Line bundles on complete flag varieties are independent of central isogeny class

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Let k be an algebraically closed field of characteristic 0. By an algebraic variety over k, we mean any quasi-projective variety over k.

Proposition 0.1. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties over k. Then f is bijective if and only if it is an isomorphism.*

Proof. Observe that a smooth quasi-projective variety over k is normal, and see [Ele]. □

Lemma 0.2. *Let $\phi : G \to G'$ be a surjective morphism of connected linear algebraic groups over k. If H is a Borel subgroup (resp. maximal torus, maximal connected unipotent subgroup) in G, then $\phi(H)$ is a Borel subgroup (resp. maximal torus, maximal connected unipotent subgroup) in G'.*

Proof. See [Hum75, §21.3, Cor. C]. □

Theorem 0.3. *Let G be a connected reductive group over k, and let B be a Borel subgroup of G. The set of unipotent elements B_u equals the commutator subgroup $\{B, B\}$ of B, and B_u is a closed, connected, nilpotent, normal subgroup of B. Moreover, B/B_u is a torus. Finally, if T is any maximal torus of G sitting in B, then $B = TB_u$ (this is a semidirect product), and the restriction of the projection $B \to B/B_u$ to T defines an isomorphism $T \simeq B/B_u$.*

Proof. See [Spr98, Cor. 6.3.3 and Thm. 6.3.5] and [Bor69, Thm. 10.6]. □

Let G be a connected semisimple linear algebraic group over k, and suppose T is a maximal torus in G. Let B be a Borel subgroup of G containing T. If λ is a character of T, then λ determines a one-dimensional irreducible representation $V_\lambda = (v_\lambda)$ of T. By Theorem 0.3, B/B_u is a torus, and there is sequence of homomorphisms $B \to B/B_u \to T$, where the second homomorphism is an isomorphism. Thus, every character of T lifts to a character λ of B. The group B acts on V_λ by $b \cdot v_\lambda = \lambda(b)^{-1} v_\lambda$ for all $b \in B$.

Theorem 0.4. *Let λ be a character of T, and, hence, of B. The set*

$$\mathcal{L}(\lambda) := G \times_B V_\lambda = G \times V_\lambda / ((g, v) \sim (gb, b^{-1} \cdot v))$$

is an algebraic variety, and it is the total space of a line bundle over the complete flag variety G/B. The morphism $\pi \colon \mathcal{L}(\lambda) \to G/B$ defining this line bundle sends $(g, v)B \mapsto gB$ for all $(g, v)B \in \mathcal{L}(\lambda)$.

Proof. See [Spr98, §8.5]. □

Remark 0.5. There is a natural G-action on $\mathcal{L}(\lambda)$ given by $h \cdot (g, v) = (hg, v)$ for all $h, g \in G$, $v \in V_\lambda$. The line bundle $\pi : \mathcal{L}(\lambda) \to G/B$ is G-*equivariant*: $\pi(h \cdot (g, v)) = h\pi(g, v)$ for all $g, h \in G$, $v \in V_\lambda$. The line bundle $\mathcal{L}(\lambda)$ is a *homogeneous line bundle*.

Remark 0.6. Let $\mathcal{B}$ be the set of Borel subgroups in G. By the discussion in [Hum75, §23.3], the map $xB \to xBx^{-1}$ defines a bijection $G/B \to \mathcal{B}$. Under the induced variety structure, we call $\mathcal{B}$ the *variety of Borel subgroups* of G.

Let G and G_1 be connected semisimple linear algebraic groups over k, with root data $\Psi = (\Sigma, \Lambda, \Sigma^\vee, \Lambda^\vee)$ and $\Psi_1 = (\Sigma_1, \Lambda_1, \Sigma_1^\vee, \Lambda_1^\vee)$, and Weyl groups $W(\Psi)$ and $W(\Psi_1)$, respectively.

Definition 0.7. (See [Ste99, Ch. 1].) A *central isogeny* of root data $f : \Psi \to \Psi_1$ is an injective group homomorphism $f : \Lambda \to \Lambda_1$ with finite cokernel such that f induces a bijection $f|_\Sigma \colon \Sigma \to \Sigma_1$, satisfying

$$f^\vee((f(\alpha))^\vee) = \alpha^\vee, \quad \alpha \in \Sigma.$$

Remark 0.8. A central isogeny of root data $f : \Psi \to \Psi_1$ induces an isomorphism of the Weyl groups,

$$W(\Psi) \to W(\Psi_1), \quad s_\alpha \mapsto s_{f(\alpha)}, \quad \alpha \in \Sigma.$$

Remark 0.9. Let $\{\alpha_i\}_{i=1}^n$ be a set of simple roots in Σ, and let $\{\lambda_i\}_{i=1}^n$ be a $\mathbb{Z}$-basis of Λ. If $f : \Psi \to \Psi_1$ is a central isogeny, then $\{f(\alpha_i)\}_{i=1}^n$ is a set of simple roots in $f(\Sigma) = \Sigma_1$, and $\{f(\lambda_i)\}_{i=1}^n$ is a $\mathbb{Z}$-basis of $f(\Lambda)$.

Definition 0.10. (See [Ste99, Ch. 1].) A *central isogeny* $\phi : G_1 \to G$ is a surjective morphism whose kernel is finite and central in G_1.

Proposition 0.11. *Let $\phi : G_1 \to G$ be an central isogeny, mapping T_1 to T. Then ϕ induces a central isogeny of root data $f : \Psi \to \Psi_1$ such that $f(\lambda) = \lambda \circ \phi|_{T_1}$ for all $\lambda \in \Lambda$.*

Proof. See [Ste99, Ch. 1]. ☐

Let G^{sc} be the connected semisimple simply-connected linear algebraic group over k with the same Dynkin type as G, and let $\Psi^{\mathrm{sc}} = (\Sigma^{\mathrm{sc}}, \Lambda^{\mathrm{sc}}, (\Sigma^{\mathrm{sc}})^\vee, (\Lambda^{\mathrm{sc}})^\vee)$ be its root datum. By [Spr98, Exercises 10.1.4(1)], there is a central isogeny $\phi : G^{\mathrm{sc}} \to G$. The group G^{sc} is called the *simply-connected cover* of G. If B^{sc} is a Borel subgroup of G^{sc} with maximal unipotent connected subgroup B_u^{sc}, then, by Lemma 0.2, $B := \phi(B^{\mathrm{sc}})$ is a Borel subgroup in G with maximal unipotent connected subgroup $B_u := \phi(B_u^{\mathrm{sc}})$. By Theorem 0.3, we can view $B^{\mathrm{sc}}/B_u^{\mathrm{sc}}$ and B/B_u as maximal tori in G^{sc} and G, respectively. Set $T^{\mathrm{sc}} := B^{\mathrm{sc}}/B_u^{\mathrm{sc}}$ and $T := \phi(T^{\mathrm{sc}}) = B/B_u$. Let $f : \Lambda \to \Lambda^{\mathrm{sc}}$ be the injective homomorphism on character lattices induced by ϕ. If $p : B \to T$ and $p^{\mathrm{sc}} : B^{\mathrm{sc}} \to T^{\mathrm{sc}}$ are the canonical projections onto the quotients, then the following diagram commutes:

$$\begin{tikzcd} B^{\mathrm{sc}} \arrow[r, "p^{\mathrm{sc}}"] \arrow[d, "\phi|_{B^{\mathrm{sc}}}"'] & T^{\mathrm{sc}} \arrow[d, "\phi|_{T^{\mathrm{sc}}}"] \\ B \arrow[r, "p"'] & T \end{tikzcd} .$$

Recall that we can lift a character of T^{sc} (resp. T) to a character of B^{sc} (resp. B) by composing the character on the right by p^{sc} (resp. p). Given a character λ of T, we have by Proposition 0.11 that $\lambda \circ \phi|_{T^{\mathrm{sc}}} = f(\lambda)$. Thus, $\lambda \circ p \circ \phi|_{B^{\mathrm{sc}}} = \lambda \circ \phi|_{T^{\mathrm{sc}}} \circ p^{\mathrm{sc}} = f(\lambda) \circ p^{\mathrm{sc}}$. From now on, we will abuse notation and denote the character $\lambda \circ p$ (resp. $f(\lambda) \circ p^{\mathrm{sc}}$) of B (resp. B^{sc}) by λ (resp. $f(\lambda)$). Thus, $\lambda \circ \phi|_{B^{\mathrm{sc}}} = f(\lambda)$.

Let $\mathcal{B}$ (resp. $\mathcal{B}^{\mathrm{sc}}$) be the variety of Borel subgroups in G (resp. G^{sc}). Following [Bor69, Prop. 11.20], we show that there is an isomorphism of flag varieties $G^{\mathrm{sc}}/B^{\mathrm{sc}} \simeq G/B$ by showing that the induced map on the variety of Borel subgroups $\phi_{\mathcal{B}^{\mathrm{sc}}} : \mathcal{B}^{\mathrm{sc}} \to \mathcal{B}$ is an isomorphism. Since ϕ is surjective, given $xBx^{-1} \in \mathcal{B}$, there is $y \in G^{\mathrm{sc}}$ such that $\phi(y) = x$. Thus, $\phi_{\mathcal{B}^{\mathrm{sc}}}(yB^{\mathrm{sc}}y^{-1}) = \phi(y)\phi(B^{\mathrm{sc}})\phi(y)^{-1} = xBx^{-1}$. To see that $\phi_{\mathcal{B}^{\mathrm{sc}}}$ is injective, we note that, since the kernel of a central isogeny is central and central elements in G^{sc} lie in B^{sc}, we have

$$\phi^{-1}(B) = B^{\mathrm{sc}} \ker \phi = B^{\mathrm{sc}}.$$

Thus, $\phi_{\mathcal{B}^{\mathrm{sc}}}$ is a bijective morphism of smooth projective varieties over k, so, by Proposition 0.1, it is an isomorphism. Since $G^{\mathrm{sc}}/B^{\mathrm{sc}} \simeq G/B$ as flag varieties, given a character λ^{sc} of T^{sc}, there is a line bundle $\mathcal{L}(\lambda^{\mathrm{sc}})$ over G/B.

Lemma 0.12. *If $\lambda^{sc} \in \Lambda^{sc}$, $\lambda \in \Lambda$, and $\lambda^{sc} = f(\lambda)$, then there is an isomorphism of line bundles $\mathcal{L}(\lambda^{sc}) \simeq \mathcal{L}(\lambda)$ over G/B.*

Proof. Let v_λ and $v_{\lambda^{\mathrm{sc}}}$ be generators of the one-dimensional irreducible representations V_λ and $V_{\lambda^{\mathrm{sc}}}$ of B and B^{sc}, respectively. Let $q : \mathcal{L}(\lambda^{\mathrm{sc}}) \to \mathcal{L}(\lambda)$ be the morphism

$$q(g, v_{\lambda^{\mathrm{sc}}}) = (\phi(g), v_\lambda), \quad g \in G^{\mathrm{sc}}.$$

To see that q is well defined, we note that

$$\begin{aligned} q(gb, \lambda^{\mathrm{sc}}(b)v_{\lambda^{\mathrm{sc}}}) &= (\phi(gb), \lambda^{\mathrm{sc}}(b)v_\lambda) \\ &\sim (\phi(g), \lambda(\phi(b)^{-1})\lambda^{\mathrm{sc}}(b)v_\lambda) = (\phi(g), \lambda(\phi(b)^{-1})\lambda(\phi(b))v_\lambda) = (\phi(g), v_\lambda). \end{aligned}$$

The morphism q is surjective because ϕ is surjective.

Now let $p : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda^{\mathrm{sc}})$ be the map

$$p(g, v_\lambda) = (h, v_{\lambda^{\mathrm{sc}}}), \quad g \in G,$$

where h is any element in $\phi^{-1}(g)$. This map is independent of the representative h. To see this, first recall that $\ker(\phi)$ is central and lies in B^{sc}. For all $c \in \ker(\phi)$, we have

$$(h, v_{\lambda^{\mathrm{sc}}}) = (hc, \lambda^{\mathrm{sc}}(c) v_{\lambda^{\mathrm{sc}}}) = (hc, \lambda(\phi(c)) v_{\lambda^{\mathrm{sc}}}) = (hc, v_{\lambda^{\mathrm{sc}}}).$$

Therefore, it does not matter which element $h \in \pi^{-1}(g)$ we choose, and p is surjective. To see that p is well-defined, we note that, for all $b \in B$, there is $b' \in \phi^{-1}(b)$ such that

$$\begin{aligned} p(gb, \lambda(b) v_\lambda) &= (hb', \lambda(b) v_{\lambda^{\mathrm{sc}}}) \\ &\sim (h, \lambda^{\mathrm{sc}}(b')^{-1} \lambda(b) v_{\lambda^{\mathrm{sc}}}) = (h, \lambda(\phi(b'))^{-1} \lambda(b) v_{\lambda^{\mathrm{sc}}}) = (h, v_{\lambda^{\mathrm{sc}}}). \end{aligned}$$

It is straightforward to verify that p is a set-theoretic inverse to q. In particular, q is a bijective morphism of algebraic varieties over k. Since G/B and $G^{\mathrm{sc}}/B^{\mathrm{sc}}$ are smooth, the line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\lambda^{\mathrm{sc}})$ are smooth. Now it follows from Proposition 0.1 that q is an isomorphism of smooth algebraic varieties over k.

It is straightforward to verify that the following diagram commutes,

$$\begin{tikzcd} \mathcal{L}(\lambda^{\mathrm{sc}}) \arrow[r, "q"] \arrow[d, "\pi^{\mathrm{sc}}"'] & \mathcal{L}(\lambda) \arrow[d, "\pi"] \\ G^{\mathrm{sc}}/B^{\mathrm{sc}} \arrow[r, "\phi_{G^{\mathrm{sc}}/B^{\mathrm{sc}}}"'] & G/B \end{tikzcd},$$

where π and π^{sc} are projections onto the first factor, and $\phi_{G^{\mathrm{sc}}/B^{\mathrm{sc}}}$ is the isomorphism of flag varieties induced by ϕ. It is also straightforward to verify that $q|_{(\phi_{G^{\mathrm{sc}}/B^{\mathrm{sc}}} \circ \pi^{\mathrm{sc}})^{-1}(gB)}$ and $q^{-1}|_{\pi^{-1}(gB)}$ are linear for all $gB \in G/B$. Therefore, q is a morphism of line bundles. $\square$

Big question: Do analogous results hold for *split* semisimple linear algebraic groups defined over *arbitrary* fields k (here, we take an *algebraic group* to be a quasi-projective reduced separated Noetherian affine smooth group scheme defined over an arbitrary field k)?

References

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