COMPONENT PLANAR GRAPHS

RAJ GANDHI

The following note describes an open problem from [GSZ21].

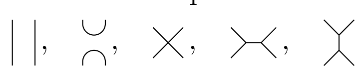
Definition 0.1. An *o–graph* is a graph that satisfies the following conditions:

- The graph lies in $\mathbb{R}^2$.
- Imagine that there exist two imaginary horizontal parallel lines in $\mathbb{R}^2$, one above the other. The graph lies between the two lines.
- All endpoints of the graph lie on one of the two horizontal lines. Moreover, on each line, the endpoints of the graph are ordered such that, if x and y are endpoints of the graph that lie on the same horizontal line, then $x < y$ if and only if x is to the left of y on this line.
- All vertices in the graph have either degree 1 or degree 3.
- The graph has no cycle of length ≤ 5.

Two o-graphs G_1 and G_2 are *isomorphic* if there is an isomorphism of graphs $\phi\colon G_1 \to G_2$ that sends the vertices in G_1 on the top (resp. bottom) horizontal line to the vertices in G_2 on the top (resp. bottom) horizontal line, and such that $x < y$ in G_1 if and only if $\phi(x) < \phi(y)$ in G_2 (i.e., ϕ respects the orderings of the vertices). A *planar o-graph* is an o-graph that is isomorphic to an o-graph whose edges intersect only at their endpoints.

Remark 0.2. When discussing connected components of o-graphs, we will view the connected components themselves as o-graphs with the orderings of the vertices induced by the orderings of the vertices of the larger o-graph.

We say that an o-graph is *component-planar* if its connected components are planar o-graphs. Let $\mathcal{F}(n, m)$ be the number of component-planar o-graphs with n endpoints that lie on the bottom horizontal line and m endpoints that lie on the top horizontal line. For example,

(0.1) 

is a complete list of the 5 component-planar o-graphs in $\mathcal{F}(2, 2)$, and there are precisely 15 component-planar o-graphs $\mathcal{F}(2, 3)$:

(0.2) 

One can count that there are 70 component-planar graphs in $\mathcal{F}(3, 3)$.

Question 0.3. *How many component-planar o-graphs are there in $\mathcal{F}(n, m)$?*

References

[GSZ21] R. Gandhi, A. Savage, and K. Zaynullin. Diagrammatics of F_4. 2021. arXiv:2107.12464.