Generating the Ring of Twisted Differential Operators of Complex Reflection Group $I_2(5)$

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Outline:

1. Reflection Groups, Root Systems, Coxeter Groups
2. Reflection Group $I_2(5)$, Twisted Differential Operators
3. Ring of Twisted Differential Operators of $I_2(5)$, Generating this Ring

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Reflection Group - Definition and Example

Definition

Let V be a vector space. Let t, α be vectors in V. Let s_α denote a linear operator. We call s_α a **Reflection**, and s_α acts on t in the following way:

$$s_\alpha t = t - 2\frac{(t,\alpha)}{(\alpha,\alpha)}\alpha$$

Definition

A group that is generated by reflections is called a **Reflection Group**.

Example

Every **Dihedral Group** is a Reflection Group.
Every Dihedral Group is generated by reflections and rotations of a regular polygon that preserve its symmetry.
Rotations can be obtained from reflections.

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Let V be a vector space. A **Root System** Φ of V is a set of vectors of V that satisfy two axioms. For all $a \in \Phi$:

1. $\Phi \cap \mathbf{c}a = \{a, -a\}, \mathbf{c} \in \mathbf{R}$
2. $s_a \Phi = \Phi$

Every root system Φ has an associated reflection group W.

Example

Consider the vector space $V = R^2$. One root system of V is the set of vectors
$\Phi = \{\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)\}$
The reflection group associated with Φ is the Dihedral Group of order $m = 4$.

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Positive, Negative, and Simple Systems

Definition

Let V be an vector space. Let $\beta = \{v_1, ..., v_n\}$ be an ordered basis of V. Let Φ be a Root System of V.

A **Positive System** Π of Φ with respect to β is the set of roots $\Pi := \{r \in \Phi | r = a_1 v_1 + ... + a_n v_n\}$ where $a_1, ..., a_n > 0$

A **Negative System** $-\Pi$ of Φ with respect to β is the set of roots $-\Pi := \{r' \in \Phi | r' = a'_1 v_1 + ... + a'_n v_n\}$ where $a'_1, ..., a'_n < 0$

A **Simple System** Δ of Φ is a vector space basis of Φ that satisfies the following conditions. Let $\Delta = \{w_1, ..., w_n\}$:

1. If $r \in \Pi$, then $r = b_1 w_1 + ... + b_n w_n$ where $b_1, ..., b_n > 0$
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Suppose $\Phi = \{\pm(1,0), \pm(1,1), \pm(0,1), \pm(-1,1)\}$. Then $\Delta = \{(-1,1),(1,0)\}$ is a simple system of Φ.

Definition

Let Φ be a root system with associated reflection group W. Let $\Delta = \{v_1, ..., v_n\}$ be a simple system of Φ. A **Simple Reflection Group** Δ_i of Δ is the set

$$\Delta_i := \{s_{v_1}, ..., s_{v_n}\}$$

We call each s_i a **Simple Reflection**.

Fact: Every reflection group is generated by a simple reflection group.

Example

A simple reflection group of the root system from above is $\Delta_i = \{s_{(-1,1)}, s_{(1,0)}\}$ where $s_{(-1,1)}$ and $s_{(1,0)}$ are called simple reflections.

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A **Coxeter Group** S is a group that has the following presentation:

$$S := \{s_{i_k} | (s_{i_1} s_{i_2})^{m(i_1, i_2)} = 1\}$$

Theorem

Let Φ be a root system with associated reflection group W and simple system Δ. Then W is generated by the following Coxeter Group S:

$$S := \{s_{i_k} | \alpha_{i_k} \in \Delta, (s_{i_1} s_{i_2})^{m(\alpha_{i_1}, \alpha_{i_2})} = 1\}$$

Proof.

Let D_n be a dihedral group of order n. Then smallest angle of that preserves the symmetry of a regular n-sided polygon is $\frac{2\pi}{n}$.
The angle between the hyperplanes of simple roots, α_1 and α_2, is $\frac{\pi}{n}$.
Therefore, a rotation through $\frac{2\pi}{n}$ is achieved by reflection $s_{\alpha_1} s_{\alpha_2}$, where n such reflections produces a rotation of 2π.
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Suppose Φ is a root system with associated reflection group W. Then for all $\beta \in \Phi$ and $s_i \in W$, $\exists$ a unique root α_j such that

$$\beta = s_i(\alpha_j)$$

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Consider the regular pentagon that lies on complex vector space V. The vertices of this pentagon are related to the golden ratio ϕ, which satisfies $\phi^2 = \phi + 1$ and $\sqrt{5} = 2\phi - 1$:

$$\tau = (\tfrac{\phi-1}{2}, \tfrac{\sqrt{\phi+2}}{2}) = \phi(1+\tau^2)$$

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$$\tau^5 = 1 = (1, 0)$$

The set $\Phi = \{\pm\tau, \pm\tau^2, \pm\tau^3, \pm\tau^4, \pm\tau^5\}$ is a root system with associated reflection group $\mathbf{W} = \mathbf{I_2(5)}$. From now on, let $(\alpha_1 = \tau^5)$, $(\alpha_2 = \tau^2)$, $(\alpha_3 = \tau)$, $(\alpha_4 = \tau^3)$, $(\alpha_5 = \tau^5)$

Introdution to Reflection Group $I_2(5)$

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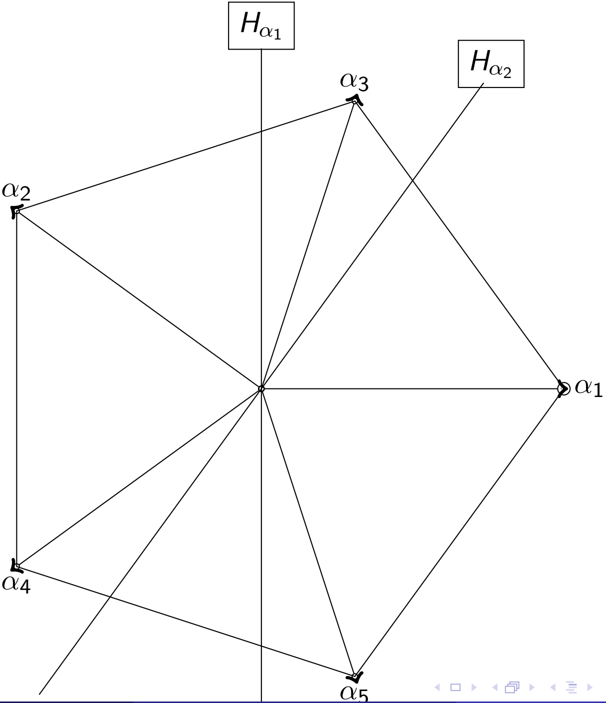
H_{α_1}
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α_2
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α_5

From Coxeter Relations, we obtain the following table:

1	1	-1	2	-2	3	-3	4	-4	5	-5
s_1	-1	1	-4	4	-5	5	-2	2	-3	3
s_2	-5	5	-2	2	-4	4	-3	3	-1	1
s_{12}	3	-3	4	-4	2	-2	5	-5	1	-1
s_{21}	5	-5	3	-3	1	-1	2	-2	4	-4
s_{121}	-3	3	-5	5	-1	1	-4	4	-2	2
s_{212}	-4	4	-3	3	-2	2	-1	1	-5	5
s_{1212}	2	-2	5	-5	4	-4	1	-1	3	-3
s_{2121}	4	-4	1	-1	5	-5	3	-3	2	-2
s_{12121}	-2	2	-1	1	-3	3	-5	5	-4	4

Polynomial and Rational Functions in Roots

Definition

Each root in Φ can be written as a linear combination of α_1 and α_2 with coefficients from the commutative ring: $R := \{a + b\Phi | a, b \in Z\}$

The ring of polynomial functions in variables $\alpha_1, ..., \alpha_5$ with integer coefficients is denoted by S.

The ring of laurent polynomals in variables $\alpha_1, ..., \alpha_5$ with integer coefficients is denoted by Q.

Example

$$s_i\left(\frac{\alpha_1^2\alpha_2+3\alpha_2\alpha_5}{\alpha_1\alpha_3^3}\right) = \frac{(s_i(\alpha_1))^2 s_i(\alpha_2)+3s_i(\alpha_2)s_i(\alpha_5)}{s_i(\alpha_1)(s_i(\alpha_3))^3}$$

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Definition

A **Twisted Differential Operator** is an operator that acts on elements of Q. There are five twisted differential operators of $I_2(5)$. They are denoted by X_i, $i = 1, ...5$. Let $q \in Q$. Then

$$X_i(q) = \frac{q - s_i q}{\alpha_i} = \frac{1}{\alpha_i}(\mathbf{1} - s_i)(q)$$

Definition: $X_i := \frac{1}{\alpha_i}(\mathbf{1} - s_i)$

Example

Action of X_1 on a polynomial function:

$$X_1(\alpha_1\alpha_2) = \frac{\alpha_1\alpha_2 - s_1(\alpha_1\alpha_2)}{\alpha_1} = \frac{\alpha_1\alpha_2 - (-\alpha_1)(-\alpha_4)}{\alpha_1} = \alpha_2 - \alpha_4$$

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A Useful Group Ring and Some Notation

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Action of X_1 on a polynomial function:

$X_1(\alpha_4) = \frac{\alpha_4 - s_1\alpha_4}{\alpha_1} = \frac{\alpha_4 + \alpha_2}{\alpha_1}$

This is an element of Q.

Definition

Let $q_1, ..., q_5$ be rational functions. Then the group ring generated by the elements $q_1\delta_{s_1}, ..., q_5\delta_{s_5}$ is denoted by $\tilde{\mathbf{Q}}[\mathbf{I}_2(\mathbf{5})]$.

In this ring, δ_{s_i} is not the same as s_i. δ_{s_i} is notation for an element of $\tilde{Q}[I_2(5)]$, whereas s_i is the action of the linear operator on a root.

From now on, we will denote δ_{s_i} by δ_i to simplify expressions.

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$X_1(\alpha_4) = \frac{\alpha_4 - s_1\alpha_4}{\alpha_1} = \frac{\alpha_4 + \alpha_2}{\alpha_1}$

This is an element of Q.

Definition

Let $q_1, ..., q_5$ be rational functions. Then the group ring generated by the elements $q_1\delta_{s_1}, ..., q_5\delta_{s_5}$ is denoted by $\tilde{\mathbf{Q}}[\mathbf{I_2}(\mathbf{5})]$.

In this ring, δ_{s_i} is not the same as s_i. δ_{s_i} is notation for an element of $\tilde{Q}[I_2(5)]$, whereas s_i is the action of the linear operator on a root.

From now on, we will denote δ_{s_i} by δ_i to simplify expressions.

From now on, we will denote $\delta_i\delta_j...$ as $\delta_{ij...}$ to simplify expressions.

Definition

Multiplication between elements of $\tilde{Q}[I_2(5)]$ is done using the following commuting rule:

$$s_i(\alpha_j)\delta_i = \delta_i\alpha_j$$

Definition

There are five unique twisted differential operators with respect to $I_2(5)$ in this group ring.

They can be viewed as elements

$X_i = \frac{1}{\alpha_i}(1 - \delta_i)$, $i = 1, ..., 5$

Multiplication between operators follows the multiplication of elements of $\tilde{Q}[I_2(5)]$ defined above.

Example

$X_1X_2 = (\frac{1}{\alpha_1}(\mathbf{1}-\delta_1))(\frac{1}{\alpha_2}(\mathbf{1}-\delta_2)) = \frac{1}{\alpha_1\alpha_2}(\mathbf{1}-\delta_2) - \frac{1}{\alpha_1}s_1(\frac{1}{\alpha_2})(\delta_1 - \delta_{12})$

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Ring of Twisted Differential Operators of $I_2(5)$

Definition

The subring of $\tilde{Q}[I_2(5)]$ generated by the elements $\{X_1, X_2, X_3, X_4, X_5\}$ is called the **Ring of Twisted Differential Operators of $\mathbf{I_2(5)}$**.

We will denote this ring $D_{I_2(5)}$.

Theorem

$$X_i^2 = 0,\ i = 1, ..., 5.$$

Proof.

$$X_i^2 = \left(\tfrac{1-\delta_i}{\alpha_i}\right)\left(\tfrac{1-\delta_i}{\alpha_i}\right) = \tfrac{1}{\alpha_i^2}(\mathbf{1}) - \tfrac{1}{\alpha_i}s_i\left(\tfrac{1}{\alpha_i}\right)(\delta_i) - \tfrac{1}{\alpha_i^2}(\delta_i) + \tfrac{1}{\alpha_i}s_i\left(\tfrac{1}{\alpha_i}\right)(\mathbf{1})$$
$$= \tfrac{1}{\alpha_i^2}(\mathbf{1}) + \tfrac{1}{\alpha_i^2}(\delta_i) - \tfrac{1}{\alpha_i^2}(\delta_i) - \tfrac{1}{\alpha_i^2}(\mathbf{1})$$
$$= 0$$

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$$\begin{aligned} X_i^2 &= (\tfrac{\mathbf{1}-\delta_i}{\alpha_i})(\tfrac{\mathbf{1}-\delta_i}{\alpha_i}) = \tfrac{1}{\alpha_i^2}(\mathbf{1}) - \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_i})(\delta_i) - \tfrac{1}{\alpha_i^2}(\delta_i) + \tfrac{1}{\alpha_i}s_i(\tfrac{1}{\alpha_i})(\mathbf{1}) \\ &= \tfrac{1}{\alpha_i^2}(\mathbf{1}) + \tfrac{1}{\alpha_i^2}(\delta_i) - \tfrac{1}{\alpha_i^2}(\delta_i) - \tfrac{1}{\alpha_i^2}(\mathbf{1}) \\ &= 0 \end{aligned}$$

□

Question: Are there relations within $D_{I_2(5)}$ that allow us to generate the ring in a simple way?

Question: Is there a way to generate $D_{I_2(5)}$ using only the elements X_1 and X_2?

Plan:

1. Express X_3, X_4, and X_5 in terms of X_1 and X_2 only.
2. Find simple relations within the ring $\langle X_1, X_2 \rangle$

From now on, we will use the following notation if it simplifies expressions:

1. $\alpha_i = i, i = 1, ..., 5$
2. For all $x \in R$, $x = [x]$
3. $X_i X_j = X_{ij}$, $i = 1, ..., 5$ and $j = 1, ..., 5$

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Theorem

The ring $D_{I_2(5)}$ is generated by $X_i, i = 1..5$, over the coefficient ring T' modulo the following relations:

1. $X_3 = \frac{[1]}{\phi([1]+2)}([3]1(X_1) + [2]2(X_2) - [3]12(X_{21} + X_{12}) + [4]1^2 2(X_{121}) + 12^2(X_{212}) - 1^2 2^2(X_{2121} - X_{1212}) + 1^3 2^2 X_{12121})$
2. $X_4 = -\frac{[1]}{\phi+2}([2]1(X_1) + 2(X_2) - 12(X_{12} + X_{21}) + 1^2 2(X_{121}))$
3. $X_5 = -\frac{[1]}{[1]+\phi^2}(1(X_1) + [2]2(X_2) - 12(X_{12} + X_{21}) + 12^2(X_{212}))$
4. $X_{12121} = X_{21212}$
5. $X_i^2 = 0$

Definition

Define the following commutative ring T:
$T := \{a + b\Phi + c\alpha_2 + d\Phi\alpha_2 + e\alpha_1 + f\Phi\alpha_1 | a, d, c, d, e, f \in Z\}$

Define $\mathbf{T}'$ as the ring of laurent polynomials in variables of T with integer coefficients.

Proof.

Fact: $\delta_1 = \mathbf{1} - (1)X_1$ and $\delta_2 = \mathbf{1} - (2)X_2$

Fact: $\alpha_3 = \phi(1 + \tau^2)$, $\alpha_4 = -\phi - \tau^2$, and $\alpha_5 = -1 - \phi\tau^2$

Fact: $X_3 = \frac{1}{\alpha_3}(\mathbf{1} - \delta_{12121})$, $X_4 = \frac{1}{\alpha_4}(\mathbf{1} - \delta_{121})$, and $X_5 = \frac{1}{\alpha_5}(\mathbf{1} - \delta_{212})$

We can use these facts to write X_3, X_4, and X_5 strictly in terms of X_1 and X_2 over the ring T'.

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$\square$

Theorem

1. A. $X_1 = \frac{[1]}{1} - \frac{[1]}{1}\delta_1$
2. B. $X_2 = \frac{[1]}{2} - \frac{[1]}{2}\delta_2$
3. C. $X_{12} = \frac{[1]}{12}(\mathbf{1}-\delta_2) + \frac{[1]}{14}(\delta_1 - \delta_{12})$
4. D. $X_{21} = \frac{[1]}{21}(\mathbf{1}-\delta_1) + \frac{[1]}{25}(\delta_2 - \delta_{21})$
5. E. $X_{212} = \frac{1+5}{12^2 5}(\mathbf{1}-\delta_2) + \frac{[1]}{124}(\delta_1 - \delta_{12}) - \frac{[1]}{235}(\delta_{21} - \delta_{212})$
6. F. $X_{121} = \frac{2+4}{1^2 24}(\mathbf{1}-\delta_1) + \frac{[1]}{125}(\delta_2 - \delta_{21}) - \frac{[1]}{134}(\delta_{12} - \delta_{121})$
7. G. $X_{1212} = \frac{14+25+45}{1^2 2^2 45}(\mathbf{1}-\delta_2) + \frac{12+23+34}{1^2 234^2}(\delta_1 - \delta_{12}) + \frac{[1]}{1235}(\delta_{212} - \delta_{21}) + \frac{[1]}{1345}(\delta_{1212} - \delta_{121})$
8. H. $X_{2121} = \frac{14+25+45}{1^2 2^2 45}(\mathbf{1}-\delta_1) + \frac{12+13+35}{12^2 35^2}(\delta_2 - \delta_{21}) + \frac{[1]}{1234}(\delta_{121} - \delta_{12}) + \frac{[1]}{2345}(\delta_{2121} - \delta_{212})$

Theorem

1. $I.\ X_{21212} = \frac{345(1+5)+235^2+134(1+5)+1^2 24}{1^2 2^3 345^2}(\mathbf{1} - \delta_2) + \frac{125+235+345+134}{1^2 2^2 34^2 5}(\delta_1 - \delta_{12}) + \frac{345+125+124+134}{12^2 3^2 45^2}(\delta_{212} - \delta_{21}) + \frac{[1]}{12345}(\delta_{1212} - \delta_{121} + \delta_{2121} - \delta_{21212})$

2. $J.X_{12121} = \frac{345(2+4)+134^2+235(2+4)+12^2 5}{1^3 2^2 34^2 5}(\mathbf{1} - \delta_1) + \frac{124+134+345+235}{1^2 2^2 345^2}(\delta_2 - \delta_{21}) + \frac{345+124+125+235}{1^2 23^2 4^2 5}(\delta_{121} - \delta_{12}) + \frac{[1]}{12345}(\delta_{2121} - \delta_{212} + \delta_{1212} - \delta_{12121})$

Direction: Determine whether $X_{12121} - X_{21212}$ is a linear combination of the other twisted differential operators of $D_{I_2(5)}$

Theorem

1. $I.\ X_{21212} = \frac{345(1+5)+235^2+134(1+5)+1^224}{1^22^33345^2}(\mathbf{1}-\delta_2) + \frac{125+235+345+134}{1^22^234^25}(\delta_1 - \delta_{12}) + \frac{345+125+124+134}{12^23^245^2}(\delta_{212} - \delta_{21}) + \frac{[1]}{12345}(\delta_{1212} - \delta_{121} + \delta_{2121} - \delta_{21212})$

2. $J.X_{12121} = \frac{345(2+4)+134^2+235(2+4)+12^25}{1^32^234^25}(\mathbf{1}-\delta_1) + \frac{124+134+345+235}{1^22^2345^2}(\delta_2 - \delta_{21}) + \frac{345+124+125+235}{1^223^24^25}(\delta_{121} - \delta_{12}) + \frac{[1]}{12345}(\delta_{2121} - \delta_{212} + \delta_{1212} - \delta_{12121})$

Direction: Determine whether $X_{12121} - X_{21212}$ is a linear combination of the other twisted differential operators of $D_{I_2(5)}$

Lemma

$$X_{12121} - X_{21212} =$$

$$\frac{[2]2^2345^2+234^25^2+1234^25+2^335^2+12^35^2-1^234^25-134^25^2-12345^2-1^234^25-1^334^2-1^324^2}{1^32^234^25^2}(\mathbf{1})+$$

$$\frac{-2345-34^25-134^2-2345-2^235-12^25-1^225-1235-1345-1^234}{1^32^234^25^2}(\delta_1)+$$

$$\frac{12^24+1234+2345+2^235+1345+345^2+1345+1^234+1^224}{1^22^3345^2}(\delta_2)+$$

$$\frac{-1234-13^24-3^245-23^25+1345+1^225+1^224+1^234}{1^22^23^245^2}(\delta_{21})+$$

$$\frac{-2345-12^24-12^25-2^235+1235+23^25+3^245+13^24}{1^22^23^24^25}(\delta_{12})+$$

$$\frac{235+345+125+124+134}{1^223^24^25}(\delta_{121})+\frac{-235-345-125-124-134}{12^23^245^2}(\delta_{212})$$

Lemma

$$X_{121} - X_{212} =$$

$$\frac{2^5+245-1^24-145}{1^22^245}(\mathbf{1})+\frac{-1-2-4}{1^224}(\delta_1)+\frac{1+2+5}{12^25}(\delta_2)+\frac{-2+3}{1234}(\delta_{12})+\frac{-1-3}{1235}(\delta_{21})+$$

$$\frac{[1]}{134}(\delta_{121})+\frac{-[1]}{235}(\delta_{212})$$

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$$\frac{-2345-34^25-134^2-2345-2^235-12^25-1^225-1235-1345-1^234}{1^32^234^25^2}(\delta_1)+$$

$$\frac{12^24+1234+2345+2^235+1345+345^2+1345+1^234+1^224}{1^22^3345^2}(\delta_2)+$$

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Lemma

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$$\frac{[1]}{134}(\delta_{121})+\frac{-[1]}{235}(\delta_{212})$$

Lemma

$$\frac{235+345+125+124+134}{12345}(X_{121} - X_{212}) = (X_{12121} - X_{21212})$$

Theorem

$$X_{12121} = X_{21212}$$

Proof.

$235 = \tau^2(\phi(1+\tau^2))(-1-\phi\tau^2) = -\phi\tau^2 - \phi^2(\tau^2)^2 - \phi(\tau^2)^2 - \phi^2(\tau^2)^3$

$345 = (\phi + \phi\tau^2)(-\phi - \tau^2)(-1-\phi\tau^2) =$
$\phi^2 + \phi\tau^2 + \phi^2\tau^2 + \phi(\tau^2)^2 + \phi^3\tau^2 + \phi^2(\tau^2)^2 + \phi^3(\tau^2)^2 + \phi^2(\tau^2)^3$

$125 = 1\tau^2(-1-\phi\tau^2) = -\tau^2 - \phi(\tau^2)^2$

$124 = 1\tau^2(-1-\phi\tau^2) = -\phi\tau^2 - (\tau^2)^2$

$134 = 1\phi(1+\tau^2)(-\phi-\tau^2) = -\phi^2 - \phi\tau^2 - \phi^2\tau^2 - \phi(\tau^2)^2$

□

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$\square$

Lemma

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$235 = \tau^2(\phi(1+\tau^2))(-1-\phi\tau^2) = -\phi\tau^2 - \phi^2(\tau^2)^2 - \phi(\tau^2)^2 - \phi^2(\tau^2)^3$

$345 = (\phi+\phi\tau^2)(-\phi-\tau^2)(-1-\phi\tau^2) =$
$\phi^2 + \phi\tau^2 + \phi^2\tau^2 + \phi(\tau^2)^2 + \phi^3\tau^2 + \phi^2(\tau^2)^2 + \phi^3(\tau^2)^2 + \phi^2(\tau^2)^3$

$125 = 1\tau^2(-1-\phi\tau^2) = -\tau^2 - \phi(\tau^2)^2$

$124 = 1\tau^2(-1-\phi\tau^2) = -\phi\tau^2 - (\tau^2)^2$

$134 = 1\phi(1+\tau^2)(-\phi-\tau^2) = -\phi^2 - \phi\tau^2 - \phi^2\tau^2 - \phi(\tau^2)^2$

□

Proof.

$$235 + 345 + 125 + 124 + 134$$
$$= -[2]\phi\tau^2 - [2]\phi(\tau^2)^2 + \phi^3\tau^2 + \phi^3(\tau^2)^2 - \tau^2 - (\tau^2)^2$$
$$= -\tau^2([2]\phi + 1) - (\tau^2)^2([2]\phi + 1) + (\phi^3)(\tau^2 + (\tau^2)^2)$$
$$= -([2]\phi + 1)(\tau^2 + (\tau^2)^2) + ([2]\phi + 1)(\tau^2 + (\tau^2)^2)$$
$$= 0$$

□

Example

$$X_{2121212} = X_{21}(X_{21212}) = X_{21}(X_{12121}) = X_2(X_1X_1)X_{2121} = 0$$

Proof.

$$
\begin{aligned}
&235 + 345 + 125 + 124 + 134 \\
&= -[2]\phi\tau^2 - [2]\phi(\tau^2)^2 + \phi^3\tau^2 + \phi^3(\tau^2)^2 - \tau^2 - (\tau^2)^2 \\
&= -\tau^2([2]\phi + 1) - (\tau^2)^2([2]\phi + 1) + (\phi^3)(\tau^2 + (\tau^2)^2) \\
&= -([2]\phi + 1)(\tau^2 + (\tau^2)^2) + ([2]\phi + 1)(\tau^2 + (\tau^2)^2) \\
&= 0
\end{aligned}
$$

□

Example

$$X_{2121212} = X_{21}(X_{21212}) = X_{21}(X_{12121}) = X_2(X_1X_1)X_{2121} = 0$$