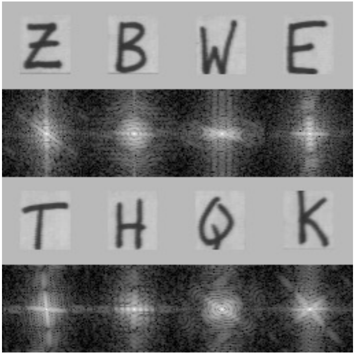
COMS20011 – Data-Driven Computer Science



2D Fourier Transform

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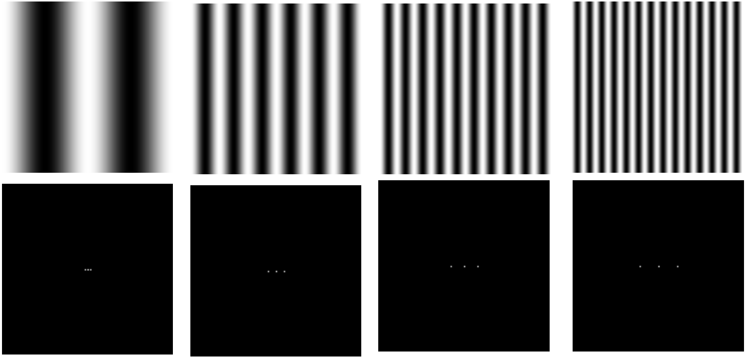
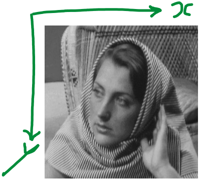


Feature Selection and Extraction

- Signal basics and Fourier Series
- 1D and **2D Fourier Transform**
- Another look at features
- Convolutions

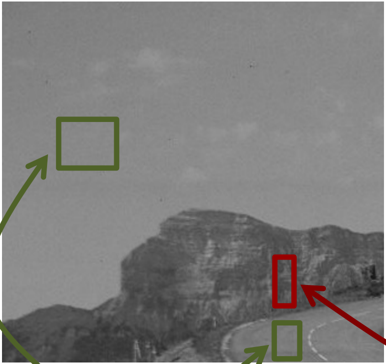
The 2D Fourier Transform



FT → straightforward extension to 2D:

- Images are functions of two variables → e.g. $f(x,y)$
- Defined in terms of *spatial frequency* → 2D frequency.
- Fourier Transform is particularly useful for characterising intensity variations across an image.
- FT identifies the *Rate of change of intensity* along each dimension.

Examples: Spatial Frequency

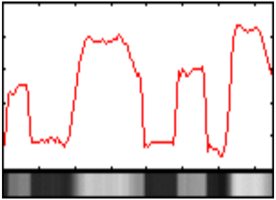


Slowly changing → low frequency

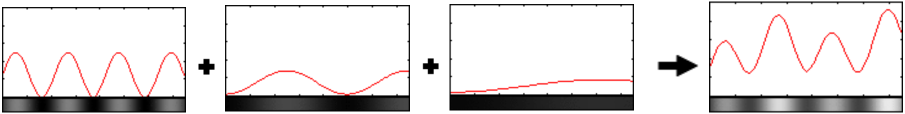
Rapidly changing → high frequency

Images are waves!? (or intuition behind FT)

Consider a single row (or column) of pixels from an image:



Add some regular waves to get one that is close to (or as good as) the image



2D Fourier Transform: Continuous Form

The Fourier Transform of a continuous function of two variables $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \quad e^{-j2\pi(ux+vy)}\, dxdy$$

Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) \quad e^{j2\pi(ux+vy)}\, dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform: Discrete Form

The FT of a discrete function of two variables, $f(x, y), x, y = 0,1,2 \ldots, N-1,$ is:

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)\ e^{-j2\pi\left(\frac{ux+vy}{N}\right)} \text{ for } u,v = 0,1,2,\ldots,N-1\,.$$

Conversely, given $F(u, v)$, we can obtain $f(x, y)$ by means of the *inverse FT:*

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\ e^{j2\pi\left(\frac{ux+vy}{N}\right)} \text{ for } x,y = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform

- The concept of the frequency domain follows from *Euler's Formula*:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x,y)$ multiplied by sines and cosines of various frequencies:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - j\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

for $u,v = 0,1,2,\ldots,N-1$.

We have transformed from a time domain to a frequency domain representation.

2D Fourier Transform

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$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

for $u,v = 0,1,2,\ldots,N-1$.

The slowest varying frequency component, i.e. when $u=0,v=0$ → average image graylevel

We have transformed from a time domain to a frequency domain representation.

2D Fourier Transform

$F(u,v)$ is a complex number & has real and imaginary parts:

$$F(u,v) = R(u,v) + jI(u,v)$$

Magnitude or *spectrum* of the FT:

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

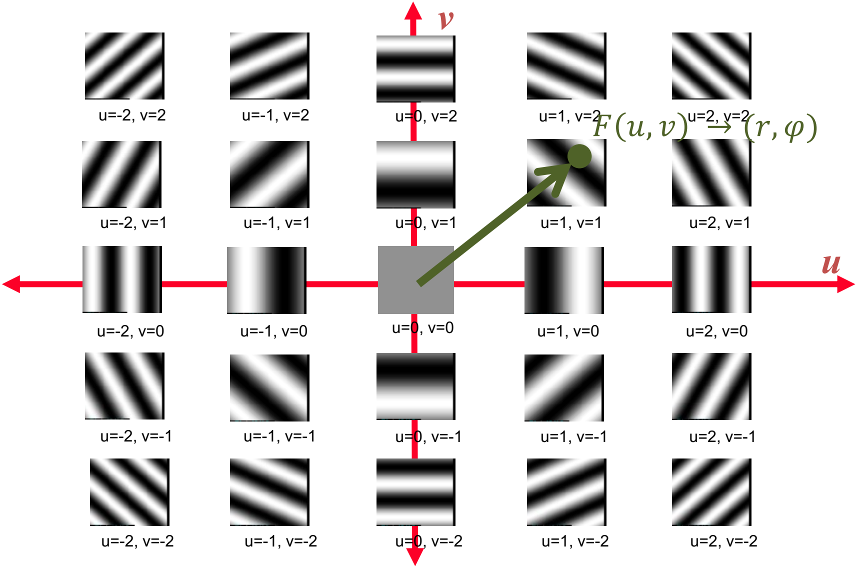
Phase angle or phase spectrum:

$$\varphi(u,v) = \tan^{-1}\frac{I(u,v)}{R(u,v)}$$

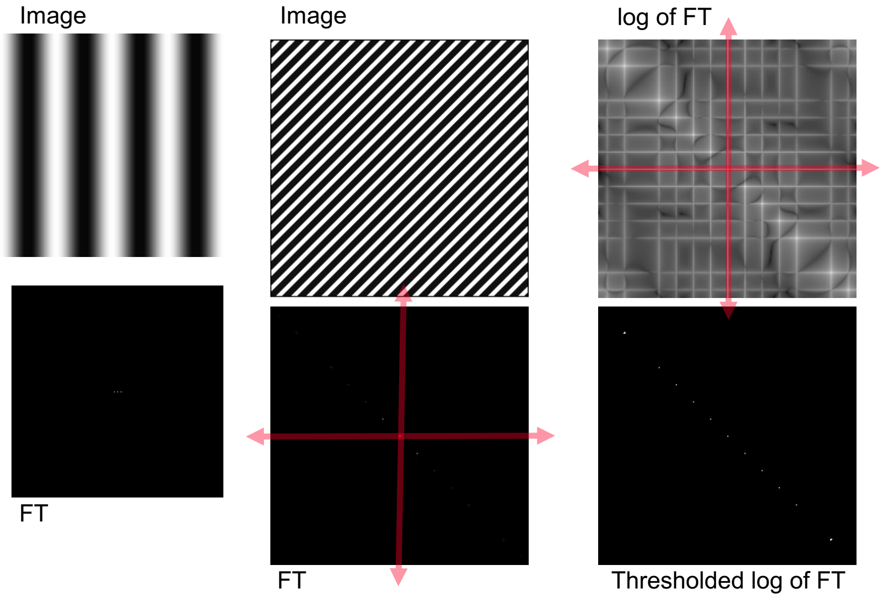
Expressing $F(u,v)$ in polar coordinates:

$$F(u,v) = |F(u,v)|e^{j\varphi(u,v)}$$

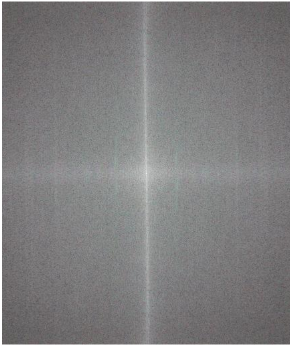
Another view: The 2D Basis Functions

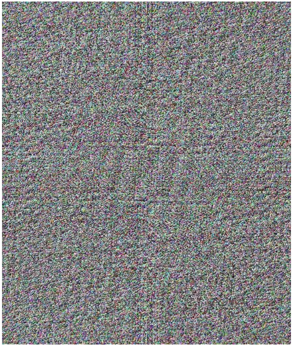


Example I: Image Analysis



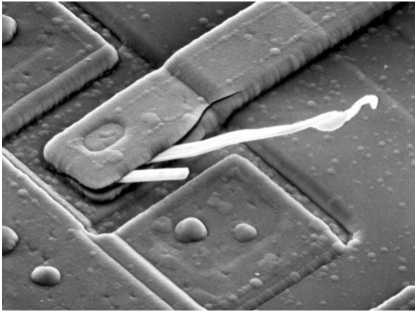
Example II: Magnitude + Phase





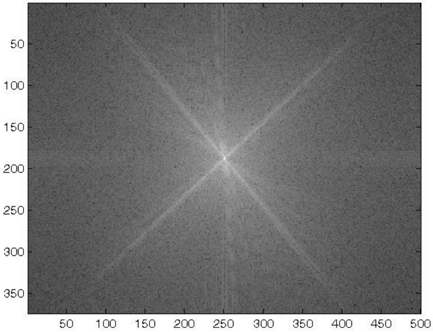
I $\log(|F(I)| + 1)$ $\varphi[\mathrm{F}(I)]$

Example IV: Interpreting the FS

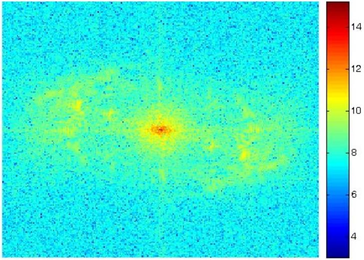


Scanning electron microscope image of an integrated circuit

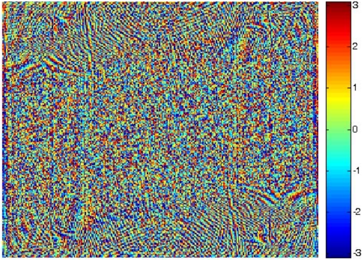
Can we interpret what the bright components mean?



Viewing Magnitude and Phase



$\log(|F(I)| + 1)$



$\varphi[\mathrm{F}(I)]$

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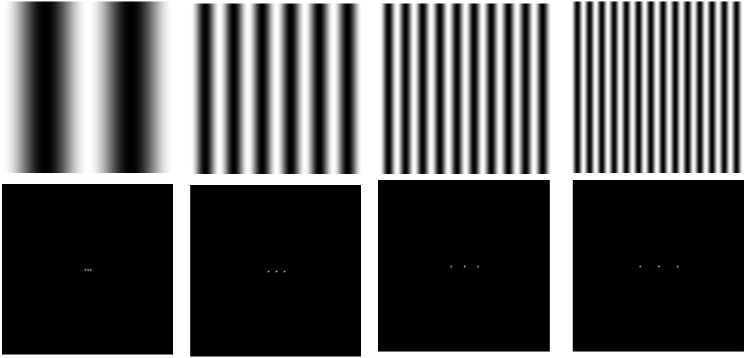

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