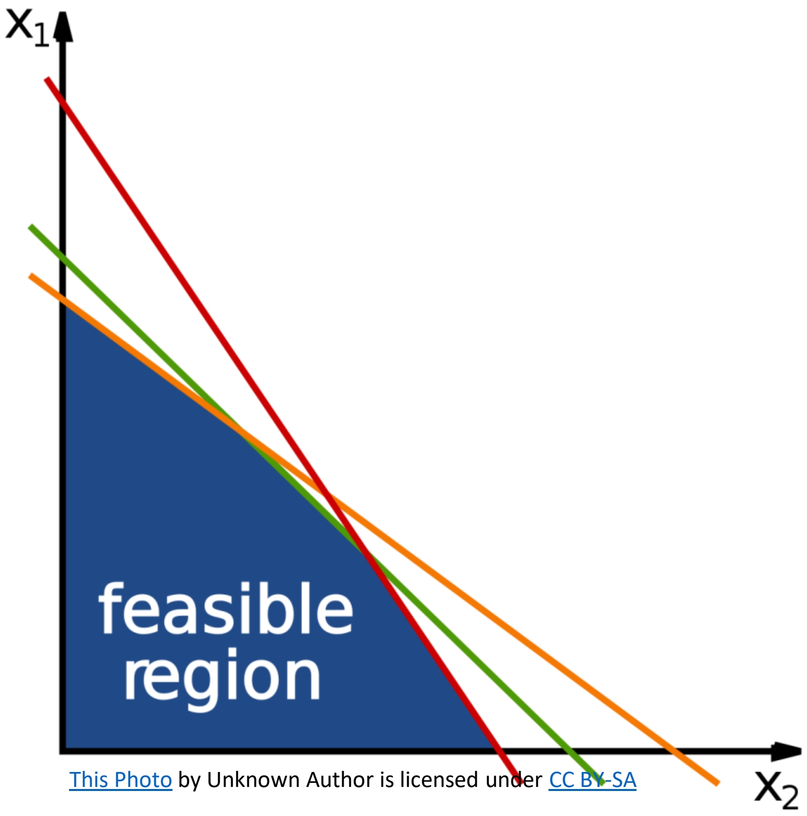

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SIT718 – Game Theory – Week 2

Delaram Pahlevani

Solving 2-Person Zero-Sum Games using LP

Linear programming allows us to address cases when both players have more than two strategies.

Let v be the value of the game, then Player I's LP for solving their ideal combination of mixed strategies $(x_1, x_2, \ldots, x_m)$ can be formulated as

$$
\begin{array}{lll}
\max & z = v & \\
\text{s.t.} & v - (a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m) & \leq 0 \\
 & v - (a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m) & \leq 0 \\
 & \vdots & \vdots \\
 & v - (a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m) & \leq 0 \\
 & x_1 + x_2 + \cdots + x_m & = 1 \\
 & x_i \geq 0, \forall i = 1, \ldots, m. & \\
 & v \text{ u.r.s.} &
\end{array}
$$

Image Source: wikipedia

Solving 2-Person Zero-Sum Games using LP

$$
\begin{array}{lll}
\min & w = v & \\
\text{s.t.} & v - (a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n) & \geq 0 \\
& v - (a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n) & \geq 0 \\
& \vdots & \vdots \\
& v - (a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n) & \geq 0 \\
& y_1 + y_2 + \cdots + y_n = 1 & \\
& y_i \geq 0, \forall j = 1, \ldots, n. & \\
& v \text{ u.r.s.} &
\end{array}
$$

Image Source: wikipedia

Example

- Let the payoff matrix of a two-person zero-sum game be:

$$v = \begin{bmatrix} -2 & 1 & -3 \\ -1 & -1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} \max \quad & z = v \\ \text{s.t.} \quad & v - (-2x_1 - x_2 + 3x_3) \leq 0 \\ & v - (x_1 - x_2) \leq 0 \\ & v - (-3x_1 + 2x_2 - x_3) \leq 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0, \forall i = 1, 2, 3. \\ & v \text{ u.r.s. (means - unrestricted sign)} \end{aligned}$$

$$\begin{aligned} \min \quad & w = v \\ \text{s.t.} \quad & v - (-2y_1 + y_2 - 3y_3) \geq 0 \\ & v - (-y_1 - y_2 + 2y_3) \geq 0 \\ & v - (3y_1 - y_3) \geq 0 \\ & y_1 + y_2 + y_3 = 1 \\ & y_i \geq 0, \forall i = 1, 2, 3. \\ & v \text{ u.r.s. (means - unrestricted sign)} \end{aligned}$$

Value of the game:
V = $-\frac{2}{11}$

$(x_1{}^*, x_2{}^*, x_3{}^*) = (\frac{3}{11}, \frac{5}{11}, \frac{3}{11})$

$(y_1{}^*, y_2{}^*, y_3{}^*) = (\frac{1}{33}, \frac{23}{33}, \frac{9}{33})$

Practice

- Let the payoff matrix of a two-person zero-sum game be:

$$V_1 = \begin{bmatrix} 3 & 1 & -3 \\ -2 & 4 & -1 \\ -5 & 6 & 2 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 3 & 2 & 5 & 4 \\ 0 & 4 & -2 & -3 \\ 2 & 3 & -2 & -4 \end{bmatrix}$$

Classic non-zero-sum games

Two prisoners awaiting trail are being kept in separate cells where they are not allowed to communicate. The prosecutor makes the same offer to both of them:

We have enough evidence to convict you both on a lesser charge. If you both plead innocent, you will be convicted and each will receive a 2-year sentence. However, if you help us, we will reward you: Admit guilt, then it'll be easier to convict your friend if he pleads innocent. He will get 5 years and we will let you go. If, however, you both plead guilty, you will both get 4 years.

Prisoner 1/Prisoner 2	Don't Confess	Confess
Don't Confess	$(-2, -2)$	$(-5, 0)$
Confess	$(0, -5)$	$(-4, -4)$

Classic non-zero-sum games

Player I has a secured return of 3 by always playing a_1, and Player II has a secured return of 3 as well by always playing A_2 . However, if they are able to "communicate", they might agree to play the game twice, with an (8,0 outcome the first time round, and (0,8) the second time round.

	A_1	A_2
a_1	$(8,0)$	$(3,3)$
a_2	$(3,3)$	$0,8)$

Classic non-zero-sum games

We can see that Player I should always use a_1. Although, knowing that would be Player I's choice, Player II will always prefer A_1. However, Player I can threaten Player II by saying "you'd better use A_2, or else I will use a_2".

	A_1	A_2
a_1	$(2, 5)$	$(5, 2)$
a_2	$(1, -3)$	$(1, -5)$

Nash Equilibrium

A strategy profile is a Nash equilibrium if no player can do better by unilaterally changing his or her strategy. Each strategy in a Nash equilibrium is a best response to all other strategies in that equilibrium.

Prisoner 1/Prisoner 2	Don't Confess	Confess
Don't Confess	$(-2, -2)$	$(-5, 0)$
Confess	$(0, -5)$	$(-4, -4)$

(-4,-4) is a Nash equilibrium point, because if either prisoner changes their strategy, then their payoff decreases from -4 to -5

Nash Equilibrium Examples

A strategy profile is a Nash equilibrium if no player can do better by unilaterally changing his or her strategy. Each strategy in a Nash equilibrium is a best response to all other strategies in that equilibrium.

	A_1	A_2
a_1	$(8,0)$	$(3,3)$
a_2	$(3,3)$	$(0,8)$

	A_1	A_2
a_1	$(2,5)$	$(5,2)$
a_2	$(1,-3)$	$(1,-5)$

(3,3)

(2,5)

Practice

	A_1	A_2
a_1	$(3, 3)$	$(11, -1)$
a_2	$(-1, 11)$	$(7, 7)$

Non-Zero Sum Games with no Nash Equilibrium

Like zero-sum games, a non-zero sum may fail to have an equilibrium point in pure strategies. Consider the following non-zero sum game:

	A_1	A_2
a_1	$(4, -2)$	$(-4, 2)$
a_2	$(-4, 2)$	$(4, -2)$