# Progressive type II censored order statistics from exponential distributions. Statistics

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# Progressive type II censored order statistics from exponential distributions

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**Abstract.** In the model of progressive type II censoring, point and interval estimation as well as relations for single and product moments are considered. Based on two-parameter exponential distributions, maximum likelihood estimators (MLEs), uniformly minimum variance unbiased estimators (UMVUEs) and best linear unbiased estimators (BLUEs) are derived for both location and scale parameters. Some properties of these estimators are shown. Moreover, results for single and product moments of progressive type II censored order statistics are presented to obtain recurrence relations from exponential and truncated exponential distributions. These relations may then be used to compute all the means, variances and covariances of progressive type II censored order statistics based on exponential distributions for arbitrary censoring schemes. The presented recurrence relations simplify those given by Aggarwala and Balakrishnan (1996).

**Key words and phrases:** Progressive type II censored order statistics, order statistics, exponential distribution, truncated exponential distribution, maximum likelihood estimation, uniformly minimum variance unbiased estimation, best linear unbiased estimation, moments, product moments, recurrence relations.

## 1 Introduction

The scheme of progressive type II censoring is of importance in life-testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment. A saving of costs and of time may be the consequence of such a sampling scheme (cf. Cohen 1963, Sen 1986).

Let us consider the following progressive type II censoring scheme: Suppose $N$ units of the same kind are placed on a lifetime test; at the first failure time of one of the units, a number of $R_1$ surviving units is randomly withdrawn from the test; at the second failure time, $R_2$ surviving units are selected at random and taken out of the experiment, and so on; finally, at the time of the $m$th failure, the remaining $R_m = N - R_1 - \cdots - R_{m-1} - m$ objects are removed. The withdrawal of units may be seen as a model describing drop-outs of units due to failures, which have causes other than the specific one under study. Progressive censoring schemes are also considered in clinical trials. Here, the drop-out of patients may be caused by migration, lack of interest or by ethical decisions. These reasons may be regarded as random withdrawals during the carrying out of the study. For a detailed discussion of progressive censoring we refer to Sen (1986). In the context of life-testing, suppose that $X^{\tilde{R}}_{1:m,N} \leq \cdots \leq X^{\tilde{R}}_{m:m,N}$ are the lifetimes of the completely observed units to fail, and that $\tilde{R} = (R_1, \ldots, R_m)$ represents the numbers of units withdrawn at these failure times. If the failure times are based on an absolutely continuous distribution function $F$ with probability density function $f$, the joint probability density function of the progressive censored failure times $X^{\tilde{R}}_{1:m,N}, \ldots, X^{\tilde{R}}_{m:m,N}$ is given by

$$f^{X^{\tilde{R}}_{1:m,N},\ldots,X^{\tilde{R}}_{m:m,N}}(x_1, \ldots, x_m) = c \prod_{j=1}^{m} f(x_j)[1 - F(x_j)]^{R_j},$$

$$-\infty < x_1 < \cdots < x_m < \infty, \quad (1)$$

where $N = m + \sum\limits_{j=1}^{m} R_j$, $m, N \in \mathbb{N}$, $R_j \in \mathbb{N}_0$, $1 \leq j \leq m$, $\tilde{R} = (R_1, \ldots, R_m)$, and

$$c = c(N, m) = \prod_{j=1}^{m} (N - \sum_{i=1}^{j-1} R_i - j + 1) = \prod_{j=1}^{m} R_j^{\sigma}$$

with $R_j^{\sigma} = \sum\limits_{k=j}^{m} (R_k + 1)$. Observe that $R_1^{\sigma} = N$.

From the representation of the joint density function it is obvious that progressive censoring can be embedded in the models of generalized order statistics and of sequential order statistics (cf. Kamps 1995, 1999). From this point of view, all the results obtained in these general models can be applied to progressive type II censoring.

In this paper, we are concerned with progressive type II censored data from exponential distributions. We consider two-parameter exponential distributions given by the density function

$$f(x) = \exp\left(-\frac{x - \mu}{\vartheta}\right), \qquad x \geq \mu,$$

with location parameter $\mu \in \mathbb{R}$ and scale parameter $\vartheta > 0$. For brevity, we write $\mathrm{Exp}(\mu, \vartheta)$ for such an exponential distribution. As pointed out by Cramer and Kamps (1997, 1998), the joint distribution (1) of a progressive type II censored sample is known as a particular Weinman multivariate exponential distribution introduced by Weinman (1966) (cf. Block 1975, Johnson and Kotz 1972). Its density function is given by

$$f^{X^{\tilde{R}}_{1:m,N},\ldots,X^{\tilde{R}}_{m:m,N}}(x_1, \ldots, x_m) = \frac{c}{\vartheta^m} \prod_{j=1}^{m} \exp\left\{-\frac{R_j + 1}{\vartheta}(x_j - \mu)\right\},$$

$$\mu \leq x_1 \leq \cdots \leq x_m.$$

Section 2 gives a survey of estimation results in the model of progressive type II censoring. The most general situation of $s$ samples with possibly different parameters

$m$ and $N$ is considered. Assuming the location parameter to be known and unknown, respectively, we show properties of the resulting estimators. At the end of the section we introduce a modified sampling scheme called *general progressive censoring* which was proposed by Balakrishnan and Sandhu (1995, 1996). We extend their estimation results by considering more than one sample. The samples have possibly different sample sizes, and they are allowed to be left censored, such that the first observation in the sample $i$ is given by the $(p_i + 1)$st order statistic. Section 3 deals with recurrence relations for single moments as well as for product moments of progressive censored order statistics. First of all, we present some general results which simplify the calculation procedure of moments w.r.t. the given censoring scheme. By this we obtain simplified versions of the recurrence relations derived by Aggarwala and Balakrishnan (1996). In Section 3.1 we are concerned with the standard exponential distribution, whereas we consider right truncated exponential distributions in Section 3.2. For further references and more information on progressive censoring we refer to Viveros and Balakrishnan (1994), Balakrishnan and Sandhu (1996) and to Aggarwala and Balakrishnan (1996, 1998).

# 2 Estimation results

In this section we assume that we have observed data from $s$ independent experiments with corresponding censoring schemes $\tilde{R}_i = (R_{i1}, \ldots, R_{im_i})$ and sample sizes $N_i$, $1 \le i \le s$. In terms of sequential order statistics, related results can be found in Cramer and Kamps (1998). The $i$th sample consists of realizations of $X^{\tilde{R}_i}_{i;1:m_i,N_i} \le \cdots \le X^{\tilde{R}_i}_{i;m_i:m_i,N_i}$, $1 \le i \le s$. We consider three estimation concepts, namely: maximum likelihood estimation, best linear unbiased estimation and uniformly minimum variance unbiased estimation. Some of the results presented in Sections 2.1 and 2.2 can be derived from those given in Cramer and Kamps (1998), since progressive type II censoring can be embedded into the framework of sequential order statistics. For details we refer to Kamps (1995, 1999) and Cramer and Kamps (1998).

## 2.1 Known location parameter

First, we consider maximum likelihood estimation of the scale parameter $\vartheta$. The result follows after some calculations from Theorem 3.1 of Cramer and Kamps (1998).

**Theorem 1.** *The MLE of $\vartheta$ is given by*

$$\vartheta^* = \frac{1}{m_\sigma} \sum_{i=1}^{s} \sum_{j=1}^{m_i} (R_{ij} + 1)(X^{\tilde{R}_i}_{i;j:m_i,N_i} - \mu)$$

*where $m_\sigma = \sum_{i=1}^{s} m_i$.*

**Remark 1.** *1. In the situation of one ordinary type II censored sample from $Exp(0, \vartheta)$-distributions described by ordinary order statistics, i.e., $s = 1$, $R_{1j} = 0$, $1 \le j \le m_1$ with $m_1 \in \{1, \ldots, N_1\}$, the results of Theorem 1 can be found in Lawless (1982,p. 102) and in Johnson et al. (1994,p. 514).*

2. *The result in terms of progressive type II censoring with $s = 1$ is given in Cohen (1995) ($\mu = 0$).*

The proof of the MLE's properties given in Theorem 2 can be carried out by a property of progressive type II censored order statistics due to Viveros and Balakrishnan (1994): The spacings of progressively type II censored order statistics from exponential distributions are independent and exponentially distributed. This property can be seen as an extension of a well-known result for ordinary order statistics established first by Sukhatme (1937). Alternatively, this assertion can be deduced from the respective result in the more general model of generalized order statistics (cf. Cramer and Kamps 1998, Kamps 1995,p. 81).

**Theorem 2.** *In the above situation, with $\vartheta^* = \vartheta^*(s)$, we find that*

1. *$\vartheta^* \sim \Gamma(m_\sigma, \vartheta/m_\sigma)$, i.e., $\vartheta^*$ is a gamma distributed random variable with parameters $m_\sigma$ and $\vartheta/m_\sigma$. Its density function is given by*

$$f_{\vartheta^*}(t) = \frac{(m_\sigma/\vartheta)^{m_\sigma}}{(m_\sigma - 1)!} t^{m_\sigma - 1} e^{-m_\sigma t/\vartheta}, \quad t \geq 0.$$

.

2. *$E(\vartheta^*)^k = \frac{(m_\sigma+k-1)!}{(m_\sigma-1)!}(\frac{\vartheta}{m_\sigma})^k$, $k \in \mathbb{N}$; in particular, $E\vartheta^* = \vartheta$ and $Var(\vartheta^*) = \frac{\vartheta^2}{m_\sigma}$. Hence, $\vartheta^*$ is an unbiased estimator of $\vartheta$.*

3. *$\vartheta^*$ is sufficient for $\vartheta$.*

4. *$(\vartheta^*(s))_s$ is strongly consistent for $\vartheta$, i.e., $\vartheta^*(s) \longrightarrow \vartheta$ a.e. w.r.t. $s \to \infty$.*

5. *$(\vartheta^*(s))_s$ is asymptotically normal, i.e., $\sqrt{m_\sigma}(\vartheta^*(s)/\vartheta - 1) \xrightarrow{d} \mathfrak{N}(0,1)$ w.r.t. $s \to \infty$.*

It can be shown that the MLE $\vartheta^*$ attains the Cramér-Rao lower bound (cf. Cramer and Kamps 1998). Since $\vartheta^*$ is unbiased and linear, it coincides with the UMVUE and the BLUE of $\vartheta$.

The distributional results of Theorem 2 can be utilized to derive confidence intervals for the parameter $\vartheta$. From the first item of the above theorem we find $\vartheta^* \sim \frac{\vartheta}{2m_\sigma}\chi^2_{2m_\sigma}$, where $\chi^2_q$ denotes the $\chi^2$-distribution with $q$ degrees of freedom. Hence, a two sided confidence interval with level $0 < \alpha < 1$ is given by

$$\left[\frac{2m_\sigma\vartheta^*}{\chi^2_{2m_\sigma}(1-\alpha/2)}, \frac{2m_\sigma\vartheta^*}{\chi^2_{2m_\sigma}(\alpha/2)}\right],$$

where $\chi^2_q(\alpha)$ is the $\alpha$-quantile of the $\chi^2_q$-distribution. A one-sided confidence interval is given by

$$\left(0, \frac{2m_\sigma\vartheta^*}{\chi^2_{2m_\sigma}(\alpha)}\right].$$

In case of one sample of ordinary order statistics, the results reduce to the representations given in, e.g., Cohen (1995).

## 2.2 Unknown location parameter

In this section, we present the MLEs of the distribution parameters. It turns out that the MLE of $\vartheta$ can be obtained from the MLE in the case of a known location parameter by replacing the location parameter $\mu$ by the corresponding MLE.

**Theorem 3.** *The simultaneous MLEs of $\mu$ and $\vartheta$ are given by*

$$\tilde{\mu} = \min\{X_{1;1:m_1,N_1}^{\tilde{R}_1}, \dots, X_{s;1:m_s,N_s}^{\tilde{R}_s}\} \text{ and}$$

$$\tilde{\vartheta} = \frac{1}{m_\sigma}\sum_{i=1}^{s}\sum_{j=1}^{m_i}(R_{ij}+1)(X_{i;j:m_i,N_i}^{\tilde{R}_i} - \tilde{\mu}), \quad \text{respectively.}$$

The MLEs $\tilde{\mu}$ and $\tilde{\vartheta}$ of the location and scale parameter possess some interesting properties.

**Theorem 4.** *The MLEs $\tilde{\vartheta}$ and $\tilde{\mu}$ are stochastically independent, and $(\tilde{\mu}, \tilde{\vartheta})$ is a complete sufficient statistic for $(\mu, \vartheta)$. Furthermore, $\tilde{\vartheta} \sim \Gamma(m_\sigma - 1, \vartheta/m_\sigma)$ and $\tilde{\mu} \sim Exp(\mu, \vartheta/N_\sigma)$, where $N_\sigma = \sum_{i=1}^{s} N_i$.*

A similar result is well-known in the framework of estimation of scale and location parameters from a normal distribution (cf., e.g., Bickel and Doksum 1977,Section 1.3). In this model, the mean and the sample variance are used as estimators for the location and the scale parameter, respectively, and they turn out to be independent. Moreover, there is an analogy in the distributions of the estimators, too. The location estimators follow the type of distribution of the underlying data whereas the scale estimators have a $\chi^2$-distribution in both settings. In case of ordinary type II censoring and one sample such a result is reported in Epstein and Sobel (1954) and David (1981,p. 153).

The preceding results can be applied to construct confidence sets for the parameters $\mu$ and $\vartheta$. Proceeding similarly to the case of an known location parameter we obtain that for a given level $0 < \alpha < 1$

$$\mathcal{K}_\mu = \left[\tilde{\mu} - \frac{m_\sigma}{(m_\sigma - 1)N_\sigma} F_{2,2(m_\sigma-1)}(1-\alpha)\tilde{\vartheta}, \tilde{\mu}\right], \qquad \text{and}$$

$$\mathcal{K}_\vartheta = \left[\frac{2m_\sigma\tilde{\vartheta}}{\chi^2_{2(m_\sigma-1)}(1-\alpha/2)}, \frac{2m_\sigma\tilde{\vartheta}}{\chi^2_{2(m_\sigma-1)}(\alpha/2)}\right]$$

are confidence intervals for $\mu$ and $\vartheta$, respectively. $F_{p,q}(1-\alpha)$ denotes the $1-\alpha$ quantile of the $F$-distribution with degrees of freedom $p$ and $q$. Since in our situation $p = 2$, we can apply the relation $F_{2,2q}(1-\alpha) = q(\alpha^{-1/q} - 1)$ (see, e.g., Epstein and Sobel 1954). For results in the case of one sample and ordinary order statistics we refer to Engelhardt (1995).

The results given in Theorem 4 lead immediately to some conclusions which are summarized in the following corollaries.

**Corollary 1.** *1. $E(\tilde{\vartheta})^k = \frac{(m_\sigma+k-2)!}{(m_\sigma-2)!}(\frac{\vartheta}{m_\sigma})^k$, $k \in \mathbb{N}$; in particular, $E\tilde{\vartheta} = \frac{m_\sigma-1}{m_\sigma}\vartheta$ and $Var(\tilde{\vartheta}) = \frac{m_\sigma-1}{m_\sigma^2}\vartheta^2$. Hence, $\hat{\vartheta} = \frac{m_\sigma}{m_\sigma-1}\tilde{\vartheta}$ is an unbiased estimator of $\vartheta$.*

2. $(\tilde{\vartheta}(s))_s$ *is strongly consistent for* $\vartheta$*, i.e.,* $\tilde{\vartheta}(s) \longrightarrow \vartheta$ *a.e. w.r.t.* $s \to \infty$*.*

3. $(\tilde{\vartheta}(s))_s$ *is asymptotically normal, i.e.,* $\sqrt{m_\sigma}(\tilde{\vartheta}(s)/\vartheta - 1) \xrightarrow{d} \mathfrak{N}(0,1)$ *w.r.t.* $s \to \infty$*.*

**Corollary 2.** 1. $E(\tilde{\mu})^k = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \left(\frac{\vartheta}{N_\sigma}\right)^j \mu^{k-j}$*; in particular,* $E\tilde{\mu} = \mu + \frac{\vartheta}{N_\sigma}$ *and* $Var(\tilde{\mu}) = \left(\frac{\vartheta}{N_\sigma}\right)^2$*. Hence, the mean squared error (MSE) is given by* $MSE(\tilde{\mu}) = 2\left(\frac{\vartheta}{N_\sigma}\right)^2$*.*

2. $(\tilde{\mu}(s))_s$ *is asymptotically unbiased, i.e.,* $E\tilde{\mu}(s) \longrightarrow \mu$ *w.r.t.* $s \to \infty$*.*

3. $(\tilde{\mu}(s))_s$ *is strongly consistent for* $\mu$*, i.e.,* $\tilde{\mu}(s) \longrightarrow \mu$ *a.e. w.r.t.* $s \to \infty$*.*

Utilizing these results we obtain the UMVUEs of the parameters. An unbiased estimator of $\vartheta$ is easy to construct from Corollary 1, i.e., $\hat{\vartheta} = \frac{m_\sigma}{m_\sigma - 1}\tilde{\vartheta}$. An unbiased estimator $\hat{\mu}$ of the location parameter $\mu$ is derived via a bias correction, i.e.,

$$\hat{\mu} = \tilde{\mu} - \hat{\vartheta}/N_\sigma.$$

As pointed out in Theorem 5, we have found the desired estimators. This extends a result given in Cohen (1995) for $s = 1$ and ordinary order statistics.

**Theorem 5.** 1. *The UMVUE of* $\vartheta$ *is given by* $\hat{\vartheta}$*. The variance of* $\hat{\vartheta}$ *is*

$$Var(\hat{\vartheta}) = \frac{\vartheta^2}{m_\sigma - 1}.$$

2. *The UMVUE of* $\mu$ *is given by* $\hat{\mu}$*. The variance of* $\hat{\mu}$ *is*

$$Var(\hat{\mu}) = \frac{m_\sigma}{(m_\sigma - 1)N_\sigma^2}\vartheta^2.$$

3. $Cov(\hat{\mu}, \hat{\vartheta}) = -\frac{1}{(m_\sigma - 1)N_\sigma}\vartheta^2.$

Asymptotic properties of the UMVUEs can be easily obtained from Corollaries 1 and 2.

In the case of a known location parameter we have seen that the MLE, the UMVUE and the BLUE of $\vartheta$ coincide. This property does not generally hold true in the case of an unknown location parameter, because the MLE $\tilde{\mu}$ of $\mu$ is nonlinear if $s \geq 2$. The BLUEs of $\mu$ and $\vartheta$ are deduced from the Gauß-Markov theorem. They are given by (cf. Cramer and Kamps 1998)

$$\begin{aligned}\vartheta_{\text{BLUE}} = \delta\Big[(N^2)_\sigma \sum_{i=1}^{s}\sum_{j=2}^{m_i} R_{ij}^\sigma \big(X_{i;j:m_i,N_i}^{\tilde{R}_i} - X_{i;j-1:m_i,N_i}^{\tilde{R}_i}\big) \\ + \sum_{i=1}^{s}((N^2)_\sigma - N_\sigma N_i)N_i X_{i;1:m_i,N_i}^{\tilde{R}_i}\Big],\end{aligned}$$

$$\mu_{\text{BLUE}} = \delta\Big[\sum_{i=1}^{s}(m_\sigma N_i - N_\sigma)N_i X_{i;1:m_i,N_i}^{\tilde{R}_i} - N_\sigma \sum_{i=1}^{s}\sum_{j=2}^{m_i} R_{ij}^\sigma (X_{i;j:m_i,N_i}^{\tilde{R}_i} - X_{i;j-1:m_i,N_i}^{\tilde{R}_i})\Big],$$

where $\delta^{-1} = m_\sigma (N^2)_\sigma - N_\sigma^2$, $(N^2)_\sigma = \sum_{i=1}^{s} N_i^2$ and $R_{ij}^\sigma = \sum_{k=j}^{m_i}(R_{ik}+1)$, $1 \le j \le m_i$, $1 \le i \le s$. The covariance matrix of the BLUEs is given by

$$\operatorname{Cov}\begin{pmatrix}\vartheta_{\text{BLUE}}\\ \mu_{\text{BLUE}}\end{pmatrix} = \frac{1}{m_\sigma (N^2)_\sigma - N_\sigma^2}\begin{pmatrix}(N^2)_\sigma & -N_\sigma\\ -N_\sigma & m_\sigma\end{pmatrix}\vartheta^2.$$

The formulae become simpler if we suppose that $m_i = m, N_i = N, R_{ij} = R_j$, which means that we have identical sample sizes and censoring schemes in the samples. In terms of the Kronecker product $\otimes$ (cf. Mardia et al. 1979,p. 459), a matrix representation of the BLUEs reads:

$$\begin{aligned}\begin{pmatrix}\vartheta_{\text{BLUE}}\\ \mu_{\text{BLUE}}\end{pmatrix} &= [(\mathbb{1}_s \otimes \beta, \mathbb{1}_{sm})'(\mathfrak{I}_s \otimes \Delta^{-1})(\mathbb{1}_s \otimes \beta, \mathbb{1}_{sm})]^{-1}(\mathbb{1}_s \otimes \beta, \mathbb{1}_{sm})'(\mathfrak{I}_s \otimes \Delta^{-1})\vec{X}\\ &= \frac{1}{s}[(\beta, \mathbb{1}_m)'\Delta^{-1}(\beta, \mathbb{1}_m))^{-1}[\mathbb{1}_s{}' \otimes ((\beta, \mathbb{1}_m)'\Delta^{-1})]\vec{X}, \end{aligned} \tag{2}$$

where $\vec{X} = (X_{1;1:m,N}^{\tilde{R}}, \ldots, X_{1;m:m,N}^{\tilde{R}}, X_{2;1:m,N}^{\tilde{R}}, \ldots, X_{s;m:m,N}^{\tilde{R}})'$, $\beta = ((R_1^\sigma)^{-1}, \sum_{j=1}^{2}(R_j^\sigma)^{-1}, \ldots, \sum_{j=1}^{m}(R_j^\sigma)^{-1})'$, $\mathbb{1}_p$ and $\mathfrak{I}_q$ denote the vector $(1,\ldots,1)'$ and the matrix $\operatorname{diag}(1,\ldots,1)$ of dimension $q$, respectively, and

$$\Delta^{-1} = \begin{bmatrix}(R_1^\sigma)^2 + (R_2^\sigma)^2 & -(R_2^\sigma)^2 & 0 & \cdots & 0\\ -(R_2^\sigma)^2 & (R_2^\sigma)^2 + (R_3^\sigma)^2 & -(R_3^\sigma)^2 & 0 & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & & & -(R_m^\sigma)^2\\ 0 & \cdots & 0 & -(R_m^\sigma)^2 & (R_m^\sigma)^2\end{bmatrix}.$$

After some lengthy calculations, the evaluation of the matrix representation (2) leads to the simplified versions:

$$\vartheta_{\text{BLUE}} = \frac{1}{m-1}\sum_{j=2}^{m}(R_j+1)(\bar{X}_{j:m,N}^{\tilde{R}} - \bar{X}_{1:m,N}^{\tilde{R}}),$$

$$\mu_{\text{BLUE}} = \frac{1}{(m-1)N}\sum_{j=1}^{m}(R_j+1)(m\bar{X}_{1:m,N}^{\tilde{R}} - \bar{X}_{j:m,N}^{\tilde{R}}),$$

where $\bar{X}_{j:m,N}^{\tilde{R}} = s^{-1}\sum_{i=1}^{s} X_{i;j:m,N}^{\tilde{R}}$, $1 \le j \le m$, $m \ge 2$.

**Remark 2.** *The following alternative representation of the BLUE of $\vartheta$ is similar to the representation of the MLE:*

$$\vartheta_{BLUE} = \frac{1}{m}\sum_{j=1}^{m}(R_j+1)(\bar{X}^{\tilde{R}}_{j:m,N} - \mu_{BLUE}).$$

*In the particular case of ordinary order statistics and $s = 1$ this yields the result of Epstein (1957). The BLUE of $\mu$ can be written as*

$$\mu_{BLUE} = \bar{X}^{\tilde{R}}_{1:m,N} - \frac{\vartheta_{BLUE}}{N}.$$

## 2.3 General progressive type II censoring

Within the model of progressive censoring, Balakrishnan and Sandhu (1995, 1996) additionally consider a left censoring of the data called *general progressive type II censoring*. They assume that the first $p$ failures are not observed, and that the first available observation is given by the $(p+1)$st ordinary order statistic. Afterwards, the progressive censoring procedure is applied as in the case without this left censoring. At the time of the $(p+1)$st failure, $R_{p+1}$ active components are withdrawn at random from the experiment. The remaining procedure is the same as described in the introduction. Let $Y^{\tilde{R}}_{p+1:m,N}, \ldots, Y^{\tilde{R}}_{m:m,N}$ denote the corresponding random variables. The density function describing the general progressive type II censoring scheme is given by

$$f^{Y^{\tilde{R}}_{p+1:m,N},\ldots,Y^{\tilde{R}}_{m:m,N}}(y_{p+1},\ldots,y_m) = c\,F^p(y_{p+1})\prod_{j=p+1}^{m} f(y_j)[1-F(y_j)]^{R_j},$$

$$-\infty < y_{p+1} < \cdots < y_m < \infty,$$

where $c = (N-p)\binom{N}{p}\prod_{j=p+2}^{m}(N+1-j-\sum_{i=p+1}^{j-1}R_i)$. This distribution can be seen as a marginal of the density function in (1) provided that the censoring scheme is given by $(R_1,\ldots,R_m)$ with $R_1 = \cdots = R_p = 0$. Since no confusion is possible in this section we write subsequently $\tilde{R} = (0,\ldots,0,R_{p+1},\ldots,R_m)$ for brevity.

Balakrishnan and Sandhu (1995, 1996) obtain the BLUEs and the MLEs of the parameters of exponential distributions in this model. In the case of a known location parameter, the likelihood equation can not be solved explicitly. In contrast to this, an explicit solution can be derived if both parameters are assumed to be unknown. We extend the set-up of Balakrishnan and Sandhu (1995, 1996) such that $s$ samples are taken into account. However, it turns out that the MLEs of the parameters are generally not explicitly available in both situations if $s \geq 2$. We consider the following sampling situation: The data is described by random variables $(Y^{\tilde{R}_i}_{i;p_i+1:m_i,N_i},\ldots,Y^{\tilde{R}_i}_{i;m_i:m_i,N_i})_{1\leq i\leq s}$ with realizations $y_{ij}$, $p_i+1 \leq j \leq m_i$, $1 \leq i \leq s$.

The log-likelihood function $l(\mu,\vartheta)$ is given by:

$$l(\mu,\vartheta;y_{ij}) = \text{const} + \sum_{i=1}^{s} p_i \log(1-\exp\{-(y_{i,p_i+1}-\mu)/\vartheta\}) - \log\vartheta\sum_{i=1}^{s}(m_i-p_i)$$

$$-\frac{1}{\vartheta}\sum_{i=1}^{s}\sum_{j=p_i+1}^{m_i}(R_{ij}+1)(y_{ij}-\mu),$$

$$\mu \le y_{i,p_i+1} \le \cdots \le y_{i,m_i} < \infty,\ 1 \le i \le s. \quad (3)$$

For a known location parameter, we obtain the following results:

**Theorem 6.** *Let the location parameter $\mu$ be known.*

1. *The MLE of the scale parameter $\vartheta$ is the unique solution of the equation*

$$\sum_{i=1}^{s}\frac{p_i(Y_{i;p_i+1:m_i,N_i}^{\tilde{R}_i}-\mu)}{\exp\{(Y_{i;p_i+1:m_i,N_i}^{\tilde{R}_i}-\mu)/\vartheta\}-1}+\vartheta\sum_{i=1}^{s}(m_i-p_i) = \sum_{i=1}^{s}\sum_{j=p_i+1}^{m_i}(R_{ij}+1)(Y_{i;j:m_i,N_i}^{\tilde{R}_i}-\mu). \quad (4)$$

2. *The BLUE of $\vartheta$ is given by*

$$\vartheta_p^* = \frac{1}{\sum_{i=1}^{s}(m_i-p_i-1+\alpha_{p_i}^2/\beta_{p_i})}\Big[\sum_{i=1}^{s}\frac{\alpha_{p_i}}{\beta_{p_i}}(Y_{i;p_i+1:m_i,N_i}^{\tilde{R}_i}-\mu) + \sum_{i=1}^{s}\sum_{j=p_i+2}^{m_i}(R_{ij}+1)(Y_{i;j:m_i,N_i}^{\tilde{R}_i}-Y_{i;p_i+1:m_i,N_i}^{\tilde{R}_i})\Big],$$

*where* $\alpha_{p_i} = \sum_{j=1}^{p_i+1}(N_i-j+1)^{-1}$, $\beta_{p_i} = \sum_{j=1}^{p_i+1}(N_i-j+1)^{-2}$, $1 \le i \le s$.

*Proof.* The condition (4) follows via differentiation of the log-likelihood function w.r.t. $\vartheta$. Denoting the left-hand side of (4) by $h$, one can show that $h$ is a strictly increasing function w.r.t. $\vartheta$. Furthermore, $\lim_{\vartheta\to 0} h(\vartheta) = 0$ and $\lim_{\vartheta\to\infty} h(\vartheta) = \infty$ such that (4) has exactly one solution. The representation of the BLUE results from the Gauß-Markov theorem. □

If the location parameter $\mu$ has to be estimated as well, the maximum likelihood estimation problem becomes solvable in the case $s = 1$ (cf. Balakrishnan and Sandhu 1996). If two or more samples are taken into consideration, the MLEs are generally not explicitly available. By partial differentiation one can derive necessary conditions for a local maximum of the likelihood function similar to (4). But, in this situation the constraint $\mu \le y_{\min} = \min_{1\le i\le s} y_{i,p_i+1}$ becomes a problem. Depending on the parameters $p_1, \ldots, p_s$, the maximum of the likelihood function can be attained on the boundary of the rectangular $I = (-\infty, y_{\min}] \times (0,\infty)$. For illustration, we consider the case of $s$ samples where only the first one is censored on the left according to the general progressive censoring scheme. Hence, we have $p_1 \geq 1$ and $p_i = 0$, $2 \leq i \leq s$. Differentiation of the log-likelihood function w.r.t. $\mu$ and $\vartheta$ leads to the equations

$$\frac{p_1}{\exp\{(Y^{\tilde{R}_1}_{1;p_1+1:m_1,N_1} - \mu)/\vartheta\} - 1} = N_\sigma - p_1$$

$$\frac{p_1(Y^{\tilde{R}_1}_{1;p_1+1:m_1,N_1} - \mu)}{\exp\{(Y^{\tilde{R}_1}_{1;p_1+1:m_1,N_1} - \mu)/\vartheta\} - 1} + \vartheta(m_\sigma - p_1) = \sum_{i=1}^{s} \sum_{j=p_i+1}^{m_i} (R_{ij}+1)(Y^{\tilde{R}_i}_{i;j:m_i,N_i} - \mu). \tag{5}$$

After some calculations we obtain the solution

$$\begin{aligned} \tilde{\vartheta}_p &= \frac{1}{m_\sigma - p_1} \sum_{i=1}^{s} \sum_{j=p_i+1}^{m_i} (R_{ij}+1)(Y^{\tilde{R}_i}_{i;j:m_i,N_i} - Y^{\tilde{R}_i}_{i;p_i+1:m_i,N_i}), \\ \tilde{\mu}_p &= Y^{\tilde{R}_1}_{1;p_1+1:m_1,N_1} + \tilde{\vartheta}_p \log(1 - p_1/N_\sigma). \end{aligned} \tag{6}$$

If $s = 1$ then (6) yields the result of Balakrishnan and Sandhu (1996). But if $s > 2$ we have to check that the point $(\tilde{\mu}_p, \tilde{\vartheta}_p)$ is admissible, i.e., $(\tilde{\mu}_p, \tilde{\vartheta}_p) \in I$. If this condition is fulfilled, we have found the MLE of $(\mu, \vartheta)$. Otherwise, the MLE of $\mu$ is given by $\tilde{\mu}_p = \min_{2 \leq i \leq s} Y^{\tilde{R}_i}_{i;1:m_i,N_i}$, since $\lim_{\vartheta \to 0,\infty} l(\mu, \vartheta; y_{ij}) = -\infty$ for arbitrary, but fixed $\mu$. The MLE of $\vartheta$ results from equation (4) with $\tilde{\mu}_p$ instead of $\mu$ by numerical computations.

The linear estimation of the parameters is easier to handle. Since the BLUEs depend only on the moments of the underlying random variables we obtain explicit expressions as in the situation of one sample. The proof follows by some lengthy calculations from the Gauß-Markov theorem.

**Theorem 7.** *Suppose that for one $i \in \{1, \ldots, s\}$ the condition $m_i \geq p_i + 2$ holds. Let $\delta_p^{-1} = \sum_{i=1}^{s} (m_i - p_i + 1)/\beta_{p_i}$ with $\alpha_{p_i} = \sum\limits_{j=1}^{p_i+1} (N_i - j + 1)^{-1}$, $\beta_{p_i} = \sum\limits_{j=1}^{p_i+1} (N_i - j + 1)^{-2}$, $1 \leq i \leq s$. Then the BLUEs of $\mu$ and $\vartheta$ are given by*

$$\begin{aligned} \mu_{p,BLUE} = \delta_p \sum_{i=1}^{s} \frac{m_i - p_i - 1}{\beta_{p_i}} Y^{\tilde{R}_i}_{i;p_i+1:m_i,N_i} \\ - \delta_p \sum_{i=1}^{s} \frac{\alpha_{p_i}}{\beta_{p_i}} \sum_{j=p_i+2}^{m_i} (R_{ij}+1)(Y^{\tilde{R}_i}_{i;j:m_i,N_i} - Y^{\tilde{R}_i}_{i;p_i+1:m_i,N_i}), \end{aligned}$$

$$\vartheta_{p,BLUE} = \delta_p \sum_{i=1}^{s} \frac{1}{\beta_{p_i}} \sum_{j=p_i+2}^{m_i} (R_{ij}+1)(Y^{\tilde{R}_i}_{i;j:m_i,N_i} - Y^{\tilde{R}_i}_{i;p_i+1:m_i,N_i}).$$

# 3 Single and product moments of progressive type II censored order statistics

The $k$th moment of the $i$th progressive type II censored order statistic $X^{\tilde{R}}_{i:m,N}$ and the product moment of the $i$th and $j$th progressive type II censored order statistics

$X^{\tilde{R}}_{i:m,N} X^{\tilde{R}}_{j:m,N}$ are given by

$$\mu^{(R_1,\ldots,R_m)^{(k)}}_{i:m,N} = E(X^{\tilde{R}}_{i:m,N})^k = c \int_{D_m} x_i^k \prod_{j=1}^{m} f(x_j)[1-F(x_j)]^{R_j}\, dx_1 \ldots dx_m$$

and

$$\mu^{(R_1,\ldots,R_m)}_{i,j:m,N} = E(X^{\tilde{R}}_{i:m,N} X^{\tilde{R}}_{j:m,N}) = c \int_{D_m} x_i x_j \prod_{k=1}^{m} f(x_k)[1-F(x_k)]^{R_k}\, dx_1 \ldots dx_m,$$

respectively, where $D_m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m | x_1 < \cdots < x_m\}$.

The following properties of the single and product moments of progressive type II censored order statistics are obtained by noticing that the $i$th failure time $X^{\tilde{R}}_{i:m,N}$ depends on the left truncated censoring scheme $(R_i, \ldots, R_m)$ only via $R_i^\sigma = \sum_{j=i}^{m} (R_j + 1)$. This fact is used to simplify the results given by Aggarwala and Balakrishnan (1996) who consider progressive type II censored order statistics based on exponential and truncated exponential distributions. Our approach makes use of the following lemma. A related result is presented in Aggarwala and Balakrishnan (1998). For further details we refer to Schenk (1998).

**Lemma 1.** *For $1 \le i \le m-1$, $t \in \mathbb{R}$ and an absolutely continuous distribution function $F$ we find*

$$\int_{D^m_{i+1}(t)} \prod_{k=i+1}^{m} f(x_k)[1-F(x_k)]^{R_k} dx_m \ldots dx_{i+1} = \left( \prod_{j=i+1}^{m} R_j^\sigma \right)^{-1} [1-F(t)]^{R^\sigma_{i+1}}, \qquad (7)$$

*where $D^m_{i+1}(t) = \{(x_{i+1}, \ldots, x_m) | t < x_{i+1} < \cdots < x_m < \infty\}$.*

*Proof.* The proof is carried out by induction on $m \ge i+1$. The case $m = i+1$ is easily checked. Hence, let (7) hold for $m-1 \ge i+1$ and for arbitrary censoring schemes $(S_1, \ldots, S_{m-1})$. Suppose that a censoring scheme $(R_1, \ldots, R_m)$ of length $m$ is given. Then we find with $S_j = R_j$, $1 \le j \le m-2$, $S_{m-1} = R_{m-1} + R_m + 1$ and $S_j^\sigma = \sum_{k=j}^{m-1} (S_k + 1)$, $1 \le j \le m-1$:

$$\begin{aligned}
& \int_{D^m_{i+1}(t)} \prod_{k=i+1}^{m} f(x_k)[1-F(x_k)]^{R_k} dx_m \ldots dx_{i+1} \\
= & \int_{D^{m-1}_{i+1}(t)} \left( \int_{x_{m-1}}^{\infty} f(x_m)[1-F(x_m)]^{R_m} dx_m \right) \prod_{k=i+1}^{m-1} f(x_k)[1-F(x_k)]^{R_k} dx_{m-1} \ldots dx_{i+1} \\
= & \frac{1}{R_m + 1} \int_{D^{m-1}_{i+1}(t)} \prod_{k=i+1}^{m-1} f(x_k)[1-F(x_k)]^{S_k} dx_{m-1} \ldots dx_{i+1} \\
= & \frac{1}{R_m^\sigma} \left( \prod_{k=i+1}^{m-1} S_k^\sigma \right)^{-1} [1-F(t)]^{S^\sigma_{i+1}}.
\end{aligned}$$

A straightforward calculation shows that $S_j^\sigma = R_j^\sigma$, $1 \le j \le m-1$, such that the latter expression can be written as $\left(\prod_{k=i+1}^{m} R_k^\sigma\right)^{-1} [1-F(t)]^{R_{i+1}^\sigma}$. This proves the assertion. $\square$

This auxiliary result leads immediately to the following theorem.

**Theorem 8.** *Let $1 \le i \le m$, $k \in \mathbb{N}_0$ and $F$ be absolutely continuous. Then the following equation holds for $0 \le j \le m-i$:*

$$\mu_{i:m,N}^{(R_1,\dots,R_m)(k)} = \mu_{i:i+j,N}^{(R_1,\dots,R_{i+j-1},R_{i+j}^\sigma-1)(k)}.$$

*In particular, we obtain for $j=0$,*

$$\mu_{i:m,N}^{(R_1,\dots,R_m)(k)} = \mu_{i:i,N}^{(R_1,\dots,R_{i-1},R_i^\sigma-1)(k)}$$

*with $R_i^\sigma - 1 \ge m-i \ge 0$, $1 \le i \le m$ and $R_i^\sigma = 1$ iff $i = m$ and $R_m = 0$.*

*Proof.* Let $0 \le j \le m-i$ be a fixed number. Writing the expectation as iterated integral, we obtain the representation

$$\mu_{i:m,N}^{(R_1,\dots,R_m)(k)} = c\int_{D_m} x_i^k \prod_{l=1}^{m} f(x_l)[1-F(x_l)]^{R_l}\,dx_m\dots dx_1$$

$$= c\int_{D_{i+j}} \left(\int_{D_{i+j+1}^m(x_{i+j})} \prod_{l=i+j+1}^{m} f(x_l)[1-F(x_l)]^{R_l}\,dx_m\dots dx_{i+j+1}\right) \times x_i^k \prod_{l=1}^{i+j} f(x_l)[1-F(x_l)]^{R_l}\,dx_{i+j}\dots dx_1.$$

Applying Lemma 1 to the innermost integral, the expression simplifies to

$$\mu_{i:m,N}^{(R_1,\dots,R_m)(k)}$$
$$= c(N,i+j)\int_{D_{i+j}} x_i^k \prod_{l=1}^{i+j-1} f(x_l)[1-F(x_l)]^{R_l} f(x_{i+j})[1-F(x_{i+j})]^{R_{i+j}^\sigma-1}\,dx_{i+j}\dots dx_1.$$

The latter integral corresponds to $\mu_{i:i+j,N}^{(R_1,\dots,R_{i+j-1},R_{i+j}^\sigma-1)(k)}$. $\square$

For the product moments an analogous result holds. The proof proceeds similar to the one of the preceding theorem and is therefore omitted.

**Theorem 9.** *Let $1 \le i \le m-1$ and $F$ be absolutely continuous. We find for $1 \le j \le m-i$:*

$$\mu_{i,i+j:m,N}^{(R_1,\dots,R_m)} = \mu_{i,i+j:i+j,N}^{(R_1,\dots,R_{i+j-1},R_{i+j}^\sigma-1)}.$$

## 3.1 Recurrence relations for moments from exponential distributions

In this section, we consider progressive type II censored order statistics based on a standard exponential distribution with probability density function $f(x) = \mathrm{e}^{-x}$, $x \geq 0$.

Now, by using Theorem 8, we derive the following recurrence relations for single moments which are simpler than those presented by Aggarwala and Balakrishnan (1996):

**Theorem 10.** *Let $k \in \mathbb{N}_0$.*

1. *For $1 \leq m \leq N$:*

$$\mu_{1:m,N}^{(R_1,\dots,R_m)^{(k+1)}} = \mu_{1:1,N}^{(R_1^\sigma-1)^{(k+1)}} = \frac{k+1}{N}\mu_{1:1,N}^{(N-1)^{(k)}} = \frac{(k+1)!}{N^{k+1}}.$$

2. *For $2 \leq i \leq m$:*

$$\begin{aligned}\mu_{i:m,N}^{(R_1,\dots,R_m)^{(k+1)}} &= \mu_{i:i,N}^{(R_1,\dots,R_{i-1},R_i^\sigma-1)^{(k+1)}}\\ &= \frac{(k+1)}{R_i^\sigma}\mu_{i:i,N}^{(R_1,\dots,R_{i-1},R_i^\sigma-1)^{(k)}} + \mu_{i-1:i-1,N}^{(R_1,\dots,R_{i-2},R_{i-1}^\sigma-1)^{(k+1)}}.\end{aligned}$$

The proof is a direct application of Theorem 8 and of Theorem 2.3 of Aggarwala and Balakrishnan (1996).

The corresponding result for product moments reads:

**Theorem 11.** *Let $1 \leq i < j \leq m$, $m \leq N$. Then:*

$$\mu_{i,j:m,N}^{(R_1,\dots,R_m)} = \frac{1}{R_j^\sigma}\mu_{i:i,N}^{(R_1,\dots,R_{i-1},R_i^\sigma-1)} + \mu_{i,j-1:j-1,N}^{(R_1,\dots,R_{j-2},R_{j-1}^\sigma-1)}.$$

The proof follows from Theorem 9 and Theorem 3.2 of Aggarwala and Balakrishnan (1996).

**Remark 3.** *Considering the case of ordinary order statistics, i.e., $R_i = 0$, $1 \leq i \leq m$, the preceding recurrence relations lead to the formulae derived by Joshi (1977, 1982) in the case of single and product moments, respectively.*

## 3.2 Recurrence relations for moments from truncated exponential distributions

Following we consider moments of progressive type II censored order statistics based on truncated standard exponential distributions with probability density function

$$f(x) = \frac{\mathrm{e}^{-x}}{P}, \quad 0 \leq x \leq P_1, \quad \text{where} \quad P = 1 - \mathrm{e}^{-P_1} > 0.$$

By analogy, results for the doubly truncated exponential distribution can be obtained as well. The results for order statistics are presented in Joshi (1979).

**Theorem 12.** *Let $k \in \mathbb{N}_0$.*

1. $\mu_{1:1,1}^{(0)^{(k+1)}} = (k+1)\mu_{1:1,1}^{(0)^{(k)}} - \left(\frac{1}{P} - 1\right) P_1^{k+1}.$

2. *Let $N \geq 2$.*

$$\mu_{1:m,N}^{(R_1,\ldots,R_m)^{(k+1)}} = \mu_{1:1,N}^{(R_1^\sigma - 1)^{(k+1)}} = \frac{(k+1)}{N}\mu_{1:1,N}^{(N-1)^{(k)}} - \left(\frac{1}{P} - 1\right)\mu_{1:1,N-1}^{(N-2)^{(k+1)}}.$$

3. *For $2 \leq i \leq m-1$,*

$$\begin{aligned}
\mu_{i:m,N}^{(R_1,\ldots,R_m)^{(k+1)}} &= \mu_{i:i,N}^{(R_1,\ldots,R_{i-1},R_i^\sigma - 1)^{(k+1)}} \\
&= \mu_{i-1:i-1,N}^{(R_1,\ldots,R_{i-2},R_{i-1}^\sigma - 1)^{(k+1)}} + \frac{1}{R_i^\sigma}\Bigg[(k+1)\mu_{i:i,N}^{(R_1,\ldots,R_{i-1},R_i^\sigma - 1)^{(k)}} \\
&\quad + \left(\frac{1}{P} - 1\right)\Bigg[\frac{c(N,i-1)}{c(N-1,i-2)}\mu_{i-1:i-1,N-1}^{(R_1,\ldots,R_{i-2},R_{i-1}^\sigma - 2)^{(k+1)}} \\
&\quad - \frac{c(N,i-1)}{c(N-1,i-1)}(R_i^\sigma - 1)\mu_{i:i,N-1}^{(R_1,\ldots,R_{i-1},R_i^\sigma - 2)^{(k+1)}}\Bigg]\Bigg].
\end{aligned}$$

*Proof.* The first recurrence relation is taken from Aggarwala and Balakrishnan (1996,Theorem 5.1). The second and third one are an application of Theorem 8 and of Theorem 5.2 and 5.7, respectively, of Aggarwala and Balakrishnan (1996). Observe that in item 3 of Theorem 12 we have $R_i^\sigma - 2 \geq m - i - 1 \geq 0$, $2 \leq i \leq m-1$. Hence, all occurring censoring schemes are well defined. The remaining case $i = m$ is given in Aggarwala and Balakrishnan (1996). □

The case of product moments is considered in the following theorem:

**Theorem 13.** *For $1 \leq i < j \leq m-1$, $m \leq N-1$:*

$$\begin{aligned}
\mu_{i,j:m,N}^{(R_1,\ldots,R_m)} &= \mu_{i,j-1:j-1,N}^{(R_1,\ldots,R_{j-2},R_{j-1}^\sigma - 1)} \\
&\quad + \frac{1}{R_j^\sigma}\Bigg[\mu_{i:i,N}^{(R_1,\ldots,R_{i-1},R_i^\sigma - 1)} - \left(\frac{1}{P} - 1\right)\Bigg[-\frac{c(N,j-1)}{c(N-1,j-2)}\mu_{i,j-1:j-1,N-1}^{(R_1,\ldots,R_{j-2},R_{j-1}^\sigma - 2)} \\
&\quad + \frac{c(N,j-1)}{c(N-1,j-1)}(R_j^\sigma - 1)\ \mu_{i,j:j,N-1}^{(R_1,\ldots,R_{j-1},R_j^\sigma - 2)}\Bigg]\Bigg].
\end{aligned}$$

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