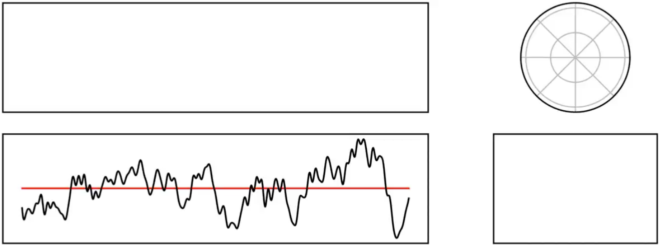
COMS20011 – Data-Driven Computer Science

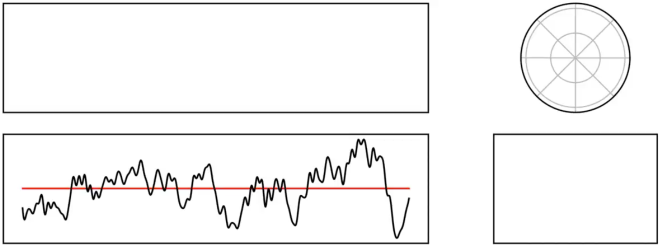


Signals & Frequencies

March 2023

Majid Mirmehdi

Next in DDCS



Feature Selection and Extraction

- **Signal basics and Fourier Series**
- 1D and 2D Fourier Transform
- Characteristics of features
- Convolutions

Typical Data Analysis Problem

Steps:

1. Pre-processing [Unit - Part 1] → Majid Mirmehdi (~10%)
2. Feature Selection [Unit - Part 3] → Majid Mirmehdi (~40%)
3. Modelling & Classification [Unit - Part 2] → Laurence Aitchison **[UD]** (~50%)

Summary: Typical Data Analysis Problem (Reminder)

1. Pre-processing
2. Feature Selection
3. Modelling & Classification

Supervised Training

Pre-processed data

Image Features → Classifier Training → Trained Classifier

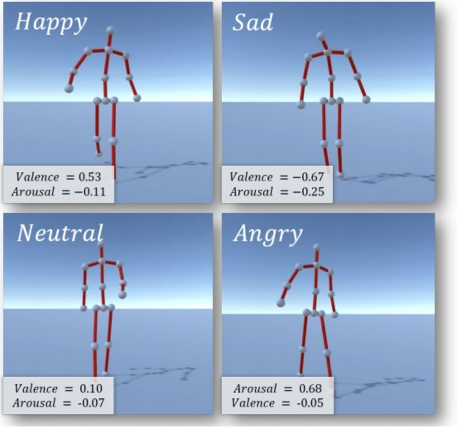
Training Labels → Classifier Training

Testing

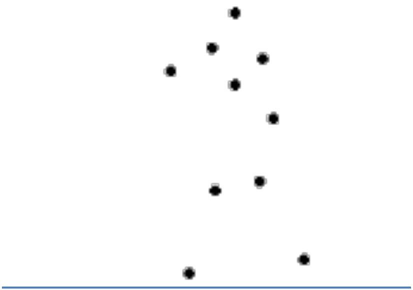
Test Data

Image Features → Trained Classifier → Prediction *Outdoor*

Features help simplify the problem



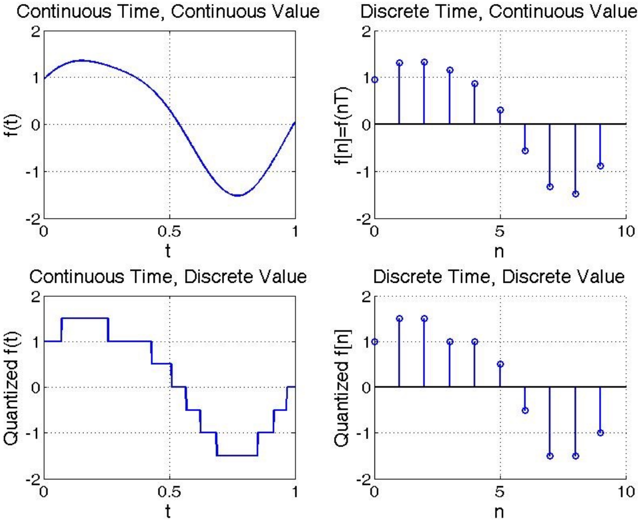
Patient with mild Parkinson’s Disease



- ➢ Even “impoverished” motion data can evoke a strong perception

Image from https://medium.com/artificial-intelligence-network/

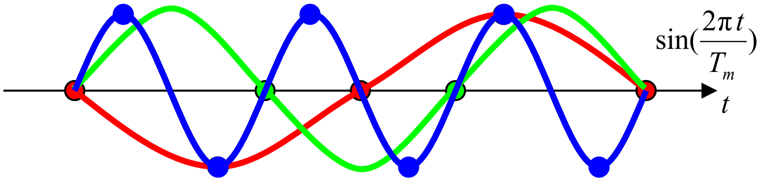
Sample and Quantise – Reminder



Nyquist-Shannon Sampling Theory - Reminder

*"An analogue signal containing components up to some maximum frequency **u** (Hz) may be completely reconstructed by regularly spread samples, provided the sampling rate is at least **2u** samples per second"*

Also referred to as the Nyquist-Shannon criterion: sampling rate s should be at least twice the highest spatial frequency u.



sampling period $T_m \leq \dfrac{1}{2u}$

equivalent to *sampling rate* $s \geq 2u$

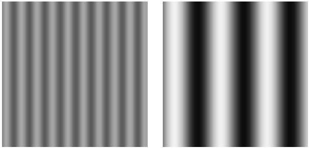
Basic Signals

AAAAATAAAAA

0000001000000

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

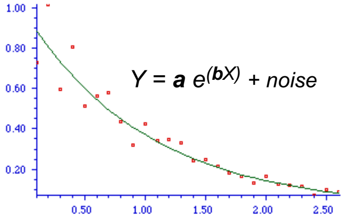
*x = sin(t) = sin(t+2**π**)*



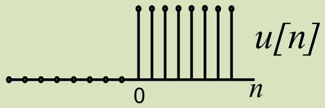
Some basic signals:

- **Unit impulse signal**
- **Unit step signal**
- **Exponential signal**
- **Periodic signal**

All signals can be represented by these basic signals!



$Y = \boldsymbol{a}\, e^{(\boldsymbol{b}X)} + noise$

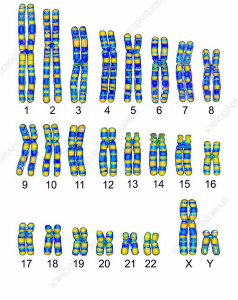


$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

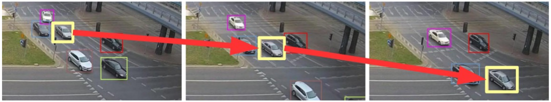
Signals as Functions

A signal is a physical quantity that is a function of one or more independent variable(s), such as space and/or time.

Data from a Gene pool



Position of a car in a video sequence



Example signals:

1D signal: $f(t)$
2D signal: $f(x,y)$
3D signal: $f(x,y,t)$ etc.

Signals as Functions

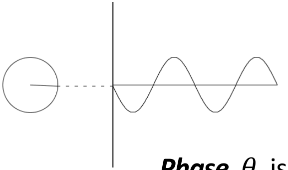
period is the time T it takes to finish one oscillation.

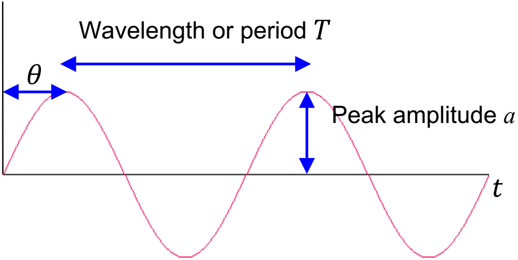
frequency $u = \frac{1}{T}$ is the number of periods per second, measured in Hz.

amplitude a is a measure of how much it changes over a single period.

Example: sine function

$$f(t) = a \sin 2\pi u t$$





Phase θ is where the signal is in the oscillation (or the angular position).

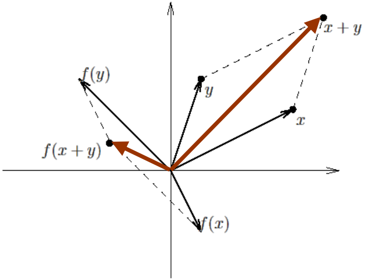
Linear Systems

For a linear system: output of the linear combination of many input signals is the same linear combination of the outputs → ***superposition***

A function f is linear if

- $f(x + y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

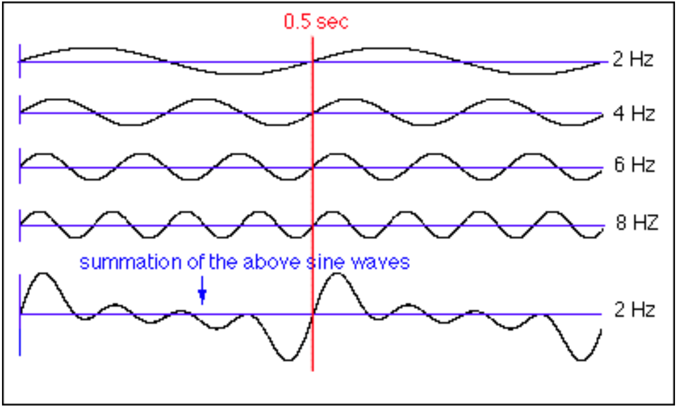
i.e., superposition holds.



Output is the sum of the system's response to these basic objects.

Example I: a simple signal

For a linear system: output of the linear combination of many input signals is the same linear combination of the outputs → ***superposition***

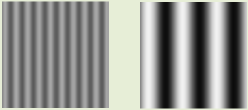


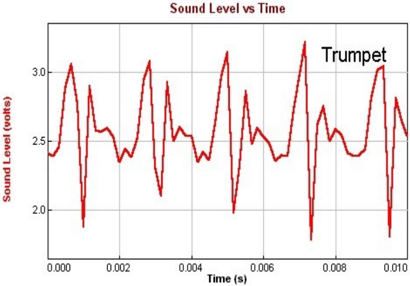
The combined frequency is the highest common factor.

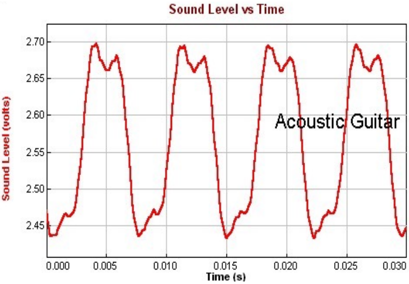
Example III: white light

How should we interpret these musical instrument signals?

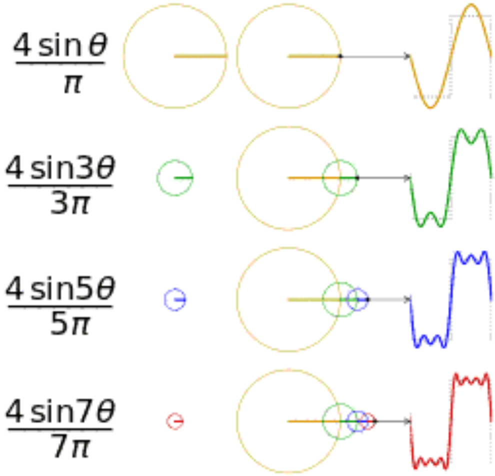
Characteristics of sound in audio signals:
High pitch - rapidly varying signal
Low pitch - slowly varying signal





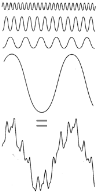


Fourier Series – A Visual Overview



Fourier Series

Trigonometric Fourier Series: Any *periodic* function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier* (1822).



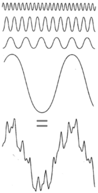
$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

Fourier Series

Trigonometric Fourier Series: Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier* (1822).



$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) \;+\; b_n \sin\left(\frac{2\pi n x}{T}\right)}$$

A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one’s frequency is an integer multiple of the fundamental frequency of the input signal.
- **a_0 is often referred to as the DC term or the average of the signal**

Fourier Series Solution

A *Fourier series* provides an equivalent representation of the function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) \; + \; b_n \sin\left(\frac{2\pi nx}{T}\right)$$

The coefficients are:

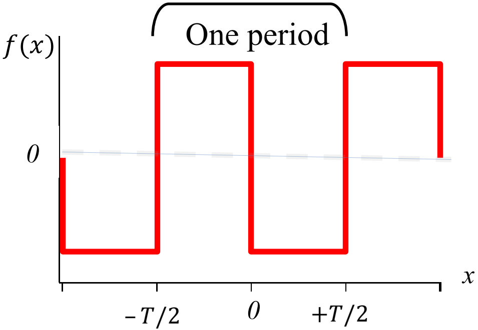
$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \cos(\frac{2\pi nx}{T}) dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \sin(\frac{2\pi nx}{T}) dx$$

Fourier Series Example: Square Wave

$f(x)$ → a square wave

$$f(x) = \begin{cases} +1 & \frac{-T}{2} \leq x < 0 \\ -1 & 0 \leq x < \frac{T}{2} \end{cases}$$



Example periodic function on $-T/2, +T/2$

Fourier Series Example: Square Wave

$f(x)$ → a square wave

$$f(x) = \begin{cases} +1 & \frac{-T}{2} \le x < 0 \\ -1 & 0 \le x < \frac{T}{2} \end{cases}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2\pi nx/T)dx$$

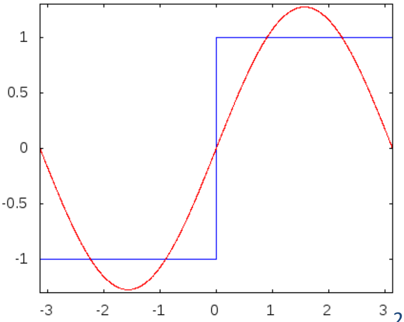
$$= \frac{2}{T}\int_{-T/2}^{0} \cos(2\pi nx/T)dx - \frac{2}{T}\int_{0}^{+T/2} \cos(2\pi nx/T)dx = \mathbf{0}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\sin(2\pi nx/T)dx$$

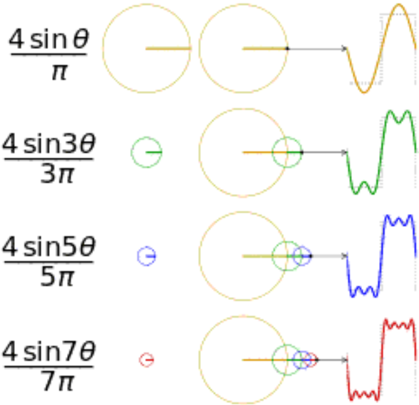
$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f(x) = \frac{4}{\pi}\cdot\sin\frac{2\pi x}{T} + \frac{4}{3\pi}\cdot\sin 3\cdot\frac{2\pi x}{T} + \frac{4}{5\pi}\cdot\sin 5\cdot\frac{2\pi x}{T} + \dots$$

n = 1,3,5,7,…



Approximating the Square Wave

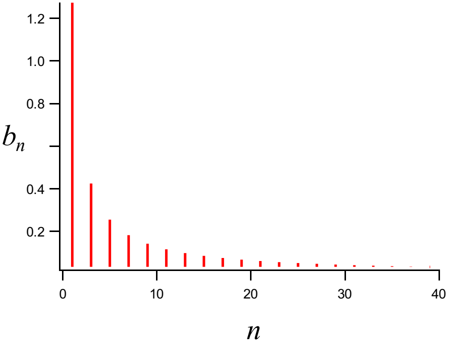


$$\theta = \frac{2\pi x}{T}$$

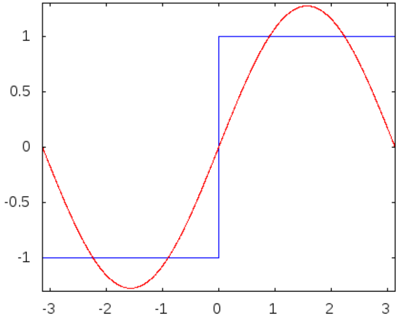
$$f(x) = \frac{4}{\pi} \cdot \sin\left(\frac{2\pi x}{T}\right) + \frac{4}{3\pi} \cdot \sin\left(3 \cdot \frac{2\pi x}{T}\right) + \frac{4}{5\pi} \cdot \sin\left(5 \cdot \frac{2\pi x}{T}\right) + \frac{4}{7\pi} \cdot \sin\left(7 \cdot \frac{2\pi x}{T}\right) + \ldots$$

Fourier Space/Domain for the Square Wave

- The set of *Fourier Space* coefficients b_n contain complete information about the function
- Although *f(x)* is periodic to infinity, b_n is negligible beyond a finite range
- Sometimes the Fourier representation is more convenient to use, or just view



Next in DDCS



Feature Selection and Extraction

- Signal basics and Fourier Series
- **1D** and 2D **Fourier Transform**
- Another look at features
- Convolutions