Mathematical Modelling

m_B = Mass of the body
m_w = Mass of the wheel
I_B = Moment of inertia of the body about the axis of wheels
I_w = Moment of inertia of the wheel
L = Height of the centre of mass of the body from the centre of the wheels
r = Radius of the wheels
T = Torque due to both the wheels

ϕ = The angular velocity of the wheel about its axis of rotation.
x = Position of centre of mass of wheel along x-axis
$\dot{x}$ = Linear velocity of centre of mass of the wheel
θ = Angle between z-axis and body of the robot
$\dot{\theta}$ = Angular velocity of the body

E_k = Kinetic Energy
E_p = Potential Energy with the plane through centre of mass of wheels as reference

Assuming the motion of wheels to be pure rolling
$x = r\phi$
$\dot{\phi} = \frac{\dot{x}}{r}$

Derivation
$E_k = (m_w + \frac{m_B}{2} + \frac{I_w}{r^2})(\dot{x})^2 + \frac{(m_B L^2 + I_B)}{2}(\dot{\theta})^2 + m_B L\dot{\theta}\dot{x}\cos(\theta)$
$E_p = m_B g L\cos(\theta)$
$\mathcal{L} = E_k - E_p = (m_w + \frac{m_B}{2} + \frac{I_w}{r^2})(\dot{x})^2 + \frac{(m_B L^2 + I_B)}{2}(\dot{\theta})^2 + m_B L\dot{\theta}\dot{x}\cos(\theta) - m_B g L\cos(\theta)$

$\frac{\partial \mathcal{L}}{\partial \theta} = m_B L\sin(\theta)(g - \dot{\theta}\dot{x})$
$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = (m_B L^2 + I_B)\ddot{\theta} + m_B L\ddot{x}\cos(\theta) - m_B L\dot{x}\dot{\theta}\sin(\theta)$

$\frac{\partial \mathcal{L}}{\partial x} = 0$
$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = (m_B + 2m_w + 2\frac{I_w}{r^2})\ddot{x} + m_B L\ddot{\theta}\cos(\theta) - m_B L\dot{\theta}^2\sin(\theta)$

By Euler-Lagrange Equation
Q = Generalised Force
q = Generalised Co-ordinates
$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) - \frac{\partial \mathcal{L}}{\partial q} = Q$

Equations of Motion
$\ddot{\theta} = \frac{m_B g L \sin(\theta) - m_B L \ddot{x} \cos(\theta) - T}{(m_B L^2 + I_B)}$
$(m_B + 2m_w + 2\frac{I_w}{r^2})\ddot{x} + m_B L \ddot{\theta} \cos(\theta) - m_B L \dot{\theta}^2 \sin(\theta) = \frac{T}{r}$

For sake of simplicity let's take $k = \frac{m_B^2 L^2}{m_B L^2 + I_B}$
From Equations of Motion

$$\ddot{x} = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - kcos(\theta)^2}.(-k.g.cos(\theta).sin(\theta) + m_B L^2 \dot{\theta}^2 sin(\theta) + (\frac{1}{r} + \frac{k.cos(\theta)}{m_B L}).T)$$

$$\bar{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

By solving $\dot{\bar{x}} = f(\bar{x}) = 0$ the equilibrium point we get are : $\begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} (Unstable), \begin{bmatrix} x \\ 0 \\ \pi \\ 0 \end{bmatrix} (Stable)$

Here x (first element of $\bar{x}$) is free to take any value.
Let's $\bar{x_o} = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} (Unstable\, Equilibrium)$

Linearising $\dot{\bar{x}} = f(\bar{x})$ about $\bar{x_o}$
$\frac{\partial \ddot{x}}{\partial x}(at\, \bar{x_o}) = 0$
$\frac{\partial \ddot{x}}{\partial \dot{x}}(at\, \bar{x_o}) = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k}.0 = 0$

$$\frac{\partial \ddot{x}}{\partial \theta}(at\,\bar{x_o}) = \frac{1}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}.(-k.g)$$

$$\frac{\partial \ddot{x}}{\partial \dot{\theta}}(at\,\bar{x_o}) = \frac{1}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}.0 = 0$$

$$\frac{\partial \ddot{x}}{\partial T}(at\,\bar{x_o}) = \frac{\dfrac{1}{r} + \dfrac{k}{m_B L}}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}$$

$$\ddot{\theta} = \frac{k.g.sin(\theta)}{m_B L} - \frac{k.\ddot{x}.cos(\theta)}{m_B L} - \frac{k.T}{(m_B L)^2}$$

$$\frac{\partial \ddot{\theta}}{\partial x}(at\,\bar{x_o}) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial \dot{x}}(at\,\bar{x_o}) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial \theta}(at\,\bar{x_o}) = \frac{k.g}{m_B L}.\frac{m_B + 2m_w + \dfrac{2I_w}{r^2}}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}$$

$$\frac{\partial \ddot{\theta}}{\partial \dot{\theta}}(at\,\bar{x_o}) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial T}(at\,\bar{x_o}) = \frac{-\,k}{(m_B L)^2} - \frac{k}{m_B L}.(\frac{1}{r} + \frac{k}{m_B L}).\frac{1}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-\,k.g}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{k.g.(m_B + 2m_w + \dfrac{2I_w}{r^2})}{m_B L.(m_B + 2m_w + \dfrac{2I_w}{r^2} - k)} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{\dfrac{1}{r} + \dfrac{k}{m_B L}}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k} \\ 0 \\ \dfrac{-k}{(m_B L)^2} - \dfrac{k}{m_B L}.(\dfrac{1}{r} + \dfrac{k}{m_B L}).(\dfrac{1}{m_B + 2m_w + \dfrac{2I_w}{r^2} - k}) \end{bmatrix}$$

State-Space Equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$