Homework 4 — Due April 16
Portfolio Optimization Via Metropolis

Ninety-seven stocks from the S&P100 were present in the index over the five year period 2012 - 2016. The accompanying file `S&P50Covariances.txt` contains the covariance matrix $\mathbf{V}$ for the 60 monthly returns of 50 of these stocks over this time period. Use this data to solve the problems below. Your starter code for this project, `Metropolis.cpp`, reads in the covariance matrix.

Use the Metropolis algorithm to solve the problems below. These problems involve finding a minimum variance portfolio subject to certain constraints. (For problems 1 and 2, there are standard quadratic programming algorithms to solve the problem. But use Metropolis, it works well.) To implement Metropolis, you must: (1) specify the state space S; (2) specify the neighbors of each state; (3) specify the energy $E(x)$ of each state $x \in S$; and (4) tinker with the temperature parameter T until the algorithm produces good results.

1 (No constraints). Find the minimum variance portfolio when both long and short positions are allowed for individual stocks. If the total portfolio is worth \$100, find the allocation to each individual stock to the nearest 0.01 dollar (1 penny). Hint: with the right definition of "neighbor" each state will have 50×49 neighboring states. Compare your portfolio to the actual minimum variance portfolio, given by $\frac{1}{c} \cdot \mathbf{V}^{-1}\mathbf{e}$, and its variance, where $\mathbf{e}_{50\times 1} = (1, 1, \ldots, 1)^T$ and $c = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}$. (Should be close.)

2 (No short positions). Find the minimum variance portfolio when short positions are not allowed for individual stocks. If the total portfolio is worth \$100, find the allocation to each individual stock to the nearest 0.01 dollar. Hint: if x is a portfolio with a short position let $\mathrm{Var}\,(x) = 1000$, then no such portfolio will be selected as optimal.

3 ('Simple' portfolios only). Let us call a \$100 portfolio of these 50 stocks 'simple' if all the stocks present in the portfolio have equal weight (e.g., 8 stocks with weight \$12.5 each). This is non-standard terminology; don't use it in an interview! Since a simple portfolio corresponds to a subset of the 50 stocks, there are 2^{50} simple portfolios of these stocks. This is a really big number — 1,125,899,906,842,624 to be precise. Among all the simple portfolios, use Metropolis (I know of no other way) to find the one that has the least return variance. As a technical matter, the empty portfolio $x_\emptyset$ with no stocks in it is a simple portfolio. Return variance makes no sense for this portfolio. (Even the notion of return makes no sense.) However, we wish to exclude it anyhow because we want our solution to have at least one stock in it! For the empty portfolio, just take $\mathrm{Var}\,(x_\emptyset) = 1000$. Hint: with the right definition of "neighbor" (which is different from problems 1 and 2) each state will have 50 neighboring states.

4 (Non-ground stable state). A state x is *stable* if $E(y) > E(x)$ for all neighbors y of x. In the context of problem 3, exhibit a stable state x whose energy $E(x)$ is not minimal, i.e., with $E(x) > E_0$. To do this, run your algorithm with zero-temperature dynamics and see where it stops improving. Verify that the resulting state is stable by computing the energy of its 50 neighbors.