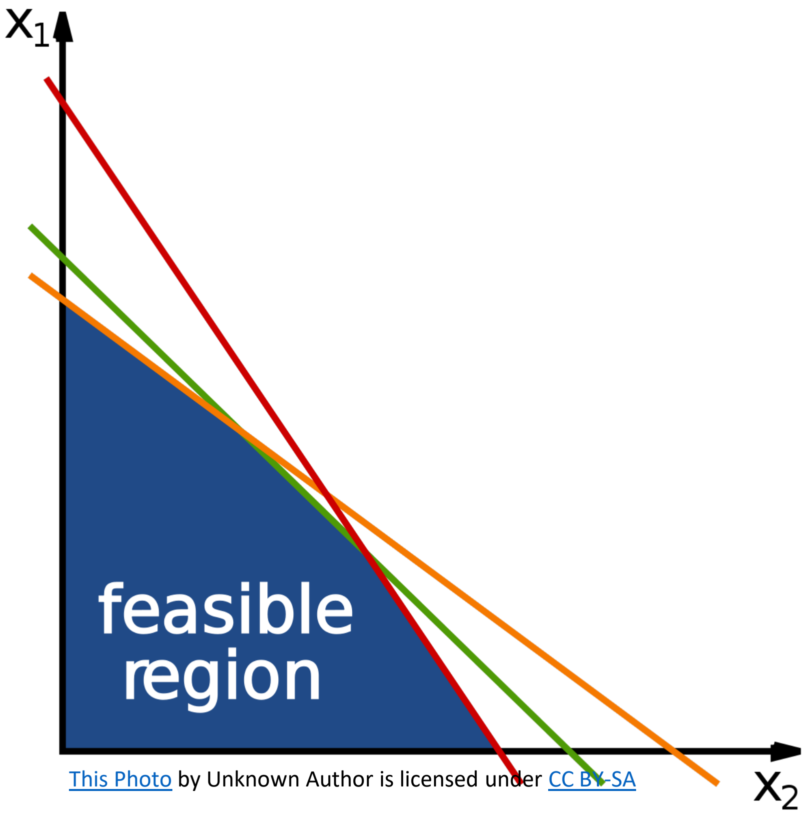

SIT718 – Introduction to Linear Programming- Week 2

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Linear Programming with 2+ Variables

Our LP formulations containing many more variables mean that we're unable to visualise them easily using the graphical method, and must resort to computational methods.

Example:

Before our toy company came into fruition, the founders were figuring out a loan policy that involves up to 12 million for all their assets. The types of loans, respective interest rates and bad-debt ratio are provided as follows:

Type of Loan	Interest Rate	Bad-Debt Ratio
Personal	18.33%	10%
Car	8.76%	7%
Home	4.62%	3%
Farm	12.5%	5%
Commercial	10%	4.5%

Example - Bank loans - continued

- Bad-debts are unrecoverable and produce no interest income.
- Ratio of bad-debts over all loans cannot be more than 4%
- At least 40% of funds must go to farm and commercial loans.
- Home loans must equal to at least 50% of the sum of personal, home and car loans.
- All loans are released at the same time.

Type of Loan	Interest Rate	Bad-Debt Ratio
Personal	18.33%	10%
Car	8.76%	7%
Home	4.62%	3%
Farm	12.5%	5%
Commercial	10%	4.5%

Example - Bank loans - continued

$$\text{Total Interest} = 0.1833(1-0.1)x_1 + 0.0876(1-0.07)x_2 + 0.0462(1-0.03)x_3 + 0.125(1-0.05)x_4 + 0.1(1-0.045)x_5$$

$$\text{Bad Debt} = 0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.045x_5$$

Therefore our objective function is:

$$\begin{aligned}\max z &= 0.16497x_1 + 0.081468x_2 + 0.044814x_3 + 0.11875x_4 \\ &\quad + 0.0955x_5 - (0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.045x_5) \\ &= 0.06497x_1 + 0.011468x_2 + 0.014814x_3 + 0.06875x_4 \\ &\quad + 0.0505x_5\end{aligned}$$

Type of Loan	Interest Rate	Bad-Debt Ratio
Personal	18.33%	10%
Car	8.76%	7%
Home	4.62%	3%
Farm	12.5%	5%
Commercial	10%	4.5%

Example - Bank loans - continued

1. Total funds should not exceed 12 million:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

2. Farm and commercial loans make up at least 40% of loans.

$$\frac{x_4 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \geq 0.4$$

$$\implies x_4 + x_5 \geq 0.4(x_1 + x_2 + x_3 + x_4 + x_5)$$

$$\implies -0.4x_1 - 0.4x_2 - 0.4x_3 + 0.6x_4 + 0.6x_5 \geq 0$$

3. Home loans make up at least 50% of of the sum of personal, car and home loans.

$$\frac{x_3}{x_1 + x_2 + x_3} \geq 0.5$$

$$\implies -0.5x_1 - 0.5x_2 + 0.5x_3 \geq 0$$

4. Bad debts should not exceed 4% of all loans.

$$0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.045x_5 \leq 0.04(x_1 + x_2 + x_3 + x_4 + x_5)$$

5. Non-negativity constraints.

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution:

$$z = 0.501;\ x_1 = x_2 = x_5 = 0;\ x_3 = x_4 = 6$$

1- Bad-debts are unrecoverable and produce no interest income.
2- Ratio of bad-debts over all loans cannot be more than 4%
3- At least 40% of funds must go to farm and commercial loans.
4- Home loans must equal to at least 50% of the sum of personal, home and car loans.
5- All loans are released at the same time.

Single Period Production Model

Hau's Clothing produces down jackets, fleece hoodies, thermo pants and beanies. If there are unmet demands, a penalty cost will be incurred.
The production time (in hours) taken in the manufacturing process (cutting, insulating, sewing, packaging) for each of the products are in the table below:

Process	Jackets	Hoodies	Pants	Beanie	Process Capacity (hrs)
Cutting	0.3	0.3	0.25	0.15	1000
Insulating	0.25	0.35	0.3	0.1	1000
Sewing	0.45	0.5	0.4	0.22	1000
Packaging	0.15	0.15	0.1	0.05	1000

The demand, profit and penalties are as follows:

	Jackets	Hoodies	Pants	Beanies
Demand	800	750	600	500
Unit Profit	$30	$40	$20	$10
Unit Penalty	$15	$24	$10	$8

Your Task:
Formulate an LP model to maximise the net profit. We'll want variables representing each of the items, and variables to represent the amount of shortage for each of the items.

Blending Crude Oils into Gasolines

	Purchase Price (per barrel)		Sales Price (per barrel)
Crude 1	$45	Gas 1	$70
Crude 2	$35	Gas 2	$60
Crude 3	$25	Gas 3	$50

Content restrictions for both oil & gas:

Crude Oil	Octane Rating	Sulphur Content
Crude 1	12	0.5%
Crude 2	6	2.0%
Crude 3	8	3.0%

Gasoline	Octane Rating	Sulphur Content
Gas 1	≥ 10	$\leq 1.0\%$
Gas 2	≥ 8	$\leq 2.0\%$
Gas 3	≥ 6	$\leq 1.0\%$

Other restrictions:

1- It costs $4 to transform one barrel of oil into one barrel of gas.
2- Purchase Limit: 5000 barrels of each type of crude oil.
3- Maximum Production: 14000 barrels per day.
4- Customer Demands:

- 3000 barrels of Gas 1
- 2000 barrels of Gas 2
- 1000 barrels of Gas 3

5- Each dollar spent on advertising increases demand by 10 barrels.

Blending Crude Oils into Gasolines

Decision Variables:
ai = the amount of dollars spent daily on advertising Gas,
i= 1,2, 3
xij = the number of barrels of Crude i used daily to produce Gas j,
i=j=1,2,3

Objective Function:
maximise the daily profit, where the profit is revenue minus costs.

Crude 1: x11+x12+x13
Crude 2: x21+x22+x23
Crude 3: x31+x32+x33

Gas 1: x11+x21+x31
Gas 2: x12+x22+x32
Gas 3: x13+x23+x33

	Purchase Price (per barrel)		Sales Price (per barrel)
Crude 1	\$45	Gas 1	\$70
Crude 2	\$35	Gas 2	\$60
Crude 3	\$25	Gas 3	\$50

- Crude 1: $x_{11} + x_{12} + x_{13}$
- Crude 2: $x_{21} + x_{22} + x_{23}$
- Crude 3: $x_{31} + x_{32} + x_{33}$

Number of barrels of each type of gas produced daily:

- Gas 1: $x_{11} + x_{21} + x_{31}$
- Gas 2: $x_{12} + x_{22} + x_{32}$
- Gas 3: $x_{13} + x_{23} + x_{33}$

Daily revenue from gas sales:

$$70(x_{11} + x_{21} + x_{31}) + 60(x_{12} + x_{22} + x_{32}) + 50(x_{13} + x_{23} + x_{33})$$

Daily cost of purchasing crude oil:

$$45(x_{11} + x_{12} + x_{13}) + 35(x_{21} + x_{22} + x_{23}) + 25(x_{31} + x_{32} + x_{33})$$

Blending Crude Oils into Gasolines

Objective Function:
maximise the daily profit, where the profit is revenue minus costs.

Daily revenue from gas sales:

$$70(x_{11}+x_{21}+x_{31})+60(x_{12}+x_{22}+x_{32})+50(x_{13}+x_{23}+x_{33})$$

Daily cost of purchasing crude oil:

$$45(x_{11}+x_{12}+x_{13})+35(x_{21}+x_{22}+x_{23})+25(x_{31}+x_{32}+x_{33})$$

Daily advertising costs:

$$a_1+a_2+a_3$$

Daily production costs:

$$4(x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23}+x_{31}+x_{32}+x_{33})$$

$$\begin{aligned}\max z = {} & 21x_{11}+11x_{12}+x_{13} \\ & +31x_{21}+21x_{22}+11x_{23} \\ & +41x_{31}+31x_{32}+21x_{33} \\ & -a_1-a_2-a_3\end{aligned}$$

Blending Crude Oils into Gasolines

- Gas 1 Demand: $x_{11} + x_{21} + x_{31} = 3000 + 10a_1$
- Gas 2 Demand: $x_{12} + x_{22} + x_{32} = 2000 + 10a_2$
- Gas 1 Demand: $x_{13} + x_{23} + x_{33} = 1000 + 10a_3$

Since we're limited to purchasing 5000 barrels for each type of crude oil, our purchase limit constraints are:

- Crude 1: $x_{11} + x_{12} + x_{13} \leq 5000$
- Crude 2: $x_{21} + x_{22} + x_{23} \leq 5000$
- Crude 3: $x_{31} + x_{32} + x_{33} \leq 5000$

Our production limit is at most 14000 barrels of gasoline in total, giving us:

$$x_{11} + x_{12} + x_{13}$$
$$+x_{21} + x_{22} + x_{23}$$
$$+x_{31} + x_{32} + x_{33} \leq 14000$$

1- It costs \$4 to transform one barrel of oil into one barrel of gas.
2- Purchase Limit: 5000 barrels of each type of crude oil.
3- Maximum Production: 14000 barrels per day.
4- Customer Demands:
- 3000 barrels of Gas 1
- 2000 barrels of Gas 2
- 1000 barrels of Gas 3

5- Each dollar spent on advertising increases demand by 10 barrels.

Blending Crude Oils into Gasolines

- Gas 1 Demand: $x_{11} + x_{21} + x_{31} = 3000 + 10a_1$
- Gas 2 Demand: $x_{12} + x_{22} + x_{32} = 2000 + 10a_2$
- Gas 1 Demand: $x_{13} + x_{23} + x_{33} = 1000 + 10a_3$

Customer Demands:
- 3000 barrels of Gas 1
- 2000 barrels of Gas 2
- 1000 barrels of Gas 3

Each dollar spent on advertising increases demand by 10 barrels.

Since we're limited to purchasing 5000 barrels for each type of crude oil, our purchase limit constraints are:

- Crude 1: $x_{11} + x_{12} + x_{13} \leq 5000$
- Crude 2: $x_{21} + x_{22} + x_{23} \leq 5000$
- Crude 1: $x_{31} + x_{32} + x_{33} \leq 5000$

Purchase Limit: 5000 barrels of each type of crude oil.

Our production limit is at most 14000 barrels of gasoline in total, giving us:

$$\begin{aligned} &x_{11} + x_{12} + x_{13} \\ &+x_{21} + x_{22} + x_{23} \\ &+x_{31} + x_{32} + x_{33} \leq 14000 \end{aligned}$$

Maximum Production: 14000 barrels per day.

Blending Crude Oils into Gasolines

$$\frac{12x_{11} + 6x_{21} + 8x_{31}}{x_{11} + x_{21} + x_{31}} \geq 10$$

$$\implies 2x_{11} - 4x_{21} - 2x_{31} \geq 0$$

With respect to octane, the constraints indicate by the octane rating what range the average octane level should be. For each gas, the average octane level can be calculated as the proportion of octane in each gas divided by the total amount of gas. For Gas 1

$$4x_{12} - 2x_{22} \geq 0$$

$$6x_{13} + 2x_{33} \geq 0$$

Similar calculations for Gas 2 and Gas 3 results in the following respective constraints

$$\frac{0.005x_{11} + 0.02x_{21} + 0.03x_{31}}{x_{11} + x_{21} + x_{31}} \leq 0.01$$

$$\implies -0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0$$

The constraints for sulphur levels in each gas have a similar calculation, except with percentages. For Gas 1, since the sulphur level must be at most 1%, we have the following constraint

$$-0.015x_{12} + 0.01x_{32} \leq 0$$

$$-0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0$$

Similar calculations for Gas 2 and Gas 3 give the following respective constraints

$$\left\{ \begin{array}{ll} x_{ij} \geqslant 0, & i,j \in \{1,2,3\} \\ a_i \geqslant 0 & i = \{1,2,3\} \end{array} \right.$$

Blending Crude Oils into Gasolines

Solution

$$z = 287750$$
$$x_{11} = 2088.9; \quad x_{12} = 2111.1; \quad x_{13} = 800$$
$$x_{21} = 777.78; \quad x_{22} = 4222.2; \quad x_{23} = 0$$
$$x_{31} = 133.33; \quad x_{32} = 3166.7; \quad x_{33} = 200$$
$$a_1 = 0; \quad a_2 = 750; \quad a_3 = 0$$

In terms of barrel production, we are to produce:

- $3,000$ barrels of Gas 1
- $9,500$ barrels of Gas 2, and
- $1,000$ barrels of Gas 3.

Transportation Models

Transportation problems involve deciding how best to allocate resources from suppliers to consumers, and can be framed into many different contexts

$$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \sum_{j=1}^{n} x_{ij} \leq s_i \quad \forall i = 1, \ldots, m \text{ (Supply Constraints)}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_i \quad \forall i = 1, \ldots, m \text{ (Demand Constraints)}$$

$$x_{ij} \geq 0 \quad \forall i = 1, \ldots, m; j = 1, \ldots, m$$

Electric Power Plants

Three electric power plants currently supply electricity to four cities, but how best do we distribute their resources to minimise costs?

	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	72
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	78
Demand (million kwh)	45	70	30	55	

Electric Power Plants - Continued

$$\begin{aligned}\min z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} \\ + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \\ + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}\end{aligned}$$

Objective Function

Plant 1: $x_{11} + x_{12} + x_{13} + x_{14} \leq 72$

Plant 2: $x_{21} + x_{22} + x_{23} + x_{24} \leq 50$

Plant 3: $x_{31} + x_{32} + x_{33} + x_{34} \leq 78$

Supply Constraint

City 1: $x_{11} + x_{21} + x_{31} \geq 45$

City 2: $x_{12} + x_{22} + x_{32} \geq 70$

City 3: $x_{13} + x_{23} + x_{33} \geq 30$

City 4: $x_{14} + x_{24} + x_{34} \geq 55$

Demand Constraint

$x_{ij} \geq 0 \forall i = 1, 2, 3; j = 1, 2, 3, 4.$

Non-negativity constraints

$x_{12} = 47, x_{13} = 25, x_{21} = 45, x_{23} = 5, x_{32} = 23, x_{34} = 55$ with the total cost being \$1484.

Optimal solution

	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	\$8	\$6	\$10	\$9	72
Plant 2	\$9	\$12	\$13	\$7	50
Plant 3	\$14	\$9	\$16	\$5	78
Demand (million kwh)	45	70	30	55	

Excess Demand & Using Dummy Demand Points

If the total supply exceeds the total demand, we can always use a dummy demand point with the demand exactly the amount of excess supply. As “shipments” to the dummy demand point is not a real shipment, the shipment cost will be zero

	City 1	City 2	City 3	City 4	Dummy	Supply (million kwh)
Plant 1	\$8	\$6	\$10	\$9	0	72
Plant 2	\$9	\$12	\$13	\$7	0	60
Plant 3	\$14	\$9	\$16	\$5	0	78
Demand (million kwh)	45	70	30	55	10	

Excess Demand & Using Dummy Demand Points

$$\min z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + \mathbf{0x_{15}}$$

$$+ 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + \mathbf{0x_{25}}$$

$$+ 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} + \mathbf{0x_{35}}$$

$$\text{s.t.} x_{11} + x_{12} + x_{13} + x_{14} + \mathbf{x_{15}} \leq 72$$

$$x_{21} + x_{22} + x_{23} + x_{24} + \mathbf{x_{25}} \leq 50$$

$$x_{31} + x_{32} + x_{33} + x_{34} + \mathbf{x_{35}} \leq 78$$

$$x_{11} + x_{21} + x_{31} \geq 45$$

$$x_{12} + x_{22} + x_{32} \geq 70$$

$$x_{13} + x_{23} + x_{33} \geq 30$$

$$x_{14} + x_{24} + x_{34} \geq 55$$

$$\mathbf{x_{15} + x_{25} + x_{35} \geq 10}$$

$$x_{ij} \geq 0 \quad \forall i = 1, \ldots, 3; j = 1, \ldots, 5$$

Optimal Solution:
X12 = 47
X13 = 25
X21 = 45
X23 = 5
X32 = 23
X34 = 55

Total Cost = $1484

Handling Shortages

If the total supply is strictly less than the total demand, then the transportation problem is infeasible

	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	60
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	70
Demand (million kwh)	45	70	30	55	
Penalty Cost (p/ million kwh)	100	120	115	130	

Example - Inventory Problem

Inventory problems can be set up as transportation problems as well, but needs a bit of work to set them up properly.

Luthier Productions produces and sells cellos to the city of Gotham. The CEO must decide how many cellos should be made during each of the next 4 quarters.
The demands for the next 4 quarters are estimated as follows:
Q1=40, Q2=60, Q3=75, Q4=25

In each quarter, Luthier Productions can produce up to 40 cellos with a production cost of $400 per cello. If extra employees are hired and overtime is incurred, they are able to make 150 more at a cost of $450 each.
A holding cost of $20 per cello is incurred for any cello that remains by the end of the quarter.
In formulating a balanced transportation problem to minimise the sum of production and inventory cost over the next four quarters, we need to identify the supply/demand points and set up a cost matrix.
In identifying the supply points, since the cost is different for overtime production of cellos, we need to treat them separately from regular production.

Example - Inventory Problem - Continued

Supply Point	Description	Amount
Point 1	Initial Inventory	$s_1 = 10$
Point 2	Q1 Regular Production	$s_2 = 40$
Point 3	Q1 Overtime Production	$s_3 = 150$
Point 4	Q2 Regular Production	$s_4 = 40$
Point 5	Q2 Overtime Production	$s_5 = 150$
Point 6	Q3 Regular Production	$s_6 = 40$
Point 7	Q3 Overtime Production	$s_7 = 150$
Point 8	Q4 Regular Production	$s_8 = 40$
Point 9	Q4 Overtime Production	$s_9 = 150$
	Total Supply	770

Demand Point	Description	Amount
Point 1	Q1 Demand	$d_1 = 40$
Point 2	Q2 Demand	$d_2 = 60$
Point 3	Q3 Demand	$d_3 = 75$
Point 4	Q4 Demand	$d_4 = 25$
Point 5	Dummy Demand	$d_5 = 770 - 200 = 570$

Inventory Problem – Matrix Cost

Supply/ Description	Q1	Q2	Q3	Q4	Dummy	Supply
1 - Initial Inv.	0	20	40	60	0	10
2 - Q1 Regular	400	420	440	460	0	40
3 - Q1 Overtime	450	470	490	510	0	150
4 - Q2 Regular	M	400	420	440	0	40
5 - Q2 Overtime	M	450	470	490	0	450
6 - Q3 Regular	M	M	400	420	0	40
7 - Q3 Overtime	M	M	450	470	0	150
8 - Q4 Regular	M	M	M	400	0	40
9 - Q4 Overtime	M	M	M	450	0	150
Demand	40	60	75	25	570	

Solving LPs with R and lpSolveAPI

$$\max z = 3s + 2t$$

$$2s + t \leq 100$$

$$s + t \leq 80$$

$$s \geq 40$$

$$s, t \geq 0$$

```
library(lpSolveAPI) # We'll want to have this at the start of every LP script for consistency
# Initialising the model with 0 initial constraints and 2 decision variables
toyCompanyModel = make.lp(0,2)
# Specifying that we want to maximise our objective function
lp.control(toyCompanyModel, sense="max")
# Setting up our objective function
set.objfn(toyCompanyModel, c(3,2))
# Adding our constraints
## 2s+t <= 100
add.constraint(toyCompanyModel, c(2,1), "<=", 100)
## s+t <= 80
add.constraint(toyCompanyModel, c(1,1), "<=", 80)
## s >= 40
add.constraint(toyCompanyModel, c(1,0), ">=", 40)
#Setting up names of constraints and variables-this is optional, make your code and models readable!
constraintNames = c("Finishing Labour Constraint", "Carpentry Labour Constraint", "Minimum Weekly Demand")
variableNames = c("soldiers", "trains")
dimnames(toyCompanyModel) = list(constraintNames, variableNames)
# Solves the model
solve(toyCompanyModel)
```

Solving LPs with R and lpSolveAPI

$$\max z = 3s + 2t$$

$$2s + t \leq 100$$

$$s + t \leq 80$$

$$s \geq 40$$

$$s, t \geq 0$$

After running the above, type the following commands in the console. The comment after each command describe what you should be seeing

```
toyCompanyModel
# You should see a representation of the model similar to how we've written it initially.

get.objective(toyCompanyModel)
# You'll obtain a single number 160, which represents the objective value z at the optimal solution.

get.variables(toyCompanyModel)
# You should obtain two numbers: 40 and 20. Both represent the optimal number of soldiers and trains to make respectively.

get.constraints(toyCompanyModel)
# You should obtain three numbers: 100, 60 and 40. These represent the right hand side of their respective constraints at our optimal solution. Notice that the first number is 100, which also matches the right hand side of the first constraint - this means that there is no 'slack', or leeway in that constraint at our optimal solution. We cannot draw the same conclusion for our second constraint, though.
```

Transportation Problem

```
electricityModel <- make.lp(7, 12)
lp.control(electricityModel, sense= "minimize")
set.objfn(electricityModel, c(8,6,10,9,9,12,13,7,14,9,16,5))
set.row(electricityModel, 1, rep(1,4), indices = c(1:4))
set.row(electricityModel, 2, rep(1,4), indices = c(5:8))
set.row(electricityModel, 3, rep(1,4), indices = c(9:12))
set.row(electricityModel, 4, rep(1,3), indices = c(1,5,9))
set.row(electricityModel, 5, rep(1,3), indices = c(2,6,10))
set.row(electricityModel, 6, rep(1,3), indices = c(3,7,11))
set.row(electricityModel, 7, rep(1,3), indices = c(4,8,12))
set.constr.type(electricityModel, c("<=",      "<=",      "<=", ">=", ">=",   ">=",      ">="))
set.rhs(electricityModel, c(72,       50,        78,        45,        70,        30,        55))
set.type(electricityModel, c(1:12),"real") # Setting all decision variables to be real numbers.
set.bounds(electricityModel, lower = rep(0, 12), upper = rep(Inf, 12))
solve(electricityModel)
electricityModel
get.objective(electricityModel)
get.variables(electricityModel)
get.constraints(electricityModel)
```