Homework 6

Volatility Estimation Via Hidden Markov Chain, Due May 12

The file `XOM5YrsDaily.txt` contains daily returns in percent for Exxon Mobil over the five year (1258 trading day) period 1/3/2012 – 12/30/2016. In this project you will estimate historical XOM price volatility over this period using a hidden Markov chain approach. The underlying (imperfectly observed) Markov chain is XOM's time-dependent volatility, σ_t. We assume it is undergoing an exponentiated random walk: $\sigma_t = \sigma_0 \exp(\alpha W_t)$. Here $0 \leq t \leq 1258$ is elapsed time in days and $(W_t : 0 \leq t \leq 1258)$ is a simple symmetric random walk on $\mathbb{Z}$ starting at $W_0 = 0$. The volatility process is not observed but we do observe a realization of XOM's daily returns (R_t), where $R_t \sim \text{Normal}\,(0, \sigma_t^2)$. Call this realization $(r_t : 1 \leq t \leq 1258)$.

Preliminaries. Let us label the states of the MC according to the value of the random walk, so $S = \{k : -1258 \leq k \leq 1258\}$. Note that at time t the only states with positive probability are

$$S_t \;=\; \{k : -t \leq k \leq \text{t in steps of } 2\},$$

e.g., $S_2 = \{-2, 0, 2\}$. For $k \in S_t$, let $s_k = \sigma_0 \exp(\alpha k)$, the daily volatility when the walk is in state k. Let D_t be the return data up to and including time t:

$$D_t \;=\; \{R_1 = r_1, \ldots, R_t = r_t\} \;=\; D_{t-1} \cap \{R_t = r_t\},$$

with $D_0 = \Omega$ (the entire sample space). For any events E and D, let $P_D[E] = P[E \,|\, D]$ so, e.g., $P_{D_0}[E] = P[E \,|\, \Omega] = P[E]$.

Estimating Volatility. To estimate the volatility for day t we may legitimately only use the data D_{t-1}. Suppose that, with our Bayesian prowess, we have calculated the numbers $p_k = P_{D_{t-1}}[W_{t-1} = k]$ for $k \in S_{t-1}$. (We have this for $t = 1$, where $p_0 = 1$ as $W_0 = 0$.) Since from time $t-1$ to time t the walk either goes up one or down one, we get that the conditional distribution of the location of the walk at time t given the data D_{t-1} is given by

$$\begin{aligned} P_{D_{t-1}}[W_t = k] \;&=\; P_{D_{t-1}}[W_{t-1} = k-1] \cdot \frac{1}{2} + P_{D_{t-1}}[W_{t-1} = k+1] \cdot \frac{1}{2} \\ &=\; \frac{p_{k-1} + p_{k+1}}{2}. \qquad (*) \end{aligned}$$

Using MAP estimation, put $\hat{k} = \text{argmax}\,(P_{D_{t-1}}[W_t = k])$, and take $\hat{\sigma}_t = s_{\hat{k}}$.

Bayesian Updating. Next we seek to update the p_ks, which reflect data through time $t-1$, to reflect the data at time t. To this end, for $k \in S_t$, put

$$\begin{aligned} \widetilde{p}_k \;=\; P_{D_t}[W_t = k] \;&=\; P_{D_{t-1}}[W_t = k \,|\, R_t = r_t] \\ &=\; P_{D_{t-1}}[R_t = r_t \,|\, W_t = k] \cdot \frac{P_{D_{t-1}}[W_t = k]}{P_{D_{t-1}}[R_t = r_t]}. \end{aligned}$$

We first calculate

$$g_k \;=\; P_{D_{t-1}}[R_t = r_t \,|\, W_t = k] \cdot P_{D_{t-1}}[W_t = k],$$

as described below, then let $Z = P_{D_{t-1}}[R_t = r_t] = \sum_k g_k$ and put each $\widetilde{p}_k = g_k/Z$.

Now $P_{D_{t-1}}[W_t = k]$ is given by $(*)$, and given $\{W_t = k\}$ we have $\sigma_t = s_k$ and hence $R_t \sim \text{Normal}\,(0, s_k^2)$. Then we have

$$P_{D_{t-1}}[R_t = r_t \,|\, W_t = k] \;=\; \frac{1}{\sqrt{2\pi} s_k} \exp(-r_t^2/2s_k^2).$$

(This is a density rather than a probability, but it gives the right answer; why?) The numbers $\widetilde{p}_k$ then become the new p_k when we advance to updating for the data at time $t+1$. In this manner we may compute the numbers $(p_k : k \in S_{t-1})$ for $t = 1$, then 2, then 3, and do forth to 1258.

Your mission is to implement this model. The model has two free parameters, σ_0 and α, which must be estimated using the data. I have done that for you — take $\sigma_0 = 0.25$ and $\alpha = 0.03$. The code `HiddenMC.cpp` will get you started. As it stands, this code reads in the time series $(r_t : 1 \leq t \leq 1258)$ from the data file `XOM5YrsDaily.txt`. Generate a (normal) histogram of the standardized returns $r_t/\hat{\sigma}_t$ and a time series showing the annualized volatility $\sqrt{252}\hat{\sigma}_t$ over time. Compare your results to the "EWMA" approach from Chapter 16.