Game Theory SIT718

Delaram Pahlevani

DEAKIN UNIVERSITY

Key concepts and ideas

Game theory is the study of mathematical models of strategic interaction between rational decision-makers.

Many situations in life involve two or more decision makers who simultaneously choose an action.

If we apply Game theory to these situations, those “decision makers” become “players”.

In this instance, the actions chosen by each player in a given situation will affect rewards earned by the other players.

Examples of Game theory

- Chicken

Two drivers on a collision course. If neither driver alters their course, there will be a collision.

- Politics

The final outcome of an election depends upon the combination of strategies selected by opposing political parties.

- Fast food sales and advertising

No company works in a vacuum and each company’s decision will affect the revenue/profit of the other.

2 Players Zero Sum Game

A two-person zero-sum game involves two players who play a game only once, where both have a number of strategies to select from.

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Zero-sum refers to the total payoff being zero. In other words, this means that whatever amount Player I wins, Player II loses the same amount.

Payoff to Player I = – Payoff to Player II.

5 + (-

2 Players Zero Sum Game (Notation)

- $m =$ number of strategies for Player I
- $n =$ number of strategies for Player II
- $A_i = i$th strategy (pure) for Player I
- $B_j = j$th strategy (pure) for Player II
- $v_{ij} =$ Payoff to Player I if he selects Strategy A_i and if Player II selects Strategy B_j
  - Payoff to Player I $= -$ Payoff to Player II

Player 2

Player II

$$\begin{bmatrix} -v_{11} & -v_{12} & -v_{1n} \\ -v_{21} & -v_{22} & -v_{2n} \\ -v_{m1} & -v_{m2} & -v_{mn} \end{bmatrix}$$

Payoff Matrix for playe

Player II

Player II's strategies

Player I

Player I's strategies

$$\begin{array}{c|c} & \begin{matrix} B_1 & B_2 & \cdots & B_n \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} \end{array}$$

2 Players Zero Sum Game

EXAMPLE

- Player I has two options: Strategies A_1 or A_2
- Player II has three options: Strategies B_1, B_2, or B_3
- If Player I uses Strategy A_1 and Player II uses Strategy B_3, then Player I wins 25 and Player II loses 25
- If Player I uses Strategy A_2 and Player II uses Strategy B_3, then Player I loses 20 and Player II wins 20

$$\begin{array}{ccc} & & \text{Player II} \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player I} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & \boxed{-4} & \boxed{25} \\ 6 & 10 & -20 \end{bmatrix} \end{array}$$

Assumptions In Game Theory

Assumptions in Game Theory:

- As with any concept in economics, there is the assumption of **rationality.**
- There is also an assumption of **maximization.**

It is assumed that players within the game are rational and will strive to maximize their payoffs in the game

The spirit of Game Theory

In reality,

- What is the most secured payoff that they can get **regardless of what the other player does**
- Each player gets the best they can **"secure"**

The spirit of game theory is to look at the worst outcomes and pick the **least worst outcome**, i.e. be as secure as possible. This can also be interpreted as **playing safe**, where we don't want to take the risk of losing more in the pursuit of larger gains.

The spirit of Game Theory

$$
\begin{array}{ccccc}
 & & & \text{Player II} & \\
 & & B_1 & B_2 & B_3 \\
\text{Player I} & \begin{array}{c} A_1 \\ A_2 \end{array} & \multicolumn{3}{c}{\begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix}}
\end{array}
$$

Ideal Scenario: Player I wins 25, Player II wins 20. This won't happen as they correspond to two different strategies for Player 1.

Maximum SECURED Payoff

In terms of playing safe, Player I would choose Strategy A_1, where we're taking the maximum of the minimum possible losses.

Mathematically, the security level for *Player I associated with Strategy* A_i is given by:

$$s_i := \min\{v_{ij} \ : \ j = 1, \ldots, n\}, \quad i = 1, \ldots, m$$

The *maximum security level* for Player I is given by:

$$s^* := \max\{s_i \ : \ i = 1, \ldots, m\}$$

The spirit of Game Theory

Player II

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player I} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix} \\ & \begin{array}{ccc} 6 & 10 & 25 \end{array} \end{array} = \begin{bmatrix} -2 & 4 & -25 \\ -6 & -10 & 20 \end{bmatrix} \begin{array}{c} -4 \\ -20 \end{array}$$

(A_1, B_1)

Maximum SECURED Payoff for Player II

$$\max_{i \le m} \min_{j \le n} a_{ij} = \min_{j \le n} \max_{i \le m} a_{ij} \equiv v,$$

- Strategy B_1, then at worst they lose 6
- Strategy B_2, then at worst they lose 10
- Strategy B_3, then at worst they lose 25

Player 1:
Min → Max
Player 2:
Max → Min

So Player II would choose Strategy B_1 since we're taking the minimum of the maximum possible losses. Recall that Player II's payoff = - Player I's payoff, hence the switch of minimum/maximum.

The spirit of Game Theory

What have we learnt so far?

- The security level of a strategy for a player is the **minimum** guaranteed payoff regardless of what strategy his opponent uses
- A player never tries to "maximize payoff"; he knows that his opponent will not let him
- Instead, a player always tries to choose, among all available strategies, the strategy that **maximizes the security level**
- In other words, a player always choose the strategy that gives the **least worst outcome**

Example 2: 2 Players Zero Sum Game

Player 1:
Min → Max
Player 2:
Max → Min

$$
\begin{array}{c|ccc|c}
 & B_1 & B_2 & B_3 & s_i \\
\hline
A_1 & 1 & 10 & 3 & 1 \\
A_2 & -2 & 5 & 1 & -2 \\
A_3 & 1 & -8 & 1 & -8 \\
\hline
t_j & 1 & 10 & 3 &
\end{array}
$$

- Maximum security level for Player I $= 1$
- Maximum security level for Player II $= 1$
- Suggested solution (A_1, B_1), i.e., Player I uses Strategy 1 and Player II uses Strategy 1 too.

Practice

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix} \end{array} \qquad \begin{array}{c} 1 \\ 2 \end{array}$$

$$\begin{array}{cc} 6 & 5 \end{array}$$

Player 1:
Min → Max
Player 2:
Max → Min

This leads us to suggest that our ideal pair of strategies would be (A_2, B_2).

Saddle Points & Pure Strategies

If there is a pair of strategies in which either player can do no better, then we have a saddle point!

- A necessary and sufficient condition for a saddle point to exist is the presence of a payoff matrix element which is both a minimum of its row and a maximum of its column.

A solution (A_i, B_j) to a 2-person zero-sum game is said to be **stable** (or, is in **equilibrium**) if:

- Player I, whilst expecting Player II to use Strategy B_j, has no incentives to choose a strategy other than A_i.
- Similarly, Player II, whilst expecting Player I to use Strategy A_i, has no incentives to choose a strategy other than B_j.
- If neither player finds an incentive to change their strategy, then we have reached an **optimal solution**.
- We also call (A_i, B_j) a **saddle point**.

Saddle Points & Pure Strategies

Let L denote the largest security level for Player I (recall that they want the $\max$ of the $\min$), and let U denote the smallest security level for Player II (recall that they want the $\min$ of the $\max$).

We call L the **lower value of the game** and call U the **upper value of the game.**

- If $U = L$, we call this the **value of the game**, and the optimal payoff for both players can be achieved by a **pure strategy.**
- if $U > L$, then a pure strategy will not result in an equilibrium and players must resort to **mixed strategies.**

Saddle Points Practice

$$
\begin{array}{ccccc}
 & A_1 & A_2 & A_3 & A_4 \\
a_1 & 5 & 2 & 1 & 3 \\
a_2 & 3 & 3 & -2 & 4 \\
a_3 & -1 & 3 & -1 & -3 \\
a_4 & 2 & -4 & 1 & -2
\end{array}
$$

Player 1:
Min → Max
Player 2:
Max → Min

Pure Strategies

- Problem

Vicky and David each have two cards, an ace and a two. They each select one of their cards, with their choices unknown to the opponent, and then they will compare the cards.
Before they compare the cards, however, Vicky gets to call "even" or "odd". Vicky wins if the sum of the face values of the selected cards is of the parity she has called, and if not, David wins. Model the game as 2-person zero-sum game, if:

1. (a) a "win" scores a single point.
2. (b) a "win" scores the face values of the selected cards.

Pure Strategies

1 even
odd

2 even
odd

Player 1:
Min → Max
Player 2:
Max → Min

- Solution

There are only 2 strategies for David, select either the "one" or the "two". Vicky, however, has 4 strategies. (1) Select the "one" and call "even". (2) Select the "two" and call "even". (3) Select the "one" and call "odd". (4) Select the "two" and call "odd". Hence, the payoff matrix is as follows.

1+1=2 David 1+2=3

Vicky

	"One"	"Two"	s_i
(Even, 1)	1	-1	-1
(Even, 2)	-1	1	-1
(Odd, 1)	-1	1	-1
(Odd, 2)	1	-1	-1
t_j	1	1	$L < U$

+1 -1

Since $L < U$, there is no saddle point for the game.

Pure Strategies

- Solution

Alternatively, if by ``winning'', one considered the amount won is the sum of face values, it is accepted correct too, and the payoff matrix is as follows.

		"One" y_1	y_2 "Two"	s_i
x_1	(Even, 1) $1+1=2$	2	-3 $1+2=3$	-3
x_2	(Even, 2)	-3	4	-3
x_3	(Odd, 1)	-2	3	-2
x_4	(Odd, 2)	3	-4	-4
	t_j	3	4	$L < U$

$3 \neq -2$

Since $L < U$, there is no saddle point for the game.

Mixed strategies

When L<U**, a pure strategy will not result in an equilibrium. A player can however *mix up* his/her strategies.**

A **mixed strategy** for Player I is a vector $x = (x_1, \ldots, x_m)$ with $x_i \geq 0$, for all i, representing the probability (i.e., portion of time) that Strategy i is used; and $x_1 + x_2 + \cdots x_m = 1$,

$\sum_{i=1}^{n} x_i = 1$

A **pure strategy** is a vector $x = (x_1, \ldots, x_m)$, with one component 1 and all other components 0. i.e., $x = (0, \ldots, 0, 1, 0, \ldots, 0)$. So if a person uses a pure strategy they use the same strategy all the time (100% of the time), which is what we do when there is a saddle point.

$\underline{x_1 + x_2 = 1}$

Mixed strategies(Graphical solutions)

Consider the following payoff matrix:

$$\begin{array}{ccc} & B_1 & B_2 \\ & y_1 & y_2 \\ A_1 \quad x_1 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$}} \\ A_2 \quad x_2 & & \end{array}$$

$6-5x_1 = 2+3x_1$

$4 = 8x_1$

$\Rightarrow x_1 = 0.5$

$x_2 = 0.5$

$x_1 + x_2 = 1 \rightarrow x_2 = 1-$

$y_1 + y_2 = 1$

Let Player I use Strategy A_1 with probability x_1 and uses Strategy A_2 with probability $1 - x_1$.

Player I's expected payoff if Player II uses Strategy B_1 is given by:

$x_1(1) + \underbrace{(1 - x_1)}_{x_2}(6) = 6 - 5x_1$

Player I's expected payoff if Player II uses Strategy B_2 is given by:

$x_1(5) + \underbrace{(1 - x_1)}_{x_2}(2) = 2 + 3x_1$

$2+3(\frac{1}{2}) = 3.5$

$(1)(x_1) + 6x_2$

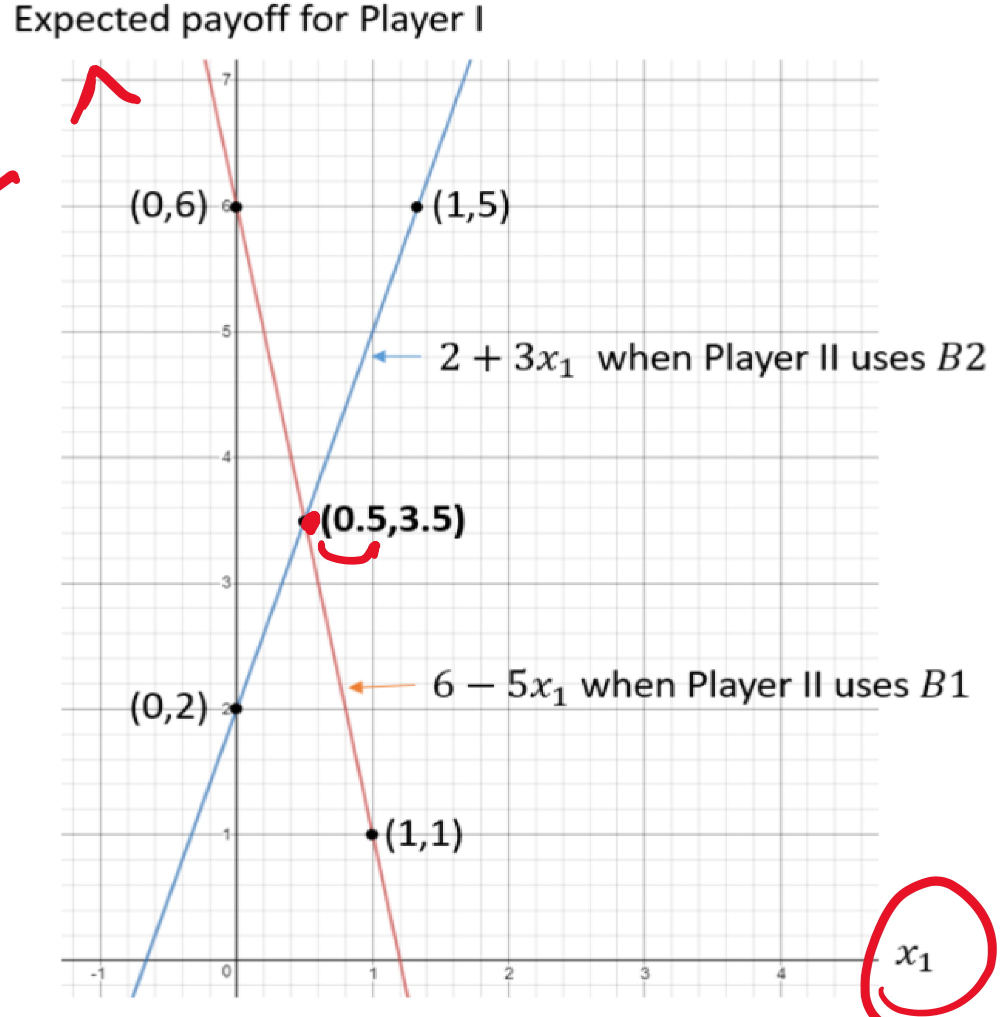
$5x_1 + 2x_2$

Mixed strategies(Graphical solutions)

Player I wants to maximise her expected payoff regardless of what the strategy made by Player II, we can use a graphical method as follows. $x_1 = 0.5$ is the optimal solution.

$x_1 = 0.5$

$x_2 = 1 - 0.5 = 0.5$



Mixed strategies(Graphical solutions)

Player II's Perspective

Let Player II uses Strategy B_1 with probability y_1 and uses Strategy B_2 with probability $1 - y_1$.

Player II's expected payoff if Player I uses Strategy A_1 is given by:

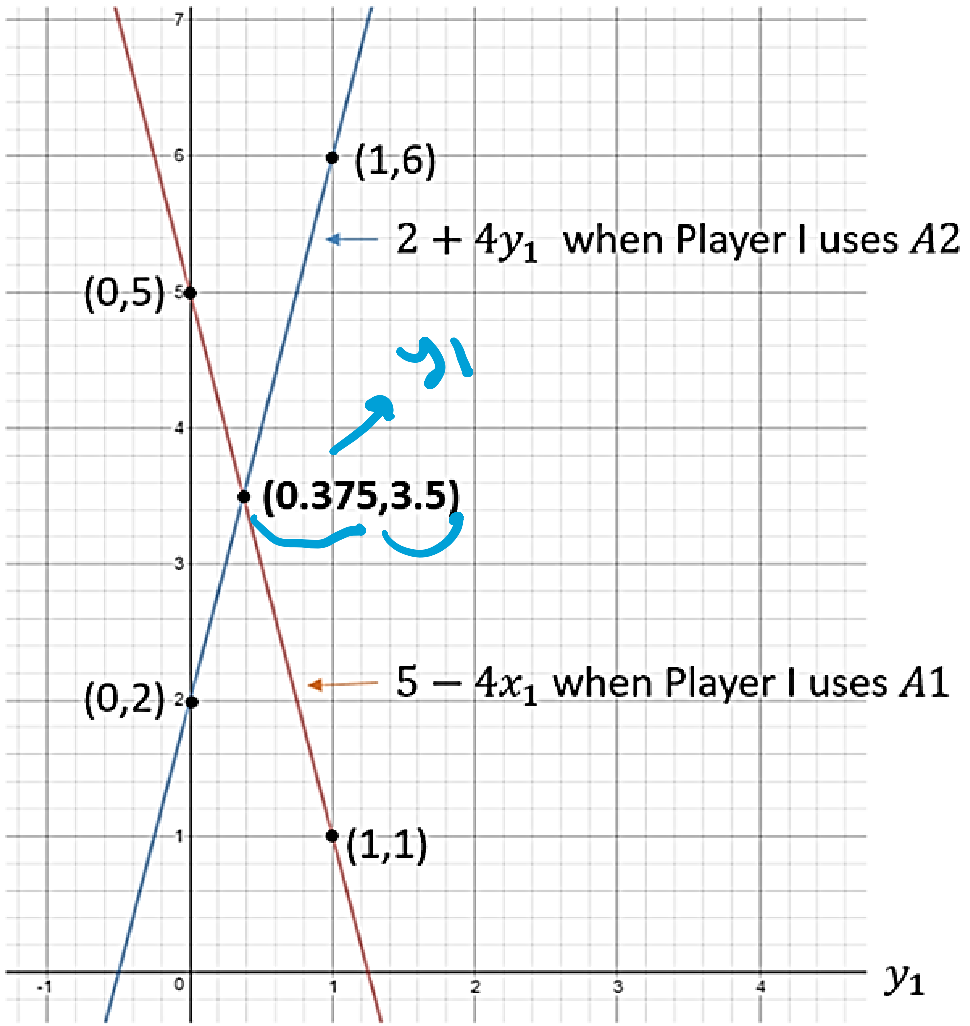
$$y_1(1) + (1 - y_1)(5) = \underbrace{5 - 4y_1}$$

Player II's expected payoff if Player I uses Strategy A_2 is given by:

$$y_1(6) + (1 - y_1)(2) = \underbrace{2 + 4y_1}$$

The graphical solution with $y_1 = 0.375$ is shown as follows.

Expected payoff for Player II



Mixed Strategies

- Problem

Consider the following 2-person zero-sum game. Does the game have a pure strategy? If not, what is the range of possible values for the value of the game, v?

$$V = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \end{array}$$

Solve the game for both players using the graphic method.

Mixed Strategies

- Solution

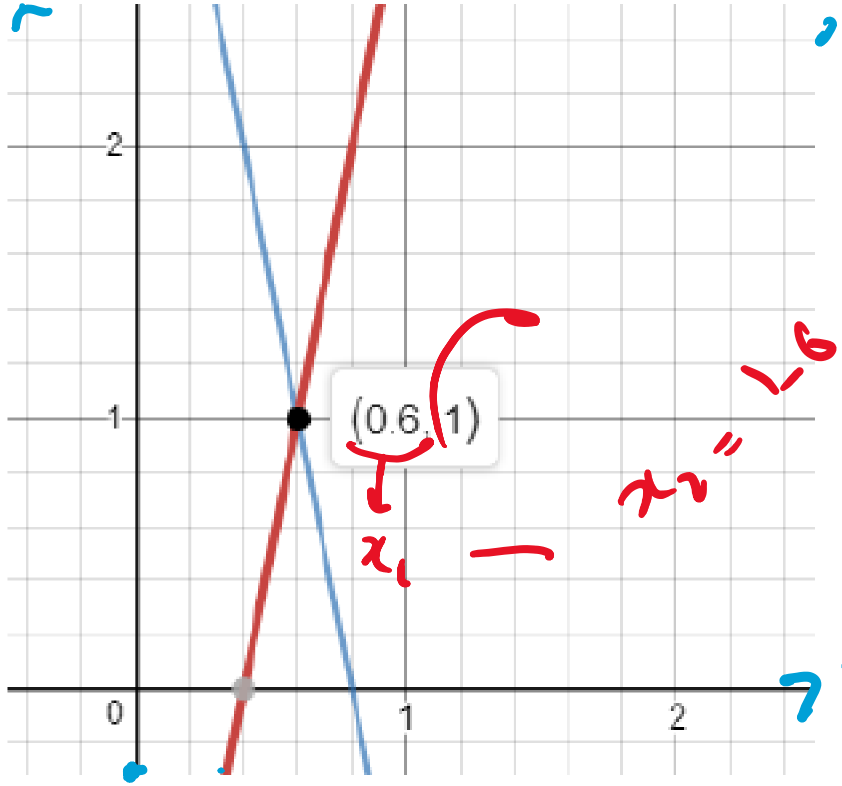
$3x_1 - 2x_2 \rightarrow 3x_1 - 2(1-x_1) = 3x_1 - 2 + 2x_1$

$z = 5x_1 \ldots$

$-x_1 + 4x_2$

$x_1 + x_2 = 1$

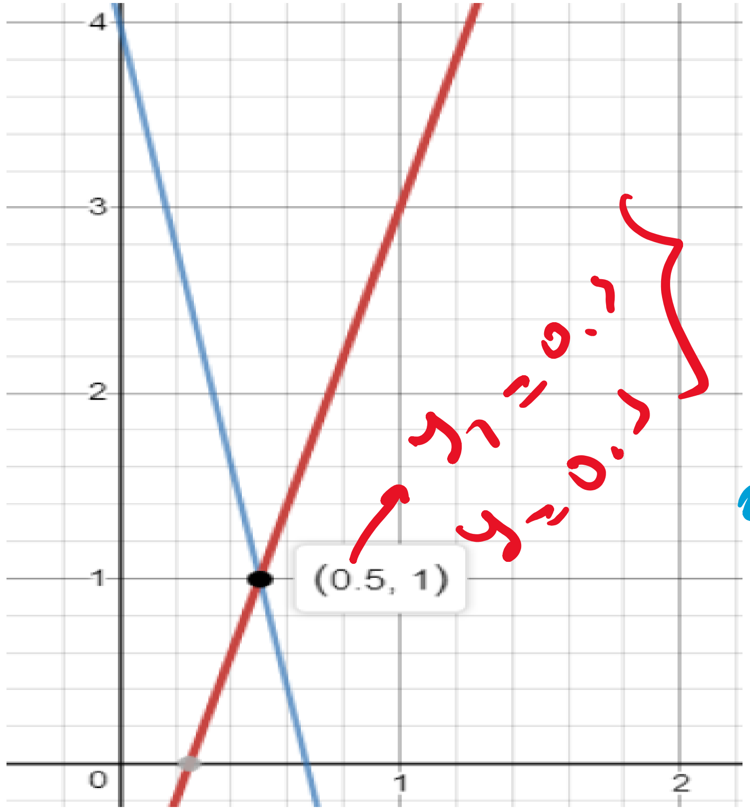
Player I



x_1

$x_2 = 1 - 0.6 = 0.4$

Player II



$y_1 = 0.5$

$y_2 = 0.5$

$-x_1 + 4(1-x_1)$

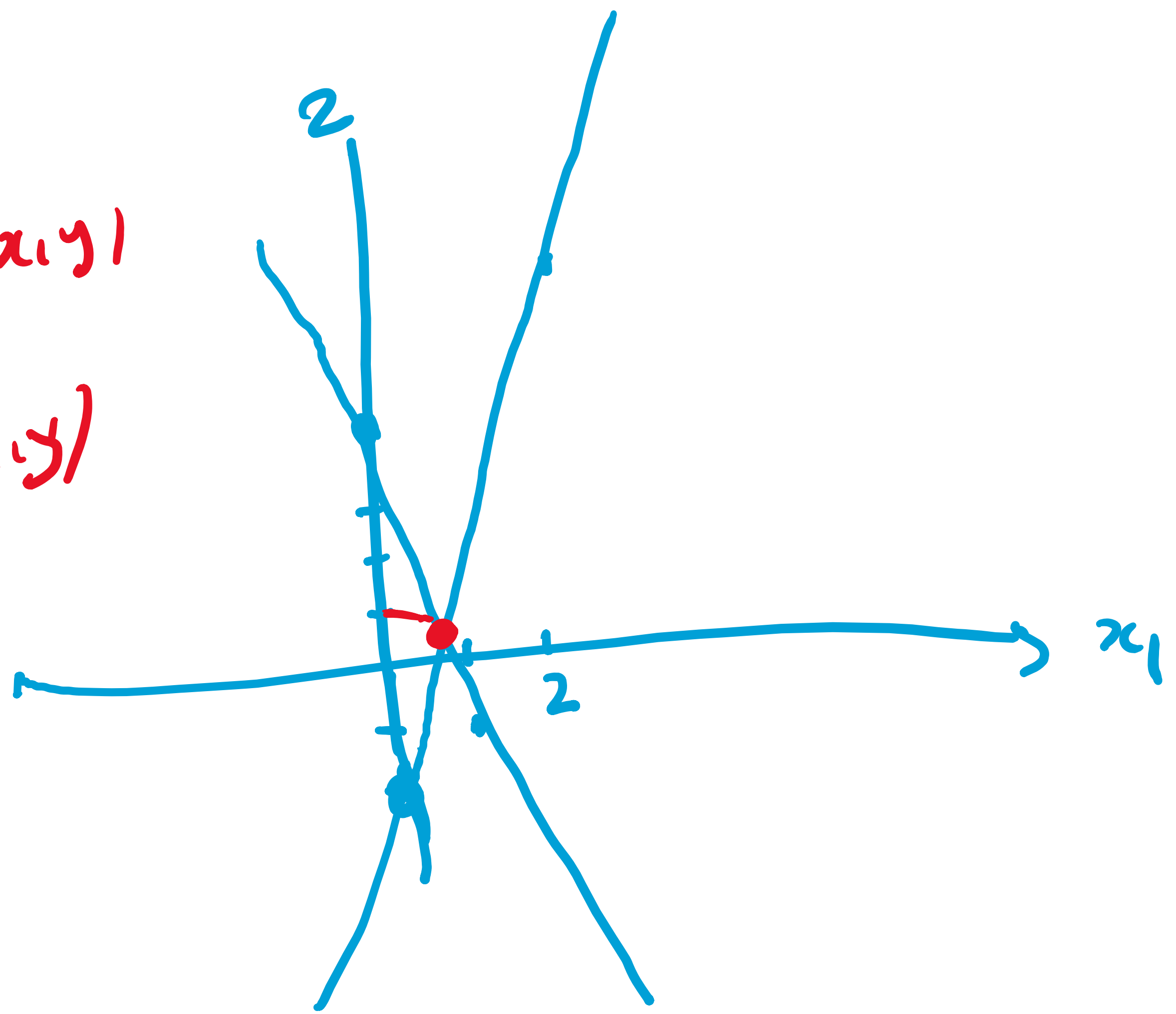
$-x_1 - 4x_1 + \ldots$

$z = -5x_1 + 4$

$$\begin{cases} 5x_1 - 2 \rightarrow (x,y) \\ -5x_1 + 4 \quad (x,y) \end{cases}$$

$$1$$

$-5+4$



THE EVOLUTION OF TRUST

look at The evolution of trust and evaluate the presentation of game theory with the benefit of the content you have engaged with this week