SIT718 Real World Analytics

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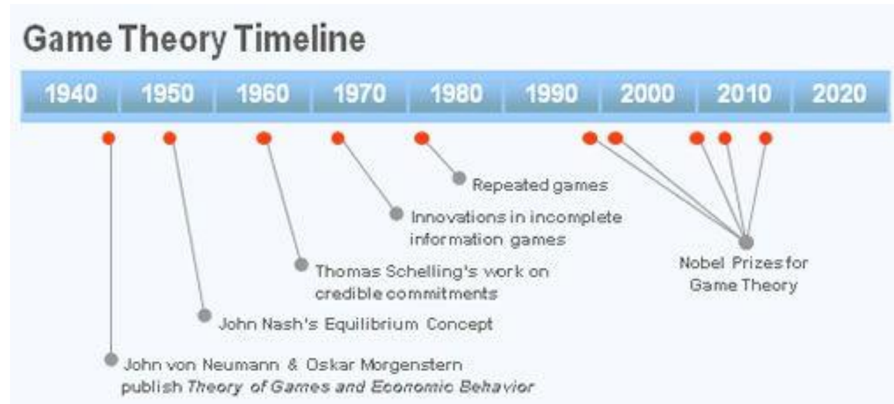
Week 9: Game Theory 1

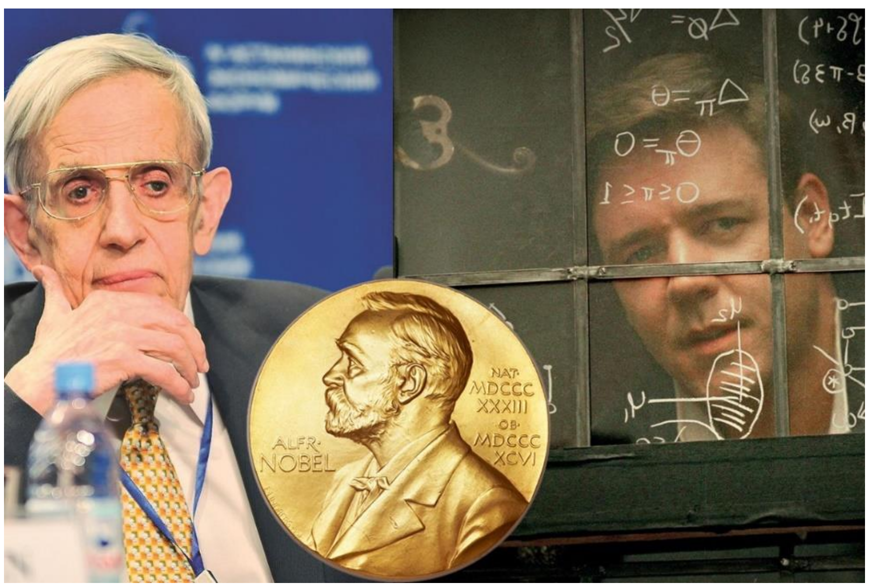
Introduction to Game Theory

- In Many situations two or more decision makers simultaneously choose an action, and the actions chosen by each player affects rewards earned by the other players.
- Examples:
  - Fast food companies - determining advertisement and pricing policy for its product and each company's decision will affect the revenue/profit of the other.
  - competitions between producers of the same product;
  - Political campaigns;
  - Marketing campaigns.
- The final outcome depends upon the combination of strategies selected by the adversaries.
- Game theory:
  Mathematical theory that deals with the general features of competitive or cooperative situations in a formal, abstract way.

Game Theory and John Nash

1994 and 2005 Nobel Prizes for Economic Sciences were given for research in game theory.





What is a Two-Person Zero-sum Game?

The simplest example is a Two-Person Zero-Sum Game.

- It involves two players (hence the term **2-person**)
- Each player has a number of strategies to select from.
- For each strategy pair chosen by the players, the total payoff is 0 (hence **zero-sum**)

$$\text{Payoff to Player I} = -\text{Payoff to Player II}$$

- Such payoffs can be presented in a matrix, which we call a **payoff matrix**)

- Both players are rational.
- Both players choose their strategies solely to promote their own welfare (no compassion for the opponent).

Reference Textbook

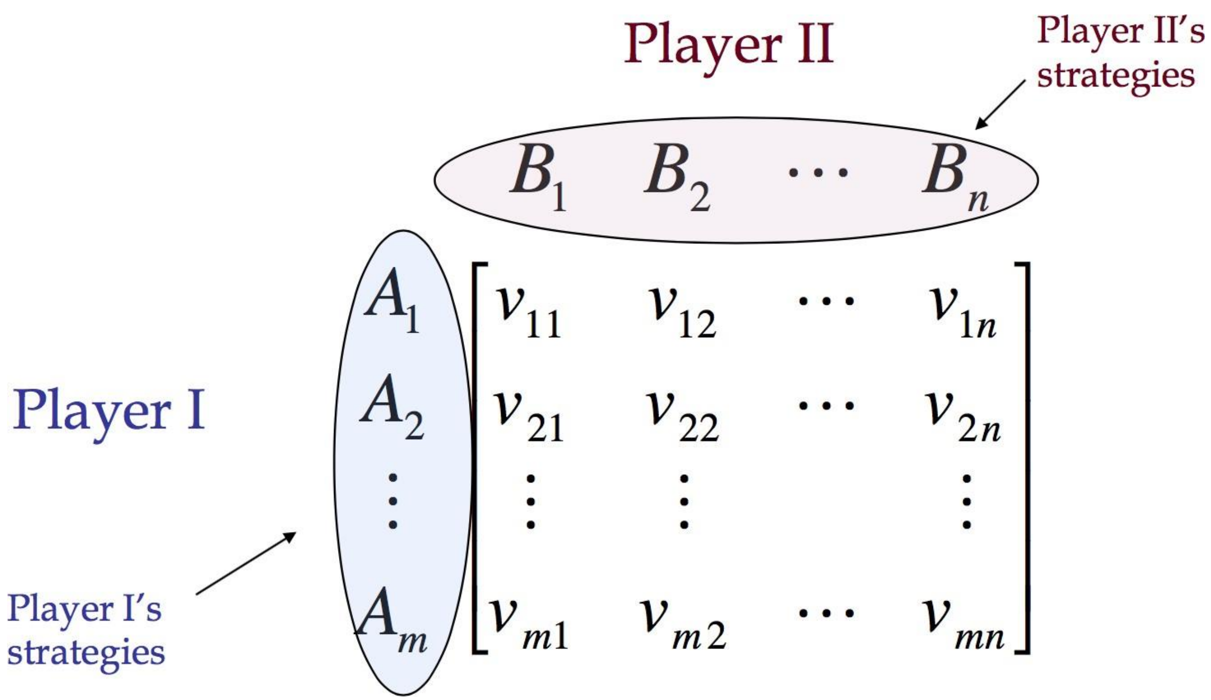
Operations Research: Applications and Algorithms by Wayne L. Winston (Chapter 14)

Notation

- m = number of strategies for Player I
- n = number of strategies for Player II
- A_i = ith strategy (pure) for Player I
- B_j = jth strategy (pure) for Player II
- v_{ij} = Payoff to Player I if he selects Strategy A_i and if Player II selects Strategy B_j
    - Payoff to Player I = −Payoff to Player II

NOTATION

The figure illustrate the mathematical notation.



EXAMPLE 1

$$\begin{array}{cc} & \text{Player II} \\ \text{Player I} & \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 2 & -4 & 25 \\ A_2 & 6 & 10 & -20 \end{array} \end{array}$$

- Player I has two options: Strategies A_1 or A_2
- Player II has three options: Strategies B_1, B_2, or B_3
- If Player I uses Strategy A_1 and Player II uses Strategy B_3, then Player I wins 25 and Player II loses 25
- If Player I uses Strategy A_2 and Player II uses Strategy B_3, then Player I loses 20 and Player II wins 20

The spirit of Game Theory

- One would like to:
  - *Maximize* possible payoff to each player
- In reality:
  - What is the most secured payoff that they can get ***regardless* of what the other player does**
  - Each player gets the best they can "**secure**"
- The spirit:
  - To look at the worst outcomes and pick **least worst outcome**, i.e., be as secure as possible
  - Not to take the risk of losing more in the pursuit of larger gains. In other words, **play safe**.

Back to Example 1

$$\begin{array}{cc|ccc} & & & \text{Player II} & \\ & & B_1 & B_2 & B_3 \\ \text{Player I} & \begin{array}{c} A_1 \\ A_2 \end{array} & \multicolumn{3}{c}{\begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix}} \end{array}$$

- **Ideal:**
  Player I wins 25, Player II wins 20 (but hey, this is not gonna happen).
- If you were Player I, without knowing what strategy Player II would choose, and if you choose Strategy A_1, you are 100% sure of one thing, and what would that be?

Back to Example 1

$$\begin{array}{cc} & \text{Player II} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player I} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix} \end{array}$$

- **Maximum SECURED Payoff for Player I:**
  If Player I chooses Strategy A_1, the worst outcome is that he loses 4 (assuming we do not know what Player II would do). If he chooses Strategy A_2, the worst outcome: he loses 20.
- Of course he would choose Strategy A_1.

Back to Example 1

$$
\begin{array}{cc|c}
 & & \text{Player II} \\
 & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\
\text{Player I} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix}
\end{array}
$$

- **Maximum SECURED Payoff for Player II:**
  - Strategy B_1: loses 6
  - Strategy B_2: loses 10
  - Strategy B_3: loses 25
- Of course Player II would choose Strategy B_1.

Maximum security levels

[*Again, this information is for those who are interested*]

- The security level for *Player I associated with Strategy A_i* is given by (Minimax strategy):

$$s_i := \min\{v_{ij} \ : j = 1, \ldots, n\}, \quad i = 1, \ldots, m$$

- The *maximum security level* for Player I is given by:

$$s^* := \max\{s_i : i = 1, \ldots, m\}$$

- The security level for *Player II associated with Strategy B_j* is given by:

$$t_j := \max\{v_{ij} \ : i = 1, \ldots, m\}, \quad j = 1, \ldots, n$$

- The *maximum security level* for Player II is given by:

$$t^* := \min\{t_j : ij = 1, \ldots, n\}$$

What have we learnt so far?

- The security level of a strategy for a player is the **minimum** guaranteed payoff regardless of what strategy his opponent uses.
- A player never tries to "maximize payoff"; he knows that his opponent will not let him.
- Instead, a player always tries to choose, among all available strategies, the strategy that **maximizes the security level**
- In other words, a player always choose the strategy that gives the **least worst outcome**. In general terms, the "pure" minimax strategy for player I is the strategy that maximizes the minimum gain of this player.

Example 2

Consider the following Two-person Zero-sum Game

$$\begin{array}{ccc}
 & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & s_i \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 10 & 3 \\ -2 & 5 & 1 \\ 1 & -8 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ -2 \\ -8 \end{array} \\
t_j & \begin{array}{ccc} 1 & 10 & 3 \end{array} &
\end{array}$$

- Player I: Determine the minimum of each row
- The Maximum of row minima is the Maximum security level for Player I $= 1$.
- Player II: Determine the maximum of each column (the payoff of Player II is negative of that for Player I)
- The minimum of row maxima is the Maximum security level for Player II $= 1$
- Suggested solution (A_1, B_1), i.e., Player I uses Strategy 1 and Player II uses Strategy 1 too.

EXAMPLE 3

A strategy is **dominated** by a second strategy if the second strategy is always at least as good (and sometimes better) regardless of what the opponent does.

Strategy		Player 2: 1	2	3
Player 1	1	1	2	4
	2	1	0	5
	3	0	1	−1

For player 1, strategy 3 is dominated by strategy 1 because the latter has larger payoffs. Eliminating strategy 3 from further consideration yields the following reduced payoff table:

	1	2	3
1	1	2	4
2	1	0	5

Player 2 now does have a dominated strategy—strategy 3, which is dominated by both strategies 1 and 2.

EXAMPLE 3

Eliminating strategy 3 of Player 2 strategy yields

	1	2
1	1	2
2	1	0

At this point, strategy 2 for player 1 becomes dominated by strategy 1 because the latter is better. Eliminating the dominated strategy leads to

	1	2
1	1	2

Strategy 2 for player 2 now is dominated by strategy 1, so strategy 2 should be eliminated. Consequently, both players should select their strategy 1.

Example 4

Consider the following Two-person Zero-sum Game

$$
\begin{array}{c|cc|c}
 & B_1 & B_2 & s_i \\
\hline
A_1 & 1 & 5 & 1 \\
A_2 & 6 & 2 & \boxed{2} \\
\hline
t_j & 6 & \boxed{5} &
\end{array}
$$

- Maximum security level for Player I $= 2$
- Maximum security level for Player II $= 5$
- Suggested solution (A_2, B_2)
- *Question: will Player II stay with Strategy 2?*

Equilibrium

A solution (A_i, B_j) to a 2-person zero-sum game is said to be **stable** (or, is in **equilibrium**) if:

- Player I, whilst expecting Player II to use Strategy B_j, has no incentives to choose a strategy other than A_i.
- Similarly, Player II, whilst expecting Player I to use Strategy A_i, has no incentives to choose a strategy other than B_j.

Equilibrium continues –

- *If neither player finds an incentive to change their strategy, then we have reached an* **optimal solution**.
- We also call (A_i, B_j) a **saddle point**.

More on Saddle Point

- Let L denote the largest security level for Player I (recall that he wants the $\max$ of the $\min$), and let U denote the smallest security level for Player II (recall that he wants the $\min$ of the $\max$).
- We call L the **lower value of the game** and call U the **upper value of the game**.
- A game that has a value of 0 is said to be a **fair game**.

More on Saddle Point

- If $U = L$, we call this the **value of the game**, and the optimal payoff for both players can be achieved by a **pure strategy**.
- if $U > L$, then a pure strategy will not result in an equilibrium and players must resort to **mixed strategies**. Some modelling problems will be considered in the workshop.

Mixed Strategies

When $L < U$, a pure strategy will not result in an equilibrium. A player can however *mix up* his/her strategies. In this case $[L, U]$ gives the range of the game. The value of the game v is within this range.

By "mixing up" their strategies, we mean, e.g.,

- Player I plays strategy A_i with probability x_i (say, 60% of the time using Strategy A_1 and 40% of the time using Strategy A_2); and
- Player II plays strategy B_j with probability y_j.

1.9 Two-person zero-sum game – graphical solution

$$\begin{array}{cc|cc} & & B_1 & B_2 \\ & & y_1 & y_2 \\ A_1 & x_1 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$}} \\ A_2 & x_2 & & \end{array}$$

Let Player I uses Strategy A_1 with probability x_1 and uses Strategy A_2 with probability $1 - x_1$,

- her expected payoff if Player II uses Strategy B_1 is given by:

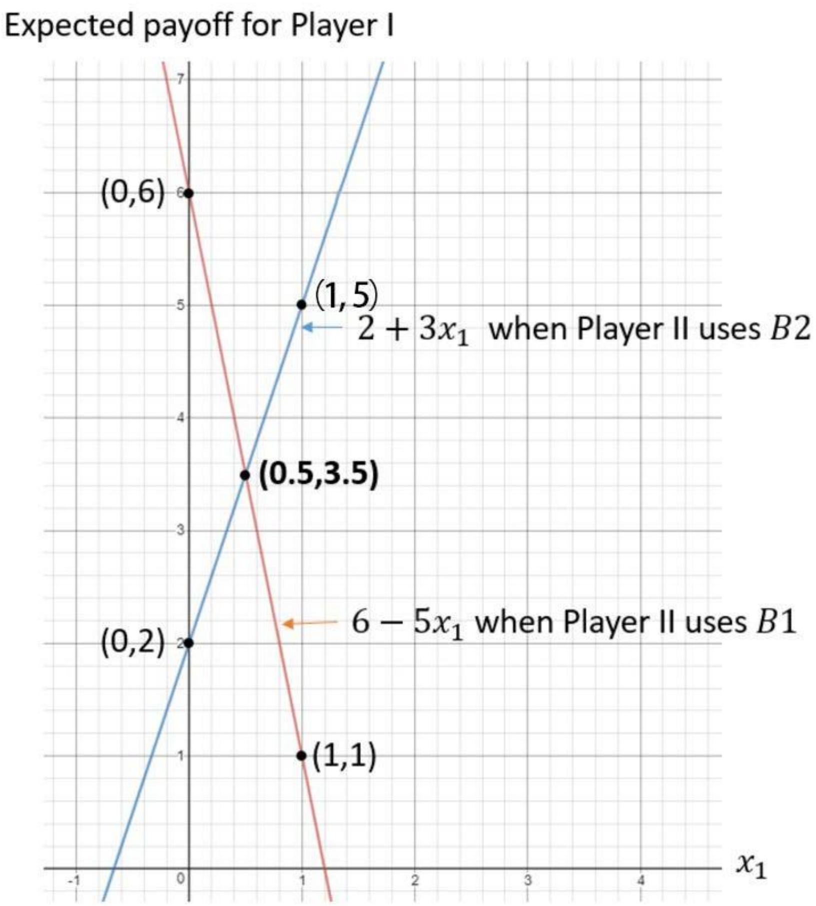
$$x_1(1) + (1 - x_1)(6) = 6 - 5x_1$$

- her expected payoff if Player II uses Strategy B_2 is given by:

$$x_1(5) + (1 - x_1)(2) = 2 + 3x_1$$

Graphical solution

Player I wants to maximise her expected payoff regardless of what the strategy made by Player II, we can use a graphical method as follows. $x_1 = 0.5$ is the optimal solution.



GRAPHICAL SOLUTION

$$\begin{array}{ccc} & & \begin{array}{cc} B_1 & B_2 \\ y_1 & y_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix} \end{array}$$

Let Player II uses Strategy B_1 with probability y_1 and uses Strategy B_2 with probability $1 - y_1$,

- her expected payoff if Player I uses Strategy A_1 is given by:

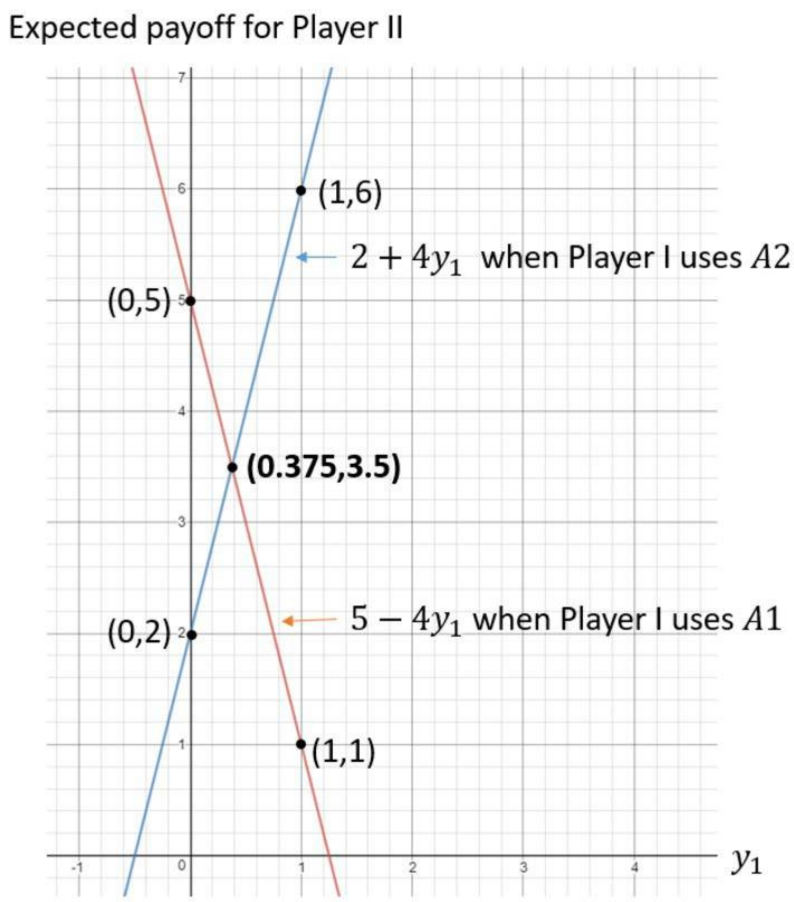
$$y_1(1) + (1 - y_1)(5) = 5 - 4y_1$$

- her expected payoff if Player I uses Strategy A_2 is given by:

$$y_1(6) + (1 - y_1)(2) = 2 + 4y_1$$

Graphical solution

The graphical solution with $y_1 = 0.375$ is shown as follows.



Graphical solution 2

Let's consider

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{bmatrix}$$

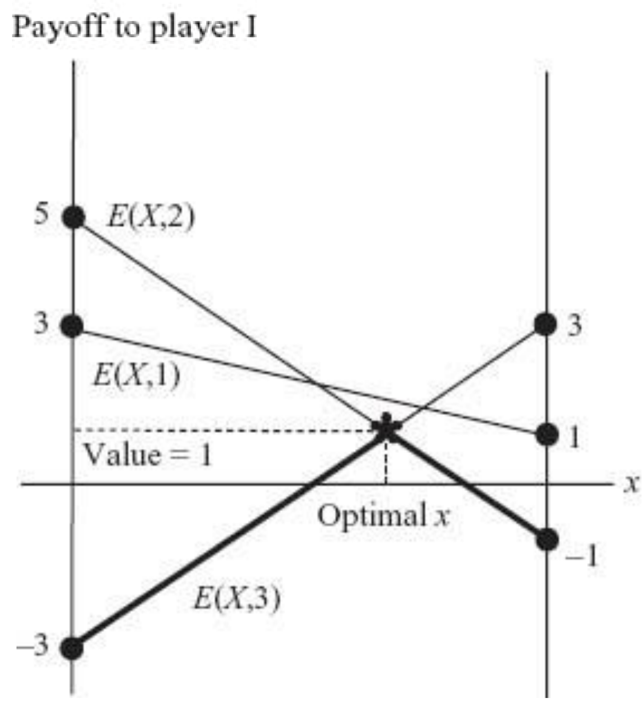
This is a 2×3 game without a saddle point in pure strategies since $L = -1$, $U = 3$

Graphical solution 2

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{bmatrix}$$

Suppose that player I uses the strategy $X = (x, 1 - x)$, then we graph the payoffs $E(X, i)$, $i = 1, 2, 3$:

$$E(X, 1) = x + 3(1 - x)$$
$$E(X, 2) = - x + 5(1 - x)$$
$$E(X, 3) = 3x - 3(1 - x)$$



The optimal strategy for player II is

$$X^* = (\frac{2}{3}, \frac{1}{3})$$

https://www.desmos.com/calculator

Graphical solution 3

Let's consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

This is a 4×2 game without a saddle point in pure strategies since $L = -1$, $U = 6$

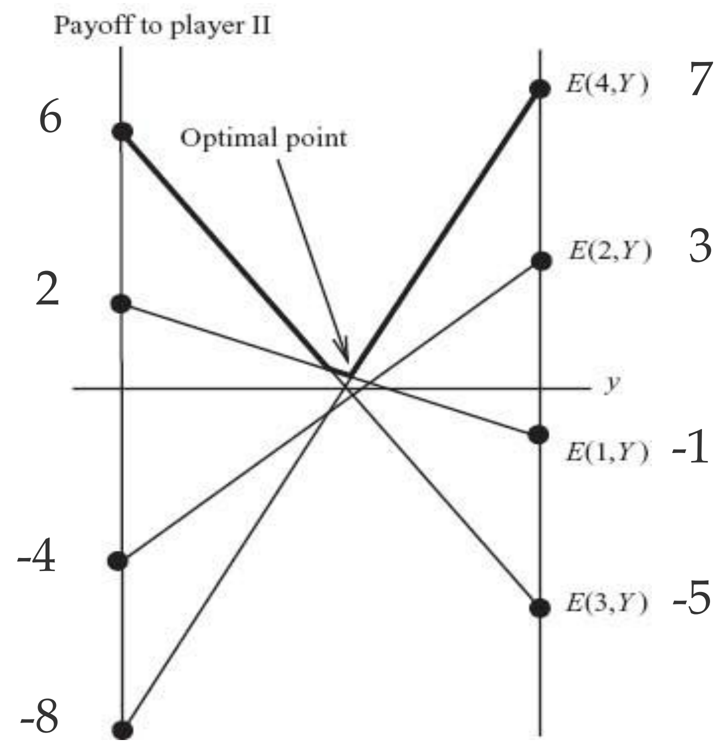
We can use the graphical solution for Player II with two decision variables.

GRAPHICAL SOLUTION 3

Suppose that player II uses the strategy $Y = (y, 1 - y)$, then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4$:

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

$$E(1, Y) = -y + 2(1 - y)$$
$$E(2, Y) = 3y - 4(1 - y)$$
$$E(3, Y) = -5y + 6(1 - y)$$
$$E(4, Y) = 7y - 8(1 - y)$$



The optimal strategy for player II is

$$Y^* = \left(\frac{5}{9}, \frac{4}{9}\right)$$