# Enhanced Correlation Coefficient Maximization: Mathematical Theory

## 1 Enhanced Correlation Coefficient (ECC)

The Enhanced Correlation Coefficient (ECC) is used to measure the alignment between two image profiles: a reference image $I_r(x)$ and a distorted image $I_w(y)$. The ECC criterion is defined as:

$$EECC(p) = \left\| \frac{\bar{\mathbf{I}}_\mathbf{r}}{||\bar{\mathbf{I}}_\mathbf{r}||} - \frac{\bar{\mathbf{I}}_\mathbf{w}(p)}{||\bar{\mathbf{I}}_\mathbf{w}(p)||} \right\|^2, \tag{1}$$

where:

- $\bar{\mathbf{I}}_\mathbf{r}$: Zero-mean version of the intensity vector for the reference image.
- $\bar{\mathbf{I}}_\mathbf{w}(p)$: Zero-mean version of the intensity vector for the warped image with parameter $p$.
- $p$: Vector of parameters controlling the geometric transformation.

The objective is to maximize the correlation defined as:

$$\phi(p) = \frac{\bar{\mathbf{I}}_\mathbf{r}^\top \bar{\mathbf{I}}_\mathbf{w}(p)}{||\bar{\mathbf{I}}_\mathbf{r}|| \, ||\bar{\mathbf{I}}_\mathbf{w}(p)||}. \tag{2}$$

This function is invariant to photometric distortions such as changes in contrast and brightness.

## 2 Geometric Transformation

The transformation of coordinates is modeled parametrically:

$$y = \psi(x; p), \tag{3}$$

where:

- $x = [x_1, x_2]^\top$: Coordinates in the reference image.
- $y = [y_1, y_2]^\top$: Coordinates in the warped image.
- $\psi(x; p)$: Parametric transformation function (e.g., affine, homography).

# 3 Optimization Formulation

The alignment problem can be formulated as:

$$\min_p E(p) = \min_p \|\mathbf{I}_r - \mathbf{I}_w(\psi(x;p))\|^2 . \tag{4}$$

To account for intensity changes, a photometric model can be included:

$$\min_{p,\theta} E(p,\theta) = \min_{p,\theta} \sum_{x \in T} |I_r(x) - \gamma \cdot I_w(\psi(x;p)) - \beta|^2 , \tag{5}$$

where $\gamma$ and $\beta$ are parameters for contrast and bias.

In ECC, we focus only on geometric components by normalizing the intensity vectors to have zero mean.

# 4 Linear Approximation

The nonlinear transformation $I_w(\psi(x;p))$ is approximated using a first-order Taylor expansion:

$$I_w(\psi(x;p)) \approx I_w(\psi(x;\tilde{p})) + J(\tilde{p})\Delta p, \tag{6}$$

where:

- $J(\tilde{p})$: Jacobian matrix computed at the current parameter $\tilde{p}$.
- $\Delta p = p - \tilde{p}$: Parameter update vector.

Thus, the intensity vector for the warped image becomes:

$$\mathbf{I}_w(p) \approx \mathbf{I}_w(\tilde{p}) + G(\tilde{p})\Delta p, \tag{7}$$

where $G(\tilde{p})$ is the Jacobian that maps parameter changes to intensity changes.

# 5 Iterative Algorithms

## 5.1 Forward Additive ECC

This method iteratively updates the parameters:

$$p_j = p_{j-1} + \Delta p, \tag{8}$$

where $\Delta p$ is obtained by maximizing the approximated objective function:

$$\phi(\Delta p|\tilde{p}) = \frac{\bar{\mathbf{I}}_{\mathbf{r}}^\top (\bar{\mathbf{I}}_{\mathbf{w}} + G\Delta p)}{\|\bar{\mathbf{I}}_{\mathbf{r}}\| \|\bar{\mathbf{I}}_{\mathbf{w}} + G\Delta p\|}. \tag{9}$$

## 5.2 Inverse Compositional ECC

In this efficient variant, the roles of the reference and warped images are interchanged. The Jacobian is computed once for the reference image:

$$p \leftarrow \psi^{-1}(x;p). \tag{10}$$

This reduces the computational complexity significantly.

# 6 Relation to SSD-Based Methods

ECC is compared to Sum of Squared Differences (SSD) methods like Lucas-Kanade:

- ECC optimizes normalized correlation, making it robust to photometric distortions.
- SSD minimizes intensity differences, which are sensitive to brightness changes.

# 7 Performance Evaluation

## 7.1 Error Metric

The squared error for iteration $j$ is defined as:

$$e(j) = \frac{1}{6}\sum_{i=1}^{3} \|\psi(x_i; p_r) - \psi(x_i; p_j)\|^2. \tag{11}$$

The Mean Squared Distance (MSD) is computed as the average error over multiple experiments.

## 7.2 Simulation Results

Simulation experiments show that ECC has:

- Faster convergence.
- Greater robustness to noise and photometric distortions.
- Lower computational complexity in the inverse compositional variant.

# 8 Conclusion

The ECC method provides an efficient and robust solution for image alignment by maximizing the enhanced correlation coefficient. It outperforms traditional SSD-based methods like Lucas-Kanade, especially under challenging conditions such as noise and photometric distortions.