Chaos: Double Pendulums

•••

Nicholas Munoz | Dr. Lepp

What is Chaos?

- Chaotic behavior can be described as the unpredictable nature of a system, non-periodic.
- Periodic systems begin to repeat or have an re-occurrence of behavior after some interval of time.
- In order to show that a system is chaotic, one of the key characteristics of chaos is the sensitivity to initial conditions. If a system has an unpredictable nature after modification of the parameters of the setup, then it shown to be sensitive to change.

Exploring the Double Pendulum

- The double pendulums exhibits varying non-chaotic and chaotic behaviors.

- Non-chaotic behavior is more likely to occur under conditions in which the pendulum experiences smaller displacement because the equations of motion become linear.

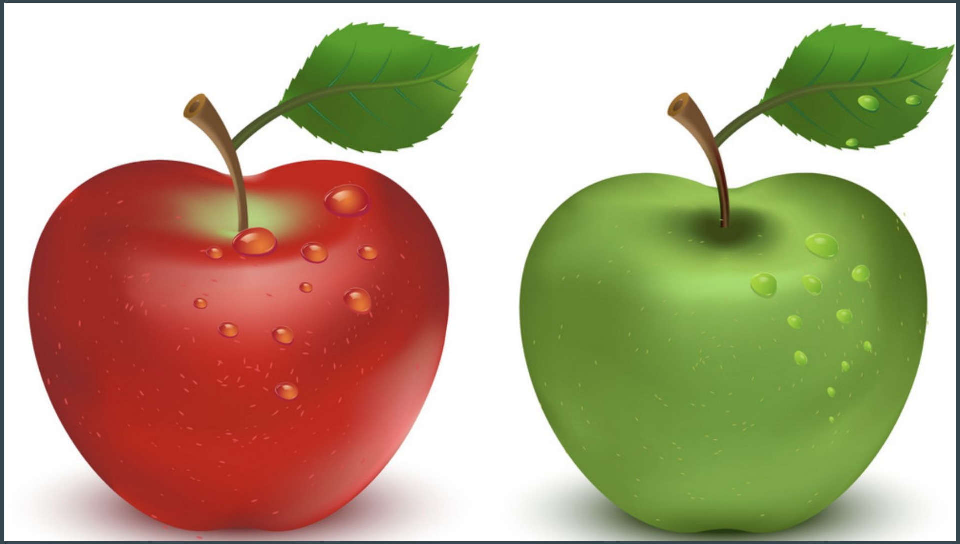
- By using python we were able to simulate the double pendulum, damped driven pendulum, and the damped driven double pendulum in order to demonstrate the sensitivity to initial conditions showing that these systems are chaotic.

Process

- Develop the equations of motion by using the Lagrangian.

- Create two near exact systems allowing for the comparison of displacement in order to be able to show the sensitivity to initial conditions between the first and second system.

- Through graphical analysis, we can show our observations in which the systems behave similarly and differently by comparing the results to see where things may begin to change.



Equations of Motion

Equations of Motion

- Primarily we can begin with a brief summarization of the equations of motion for the double pendulum.

- Introducing damping and driving into a single pendulum system will show how these terms play an importance in creating chaotic behavior which allows us to explore how these can additionally be introduced into the double pendulum system.

Position

$$x_1 = l_1 sin(\theta_1)$$

$$y_1 = -l_1 cos(\theta_1)$$

$$x_2 = x_1 + l_2 sin(\theta 2)$$

$$y_2 = y_1 - l_2 cos(\theta_2)$$

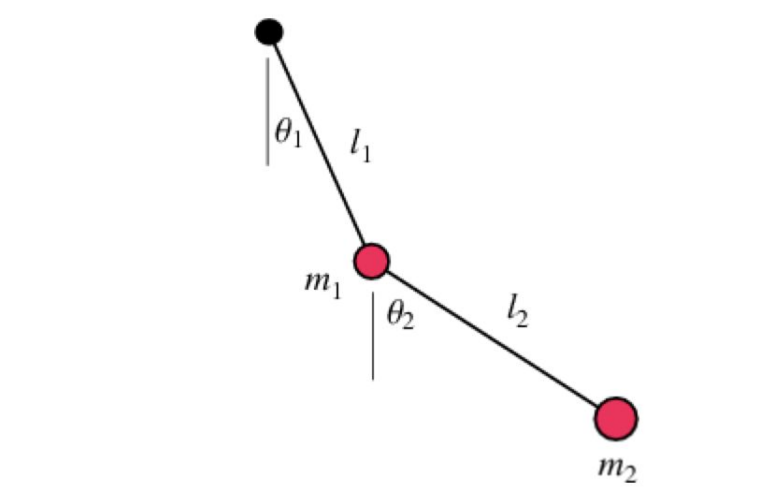
Velocity

$$\dot{x}_1 = l_1 \dot{\theta}_1 cos(\theta_1)$$

$$\dot{y}_1 = l_1 \dot{\theta}_1 sin(\theta_1)$$

$$\dot{x}_2 = \dot{x}_1 + l_2 \dot{\theta}_2 cos(\theta_2)$$

$$\dot{y}_2 = \dot{y}_1 + l_2 \dot{\theta}_2 cos(\theta_2)$$



Lagrangian

- Generalized approach to avoid constraints by using energy to derive the equations of motion.
- Here, we substitute all of the positions and velocities previously into the kinetic and potential energy equations.

$$T = 1/2(m_1(\dot{x}_1^2 + \dot{y_1}^2) + m_2(\dot{x}_2^2 + \dot{y_2}^2))$$

$$U = 1/2m_1gy_1 + 1/2m_2gy_2$$

$$T = 1/2(m_1 + m_2)l_1^2\dot{\theta_1}^2 + 1/2(m_1 + m_2)l_2^2\dot{\theta_2}^2$$

$$+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2cos(\theta_1 - \theta_2)$$

$$U = -g((m_1 + m_2)l_1cos(\theta_1) + m_2l_2cos(\theta_2)$$

$$L = T - U$$

Euler - Lagrange Equation

Utilizing the Euler-Lagrange equation we can create two coupled second order differential equations that are then reduced to four first order differential equations that we can integrate to arrive at the equations of motion.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{for } q_i = \theta_1, \theta_2.$$

Second Order Differentials

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0$$

First Order Differentials

$$z_1 = \dot{\theta}_1$$

$$z_2 = \dot{\theta}_2$$

$$\dot{z}_1 = \frac{m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) - m_2 \sin(\theta_1 - \theta_2)[l_1 z_1^2 \cos(\theta_1 - \theta_2) + l_2 z_2^2] - (m_1 + m_2) g \sin\theta_1}{l_1[m_1 + m_2 \sin^2(\theta_1 - \theta_2)]},$$

$$\dot{z}_2 = \frac{(m_1 + m_2)[l_1 z_1^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 + g \sin\theta_1 \cos(\theta_1 - \theta_2)] + m_2 l_2 z_2^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2)}{l_2[m_1 + m_2 \sin^2(\theta_1 - \theta_2)]}$$

Coding

- SciPy’s library has an integration function which passes in the jacobian, the derivative function, that contains the four first order ODE’S, a starting point, and the parameters of the problem.

- MATPLOT library was used to create all of the plots.

- The imageio library was used to collect all of the images the code produced and turned them into a gif animation for the double pendulum.

Double Pendulum

Chaotic Vs. Non-Chaotic

Chaotic

Initial Conditions

Length of Rods = 1 (m)

Mass = 1 (kg)

Time = 30 (s)

Angle 1 = 3pi/4 (radians)

Angular Velocity 1 = 0 (radian/s)

Angle 2 = 3pi/7 (radians)

Angular Velocity 2 = 0 (radians/s)

Angle 1 “Sensitivity” = 1e-5 (radians)

Non-Chaotic

Initial Conditions

Length of Rods = 1 (m)

Mass = 1 (kg)

Time = 30 (s)

Angle 1 = 3pi/40 (radians)

Angular Velocity 1 = 0 (radian/s)

Angle 2 = 3pi/70 (radians)

Angular Velocity 2 = 0 (radians/s)

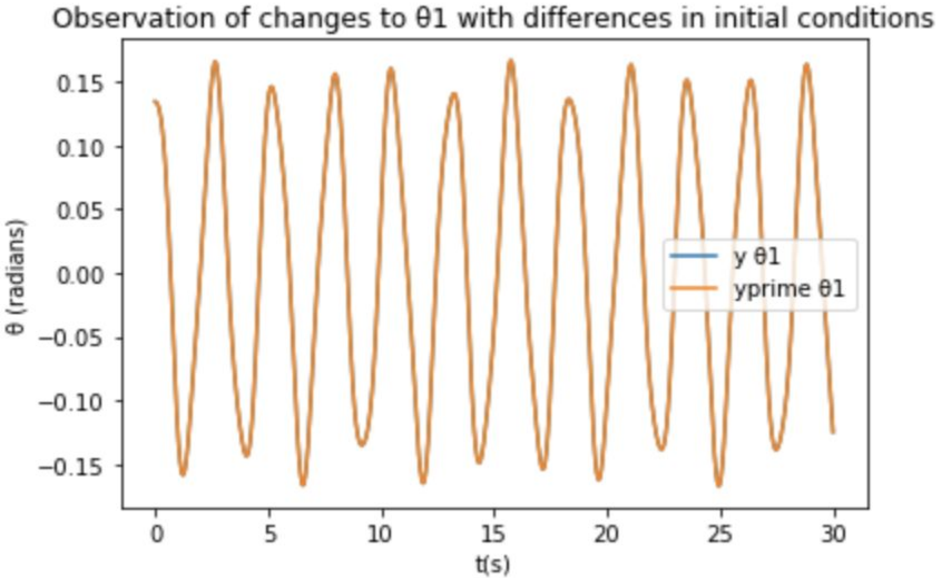
Angle 1 “Sensitivity” = 1e-5 (radians)

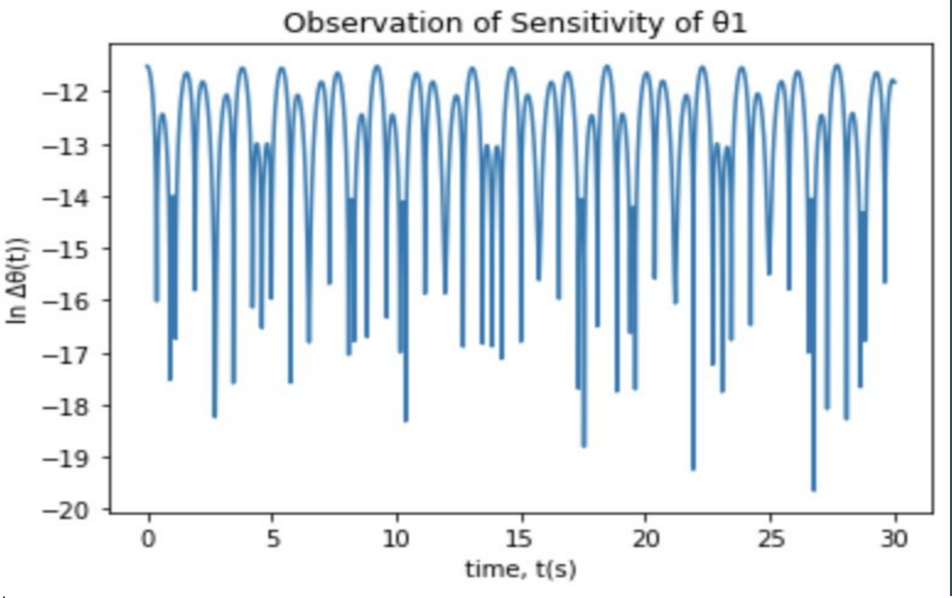
Chaotic vs Non-chaotic

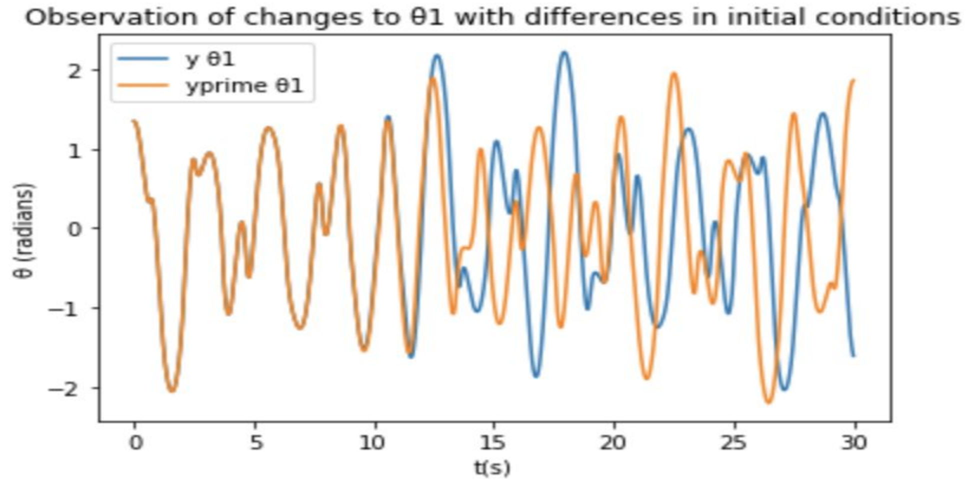
- By doing a factor of 10 reduction we can work with small angles to show the non-chaotic behavior of the double pendulum. At any small angle the equations of motion become nearly linear.

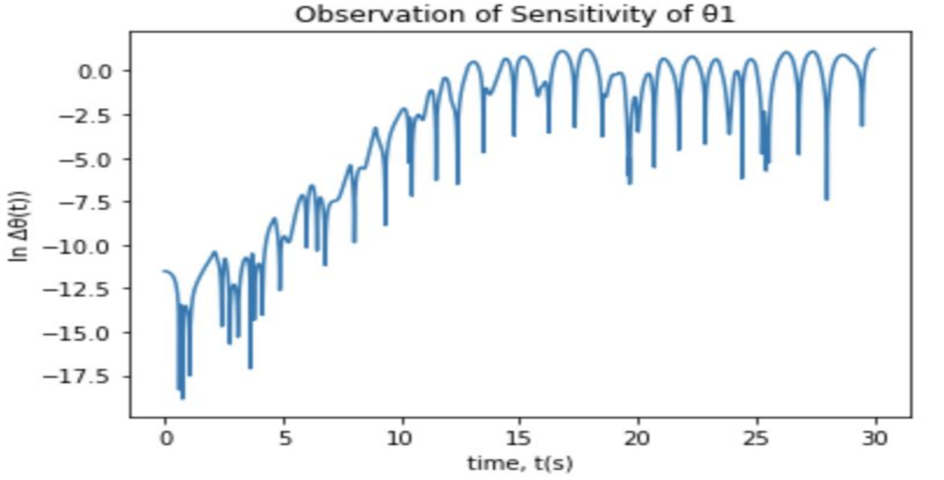
- In our simulation our starting point ‘y0’ is expressed as a 4-dimensional array storing the first angle | angular velocity of the first pendulum and respectively the second angle | angular velocity of the second pendulum.

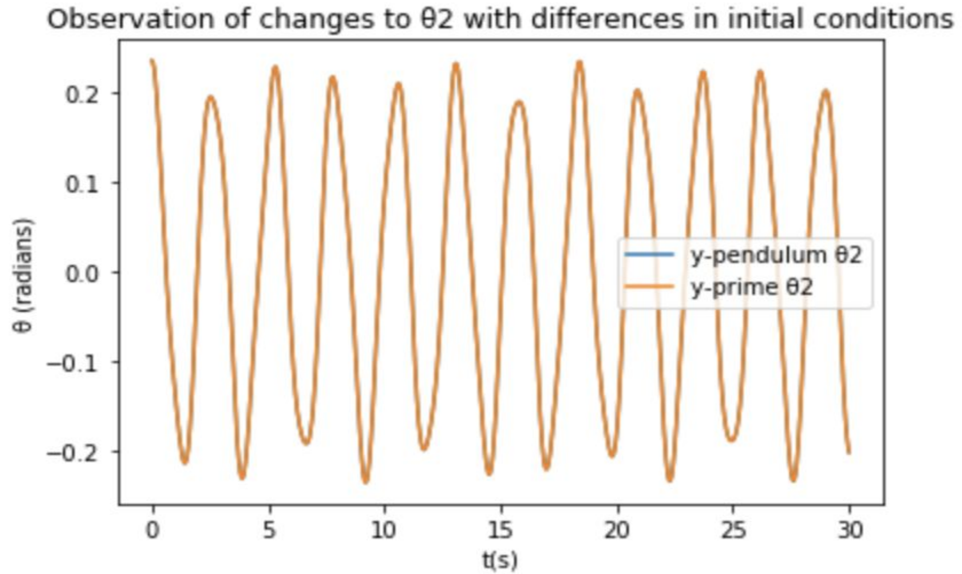
- To show the sensitivity of the chaotic system and to see effect on the non-chaotic system our second simulation of the original initial conditions with a small change to the angle of the first pendulum will be a value of 1e-5.

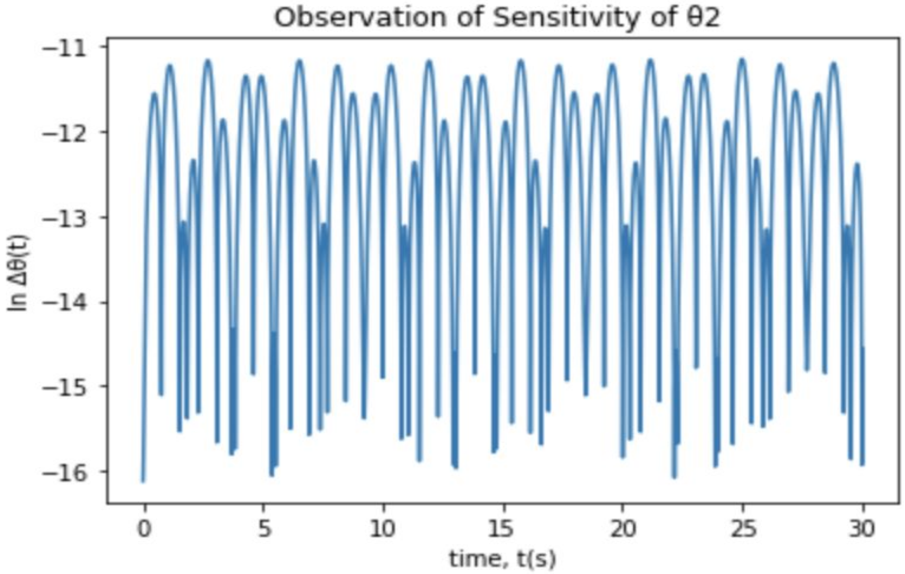

Observation of changes to θ1 with differences in initial conditions
y θ1
yprime θ1
θ (radians)
t(s)

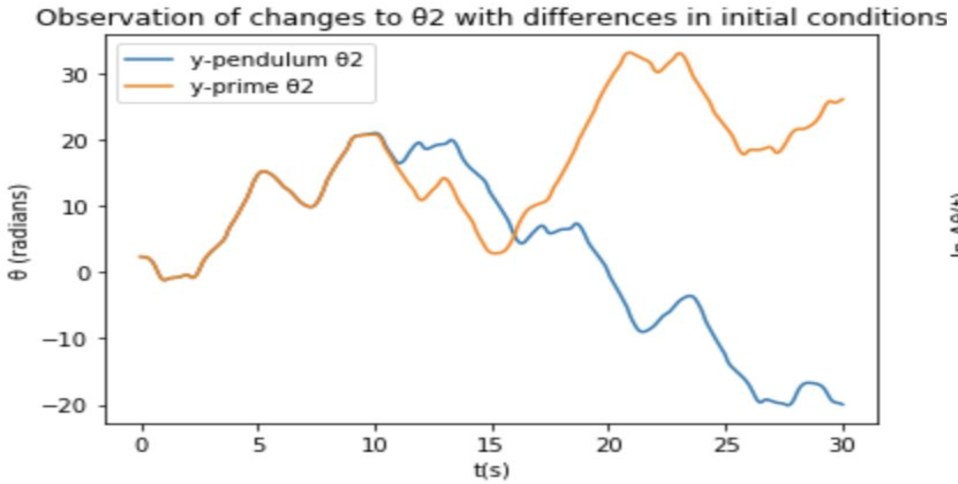

Observation of Sensitivity of θ1
ln Δθ(t)
time, t(s)

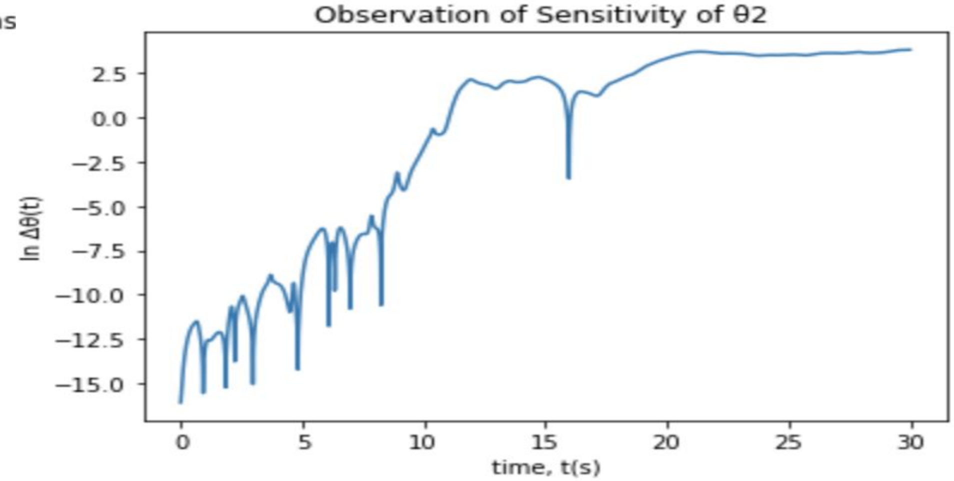

Observation of changes to θ1 with differences in initial conditions
y θ1
yprime θ1
θ (radians)
t(s)

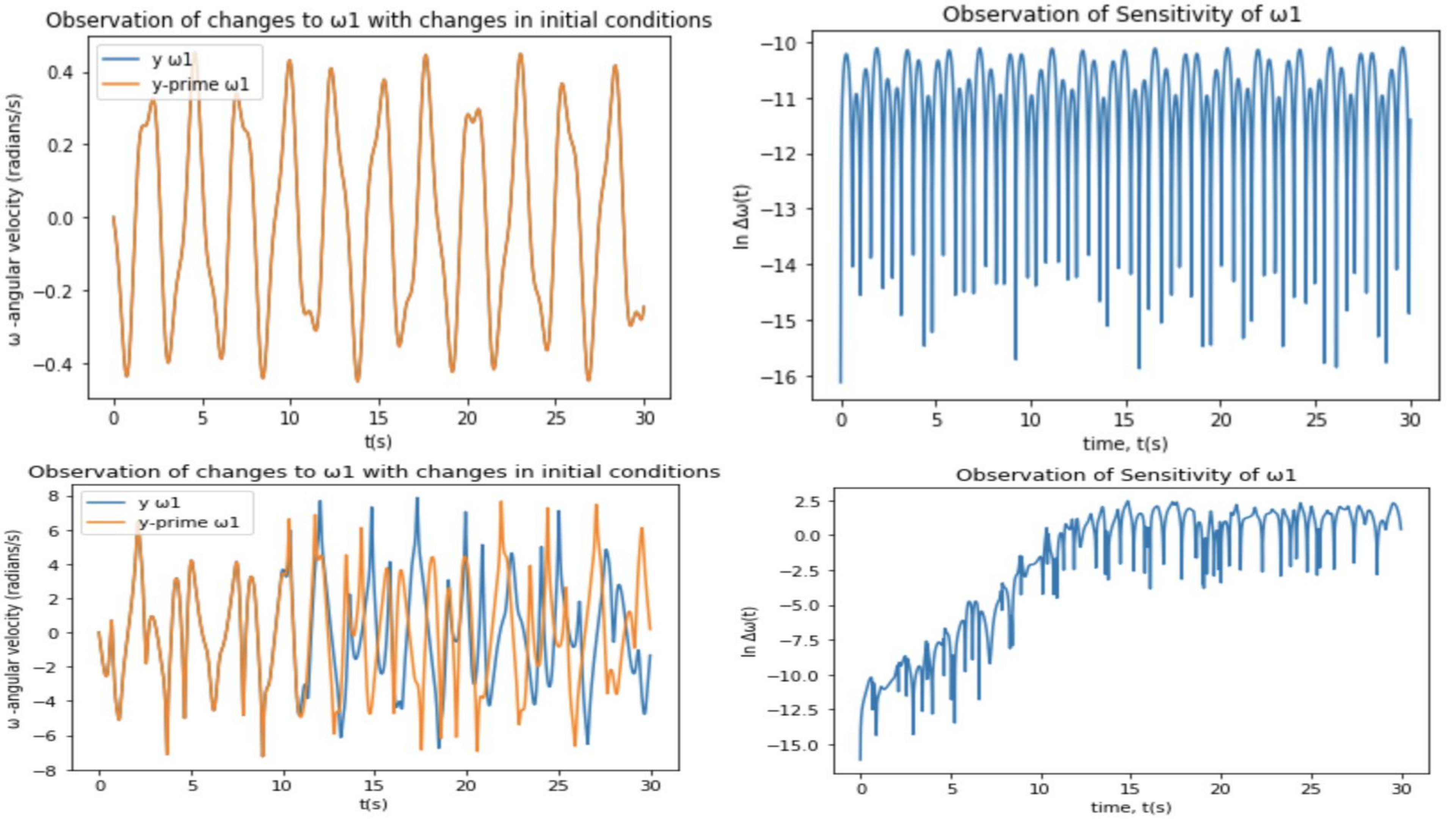

Observation of Sensitivity of θ1
ln Δθ(t)
time, t(s)

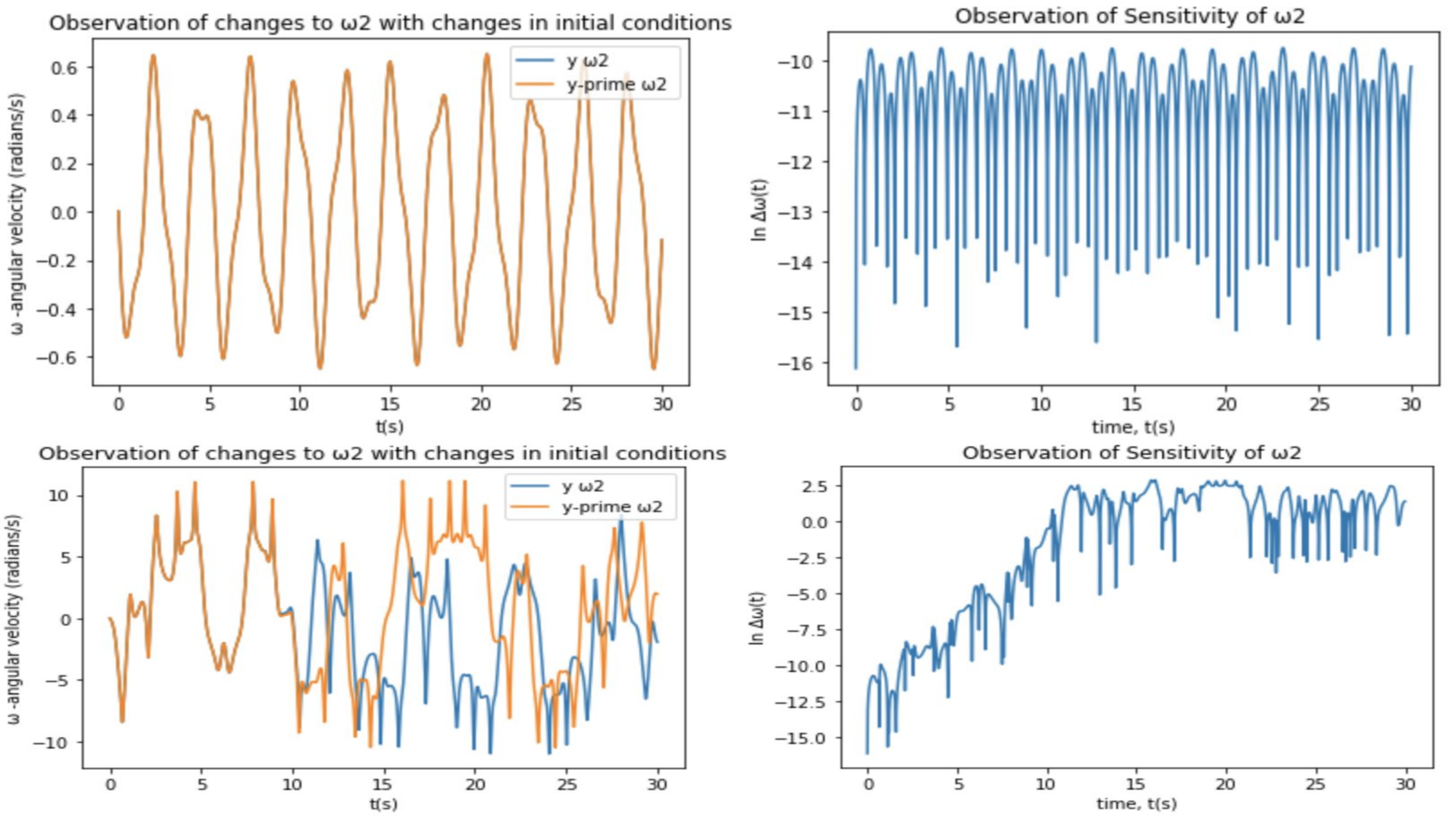

Observation of changes to θ2 with differences in initial conditions
y-pendulum θ2
y-prime θ2
θ (radians)
t(s)


Observation of Sensitivity of θ2
ln Δθ(t)
time, t(s)


Observation of changes to θ2 with differences in initial conditions
y-pendulum θ2
y-prime θ2
θ (radians)
t(s)

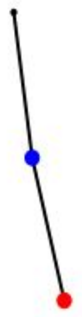

Observation of Sensitivity of θ2
ln Δθ(t)
time, t(s)


Observation of changes to ω1 with changes in initial conditions
y ω1
y-prime ω1
ω -angular velocity (radians/s)
t(s)
Observation of Sensitivity of ω1
ln Δω(t)
time, t(s)
Observation of changes to ω1 with changes in initial conditions
y ω1
y-prime ω1
ω -angular velocity (radians/s)
t(s)
Observation of Sensitivity of ω1
ln Δω(t)
time, t(s)


Observation of changes to ω2 with changes in initial conditions
y ω2
y-prime ω2
ω -angular velocity (radians/s)
t(s)
Observation of Sensitivity of ω2
ln Δω(t)
time, t(s)
Observation of changes to ω2 with changes in initial conditions
y ω2
y-prime ω2
ω -angular velocity (radians/s)
t(s)
Observation of Sensitivity of ω2
ln Δω(t)
time, t(s)

Chaotic

Non-Chaotic



Large Displacement

Small Displacement

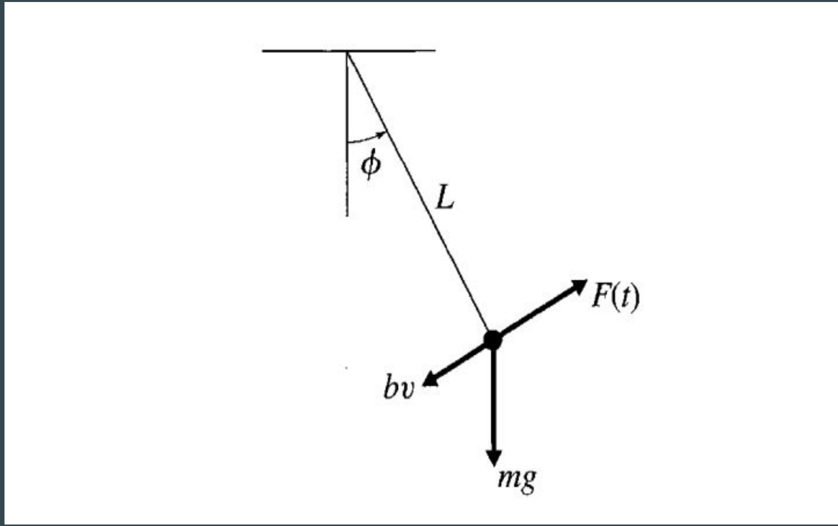
Damped Driven Pendulum

Damped Driven Pendulum

- Here two key terms are added: damping and driving forces.

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t.$$

- Damping: Takes energy out of the system.
- Driving: Adds energy to the system.



Initial Conditions

Length of Rods = 1 (m)

Mass = 1 (kg)

Time = 30 (s)

Angle = 2.4 (radians)

Angular Velocity = 0 (radians/s)

Damping Coefficient = 0

Drive Frequency = 1.0 (s-1)

Drive Coefficient = 1.0

Sensitivity Adjustments

Length of Rods = Unchanged

Mass = Unchanged

Time = Unchanged

Angle = Unchanged

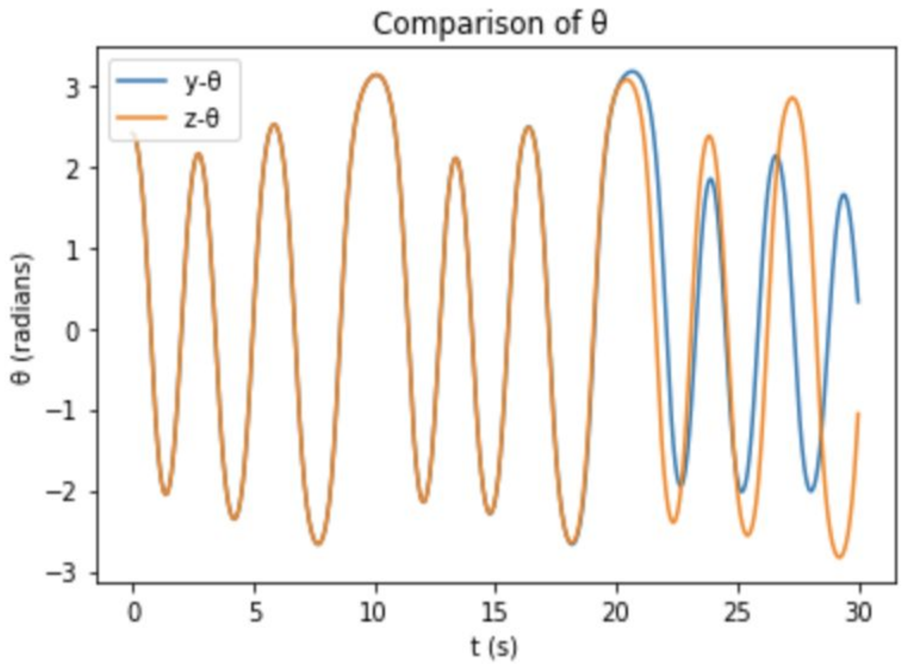
Angular Velocity = Unchanged

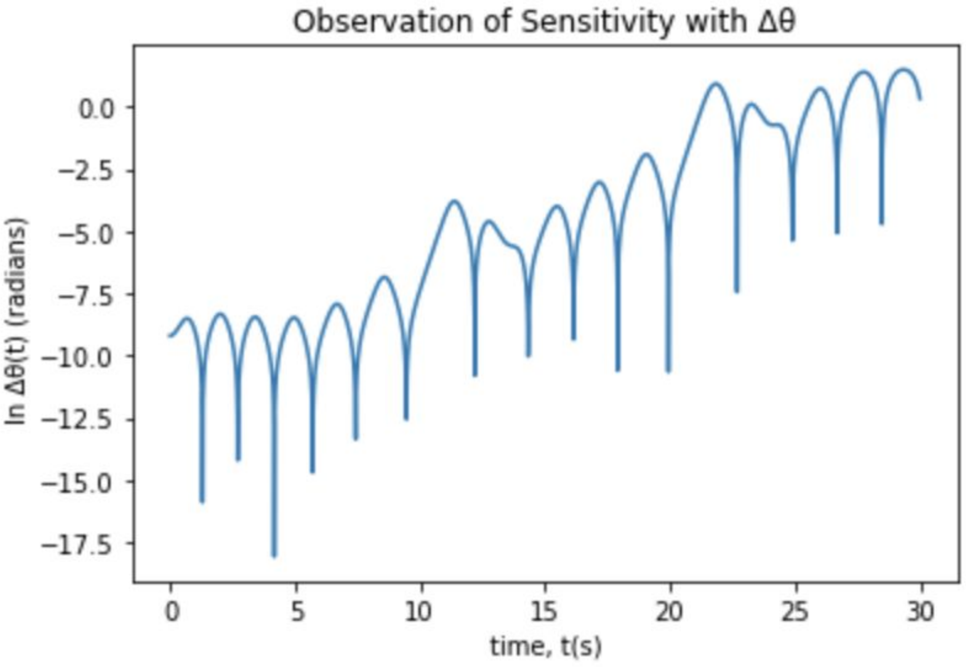
Angle "Sensitivity" = +1e-4 (radians)

Damping Coefficient = +0.23

Drive Frequency = +0.8 (s-1)

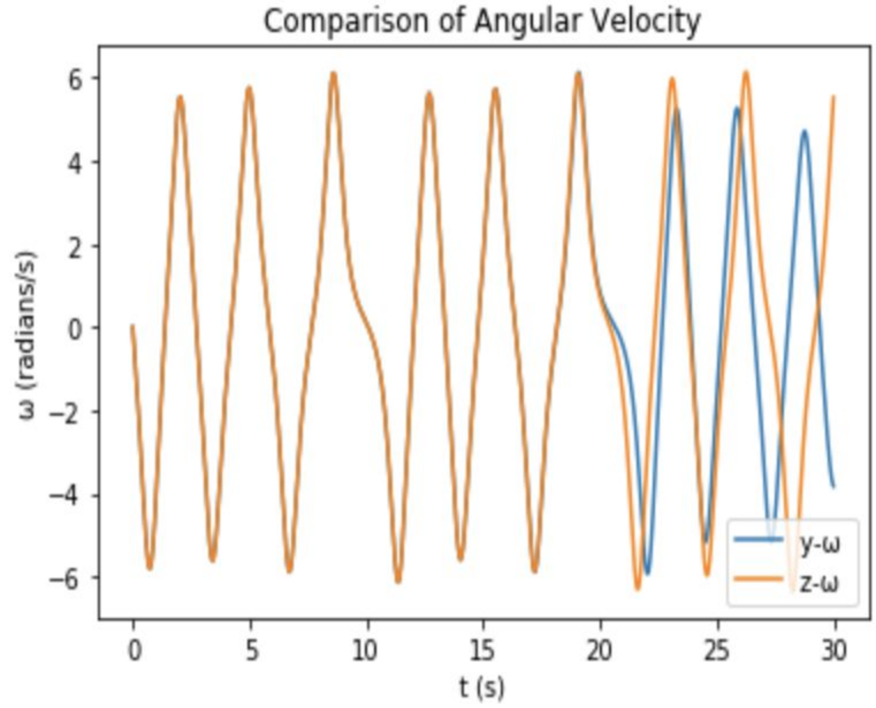
Drive Coefficient = +0.5

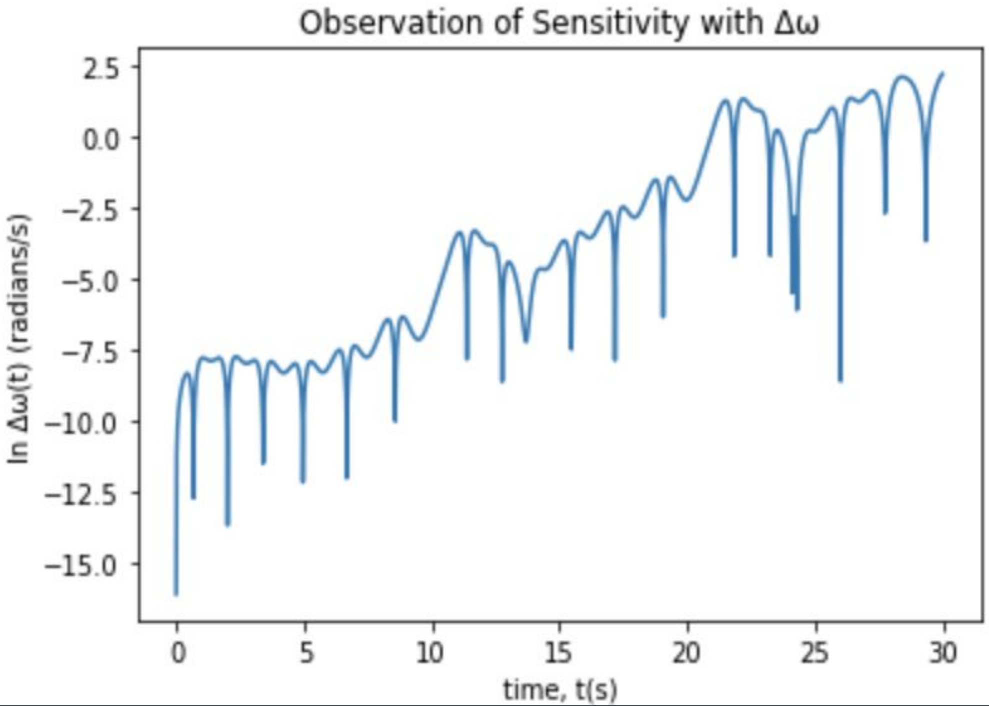




- Angles diverge begin to clearly diverge at ~ t = 19s.

- Exponential growth shows that the values differing between the two angles of the pendulum start to grow apart in magnitude and change direction.





- Similar behavior
- Exponential Growth

Damped Driven Double Pendulum

Initial Conditions

Length of Rods = 1 (m)

Mass = 1 (kg)

Time = 30 (s)

Angle 1 = 3.4 (radians)

Angular Velocity 1 = 0 (radian/s)

Angle 2 = 3.4 (radians)

Angular Velocity 2 = 0 (radians/s)

Drive Frequency = 1.5 (s-1)

Driving Coefficient = 1.5

Damping Coefficient = 0

Sensitivity Adjustments

Length of Rods = Unchanged

Mass = Unchanged

Time = Unchanged

Angle 1 = 3.4(radians)

Angular Velocity 1 = Unchanged

Angle 2 = Unchanged

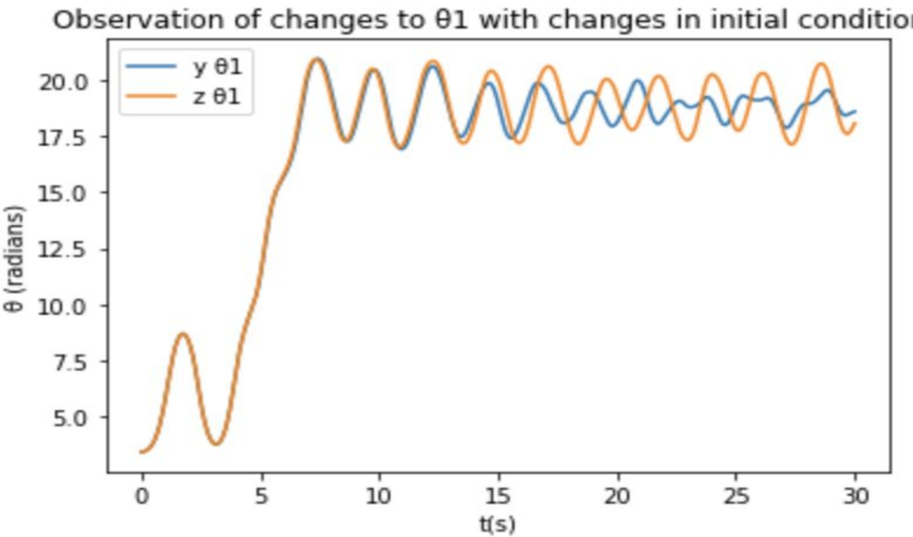
Angular Velocity 2 = Unchanged

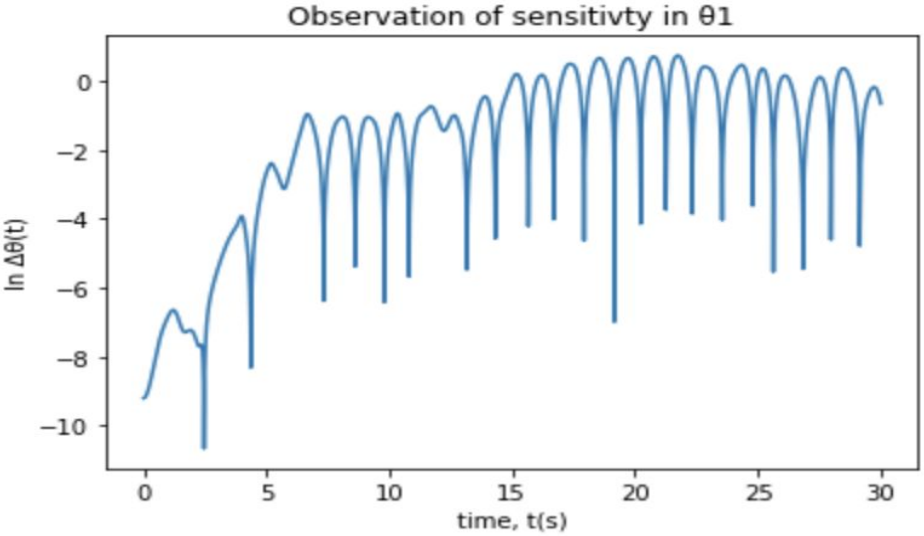
Angle 1 “Sensitivity” = +1e-4 (radians)

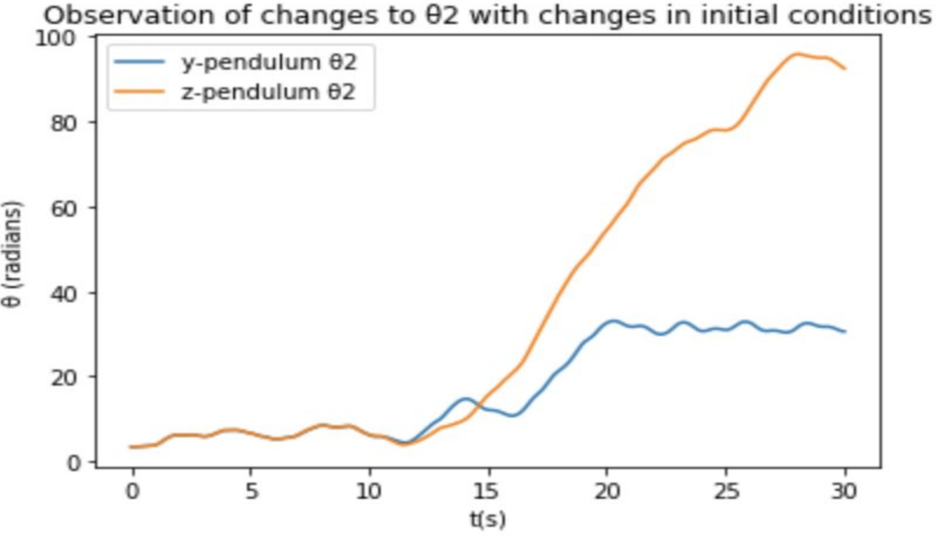
Drive Frequency = +0.5 (s-1)

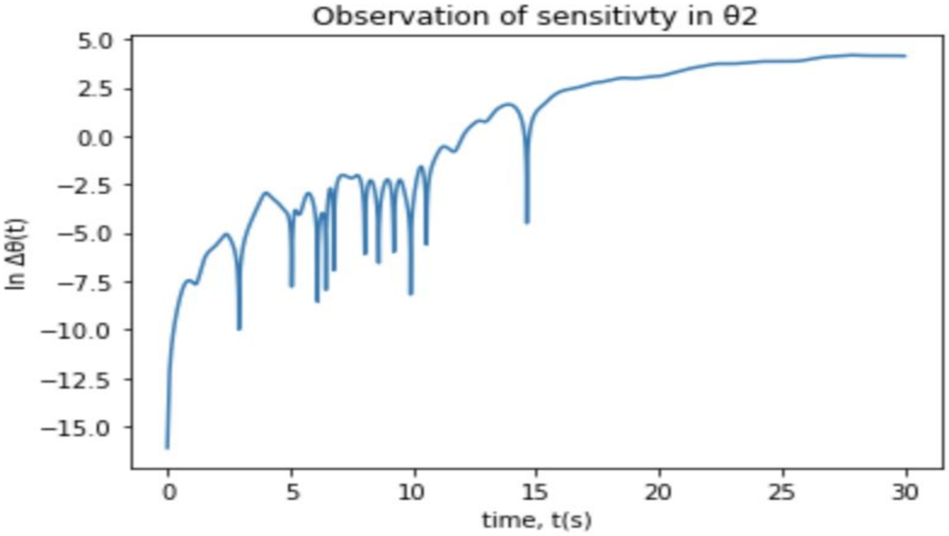
Driving Coefficient = +0.5

Damping Coefficient = +0.1


Observation of changes to θ1 with changes in initial conditions
y θ1
z θ1
θ (radians)
t(s)


Observation of sensitivty in θ1
ln Δθ(t)
time, t(s)


Observation of changes to θ2 with changes in initial conditions
y-pendulum θ2
z-pendulum θ2
θ (radians)
t(s)


Observation of sensitivty in θ2
ln Δθ(t)
time, t(s)

Application

Applications

- Biomechanics & Kinesiology

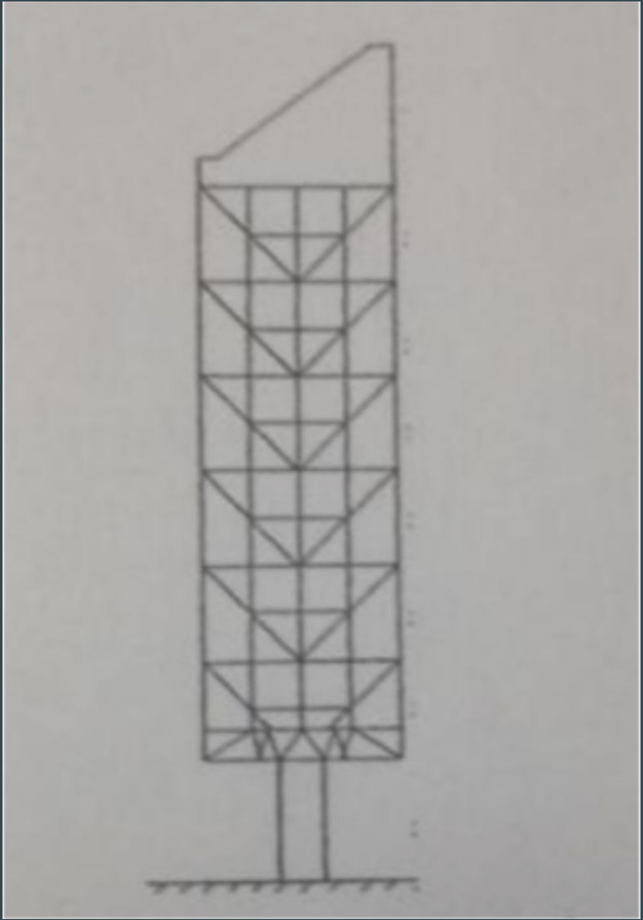
  Explore how forces act on or are caused by limbs.

- Robotics

  Robots are being designed to mimic living species the applications of biomechanics also fall into place here to design complex life-like robots

- Architecture

  Driving & Damping Mechanisms are used to preserve building integrity from external environmental forces like winds or earthquakes.



Conclusion

- At small angles the double pendulum experiences smaller displacements and the equations of motion become linear and chaotic motion is not displayed.

- With larger angles and introducing driving and damping the complexity of the double pendulum and the equations of motion becomes non-linear showing that it can be sensitive to the initial conditions which is a key characteristic to determine that it can have an unpredictable behavior indicating its chaotic nature.

- With the limited time we had we were able to explore what we could of the chaotic regimes of the double pendulum but did not have sufficient time to further explore the applications of the double pendulum.

Questions