The formal group ring and real finite reflection groups

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Root systems

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$$s_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha, \quad v \in V.$$

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Definition

A *root system* Σ in V is a finite set of nonzero vectors in V satisfying the conditions:

1. $\Sigma \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Sigma$;
2. $s_\alpha(\Sigma) = \Sigma$ for all $\alpha \in \Sigma$;
3. The roots $\alpha \in \Sigma$ generate V.

Note: given $\alpha, \beta \in \Sigma$, we do not require that $s_\alpha(\beta) = \beta - n\alpha$ for some $n \in \mathbb{Z}$.

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Note: given $\alpha, \beta \in \Sigma$, we do not require that $s_\alpha(\beta) = \beta - n\alpha$ for some $n \in \mathbb{Z}$. The group W generated by the reflections s_α, $\alpha \in \Sigma$, is the *real finite reflection group* of Σ.

Definition

A subset $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of Σ is a *simple system* of Σ if it is an $\mathbb{R}$-basis of V, and if every root $\alpha \in \Sigma$ can be written as an $\mathbb{R}$-linear combination of elements in Δ with all coefficients nonnpositive or all coefficients nonnegative. We call s_{α_i} a *simple reflection*.

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Can we find a root system Σ in V whose real finite reflection group is W, and a simple system Δ of Σ, such the following property holds?

Let $\alpha \in \Sigma$ be any root. By definition of Δ, there exist unique elements $c_i^\alpha \in \mathbb{R}$ such that $\alpha = c_1^\alpha \alpha_1 + \cdots + c_n^\alpha \alpha_n$. Let $\mathcal{R}$ be the subring of $\mathbb{R}$ generated by the elements c_i^α over all $i = 1, \ldots, n$ and $\alpha \in \Sigma$

Property

The subring $\mathcal{R}$ a free finitely-generated $\mathbb{Z}$-module with a power basis (i.e., a basis of the form $\{1, \beta, \beta^2, \ldots, \beta^{l-1}\}$, $l \geqslant 1$, where $\beta \in \mathcal{R}$).

One can show that $\mathcal{R}$ is the unital subring of $\mathbb{R}$ generated by the elements $\alpha_i^\vee(\alpha_j) := 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ over all pairs of simple roots $\alpha_i, \alpha_j \in \Delta$.

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If $W = I_2(m)$ is a dihedral group of order $2m$, $m \geqslant 3$, then we can choose (Σ, Δ) such that $\mathcal{R} = \mathbb{Z}[2\cos\left(\frac{\pi}{m}\right)]$.

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Example

If $W = H_3$ or $W = H_4$, then we can choose (Σ, Δ) such that $\mathcal{R} = \mathbb{Z}[\tau]$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden section. It is a root of $x^2 - x - 1$.

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Definition

Fix a power basis $\{e_i\}$ of $\mathcal{R}$. Let Λ be the $\mathcal{R}$-module generated Σ. Then Λ is a free finitely-generated $\mathbb{Z}$-module with basis $\{e_i\alpha_j\}$.

Formal group laws

Definition

A one-dimensional commutative formal group law (FGL) (R, F) over a commutative unital ring R is a power series $F(u, v) \in R[\![u, v]\!]$ satisfying the following axioms:

1. $F(u, 0) = F(0, u) = u \in R[\![u]\!]$;
2. $F(u, v) = F(v, u)$;
3. $F(u, F(v, w)) = F(F(u, v), w) \in R[\![u, v, w]\!]$.

A *morphism* $f\colon (R, F) \to (R, F')$ of FGLs over R is a power series $f(u) \in R[\![u]\!]$ such that $f(F(u, v)) = F'(f(u), f(v))$ and $f(0) = 0$.

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Let $(\mathbb{C}, F)$ be an FGL. Suppose $(\mathbb{C}, F_a)$ is the additive formal group law over $\mathbb{C}$, i.e., $F_a(u, v) = u + v$. There are isomorphisms of FGLs $\log_F : (\mathbb{C}, F) \to (\mathbb{C}, F_a)$ and $\exp_F : (\mathbb{C}, F_a) \to (\mathbb{C}, F)$ called the *logarithm* and *exponential* of $(\mathbb{C}, F)$, i.e., $\exp_F(\log_F(u)) = \log_F(\exp_F(u)) = u$.

Definition

Let R be a subring of $\mathbb{C}$ containing the coefficients in the series $F(u, v)$ and the coefficients in the logarithm and exponential of $(\mathbb{C}, F)$. We call R an *ample ring* with respect to $(\mathbb{C}, F)$. Thus, we can view (R, F) as a formal group law with a logarithm and exponetial.

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Example

The *additive* FGL (R, F_a) over R is $F_a(x, y) = x + y$.
If R is an ample ring with respect to $(\mathbb{C}, F_a)$, then the logarithm of (R, F_a) is $\log_{F_a}(x) = x$, and the exponential of (R, F_a) is $\exp_{F_a}(x) = x$.

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Example

The *multiplicative* FGL (R, F_m) over R is $F_m(x,y) = x + y + xy$.
If R is an ample ring with respect to $(\mathbb{C}, F_m)$, then the logarithm and exponential series of (R, F_m) are given by the formulas

$$\log_{F_m}(x) = \log(1+x) = \sum_{i \geqslant 1} (-1)^{i-1} \frac{x^i}{i}; \quad \exp_{F_m}(x) = \exp(x) - 1 = \sum_{i \geqslant 1} \frac{x^i}{i!}.$$

Formal group ring

Assumption

If Σ is *noncrystallographic*, then R is an ample ring with respect to an FGL $(\mathbb{C}, F)$, such that R contains $\mathcal{R}$. If Σ is *crystallographic*, then R is a subring of $\mathbb{C}$, and (R, F) is an FGL.

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Definition

Set $R[\![x_\Lambda]\!] := R[\![x_\lambda]\!]_{\lambda \in \Lambda}$, and let

$$f_{i,j} = \begin{cases} e_i \log_F(x_{\alpha_j}) - \log_F(x_{e_i \alpha_j}), & \Sigma \text{ noncrystallographic;} \\ 0, & \Sigma \text{ crystallographic.} \end{cases}$$

Let $\mathcal{J}_F$ be the closure of the ideal in $R[\![x_\Lambda]\!]$ generated by

$$x_0 \quad \text{and} \quad x_{\lambda_1+\lambda_2} - (x_{\lambda_1} +_F x_{\lambda_2}) \quad \text{and} \quad f_{i,j}; \quad \lambda_1, \lambda_2 \in \Lambda; e_i \in B; \alpha_j \in \Delta.$$

The quotient $R[\![\Lambda]\!]_F := R[\![x_\Lambda]\!]_F / \mathcal{J}_F$ is the *formal group ring*.

Example

Let $S^i_R(\Lambda)$ be the i-th symmetric power of the R-module $R \otimes_{\mathcal{R}} \Lambda$, and set $(S^*_R(\Lambda))^\wedge := \prod_{i=0}^{\infty} S^i_R(\Lambda)$. There is an R-algebra isomorphism

$$R[\![\Lambda]\!]_{F_a} \simeq (S^*_R(\Lambda))^\wedge.$$

Example

Let $S_R^i(\Lambda)$ be the i-th symmetric power of the R-module $R \otimes_{\mathcal{R}} \Lambda$, and set $(S_R^*(\Lambda))^\wedge := \prod_{i=0}^{\infty} S_R^i(\Lambda)$. There is an R-algebra isomorphism

$$R[\![\Lambda]\!]_{F_a} \simeq (S_R^*(\Lambda))^\wedge.$$

Proposition

The following properties hold in $R[\![\Lambda]\!]_F$:

1. There is a well-defined W-action on $R[\![\Lambda]\!]_F$ given by $w(x_\lambda) = x_{w(\lambda)}$.
2. $R[\![\Lambda]\!]_F$ is an integral domain.
3. x_{α_j} divides $x_{e_i\alpha_j}$ in $R[\![\Lambda]\!]_F$ for all $e_i \in B$ and $\alpha_j \in \Delta$.

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3. x_{α_j} divides $x_{e_i\alpha_j}$ in $R[\![\Lambda]\!]_F$ for all $e_i \in B$ and $\alpha_j \in \Delta$.

Corollary

For any $u \in R[\![\Lambda]\!]_F$ and root $\alpha \in \Sigma$, the element $u - s_\alpha(u)$ is divisible by x_α in $R[\![\Lambda]\!]_F$.

Formal Demazure operators

Definition

For each root $\alpha \in \Sigma$, we define a *formal Demazure operator* Δ_α on $R[\![\Lambda]\!]_F$ by the formula

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{x_\alpha}, \quad u \in R[\![\Lambda]\!]_F.$$

We set $\Delta_i := \Delta_{\alpha_i}$ for $\alpha_i \in \Delta$.

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Definition

Let $\mathcal{D}_{(R,F)}(\Lambda)$ be the subalgebra of R-linear endomorphisms of $R[\![\Lambda]\!]_F$ generated by the formal Demazure operators Δ_α for all roots α, and by multiplication by elements of $R[\![\Lambda]\!]_F$.

Given $q \in R[\![\Lambda]\!]_F$, let q^* be the corresponding multiplication operator in $\mathcal{D}_F$. Given $w \in W$, let $w = s_{i_1} \cdots s_{i_r}$ be a reduce decomposition of w. We call $I_w := (\alpha_{i_1}, \ldots, \alpha_{i_r})$ a reduced sequence of w. Fix reduced sequences $\{I_w\}_{w \in W}$, and set $\Delta_{I_w} := \Delta_{i_1} \circ \cdots \circ \Delta_{i_r}$, if $I_w := (\alpha_{i_1}, \ldots, \alpha_{i_r})$.

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Theorem

The elements $q^* \in R[\![\Lambda]\!]_F$ and the formal Demazure operators $\Delta_i = \Delta_{\alpha_i}$, where $\alpha_i \in \Delta$, satisfy the following relations:

1. $\Delta_i \circ q^* = \Delta_i(q) + (s_i(q))^* \circ \Delta_i$;
2. $\Delta_i^2 = \kappa_i^* \circ \Delta_i$, where $\kappa_i := \frac{1}{x_{\alpha_i}} + \frac{1}{x_{-\alpha_i}} \in R[\![\Lambda]\!]_F$;
3. $\underbrace{\Delta_i \circ \Delta_j \circ \Delta_i \cdots}_{m_{i,j}\text{-times}} - \underbrace{\Delta_j \circ \Delta_i \circ \Delta_j \cdots}_{m_{i,j}\text{-times}} = \sum_{w < w_0^{i,j}} \left(\kappa_{i,j}^w\right)^* \circ \Delta_{I_w}, \quad \kappa_{i,j}^w \in R[\![\Lambda]\!]_F.$

Here $w_0^{i,j} := \underbrace{s_i s_j s_i \cdots}_{m_{i,j}\text{-times}}$, and the ordering $<$ is with respect to the Bruhat order on W. These relations, together with the ring law in $R[\![\Lambda]\!]_F$ and the fact that the Δ_i are R-linear form a complete set of relations in $\mathcal{D}_F$.

Set $y_i := \frac{1}{x_{\alpha_i}}$ and $s^{(k)}_{i,j,\dots} := \underbrace{s_i s_j s_i \cdots}_{k\text{-times}}$ and $\Delta^{(k)}_{i,j,\dots} := \underbrace{\Delta_i \circ \Delta_j \circ \Delta_i \cdots}_{k\text{-times}}$, for $k \geqslant 0$.

For $k_1 \leqslant k_2$, define the operator

$$S^{(k_1,k_2)}_{i,j}(u) := s^{(k_1)}_{i,j,\dots}(u) + s^{(k_1+1)}_{i,j,\dots}(u) + \cdots + s^{(k_2)}_{i,j,\dots}(u).$$

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Corollary

The difference $\Delta^{(m_{i,j})}_{i,j,\dots} - \Delta^{(m_{i,j})}_{j,i,\dots}$ can be written as a linear combination

$$\Delta^{(m_{i,j})}_{i,j,\dots} - \Delta^{(m_{i,j})}_{j,i,\dots} = \sum_{k=1}^{m_{i,j}-2} \left(\kappa^{(k)}_{i,j} \Delta^{(k)}_{i,j,\dots} - \kappa^{(k)}_{j,i} \Delta^{(k)}_{j,i,\dots}\right), \quad \kappa^{(k)}_{i,j} \in R[\![\Lambda]\!]_F.$$

For odd $m_{i,j}$:

$$\kappa^{(m_{i,j}-2)}_{j,i} = S^{(0,m_{i,j}-2)}_{j,i}(y_i y_j) - y_i s^{(m_{i,j}-2)}_{j,i,\dots}(y_i);$$

$$\kappa^{(m_{i,j}-3)}_{i,j} = -y_j\{s_i(y_i y_j) + \left[S^{(2,m_{i,j}-3)}_{i,j} - S^{(2,m_{i,j}-3)}_{j,i}\right](y_i y_j) \\ - s^{(m_{i,j}-2)}_{j,i,\dots}(y_i y_j) + y_i s^{(m_{i,j}-2)}_{j,i,\dots}(y_i) - s^{(m_{i,j}-3)}_{i,j,\dots}(y_i) s^{(m_{i,j}-2)}_{i,j,\dots}(y_j)\}.$$