The determination of the elastic field of an ellipsoidal inclusion, and related problems

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It is supposed that a region within an isotropic elastic solid undergoes a spontaneous change of form which, if the surrounding material were absent, would be some prescribed homogeneous deformation. Because of the presence of the surrounding material stresses will be present both inside and outside the region. The resulting elastic field may be found very simply with the help of a sequence of imaginary cutting, straining and welding operations. In particular, if the region is an ellipsoid the strain inside it is uniform and may be expressed in terms of tabulated elliptic integrals. In this case a further problem may be solved. An ellipsoidal region in an infinite medium has elastic constants different from those of the rest of the material; how does the presence of this inhomogeneity disturb an applied stress-field uniform at large distances? It is shown that to answer several questions of physical or engineering interest it is necessary to know only the relatively simple elastic field inside the ellipsoid.

1. Introduction

In the physics of solids a number of problems present themselves in which the uniformity of an elastic medium is disturbed by a region within it which has changed its form or which has elastic constants differing from those of the remainder. Some of these problems may be solved for a region of arbitrary shape. Others are intractable unless the region is some form of ellipsoid. Fortunately, the general ellipsoid is versatile enough to cover a wide variety of particular cases. It is the object of this paper to develop a simple method of solving these problems.

When a twin forms inside a crystal the material is left in a state of internal stress, since the natural change of shape of the twinned region is restrained by its surroundings. A similar state of strain arises if a region within the crystal alters its unconstrained form because of thermal expansion, martensitic transformation, precipitation of a new phase with a different unit cell, or for some other reason. These examples suggest the following general problem in the theory of elasticity.

The transformation problem

A region (the 'inclusion') in an infinite homogeneous isotropic elastic medium undergoes a change of shape and size which, but for the constraint imposed by its surroundings (the 'matrix'), would be an arbitrary homogeneous strain. What is the elastic state of inclusion and matrix?

We shall solve this problem with the help of a simple set of imaginary cutting, straining and welding operations. Cut round the region which is to transform and remove it from the matrix. Allow the unconstrained transformation to take place. Apply surface tractions chosen so as to restore the region to its original form, put it back in the hole in the matrix and rejoin the material across the cut. The stress is now zero in the matrix and has a known constant value in the inclusion. The applied surface tractions have become built in as a layer of body force spread over the interface between matrix and inclusion. To complete the solution this unwanted layer is removed by applying an equal and opposite layer of body force; the additional elastic field thus introduced is found by integration from the expression for the elastic field of a point force.

So far nothing has been assumed about the shape of the inclusion. However, we shall find that if it is an ellipsoid the stress within the inclusion is uniform. This fact enables us to use the solution of the transformation problem as a convenient stepping-stone in solving a second set of elastic problems. Superimpose on the whole solid a uniform stress which just annuls the stress in the inclusion. The removal of the unstressed inclusion to leave a hole with a stress-free surface is then a mere formality, and we have solved the problem of the perturbation of a uniform stress field by an ellipsoidal cavity. More generally, suppose that the uniform applied stress does not annul the stress in the inclusion. Then the stress and strain in the inclusion are not related by the Hooke law of the material, since part of the strain arises from a non-elastic twinning or other transformation with which no stress is associated. The stress and strain are, however, related by the Hooke law of some hypothetical material, and the transformed ellipsoid may be replaced by an ellipsoid of the hypothetical material which has suffered the same total strain, but purely elastically. We have thus solved the following problem.

The inhomogeneity problem

An ellipsoidal region in a solid has elastic constants differing from those of the remainder (if, in particular, the constants are zero within the ellipsoid we have the case of a cavity). How is an applied stress, uniform at large distances, disturbed by this inhomogeneity?

The strain in the inclusion or inhomogeneity may be found explicitly in terms of tabulated elliptic integrals. The elastic field at large distances is also easy to determine. The field at intermediate points is more complex, but for many purposes we do not need to know it. In fact, knowing only the uniform strain inside the ellipsoid we can find the following items of physical or engineering interest:

(i) The elastic field far from an inclusion.

(ii) All the stress and strain components at a point immediately outside the inclusion.

(iii) The total strain energy in matrix and inclusion.

(iv) The interaction energy of the elastic field of the inclusion with another elastic field.

(v) The elastic field far from an inhomogeneity.

(vi) All the stress and strain components at a point immediately outside the inhomogeneity. (This solves the problem of stress concentration.)

(vii) The interaction energy of the inhomogeneity with an elastic field.

(viii) The change in the gross elastic constants of a material when a dilute dispersion of ellipsoidal inhomogeneities is introduced into it.

Problems (i) to (iv) can also be solved for an inclusion of arbitrary shape, (i) and (iv) trivially, (ii) and (iii) if we can evaluate the necessary integrals. They differ, of course, from the problems considered by Nabarro (1940) and Kröner (1954) in which the inclusion breaks away from the matrix. Problems (v) to (viii) can only be solved for the ellipsoid. They each have an analogue in the theory of slow viscous flow.

Many particular cases of these problems have been discussed. Robinson (1951) gives references to earlier work; see also Shapiro (1947), Sternberg, Eubanks & Sadowsky (1951). Apart from some increase in generality (we consider shear transformations and the disturbance of an arbitrary shear stress by an ellipsoidal inhomogeneity) our treatment is, perhaps, rather simpler and more direct than the orthodox method. Nowhere do we have to introduce ellipsoidal co-ordinates, search for suitable stress functions or match stress and displacement at an interface. Indeed, we do not even use the equations of elastic equilibrium explicitly except in certain of the applications (i) to (viii).

2. The general inclusion

We employ the usual suffix notation. A repeated suffix is summed over the values 1, 2, 3 and suffixes preceded by a comma denote differentiation:

$$u_{i,j} = \partial u_i/\partial x_j, \quad \phi_{,ik} = \partial^2\phi/\partial x_i \partial x_k.$$

The elastic displacement u_i, strain e_{ij} and stress p_{ij} are related by

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \tag{2·1}$$

$$p_{ij} = \lambda e_{mm}\delta_{ij} + 2\mu e_{ij} \tag{2·2}$$

in an isotropic medium with Lamé constants λ, μ. When a particular set of elastic functions are distinguished by an affix (e.g. u_i^C, e_{ij}^C, p_{ij}^C), it is to be understood that they are related by (2·1) and (2·2). It is often convenient to split a second-order tensor f_{ij} into its scalar and so-called deviatoric parts:

$$f_{ij} = {}'f_{ij} + \tfrac{1}{3}f\delta_{ij},$$

where $$f = f_{mm} \quad \text{and} \quad {}'f_{ij} = f_{ij} - \tfrac{1}{3}f\delta_{ij}.$$

Thus, for example, (2·2) may be written

$$p = 3\kappa e, \quad {}'p_{ij} = 2\mu\, {}'e_{ij} \quad (\kappa = \lambda + \tfrac{2}{3}\mu), \tag{2·3}$$

and the inversion

$$e = p/3\kappa, \quad {}'e_{ij} = {}'p_{ij}/2\mu$$

is immediate, whereas to find e_{ij} in terms of p_{ij} from (2·2) is more difficult. For two tensors f_{ij}, g_{ij} we also have the convenient relation $f_{ij}g_{ij} = \frac{1}{3}fg + {}'f_{ij}\,{}'g_{ij}$; there are no cross-terms between the scalar and deviatoric parts. The elastic energy density is thus

$$\tfrac{1}{2}p_{ij}e_{ij} = \tfrac{1}{2}(\kappa e^2 + 2\mu\, {}'e_{ij}\,{}'e_{ij}) = \frac{1}{2}\left(\frac{1}{9\kappa}p^2 + \frac{1}{2\mu}\,{}'p_{ij}\,{}'p_{ij}\right). \tag{2·4}$$

Following Robinson (1951) we shall give the name 'stress-free strain' to the uniform transformation strain e_{ij}^T which the inclusion would undergo in the absence of the matrix. The main problem is to find the 'constrained strain' e_{ij}^C in the inclusion when it transforms while it is embedded in the matrix and also the strain set up in the matrix, which we shall also call e_{ij}^C. Let S be the surface separating matrix and inclusion, n_i its outward normal and $\mathrm{d}S_i = n_i\,\mathrm{d}S$ the product of the normal and an element of S. We now carry out the steps outlined in the Introduction.

I. Remove the inclusion and allow it to undergo the stress-free strain e_{ij}^T without altering its elastic constants. Let

$$p_{ij}^T = \lambda e^T \delta_{ij} + 2\mu e_{ij}^T$$

be the stress derived from e_{ij}^T by Hooke's law. At this stage the stress in the inclusion and matrix is zero.

II. Apply surface tractions $-p_{ij}^T n_j$ to the inclusion. This brings it back to the shape and size it had before transformation. Put it back in the matrix and reweld across S. The surface forces have now become a layer of body force spread over S.

III. Let these body forces relax, or, what comes to the same thing, apply a further distribution $+p_{ij}^T n_j$ over S. The body is now free of external force but in a state of self-stress because of the transformation of the inclusion.

Since the displacement at $\mathbf{r}$ due to a point-force F_i at $\mathbf{r}'$ is (Love 1927)

$$U_j(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi\mu}\frac{F_j}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{16\pi\mu(1-\sigma)} F_l \frac{\partial^2}{\partial x_l \partial x_j} |\mathbf{r}-\mathbf{r}'|, \tag{2·5}$$

the displacement impressed on the material in stage III is

$$u_i^C(\mathbf{r}) = \int_S \mathrm{d}S_k p_{jk}^T U_j(\mathbf{r}-\mathbf{r}'), \tag{2·6}$$

where σ is Poisson's ratio. It will be convenient to take the state of the material at the conclusion of stage II as a state of zero displacement. This is reasonable, since the stress and strain in the matrix are then zero and the inclusion, though not stress-free, has just the geometrical form which it had before the transformation occurred. u_i^C is then the actual displacement in the matrix and inclusion. The strain in matrix or inclusion is

$$e_{ij}^C = \tfrac{1}{2}(u_{i,j}^C + u_{j,i}^C).$$

The stress in the matrix is derived from e_{ij}^C by Hooke's law:

$$p_{ij}^C = \lambda e^C \delta_{ij} + 2\mu e_{ij}^C.$$

On the other hand, the inclusion had a stress $-p_{ij}^T$ even before stage III, so that the stress in it is

$$p_{ij}^I = p_{ij}^C - p_{ij}^T = \lambda(e^C - e^T)\,\delta_{ij} + 2\mu(e_{ij}^C - e_{ij}^T), \tag{2·7}$$

where, according to our general convention, p_{ij}^C is the stress derived by Hooke's law from the strain e_{ij}^C in the inclusion.

By using Gauss's theorem and the equivalence of $\partial/\partial x_i$ and $-\partial/\partial x_i'$ when acting on $|\mathbf{r}-\mathbf{r}'|$, (2·7) may be made to read

$$u_i^C = \frac{1}{16\pi\mu(1-\sigma)} p_{jk}^T \psi_{,ijk} - \frac{1}{4\pi\mu} p_{ik}^T \phi_{,k}, \tag{2·8}$$

where

$$\phi = \int_V \frac{\mathrm{d}v}{|\mathbf{r}-\mathbf{r}'|} \quad \text{and} \quad \psi = \int_V |\mathbf{r}-\mathbf{r}'|\,\mathrm{d}v$$

are the ordinary Newtonian potential and the biharmonic potential of attracting matter of unit density filling the volume V bounded by S. Evidently

$$\nabla^2\psi = 2\phi \tag{2·9}$$

and

$$\nabla^4\psi = 2\nabla^2\phi = \begin{cases} -8\pi \text{ inside } S, \\ 0 \text{ outside } S. \end{cases}$$

Generally we must know both ψ and ϕ. However, if we are interested only in the dilatation in the material, it is enough to know ϕ:

$$e^C = -\frac{1-2\sigma}{8\pi\mu(1-\sigma)} p^T_{ik}\phi_{,ik}. \tag{2·10}$$

Again, if e^T_{ij} is a pure dilatation $\frac{1}{3}e^T\delta_{ij}$, then

$$e^C_{il} = -\frac{1}{4\pi}\frac{1+\sigma}{3(1-\sigma)} e^T\phi_{,il},$$

a result due to Crum (Nabarro 1940). In this case the dilatation is $e^T(1+\sigma)/3(1-\sigma)$ in the inclusion and zero in the matrix. Thus, for example, with $\sigma = \frac{1}{3}$, the constraint of the matrix reduces the free expansion of the inclusion by a factor $\frac{2}{3}$.

The second derivatives of a potential function satisfying $\nabla^2 U = -4\pi\rho$ undergo a jump $\Delta U_{,ij} = -4\pi\Delta\rho n_i n_j$ on crossing a surface (with normal n_i) across which the density jumps by $\Delta\rho$ (Poincaré 1899). (This is perhaps more familiar in the form: the jump in attraction across a double layer is equal to its moment. In our problem $-\phi_{,i}$ is the potential of a double layer over S with unit moment directed along the x_i axis and $\phi_{,ij}$ is the corresponding force.) This gives for ϕ the expression

$$\phi_{,ij}(\text{out}) - \phi_{,ij}(\text{in}) = 4\pi n_i n_j \tag{2·11}$$

for the difference at adjacent points just inside and outside S. Applying the same argument in turn to $\psi_{,ij}$, which is the potential derived from the density $-2\phi_{,ij}/4\pi$, we have

$$\psi_{,ijkl}(\text{out}) - \psi_{,ijkl}(\text{in}) = 8\pi n_i n_j n_k n_l. \tag{2·12}$$

From (2·11), (2·12) and (2·8) we can find the stresses and strains just outside the inclusion from their values at an adjacent point just inside without having to solve the exterior problem at all. We easily find that

$$e^C(\text{out}) = e^C(\text{in}) - \frac{1}{3}\frac{1+\sigma}{1-\sigma}e^T - \frac{1-2\sigma}{1-\sigma}\,{}'e^T_{ij}n_i n_j$$

and

$$
\begin{aligned}
{}'e^C_{il}(\text{out}) = {}'e^C_{il}(\text{in}) + \frac{1}{1-\sigma}\,{}'e^T_{jk}n_j n_k n_i n_l - {}'e^T_{ik}n_k n_l - {}'e^T_{lk}n_k n_i \\
+ \frac{1-2\sigma}{3(1-\sigma)}\,{}'e^T_{jk}n_j n_k \delta_{il} - \frac{1}{3}\frac{1+\sigma}{1-\sigma}e^T(n_i n_l - \tfrac{1}{3}\delta_{il}).
\end{aligned} \tag{2·13}
$$

The C quantities are related by (2·2) and so are the T quantities. Thus either or both sides of these equations may be expressed in terms of stress without trouble. This solves problem (ii).

We can find a convenient alternative form for u^C_i by noting that (2·5) may be written as

$$16\pi\mu(1-\sigma)\,U_i = \frac{F_j}{|\mathbf{r}-\mathbf{r}'|}\left[(3-4\sigma)\,\delta_{ij} + \frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{r}-\mathbf{r}'|^2}\right]. \tag{2·14}$$

Inserting this in (2·6) and using Gauss's theorem to convert to a volume integral we find

$$u^C_i(\mathbf{x}) = \frac{p_{jk}}{16\pi\mu(1-\sigma)}\int_V \frac{\mathrm{d}v}{r^2} f_{ijk}(\mathbf{l}) = \frac{e^T_{jk}}{8\pi(1-\sigma)}\int_V \frac{\mathrm{d}v}{r^2} g_{ijk}(\mathbf{l}), \tag{2·15}$$

where r and $\mathbf{l} = (l_1, l_2, l_3)$ are the length and direction of the line drawn from the volume element $\mathrm{d}v$ to the point of observation $\mathbf{x}$ and

$$f_{ijk} = (1-2\sigma)(\delta_{ij}l_k + \delta_{ik}l_j) - \delta_{jk}l_i + 3l_il_jl_k, \tag{2·16}$$

$$g_{ijk} = (1-2\sigma)(\delta_{ij}l_k + \delta_{ik}l_j - \delta_{jk}l_i) + 3l_il_jl_k. \tag{2·17}$$

For points $\mathbf{x}$ remote from the inclusion we may take everything except $\mathrm{d}v$ outside the sign of integration to obtain

$$u_i^C(\mathbf{x}) = Vp_{jk}^T f_{ijk}/16\pi\mu(1-\sigma)\,r^2 = Ve_{jk}^T g_{ijk}/8\pi(1-\sigma)\,r^2, \tag{2·18}$$

where r and $\mathbf{l}$ are now the distance and direction of $\mathbf{x}$ from the inclusion. This solves problem (i).

The strain energy density in the inclusion is $\frac{1}{2}p_{ij}^I e_{ij}^I$, where e_{ij}^I is the strain derived from p_{ij}^I by Hooke's law. By (2·7) the elastic energy in the inclusion is thus

$$\frac{1}{2}\int_V p_{ij}^I(e_{ij}^C - e_{ij}^T)\,\mathrm{d}v. \tag{2·19}$$

The elastic energy in the matrix is

$$-\frac{1}{2}\int_S p_{ij}^C u_i^C\,\mathrm{d}S_j = -\frac{1}{2}\int_S p_{ij}^I u_i^C\,\mathrm{d}S_j = -\frac{1}{2}\int_V p_{ij}^I e_{ij}^C\,\mathrm{d}v. \tag{2·20}$$

The first member exhibits it as the work done in setting up the elastic field by applying suitable forces to the surface S; the sign is correct if the normal points from inclusion to matrix. The second follows because displacement and normal traction are continuous across S. The third follows from Gauss's theorem, the equilibrium equation $p_{ij,j}^I = 0$ and the symmetry condition $p_{ij}^I = p_{ji}^I$. The total strain energy in matrix and inclusion is thus

$$E_{\text{el.}} = -\frac{1}{2}\int_V p_{ij}^I e_{ij}^T\,\mathrm{d}v. \tag{2·21}$$

In the special case where e_{ij}^T is a uniform expansion we have at once from (2·10) and (2·21) $E_{\text{el.}} = 2\mu(e^T)^2V(1+\sigma)/9(1-\sigma)$, whatever be the shape of the cavity, as pointed out by Crum (Nabarro 1940).

The interaction energy of the elastic field u_i^C with another field u_i^A is (Eshelby 1951, 1956)

$$E_{\text{int.}} = \int_\Sigma (p_{ij}^C u_i^A - p_{ij}^A u_i^C)\,\mathrm{d}S_j \tag{2·22}$$

taken over any surface Σ enclosing the inclusion. Let us take Σ to be a surface just outside S. Once again, since u_i^C and the normal stress are continuous across S, (2·22) can be converted into an integral over a surface just inside S, and hence into a volume integral over the inclusion:

$$E_{\text{int.}} = \int_V (p_{ij}^I e_{ij}^A - p_{ij}^A e_{ij}^C)\,\mathrm{d}v.$$

The second term in the integrand is equal to $-p_{ij}^C e_{ij}^A$, so that

$$E_{\text{int.}} = -\int_V p_{ij}^T e_{ij}^A\,\mathrm{d}v = -\int_V p_{ij}^A e_{ij}^T\,\mathrm{d}v = -\int_\Sigma p_{ij}^A u_i^T\,\mathrm{d}S_j. \tag{2·23}$$

This solves problem (iv). The same result is reached by evaluating (2·22) over a large sphere using the remote field (2·18). It is fortunate that we need only e^T_{ij} and not e^C_{ij}. In fact the last member of (2·23) has formally the appearance of being the work done against the external field in 'blowing up' the inclusion (regarded as rigid) to a final shape specified by e^T_{ij}. It is perhaps not obvious that this should be so, since the inclusion is not rigid and its final shape is described by a displacement u^C_i which may be quite complicated (e.g. it produces a barrel or pincushion distortion of the cubical inclusion which Cochardt, Schoek & Wiedersich (1955) consider). If we regard the inclusion as capable of moving through the matrix, as in the elastic model of a substitutional atom,

$$F_l = -\partial E_{\text{int.}}/\partial \xi_l \tag{2·24}$$

is the 'force' on the inclusion, where ξ_l is a vector specifying its position.

Let $E_{\text{trans.}}$ be the change of internal energy when the inclusion transforms in the absence of the matrix. Consider the sum

$$E = E_{\text{trans.}} + E_{\text{el.}} + E_{\text{int.}}.$$

Give their adiabatic values to λ, μ, κ and suppose that the constrained transformation occurs without any heat flow. Then E can be interpreted indifferently as the enthalpy change of the inclusion, the enthalpy change of the body (inclusion plus matrix) or the change of internal energy of the body and loading mechanism regarded as a single thermodynamic system. There is a similar interpretation for an isothermal process if we read 'Helmholtz free energy' for 'internal energy' and 'Gibbs free energy' for 'enthalpy' and give λ, μ, κ their isothermal values.

Since problems (v) to (viii) can only be solved for an ellipsoid their discussion is deferred to § 4.

As a simple example of the use of (2·18), suppose that we need the field at large distances from a dislocation loop of area A in the x_1x_2 plane with its Burgers vector along the positive x_3 axis. We have to insert a sheet of material of area A and thickness b. One way to do this is to cut out a disk of area A and height h, give it a permanent strain $e^T_{33} = b/h$ to increase its height by b and then force it back into the cavity. In (2·18) we have to put $V = Ah$, $e^T_{33} = b/h$ and the other e^T_{ij} equal to zero. Thus

$$u_i = bA\, g_{i33}/8\pi(1-\sigma)\, r^2.$$

Suppose next that the Burgers vector lies in the plane of the loop and, say, along the x_1 axis. We now give the disk a permanent shear $e^T_{13} = \frac{1}{2}b/h$, which gives its upper and lower surfaces a relative offset b, and re-insert it in the matrix. In the limit $h \to 0$ we have a displacement discontinuity b across the loop. Putting $V = Ah$, $e^T_{13} = e^T_{31} = \frac{1}{2}b/h$ and the other e^T_{ij} zero in (2·18) we get

$$u_i = bAg_{i13}/4\pi(1-\sigma)\, r^2, \tag{2·25}$$

reproducing a result of Nabarro's (1951). It is perhaps not quite clear that the restraint of the matrix will not reduce the offset to something less than b. Actually this is not true in the limit $h \to 0$. But we can see that (2·25) is correct by inserting $Ve^T_{13} = \frac{1}{2}Ab$ in (2·23); this gives $E_{\text{int.}} = -bAp^A_{13}$, which is the correct interaction energy for such a loop (Nabarro 1952). Indeed by the same argument we can find the remote field of an arbitrary loop of area A, normal n_i and Burgers vector b_i. The interaction energy is $-b_i p_{ij}^A n_j A$ for any p_{ij}^A. Equation (2·23) then shows that $Ve_{ij}^T = \frac{1}{2}(b_i n_j + b_j n_i)$ and (2·18) gives

$$u_i = Ab_i n_k g_{ijk}/8\pi(1-\sigma)\,r^2.$$

There is, in fact, a more general connexion with dislocation theory. The stress-free strain in the inclusion may always be imagined to be (or may actually be) the result of plastic deformation. A set of dislocation loops (with equal Burgers vectors) expanding from zero size on a close set of equally-spaced planes will give a shear if their Burgers vectors lie in the planes, or an extension perpendicular to the planes if their Burgers vectors are at right angles to the planes. (In the latter case their movement is non-conservative.) If the deformation occurs in the absence of the matrix these loops will, so to speak, disappear into free space. But if the inclusion is embedded in the matrix the dislocations will lodge in the surface S separating matrix from inclusion. S then becomes the discontinuity surface of a Somigliana (1914, 1915) dislocation. In this generalized type of dislocation there is a variable discontinuity d_i of displacement across S. In our model, d_i makes itself felt through the gaps and interpenetrations of matter which we should find if we tried to re-insert the transformed inclusion into the hole in the matrix without pulling it back to its original shape. It is easy to see that d_i has the value $\frac{1}{2}e_{ij}^T x_j'$ at a point x_j' of S and hence to verify, after some manipulation, that our expression for u_i^C agrees with Somigliana's.

3. The ellipsoidal inclusion

In discussing the elastic field inside an inclusion it is convenient to redefine the l_i in (2·16), (2·17) to be the direction cosines of a line drawn *from* the point of observation $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ *to* the volume element $\mathrm{d}v$. This involves changing the sign of the integrals in (2·15). Let us first integrate over an elementary cone $\mathrm{d}\omega(\mathbf{l})$ centred on the direction $\mathbf{l} = (l_1, l_2, l_3) = (l, m, n)$ with its vertex at $\mathbf{x}$. It gives a contribution $r(\mathbf{l})\,\mathrm{d}\omega$ to $\int \mathrm{d}v/r^2$. Thus

$$8\pi\mu(1-\sigma)\,u_i(\mathbf{x}) = -e_{jk}^T \int_{4\pi} r(\mathbf{l})\,d\omega(\mathbf{l})\,g_{ijk}(\mathbf{l}), \tag{3·1}$$

which gives the displacement at $\mathbf{x}$ in terms of an angular integration over the polar diagram $r = r(l, m, n)$ of the surface S as viewed from $\mathbf{x}$.

More briefly we could go directly from (2·6) to (3·1) by writing $r^{-1} = \frac{1}{2}\nabla^2 r$ in (2·14) applying Stokes's theorem in the form

$$\int_S w_{\ldots j,l}\,\mathrm{d}S_j = \int_S w_{\ldots i,i}\,\mathrm{d}S_l$$

and noting that $\mathrm{d}\omega = n_i r_i\,\mathrm{d}S/r^3$.

For the ellipsoid

$$X^2/a^2 + Y^2/b^2 + Z^2/c^2 = 1$$

$r(\mathbf{l})$ is the positive root of

$$(x+rl)^2/a^2 + (y+rm)^2/b^2 + (z+rn)^2/c^2 = 1,$$

that is, $$r(\mathbf{l}) = -f/g + (f^2/g^2 + e/g)^{\frac{1}{2}}, \tag{3·2}$$

where $$g = l^2/a^2 + m^2/b^2 + n^2/c^2 \tag{3·3}$$

and $$f = lx/a^2 + my/b^2 + nz/c^2, \quad e = 1 - x^2/a^2 - y^2/b^2 - z^2/c^2.$$

The sign of the square root is evidently correct, since e is positive if $\mathbf{x}$ is within the ellipsoid. In any case, we may omit this term when (3·2) is inserted in (3·1) since it is even in $\mathbf{l}$, whilst g_{ijk} is odd. To retain the advantages of suffix notation we introduce the 'vector' $$\lambda_1 = l/a^2, \quad \lambda_2 = m/b^2, \quad \lambda_3 = n/c^2.$$

Then $$u_i^C(\mathbf{x}) = \frac{x_m e_{jk}^T}{8\pi(1-\sigma)} \int_{4\pi} \frac{\lambda_m g_{ijk}}{g} \mathrm{d}\omega \tag{3·4}$$

and the strains $$e_{il}^C(\mathbf{x}) = \frac{e_{jk}^T}{16\pi(1-\sigma)} \int_{4\pi} \frac{\lambda_i g_{ljk} + \lambda_l g_{ijk}}{g} \mathrm{d}\omega \tag{3·5}$$

are uniform and depend only on the shape of the ellipsoid. The same is also true for an anisotropic medium. (This verifies a hypothesis of Frank's (private communication).) For it can easily be shown (see, for example, Eshelby 1951) that (2·5) has then to be replaced by $$U_j(\mathbf{r}) = F_i D_{ij}(\mathbf{l})/r,$$

where the functions of direction D_{ij} cannot generally be found in finite form. A repetition of the argument will evidently lead to an expression like (3·4), but with the g_{ijk} no longer given by (2·17).

It is convenient to write the relation between the constrained and stress-free strains in the inclusion in the form
$$e_{il}^C = S_{ilmn} e_{mn}^T. \tag{3·6}$$

From the symmetry of the problem it is clear that the S_{ijkl} have some of the properties of the elastic coefficients of an orthorhombic crystal with its axes parallel to the axes of the ellipsoid, though relations of the form $S_{1122} = S_{2211}$ are not valid. Coefficients coupling an extension and a shear ($S_{1112}, S_{1123}, S_{2311} \ldots$) or one shear to another ($S_{1223} \ldots$) are zero. In fact, (3·5) vanishes if any one of l, m, n appears raised to an odd power in the integrand. The reduction of surface integrals of the type $\int l^{2i} m^{2j} n^{2k} \mathrm{d}\omega/g$ to simple integrals has been given by Routh (1892). We find

$$\left.\begin{aligned} S_{1111} &= Qa^2 I_{aa} + RI_a, \\ S_{1122} &= Qb^2 I_{ab} - RI_a, \\ S_{1212} &= Q\tfrac{1}{2}(a^2+b^2) I_{ab} + R\tfrac{1}{2}(I_a + I_b), \end{aligned}\right\} \tag{3·7}$$

$$\left.\begin{aligned} &\text{where} \quad Q = \frac{3}{8\pi(1-\sigma)}, \quad R = \frac{1-2\sigma}{8\pi(1-\sigma)}, \quad \tfrac{1}{3}Q + R = \frac{1}{4\pi}, \\ &\text{and} \quad I_a = \int \frac{l^2}{a^2} \frac{\mathrm{d}\omega}{g} = 2\pi abc \int_0^\infty \frac{\mathrm{d}u}{(a^2+u)\Delta}, \\ &I_{aa} = \int \frac{l^4}{a^4} \frac{\mathrm{d}\omega}{g} = 2\pi abc \int_0^\infty \frac{\mathrm{d}u}{(a^2+u)^2\Delta}, \\ &I_{ab} = \int \frac{l^2}{a^2} \frac{m^2}{b^2} \frac{\mathrm{d}\omega}{g} = \tfrac{2}{3}\pi abc \int_0^\infty \frac{\mathrm{d}u}{(a^2+u)(b^2+u)\Delta}, \end{aligned}\right\} \tag{3·8}$$

with $$\Delta = (a^2+u)^{\frac{1}{2}} (b^2+u)^{\frac{1}{2}} (c^2+u)^{\frac{1}{2}}.$$

The remaining coefficients are found by simultaneous cyclic interchange of $(1, 2, 3)$, (a, b, c), (l, m, n). I_a, I_b, I_c occur as coefficients in the expression

$$\phi = \tfrac{1}{2}(a^2-x^2)\,I_a+\tfrac{1}{2}(b^2-y^2)\,I_b+\tfrac{1}{2}(c^2-z^2)\,I_c$$

for the Newtonian potential within an ellipsoid of unit density. We have† (Kellogg 1929)

$$I_a = \frac{4\pi abc}{(a^2-b^2)\,(a^2-c^2)^{\frac{1}{2}}}(F-E),$$

$$I_c = \frac{4\pi abc}{(b^2-c^2)\,(a^2-c^2)^{\frac{1}{2}}}\left\{\frac{b(a^2-c^2)^{\frac{1}{2}}}{ac}-E\right\}, \tag{3·9}$$

where $F = F(\theta, k)$ and $E = E(\theta, k)$ are elliptic integrals of the first and second kinds of amplitude and modulus

$$\theta = \sin^{-1}(1-c^2/a^2)^{\frac{1}{2}}, \quad k = (a^2-b^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}}$$

and it is assumed that $$a > b > c.$$

The relations

$$I_a+I_b+I_c = 4\pi, \tag{3·10}$$

$$I_{aa}+I_{ab}+I_{ac} = 4\pi/3a^2, \tag{3·11}$$

$$a^2I_{aa}+b^2I_{ab}+c^2I_{ac} = I_a \tag{3·12}$$

follow from the ω integrals when we use the definition (3·3) of g and the relation $l^2+m^2+n^2 = 1$. Again, if we split the factor $(a^2+u)^{-1}\,(b^2+u)^{-1}$ in the u integral for I_{ab} into partial fractions we have $3(a^2-b^2)\,I_{ab} = I_b-I_a$. Thus when I_a, I_c have been calculated from (3·9) we have for I_b

$$I_b = 4\pi-I_a-I_c$$

and the remaining quantities are found from

$$I_{ab} = (I_b-I_a)/3(a^2-b^2), \tag{3·13}$$

$$I_{aa} = 4\pi/3a^2-I_{ab}-I_{ac} \tag{3·14}$$

and their cyclic counterparts.

For the oblate spheroid $(a = b > c)$ with

$$I_a = I_b = \frac{2\pi a^2 c}{(a^2-c^2)^{\frac{3}{2}}}\left\{\cos^{-1}\frac{c}{a}-\frac{c}{a}\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}\right\} \tag{3·15}$$

the relation (3·13) fails, though not its analogues for I_{bc} or I_{ac}. But from the u integrals for I_{aa} and I_{ab} it is clear that $I_{aa} = 3I_{ab}$ and we may use (3·14). For the prolate spheroid $(b = c < a)$ we have

$$I_b = I_c = \frac{2\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}}\left\{\frac{a}{c}\left(\frac{a^2}{c^2}-1\right)^{\frac{1}{2}}-\cosh^{-1}\frac{a}{c}\right\}, \tag{3·16}$$

and the remaining quantities may be determined similarly.

† Osborn (1945) has given curves for these quantities as functions of b/a and c/a. In his notation $I_a = L$, $I_b = M$, $I_c = N$.

For the elliptic cylinder $x^2/a^2+y^2/b^2 = 1$, $c\rightarrow\infty$ we have the simple results

$$I_a = 4\pi b/(a+b), \quad I_b = 4\pi a/(a+b), \quad I_c = 0,$$
$$I_{ab} = 4\pi/3(a+b)^2, \quad I_{aa} = 4\pi/3a^2-I_{ab}, \quad I_{bb} = 4\pi/3b^2-I_{ab}. \tag{3·17}$$

I_{ac}, I_{bc} and I_{cc} are zero. However, it is clear from (3·7) that what we really need is the limit of their products with c^2. In this sense (3·12) and (3·13) give

$$c^2I_{ac} = \tfrac{1}{3}I_a, \quad c^2I_{bc} = \tfrac{1}{3}I_b, \quad c^2I_{cc} = 0. \tag{3·18}$$

The uniform rotation $\varpi^C_{il} = \frac{1}{2}(u^C_{i,l}-u^C_{l,i})$ in the inclusion may be written in a form

$$\varpi^C_{il} = \Pi_{iljk}e^T_{jk} \tag{3·19}$$

analogous to (3·6). The only non-zero components are Π_{1212}, Π_{2323}, Π_{3131} where, for example,

$$\Pi_{3131} = -\Pi_{1331} = (I_a-I_c)/8\pi. \tag{3·20}$$

These results are only valid in a co-ordinate system whose axes are parallel to the principal axes of the ellipsoid. For any other system there are still relations of the form (3·6), (3·19) and the new coefficients S_{ijkl}, Π_{ijkl} must be found by the usual law for transforming tensors.

Problems (i) to (iv) of § 1 are solved as for the inclusion of arbitrary shape (§ 2). The only simplification is that, since e^C_{ij} is uniform within the inclusion, (2·21) becomes

$$E_{\text{el.}} = -\tfrac{1}{2}Vp^I_{ij}e^T_{ij}, \quad V= \tfrac{4}{3}\pi abc, \tag{3·21}$$

and similarly (2·23) becomes

$$E_{\text{int.}} = -Vp^A_{ij}e^T_{ij} \tag{3·22}$$

if the applied field p^A_{ij} is also uniform. The field immediately outside the inclusion is found from (2·13) using the expressions

$$n_1 = x/a^2h, \quad n_2 = y/b^2h, \quad n_3 = z/c^2h, \quad h^2 = x^2/a^4+y^2/b^4+z^2/c^4 \tag{3·23}$$

for the components of the normal to an ellipsoid at the point x, y, z.

We have seen that usually it is sufficient to know only the elastic field within, just outside and far from the inclusion. The field at any point outside the inclusion can, of course, be found from (2·8) if we know the potentials ϕ and ψ. The expression for ϕ is well known (Kellogg 1929). Dirichlet (1839) calculated the exterior potential of an ellipsoid when the law of attraction is the inverse pth power of the distance. His derivation is only valid for $2 \leqslant p < 3$ and so does not cover the biharmonic case $p = 0$. However, his calculation of the force $-\operatorname{grad}\psi$ is valid for $p = 0$, and the derivatives are all we need to know. His result is

$$\frac{\partial\psi}{\partial x} = x\pi abc\int_\lambda^\infty \frac{U\,\mathrm{d}u}{(a^2+u)\,\Delta}, \quad \frac{\partial\psi}{\partial y} = \dots,$$

where
$$U = 1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}$$

and λ is the positive root of $U(u) = 0$. The integral can be reduced to elliptic integrals by the same substitutions as serve for ϕ. (For the details see, for example, Byrd & Friedman (1954).) For an external point ϕ and ψ are respectively first and second degree polynomials in x^2, y^2, z^2 whose coefficients are integrals of the type (3·8) with the lower limit changed from 0 to λ. These coefficients can be made to depend on the integrals (3·9) with argument

$$\theta = \sin^{-1}(a^2-c^2)^{\frac{1}{2}}/(a^2+\lambda)^{\frac{1}{2}}$$

and modulus

$$k = (a^2-b^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}}.$$

They depend on x, y, z through λ.

4. The ellipsoidal inhomogeneity

The inhomogeneity problem for the ellipsoid is solved in the way outlined in § 1. On the elastic field e^C_{ij} due to an ellipsoidal inclusion with arbitrary e^T_{ij} we superimpose a uniform strain e^A_{ij}. The deformation of the surface of the inclusion is specified by the strain $e^C_{ij}+e^A_{ij}$. Because a part e^T_{ij} of this strain is not associated with any stress (compare equation (2·7)) the uniform stress in the inclusion is given by applying Hooke's law not to $e^C_{ij}+e^A_{ij}$, but rather to $e^C_{ij}+e^A_{ij}-e^T_{ij}$. In the notation of (2·3) the strain in the inclusion is

$$e = e^C+e^A, \quad 'e_{ij} = 'e^C_{ij}+'e^A_{ij}, \tag{4·1}$$

but the stress in it is

$$p = 3\kappa(e^C+e^A-e^T), \quad 'p_{ij} = 2\mu('e^C_{ij}+'e^A_{ij}-'e^T_{ij}). \tag{4·2}$$

Take an ellipsoid the same shape and size as the untransformed inclusion and made of an isotropic material with elastic constants λ_1, μ_1, $\kappa_1 = \lambda_1+\frac{2}{3}\mu_1$ different from those of the matrix and inclusion. Subject this ellipsoid to the strain (4·1). If this treatment develops the stress (4·2) in it, it may be used to replace the inclusion with continuity of displacement and surface traction across the interface. We can always ensure that the correct stress is developed by choosing λ_1, μ_1 suitably. It is only necessary that they should satisfy the relations

$$\kappa_1(e^C+e^A) = \kappa(e^C+e^A-e^T) \tag{4·3}$$

and

$$\mu_1('e^C_{ij}+'e^A_{ij}) = \mu('e^C_{ij}+'e^A_{ij}-'e^T_{ij}). \tag{4·4}$$

Actually, it is the uniform applied field e^A_{ij} and the elastic constants of the inhomogeneity which are prescribed, and (4·3), (4·4) are equations which have to be solved for e^T_{ij} in terms of e^A_{ij}, λ_1, μ_1 after eliminating e^C_{ij} with the help of the relation $e^C_{ij} = S_{ijkl}e^T_{kl}$. Equations (4·3) and (4·4) are not as simple as they appear, since each of e^C, $'e^C_{ij}$ depends on both e^T and $'e^T_{ij}$. However, for the shear components the solution is immediate, since the S_{ijkl} do not couple different shears:

$$e^T_{13} = \frac{\mu-\mu_1}{2(\mu_1-\mu)\,S_{1313}+\mu}\,e^A_{13}, \ldots. \tag{4·5}$$

On the other hand, for the components e^T_{11}, e^T_{22}, e^T_{33} we have to solve the three simultaneous equations

$$\begin{aligned}(\lambda_1-\lambda)\,S_{mmpq}\,e^T_{pq}+2(\mu_1-\mu)\,S_{ijpq}\,e^T_{pq}+\lambda e^T+2\mu e^T_{ij}\\ = (\lambda-\lambda_1)\,e^A+2(\mu-\mu_1)\,e^A_{ij}, \quad ij = 11, 22, 33.\end{aligned}$$

(Only e^T_{11}, e^T_{22}, e^T_{33} appear in the pq summations since, e.g. $S_{1112} = 0$.)

From our derivation it is clear that e^T_{ij} found in this way is the stress-free strain of a certain transformed inclusion which, with the given applied field e^A_{ij}, could replace the inhomogeneity without altering the stress or displacement anywhere. We shall call this imaginary transformed inclusion the 'equivalent inclusion'. Outside the inclusion the elastic field u_i, e_{ij}, p_{ij} is the sum of the applied field u^A_i, e^A_{ij}, p^A_{ij} and the field u^C_i, e^C_{ij}, p^C_{ij} of the equivalent inclusion. It is convenient to regard the latter as the field 'of' or 'due to' the inhomogeneity. It measures the perturbation of the applied field by the inhomogeneity and may be found from e^T_{ij} by the methods of §§ 2, 3. In particular the remote field of the inhomogeneity follows at once from (2·18). Inside the inhomogeneity the total strain is

$$e = \frac{\kappa}{\kappa - \kappa_1} e^T, \quad 'e_{ij} = \frac{\mu}{\mu - \mu_1}\,'e^T_{ij}, \tag{4·6}$$

by (4·3) and (4·4). The field immediately outside the inhomogeneity is found from (2·13) and (3·23).

The question of the interaction of a stress field and an elastic inhomogeneity has been discussed elsewhere (Eshelby 1951, 1956). We shall need the following result. If the initial elastic constants κ, μ of a body loaded by surface tractions are changed to arbitrary functions of position $\kappa^*(\mathbf{x})$, $\mu^*(\mathbf{x})$ the surface tractions being held constant, then the total energy of the system increases by

$$E_{\text{int.}} = -\frac{1}{2}\int \{(\kappa - \kappa^*)\, ee^* + 2(\mu - \mu^*)\,'e_{ij}\,'e^*_{ij}\}\,\mathrm{d}v, \tag{4·7}$$

where e_{ij}, e^*_{ij} are the strains before and after the change. By total energy we mean the sum of the elastic energy of the body and the potential energy of the loading mechanism. $E_{\text{int.}}$ is by definition the interaction energy of the applied stress and the inhomogeneity described by the non-uniform elastic constants κ^*, μ^*. The increase of elastic energy arising from the change is

$$\Delta E = -E_{\text{int.}}. \tag{4·8}$$

Equation (4·7) is valid also if e_{ij} is the strain arising from sources of internal stress or because the material is strained by rigid grips, but in this case (4·8) is replaced by

$$\Delta E = +E_{\text{int.}}. \tag{4·9}$$

In the present case the change of elastic constants is confined to the interior of the ellipsoid and e_{ij}, e^*_{ij} are uniform there. Thus

$$\begin{aligned} E_{\text{int.}} &= -\tfrac{1}{2}V\{(\kappa - \kappa_1)\, e^A(e^A + e^C) + 2(\mu - \mu_1)\,'e^A_{ij}('e^A_{ij} + 'e^C_{ij})\} \\ &= -\tfrac{1}{2}V(\kappa e^A e^T + 2\mu\,'e^A_{ij}\,'e^T_{ij}) = -\tfrac{1}{2}V p^A_{ij} e^T_{ij} \end{aligned} \tag{4·10}$$

by (4·8) and (2·3) (V is the volume of the ellipsoid). This solves problem (vii). It will be noticed that (4·10) is just half the expression (3·22) for the equivalent inclusion. The physical interpretation is as follows. Equation (4·10) is still approximately correct if p^A_{ij} is a slowly varying function of position. The 'force' on the inhomogeneity regarded as an elastic singularity is again given by (2·24). F_l depends only on the remote field of the singularity and not at all on whether its field is permanent (internal stress) or merely 'induced' in an inhomogeneity by an elastic field. The factor $\frac{1}{2}$ ensures that this is so, for clearly (4·10) varies with position twice as fast when e_{ij}^T is a linear function of e_{ij}^A as it does when e_{ij}^T is constant.

The effective bulk elastic constants for a material containing a uniform dispersion of ellipsoidal inhomogeneities (not necessarily all of the same form or orientation) may be calculated as follows. Consider a specimen of unit volume. Let E_0 be its elastic energy when it is free of inhomogeneities and certain surface tractions produce a uniform stress p_{mn}^A in it. If we introduce the inhomogeneities, keeping the surface tractions constant, the elastic energy is augmented by $-\Sigma E_{\text{int.}}(p_{mn}^A)$, the sum of the interaction energies of all the inhomogeneities with the particular stress p_{mn}^A (compare equation (4·8)). Since E_0 and $E_{\text{int.}}$ are both quadratic functions of p_{mn}^A we may write

$$\tfrac{1}{2}s_{ijkl}p_{ij}^A p_{kl}^A = E_0 - \Sigma E_{\text{int.}}(p_{mn}^A). \tag{4·11}$$

The s_{ijkl} are then the elastic compliance constants which would be inferred from the work done in applying the given type of loading. To find the individual components of s_{ijkl} we put, say, $p_{11}^A = 1, p_{22}^A = 1, p_{ij}^A = 0$ otherwise, and obtain s_{1122}, and so forth. Unless the ellipsoids are spheres, or have their orientations distributed at random, the effective constants s_{ijkl} will be anisotropic. The effective elastic moduli c_{ijkl} may be found from the relation $c_{ijkl}s_{klmn} = \delta_{im}\delta_{jn}$. It would not do to find them directly by equating the right-hand side of (4·11) to $\frac{1}{2}c_{ijkl}e_{ij}^A e_{kl}^A$, since that equation was derived by considering a process at constant load. Rather, we must consider a unit volume given a uniform macroscopic strain by a rigid framework which keeps the surface displacements fixed when the inhomogeneities are introduced. This leads to

$$\tfrac{1}{2}c_{ijkl}e_{ij}^A e_{kl}^A = E_0 + \Sigma E_{\text{int.}}(e_{mn}^A). \tag{4·12}$$

(For the difference in sign compare (4·8) and (4·9).)†

5. Discussion

It has been shown how the problems listed in § 1 may be solved for the ellipsoid. Explicit solutions for the general case would be very clumsy and in this section only a few special cases are considered.

For a sphere of radius a, we have at once from (3·10), (3·11) and the symmetry of the problem, $I_a = I_b = I_c = 4\pi/3$ and $I_{aa} = I_{bb} = 3I_{ab} = \ldots = 4\pi/5a^2$. We find the following expressions for the constrained strain e_{ij}^C inside the transformed sphere in terms of the stress-free strain e_{ij}^T:

$$e^C = \alpha e^T, \quad 'e_{ij}^C = \beta' e_{ij}^T,$$

where

$$\alpha = \frac{1}{3}\frac{1+\sigma}{1-\sigma}, \quad \beta = \frac{2}{15}\frac{4-5\sigma}{1-\sigma}.$$

For a spherical inhomogeneity with elastic constants κ_1, μ_1 in an applied field e_{ij}^A the equivalent e_{ij}^T are given by

$$e^T = Ae^A, \quad 'e_{ij}^T = B'e_{ij}^A,$$

† Failure to appreciate this point led to an error in a previous paper (Eshelby 1955). It may be rectified by changing the sign of the right-hand side of the equation for ΔE on p. 488, col. 2, line 5. In addition a factor r^{-6} is missing from the right-hand side of equation (4).

where
$$A = \frac{\kappa_1-\kappa}{(\kappa-\kappa_1)\alpha-\kappa} = \frac{\kappa-\kappa_1}{\kappa}\frac{4\mu+3\kappa}{4\mu+3\kappa_1}$$
and
$$B = \frac{\mu_1-\mu}{(\mu-\mu_1)\beta-\mu}.$$
The interaction energy is†
$$E_{\text{int.}} = \tfrac{1}{2}V\left\{\frac{A}{9\kappa}p^A p^A + \frac{B}{2\mu}{}'p^A_{ij}{}'p^A_{ij}\right\}.$$

By (2·13) the stress just outside the inhomogeneous sphere is easily found to be

$$p = p^A - \frac{1+\sigma}{1-\sigma}B'p^A_{ij}n_i n_j,$$
$$'p_{il} = (1+\beta B)'p^A_{il} - B('p^A_{ik}n_k n_l + 'p^A_{lk}n_k n_i) + \frac{B}{1-\sigma}'p^A_{jk}n_j n_k n_i n_l$$
$$+\frac{1-2\sigma}{3(1-\sigma)}B'p^A_{jk}n_j n_k \delta_{il} - \frac{1-2\sigma}{3(1-\sigma)}Ap^A(n_i n_l - \tfrac{1}{3}\delta_{il}).$$

In particular, for the stress at the surface of a spherical cavity ($\kappa_1 = 0$, $\mu_1 = 0$) perturbing a uniform stress-field p^A_{ij} we find

$$p_{il} = \frac{15}{7-5\sigma}\left\{(1-\sigma)(p^A_{il} - p^A_{ik}n_k n_l - p^A_{lk}n_k n_i) + p^A_{jk}n_j n_k n_i n_l \right.$$
$$\left. - \sigma p^A_{jk}n_j n_k \delta_{il} + \frac{1-5\sigma}{10}p^A(n_i n_l - \delta_{il})\right\}.$$

The expression given by Landau & Lifshitz (1954) is clearly incorrect, since the surface traction $p_{il}n_l$ formed from it does not vanish.

According to the discussion leading to (4·7), the energy density of a body containing a volume fraction v of inhomogeneous spheres is

$$\frac{1}{2}\left\{\frac{1}{9\kappa}(1+Av)p^A p^A + \frac{1}{2\mu}(1+Bv)'p^A_{ij}{}'p^A_{ij}\right\},$$

and so the effective bulk elastic constants are

$$\kappa_{\text{eff.}} = \kappa/(1+Av), \quad \mu_{\text{eff.}} = \mu/(1+Bv).$$

Since we have neglected the interaction between the spheres these expressions will only be valid for small v and we may equally well write

$$\kappa_{\text{eff.}} = \kappa(1-Av), \quad \mu_{\text{eff.}} = \mu(1-Bv).$$

When we let κ_1, μ_1 approach zero or infinity we recover known results for a material containing empty spherical cavities (Mackenzie 1950) or rigid and incompressible spherical inclusions (Hashin 1955). For arbitrary κ_1, μ_1 the expression for $\kappa_{\text{eff.}}$ agrees with Bruggemann's (1937). Bruggemann's expression for $\mu_{\text{eff.}}$ is independent of the Poisson's ratio of the matrix and can hardly be correct. It is derived by considering the perturbation of the non-uniform elastic field in a sphere twisted in pure torsion when a spherical inclusion is introduced at the centre. This is obviously not typical of

* There is an error in a previous paper (Eshelby 1951); the expressions for δ, $\mathbf{F}(K'=0)$, $\mathbf{F}(K'=\infty)$ on page 104 should all be multiplied by $4\mu/3K$.

the effect of an inhomogeneity at an arbitrary point in the sphere. A well-known analogy (Goodier 1936) between problems in elasticity and viscosity enables us to interpret $\mu_{\text{eff.}}$ as the effective viscosity of a suspension of rigid spheres in a liquid of viscosity μ provided we put $\mu_1 = \infty$, $\sigma = \frac{1}{2}$. This gives Einstein's expression $\mu_{\text{eff.}} = (1+2{\cdot}5v)\mu$. Kynch (1956) has discussed the value of v above which the mutual interaction of the spheres makes this expression inaccurate. Evidently $\kappa_{\text{eff.}}, \mu_{\text{eff.}}$ will be subject to a similar limitation.

The problem of an ellipsoidal inclusion which has undergone a simple shear is of interest in connexion with twinning and martensitic or other diffusionless transformations. Suppose, then, that an ellipsoidal region of volume V undergoes a pure shear transformation in which $e^T_{13} = e^T_{31}$ are the only non-zero components of e^T_{ij}. Then (3·21) and (2·7) give

$$E_{\text{el.}} = 2\gamma\mu V(e^T_{13})^2 \tag{5·1}$$

with

$$\gamma = 1 - 2S_{1313} \tag{5·2}$$

for the total elastic energy in matrix and inclusion. $2\mu V(e^T_{13})^2$ is the energy necessary to pull the inclusion back to its original shape in the absence of the matrix. Alternatively, it is the energy we should find if the inclusion transformed whilst embedded in an imaginary rigid matrix. Thus we may regard γ as a measure of the extent to which the matrix is able to accommodate the transformation. It also describes the partition of the total strain energy between matrix and inclusion, for from (2·19) and (2·20) we have

$$\frac{\text{energy in matrix}}{\text{energy in inclusion}} = \frac{1-\gamma}{\gamma},$$

so that for good accommodation (small γ) most of such energy as remains is in the matrix. We also have from (2·7) the value

$$p^I_{13}/(-p^T_{13}) = \gamma$$

for the ratio between the actual stress in the inclusion to the value it would have if the transformation occurred in a rigid matrix.

For an oblate spheroid we have

$$\gamma = \frac{2-\sigma}{1-\sigma}\frac{I_a}{4\pi} - \tfrac{3}{8}Q\frac{c^2}{a^2-c^2}(4\pi - 3I_a)$$

with I_a given by (3·15). For a sphere $\gamma = (7-5\sigma)/15(1-\sigma)$ and for a needle $(b = c \ll a)$, $\gamma = \frac{1}{2}$. The values for sphere and needle are about the same if Poisson's ratio is in the neighbourhood of $\frac{1}{3}$.

If the inclusion is a thin plate in the form of an ellipsoid with its c axis much less than its a and b axes we have

$$\gamma = \eta c/b,$$

where η has the following values:

$$\left.\begin{aligned} \eta &= E(k) + \frac{\sigma}{1-\sigma}\frac{K(k)-E(k)}{k^2}; && a>b,\ k=(1-b^2/a^2)^{\frac{1}{2}},\ k'=b/a;\\ &= \frac{E(k)}{k'} + \frac{\sigma}{1-\sigma}\frac{1}{k'}\frac{E(k)-k'^2K(k)}{k^2}; && b>a,\ k=(1-a^2/b^2)^{\frac{1}{2}},\ k'=a/b;\\ &= \pi(2-\sigma)/4(1-\sigma); && a=b. \end{aligned}\right\} \tag{5·3}$$

Here $E(k)$ and $K(k)$ are complete elliptic integrals. It follows that for a very thin plate γ approaches zero and there is complete accommodation. If the operative shear is e^T_{12} instead of e^T_{13}, tending to deform the plate in its own plane, the corresponding accommodation factor $1-2S_{1212}$ approaches unity as the thickness of the plate decreases and there is no accommodation. We may compare these results with the case where e^T_{ij} is a pure dilatation. Then, as we saw in § 2,

$$E_{\text{el.}} = 2\mu V(e^T)^2(1+\sigma)/9(1-\sigma)$$

whatever is the shape of the inclusion. In a rigid matrix the energy would be $\frac{1}{2}\kappa V(e^T)^2$ and so the accommodation factor is always $2(1-2\sigma)/3(1-\sigma)$ or $\frac{1}{2}$ for $\sigma = \frac{1}{3}$.

As a rough illustration of how these results might be used, consider the formation of martensite in iron. Zener (1946) has shown that the thermodynamics of the process suggest that a strain energy of 290 cal is associated with each mole transformed. Suppose that the transformation involves a 5 % volume expansion and a 10° shear, so that $e^T = 0\cdot05$, $e^T_{13} = 0\cdot009$. It is easily seen that the total strain energy is the sum of the values it would have if the dilatation or shear acted alone. With $\mu = 8\times10^{11}\,\text{dyn/cm}^2$, $\sigma = \frac{1}{4}$, and V one molar volume, the shape-independent dilatational contribution to the energy is 25 cal. This leaves 265 cal for the shear contribution. The quantity $2\mu V(e^T_{13})^2$ has the value 1900 cal. If the transformed region is supposed to be an ellipsoid the accommodation factor is thus $\gamma = 265/1900 = 0\cdot14$, and this tells us something about its shape. For example, if it is assumed to be a circular disk, (5·3) shows that its thickness/diameter ratio must be 0·08. In the presence of an applied stress the free energy change associated with the transformation becomes $E_{\text{el.}}+E_{\text{int.}}$ instead of $E_{\text{el.}}$ For the case we have been considering equation (3·22) gives $E_{\text{int.}} \sim 8\cdot10^{-9}(-\frac{1}{3}p^A) - 3\cdot10^{-8}p^A_{13}$ cal/mole if the applied field p^A_{ij} is measured in dynes/cm².

For a cavity, the equations (4·3), (4·4) for the ellipsoidal inhomogeneity simplify to

$$e^T_{ij} - e^C_{ij} = e^T_{ij} - S_{ijkl}e^T_{kl} = e^A_{ij}; \tag{5·4}$$

we shall only consider this case.

Suppose that an ellipsoidal cavity is perturbing a simple shear $e^A_{13} = S/2\mu$. We have at once for the equivalent stress-free strain, putting $\mu_1 = 0$ in (4·5),

$$e^T_{13} = e^A_{13}/\gamma, \tag{5·5}$$

with the notation of (5·2). The interaction energy is

$$E_{\text{int.}} = -\tfrac{1}{2}Vp^A_{ij}e^T_{ij} = -VS^2/2\mu\gamma. \tag{5·6}$$

If we let the c axis of the ellipsoid become very small we have an elliptical crack. From (5·3) it is clear that $V\gamma$ remains finite as c approaches zero and the interaction energy of the crack with the applied shear stress S is

$$E_{\text{int.}} = -2\pi ab^2S^2/3\mu\eta.$$

Consider next the displacement of the faces of the crack. If the c axis is still finite, the displacement of a point x_i at the surface of the cavity is

$$u^C_i = (e^C_{ij} + \varpi^C_{ij})x_j$$

plus the displacement u_i^A due to the applied field. We suppose that $u_i^A = 0$ in the plane of the crack. If we evaluate u_i^C from (3·6) and (3·19) and pass the limit $c \to 0$ and use (5·5) we find that in the plane of the crack

$$\pm u_1^C = (bS/\mu\eta)(1 - x_1^2/a^2 - x_2^2/b^2)^{\frac{1}{2}} \equiv \tfrac{1}{2}\Delta u_1, \tag{5·7}$$
$$u_2^C = 0, \quad u_3^C = \alpha x_1, \quad \alpha = \pi(1-2\sigma)\, b e_{13}^A/4(1-\sigma)\, a\eta;$$

the + and − signs refer to the upper and lower faces of the crack. Thus the plane of the crack is tilted through an angle α, but it remains a plane. The relative displacement Δu_1 of the faces is everywhere parallel to the x_1 axis and has an ellipsoidal distribution.

There is a problem in dislocation theory closely related to the theory of the sheared crack. Under the influence of a stress $p_{13}^A = S$ dislocation loops expand in the x_1x_2 plane from a source at the origin and pile up against an elliptical barrier until their back-stresses annul p_{13}^A at the source. What is their distribution if each loop is in equilibrium under the combined action of the other loops and the applied stress? In the limit of a large number of loops with very small Burgers vectors the crack and dislocation problems coincide (cf. Leibfried 1954). If the source has its Burgers vector parallel to the x_1 axis the dislocations are of pure edge type where they cross the x_1 axis and the number of them crossing a length dx_1 is $dx_1\, \partial\Delta u_1(x_1, x_1 = 0)/\lambda\, \partial x_1$. Where they cross the x_2 axis they are of pure screw type and their density is $\partial\Delta u_1(x_1 = 0,\ x_2)/\lambda\, \partial x_2$. The interaction energy of the loops is given by (5·6). In diagrams the tip of an array of piled-up dislocations is often dráwn curling up or down. The remarks following (5·7) do not support this.

As a further example consider a spheroidal cavity ($b = c$) in a material subject to a simple tension stress T. If the a axis coincides with the direction of T we need only know e_{11}^T to find the interaction energy. The non-zero components of e_{ij}^A are e_{11}^A, $e_{22}^A = e_{33}^A = -\sigma e_{11}^A$. From (5·4) we have

$$e_{11}^T = \epsilon e_{11}^A, \quad \text{where} \quad \epsilon = \frac{(1 - S_{33} - S_{23}) - 2\sigma S_{13}}{(1 - S_{33} - S_{23})(1 - S_{11}) - 2S_{13}S_{31}}, \tag{5·8}$$

with the abbreviated notation $S_{11} = S_{1111}$, $S_{13} = S_{1133}$ ....

On the other hand, if the a axis is at right angles to the direction of T we only need to know e_{33}^T. The non-vanishing applied strains are e_{33}^A, $e_{11}^A = e_{22}^A = -\sigma e_{33}^A$ and we find

$$e_{33}^T - e_{22}^T = \frac{1+\sigma}{1 + S_{23} - S_{33}} e_{33}^A, \quad e_{33}^T + e_{22}^T = \frac{(1-\sigma)(1-S_{11}) - 2\sigma S_{31}}{(1 - S_{33} - S_{22})(1 - S_{11}) - 2S_{13}S_{31}} e_{33}^A \tag{5·9}$$

or, say,
$$e_{33}^T = \zeta e_{33}^A.$$

The interaction energies are respectively

$$E_{\text{int.}}(\|) = -\tfrac{1}{2} V\epsilon T^2/E \tag{5·10}$$

and
$$E_{\text{int.}}(\perp) = -\tfrac{1}{2} V\zeta T^2/E, \tag{5·11}$$

where E is Young's modulus.

If the direction of T remains unaltered and the cavity changes from the parallel to the perpendicular orientation, the interaction energy changes from (5·10) to (5·11).

The parallel or transverse orientation is energetically favourable according as the spheroid is oblate or prolate. We may take the case of the prolate spheroid as illustrating the orienting effect of an applied stress on a di-vacancy in a metal (A. Seeger, private communication). For $a/c = 2$, $\sigma = \frac{1}{3}$ we find $\epsilon = 2{\cdot}24$, $\zeta = 5{\cdot}88$.

When a approaches zero the numerator of ϵ becomes $1-2\sigma$ and the denominator approaches zero as $a\pi(1-2\sigma)/4c(1-\sigma^2)$. The product ϵV in (5·10) remains finite and we reproduce Sack's (1946) value $-8c^3(1-\sigma^2)T^2/2E$ for the interaction energy of a penny-shaped crack in tension.

The total strain at the surface of the cavity $e^C_{ij}+e^A_{ij}$ is, according to (5·4), given by the right-hand sides of (2·13) with e^C, $'e^C_{ij}$ replaced by e^T, $'e^T_{ij}$. The normal is given by (3·23) and the stress concentration can be found from (2·3). For the spheroid in tension we must supplement (5·8) by

$$e^T_{22} = e^T_{33} = e^A_{11}(\epsilon - \epsilon S_{11} - 1)/2S_{13}$$

and (5·9) by

$$(1-S_{11})\,e^T_{11} = S_{12}(e^T_{22}+e^T_{33}) - \sigma e^A_{33}.$$

These results actually apply to a quite general state of triaxial stress symmetrical about the polar axis of the spheroid, for in the applied strain

$$e^A_{22} = e^A_{33} = -\sigma e^A_{11}, \quad e^A_{12} = e^A_{23} = e^A_{31} = 0,$$

we may take σ to be any number unconnected with Poisson's ratio. (The σ implicit in the S_{ij} must, of course, be put equal to Poisson's ratio.) The stress concentration about an ellipsoid in shear is found similarly from (5·5).

Two-dimensional problems involving an infinite elliptic cylinder can be dealt with similarly, using (3·17) and (3·18). The interaction energy per unit length is

$$E_{\text{int.}} = -\tfrac{1}{2}Ap^A_{ij}e^T_{ij},$$

where A is the cross-sectional area of the cylinder. The passage to the limit $b \to 0$ or $a \to 0$ is in this case very easy and we can derive well-known results for cracks in plane strain tension or shear. Another simple case is that of a crack joining the points $x = \pm a$, $y = 0$ perturbing a uniform stress $p^A_{23} = S$. Both before and after the introduction of the crack there is a state of anti-plane strain in which the displacement is everywhere perpendicular to the xy plane. The interaction energy and relative shift of the faces of the crack are

$$E_{\text{int.}} = -\pi a^2 S^2/2\mu \quad \text{and} \quad \Delta u_3 = (2S/\mu)\,(a^2-x^2)^{\frac{1}{2}}. \tag{5·12}$$

Several writers have derived approximate expressions for the reduction of energy by a crack (or an array of dislocations simulating a crack) by supposing that the applied stress is effectively relaxed to zero in a region about the crack whose dimensions are of the order of the width of the crack. (In the same sense we might say that in (5·10) or (5·11) the 'energy drainage volume' of the cavity was ϵ or ζ times its geometrical volume.) This method gives correct results, but the logic behind it is not clear. If the applied stress is maintained by constant external loads, the elastic energy is *increased* by a certain amount when the crack is introduced (compare equation (4·8)). At the same time, the loading mechanism has to expend twice this amount of work. Thus the decrease of total energy ($-E_{\text{int.}}$) is numerically equal to the increase of elastic energy. On the other hand, if the applied stress is due to sources of internal strain, or if the body is strained by rigid clamps, the elastic energy (now the total energy) clearly *decreases* (compare equation (4·9)). But even here the decrease is in no simple sense located near the crack. This may easily be shown explicitly for the case of the crack in anti-plane strain (equation (5·12)). In terms of the elliptic co-ordinates ξ, η defined by

$$x = a\cosh\xi\cos\eta, \quad y = a\sinh\xi\sin\eta,$$

the displacement is

$$u_3 = (aS/\mu)\cosh\xi\sin\eta. \tag{5·13}$$

For, when ξ is large enough for hyperbolic sine and cosine to be indistinguishable it reduces to the uniform state of shear

$$u_3 = Sy/\mu = (aS/\mu)\sinh\xi\sin\eta, \tag{5·14}$$

whilst the traction on a curve ξ = const. is proportional to $\partial u_3/\partial\xi$ and so vanishes on the limiting ellipse $\xi = 0$ defining the crack. The energy in the small rectangle $d\xi\, d\eta$ at ξ, η is $\frac{1}{2}\mu\{(\partial u_3/\partial\xi)^2 + (\partial u_3/\partial\eta)^2\}\, d\xi\, d\eta$. If we evaluate this using the displacement without ((5·14)) and with ((5·13)) the crack, we find that the change of energy density ΔE at any point due to the introduction of the crack is given by

$$\Delta E\, d\xi\, d\eta = \tfrac{1}{2}(S^2/\mu)\cos 2\eta\, d\xi\, d\eta. \tag{5·15}$$

We may say that there is stress relaxation between the hyperbolas $\eta = \frac{1}{4}\pi, \frac{3}{4}\pi$ and stress concentration outside them. The integral of (5·15) over any ellipse with the ends of the crack for foci is precisely zero. By judiciously deforming the ellipse we can find a curve within which the energy 'relaxation' is positive or negative. Attempts to evaluate interaction energies in this way lead not only to errors of sign (which may be corrected by common sense), but also to incorrect numerical factors.

The problem of a rigid and incompressible ellipsoidal inhomogeneity is also relatively simple, since (4·3) and (4·4) reduce to

$$S_{ijkl}e^T_{kl} = -e^A_{ij}.$$

From the solution Goodier's (1936) analogy enables us to find the perturbation of the slow motion of a viscous fluid when a solid ellipsoid is immersed in it. We have only to put $\sigma = \frac{1}{2}$ in the matrix and interpret μ, u_i, e_{ij} and p_{ij} as viscosity, velocity, rate of strain and stress. The energy density becomes half the rate of dissipation of energy per unit volume. Equation (4·11) or (4·12) enables us to find the viscosity of a dilute suspension of ellipsoids. $E_{\text{int.}}$ is positive for a rigid inclusion and so the viscosity is increased. Equation (4·11) now states that a viscometer working at constant load will produce a lower rate of deformation and so will dissipate less energy, whilst equation (4·12) states that a viscometer working at constant speed will have to work harder to maintain a prescribed rate of strain.

For a single immersed ellipsoid the increase in the rate of energy dissipation is clearly twice $E_{\text{int.}}$ for the related elastic problem. The calculation is much simplified by the fact that for $\sigma = \frac{1}{2}$, $R = 0$ in (3·7), whilst the dilatations e^A, e^T, e^C are all zero. We can easily verify, for example, Jeffery's (1922) expression for the energy dissipation by a prolate spheroid, as amended by Eisenschitz (1933). To find the viscosity o a dispersion of spheroids it is necessary to decide what orientation they will take up The elastic analogy suggests (though it does not prove) that they will ultimatel orient themselves so as to minimize the energy dissipated. This agress with Jeffery's hypothesis, verified experimentally by Taylor (1923).

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