

The Assignment Problem (cont.)

x_{ij} can have only values zero or one.

- One if assignee i performs task j.
- Zero if not.



Minimize $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for } j = 1, 2, \ldots, n,$$

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j$$

$$(x_{ij} \text{ binary}, \qquad \text{for all } i \text{ and } j).$$

The Assignment Problem (cont.)

The table below provides the travel time (in hours) required for the driver to complete that single job. Each driver can only be assigned to one job, and a single job can only be assigned to one driver. Realistically this would come from a scenario where all drivers start their jobs at the same time."

Driver	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

The LP can then be formulated using what we call a binary variable - a decision variable that can only be 0 or 1.

The Assignment Problem (cont.)

Let $x_{ij} = 1$ if driver i is assigned to job j, $x_{ij} = 0$ otherwise

Our objective function is to now minimise the total amount of time spent on these jobs.

$$\begin{aligned}
\min z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} \\
& + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\
& + 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} \\
& + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44}
\end{aligned}$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$

$$x_{13} + x_{23} + x_{33} + x_{44} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 1$$

$$x_{ij} \in \{0, 1\}, \forall i, j = 1, 2, 3, 4$$

The Assignment Problem (cont.)

By solving this LP (which we'll get you to do in R in the next section), we'll obtain the following solution.

$$x_{12} = x_{24} = x_{33} = x_{41} = 1$$

and all other $x_{ij} = 0$. Translating this back to the given scenario means that:

- Driver 1 is assigned to Job 2.
- Driver 2 is assigned to Job 4.
- Driver 3 is assigned to Job 3.
- Driver 4 is assigned to Job 1.

This solution can be represented as a matrix where in the same way, the rows refer to the driver and the columns refer to the job:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This results in $5 + 5 + 3 + 2 = 15$ hours of time spent driving

Solving LP with R

BLENDING CRUDE OILS INTO GASOLINES
mapping table

	Index	1	2	3	4	5	6	7	8	9	10	11	12		
	Variable					x						a			
		11	12	13	21	22	23	31	32	33	1	2	3		
	obj	21	11	1	31	21	11	41	31	21	-1	-1	-1		
															target
	1	1			1			1			-10			=	3000
	2		1			1			1			-10		=	2000
	3			1			1			1			-10	=	1000
	4	1	1	1										<=	5000
	5				1	1	1							<=	5000
	6							1	1	1				<=	5000
constrants	7	1	1	1	1	1	1	1	1	1				<=	14000
	8	2			-4			-2						>=	0
	9		4			-2								>=	0
	10			6						2				>=	0
	11	-0.005			0.01			0.02						<=	0
	12		-0.015						0.01					<=	0
	13			-0.005			0.01			0.02				<=	0

Solving LP with R (cont.)

```
library(lpSolveAPI)

lprec <- make.lp(13, 12) # 12 variables and 13 constraints

lp.control(lprec, sense= "maximize")

# set objective function
set.objfn(lprec, c(21,  11, 1,  31, 21, 11, 41, 31, 21, -1, -1, -1))

# set constraints
set.row(lprec, 1, c(1,1,1,-10), indices = c(1,4,7,10))
set.row(lprec, 2, c(1,1,1,-10), indices = c(2,5,8,11))
set.row(lprec, 3, c(1,1,1,-10), indices = c(3,6,9,12))
set.row(lprec, 4,rep(1,3), indices =c(1:3))
set.row(lprec, 5,rep(1,3), indices =c(4:6))
set.row(lprec, 6,rep(1,3), indices =c(7:9))
set.row(lprec, 7,rep(1,9), indices =c(1:9))
set.row(lprec, 8,c(2,-4,-2), indices =c(1,4,7))
set.row(lprec, 9,c(4,-2), indices =c(2,5))
set.row(lprec, 10,c(6,2), indices =c(3,9))
set.row(lprec, 11,c(-0.005,0.01,0.02), indices =c(1,4,7))
set.row(lprec, 12,c(-0.015,0.01), indices =c(2,8))
set.row(lprec, 13,c(-0.005,0.01,0.02), indices =c(3,6,9))


set.rhs(lprec, c(3000,2000,1000,5000,5000,5000, 14000,0,0,0,0,0,0))


set.constr.type(lprec, c("=", "=",  "=", "<=",  "<=", "<=", "<=", ">=",
                         ">=",  ">=", "<=", "<=", "<="))

set.type(lprec, c(1:12),"real")

set.bounds(lprec, lower = rep(0, 12), upper = rep(Inf, 12))
```

Solving LP with R (cont.)

```
> solve(lprec)
[1] 0
> objvalue<-get.objective(lprec)
> objvalue
[1] 287750
> solution<-get.variables(lprec)
> solution
 [1] 2222.2222 2111.1111  666.6667  444.4444 4222.2222  333.3333  333.3333
 [8] 3166.6667    0.0000    0.0000  750.0000    0.0000
> sum(solution[c(1,4,7)])
[1] 3000
> sum(solution[c(2,5,8)])
[1] 9500
> sum(solution[c(3,6,9)])
[1] 1000
```

Variable	x									a		
	11	12	13	21	22	23	31	32	33	1	2	3
	2222.22	2111.111	666.6667	444.4444	4222.222	333.3333	333.3333	3166.667	0	0	750	0

Therefore we should spend $750 on advertising for Gas 2, and no money for the other two gases.

In terms of barrel production, we are to produce:

- $3,000$ barrels of Gas 1
- $9,500$ barrels of Gas 2, and
- $1,000$ barrels of Gas 3.

Solving LP with R (cont.)

Assignment Problem

Driver	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

mapping table

	Index			
machine	Job1	Job2	Job3	Job4
1	1	2	3	4
2	5	6	7	8
3	9	10	11	12
4	13	14	15	16

Solving LP with R (cont.)

```
lprec <- make.lp(8, 16) # 16 variables and 8 constraints

lp.control(lprec, sense= "minimize")

# set objective function
set.objfn(lprec, c(14, 5, 8, 7,2, 12, 6,5,7, 8, 3, 9,2, 4, 6, 10))

# set constraints
set.row(lprec, 1, rep(1,4), indices = c(1:4))
set.row(lprec, 2, rep(1,4), indices = c(5:8))
set.row(lprec, 3, rep(1,4), indices = c(9:12))
set.row(lprec, 4, rep(1,4), indices =c(13:16))
set.row(lprec, 5, rep(1,4), indices =seq(from = 1, to = 13, by = 4))
set.row(lprec, 6, rep(1,4), indices =seq(from = 2, to = 14, by = 4))
set.row(lprec, 7, rep(1,4), indices =seq(from = 3, to = 15, by = 4))
set.row(lprec, 8, rep(1,4), indices =seq(from = 4, to = 16, by = 4))


set.rhs(lprec, rep(1,8))


set.constr.type(lprec, rep("=", 8))

set.type(lprec, c(1:16),"binary")

set.bounds(lprec, lower = rep(0, 16), upper = rep(1, 16))
```

Solving LP with R (cont.)

```
> solve(lprec)
[1] 0
>
> objvalue<-get.objective(lprec)
> objvalue
[1] 15
> solution<-get.variables(lprec)
> solution<- matrix(solution, nrow = 4, ncol = 4,byrow = TRUE) # reshape the varible values
> solution
     [,1] [,2] [,3] [,4]
[1,]    0    1    0    0
[2,]    0    0    0    1
[3,]    0    0    1    0
[4,]    1    0    0    0
```