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Robust estimation under progressive censoring

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Abstract

For progressively censored failure time data, the influence function and the breakdown point of robust M-estimators are derived. The most robust and the optimal robust estimators are also developed. The optimal members within two classes of ψ-functions are characterized. The first optimality result is the censored data analogue of the general optimality result. The second result pertains to a restricted class of ψ-functions. The usefulness of the two classes of ψ-functions is examined and it was found that the breakdown point and efficiency of the restricted class of optimal estimators compare favorably with those of the corresponding general optimal robust estimators. From the computational point of view, the restricted class of optimal ψ-functions are readily obtainable from the general optimal ψ-functions in the uncensored case. A data set illustrates the optimal robust estimators for the parameters of the extreme value distribution.

Keywords: Breakdown point; Influence function; James-type M-estimators; M-estimators; Most robust estimators; Optimal robust estimators; Progressive censoring

1. Introduction

The statistical analysis of what is variously referred to as failure time, lifetime or survival data has been widely developed, especially in the biomedical and engineering sciences. Applications of survival data statistical analysis range from research involving

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human diseases to investigations in the endurance of manufactured items. Survival data often come with a special feature. It is known as censoring and occurs when exact survival times are known only for a portion of the individuals or items under study. The complete survival times are not observed by the experimenter either intentionally or unintentionally.

In this article, we consider a general scheme of censoring which is known as progressive Type II right censoring. In this scheme of censoring, n units are placed on a life-testing experiment and only m are completely observed until failure. The censoring occurs progressively in m stages. These m stages are failure times of m completely observed units. At the time of the first failure (the first stage), r_1 of the $n-1$ surviving units are randomly withdrawn (censored intentionally) from the experiment, r_2 of the $n-2-r_1$ surviving units are withdrawn at the time of the second failure (the second stage) and so on. Finally, at the time of the mth failure (the mth stage), all the remaining $r_m = n-m-r_1-\cdots-r_{m-1}$ surviving units are withdrawn. We will refer this to as progressively Type-II right censoring scheme $(r_1, r_2, \ldots, r_m)$. It is clear that this scheme includes the conventional Type-II right censoring scheme (when $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n-m$) and complete sampling scheme (when $n = m$ and $r_1 = r_2 = \cdots = r_m = 0$). The ordered survival data which arise from such progressively Type-II right censoring scheme are called progressively Type-II right censored order statistics. For the theory, methods and applications of progressive censoring, readers are referred to the book by Balakrishnan and Aggarwala (2000).

Since the publication of the book by Balakrishnan and Aggarwala (2000), considerable amount of research work has been carried out on progressive censoring methodology. Balasooriya et al. (2000) and Balasooriya and Balakrishnan (2000) studied progressively censored reliability sampling plans for Weibull and lognormal distributions, respectively. Ng et al. (2002) discussed the estimation of parameters from progressively censored data using the EM-algorithm, while Ng et al. (2003) used this approach to determine optimal progressive censoring schemes and reliability sampling plans for the Weibull distribution. Balakrishnan et al. (2001) and Balakrishnan and Lin (2002) discussed inferential procedures for exponential distribution, while Balakrishnan et al. (2003a) discussed for normal distribution based on progressively Type-II censored samples. Recently, Balakrishnan et al. (2002, 2003b) proposed goodness-of-fit tests for testing exponentiality and a general location-scale distribution, respectively, based on progressively Type-II censored samples.

Let $X_1, \ldots, X_n$ denote the failure times of n independent units placed on a life-test experiment. Assume these come from a common continuous cumulative distribution function $F(x, \theta)$ and probability density function $f(x, \theta)$ involving a possibly vector-valued parameter θ. Let $Y_1 \leqslant \cdots \leqslant Y_m$ denote the above mentioned m progressively Type-II right censored order statistics. The purpose of this paper is to develop robust M-estimates for θ based on progressively Type-II right censored samples. Robust statistical methods protect estimates against outliers. For a specific level of protection, an optimal estimate is produced using the approaches by Huber (1981) and Hampel et al. (1986). Usually, the parametric analysis of Type-II censored data is not much sensitive to outliers on the right. But outliers on the left can have big influence on the maximum likelihood (ML) estimates of the parameters. For robust statistical procedures in the uncensored case, readers are referred to three books—Huber (1981), Hampel et al. (1986) and Jureckova and Sen (1996).

The log-likelihood function of the m progressively Type-II right censored order statistics $Y_1 \leqslant \cdots \leqslant Y_m$, except for constant term, is

$$\sum_{i=1}^{m} \rho_1(Y_i, \theta) + \sum_{i=1}^{m} r_i \rho_2(Y_i, \theta)$$

in which $\rho_1(x, \theta) = \log f(x, \theta), \rho_2(x, \theta) = \log(1 - F(x, \theta))$. The ML estimate of θ is obtained by solving the likelihood equation

$$\sum_{i=1}^{m} \rho_1'(Y_i, \theta) + \sum_{i=1}^{m} r_i \rho_2'(Y_i, \theta) = 0 \tag{1}$$

in which f' denotes the derivative of a function f with respect to θ. ML estimators are efficient if the assumed model distribution is appropriate but they have undesirable properties of being very sensitive to outlying observations. In a robust M-estimation procedure, ρ_1', ρ_2' in (1) are replaced by functions ψ_1, ψ_2. Then one solves

$$\sum_{i=1}^{m} \psi_1(Y_i, \theta) + \sum_{i=1}^{m} r_i \psi_2(Y_i, \theta) = 0. \tag{2}$$

The functions ψ_1 and ψ_2 are appropriately chosen in order to get various kinds of robust estimates of θ. The extent by which a outlier influence the estimator is specified by what is called the gross-error sensitivity. The estimator which minimises the gross-error-sensitivity is called the most robust estimator. There is a tradeoff between efficiency and resistance against outliers. For a specific level of protection, the estimator which maximizes the efficiency among all estimators providing that level of protection is called the optimal robust estimator. Theory and methods for finding the most robust and the optimal robust estimators for the uncensored case are available in Hampel et al. (1986). Akritas et al. (1993) considered the case of Type-II censored observations.

By noting that ρ_2' in the likelihood Eq. (1) is given by

$$\rho_2'(y, \theta) = E[\rho_1'(Y, \theta) | Y > y],$$

one can have a more specialized class of robust M-estimators for the censored data case. In this specialized class, the robust M-estimators are obtained from (2) with

$$\psi_2(y, \theta) = E[\psi_1(Y, \theta) | Y > y]$$

and $E[\psi_1(Y, \theta)] = 0$. Similar estimators were proposed by James (1986) in the context of randomly censored data and those would be referred as James-type M-estimators in this article. In the next section, the influence function and the breakdown point are obtained. The most robust estimators and the optimal robust estimators are derived in Section 3. In Section 4, numerical illustrations involving extreme value distribution is provided. We make some concluding remarks in Section 5. Section 6 is the appendix where we provide a list of notations used in this article and proofs of some results.

2. Properties of the M-estimator

In the first subsection, the influence function of a robust M-estimator of an one-dimensional parameter θ is developed. Next the gross-error-sensitivity is defined using that influence function. In the second subsection, we derive the breakdown point of the location estimates.

2.1. The influence function

Let us consider the case when θ is a one-dimensional parameter. We will first derive the functional form $T(F)$ of an M-estimator for θ. In the scheme of progressive Type-II right censoring, the censoring occurs in m stages unlike the usual Type-II right censoring where censoring occurs in only one stage. In the later scheme, we do not censor until the failure of the sth observation. Therefore, the proportion of the uncensored observation is s/n and it is assumed that $s/n \to p$ and one defines $Q_p(F)$ as the pth quantile of F. In the progressive censoring, the proportions of uncensored observations are defined in m stages. Let us define these m proportions as $p_1, p_2, \ldots, p_m$ and their corresponding quantiles as $Q_{p_1}(F), Q_{p_2}(F), \ldots, Q_{p_m}(F)$. We will write $Q_{p_1}, Q_{p_2}, \ldots, Q_{p_m}$ for $Q_{p_1}(F), Q_{p_2}(F), \ldots, Q_{p_m}(F)$. Suppose in a progressive censoring, r_i items are randomly withdrawn after s_i (additional) failures; $i = 1, 2, \ldots, m$. Let us assume that $s_i/n \to p_i^*$; $i = 1, 2, \ldots, m$. Then the following holds:

$$
\begin{aligned}
&\frac{s_1}{n} \to \int_{-\infty}^{Q_{p_1}} f(x)\,\mathrm{d}x = p_1,\\
&\frac{s_2}{n - r_1 - s_1} \to \int_{Q_{p_1}}^{Q_{p_2}} \frac{f(x)}{\bar{F}(Q_{p_1})}\,\mathrm{d}x,\\
&\ldots\\
&\frac{s_m}{n - \sum_{i=1}^{m-1} r_i - \sum_{i=1}^{m-1} s_i} \to \int_{Q_{p_{m-1}}}^{Q_{p_m}} \frac{f(x)}{\bar{F}(Q_{p_{m-1}})}\,\mathrm{d}x.
\end{aligned} \tag{3}
$$

Divide both sides of Eq. (2) by n and observing that

$$
\begin{aligned}
&\frac{\psi_1(Y_i,\theta)}{n} = \frac{\psi_1(Y_i,\theta)}{n - r_1 - s_1}\,\frac{n - r_1 - s_1}{n}; \qquad i = s_1 + 1, \ldots, s_1 + s_2,\\
&\ldots\\
&\frac{\psi_1(Y_i,\theta)}{n} = \frac{\psi_1(Y_i,\theta)}{n - \sum_{i=1}^{m-1} r_i - \sum_{i=1}^{m-1} s_i}\,\frac{n - \sum_{i=1}^{m-1} r_i - \sum_{i=1}^{m-1} s_i}{n};\\
&\quad i = \sum_{i=1}^{m-1} s_i + 1, \ldots, \sum_{i=1}^{m} s_i,
\end{aligned}
$$

it can be shown using (3) that

$$\sum_{i=1}^{s_1} \frac{\psi_1(Y_i,\theta)}{n} \to \int_{-\infty}^{Q_{p_1}} \psi_1(y)\,\mathrm{d}F(y),$$

$$\sum_{i=s_1+1}^{s_1+s_2} \frac{\psi_1(Y_i,\theta)}{n} \to \frac{\int_{Q_{p_1}}^{Q_{p_2}} \psi_1(y)\,\mathrm{d}F(y)}{1-p_1}(1-q_1-p_1^*),$$

$$\sum_{i=s_1+s_2+1}^{s_1+s_2+s_3} \frac{\psi_1(Y_i,\theta)}{n} \to \frac{\int_{Q_{p_2}}^{Q_{p_3}} \psi_1(y)\,\mathrm{d}F(y)}{1-p_2}(1-q_1-q_2-p_1^*-p_2^*),$$

$$\ldots$$

$$\sum_{i=s_1+\cdots+s_{m-1}+1}^{s_1+\cdots+s_m} \frac{\psi_1(Y_i,\theta)}{n} \to \frac{\int_{Q_{p_{m-1}}}^{Q_{p_m}} \psi_1(y)\,\mathrm{d}F(y)}{1-p_{m-1}} \left(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*\right) \tag{4}$$

where $r_i/n \to q_i$, $i = 1,2,\ldots,m$. Note that $\sum_{i=1}^m q_i + \sum_{i=1}^m p_i^* = 1$. Using (4), the functional form $T(F)$ of the M-estimator of θ corresponding to (2) would be defined as the solution to

$$\begin{aligned}
&\int_{-\infty}^{Q_{p_1}} \psi_1(y,T(F))\,\mathrm{d}F(y) + \frac{(1-q_1-p_1^*)}{1-p_1}\int_{Q_{p_1}}^{Q_{p_2}} \psi_1(y,T(F))\,\mathrm{d}F(y)\\
&\quad + \frac{(1-q_1-q_2-p_1^*-p_2^*)}{1-p_2}\int_{Q_{p_2}}^{Q_{p_3}} \psi_1(y,T(F))\,\mathrm{d}F(y) + \cdots\\
&\quad + \frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\int_{Q_{p_{m-1}}}^{Q_{p_m}} \psi_1(y,T(F))\,\mathrm{d}F(y)\\
&\quad + q_1\psi_2(Q_{p_1},T(F)) + q_2\psi_2(Q_{p_2},T(F)) \cdots + q_m\psi_2(Q_{p_m},T(F)) = 0.
\end{aligned} \tag{5}$$

It is clear from (3) that there exists some relation between $p_i^*, q_i, p_i;\ i=1,2,\ldots,m$. For example, consider the second line of (3). The expression in the right side of the arrow equals $(p_2-p_1)/(1-p_1)$ while it can be shown that the expression $s_2/(n-r_1-s_1)$ in the left side of the arrow approaches to $p_2^*/(1-q_1-p_1^*)$. Therefore, $(p_2-p_1)/(1-p_1)$ and $p_2^*/(1-q_1-p_1^*)$ must be equal to each other. Similar arguments hold for the following lines of (3). The relations between $p_i^*, q_i, p_i;\ i=1,2,\ldots,m$ are therefore given by

$$
\begin{aligned}
&p_1 = p_1^*\\
&p_2 = p_1 + p_2^* \cdot \frac{1-p_1}{1-q_1-p_1^*},\\
&p_3 = p_2 + p_3^* \cdot \frac{1-p_2}{1-q_1-q_2-p_1^*-p_2^*},\\
&\dots\\
&p_m = p_{m-1} + p_m^* \cdot \frac{1-p_{m-1}}{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}.
\end{aligned} \tag{6}
$$

In the following lemma, we provide another relation between $p_i^*,\ q_i, p_i;\ i=1,2,\dots,m$ which would be helpful subsequently.

Lemma 2.1. *$p_i^*, q_i, p_i;\ i=1,2,\dots,m$ satisfy*

$$1-p_l = (1-p_{l-1})\frac{1-\sum_{i=1}^{l-1} q_i - \sum_{i=1}^{l} p_i^*}{1-\sum_{i=1}^{l-1} q_i - \sum_{i=1}^{l-1} p_i^*}; \quad l=2,\dots,m. \tag{7}$$

Also, for $l=m$,

$$1-p_m = (1-p_{m-1})\frac{q_m}{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}; \tag{8}$$

Proof. The relation in (6) is given by

$$p_l = p_{l-1} + p_l^* \cdot \frac{1-p_{l-1}}{1-\sum_{i=1}^{l-1} q_i - \sum_{i=1}^{l-1} p_i^*}; \quad l=2,\dots,m.$$

Subtracting both sides of the above equation from 1 and simplifying the expressions one gets (7). For $l=m$, (8) holds since $q_m = 1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m} p_i^*$. Hence Lemma 2.1 is proved. □

The influence function of $T(F)$ will be denoted by $IF(x;T,F)$ and it is the measurement of how one particular outlying observation affects the estimate infinitesimally. Mathematically, $IF(x;T,F)$ is defined as the derivative of $T(F^s)$ with respect to s, evaluated at $s=0$, where $F^s = F + s(\Delta_x - F)$, and Δ_x is the point mass function at x. Therefore,

$$IF(x;T,F) = \left.\frac{\delta T(F^s)}{\delta s}\right|_{s=0}.$$

In order to obtain the mathematical expression for $IF(x;T,F)$, one substitutes F^s in place of F in (5), differentiate with respect to s at s=0 and finally solves for $IF(x;T,F)$.

Differentiating (5) with respect to $Q_{p_1}, Q_{p_2}, \ldots, Q_{p_m}$, one gets by using (7),

$$
\begin{aligned}
&\frac{q_1}{1-p_1}\,\psi_1(Q_{p_1})f(Q_{p_1}) + q_1\psi_2'(Q_{p_1}) = 0\\
&\frac{q_2}{1-p_2}\,\psi_1(Q_{p_2})f(Q_{p_2}) + q_2\psi_2'(Q_{p_2}) = 0\\
&\ldots\\
&\frac{q_m}{1-p_m}\,\psi_1(Q_{p_m})f(Q_{p_m}) + q_m\psi_2'(Q_{p_m}) = 0. \qquad (9)
\end{aligned}
$$

Using the identities in (9), we get the expression for $IF(x;T,F)$ as

$$IF(x;T,F) = -\frac{N(x;T,F)}{D_1(x;T,F)} \qquad (10)$$

in which

$$
\begin{aligned}
N(x;T,F) = {} & \psi_1(x,T(F)))I(x < Q_{p_1})\\
& + \frac{(1-q_1-p_1^*)}{1-p_1}\,\psi_1(x,T(F))I(Q_{p_1} < x < Q_{p_2})\\
& + \frac{(1-q_1-q_2-p_1^*-p_2^*)}{1-p_2}\,\psi_1(x,T(F))I(Q_{p_2} < x < Q_{p_3}) + \cdots\\
& + \frac{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}{1-p_{m-1}}\,\psi_1(x,T(F))I(Q_{p_{m-1}} < x < Q_{p_m})\\
& + q_1\psi_2(Q_{p_1},T(F)) + q_2\psi_2(Q_{p_2},T(F)) + \cdots\\
& + q_m\psi_2(Q_{p_m},T(F)), \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
D_1(x;T,F) = {} & \int_{-\infty}^{Q_{p_1}} \psi_1'(y,T(F))\,\mathrm{d}F(y)\\
& + \frac{(1-q_1-p_1^*)}{1-p_1}\int_{Q_{p_1}}^{Q_{p_2}} \psi_1'(y,T(F))\,\mathrm{d}F(y)\\
& + \frac{(1-q_1-q_2-p_1^*-p_2^*)}{1-p_2}\int_{Q_{p_2}}^{Q_{p_3}} \psi_1'(y,T(F))\,\mathrm{d}F(y) + \cdots\\
& + \frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\int_{Q_{p_{m-1}}}^{Q_{p_m}} \psi_1'(y,T(F))\,\mathrm{d}F(y)\\
& + q_1\psi_2'(Q_{p_1},T(F)) + q_2\psi_2'(Q_{p_2},T(F)) + \cdots + q_m\psi_2'(Q_{p_m},T(F)). \qquad (12)
\end{aligned}
$$

If the estimator is Fisher Consistent at the hypothesized model, for $F=F_\theta$, (5) can be written as

$$\begin{aligned}
&\int_{-\infty}^{Q_{p_1}(F_\theta)} \psi_1(y,\theta)\,\mathrm{d}F_\theta(y)\\
&\quad+\frac{(1-q_1-p_1^*)}{1-p_1}\int_{Q_{p_1}(F_\theta)}^{Q_{p_2}(F_\theta)} \psi_1(y,\theta)\,\mathrm{d}F_\theta(y)\\
&\quad+\frac{(1-q_1-q_2-p_1^*-p_2^*)}{1-p_2}\int_{Q_{p_2}(F_\theta)}^{Q_{p_3}(F_\theta)} \psi_1(y,\theta)\,\mathrm{d}F_\theta(y)+\cdots\\
&\quad+\frac{(1-\sum_{i=1}^{m-1} q_i-\sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\int_{Q_{p_{m-1}}(F_\theta)}^{Q_{p_m}(F_\theta)} \psi_1(y,\theta)\,\mathrm{d}F_\theta(y)\\
&\quad+q_1\psi_2(Q_{p_1}(F_\theta),\theta)+q_2\psi_2(Q_{p_2}(F_\theta),\theta)+\cdots+q_m\psi_2(Q_{p_m}(F_\theta),\theta)=0. \qquad (13)
\end{aligned}$$

Differentiating (13) with respect to θ and using (9), an alternative expression for the denominator (12) of $IF(x;T,F)$ in (10) can be obtained. Therefore,

$$IF(x;T,F_\theta)=\frac{N(x;T,F_\theta)}{D_2(x;T,F_\theta)}$$

in which $N(x;T,F_\theta)$ is the expression (11) with $T(F)$ replaced by θ and

$$\begin{aligned}
D_2(x;T,F_\theta)=&\int_{-\infty}^{Q_{p_1}} \psi_1(y,\theta)\Lambda(y,\theta)\,\mathrm{d}F_\theta(y)\\
&+\frac{(1-q_1-p_1^*)}{1-p_1}\int_{Q_{p_1}}^{Q_{p_2}} \psi_1(y,\theta)\Lambda(y,\theta)\,\mathrm{d}F_\theta(y)\\
&+\frac{(1-q_1-q_2-p_1^*-p_2^*)}{1-p_2}\int_{Q_{p_2}}^{Q_{p_3}} \psi_1(y,\theta)\Lambda(y,\theta)\,\mathrm{d}F_\theta(y)\\
&+\cdots\\
&+\frac{(1-\sum_{i=1}^{m-1} q_i-\sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\int_{Q_{p_{m-1}}}^{Q_{p_m}} \psi_1(y,\theta)\Lambda(y,\theta)\,\mathrm{d}F_\theta(y)
\end{aligned}$$

where $\Lambda(x,\theta)=\delta/\delta\theta[\log f(x,\theta)]$.

The gross-error-sensitivity γ is defined in terms of the influence function as

$$\gamma=\sup_x\{|IF(x;T,F)|\}$$

and it is one measure of robustness.

2.2. Breakdown point

We derive the breakdown point of location estimates in the following theorem by adapting the arguments of Huber (1981) (Theorem 2.6, p. 54) to the context of progressive censoring. Proof is in Appendix A. The estimate of the location parameter corresponds to the functional $T(F)$ in (5) with $\psi_i(y,\theta) = \psi_i(y-\theta)$; $i = 1,2$.

Theorem 2.2. *Let ψ_i; $i = 1,2$ be monotone increasing but not necessarily continuous functions such that ψ_1 takes values of both signs. Then the breakdown point of $T(F,\psi_1,\psi_2)$ is*

$$\min\left(\frac{-\psi_1(-\infty)}{\psi_1(\infty)-\psi_1(-\infty)}, \frac{\psi_1(\infty)\sum_{i=1}^m p_i^* + \psi_2(\infty)(1-\sum_{i=1}^m p_i^*)}{\psi_1(\infty)-\psi_1(-\infty)}\right).$$

The next corollary provides the breakdown point formula for the special class of James-type M-estimators.

Corollary 2.2. *The breakdown point for the James-type M-estimators of location is given as*

$$\min\left(\frac{-\psi_1(-\infty)}{\psi_1(\infty)-\psi_1(-\infty)}, \frac{\psi_1(\infty)}{\psi_1(-\infty)-\psi_1(\infty)}\right).$$

Proof. For James-type M-estimators $\psi_2(x) = E[\psi_1(X)|X > x]$. Therefore, $\psi_2(\infty) = \psi_1(\infty)$. The rest follows by substituting $\psi_1(\infty)$ in place of $\psi_2(\infty)$ in the breakdown point obtained in Theorem 2.1.

Remark 2.1. It is to be noted that the breakdown point formula in Corollary 2.1 is same as the uncensored data breakdown point of the M-estimator for the location. It then follows that the breakdown point of James-type M-estimators of location in independent of the degree of censoring. In other words, censoring can not improve the breakdown point of the James-type M-estimators of location.

Remark 2.2. From Theorem 3.1 of the next section, it can be seen that for symmetric distributions with $\Lambda(\infty) = \infty$, the optimal estimators within both classes satisfy $\psi_2(\infty) = \psi_1(\infty)$ and $\psi_1(\infty) = -\psi_1(-\infty)$. It follows that for symmetric distributions, the optimal estimators within both classes have breakdown point 0.5. The breakdown points of the optimal robust estimators at the extreme value distribution for different degrees of censoring and different bounds on gross-error sensitivities b are provided in Table 1 in Section 4.

3. Robust estimators

The ψ-functions corresponding to the most robust estimator are derived in the first subsection. The optimal robust estimators within two classes of ψ-functions are characterized in subsection two. The first optimality result is the progressively censored data analogue of the optimality result described in Hampel et al. (1986). The second result pertains to a restricted class of ψ-functions which is the analogue of the class of ψ-functions considered in James (1986) for randomly censored data.

3.1. The most robust estimator

The most robust estimator is the estimator which minimizes the gross-error-sensitivity. For the uncensored case, the most robust estimator for location is the ML estimator under the double exponential distribution, which is $\hat{\theta} = \text{median}(F)$. For conventional Type II right censored data, it is shown in Akritas et al. (1993) that the most robust estimator for the location parameter corresponds to $\psi_1^0(x) = \text{sign}(x)$ and $\psi_2^0(x) = 1$. In that situation, if proportion of uncensored observation $p > 0.5$ then, (since in that case $\theta < Q_p$), the most robust estimator turns out again to be $\hat{\theta} = \text{median}(F)$. Under progressive censoring, since censoring occurs in multiple stages and at each of the failure times, the most robust estimator for location cannot be equal to median(F). In case of progressive right censoring, $\psi_1^0(x) = \text{sign}(x)$ and $\psi_2^0(x) = 0$ provides the most robust estimator for the special case when $p_1^* = \cdots = p_m^*$. For this choice, it is shown below that if $Q_{p_i} \leqslant \theta < Q_{p_{i+1}}$ for some i then $\hat{\theta} = F^{-1}(p_i)$ as expected.

For simplicity, let us assume that $m = 4$ and consider the cases

(i) $Q_{p_1} \leqslant Q_{p_2} < \theta < Q_{p_3} \leqslant Q_{p_4}$ and
(ii) $Q_{p_1} \leqslant Q_{p_2} = \theta < Q_{p_3} \leqslant Q_{p_4}$.

In both cases, we will show that $\hat{\theta} = F^{-1}(p_2)$. For general m, the proof follows in the similar lines.

Case (i): $Q_{p_1} \leqslant Q_{p_2} < \theta < Q_{p_3} \leqslant Q_{p_4}$: For the choice of $\psi_1^0(x)=\text{sign}(x)$ and $\psi_2^0(x)= 0$, the defining equation (5) for the M-estimator reduces to

$$\begin{aligned}
&\int_{-\infty}^{Q_{p_1}} -\mathrm{d}F(x) + \frac{1-q_1-p_1^*}{1-p_1} \int_{Q_{p_1}}^{Q_{p_2}} -\mathrm{d}F(x) \\
&\quad + \frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2} \int_{Q_{p_2}}^{\theta} -\mathrm{d}F(x) \\
&\quad + \frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2} \int_{\theta}^{Q_{p_3}} \mathrm{d}F(x) \\
&\quad + \frac{1-q_1-q_2-q_3-p_1^*-p_2^*-p_3^*}{1-p_3} \int_{Q_{p_3}}^{Q_{p_4}} \mathrm{d}F(x) = 0. \qquad (14)
\end{aligned}$$

Simplifying (14) and using relation (6) one obtains $\hat{\theta}=F^{-1}(p_2)$ when $p_1^*=p_2^*=p_3^*=p_4^*$.

Case (ii): $Q_{p_1} \leqslant Q_{p_2}=\theta < Q_{p_3} \leqslant Q_{p_4}$: For the choice of $\psi_1^0(x)=\text{sign}(x)$ and $\psi_2^0(x)= 0$, the defining equation (5) for the M-estimator reduces to

$$\begin{aligned}
&\int_{-\infty}^{Q_{p_1}} -\mathrm{d}F(x) + \frac{1-q_1-p_1^*}{1-p_1} \int_{Q_{p_1}}^{\theta} -\mathrm{d}F(x) + \frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2} \int_{Q_{p_2}}^{Q_{p_3}} \mathrm{d}F(x) \\
&\quad + \frac{1-q_1-q_2-q_3-p_1^*-p_2^*-p_3^*}{1-p_3} \int_{Q_{p_3}}^{Q_{p_4}} \mathrm{d}F(x) = 0. \qquad (15)
\end{aligned}$$

Again, simplifying (15) and using relation (6) one obtains $\hat{\theta} = F^{-1}(p_2)$ when $p_1^* = p_2^* = p_3^* = p_4^*$.

3.2. Optimal robust estimator

In this subsection, we present the optimal robust estimators for two classes of ψ-functions. One class of ψ-functions is the progressively censored data version of the result described in Hampel et al. (1986) for uncensored data and Akritas et al. (1993) for conventional Type II right censored data. The optimal members within this class of estimators will be referred to as the optimal robust estimators. The second class of ψ-functions belongs to a restricted class which is the analogue to the class considered in James (1986) for randomly censored data and in Akritas et al. (1993) for conventional Type II right censored data. The optimal estimators within this class of estimators will be referred to as Optimal James-type estimators. We examine the usefulness of these two classes of ψ-functions. The breakdown point and efficiency of the optimal James-type estimators compare favorably with those of the corresponding optimal robust estimators. From the computational point of view, the optimal James-type estimators are easier to obtain since the corresponding ψ-functions are readily obtainable from the optimal ψ-functions in the uncensored case.

We will consider the case of one-dimensional parameter. It is assumed that the underlying distribution F_0 is a member of the parametric family $F(x,\theta)$; $\theta \in \Theta$ where Θ is an open and convex subset of the real line. Also, assume that each $F(x,\theta)$ has a strictly positive density $f(x,\theta)$ with respect to some measure λ. Let θ_0 be defined by $F(x,\theta_0) = F_0(x)$ and we will set $Q_{p_i} = Q_{p_i}(F_0)$; $i = 1, 2, \ldots, m$, $s(x) = s(x,\theta_0)$ and $\Lambda(x) = \Lambda(x,\theta_0)$. $s(x)$ is defined by

$$
\begin{aligned}
s(x) &= [\Lambda(x)] I(x \leqslant Q_{p_1}) \\
&\quad + \left[\frac{1 - q_1 - p_1^*}{1 - p_1}\Lambda(x)\right] I(Q_{p_1} < x \leqslant Q_{p_2}) \\
&\quad + \left[\frac{1 - q_1 - q_2 - p_1^* - p_2^*}{1 - p_2}\Lambda(x)\right] I(Q_{p_2} < x \leqslant Q_{p_3}) + \cdots \\
&\quad + \left[\frac{1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}{1 - p_{m-1}}\Lambda(x)\right] I(Q_{p_{m-1}} < x \leqslant Q_{p_m}) + C_\lambda
\end{aligned} \tag{16}
$$

in which $C_\lambda = C_\lambda(q_1, \ldots, q_m, p_1, \ldots, p_m) = q_1 E[\Lambda(x)|x > Q_{p_1}] + q_2 E[\Lambda(x)|x > Q_{p_2}] + \cdots + q_m E[\Lambda(x)|x > Q_{p_m}]$. $s(x)$ satisfies $\int s(x)\,\mathrm{d}F_0(x) = 0$ (we show it in the following Lemma 3.1) and $0 < \int s^2(x)\,\mathrm{d}F_0(x) < \infty$.

Lemma 3.1. *$s(x)$ given by (16) satisfies $\int s(x)\,\mathrm{d}F_0(x) = 0$.*

Proof. The integral $\int s(x)\,\mathrm{d}F_0(x)$ is given by the following expression

$$
\int s(x)\,\mathrm{d}F_0(x) = \int_{-\infty}^{Q_{p_1}} \Lambda(x)\,\mathrm{d}F_0(x) + q_1 \frac{f(Q_{p_1})}{1 - p_1}
$$

$$+\frac{1-q_1-p_1^*}{1-p_1}\int_{Q_{p_1}}^{Q_{p_2}}\Lambda(x)\,\mathrm{d}F_0(x)+q_2\frac{f(Q_{p_2})}{1-p_2}+\cdots$$
$$+\frac{(1-\sum_{i=1}^{m-1}q_i-\sum_{i=1}^{m-1}p_i^*)}{1-p_{m-1}}\int_{Q_{p_{m-1}}}^{Q_{p_m}}\Lambda(x)\,\mathrm{d}F_0(x)+q_m\frac{f(Q_{p_m})}{1-p_m}. \tag{17}$$

We will show that $\int s(x)\,\mathrm{d}F_0(x)=0$ when $m=3$ for simplicity. For general m the proof follows in the similar line. When $m=3$, (17) reduces to

$$-f(Q_{p_1})+q_1\frac{f(Q_{p_1})}{1-p_1}+\frac{1-q_1-p_1^*}{1-p_1}[f(Q_{p_1})-f(Q_{p_2})]+q_2\,\frac{f(Q_{p_2})}{1-p_2}$$
$$+\frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2}[f(Q_{p_2})-f(Q_{p_3})]+q_3\frac{f(Q_{p_3})}{1-p_3}. \tag{18}$$

Since from (7) and (8) it follows that

$$\frac{q_3}{1-p_3}=\frac{1-q_1-q_2-p_1^*-p_2^*-p_3^*}{1-p_3}=\frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2},$$
$$\frac{1-q_1-p_1^*-p_2^*}{1-p_2}=\frac{1-q_1-p_1^*}{1-p_1},$$

it can be seen easily that (18) reduces to zero.

In the next theorem, we present the optimal robust estimators as well as optimal James-type estimators.

Theorem 3.2. *Suppose above conditions hold. Let $b>0$ be some constant. Then there exist real numbers a, $\tilde{a}$ such that*

$$\Psi_b(x)=[s(x)-a]_{-b}^{b}\quad \textit{and}$$
$$\tilde{\Psi}_b(x)=[\Lambda(x)-\tilde{a}]_{-b}^{b}I(x\leqslant Q_{p_1})$$
$$+\frac{1-q_1-p_1^*}{1-p_1}[\Lambda(x)-\tilde{a}]_{-b}^{b}I(Q_{p_1}<x\leqslant Q_{p_2})+\cdots$$
$$+\frac{(1-\sum_{i=1}^{m-1}q_i-\sum_{i=1}^{m-1}p_i^*)}{1-p_{m-1}}[\Lambda(x)-\tilde{a}]_{-b}^{b}I(Q_{p_{m-1}}<x\leqslant Q_{p_m})$$
$$+q_1E\{[\Lambda(x)-\tilde{a}]_{-b}^{b}|x>Q_{p_1}\}$$
$$+q_2E\{[\Lambda(x)-\tilde{a}]_{-b}^{b}|x>Q_{p_2}\}+\cdots$$
$$+q_mE\{[\Lambda(x)-\tilde{a}]_{-b}^{b}|x>Q_{p_m}\} \tag{19}$$

satisfy $\int \Psi_b(x)\,\mathrm{d}F_0(x)=0$, $\int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x)=0$, $d(\Psi_b)>0$ *and* $d(\tilde{\Psi}_b)>0$ *where*

$$\begin{aligned} d(\Psi_b) = & \int_{-\infty}^{Q_{p_1}} \Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x) \\ & + \int_{Q_{p_1}}^{Q_{p_2}} \frac{1-q_1-p_1^*}{1-p_1}\,\Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x)+\cdots \\ & + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \frac{(1-\sum_{i=1}^{m-1} q_i-\sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\,\Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x) \end{aligned}$$

and $\Psi_b(x)$ *in* $d(\Psi_b)$ *is replaced by* $\tilde{\Psi}_b(x)$ *in* $d(\tilde{\Psi}_b)$. *Function* Ψ_b *minimises*

$$\frac{\int \Psi^2\,\mathrm{d}F_0}{[d(\Psi)]^2} \tag{20}$$

among all

$$\begin{aligned} \Psi(x) = & [\psi_1(x)+C_\psi]I(x\leqslant Q_{p_1}) \\ & +[\psi_2(x)+C_\psi]I(Q_{p_1}<x\leqslant Q_{p_2}) \\ & +[\psi_3(x)+C_\psi]I(Q_{p_2}<x\leqslant Q_{p_3}) \\ & +\cdots \\ & +[\psi_m(x)+C_\psi]I(Q_{p_{m-1}}<x\leqslant Q_{p_m}) \end{aligned}$$

that satisfy

$$\int \Psi(x)\,\mathrm{d}F_0(x)=0, \tag{21}$$

$$d(\Psi)\neq 0 \tag{22}$$

and

$$\sup_x \left|\frac{\Psi(x)}{d(\Psi)}\right| < \frac{b}{d(\Psi_b)}. \tag{23}$$

Here $C_\psi = C_\psi(q_1,\ldots,q_m,p_1,\ldots,p_m) = q_1E[\psi_1(x)|x>Q_{p_1}] + q_2E[\psi_1(x)|x>Q_{p_2}] + \cdots + q_mE[\psi_1(x)|x>Q_{p_m}]$ and $d(\Psi)$ is obtained by replacing $\Psi(x)$ in place of $\Psi_b(x)$ in $d(\Psi_b)$ given earlier. Function $\tilde{\Psi}_b$ minimises (20) among all functions Ψ as given above which satisfy (21), (22) and (23) with $d(\Psi_b)$ replaced by $d(\tilde{\Psi}_b)$ and additionally $\int \psi_1(x)\,\mathrm{d}F_0(x)=0$. Any other solution to the above extremal problems coincides with a non-zero multiple of Ψ_b or $\tilde{\Psi}_b$ almost everywhere with respect to F_0.

Proof. In Appendix A. □

In the following lemma, we show that the optimal James-type Ψ-functions are obtained from the optimal Ψ-functions in the uncensored case by showing that

$$\int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x) = \int [\Lambda(x) - \tilde{a}]_{-b}^{b}\,\mathrm{d}F_0(x).$$

Therefore, $\tilde{a}$ in $\tilde{\Psi}_b(x)$ is determined from $\int [\Lambda(x) - \tilde{a}]_{-b}^{b}\,\mathrm{d}F_0(x) = 0$.

Lemma 3.3. $\int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x) = \int [\Lambda(x) - \tilde{a}]_{-b}^{b}\,\mathrm{d}F_0(x)$.

Proof. $\tilde{\Psi}_b(x)$ given in (19) simplifies to (with $g(x) = [\Lambda(x) - \tilde{a}]_{-b}^{b}$)

$$\begin{aligned}
\tilde{\Psi}_b(x) = {} & g(x) I(x \leqslant Q_{p_1}) + q_1 \frac{\int_{Q_{p_1}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1 - p_1} \\
& + \frac{1 - q_1 - p_1^*}{1 - p_1} g(x) I(Q_{p_1} < x \leqslant Q_{p_2}) + q_2 \frac{\int_{Q_{p_2}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1 - p_2} + \cdots \\
& + \frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}} g(x) I(Q_{p_{m-1}} < x \leqslant Q_{p_m}) \\
& + q_m \frac{\int_{Q_{p_m}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1 - p_m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x) \\
& \quad = \int_{-\infty}^{Q_{p_1}} g(x)\,\mathrm{d}F_0(x) + \frac{q_1}{1 - p_1} \int_{Q_{p_1}}^{\infty} g(x)\,\mathrm{d}F_0(x) \\
& \quad + \frac{1 - q_1 - p_1^*}{1 - p_1} \int_{Q_{p_1}}^{Q_{p_2}} g(x)\,\mathrm{d}F_0(x) + \frac{q_2}{1 - p_2} \int_{Q_{p_2}}^{\infty} g(x)\,\mathrm{d}F_0(x) \\
& \quad + \frac{1 - q_1 - q_2 - p_1^* - p_2^*}{1 - p_2} \int_{Q_{p_2}}^{Q_{p_3}} g(x)\,\mathrm{d}F_0(x) + \frac{q_3}{1 - p_3} \int_{Q_{p_3}}^{\infty} g(x)\,\mathrm{d}F_0(x) \\
& \quad + \cdots \\
& \quad + \frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}} \int_{Q_{p_{m-1}}}^{Q_{p_m}} g(x)\,\mathrm{d}F_0(x) \\
& \quad + \frac{q_m}{1 - p_m} \int_{Q_{p_m}}^{\infty} g(x)\,\mathrm{d}F_0(x).
\end{aligned}$$

Rearranging the terms in the right hand side of the above equation,

$$\begin{aligned}
&\int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x)\\
&\quad = \int_{-\infty}^{Q_{p_1}} g(x)\,\mathrm{d}F_0(x) + \int_{Q_{p_1}}^{Q_{p_2}} \left(\frac{q_1}{1-p_1} + \frac{1-q_1-p_1^*}{1-p_1}\right) g(x)\,\mathrm{d}F_0(x)\\
&\qquad + \int_{Q_{p_2}}^{Q_{p_3}} \left(\frac{q_1}{1-p_1} + \frac{q_2}{1-p_2} + \frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2}\right) g(x)\,\mathrm{d}F_0(x) + \cdots\\
&\qquad + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \left(\frac{q_1}{1-p_1} + \cdots + \frac{q_{m-1}}{1-p_{m-1}}\right.\\
&\qquad \left. + \frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\right) g(x)\,\mathrm{d}F_0(x)\\
&\qquad + \int_{Q_{p_m}}^{\infty} \left(\frac{q_1}{1-p_1} + \frac{q_2}{1-p_2} + \cdots + \frac{q_m}{1-p_m}\right) g(x)\,\mathrm{d}F_0(x).
\end{aligned} \tag{24}$$

In the sequel, we now show

$$\frac{q_1}{1-p_1} + \frac{q_2}{1-p_2} + \cdots + \frac{q_m}{1-p_m} = 1.$$

(8) implies

$$\frac{q_m}{1-p_m} = \frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}. \tag{25}$$

Adding $q_{m-1}/(1-p_{m-1})$ to both sides of (25) one gets

$$\frac{q_m}{1-p_m} + \frac{q_{m-1}}{1-p_{m-1}} = \frac{(1-\sum_{i=1}^{m-2} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}.$$

Continuing in the above fashion and using (7) in every step, one gets

$$\frac{q_m}{1-p_m} + \frac{q_{m-1}}{1-p_{m-1}} + \cdots + \frac{q_2}{1-p_2} = \frac{1-q_1-p_1^*-p_2^*}{1-p_2}. \tag{26}$$

The right hand side of (26) equals

$$\frac{1-q_1-p_1^*-p_2^*}{1-p_2} = \frac{1-q_1-p_1}{1-p_1} \tag{27}$$

since $p_1 = p_1^*$. Using (26) and (27), it can be seen that

$$\frac{q_m}{1-p_m} + \frac{q_{m-1}}{1-p_{m-1}} + \cdots + \frac{q_1}{1-p_1} = \frac{1-q_1-p_1}{1-p_1} + \frac{q_1}{1-p_1} = 1. \tag{28}$$

Using (8) in (28), one also gets

$$\frac{q_1}{1-p_1}+\frac{q_2}{1-p_2}+\cdots+\frac{q_{m-1}}{1-p_{m-1}}+\frac{(1-\sum_{i=1}^{m-1}q_i-\sum_{i=1}^{m-1}p_i^*)}{1-p_{m-1}}=1. \qquad (29)$$

Substituting $m=2,3$ in (29) one gets respectively,

$$\frac{q_1}{1-p_1}+\frac{1-q_1-p_1^*}{1-p_1}=1, \qquad (30)$$

$$\frac{q_1}{1-p_1}+\frac{q_2}{1-p_2}+\frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2}=1. \qquad (31)$$

Using (28)–(30) and (31), (24) reduces to

$$\begin{aligned}\int \tilde{\Psi}_b(x)\,\mathrm{d}F_0(x) &= \int_{\infty}^{Q_{p_1}} g(x)\,\mathrm{d}F_0(x)+\int_{Q_{p_1}}^{Q_{p_2}} g(x)\,\mathrm{d}F_0(x)+\cdots\\ &\quad+\int_{Q_{p_m}}^{\infty} g(x)\,\mathrm{d}F_0(x)\\ &=\int [\Lambda(x)-\tilde{a}]_{-b}^{b}\,\mathrm{d}F_0(x).\end{aligned}$$

Hence Lemma 3.3 is proved. □

4. Numerical illustrations

The location estimation from an extreme value distribution with density

$$f(y;\mu,\sigma)=(1/\sigma)\exp[(y-\mu)/\sigma]\cdot\exp[-\exp(y-\mu)/\sigma]$$

at $\sigma=1$ is considered. Joint estimation of location and scale is not considered in this article for the sake of simplicity. However, the advantage of robust estimation is in general more pronounced when joint estimation of location and scale is considered. For robust estimation, three bounds of the gross error sensitivity and eight different degrees of progressive censoring are considered.

In that setting, comparisons of the breakdown points and asymptotic variances for optimal robust M-estimators and James-type estimators are provided in Tables 1 and 2, respectively. In both tables, b is the bound on the ψ function, $P=\sum_{i=1}^{3} p_i^*$ is the total proportion of the observed or uncensored data in $m=3$ stages and p_3 denotes the highest adjusted proportion of the uncensored data. Three bounds $b=1$, $b=2$ and $b=3$ are considered in three columns. Four different values $P=0.6$, $P=0.7$, $P=0.8$ and $P=0.9$ of total proportion of the uncensored data are considered. Under each of these above four P values, two different sets of p_1^*, p_2^* and p_3^* of uncensored proportions and q_1, q_2 and q_3 of censored proportions are considered. For each of these eight different levels of progressive censoring, p_3 is computed and all these are reported in the first eight rows of both the tables. The last row of both the tables correspond to the uncensored case.

Table 1
Breakdown points of the optimal robust estimators for the location parameter of an extreme value distribution for progressively censored data

	$b=1$	$b=2$	$b=3$
$P=0.6$; $p_3=0.6889$; ($p_1^*=0.3$, $p_2^*=0.2$, $p_3^*=0.1$ $q_1=0.1, q_2=0.1, q_3=0.2$)	$\frac{1}{2}$ ($a \simeq 0$)	$\frac{1}{3}$ ($a \simeq 0$)	$\frac{1}{4}$ ($a \simeq 0$)
$P=0.6$; $p_3=0.8393$; ($p_1^*=0.1$, $p_2^*=0.2$, $p_3^*=0.3$ $q_1=0.2, q_2=0.1, q_3=0.1$)	$\frac{1}{2}$ ($a \simeq 0$)	$\frac{1}{3}$ ($a \simeq 0$)	$\frac{1}{4}$ ($a \simeq 0$)
$P=0.7$; $p_3=0.7487$; ($p_1^*=0.3$, $p_2^*=0.3$, $p_3^*=0.1$ $q_1=0.05, q_2=0.05, q_3=0.2$)	$\frac{1}{2}$ ($a \simeq 0$)	$\frac{1}{3}$ ($a \simeq 0$)	$\frac{1}{4}$ ($a \simeq 0$)
$P=0.7$; $p_3=0.9267$; ($p_1^*=0.1$, $p_2^*=0.2$, $p_3^*=0.4$ $q_1=0.15, q_2=0.1, q_3=0.05$)	0.4998 ($a=-0.000612$)	$\frac{1}{3}$ ($a \simeq 0$)	$\frac{1}{4}$ ($a \simeq 0$)
$P=0.8$; $p_3=0.8636$; ($p_1^*=0.4$, $p_2^*=0.3$, $p_3^*=0.1$ $q_1=0.05, q_2=0.05, q_3=0.1$)	$\simeq \frac{1}{2}$ ($a \simeq 0$)	$\simeq \frac{1}{3}$ ($a \simeq 0$)	$\simeq \frac{1}{4}$ ($a \simeq 0$)
$P=0.8$; $p_3=0.9386$; ($p_1^*=0.1$, $p_2^*=0.2$, $p_3^*=0.5$ $q_1=0.1, q_2=0.05, q_3=0.05$)	0.4973 ($a=-0.010841$)	$\simeq \frac{1}{3}$ ($a \simeq 0$)	$\simeq \frac{1}{4}$ ($a \simeq 0$)
$P=0.9$; $p_3=0.9426$; ($p_1^*=0.4$, $p_2^*=0.3$, $p_3^*=0.2$ $q_1=0.025, q_2=0.025, q_3=0.05$)	0.4954 ($a=-0.018343$)	$\simeq \frac{1}{3}$ ($a \simeq 0$)	$\simeq \frac{1}{4}$ ($a \simeq 0$)
$P=0.9$; $p_3=0.9718$; ($p_1^*=0.2$, $p_2^*=0.3$, $p_3^*=0.4$ $q_1=0.025, q_2=0.025, q_3=0.05$)	0.4871 ($a=-0.050322$)	0.33074 ($a=-0.011979$)	$\simeq \frac{1}{4}$ ($a \simeq 0$)
$P=1.0$	0.4569 ($a=-0.1586$)	0.321 ($a=-0.05247$)	0.246 ($a=-0.01866$)

In Table 1, a denotes the centering constant. As mentioned in Remark 2.1, the breakdown points for the optimal James-type estimators is the same as the uncensored data breakdown points of the M-estimators for location. Therefore, the breakdown points for the optimal James-type estimators will be found in the last row of Table 1. It is clear from this table that even with significant amount of censoring, the breakdown points of James-type optimal robust M-estimators of location for an extreme value distribution do not differ much from that of optimal robust M-estimators. The asymptotic variances of James-type optimal robust M-estimators and the optimal robust M-estimators of the location parameter of the extreme value distribution are provided in Table 2 in the same

Table 2
Asymptotic variances of the two types of optimal robust estimators of the location parameter of an extreme value distribution for progressively censored data

		$b=1$	$b=2$	$b=3$
$P=0.6$; $p_3=0.6889$	James-type	2.4585	2.1844	2.1221
$(p_1^*=0.3, p_2^*=0.2, p_3^*=0.1$	Optimal robust	2.0914	2.0914	2.0914
$q_1=0.1, q_2=0.1, q_3=0.2)$;	Efficiency	0.8507	0.9574	0.9855
$P=0.6$; $p_3=0.8393$	James-type	4.3015	3.6676	3.5116
$(p_1^*=0.1, p_2^*=0.2, p_3^*=0.3$	Optimal robust	3.4328	3.4328	3.4328
$q_1=0.2, q_2=0.1, q_3=0.1)$	Efficiency	0.7980	0.9360	0.9776
$P=0.7$; $p_3=0.7487$	James-type	2.7119	2.2592	2.1571
$(p_1^*=0.3, p_2^*=0.3, p_3^*=0.1$	Optimal robust	2.1071	2.1071	2.1071
$q_1=0.05, q_2=0.05, q_3=0.2)$	Efficiency	0.7770	0.9327	0.9768
$P=0.7$; $p_3=0.9267$	James-type	3.5793	3.3016	3.1987
$(p_1^*=0.1, p_2^*=0.2, p_3^*=0.4$	Optimal robust	3.1431	3.1450	3.1450
$q_1=0.15, q_2=0.1, q_3=0.05)$	Efficiency	0.8781	0.9526	0.9832
$P=0.8$; $p_3=0.8636$	James-type	3.1385	2.6297	2.5005
$(p_1^*=0.4, p_2^*=0.3, p_3^*=0.1$	Optimal robust	2.4353	2.4353	2.4353
$q_1=0.05, q_2=0.05, q_3=0.1)$	Efficiency	0.7760	0.9261	0.9739
$P=0.8$; $p_3=0.9386$	James-type	2.9212	2.7122	2.6417
$(p_1^*=0.1, p_2^*=0.2, p_3^*=0.5$	Optimal robust	2.6036	2.6044	2.6044
$q_1=0.1, q_2=0.05, q_3=0.05)$	Efficiency	0.8913	0.9603	0.9859
$P=0.9$; $p_3=0.9426$	James-type	2.4518	2.2672	2.2062
$(p_1^*=0.4, p_2^*=0.3, p_3^*=0.2$	Optimal robust	2.1768	2.1740	2.1740
$q_1=0.025, q_2=0.025, q_3=0.05)$	Efficiency	0.8878	0.9589	0.9854
$P=0.9$; $p_3=0.9718$	James-type	2.0709	1.8957	1.8699
$(p_1^*=0.2, p_2^*=0.3, p_3^*=0.4$	Optimal robust	1.9362	1.8560	1.8558
$q_1=0.025, q_2=0.025, q_3=0.05)$	Efficiency	0.9349	0.9791	0.9924

setting as that of Table 1. The efficiency of James-type estimators are also computed in each situation. As expected, the above efficiency is always less than 1 but improves significantly as b increases and when proportion of censoring is generally small.

It was demonstrated in the previous section that the optimal ψ-function within the James-class is equivalent to the optimal ψ-function in the uncensored case and therefore it is simpler to compute. As it is seen from Tables 1 and 2 that this simplicity can be enjoyed when value of gross error sensitivity b is moderate and proportion of censoring is not high.

Example 1. In this example, we illustrate robust estimation of location using the data in Viveros and Balakrishnan (1994). The data come from a progressively censored sample generated from the log-times to breakdown data on insulating fluid tested at

Table 3
Progressively censored data given in Viveros and Balakrishnan (1994)

i	1	2	3	4	5	6	7	8
Y_i	−1.6608	−0.2485	−0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
r_i	0	0	3	0	3	0	0	5

Table 4
The changes of ORE and MLE of the location parameter in the presence of outliers for the original data given in Viveros and Balakrishnan (1994)

		ORE	MLE
(i)	Original data	2.307	2.206
(ii)	$1.9947 \to 3.9894$	1.936	3.7862
	$1.9947 \to 7.9788$	1.936	7.6927
	$1.9947 \to 11.9682$	1.936	11.6805
	$1.9947 \to 19.947$	1.936	19.6593
(iii)	$1.9947 \to 3.9894$	1.890	3.8828
	$1.8718 \to 3.7436$		
	$1.9947 \to 7.9788$	1.890	7.7893
	$1.8718 \to 7.4872$		
	$1.9947 \to 11.9682$	1.890	11.2421
	$1.8718 \to 11.2308$		
	$1.9947 \to 19.947$	1.890	19.7069
	$1.8718 \to 18.718$		
(iv)	$1.9947 \to 3.9894$	2.3162	3.9297
	$1.8718 \to 3.7436$		
	$1.5789 \to 3.1578$		
	$1.9947 \to 7.9788$	2.3162	7.8173
	$1.8718 \to 7.4872$		
	$1.5789 \to 6.3156$		
	$1.9947 \to 11.9682$	2.3162	11.7699
	$1.8718 \to 11.2308$		
	$1.5789 \to 9.4734$		
	$1.9947 \to 19.947$	2.3162	19.7094
	$1.8718 \to 18.718$		
	$1.5789 \to 15.789$		

34 KV and are presented in Table 3. It is assumed that these data come from an extreme value distribution with the scale parameter $\sigma = 1$. It is to be noted that the value $\sigma = 1$ does not differ much from its ML estimate (MLE) of 1.026 computed in Ng et al. (2002) and Viveros and Balakrishnan (1994) by two different procedures. The optimal robust estimate (ORE) of location is computed to be 2.307 while its MLE

Table 5
OREs and MLEs of location parameter for the progressively censored data using $m = 30$ stage when the uncensored data is given in David (1952)

	ORE	MLE
$P = 0.6$ $(s_1 = 20, s_2 = 18, s_3 = 15, s_4 = 15, s_5 = \cdots = s_{30} = 7$ $r_1 = \cdots = r_{29} = 5, r_{30} = 22)$	49.356	49.246
$P = 0.6$ $(s_1 = \cdots = s_{26} = 7, s_{27} = 15, s_{28} = 15, s_{29} = 18, s_{30} = 20$ $r_1 = 22, r_2 = \cdots = r_{30} = 5)$	56.272	55.993
$P = 0.7$ $(s_1 = 30, s_2 = 28, s_3 = 26, s_4 = 25, s_5 = \cdots = s_{30} = 7$ $r_1 = \cdots = r_{29} = 4, r_{30} = 10)$	51.616	51.618
$P = 0.7$ $(s_1 = \cdots = s_{26} = 7, s_{27} = 25, s_{28} = 26, s_{29} = 28, s_{30} = 30$ $r_1 = 10, r_2 = \cdots = r_{30} = 4)$	58.761	58.334
$P = 0.8$ $(s_1 = 42, s_2 = 38, s_3 = 36, s_4 = 35, s_5 = \cdots = s_{30} = 7$ $r_1 = \cdots = r_{29} = 2, r_{30} = 26)$	51.418	51.418
$P = 0.8$ $(s_1 = \cdots = s_{26} = 7, s_{27} = 35, s_{28} = 36, s_{29} = 38, s_{30} = 42$ $r_1 = 26, r_2 = \cdots = r_{30} = 2)$	61.447	60.775
$P = 0.9$ $(s_1 = 54, s_2 = 48, s_3 = 46, s_4 = 45, s_5 = \cdots = s_{30} = 7$ $r_1 = \cdots = r_{29} = 1, r_{30} = 13)$	55.318	55.360
$P = 0.9$ $(s_1 = \cdots = s_{26} = 7, s_{27} = 45, s_{28} = 46, s_{29} = 48, s_{30} = 54$ $r_1 = 13, r_2 = \cdots = r_{30} = 1)$	63.569	62.949
$P = 1.0$	64.578	64.491

is 2.206. For the optimal robust estimator the gross error sensitivity bound b was taken as $b = 1$ and then the centering constant a in this case was computed to be $a \simeq 0$. To demonstrate the benefits of using the robust estimates, Table 4 shows the effect of replacing one or more of the highest observations by even higher values. Multiple outliers (from 1 to 3) are created from the progressively censored data in Table 3 with varying degree of slippage by multiplying the original value by $k = 2, 4, 6, 10$. Blocks (ii), (iii), (iv) in Table 4 correspond to the situations where there are 1, 2 and 3 outliers respectively. Within each of the (ii), (iii) and (iv) numbered blocks, four different rows correspond to the value $k = 2, 4, 6, 10$ respectively. When the highest data

Table 6
OREs and MLEs of location parameter for the progressively censored data using $m = 40$ stages when the uncensored data is given in David (1952)

	ORE	MLE
$P = 0.6$ $(s_1 = 10, s_2 = 9, s_3 = 8, s_4 = 7, s_5 = \cdots = s_{40} = 6,$ $r_1 = \cdots = r_{39} = 4, r_{40} = 11)$	52.237	52.050
$P = 0.6$ $(s_1 = \cdots = s_{36} = 6, s_{37} = 7, s_{38} = 8, s_{39} = 9, s_{40} = 10,$ $r_1 = 11, r_2 = \cdots = r_{40} = 4)$	56.400	56.166
$P = 0.7$ $(s_1 = 20, s_2 = 19, s_3 = 18, s_4 = 18, s_5 = \cdots = s_{40} = 6,$ $r_1 = \cdots = r_{39} = 3, r_{40} = 9)$	52.871	52.826
$P = 0.7$ $(s_1 = \cdots = s_{36} = 6, s_{37} = 18, s_{38} = 18, s_{39} = 19, s_{40} = 20,$ $r_1 = 9, r_2 = \cdots = r_{40} = 3)$	60.683	60.333
$P = 0.8$ $(s_1 = 33, s_2 = 29, s_3 = 28, s_4 = 27, s_5 = \cdots = s_{40} = 6,$ $r_1 = \cdots = r_{39} = 2, r_{40} = 6)$	58.114	58.064
$P = 0.8$ $(s_1 = \cdots = s_{36} = 6, s_{37} = 27, s_{38} = 28, s_{39} = 29, s_{40} = 33,$ $r_1 = 6, r_2 = \cdots = r_{40} = 2)$	61.629	60.778
$P = 0.9$ $(s_1 = 45, s_2 = 39, s_3 = 38, s_4 = 37, s_5 = \cdots = s_{40} = 6,$ $r_1 = \cdots = r_{39} = 1, r_{40} = 3)$	60.699	60.474
$P = 0.9$ $(s_1 = \cdots = s_{36} = 6, s_{37} = 37, s_{38} = 38, s_{39} = 39, s_{40} = 45,$ $r_1 = 13, r_2 = \cdots = r_{30} = 1)$	63.734	62.948
$P = 1.0$	64.578	64.491

point 1.9947 is replaced by any outlier considered, the optimal robust M-estimate is changed from 2.307 to 1.936 whereas the corresponding MLEs are recorded in the last column of the Table 4. Similarly, in the case of 2 and 3 outliers, the optimal robust M-estimates are 1.890 and 2.3162 respectively, while corresponding MLEs are again recorded in the last column of the Table 4.

The breakdown point of the optimal robust estimator of location in this case was computed to be 0.5 with $b = 1$. Therefore, the optimal robust estimator can handle upto 50% contaminated data. With $n = 19$ observations, therefore, it should be able to handle upto 9 outliers. Also, note that there are 8 observations which are greater than or equal to $Y_6 = 1.5789$ whereas 12 observations are greater than or equal to $Y_5 = 1.0224$. It can

Table 7
The changes of ORE and MLE of the location parameter in the presence of outliers for progressively censored data $Y_1,\ldots,Y_{375}$ corresponding to $m=40, P=0.9, s_1=45, s_2=39, s_3=38, s_4=37, s_5=\cdots=s_{40}=6, r_1=\cdots=r_{39}=1, r_{40}=3$ when the original data is given in David (1952)

		ORE	MLE
(i)	$Y_1,\ldots,Y_{375}$	60.699	60.474
(ii)	$Y_{375}\to 2Y_{375}$	59.373	115.834
	$Y_{375}\to 4Y_{375}$	59.373	203.751
(iii)	$(Y_{333},\ldots Y_{375})\to(2Y_{333},\ldots 2Y_{375})$	58.734	121.539
	$(Y_{333},\ldots Y_{375})\to(4Y_{333},\ldots 4Y_{375})$	58.734	215.762
(iv)	$(Y_{291},\ldots Y_{375})\to(2Y_{291},\ldots 2Y_{375})$	61.257	127.892
	$(Y_{291},\ldots Y_{375})\to(4Y_{291},\ldots 4Y_{375})$	61.257	223.523

be seen from Table 4 that the optimal robust estimator is stable when there are upto eight right-side outliers (changing observations 1.9947, 1.8718 and 1.5789 to higher numbers produce eight right-side outliers). However, this is no longer the case when $Y_5=1.0224$ is changed to a be a high outlier and it does not contradict the breakdown point calculation.

Example 2. In this example, we use the data given in Davis (1952). Lifetimes (in hours) of 417 forty-watt 110-V internally frosted incandescent lamps are recorded in the paper. For computational simplicity, we converted the unit of lifetimes (in hours) to lifetimes (in days). We create progressively censored samples from this uncensored data using $m=30$ and 40 stages. For each of these two choices, different proportions of uncensored data $P=0.6, 0.7, 0.8, 0.9$ are considered. Also, for a fixed number of stages in the scheme and for a fixed proportion P of uncensored data, two levels of censoring (one reflects comparatively delayed censoring than the other) are considered. When $m=30$ stages are used, Table 5 records the computed values of MLEs and OREs of location for the above mentioned eight different censoring schemes while the same is recorded in Table 6 for $m=40$. For the optimal robust estimator, the gross error sensitivity bound b was taken as $b=3$.

The breakdown point of the optimal robust estimator of location is either exactly or approximately equal to 0.25 when $b=3$. Therefore, the optimal robust estimator can handle upto 25% contamination in the data. Each of the sixteen situations considered above supports this breakdown point calculation as the optimal robust estimator can handle upto approximately 104 (which is 25% of 417) outliers. We illustrate this phenomenon in Table 7 using the case of $m=40$, $P=0.9$ and when the delayed censoring scheme ($s_1=45, s_2=39, s_3=38, s_4=37, s_5=\cdots=s_{40}=6, r_1=\cdots=r_{39}=1, r_{40}=3$) was used. Table 7 shows the effect of increasing one or more of the highest observations. Multiple outliers are created from the progressively censored data $Y_1,\ldots,Y_{375}$ with various degree of slippage by multiplying the original values by $k=2,4$. It was seen from

Table 7 that the optimal robust estimator is stable when there are upto approximately 104 large outliers (changing observations $Y_{291}, Y_{292}, \ldots, Y_{375}$ to higher numbers produce 104 large outliers). As seen in Example 1, corresponding MLEs are adversely affected, which are recorded in the last column of Table 7.

5. Conclusions

In this paper, by considering the class of ψ-functions due to Hampel et al. (1986), we have derived the optimal robust estimators under progressively Type-II censored samples. Next, by considering the class of ψ-functions due to James (1986), we have derived the optimal James-type estimators under progressively Type-II censored samples. We have compared the efficiency of the optimal James-type estimators with that of the corresponding optimal robust estimators, and have shown that they compare favorably in addition to possessing computational ease. Finally, we have presented two illustrative examples to display the robustness features of these estimators for the case of extreme value distribution when the underlying progressively Type-II censored samples do contain a few outliers.

Appendix A. List of Notations and proofs of results

List of Notations

- $X_1, \ldots, X_n$: n independent units placed on a life-test experiment
- m: stages where units are randomly withdrawn
- $Y_1 \leqslant \cdots \leqslant Y_m$: m progressively Type-II right censored order statistics
- $F(x, \theta)$: continuous cumulative distribution function of X
- $f(x, \theta)$: probability density function of X
- $\rho_1(x, \theta)$: $\log f(x, \theta)$
- $\rho_2(x, \theta)$: $\log(1 - F(x, \theta))$
- $\psi_1(x, \theta)$: replacement of $\rho_1'(x, \theta)$ to get robust estimate of θ
- $\psi_2(x, \theta)$: replacement of $\rho_2'(x, \theta)$ to get robust estimate of θ
- $T(F)$: a functional which is a M-estimator of θ
- s_i: additional failures before the ith stage
- r_i: units randomly withdrawn at the ith stage
- p_i^*: $\lim s_i/n$, proportion of additional uncensored units before the ith stage
- P: $\sum_{i=1}^{m} p_i^*$, total proportion of uncensored units
- q_i: $\lim r_i/n$, proportion of censored units at the ith stage
- p_i: proportion of adjusted uncensored units at the ith stage
- Q_{p_i}: p_ith quantile of F
- $IF(x; T, F)$: $\delta T(F^s)/\delta s|_{s=0}$, where $F^s = F + s(\Delta_x - F)$, and Δ_x is the point mass function at x
- $\Lambda(x, \theta)$: $\frac{\delta}{\delta\theta} \log f(x, \theta)$
- C_λ: $q_1 E[\Lambda(x)|x > Q_{p_1}] + q_2 E[\Lambda(x)|x > Q_{p_2}] + \cdots + q_m E[\Lambda(x)|x > Q_{p_m}]$

- $s(x)$: $[\Lambda(x)]I(x \leqslant Q_{p_1}) + [\frac{1-q_1-p_1^*}{1-p_1}\Lambda(x)]I(Q_{p_1} < x \leqslant Q_{p_2}) + [\frac{1-q_1-q_2-p_1^*-p_2^*}{1-p_2}\Lambda(x)]$ $I(Q_{p_2} < x \leqslant Q_{p_3}) + \cdots + [\frac{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}{1-p_{m-1}}\Lambda(x)]I(Q_{p_{m-1}} < x \leqslant Q_{p_m}) + C_\lambda$
- $[h(x)]_{-b}^{b}$: $\begin{cases} -b & \text{if } h(x) < -b \\ h(x) & \text{if } -b < h(x) < b \\ b & \text{if } h(x) > b \end{cases}$
- $\Psi_b(x)$: ψ-function for optimal robust estimator $= [s(x) - a]_{-b}^{b}$
- $g(x)$: $[\Lambda(x) - \tilde{a}]_{-b}^{b}$
- $\tilde{\Psi}_b(x)$: ψ-function for James-type optimal robust estimator $= g(x)I(x \leqslant Q_{p_1})$ $+\ q_1\frac{\int_{Q_{p_1}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1-p_1} + \frac{1-q_1-p_1^*}{1-p_1}g(x)I(Q_{p_1} < x \leqslant Q_{p_2}) + q_2\frac{\int_{Q_{p_2}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1-p_2} + \cdots +$ $\frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}g(x)I(Q_{p_{m-1}} < x \leqslant Q_{p_m}) + q_m\frac{\int_{Q_{p_m}}^{\infty} g(x)\,\mathrm{d}F_0(x)}{1-p_m}$.

Proof of Theorem 2.2. Since the parameter is a location parameter, we will assume that $\theta = 0$ and so $T(F_0) = 0$. We will set

$$\begin{aligned}\lambda(t;F) = \int_{-\infty}^{\infty} & [\psi_1(y-t)I(y < Q_{p_1}) + q_1\psi_2(Q_{p_1})I(y = Q_{p_1}) \\ & + \frac{(1-q_1-p_1^*)}{1-p_1}\psi_1(y-t)I(Q_{p_1} < y < Q_{p_2}) + q_2\psi_2(Q_{p_2})I(y = Q_{p_2}) \\ & + \cdots + \frac{(1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1-p_{m-1}}\psi_1(y-t)I(Q_{p_{m-1}} < y < Q_{p_m}) \\ & + q_m\psi_2(Q_{p_m})I(y \geqslant Q_{p_m})]\,\mathrm{d}F(y).\end{aligned}$$

$\lambda(t;F)$ increases as F gets stochastically larger but is decreasing in t. The stochasticaly largest member of the ε-Levy neighbourhood of F_0 is $F_1(x) = \max\{F_0(x-\varepsilon) - \varepsilon, 0\}$. It is an improper distribution with mass ε at $+\infty$. The stochasticaly smallest member is $F_2(x) = \min\{F_0(x+\varepsilon) + \varepsilon, 1\}$; it is an improper distribution with mass ε at $-\infty$. We will consider the following $m+1$ cases where the proportion of contamination, ε, satisfies (1) $\varepsilon > 1 - p_1$, (2) $1 - p_2 < \varepsilon < 1 - p_1$, (3) $1 - p_3 < \varepsilon < 1 - p_2, \cdots (m)$ $1 - p_m < \varepsilon < 1 - p_{m-1}, (m+1) \varepsilon < 1 - p_m$.

Let us consider the first case (1) $\varepsilon > 1 - p_1$. Here $p_1 > 1 - \varepsilon$ and this implies $p_1 = F_1(Q_{p_1}) = F_0(Q_{p_1} - \varepsilon) - \varepsilon > 1 - \varepsilon = F_0(\infty - \varepsilon) - \varepsilon$. The inequality in the last sentence, in turn, implies that $Q_{p_1}(F_1) = \infty$. Therefore, $\lambda(t;F_1)$ in this case reduces to

$$\lambda(t;F_1) = \int_{F_0^{-1}(\varepsilon)}^{\infty} \psi_1(x + \varepsilon - t)\,\mathrm{d}F_0 + \varepsilon\psi_1(\infty).$$

So, $\lim_{t\to\infty}\lambda(t;F_1) = \psi_1(-\infty)(1-\varepsilon) + \varepsilon\psi_1(\infty) < 0$ whenever

$$\varepsilon < \frac{-\psi_1(-\infty)}{\psi_1(\infty) - \psi_1(-\infty)}. \tag{A.1}$$

On the other hand, $\lim_{t\to-\infty}\lambda(t;F_2)=\varepsilon\psi_1(-\infty)+(\sum_{i=1}^m p_i^*-\varepsilon)\psi_1(\infty)+(1-\sum_{i=1}^m p_i^*)$ $\psi_2(\infty)$ using (6) and the equation $\sum_{i=1}^m q_i=1-\sum_{i=1}^m p_i^*$. Now $\lim_{t\to-\infty}\lambda(t;F_2)>0$ whenever

$$\varepsilon<\frac{\psi_1(\infty)\sum_{i=1}^m p_i^*+\psi_2(\infty)(1-\sum_{i=1}^m p_i^*)}{\psi_1(\infty)-\psi_1(-\infty)}. \tag{A.2}$$

Next we consider the second case (2) $1-p_2<\varepsilon<1-p_1$. Here $p_2>1-\varepsilon$ and this implies $p_2=F_1(Q_{p_2})=F_0(Q_{p_2}-\varepsilon)-\varepsilon>1-\varepsilon=F_0(\infty-\varepsilon)-\varepsilon$. The inequality in the last sentence, in turn, implies that $Q_{p_2}(F_1)=\infty$. In this case, $Q_{p_1}<F_0^{-1}(\varepsilon)<Q_{p_2}$. Therefore, $\lambda(t;F_1)$ in this case reduces to

$$\lambda(t;F_1)=\int_{F_0^{-1}(\varepsilon)}^{\infty}\frac{1-q_1-p_1^*}{1-p_1}\psi_1(x+\varepsilon-t)\,\mathrm{d}F_0+\frac{1-q_1-p_1^*}{1-p_1}\varepsilon\psi_1(\infty).$$

So, $\lim_{t\to\infty}\lambda(t;F_1)=[\psi_1(-\infty)(1-\varepsilon)+\varepsilon\psi_1(\infty)](1-q_1-p_1^*)/(1-p_1)<0$ again gives us (A.1). Consideration of $\lim_{t\to-\infty}\lambda(t;F_2)>0$ again gives us (A.2). Using similar arguments as in case (2), it can be shown that ε-contamination in the right and ε-contamination in the left produce (A.1) and (A.2) respectively in the cases (3) through (m). For the case $(m+1)$ when $\varepsilon<1-p_m$, the functional is not affected by ε-contamination on the right. (A.2) is again obtained considering the contamination to the left. Theorem is then proved using inequalities in (A.1) and (A.2).

Proof of Theorem 3.2. The proof for the unrestricted case is given in detail. The proof for the James-class of estimators follows in similar lines. It is easily seen that there exists an a such that $\int\Psi_b(x)\,\mathrm{d}F_0(x)=0$. In order to show $d(\Psi_b)>0$, let us first assume that $|a|\leqslant b$. Let $P=\{x\colon s(x)>0\}$, $N=\{x\colon s(x)<0\}$, $P_1=P\cap\{x<Q_{p_1}\}$, $P_2=P\cap\{Q_{p_1}<x\leqslant Q_{p_2}\},\ldots,P_m=P\cap\{Q_{p_{m-1}}<x\leqslant Q_{p_m}\}$ and $N_1=N\cap\{x<Q_{p_1}\}$, $N_2=N\cap\{Q_{p_1}<x\leqslant Q_{p_2}\},\ldots,N_m=N\cap\{Q_{p_{m-1}}<x\leqslant Q_{p_m}\}$. Write

$$\begin{aligned}d(\Psi_b)&=\int_{-\infty}^{Q_{p_1}}\Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x)+\int_{Q_{p_1}}^{Q_{p_2}}\frac{1-q_1-p_1^*}{1-p_1}\Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x)\\&\quad+\cdots+\int_{Q_{p_{m-1}}}^{Q_{p_m}}\frac{(1-\sum_{i=1}^{m-1}q_i-\sum_{i=1}^{m-1}p_i^*)}{1-p_{m-1}}\Psi_b(x)\Lambda(x)\,\mathrm{d}F_0(x)\\&=\int_{-\infty}^{Q_{p_1}}[\Psi_b(x)+a]\Lambda(x)\,\mathrm{d}F_0(x)\\&\quad+\int_{Q_{p_1}}^{Q_{p_2}}\frac{1-q_1-p_1^*}{1-p_1}[\Psi_b(x)+a]\Lambda(x)\,\mathrm{d}F_0(x)\\&\quad+\cdots+\int_{Q_{p_{m-1}}}^{Q_{p_m}}\frac{(1-\sum_{i=1}^{m-1}q_i-\sum_{i=1}^{m-1}p_i^*)}{1-p_{m-1}}[\Psi_b(x)+a]\Lambda(x)\,\mathrm{d}F_0(x)\end{aligned}$$

$$= \int_{P_1} \min\{s(x), a+b\} s(x)\, \mathrm{d}F_0(x) + \int_{P_2} \min\{s(x), a+b\} s(x)\, \mathrm{d}F_0(x)$$
$$+ \cdots + \int_{P_m} \min\{s(x), a+b\} s(x)\, \mathrm{d}F_0(x)$$
$$+ \int_{N_1} \max\{s(x), a-b\} s(x)\, \mathrm{d}F_0(x) + \int_{N_2} \max\{s(x), a-b\} s(x)\, \mathrm{d}F_0(x)$$
$$+ \cdots + \int_{N_m} \max\{s(x), a-b\} s(x)\, \mathrm{d}F_0(x). \tag{A.3}$$

It can be noted that all the terms on the right hand side are non-negative. Therefore, $d(\Psi_b) > 0$. Suppose, if possible, $d(\Psi_b) = 0$. Then all the quantities on the right hand side of (A.3) are zero. Now $|a| \leqslant b$ implies that at least one of the relations $a+b > 0$ and $a-b < 0$ is always true. When $a+b > 0$, it follows that $\min\{s(x), a+b\} s(x) > 0$ for $x \in P_1, P_2, \ldots, P_m$. Therefore, the first m terms on the right hand side of (A.3) are zero imply that

$$\int_{P_1} \mathrm{d}F_0(x) = \int_{P_2} \mathrm{d}F_0(x) = \cdots = \int_{P_m} \mathrm{d}F_0(x) = 0. \tag{A.4}$$

Since $\int s(x)\, \mathrm{d}F_0(x) = 0$, (A.4) implies that

$$\int_{N_1} \mathrm{d}F_0(x) = \int_{N_2} \mathrm{d}F_0(x) = \cdots = \int_{N_m} \mathrm{d}F_0(x) = 0.$$

This contradicts $\int s(x)^2\, \mathrm{d}F_0(x) > 0$. Similarly when $a - b < 0$, $d(\Psi_b) = 0$ contradicts $\int s(x)^2\, \mathrm{d}F_0(x) > 0$. Thus for $|a| \leqslant b$, $d(\Psi_b) > 0$.

Next let $a > b$. Observe that $\Psi_b(x) > -b$ implies that $s(x) > a - b > 0$. Thus

$$d(\Psi_b) = \int_{-\infty}^{Q_{p_1}} \Psi_b(x) \Lambda(x)\, \mathrm{d}F_0(x) + \int_{Q_{p_1}}^{Q_{p_2}} \frac{1 - q_1 - p_1^*}{1 - p_1} \Psi_b(x) \Lambda(x)\, \mathrm{d}F_0(x)$$
$$+ \cdots + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}} \Psi_b(x) \Lambda(x)\, \mathrm{d}F_0(x)$$
$$= \int_{-\infty}^{Q_{p_1}} [\Psi_b(x) - (-b)] \Lambda(x)\, \mathrm{d}F_0(x)$$
$$+ \int_{Q_{p_1}}^{Q_{p_2}} \frac{1 - q_1 - p_1^*}{1 - p_1} [\Psi_b(x) - (-b)] \Lambda(x)\, \mathrm{d}F_0(x)$$
$$+ \cdots + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}} [\Psi_b(x) - (-b)] \Lambda(x)\, \mathrm{d}F_0(x)$$
$$= \int_{P_1} [\Psi_b(x) - (-b)] s(x)\, \mathrm{d}F_0(x) + \int_{P_2} [\Psi_b(x) - (-b)] s(x)\, \mathrm{d}F_0(x)$$
$$+ \cdots + \int_{P_m} [\Psi_b(x) - (-b)] s(x)\, \mathrm{d}F_0(x)$$

is non-negative. We now show that $d(\Psi_b) > 0$ by showing that $\Psi_b(x)$ cannot be equal to $-b$ for all $x \in P_1, P_2, \ldots, P_m$ since $\int \Psi_b(x)\,\mathrm{d}F_0(x) = -b \int \mathrm{d}F_0(x) - u = -b - u$ for some $u > 0$ and $\int \Psi_b(x)\,\mathrm{d}F_0(x) \neq 0$ in that case. Similarly, when $a < b$, it can be shown that $d(\Psi_b) > 0$.

Next, we prove the optimality and uniqueness of Ψ_b. Consider $\Psi(x)$ which satisfy (21), (22) and (23). Without loss of generality, assume that $d(\Psi) = d(\Psi_b)$. So, only the numerator of (20) is to be minimized. Using (21), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{Q_{p_1}} [\Lambda(x) + C_\lambda - a - \Psi_1(x) - C_\psi]^2\,\mathrm{d}F_0(x) \\
&\quad + \int_{Q_{p_1}}^{Q_{p_2}} \left[\frac{1 - q_1 - p_1^*}{1 - p_1}\,\Lambda(x) + C_\lambda - a - \Psi_2(x) - C_\psi\right]^2 \mathrm{d}F_0(x) + \cdots \\
&\quad + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \left[\frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}}\,\Lambda(x) + C_\lambda - a - \Psi_m(x) - C_\psi\right]^2 \mathrm{d}F_0(x) \\
&\quad = \int_{-\infty}^{Q_{p_1}} [\Lambda(x) - a - \Psi_1(x)]^2\,\mathrm{d}F_0(x) \\
&\qquad + \int_{Q_{p_1}}^{Q_{p_2}} \left[\frac{1 - q_1 - p_1^*}{1 - p_1}\,\Lambda(x) - a - \Psi_2(x)\right]^2 \mathrm{d}F_0(x) + \cdots \\
&\qquad + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \left[\frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}}\,\Lambda(x) - a - \Psi_m(x)\right]^2 \mathrm{d}F_0(x) \\
&\qquad + (C_\lambda - C_\psi)^2 - 2a(C_\lambda - C_\psi) \\
&\quad = \int_{-\infty}^{Q_{p_1}} [\Lambda(x) - a]^2\,\mathrm{d}F_0(x) + \int_{-\infty}^{Q_{p_1}} \Psi_1^2(x)\,\mathrm{d}F_0(x) \\
&\qquad + \int_{Q_{p_1}}^{Q_{p_2}} \left[\frac{1 - q_1 - p_1^*}{1 - p_1}\,\Lambda(x) - a\right]^2 \mathrm{d}F_0(x) + \int_{Q_{p_1}}^{Q_{p_2}} \Psi_2^2(x)\,\mathrm{d}F_0(x) + \cdots \\
&\qquad + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \left[\frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}}\,\Lambda(x) - a\right]^2 \mathrm{d}F_0(x) \\
&\qquad + \int_{Q_{p_{m-1}}}^{Q_{p_m}} \Psi_m^2(x)\,\mathrm{d}F_0(x) + (C_\lambda - C_\psi)^2 - 2a(C_\lambda - C_\psi) - 2d(\Psi_b)
\end{aligned}
$$

in which C_λ and C_ψ are given earlier. Hence to minimize the numerator of (20), it is sufficient to minimize the second equation. Under the restriction, $|\Psi| \leqslant b$, the left hand side is minimized by

$$\Psi_1(x) = [\Lambda(x) - a]_{-b}^{b},$$

$$\Psi_2(x) = \left[\frac{1 - q_1 - p_1^*}{1 - p_1}\Lambda(x) - a\right]_{-b}^{b},$$

$$\dots$$

$$\Psi_m(x) = \left[\frac{(1 - \sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*)}{1 - p_{m-1}}\Lambda(x) - a\right]_{-b}^{b}$$

almost everywhere with respect to F_0. This completes the proof of the theorem.

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