CLRS

EXERCISE-SOL^n

G. RAJVARDHAN
GROUP-1

3.2-1: Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $\max\{f(n), g(n)\} = \Theta(f(n)+g(n))$.

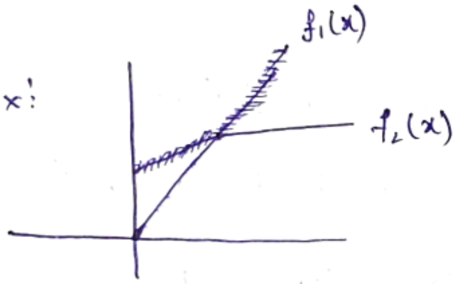
Sol^n: Given: $f(n)$ & $g(n)$ are asymptotically non-negative which means as its input value approaches infinity, its output values remain non-negative.

by definition we know that $\max\{a,b\} \geq a$

(or)

$\max\{a,b\} \geq b$

Ex:



here the shaded part is $\max\{f_1(x), f_2(x)\}$

during some interval it is equal to f_1 & f_2 in some other.

$\Rightarrow$

$$\max\{a,b\} \geq a \quad -(1)$$

$$\max\{a,b\} \geq b \quad -(2)$$

adding the two inequalities:

$$2\max\{a,b\} \geq a+b$$

$$\boxed{\max\{a,b\} \geq \frac{1}{2}(a+b)} \quad -(3)$$

since $\max\{a,b\}$ is always less than $(a+b)$ in any interval because at any given interval $\max\{a,b\}$ could be 'a' or 'b'.

So; $\max\{a, b\} \leq a+b$

↳ equality holds when one of the two functions are 'zero' at some intervals.

So; $$\boxed{\max\{f(n), g(n)\} \leq f(n)+g(n)} \quad -(4)$$

So, by definition of O-(Big Oh)

for a function $f(n)$ and $g(n)$

$f(n) = O(g(n))$: there exists positive constants c and n_0 such that

$$0 \leq f(n) \leq c \cdot g(n) \quad \forall n > n_0$$

from inequality (4)

$$\underbrace{0 \leq}_{} \max\{f(n), g(n)\} \leq f(n) + g(n)$$

↳ as it is asymptotically non-negative (from given)

here the constant $c = 1$.

$\therefore$ $$\boxed{\max\{f(n), g(n)\} = O(f(n)+g(n))}$$

from the definition of 'Ω':

for functions $f(n)$ & $g(n)$

$f(n) = \Omega(g(n))$: there exists positive constants c & $\forall\, n_0 \geq n_0$; $n_0 > 0$

$$0 \leq c.g(n) \leq f(n)$$

from inequality (3): $\max\{f(n), g(n)\} \geq \frac{1}{2}(f(n)+g(n))$

$$0 \leq \frac{1}{2}(f(n)+g(n)) \leq \max\{f(n), g(n)\}$$

$$\Rightarrow \quad \max\{f(n), g(n)\} = \Omega(f(n)+g(n))$$

from definition of Θ:

for functions $f(n)$ & $g(n)$

$f(n) = \Theta(g(n))$: there exists $c_1, c_2 > 0$ & n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \forall\, n \geq n_0$$

$$\Rightarrow \quad 0 \leq \frac{1}{2}(f(n)+g(n)) \leq \max\{f(n), g(n)\} \leq f(n)+g(n)$$

$$\Rightarrow \quad \boxed{\max\{f(n), g(n)\} = \Theta(f(n)+g(n))}$$

Proved !! 😊 !!

3.2.2: Explain why "The running time for Algorithm A is atleast $O(n^2)$" is meaningless.

Soln: $O(g(n)) = f(n)$ loosely says that f grows no faster than $g(n)$ {source: vazirani}

Which means an asymptotic upper bound

i.e; O-nototation give an upper bound on function to within a constant factor

But in the statement it says atleast $O(n^2)$ indicating that it is the $(O(n^2))$ is the least / Lower bound for the running time of the algorithm A.

which contradicts with the definition

More correct way would be '$\Omega(n^2)$' in place of '$O(n^2)$'.

3.2.3: Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

Ans) Yes and No,

$2^{n+1} = 2^n \cdot 2^1 = 2 \cdot 2^n$

since for denoting 2^{n+1} as $O(2^n)$

we need to have some constant c

which satisfies

$\Rightarrow 2^{n+1} \leq c \cdot 2^n \Rightarrow 2 \cdot 2^n \leq c \cdot 2^n$

so for $c \geq 2$

the inequality holds.

So; $\forall c \geq 2 \ \exists n_0$ s.t $\forall n \geq n_0$

$$\boxed{2^{n+1} = O(2^n)}$$

* 2^{2n} is equal to $(2^2)^n = (4^n)$

since 4^n & 2^n are exponential founctions

for $2^{2n} / 4^n$ to be equal to $O(2^n)$

4^n should satisfy equality

$\Rightarrow 4^n \leq c \cdot 2^n$ where c is a constant $c > 0$

solving

$\Rightarrow n \log 4 \leq \log c + n \log 2$

$2n \log 2 \leq \log c + n \log 2$

$n \log 2 \leq \log c$

$\boxed{2^n \leq c} \rightarrow$ here we get c be a function which contradicts with our original assumption so,

$$\boxed{2^{2n} \neq O(2^n)}$$

3.2.4: Thm 3.1: for any two functions $f(n)$ & $g(n)$ we have $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ Prove:

Given: The biconditional / double implication

$$p \Leftrightarrow q$$

p: $f(n) = \Theta(g(n))$ & q: $f(n) = O(g(n)) \wedge f(n) = \Omega(g(n))$

Proving $p \Leftrightarrow q$ involves

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

Proof-1: $(p \rightarrow q)$:

By direct proof: Assuming 'A' is true:

if $f(n) = \Theta(g(n))$

then for some constants $c_1, c_2 > 0$

$$0 \leq \underbrace{c_1 g(n) \leq f(n)}_{1} \underbrace{\leq c_2 (g(n))}_{2}$$

Now, for ①:

$$0 \le c_1 g(n) \le f(n) \qquad c_1 > 0$$

which is the definition of

$$\Omega(g(n)) = f(n)$$

Now for ②:

$$f(n) \le c_2 g(n) \qquad c_2 > 0$$

which is the definition of

$$O(g(n)) = f(n)$$

$\Rightarrow \quad f(n) = \Omega(g(n)) \wedge f(n) = O(g(n))$

$\Rightarrow \quad q$ is true.

$\Rightarrow \quad \boxed{p \rightarrow q \text{ is true}}$

Proof-2: $q \longrightarrow p$

By direct proof: Assuming 'q' is true:

$\Rightarrow \quad f(n) = \Omega(g(n)) \wedge f(n) = O(g(n))$

$\Rightarrow$ for some constant $c_1 > 0$ & $\forall n \ge n_0 ; n_0 > 0$

$\Rightarrow \quad c_1 g(n) \le f(n)$

for some constant $c_2 > 0$ & $\forall n \ge n_0 \quad n_0 > 0$

$f(n) \le c_2 g(n)$

$\Rightarrow \quad 0 \le c_1 g(n) \le f(n) \le c_2 g(n)$

$$c_1, c_2 > 0$$

which is the definition of $f(n) = \Theta(g(n))$

$\Rightarrow$ p is true

$\Rightarrow$ $\boxed{q \to p \text{ is also true}}$

Now we have

$(p \to q) \wedge (q \to p)$ as true

$\Rightarrow$ $p \Leftrightarrow q$ is true.

hence our thm is true & proved.

3.2-5: Prove that the running time of an algorithm is $\Theta(g(n))$ iff its worst case is $O(g(n))$ and its best case running time is $\Omega(g(n))$.

A) source: internet:

If T_w is the worst-case running time

T_b is the best-case running time

we can conclude from the given,

$0 \leq c_1 g(n) \leq T_b(n)$ for $n \geq n_b$

$\hookrightarrow \Omega(g(n))$

and

$0 \leq T_w(n) \leq c_2(g(n))$ for $n \geq n_w$

combining the inequalities we get:

$$0 \leq c_1 g(n) \leq T_b(n) \leq T_w(n) \leq c_2 g(n)$$

since running time is bound between T_b and T_w and the above is the definition of Θ-notation we can conclude that

Algorithm's running time if $\Theta(g(n))$.

3.2-6: Prove that $o(g(n)) \cap \omega(g(n))$ is empty set $= \emptyset$.

sol^n: by definitions of $o()$ & $\omega()$

↳ Say function $f(n)$

$\Rightarrow$ $o(g(n)) = f(n)$ iff

* $c\, g(n) > f(n)$ $\forall c > 0$ & $n_0 > 0$ $\forall n \geq n_0$ — (1)

* $0 \leq c\, g(n) < f(n)$ $\forall c > 0$ & $\forall n \geq n_0$ $n_0 > 0$ — (2)

from (1) & (2) which independently says that inequality holds for all constants,

at some point we may arrive for same constants & same n_0's

which generates a contradiction

$\Rightarrow$ there exists no-function which satisfies

$f(n) = o(g(n)) \wedge f(n) = \omega(g(n))$

3.2-7: For a given function $g(n,m)$, we denote by $O(g(n,m))$ the set of functions:

$$O(g(n,m)) = \{ f(n,m) : \text{there exists positive constants } c, n_0 \text{ and } m_0 \text{ such that } 0 \le f(n,m) \le c\,g(n,m) \quad \forall\, n \ge n_0 \vee m \ge m_0 \}$$

$\rightarrow$(or)

A) Ω-notation:

$$\Omega(g(n,m)) = \{ f(n,m) \mid \exists_{c>0}\ \exists_{n_0>0}\ \exists_{m_0>0}$$

s.t $\quad 0 \le c\,g(n,m) \le f(n,m)$

$$\forall\, n \ge n_0 \vee \forall\, m \ge m_0$$

Θ-notation:

$$\Theta(g(n,m)) = \{ f(n,m) \mid \exists_{c_1>0}\ \exists_{c_2>0}\ \exists_{m_0>0}\ \exists_{n_0>0}$$

S.t

$$0 \le c_1 g(n,m) \le f(n,m) \le c_2 g(n,m)$$

$$\forall\, n \ge n_0 \vee \forall\, m \ge m_0$$

3.3.1: $f(n)$ & $g(n)$ are monotically increasing functions

show $f(n)+g(n)$ & $f(g(n))$ are also monotonic.

if $g(n)$ & $f(n)$ are in addition non-negative, then

$f(n)\cdot g(n)$ is monotic.

Proof: Given: $f(n)$ & $g(n)$ are monotically increasing functions.

$\Rightarrow$ $m \geq n$

$f(m) \geq f(n)$ — (1)

$g(m) \geq g(n)$ — (2)

adding (1) & (2)

$\Rightarrow$ $\boxed{f(m)+g(m) \geq f(n)+g(n)}$ — (3)

ineq (3) proves that $f(m)+g(m) / f(n)+g(n)$ is monotonically increasing.

Since: for $\forall\, m \geq n$

$g(m) \geq g(n)$

$\Rightarrow$ from given $f(n)$ is monotonically increasing

$\Rightarrow$ $\boxed{f(g(m)) \geq f(g(n))}$

Given: $f(n)$ & $g(n)$ are addition non-negative

meaning we are performing addition with non-negative numbers

$\Rightarrow \quad \forall\ m \geq n$

$$f(m) \geq f(n) \quad - (4)$$

$$g(m) \geq g(n) \quad - (5)$$

multiplying (4) & (5)

$$\Rightarrow \quad f(m) \cdot g(m) \geq f(n) \cdot g(n)$$

$\Rightarrow$ $f(n) \cdot g(n)$ is monotonically increasing. //

3.3.2: prove that $\lfloor \alpha n \rfloor + \lceil (1-\alpha) n \rceil = n$ for $n \in \mathbb{Z}$ & $\alpha \in \mathbb{R}$ and $0 \leq \alpha \leq 1$

Sol^n: $\Rightarrow$ LHS:

$$\lfloor \alpha n \rfloor + \lceil n - n\alpha \rceil$$

$$\lfloor \alpha n \rfloor + \lceil n + (-n\alpha) \rceil$$

given n is integer;

$$\lceil I + m \rceil = I + \lceil m \rceil \quad I \in \mathbb{Z}^+$$

$$\Rightarrow \lfloor \alpha n \rfloor + n + \lceil -n\alpha \rceil \qquad \lceil -a \rceil = -\lfloor a \rfloor$$

$$\Rightarrow \cancel{\lfloor \alpha n \rfloor} + n - \cancel{\lfloor n\alpha \rfloor} = n = RHS$$

Proved.

3.3.3: Use $1+x \leq e^x$ or other means to show that $(n + o(n))^k = \Theta(n^k)$ for any real constant k. Conclude that $\lceil n \rceil^k = \Theta(n^k)$ and $\lfloor n \rfloor^k = \Theta(n^k)$

Sol^n: Given,

$$(n + o(n))^k$$

$$\mapsto \quad o(n) = \{ f(n) : \exists c_1 > 0 \; \exists n_0 > 0 \; \forall n \geq n_0 \quad f(n) < c.n \}$$

$$\Rightarrow (n + o(n))^k < (n + c_1 n)^k$$

$$(n + o(n))^k < (n(1 + c_1))^k$$

$$(n + o(n))^k < n^k (1 + c_1)^k = c_2^k n^k$$

$$= O(n^k) \quad -(1)$$

Now;

$$\Rightarrow (n + o(n))^k \geq (n+0)^k$$

$$(n + o(n))^k \geq n^k$$

$$= \Omega(n^k) \quad -(2)$$

from (1) & (2)

$$\Rightarrow (n + o(n))^k = \Theta(n^k)$$

(ii) $n-1 < \lfloor n \rfloor \leq n$ (floor)

$$\longrightarrow \quad n-1 < \lfloor n \rfloor \leq n$$

$$\Rightarrow \frac{n}{2} \leq n-1 < \lfloor n \rfloor \leq n$$

$$\Rightarrow \left(\frac{n}{2}\right)^k \leq (\lfloor n \rfloor)^k \leq n^k$$

$$\Rightarrow \left(\frac{1}{2}\right)^k n^k \leq (\lfloor n \rfloor)^k \leq n^k \Rightarrow \underbrace{c\,n^k \leq (\lfloor n \rfloor)^k \leq n^k}_{\text{which is the definition of } \Theta(n^k)}$$

* $n \leq \lceil n \rceil < n+1$ (ceil)

$$n^k \leq (\lceil n \rceil)^k \leq (2n)^k$$

$$\underbrace{n^k \leq (\lceil n \rceil)^k \leq 2^k n^k}_{\text{which is the definition of}} \ \Theta(n^k)$$

Exercise

3.3.4: prove:

$$n! = o(n^n)$$

$$n! = \omega(2^n)$$

$$\log(n!) = \Theta(n\log n)$$

Soln:

$n! \leq n^n$

↳ $n(n-1)(n-2) \ldots 1 \leq n.n.n \ldots n$

for every term in factorial we are multiplying by 'n' on RHS.

$\Rightarrow \forall n \geq n_0 \quad \exists c>0 \quad \exists n_0 > 0$

$$\boxed{n! < n^n . c}$$

$$\Rightarrow \boxed{n! = o(n^n)}$$

$n! = \omega(2^n)$

↳ $n! = n(n-1)(n-2) \ldots 1$

$2^n = 2.2.2 \ldots .2$

$\Rightarrow \quad n! > 2^n$

as forevery term in factorial we are multiplying by 2 on RHS which is less than the $(n-k)$ $(k \neq n-2, n-1)$

$\Rightarrow \quad \exists c>0 \quad \exists n_0>0 \quad \forall n \geq n_0$

$$\boxed{n! = \omega(2^n)}$$

* $\log(n!) = \log(n.(n-1).(n-2)\ldots\ldots)$

we know than $n! \leq n^n$ $\exists n_0 > 0$ $\forall n \geq n_0$

taking log on both sides;

$$\log(n!) \leq \log(n^n) \quad - (1)$$

stirling's approximation: $\boxed{n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}}$

$$\boxed{\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}}$$

But;

$$\log(n) + \log(n-1) + \ldots \log(1) \geq \log\left(\frac{n}{2}\right) + \log\left(\frac{n}{2}+1\right) + \ldots \log(n)$$

$$\log(n!) \geq \log\left(\frac{n}{2}\right) + \log\left(\frac{n}{2}\right) + \ldots\ldots \log\left(\frac{n}{2}\right)$$

$$\log(n!) \geq \frac{n}{2}\log\left(\frac{n}{2}\right) \quad - (2)$$

from eq 1 & 2

$$\boxed{\log(n!) = \Theta(n\log n)}$$

3.3.5: Is function $\lceil \log n \rceil !$ polynomially bounded?
is $\lceil \log(\log n) \rceil !$ " " ?

A) $f(n)$ is polynomially bounded if $\exists c > 0$ $\exists k > 0$ $\exists n_0 > 0$

$\forall n \geq n_0$ $\boxed{f(n) \leq c \cdot n^k}$

~~Lemma~~

$$\log(f(n)) \leqslant c.k \log n \longrightarrow O(\log n)$$

we need to compare $\lceil \log n \rceil!$ & $\log n$ growth.

from previous exercise Q8, $\log n! = \Theta(n \log n)$

$$\Rightarrow \log(\lceil \log n \rceil !) = \Theta(\lceil \log n \rceil \log \lceil \log n \rceil)$$

$$\Rightarrow \log(\lceil \log n \rceil !) = \Theta(\underline{\log n . \log(\log n)})$$

& $\log n$,

asymptotically / ultimate behaviour
as it approaches a particular limit

two functions are said to be asymptotic if they reach/ approach same Limit.

$$\log n . \log(\log n) > \log n \quad \forall n > 4$$

$$\Rightarrow \qquad = \omega(\log n)$$

$$\Rightarrow \log(\lceil \log n \rceil !) = \omega(\log n)$$

$\Rightarrow$ it is not polynomially bounded

3.3-6: which one is asymptotically Larger: $\log(\log^* n)$ or $\log^*(\log n)$?

Soln: Let $\log^* n = x$ → no. of iterations

$\log^*(\log n) = x-1$ → reduces the iteration by 1.

$\log(\log^* n) = \log(x)$

$\Rightarrow \quad x-1 > \log x$

$$\boxed{\log^*(\log n) > \log(\log^* n)}$$

3.3.7: Show that golden ratio ϕ & its conjugate $\hat{\phi}$ both satify the equation $\boxed{x^2 = x+1}$

Ans: $\phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}$

↳ substituting them in the equation $\boxed{x^2 = x+1}$

$\rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{1+\sqrt{5}}{2}\right)+1 \qquad \Rightarrow \left(\frac{1-\sqrt{5}}{2}\right)^2 = \left(\frac{1-\sqrt{5}}{2}\right)+1$

$\rightarrow \frac{1}{4}+\frac{5}{4}+\frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}+2}{2} = \frac{3+\sqrt{5}}{2} \qquad \frac{1}{4}+\frac{5}{4}$

LHS = RHS

3.3.8: Prove by induction that the i^{th} Fibonacci number satifies the equation $F_i = (\phi^i - \hat{\phi}^i)/\sqrt{5}$

Ans: Base case: for $i = 1$

$F_1 = 1;$

$\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \Rightarrow \frac{\phi - \hat{\phi}}{\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} \times \frac{1}{\sqrt{5}} = 1$

Induction hypothesis: Assumption that $F_i = \dfrac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ is true for all $k \geq 1$

$$\Rightarrow \quad F_K = \frac{\phi^K - \hat{\phi}^K}{\sqrt{5}} ; \quad F_{K-1} = \frac{\phi^{K-1} - \hat{\phi}^{K-1}}{\sqrt{5}}$$

$\Rightarrow F_{K+1} = F_K + F_{K-1}$ [By definition]

$$\Rightarrow \boxed{F_{K+1} = \frac{\phi^K - \hat{\phi}^K}{\sqrt{5}} + \frac{\phi^{K-1} - \hat{\phi}^{K-1}}{\sqrt{5}}}$$

$$= \frac{\phi^K - \hat{\phi}^K}{\sqrt{5}} + \frac{\frac{\phi^K}{\phi} - \frac{\hat{\phi}^K}{\hat{\phi}}}{\sqrt{5}}$$

$$= \frac{\phi^K - \hat{\phi}^K}{\sqrt{5}} + \frac{\hat{\phi}\,\phi^K - \phi\,\hat{\phi}^K}{\underbrace{\phi\hat{\phi}}_{-1}} \times \frac{1}{\sqrt{5}}$$

$\Rightarrow$ taking common terms;

$$\boxed{\phi\hat{\phi} = -1}$$

$$\cancel{\frac{\phi^K - \hat{\phi}^K}{\sqrt{5}}} \quad \frac{\phi^K - \hat{\phi}^K}{\sqrt{5}} + \frac{\hat{\phi}^K \phi - \hat{\phi}\,\phi^K}{\sqrt{5}}$$

$$\Rightarrow \frac{\phi^K(1 - \hat{\phi}) - \hat{\phi}^K(1 - \phi)}{\sqrt{5}}$$

$$\boxed{\phi + \hat{\phi} = 1}$$

$$\Rightarrow \frac{\phi^{K+1} - \hat{\phi}^{K+1}}{\sqrt{5}}$$

$$\boxed{F_{K+1} = \frac{\phi^{K+1} - \hat{\phi}^{K+1}}{\sqrt{5}}}$$

Proved //

3.3.9 Show that $K \log K = \Theta(n)$ implies $K = \Theta(n/\log n)$ 5

Ans: By using the symmetry property of Θ-notation;

$$K \ln K = \Theta(n) \Rightarrow \boxed{n = \Theta(K \ln K)} \quad -(1)$$

$\Rightarrow$ taking log;

$$\ln(n) = \Theta(\ln(K \ln K))$$

$$\ln(n) = \Theta(\ln K + \ln(\ln K))$$

$$\boxed{\ln(n) = \Theta(\ln K)} \quad -(2)$$

$(1)/(2) \Rightarrow$

$$\frac{n}{\ln(n)} = \frac{\Theta(K \ln K)}{\Theta(\ln K)}$$

$$\frac{n}{\ln(n)} = \Theta(K)$$

again by symmetry propertery:

$$\boxed{K = \Theta\left(\frac{n}{\ln n}\right)}$$