Formal Affine Demazure Algebras

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A Bit of Motivation

$$X = \{\text{Smooth quasi-projective variety over field } k\}.$$
$$k(X) = \{\text{Function field of } X\}.$$
$$W = \{\text{Irreducible closed subvariety of } X \text{ of codimension } (r-1) \text{ in } X\}.$$
$$D = \{\text{Irreducible closed subvariety of } W \text{ of codimension 1 in } W\}.$$

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Let $f \in k(X)^{\times}$. "Restricting" f to D, possible to write $f = ut^n$, u unit, $n \in \mathbb{Z}$.

Homomorphism. $\quad \mathrm{ord}_D : k(X)^{\times} \to \mathbb{Z}, \quad f = ut^n \mapsto n.$

Finite formal sum. $\quad \mathrm{div}(f) := \sum_D \mathrm{ord}_D(f) D.$

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Finite formal sum. $\quad \mathrm{div}(f) := \sum_D \mathrm{ord}_D(f) D.$

$$Z^r := \{\text{Free abelian group on irreducible closed codim. } r \text{ subvarieties in } X\}.$$

$$B^r := \{\text{Subgroup of } Z^r \text{ generated by elements of the form } \mathrm{div}(f)\}.$$

$$CH^r(X) := Z^r/B^r.$$

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CH^* can be viewed as a functor:
$\{$Category of smooth quasi-projective varieties over $k\} \to$
$\{$Category of graded commutative rings with unit$\}$.

It satisfies various "functorial" properties:

1. *(Flat pullbacks)*: Let $f : X \to Y$ be a *flat* morphism of varieties. Let $W \subseteq Y$ be a smooth closed subvariety. Define the *pullback* $f^* : CH^i(Y) \to CH^i(X)$ by $f^*([W]) = [f^{-1}(W)]$. By flatness, it preserves the graded degree i.

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2. *(Proper push-forwards)*: Let $f : X \to Y$ be a *proper* morphism of varieties, and suppose $Z \subseteq X$ is a smooth closed subvariety. Then there is a *push-forward* $f_* : CH^{\dim X - i}(X) \to CH^{\dim Y - i}(Y)$ given by

$$f_*([Z]) = \begin{cases} 0, & \text{if } \dim f(Z) < \dim Z, \\ [k(Z) : k(f(Z))][f(Z)], & \text{if } \dim f(Z) = \dim Z \end{cases}.$$

1. *(Localization Exact Sequence)*: Let $i : Y \subseteq X$ be a smooth closed subvariety, and let $j : U = X \setminus Y \subseteq X$ be its open complement in X. Then

$$CH^{\dim Y - r}(Y) \xrightarrow{i_*} CH^{\dim X - r}(X) \xrightarrow{j^*} CH^{\dim X - r}(U) \longrightarrow 0$$

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2. (Chern classes): Let $VB(X)$ be set of isomorphism classes of vector bundles over X (there is a map $[U] \to [V]$ in $VB(X)$ whenever there is a morphism $V \to U$). The *Chern class* of X with respect to CH^* is a collection of set maps $c_i^{CH^*} : VB(X) \to CH^*(X)$ satisfying naturality and various properties, where we view $CH^*(X)$ as a set here.

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Let L_1 and L_2 be two line bundles over X. Then

$$c_1^{CH^*}(L_1 \otimes L_2) = c_1^{CH^*}(L_1) + c_1^{CH^*}(L_2).$$

Oriented Cohomology Theories: Definition

$$\mathcal{V} = \{\text{the category of smooth quasi-projective varieties over } k\}.$$
$$\mathcal{R} = \{\text{the category of graded commutative unital rings}\}.$$
$$h^* : \mathcal{V} \to \mathcal{R} = \{\text{contravariant functor satisfying various axioms}\}.$$

Note, given $f : X \to Y$ in $\mathcal{V}$, the existence of a pullback $f^* : h^i(Y) \to h^i(X)$ is implied by the contravariance of h^* (i.e., $f^* := h^*(f)$).

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1. Give a *projective* morphism $f : X \to Y$ in $\mathcal{V}$, there is a homomorphism of graded $h^*(Y)$-modules

$$f_* : h^{\mathrm{dimX-j}}(X) \to h^{\mathrm{dim}Y-j}(Y)$$

called the *push-forward*. Here $h^*(X)$ is an $h^*(Y)$-module via the pullback f^*.

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2. Let VB be the contravariant "vector bundles" functor, so $VB(X)$ is the set of isomorphism classes of vector bundles over X. There exist natural transformations $c_i^{h^*} : VB \to h^*$ called *Chern classes* satisfying various propertes, where h^* is viewed as a set-theoretic functor here.

Recall: Given line bundles L_1 and L_2 over X, the Chern classes for Chow theory satisfy

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Definition

A one-dimensional commutative formal group law over commutative ring R is a power series $F(u, v) = \sum_{i,j \geqslant 0} a_{i,j} u^i v^j \in R[\![u, v]\!]$ satisfying:

1. $F(0, u) = F(u, 0) = u \in R[\![u]\!]$,
2. $F(u, v) = F(v, u)$, and
3. $F(u, F(v, w)) = F(F(u, v), w)$.

Note:

(1) implies that $a_{0,0} = 0$, and $a_{1,0} = a_{0,1} = 1$.

(2) implies that $a_{i,j} = a_{j,i}$ for all i, j.

Example

The *additive* formal group law over R is

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The *multiplicative-periodic* formal group law over R is

$$F(u,v) = u + v - \beta uv,$$

where $\beta \in R^{\times}$.

Example

Let E be the elliptic curve

$$E: \quad v = u^3 + a_1 uv + a_2 u^2 v + a_3 v^2 + a_4 uv^2 + a_6 v^3.$$

The *elliptic* formal group law over $R = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ is

$$F(u, v) = u + v - a_1 uv - a_2(u^2 v + uv^2) + O(4).$$

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Let $\mathbb{L}$ be the *Lazard* ring, i.e., the commutative ring on generators $a_{i,j}$, $i, j \in \mathbb{N}_{>0}$, subject to the relations imposed by the axioms of the formal group law. Then

$$F_U(u, v) = u + v + \sum_{i,j \geqslant 1} a_{i,j}u^iv^j$$

is the *universal* formal group law.

F_U is *universal* in the following sense:

Lemma

Let F be an arbitrary formal group law over R. Then there is a unique ring homomorphism

$$f : \mathbb{L} \to R$$

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Quillen's formula: Let L_1 and L_2 be two line bundles over X. Let h^* be any oriented cohomology theory. Then

$$c_1^{h^*}(L_1 \otimes L_2) = F(c_1^{h^*}(L_1), c_1^{h^*}(L_2)),$$

where F is a formal group law over $R = h^*(\mathrm{pt})$, $\mathrm{pt} := Spec(k)$.

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Algebraic K-theory corresponds to the *multiplicative-period* formal group law over $\mathbb{Z}[\beta, \beta^{-1}]$.

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Answer: Yes. It is called *algebraic cobordism* and denoted Ω^*.

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Theorem (Levine-Morel (2007))

Let h^* be an oriented cohomology theory over k. Then there is a unique morphism of oriented cohomology theories $\Omega^* \to h^*$ (i.e., a natural transformation that commutes with push-forwards), and the formal group law corresponding to Ω^* is the universal formal group law F_U over the Lazard ring $\mathbb{L}$.

Question: Given a formal group law F, can we construct an oriented cohomology theory?

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Answer: Yes. Let F be a formal group law over R. Then there is an oriented cohomology theory

$$h^*(-)_F = \Omega^*(-) \otimes_{\mathbb{L}} R,$$

where R is an $\mathbb{L}$-module via the universal morphism $\mathbb{L} \to R$.
A theory constructed in this way is called a *free oriented cohomology theory*.

Detour

Definition

An *algebraic group* G over k is an object that is simultaneously a group and an algebraic variety over k, such that the group operations

$$\mu : G \times G \to G, \quad \mu(x,y) = xy, \quad \text{and} \quad \iota : G \to G, \quad \iota(x) = x^{-1},$$

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Example

The *multiplicative group* $\mathbf{G_m}$ is the affine open $K^* \subseteq \mathbb{A}^1_k$ with group operations $\mu(x,y) = xy$ and $\mu(x) = x^{-1}$.

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There is a finite subset of $\Lambda(G)$ called a *root system* and denoted Φ, which corresponds to the *adjoint representation* of G. The lattice generated by Φ is called the *root lattice* Λ_r and is contained in $\Lambda(G)$.

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Theorem

Let G be split connected reductive algebraic group over a field k containing split maximal torus T. Then G is completely determined by the quadruple $(\Lambda(T), \Phi, \Lambda(T)^\vee, \Phi^\vee)$ called the *root datum* of G, where $\Lambda(G)^\vee$ and $\Phi^\vee$ are the *duals* of $\Lambda(G)$ and Φ, respectively.

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The *formal group algebra* is the (complete) quotient:

$$\mathcal{S} := R[\![x_\Lambda]\!]/\mathcal{J}_F.$$

Note: This definition of is purely algebraic, and independent of all axioms of the oriented cohomology theory.

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Set $\mathcal{Q}_W := \mathcal{Q} \otimes_R R[W]$ as an R-module, with multiplication

$$(q\delta_w) \cdot (q'\delta_{w'}) = qw(q')\delta_{ww'}, \quad w, w' \in W \text{ and } q, q' \in \mathcal{Q},$$

and extended by linearity. Here, δ_w is the element in $R[W]$ corresponding to $w \in W$, and $\mathcal{Q}_W$ is the *localized twisted formal group algebra*.

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Definition

Let $\alpha \in \Phi$. Define a *Demazure element*

$$X_\alpha = \frac{1}{x_\alpha}(1 - \delta_{s_\alpha}).$$

The *formal affine Demazure algebra* $\mathbf{D}_F$ is the R-subalgebra of $\mathcal{Q}_W$ generated by $\mathcal{S}$ and the Demazure elements X_α, $\alpha \in \Phi$.

For $w \in W$, fix a reduced decomposition $w = s_{\alpha_{i_1}} \circ \cdots \circ s_{\alpha_{i_m}}$ in simple roots. Set $I_w := (\alpha_{i_1}, \ldots, \alpha_{i_m})$. Define

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Theorem (Calmes-Zainoulline-Zhong (2016))

The formal affine Demazure algebra $\mathbf{D}_F$ is a free $\mathbb{S}$-module with basis X_{I_w}, $w \in W$.

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Research Question: Can this theorem be extended to arbitrary Coxeter groups (i.e., to arbitrary complex reflection groups)?

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Work in Progress: Extend the theorem to all real finite reflection groups.

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Theorem

For each character $\lambda \in \Lambda$ of $T \subseteq G$, there is a canonical line bundle $L(\lambda)$ over the projective variety G/B.

We would like to compute the oriented cohomology $h^*(G)$. To do this, we will use knowledge of the oriented cohomology $h^*(G/B)$.

Dual of the Formal Affine Demazure Algebra

There is a *coproduct* on the formal affine Demazure algebra $\mathbf{D}_F$, which induces a *commutative product* on the dual $\mathbf{D}_F^* := \mathrm{Hom}(\mathbf{D}_F, \mathcal{S})$.

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Let I be the ideal in $\mathbf{D}_F^*$ generated by multiplication by elements x_λ, $\lambda \in \Lambda$. Let $\pi : \mathbf{D}_F^* \to \mathbf{D}_F^*/I$ be the natural projection.

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There is a *coproduct* on the formal affine Demazure algebra $\mathbf{D}_F$, which induces a *commutative product* on the dual $\mathbf{D}_F^* := \mathrm{Hom}(\mathbf{D}_F, \mathcal{S})$.

There is an R-algebra map $c_{\mathcal{S}} : \mathcal{S} \to \mathbf{D}_F^*$ defined by $x_\lambda \mapsto \mathrm{ev}_{x_\lambda}$.

Let h^* be the oriented cohomology theory corresponding to F. There is another R-algebra map $\mathfrak{c} : \mathcal{S} \to h^*(G/B)$ defined by $x_\lambda \mapsto c_1^{h^*}(L(\lambda))$.

Let I be the ideal in $\mathbf{D}_F^*$ generated by multiplication by elements x_λ, $\lambda \in \Lambda$. Let $\pi : \mathbf{D}_F^* \to \mathbf{D}_F^*/I$ be the natural projection.

Theorem (Calmes-Petrov-Zainoulline (2013))

There is an R-algebra isomorphism $\theta : h^*(G/B) \to \mathbf{D}_F^*/I$, such that $\mathfrak{c} = \theta \circ \pi \circ c_{\mathcal{S}}$.

Theorem (Corollary of Gille-Zainoulline (2012))

There are R-algebra isomorphisms

$$h^*(G) \simeq h^*(G/B)/(\mathrm{im}\mathfrak{c}) \simeq \mathbf{D}^*/(I + (\mathrm{im}(c_{\mathfrak{S}}))).$$

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Research Problem: Compute $\mathbf{D}^*/(I + (\mathrm{im}(c_{\mathcal{S}})))$ for real finite reflection groups (gives *virtual* cohomology: no algebraic groups can actually be constructed from real root systems of types H and $I_2(m)$, $m \geqslant 7$).

Progress: Given an *adjoint* algebraic group, i.e., the root lattice Λ_r equals the character lattice Λ, of *rank* two, i.e., the root system generates a two-dimensional vector space, we have:

$$\Omega^*(A_1 \times A_1) = \mathbb{L}[x, y]/(2x, 2y)$$

$$\Omega^*(A_2) = \mathbb{L}[x]/(3x, x^3)$$

$$\Omega^*(B_2) = \mathbb{L}[x]/(x^4, 2x^2, 2x - a_{11}x^2)$$

$$\Omega^*(G_2) = \mathbb{L}[x]/(x^2, 2x, a_{11}x).$$