Computational Neuroscience 1.1

PHPH20007

`github.com/conorhoughton/PHPH20007`

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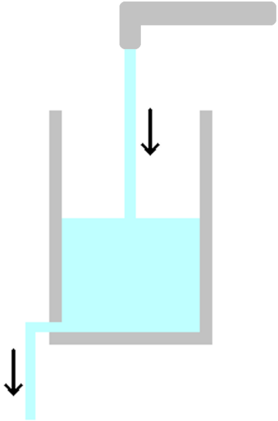
Modelling

Modelling helps us get more out of data: it matches what we measure to what we want to know. Sometimes it suggests what the brain might be doing and how it is doing it.

The leaky integrate and fire model

$$\tau \frac{dV}{dt} = E_l - V + R_m I_e$$

A leaky bucket



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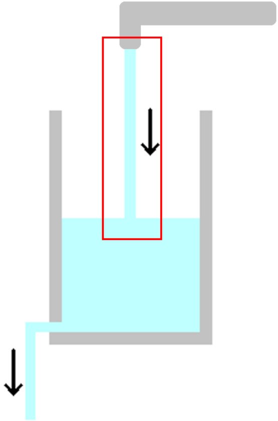

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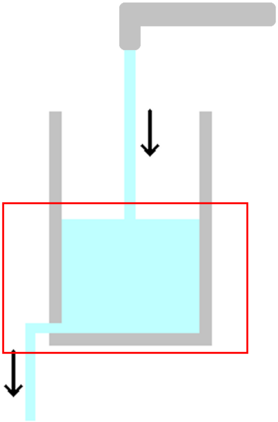

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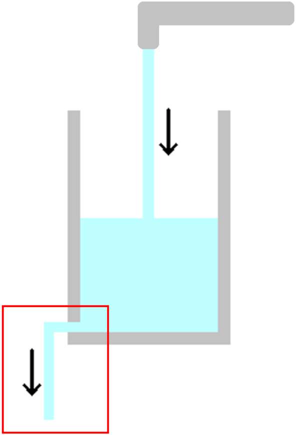
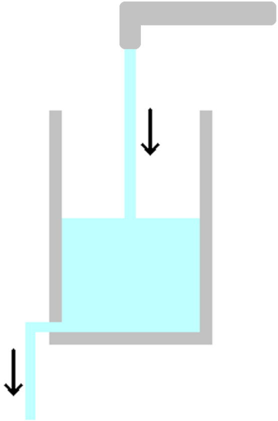


A leaky bucket - increased inflow



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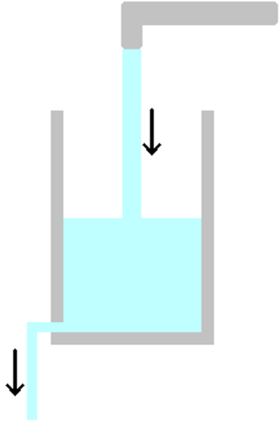

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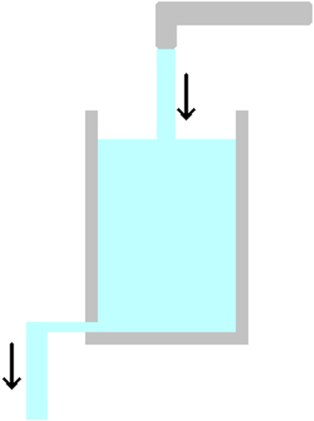
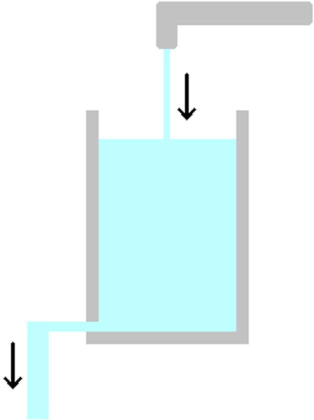
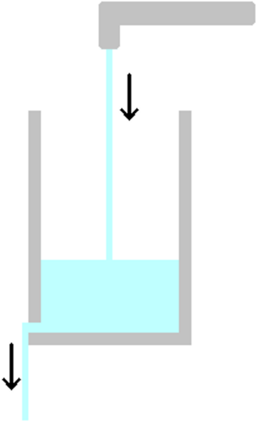


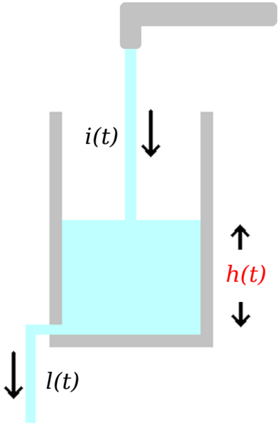
A leaky bucket - decreased inflow



A leaky bucket - decreased inflow

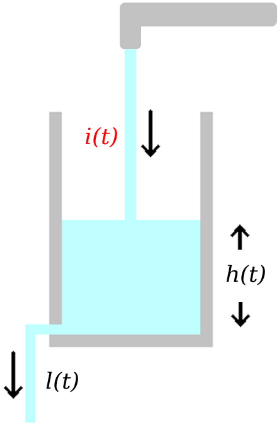


A leaky bucket - notation



$h(t)$ is the height of the water.

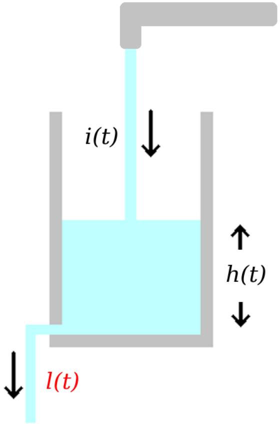
A leaky bucket - notation



$i(t)$ is the rate of flow into the bucket.

It is a *rate* so it would be measured in volume per time, for example, Ls^{-1}.

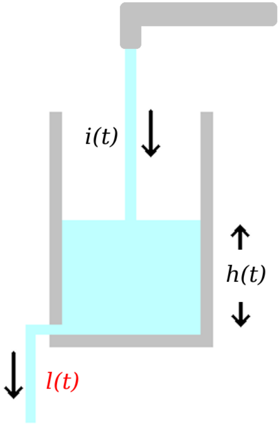
A leaky bucket - notation



$l(t)$ is the rate of flow out of the bucket.

It is a *rate* so it would be measured in volume per time, for example, Ls^{-1}.

A leaky bucket - leak



The speed of the leak depends on the height of the water so

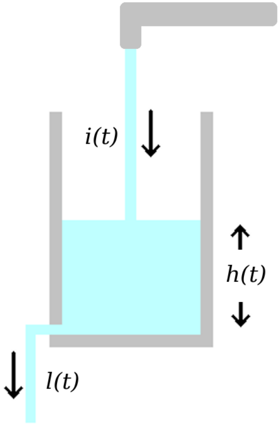
$$l(t) \propto h(t)$$

or, adding a constant of proportionality

$$l(t) = Gh(t)$$

where G depends on some physics stuff we aren't interested in here.

A leaky bucket - net flow



The net flow into the bucket is therefore

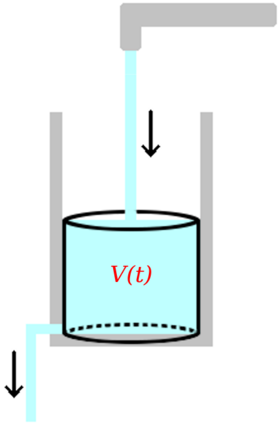
$$i(t) - l(t)$$

or,

$$i(t) - Gh(t)$$

and, remember this is a flow, so it is measured in volume per time.

A leaky bucket - volume



The net flow changes the volume of water.

The volume depends on the height:

$$V(t) = Ch(t)$$

where C is a constant.

In the case of a cylindrical bucket, using the formula for the volume of a cylinder tells us that $C = \pi r^2$ where r is the radius of the bucket; but we'll just call it C.

Equation for a leaky bucket

The rate of change of the volume is equal to the net inflow. The rate of change is given by the derivative:

$$\frac{dV}{dt} = i - Gh$$

Equation for a leaky bucket

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$$\frac{dV}{dt} = i - Gh$$

Notice that when we want to emphasis that something changes with time we write the brackets t, like $h(t)$ but at other times we leave it out to stop making equations look to cluttered!.

Equation for a leaky bucket

$$\frac{dV}{dt} = i - Gh$$

Substituting $V = Ch$ we get

$$\frac{d\,Ch}{dt} = i - Gh$$

Since C is a constant this means

$$C\frac{dh}{dt} = i - Gh$$

This is our basic equation for the height of the water in a leaky bucket.

Equation for a leaky bucket

Let's rewrite this equation as

$$\tau \frac{dh}{dt} = \frac{1}{G} i - h$$

where $\tau = C/G$.

This notation is used because τ is a time-scale and τ is the Greek equivalent of t. If you work out all the units for the constants C and G you can see τ has units of time; it also balances the *per time* in the derivative.

Constant input

First lets looks at what happens when the water is flowing in at a constant rate, that is when i doesn't change with time; lets write

$$i = \bar{\imath}$$

when the input is constant.

Behaviour of the equation

$$\tau \frac{dh}{dt} = \frac{1}{G}\bar{\imath} - h$$

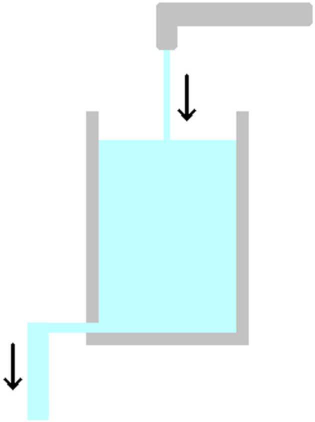
If $h = \bar{\imath}/G$ then $dh/dt = 0$. This is the equilibrium where the inflow is equal to the outflow.

Behaviour of the equation

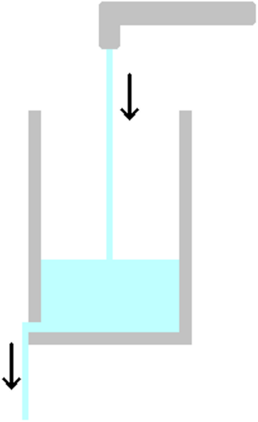
$$\tau \frac{dh}{dt} = \frac{1}{G}\bar{\text{I}} - h$$

If $h > \bar{\text{I}}/G$ then $dh/dt < 0$ and the level of the water falls; so if the water height is too high it decreases towards equilibrium.

A leaky bucket - decreased inflow



A leaky bucket - decreased inflow



Behaviour of the equation

$$\tau \frac{dh}{dt} = \frac{1}{G}\bar{\imath} - h$$

Conversely if $h < \bar{\imath}/G$ then $dh/dt > 0$ and the level of the water rises; so if the water height is too low it increases towards equilibrium.

Behaviour of the equation

$$\tau \frac{dh}{dt} = \frac{1}{G}\bar{\imath} - h$$

Notice also that the more $h < \bar{\imath}/G$ the more $dh/dt > 0$ and *visa versa* for $h > \bar{\imath}/G$: the further the height is from the equilibrium, the faster it goes towards equilibrium.

Solution of the equation

In fact we can solve the equation:

$$h(t) = [h(0) - \bar{\imath}/G]e^{-t/\tau} + \bar{\imath}/G$$

Solving the equation isn't hard but we won't look at it here.

Solution of the equation

$$h(t) = [h(0) - \bar{\imath}/G]e^{-t/\tau} + \bar{\imath}/G$$

where $h(0)$ is the initial value of h; that is the value it starts at.

Solution of the equation

$$h(t) = [h(0) - \bar{\imath}/G]e^{-t/\tau} + \bar{\imath}/G$$

Here

$$e^{-t/\tau}$$

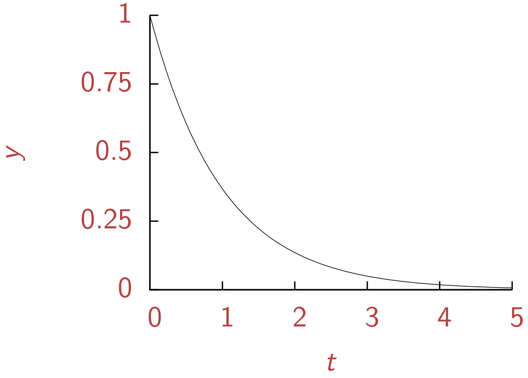
is the exponential function; sometimes it is written in another form:

$$\exp\left(-\frac{t}{\tau}\right)$$

and is one of the standard mathematical functions like sine and cosine.

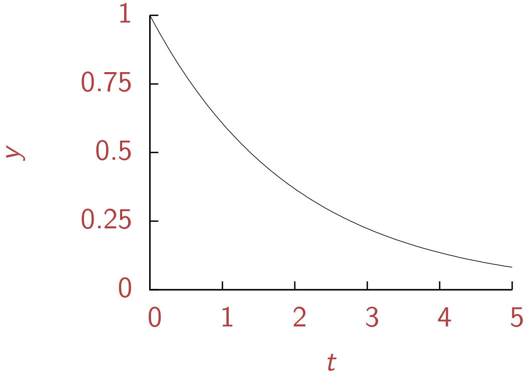
Exponential

$$y = e^{-t}$$



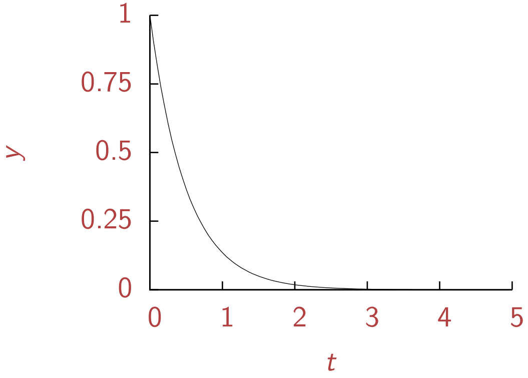
Exponential

$$y = e^{-t/2}$$



Exponential

$$y = e^{-2t}$$



Solution of the equation

$$h(t) = [h(0) - \bar{\imath}/G]e^{-t/\tau} + \bar{\imath}/G$$

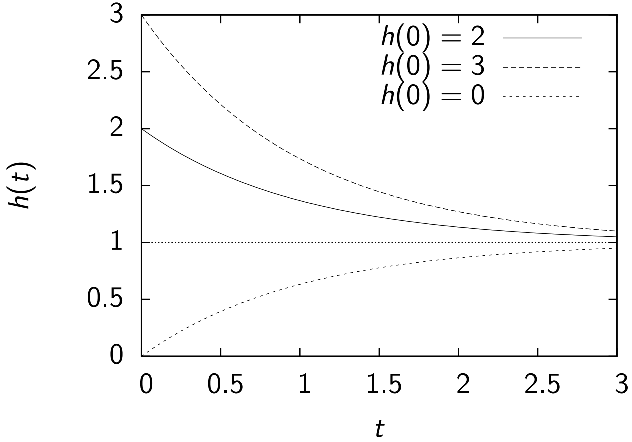
and the

$$e^{-t/\tau}$$

in the first term decays away and the height approaches its equilibrium value of $\bar{\imath}/G$.

Exponential relaxation

These dynamics are very common and are sometimes called *exponential relaxation to the equilibrium*. Here are some examples with $\bar{\imath} = G$, $\tau = 1$ and three different initial values $h(0)$.



Variable input

If the input i changes with time it is often still possible to solve the equation, but if you have have a multi-neuron model this might not be possible, so it is often easier to solve *numerically*; that is work out the value step by step. We will look at that but first we should note we have an idea of what happens from considering the constant case. The solution *chases* the input.

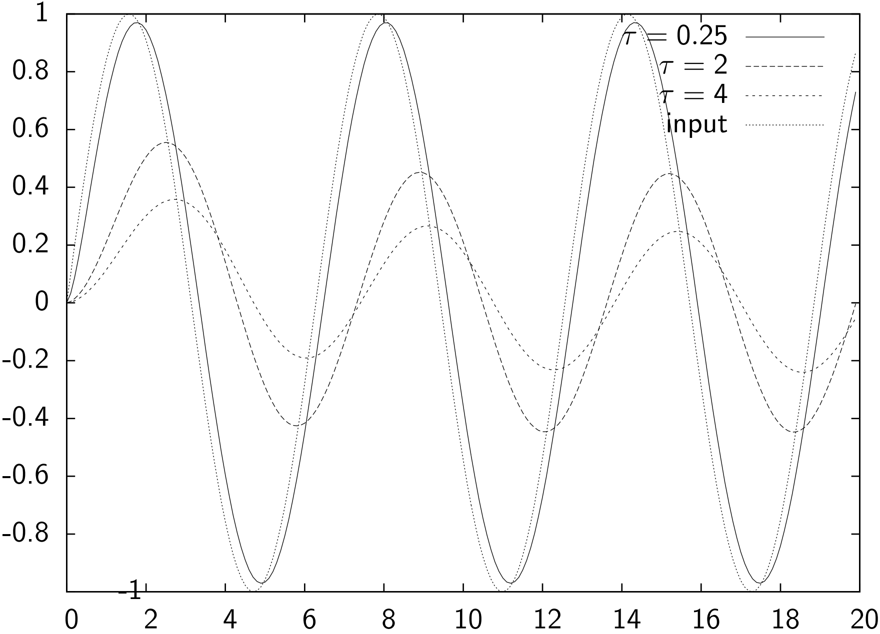
Variable input

Lets look at an example where the input is a sine wave

$$i(t) = \sin t$$

and the equation is evolved with $h(0) = 0$ and three different τ values.

Variable input

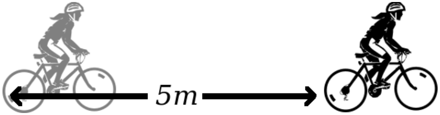


Numerical solution

This means we work out the values of the solution approximately on a computer without solving the equation explicitly.

Numerical solution

Numerical solution



Numerical solution

$$\frac{df(t)}{dt} = F(f, t)$$

This includes our bucket case if we replace f with h and $F(f, t)$ with $(i/G - h)/\tau$.

Euler's approximation

$$f(t + \delta t) \approx f(t) + \delta t F(f, t)$$

Euler's approximation

$$f(t + \delta t) \approx f(t) + \delta t F(f, t)$$

remembering that

$$\frac{df(t)}{dt} = F(f, t)$$

Euler's approximation

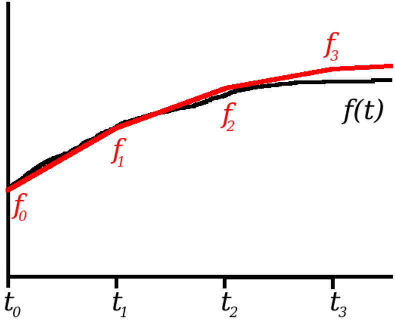
$$f(t+\delta t) = f(t) + \delta t F(f,t) + O(\delta t^2)$$

Euler's approximation

$$t_n = t_0 + n\delta t$$

and

$$f(t_n) \approx f_n$$



Numerical solution

$$f_{n+1} = f_n + \delta t F(f_n, t_n)$$

Numerical solution

$$h_{n+1} = h_n + \frac{[i(t_n)/G - h_n]\delta t}{\tau}$$

Summary

- Worked out the equation for the height of water in a bucket.
- Looked at its solution for constant input.
- Gained some intuition as to how it behaves for variable input.
- Learned how to solve it numerical using the Euler approximation.