LEARNING OBJECTIVES

At the end of this section, you should be able to do the following:

- Explain why it is not appropriate to conduct multiple independent t tests to compare the means of more than two independent groups
- Apply one-way ANOVA to test the difference between means of several groups or compare populations each containing several levels or subgroups.
- Run post hoc test (multiple comparisons) to determine which groups are different.

WHY ANOVA?

- In the previous topic, we looked at how to compare the means of two independent groups
- In this topic, we will learn how to compare the means of more than two independent groups
- So why not just perform multiple two independent sample t-tests?
  - For *K* independent groups there are K(K-1)/2 possible pairs.
  - If you had 5 independent groups, that would equal 5(5-1)/2 =10 independent t tests!
  - And those 10 independent t-test would not give us information about the intendent variable overall.
  - Greater chance of making type I error: multiple pair-wise comparison means the error compounds with each t-test

THE COMPLETELY RANDOMISED DESIGN: ONE WAY ANOVA

ANOVA (ANalysis Of VAriance)

- The one-way analysis of variance is used to test the claim that three or more population means are equal
- This is an extension of the two independent samples t-test
- The response variable is the variable we are comparing
- The factor variable is the categorical variable being used to define the groups (we will assume k samples (groups))
- The one-way is because each value is classified in exactly one way (examples include comparisons by gender, race, political party, color, etc.)

HYPOTHESIS OF ONE-WAY ANOVA

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_c$$

- All population means are equal; (no variation in means between groups)

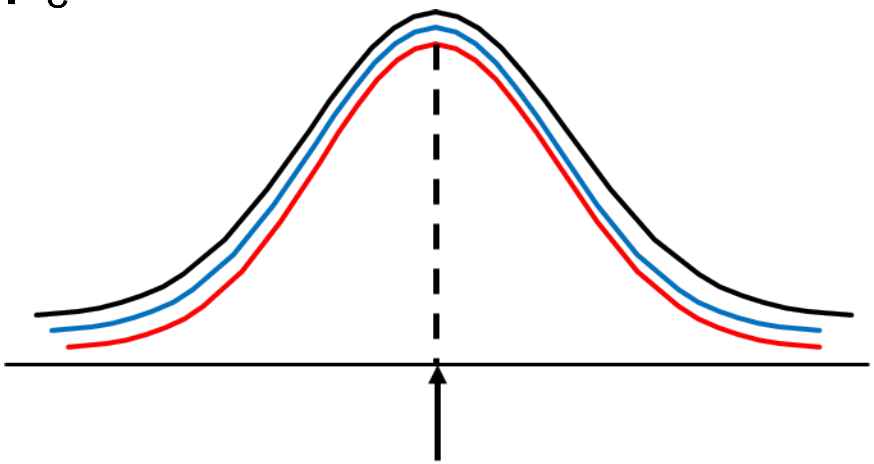
$$H_1 : \text{Not all of the population means are the same}$$

- At least one population mean is different; This does not mean that all population means are different (some pairs may be the same).

ONE-FACTOR ANOVA (1 OF 2)

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_c$$

- All means are the same: the null hypothesis is true

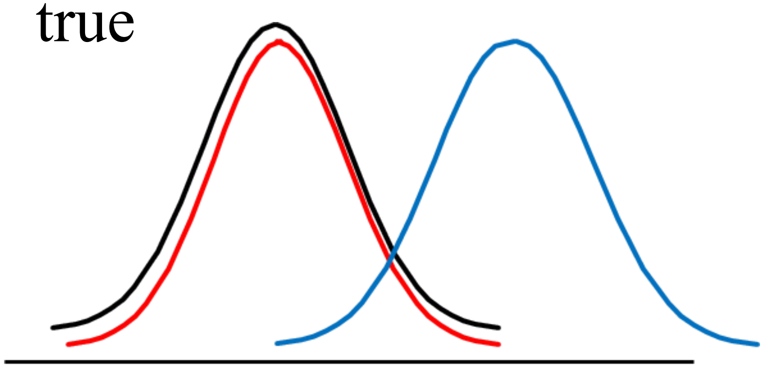


$$\mu_1 = \mu_2 = \mu_3$$

ONE-FACTOR ANOVA (2 OF 2)

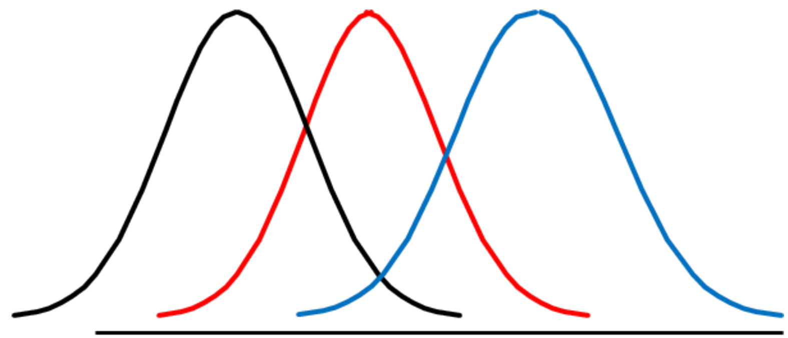
H_1 : Not all μ_j are the same

- At least one mean is different: The null hypothesis is NOT true



$$\mu_1 = \mu_2 \neq \mu_3$$

Is one mean so far away from the other two that it is likely not from the same population?



$$\mu_1 \neq \mu_2 \neq \mu_3$$

Or all three are so far apart that they all likely come from different populations?

ONE-WAY ANOVA : EXAMPLE

Suppose we want to compare three sample means to see if a difference exists somewhere among them:

- MIS770 classroom is divided into three rows: front, middle, and back
- The professor noticed that the further the students were from her, the more likely they were to miss class or use video sharing platforms like TikTok during class
- She particularly wanted to see if the students further away did worse on the exams

ONE-WAY ANOVA : EXAMPLE…

- A random sample of the students in each row was taken (samples are small for illustration)
- The marks for those students on the second exam was recorded
  - Front: 82, 83, 97, 93, 55, 67, 53
  - Middle: 83, 78, 68, 61, 77, 54, 69, 51, 63
  - Back: 38, 59, 55, 66, 45, 52, 52, 61

ONE-WAY ANOVA : EXAMPLE…

The summary statistics for the grades of each row are shown in the table below

Row	Front	Middle	Back
Sample size	7	9	8
Mean	75.71	67.11	53.50
St. Dev	17.63	10.95	8.96
Variance	310.90	119.86	80.29

UNDERSTANDING ONE-ONE WAY ANOVA

Variation

- Variation is the sum of the squares of the deviations between a value and the mean of the value
- Sum of Squares is abbreviated by SS and often followed by a variable in parentheses such as SS(B)etween or SS(W)ithin so we know which sum of squares we're talking about

UNDERSTANDING ONE-ONE WAY ANOVA

Are all of the values identical?

- No, so there is some variation in the data
- This is called the total variation
- Denoted SS (T) for the total Sum of Squares (variation)
- Sum of Squares is another name for Variation

UNDERSTANDING ONE-ONE WAY ANOVA

Are all of the sample means identical?

- No, so there is some variation between the groups
- This is called the between group variation
- Sometimes called the variation due to the factor
- Denoted SS(B) for Sum of Squares (variation) between the groups

UNDERSTANDING ONE-ONE WAY ANOVA

Are each of the values with each group identical?

- No, so there is some variation within the groups
- This is called the within group variation
- Sometimes called the error variation
- Denoted SS(W) for Sum of Squares (variation) within the groups

UNDERSTANDING ONE-ONE WAY ANOVA

Step-1 : Computing Grand Mean

- Strategies for computing Grand Mean
- The Grand Mean is the average of all the values when the factor is ignored
- It is a weighted average of the individual sample means

$$\overline{\overline{x}} = \frac{\sum_{i=1}^{k} n_i \overline{x}_i}{\sum_{i=1}^{k} n_i}$$

$$\overline{\overline{x}} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2 + \cdots + n_k\overline{x}_k}{n_1 + n_2 + \cdots + n_k}$$

Group	Data				Means
1	X_{11}	X_{12}	$\cdots$	X_{1n_1}	$\overline{X}_{1\cdot}$
2	X_{21}	X_{22}	$\cdots$	X_{2n_2}	$\overline{X}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	X_{m1}	X_{m2}	$\cdots$	X_{mn_m}	$\overline{X}_{m\cdot}$
		Grand Mean			$\overline{X}_{\cdot\cdot}$

UNDERSTANDING ONE-ONE WAY ANOVA

Let us use summary statistics to complete the computations

Row	Front	Middle	Back
Sample size	7	9	8
Mean	75.71	67.11	53.50
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Variance	310.90	119.86	80.29

UNDERSTANDING ONE-ONE WAY ANOVA

Lets use summary statistics to complete the computations

$$\bar{\bar{x}} = \frac{7(75.71)+9\ (67.11)+8\ (53.50)}{7+9+8}$$

$$\bar{\bar{x}} = \frac{1562}{24}$$

$$\bar{\bar{x}} = 65.08$$

The grand mean for our example is 65.08

UNDERSTANDING ONE-ONE WAY ANOVA

Step 2: Compute SS(B)

- The between group variation is the variation between each sample mean and the grand mean
- Each individual variation is weighted by the sample size
- General Formula

$$SS(B) = \sum_{i=1}^{k} n_i \left(\bar{x}_i - \bar{\bar{x}}\right)^2$$

- Expanded Formula

$$SS(B) = n_1\left(\bar{x}_1 - \bar{\bar{x}}\right)^2 + n_2\left(\bar{x}_2 - \bar{\bar{x}}\right)^2 + \cdots + n_k\left(\bar{x}_k - \bar{\bar{x}}\right)^2$$

UNDERSTANDING ONE-ONE WAY ANOVA

- Means for groups:
  - Front = 75.71
  - Middle = 67.11
  - Back = 53.50
- Grand Mean = 65.08
- Compute using expanded formula

$$SS(B) = n_1\left(\bar{x}_1 - \bar{\bar{x}}\right)^2 + n_2\left(\bar{x}_2 - \bar{\bar{x}}\right)^2 + \cdots + n_k\left(\bar{x}_k - \bar{\bar{x}}\right)^2$$

$$SS(B) = 7(75.71 - 65.08)^2 + 9(67.11 - 65.08)^2 + 8(53.50 - 65.08)^2$$

- SS (B) = 1901.5 ≈ 1902
- The Between Group Variation for our example is SS(B)=1902

UNDERSTANDING ONE-ONE WAY ANOVA

Step3: ComputeSS(W)

- Computing SS(W), Within Group Variation is the weighted total of the individual variations
- The weighting is done with the df; the df for each sample n – 1
- Front: n = 7; df = 6; Variance = 310.90
- Middle: n = 9; df = 8; Variance = 119.86
- Back: n = 8; df = 7; Variance = 80.29
- General Formula $SS(W) = \sum_{i=1}^{k} df_i s_i^2$
- Expanded Formula

$$SS(W) = df_1 s_1^2 + df_2 s_2^2 + \cdots + df_k s_k^2$$

UNDERSTANDING ONE-ONE WAY ANOVA

- The Within Group Variation is the weighted total of the individual variations

$$SS(W) = 6(310.90) + 8(119.86) + 7(80.29)$$

$$SS(W) = 3386.31 \approx 3386$$

- The within group variation for our example is 3386

UNDERSTANDING ONE-ONE WAY ANOVA

- After filling in the sum of squares, we have …

Source	SS	df	MS	F	p
Between	1902				
Within	3386				
Total	5288				

UNDERSTANDING ONE-ONE WAY ANOVA

Degrees of Freedom for the ANOVA

- The between group df is one less than the number of groups
- We have three groups, so df(B) = 2
- The within group df is the sum of the individual df's of each group
- The sample sizes are 7, 9, and 8
- df(W) = 6 + 8 + 7 = 21
- The total df is one less than the sample size
- df(Total) = 24 – 1 = 23

UNDERSTANDING ONE-ONE WAY ANOVA

Filling in the degrees of freedom gives this table…

Source	SS	df	MS	F	p
Between	1902	2			
Within	3386	21			
Total	5288	23			

UNDERSTANDING ONE-ONE WAY ANOVA

Step 4: Compute Mean Squares (Variances)

- The variances are also called the Mean of the Squares and abbreviated by MS, often with an accompanying variable MS(B) or MS(W)
- They are an average squared deviation from the mean and are found by dividing the variation by the degrees of freedom
- MS = SS / df

$$Variance = \frac{Variation}{df}$$

UNDERSTANDING ONE-ONE WAY ANOVA

• MS(B)	= 1902 / 2	= 951.0
• MS(W)	= 3386 / 21	= 161.2
• MS(T)	= 5288 / 23	= 229.9

– Notice that the MS(Total) is NOT the sum of MS(Between) and MS(Within).

– This works for the sum of squares SS(Total), but not the mean square MS(Total)

– The MS(Total) isn't usually shown

UNDERSTANDING ONE-ONE WAY ANOVA

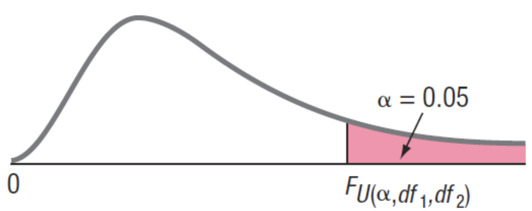
- **Completing the MS gives …**

Source	SS	df	MS	F	p
Between	1902	2	951.0		
Within	3386	21	161.2		
Total	5288	23	229.9		

UNDERSTANDING ONE-ONE WAY ANOVA

Step 5: F-Statistic and Effect

- If the F-statistic is large we reject that the effect is "zero" in favor of the alternative tha the effect of the factor is non-zero
- If computed F > 3.47 we reject the null hypothesis



Critical Value F (5%, 2,21)

Denominator df_2	1	2
1	161.40	199.50
2	18.51	19.00
3	10.13	9.55
4	7.71	6.94
5	6.61	5.79
6	5.99	5.14
7	5.59	4.74
8	5.32	4.46
9	5.12	4.26
10	4.96	4.10
11	4.84	3.98
12	4.75	3.89
13	4.67	3.81
14	4.60	3.74
15	4.54	3.68
16	4.49	3.63
17	4.45	3.59
18	4.41	3.55
19	4.38	3.52
20	4.35	3.49
21	4.32	3.47
22	4.30	3.44
23	4.28	3.42
24	4.26	3.40

UNDERSTANDING ONE-ONE WAY ANOVA

- F test statistic:
  - Is the ratio of two sample variances
  - The MS(B) and MS(W) are two sample variances and that's what we divide to find F.
  - F = MS(B) / MS(W)
- Computed F = 951.0 / 161.2 = 5.9

UNDERSTANDING ONE-ONE WAY ANOVA

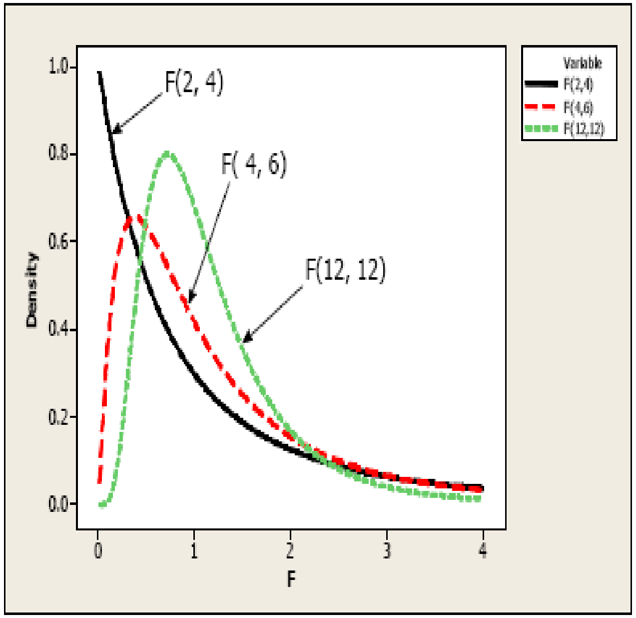
- Adding F to the table …

Source	SS	df	MS	F	p
Between	1902	2	951.0	5.9	
Within	3386	21	161.2		
Total	5288	23	229.9		

UNDERSTANDING ONE-ONE WAY ANOVA

- The F test is a right-tail test
- The CV from the table is: 3.47
  - $F_{2,21}$
    - where df(B) is numerator and df(W) is denominator

- The p-value is the area to the right of the test statistic
- $P(F_{2,21} > 5.9) = 0.009$



α	F
0.05	3.47
0.25	4.42
0.01	5.78
0.005	6.89

UNDERSTANDING ONE-ONE WAY ANOVA

- Completing the table with the p-value

Source	SS	df	MS	F	p
Between	1902	2	951.0	5.9	0.009
Within	3386	21	161.2		
Total	5288	23	229.9		

UNDERSTANDING ONE-ONE WAY ANOVA

- The p-value is 0.009, which is less than the significance level of 0.05, so we reject the null hypothesis.
- The null hypothesis is that the means of the three rows in class were the same, but we reject that, so at least one row has a different mean.
- There is enough evidence to support the claim that there is a difference in the mean scores of the front, middle, and back rows in class.
- The ANOVA doesn’t tell which row is different, you would need to look at confidence intervals or run post hoc tests to determine that

TUKEY-KRAMER PROCEDURE

- Tells which population means are significantly different
- e.g. $\mu 1 = \mu 2 \neq \mu 3$
- Done after rejection of equal means in ANOVA
- Allows paired comparisons
- Compare absolute mean differences with critical range
- Critical range: In the Tukey-Kramer method, <u>the value</u> above which differences in means are significant

From table

$$\text{Critical Range} = Q_u \sqrt{\frac{\text{MSW}}{2}\left(\frac{1}{n_j} + \frac{1}{n_{j'}}\right)}$$

Sample 2 size

Sample 1 size

- where Qu is the upper-tail critical value from a Studentised range distribution having c degrees of freedom in the numerator and n – c degrees of freedom in the denominator.

Number of groups

TUKEY-KRAMER PROCEDURE

c degrees of freedom in the numerator = 3

Anova: Single Factor						
SUMMARY						
Groups	*Count*	*Sum*	*Average*	*Variance*		
Front	7	530	75.71429	310.9048		
Middle	9	604	67.11111	119.8611		
Back	8	428	53.5	80.28571		
ANOVA						
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	1901.516	2	950.7579	5.896056	0.009284	3.4668
Within Groups	3386.317	21	161.2532			
Total	5287.833	23				

n – c degrees of freedom in the denominator = 24-3 = 21

Q STATISTIC

Upper 5% points ($\alpha = 0.05$)

Numerator degrees of freedom

Denominator degrees of freedom	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	18.00	27.00	32.80	37.10	40.40	43.10	45.40	47.40	49.10	50.60	52.00	53.20	54.30	55.40	56.30	57.20	58.00	58.80	59.60
2	6.09	8.30	9.80	10.90	11.70	12.40	13.00	13.50	14.00	14.40	14.70	15.10	15.40	15.70	15.90	16.10	16.40	16.60	16.80
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.15	10.35	10.52	10.69	10.84	10.98	11.11	11.24
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66	8.79	8.91	9.03	9.13	9.23
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.09	7.17
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11	6.20	6.27	6.34	6.40	6.47
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.99	6.06	6.14	6.20	6.26	6.33
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.62	5.71	5.80	5.88	5.95	6.03	6.09	6.15	6.21
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	6.00	6.05	6.11
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.72	5.79	5.85	5.92	5.97	6.03
15	3.01	3.67	4.08	4.37	4.60	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.58	5.65	5.72	5.79	5.85	5.90	5.96
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.72	5.79	5.84	5.90
17	2.98	3.63	4.02	4.30	4.52	4.71	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.55	5.61	5.68	5.74	5.79	5.84
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.32	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20 21?	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.50	5.54	5.59
30	2.89	3.49	3.84	4.10	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.48
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.91	4.98	5.05	5.11	5.16	5.22	5.27	5.31	5.36
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.16	5.20	5.24
120	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.48	4.56	4.64	4.72	4.78	4.84	4.90	4.95	5.00	5.05	5.09	5.13
∞	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80	4.85	4.89	4.93	4.97	5.01

continued

TUKEY-KRAMER PROCEDURE

Tukey Kramer Multiple Comparisons

Group	Sample Mean	Sample Size
1 = Front	75.71	7
2 = Middle	67.11	9
3 = Back	53.5	8

Comparison	Absolute Difference	Std. Error of Difference	Critical Range	Results
Group 1 to Group 2	8.6	4.52506248	16.2	**Means are not different**
Group 1 to Group 3	22.21	4.64714774	16.637	**Means are different**
Group 2 to Group 3	13.61	4.363079	15.62	**Means are not different**

Other Data	
Level of significance	0.05
Numerator d.f.	3
Denominator d.f.	21
MSW	161.25
Q Statistic	3.58

ANOVA ASSUMPTIONS

Assumptions

- The data are randomly sampled
- The variances of each population are assumed equal
- The populations are normally distributed