

# *Contents*

<b>Preface</b>	<b>v</b>
<b>Chapter 1. General theory of eigenstrains</b>	<b>1</b>
1. Definition of eigenstrains	1
2. Fundamental equations of elasticity	3
Hooke's law	3
Equilibrium conditions	5
Compatibility conditions	6
3. General expressions of elastic fields for given eigenstrain distributions	7
Periodic solutions	7
Method of Fourier series and Fourier integrals	9
Method of Green's functions	11
Isotropic materials	13
Cubic crystals	14
Hexagonal crystals (transversely isotropic)	14
4. Exercises of general formulae	15
A straight screw dislocation	15
A straight edge dislocation	18
Periodic distribution of cuboidal precipitates	20
5. Static Green's functions	21
Isotropic materials	22
* Anisotropic materials	25
* Transversely isotropic materials	26
* Kröner's formula	31
* Derivatives of Green's functions	32
* Two-dimensional Green's function	34
6. Inclusions and inhomogeneities	38
Inclusions	38
Inhomogeneities	40
* Effect of isotropic elastic moduli on stress	42

7.	Dislocations	44
	Volterra and Mura formulas	45
	* The Indenbom and Orlov formula	48
	* Disclinations	49
8.	Dynamic solutions	53
	Uniformly moving edge dislocation	55
	Uniformly moving screw dislocation	57
*9.	Dynamic Green's functions	57
	Isotropic materials	61
	Steady State	64
*10.	Incompatibility	65
	* Riemann-Christoffel curvature tensor	71
 <b>Chapter 2. Isotropic inclusions</b>		 74
11.	Eshelby's solution	74
	Interior points	75
	Sphere	79
	Elliptic cylinder	80
	Penny-shape	81
	Flat ellipsoid	83
	Oblate spheroid	84
	Prolate spheroid	84
	Exterior points	84
	Thermal expansion with central symmetry	88
*12.	Ellipsoidal inclusions with polynomial eigenstrains	89
	* The I-integrals	92
	* Sphere	93
	* Elliptic cylinder	94
	* Oblate spheroid	94
	* Prolate spheroid	94
	* Elliptical plate	95
	* The Ferrers and Dyson formula	95
13.	Energies of inclusions	97
	Elastic strain energy	97
	Interaction energy	99
	Strain energy due to a spherical inclusion	101
	Elliptic cylinder	101
	Penny-shaped flat ellipsoid	101
	Spheroid	102
*14.	Cuboidal inclusions	104

15. Inclusions in a half space	110
Green's functions	110
Ellipsoidal inclusion with a uniform dilatational eigenstrain	114
* Cuboidal inclusion with uniform eigenstrains	121
* Periodic distribution of eigenstrains	121
Joined half-spaces	123
<b>Chapter 3. Anisotropic inclusions</b>	129
16. Elastic field of an ellipsoidal inclusion	129
17. Formulae for interior points	133
Uniform eigenstrains	134
Spheroid	137
Cylinder (elliptic inclusion)	141
Flat ellipsoid	143
Eigenstrains with polynomial variation	144
Eigenstrains with a periodic form	144
*18. Formulae for exterior points	149
Examples	156
19. Ellipsoidal inclusions with polynomial eigenstrains in anisotropic media	158
Special cases	160
*20. Harmonic eigenstrains	161
21. Periodic distribution of spherical inclusions	165
<b>Chapter 4. Ellipsoidal inhomogeneities</b>	177
22. Equivalent inclusion method	178
Isotropic materials	181
Sphere	183
Penny shape	184
Rod	185
Anisotropic inhomogeneities in isotropic matrices	187
Stress field for exterior points	187
23. Numerical calculations	188
Two ellipsoidal inhomogeneities	192
*24. Impotent eigenstrains	198
25. Energies of inhomogeneities	204
Elastic strain energy	204
Interaction energy	208
Columnetti's theorem	211
Uniform plastic deformation in a matrix	213
Energy balance	215

26.	Precipitates and martensites	218
	Isotropic precipitates	219
	Anisotropic precipitates	220
	Incoherent precipitates	226
	Martensitic transformation	229
	Stress orienting precipitation	237
<b>Chapter 5.</b>	<b>Cracks</b>	<b>240</b>
27.	Critical stresses of cracks in isotropic media	240
	Penny-shaped cracks	240
	Slit-like cracks	242
	Flat ellipsoidal cracks	244
	Crack opening displacement	247
28.	Critical stresses of cracks in anisotropic media	248
	Uniform applied stress	248
	Non-uniform applied stress	253
	* $II$ integrals for a penny-shaped crack	255
	* $II$ integrals for cubic crystals	255
	* $II$ integrals for transversely isotropic materials	257
29.	Stress intensity factor for a flat ellipsoidal crack	260
	Uniform applied stresses	264
	Non-uniform applied stresses	268
30.	Stress intensity factor for a slit-like crack	271
	Uniform applied stresses	272
	Non-uniform applied stresses	274
	Isotropic materials	274
31.	Stress concentration factors	277
	Simple tension	278
	Pure shear	279
32.	Dugdale-Barenblatt cracks	280
	BCS model	288
	Penny shaped crack	292
*33.	Stress intensity factor for an arbitrarily shaped plane crack	297
	* Numerical examples	305
34.	Crack growth	307
	Energy release rate	307
	The J-integral	311
	Fatigue	314
	Dynamic crack growth	319

<b>Chapter 6. Dislocations</b>	<b>324</b>
35. Displacement fields	324
Parallel dislocations	325
A straight dislocation	327
36. Stress fields	327
Dislocation segments	328
Willis' formula	333
The Asaro et al. formula	334
Dislocation loops	335
37. Dislocation density tensor	338
Surface dislocation density	341
Impotent distribution of dislocations	343
38. Dislocation flux tensor	345
Line integral expression of displacement and plastic distortion fields	348
The elastic field of moving dislocations wave equations of tensor potentials.	351
Wave equations of tensor potentials	352
39. Energies and forces	353
Dynamic consideration	354
40. Plasticity	361
Mathematical theory of plasticity	361
Dislocation theory	363
Plane strain problems	365
Beams and cylinders	373
41. Dislocation model for fatigue crack initiation	379
<b>Chapter 7. Material properties and related topics</b>	<b>388</b>
42. Macroscopic average	388
Average of internal stresses	388
Macroscopic strains	389
Tanaka-Mori's theorem	390
Image stress	393
Random distribution of inclusions-Mori and Tanaka's theory	394
43. Work-hardening of dispersion hardened alloys	398
Work-hardening in simple shear	398
Dislocations around an inclusion	402
Uniformity of plastic deformation	405
44. Diffusional relaxation of internal and external stresses	406
Relaxation of the internal stress in a plastically deformed dispersion strengthened alloy	407

Diffusional relaxation process, climb rate of an Orowan loop	408
Recovery creep of a dispersion strengthened alloy	412
Interfacial diffusional relaxation	414
45. Average elastic moduli of composite materials	421
The Voigt approximation	421
The Reuss approximation	424
Hill's theory	426
Eshelby's method	428
Self-consistent method	430
Upper and lower bounds	433
Other related works	437
46. Plastic behavior of polycrystalline metals and composites	439
Taylor's analysis	439
Self-consistent method	443
Embedded weakened zone	448
47. Viscoelasticity of composite materials	449
Homogeneous inclusions	449
Inhomogeneous inclusions	452
Waves in an infinite medium	453
48. Elastic wave scattering	455
Dynamic equivalent inclusion method	459
Green's formula	460
49. Interaction between dislocations and inclusions	463
Inclusions and dislocations	463
Cracks in two-phase materials	471
50. Eigenstrains in lattice theory	477
A uniformly moving screw dislocation	480
51. Sliding inclusions	484
Shearing Eigenstrains	486
Spheroidol inhomogeneous inclusions	488
52. Recent developments	492
Inclusions, precipitates, and composites	492
Half-spaces	494
Non-elastic matrices	494
Cracks and inclusions	495
Sliding and debonding inclusions	497
Dynamic cases	497
Miscellaneous	498

<i>Contents</i>	xiii
<b>Appendix 1</b>	499
Einstein summation convention	499
Kronecker delta	499
Permutation tensor	499
<b>Appendix 2</b>	501
The elastic moduli for isotropic materials	501
<b>Appendix 3</b>	505
Fourier series and integrals	505
Dirac's delta function and Heaviside's unit function	507
Laplace transform	508
<b>Appendix 4</b>	510
Dislocations pile-up	510
<b>References</b>	513
<b>Author index</b>	572
<b>Subject index</b>	582





# *General theory of eigenstrains*

The definition of eigenstrains is given first. Then the associated general solutions for elastic fields for given eigenstrains are expressed by Fourier integrals and Green's functions. Some details of calculations for Green's functions are described for static and dynamic cases.

As fundamental formulae for the subsequent chapters, general expressions of elastic fields are given for inclusions, dislocations, and disclinations. The stress discontinuity on boundaries of inclusions and the incompatibility of eigenstrains are discussed as general theories.

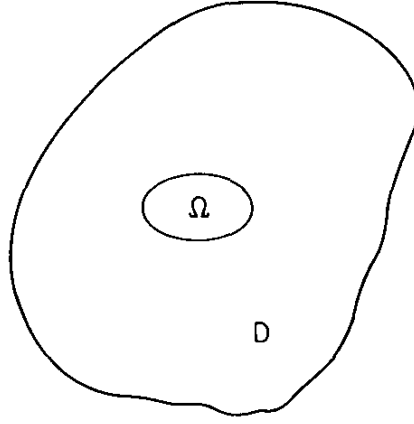
Throughout this work, a fixed rectangular Cartesian coordinate system with coordinate axes  $x_i$ ,  $i = 1, 2, 3$ , is used.

## **1. Definition of eigenstrains**

'Eigenstrain' is a generic name given by the author to such nonelastic strains as thermal expansion, phase transformation, initial strains, plastic strains, and misfit strains. 'Eigenstress' is a generic name given to self-equilibrated internal stresses caused by one or several of these eigenstrains in bodies which are free from any other external force and surface constraint. The eigenstress fields are created by the incompatibility of the eigenstrains.

This new English terminology was adapted from the German 'Eigenspannungen und Eigenspannungsquellen,' which is the title of H. Reissner's paper (1931) on residual stresses. Eshelby (1957) referred to eigenstrains as stress-free transformation strains in his celebrated paper which has stimulated the present author to work on inclusion and dislocation problems. The term 'elastic polarization' was used by Kröner (1958) for eigenstrains in a slightly different context—when the nonhomogeneity of polycrystal deformation is under consideration.

Engineers have used the term 'residual stresses' for the self-equilibrated internal stresses when they remain in materials after fabrication or plastic deformation. Eigenstresses are called thermal stresses when thermal expansion

Fig 1.1 Inclusion  $\Omega$ 

is a cause of the corresponding elastic fields. For example, when a part  $\Omega$  of a material (Fig. 1.1) has its temperature raised by  $T$ , thermal stress  $\sigma_{ij}$  is induced in the material  $D$  by the constraint from the part which surrounds  $\Omega$ . The thermal expansion  $\alpha T$ , where  $\alpha$  is the linear thermal expansion coefficient, constitutes the thermal expansion strain,

$$\epsilon_{ij}^* = \delta_{ij} \alpha T, \quad (1.1)$$

where  $\delta_{ij}$  is the Kronecker delta (see Appendix 1). The thermal expansion strain is the strain caused when  $\Omega$  can be expanded freely with the removal of the constraint from the surrounding part.

The actual strain is then the sum of the thermal and elastic strains. The elastic strain is related to the thermal stress by Hooke's law. The thermal expansion strain (1.1) is a typical example of an eigenstrain. In the elastic theory of eigenstrains and eigenstresses, however, it is not necessary to attribute  $\epsilon_{ij}^*$  to any specific source. The source could be phase transformation, precipitation, plastic deformation or a fictitious source necessary for the equivalent inclusion method (to be discussed in Section 22).

When an eigenstrain  $\epsilon_{ij}^*$  is prescribed in a finite subdomain  $\Omega$  in a homogeneous material  $D$  (see Fig. 1.1) and it is zero in the matrix  $D-\Omega$ , then  $\Omega$  is called an inclusion. The elastic moduli of the material are assumed to be homogeneous when inclusions are under consideration.

If a subdomain  $\Omega$  in a material  $D$  has elastic moduli different from those of the matrix, then  $\Omega$  is called an inhomogeneity. Applied stresses will be disturbed by the existence of the inhomogeneity. This disturbed stress field will be simulated by an eigenstress field by considering a fictitious eigenstrain  $\epsilon_{ij}^*$  in  $\Omega$  in a *homogeneous* material.

When  $\Omega$  in Fig. 1.1 is a plane embedded in a three-dimensional material  $D$  and  $\epsilon_{ij}^*$  is given on  $\Omega$  as a plastic strain caused by a finite slip  $\mathbf{b}$ , the boundary

of  $\Omega$  is called a dislocation loop. If  $\epsilon_{ij}^*$  is created by a rigid rotation of plane  $\Omega$  by  $\omega$ , the boundary of  $\Omega$  is called a disclination loop.

## 2. Fundamental equations of elasticity

In this section the field equations for the elasticity theory will be reviewed with particular reference to solving eigenstrain problems. These problems consist of finding displacement  $u_i$ , strain  $\epsilon_{ij}$ , and stress  $\sigma_{ij}$  at an arbitrary point  $x(x_1, x_2, x_3)$  when a free body  $D$  is subjected to a given distribution of eigenstrain  $\epsilon_{ij}^*$ . A free body is one which is free from any external surface or body force.

### *Hooke's law*

For infinitesimal deformations considered in this book, the total strain  $\epsilon_{ij}$  is regarded as the sum of elastic strain  $e_{ij}$  and eigenstrain  $\epsilon_{ij}^*$ ,

$$\epsilon_{ij} = e_{ij} + \epsilon_{ij}^*. \quad (2.1)$$

The total strain must be compatible,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (2.2)$$

where  $u_{i,j} = \partial u_i / \partial x_j$ .

The elastic strain is related to stress  $\sigma_{ij}$  by Hooke's law;

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^*) \quad (2.3)$$

or

$$\sigma_{ij} = C_{ijkl}(u_{k,l} - \epsilon_{kl}^*), \quad (2.4)$$

where  $C_{ijkl}$  are the elastic moduli (constants) (see Appendix 2), and the summation convention for the repeated indices is employed (see Appendix 1). Since  $C_{ijkl}$  is symmetric ( $C_{ijlk} = C_{ijkl}$ ), we have  $C_{ijkl}u_{l,k} = C_{ijkl}u_{k,l}$ . In the domain where  $\epsilon_{ij}^* = 0$ , (2.4) becomes

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = C_{ijkl}u_{k,l}. \quad (2.5)$$

The inverse expression of (2.3) is

$$\epsilon_{ij} - \epsilon_{ij}^* = C_{ijkl}^{-1}\sigma_{kl}, \quad (2.6)$$

where  $C_{ijkl}^{-1}$  is the elastic compliance.

For isotropic materials, (2.3) and (2.6) can be written as

$$\begin{aligned}\sigma_{ij} &= 2\mu(\epsilon_{ij} - \epsilon_{ij}^*) + \lambda\delta_{ij}(\epsilon_{kk} - \epsilon_{kk}^*), \\ \epsilon_{ij} - \epsilon_{ij}^* &= \{ \sigma_{ij} - \delta_{ij}\sigma_{kk}\nu/(1+\nu) \} / 2\mu,\end{aligned}\tag{2.7}$$

where  $\lambda$  and  $\mu$  are the Lamé constants, and  $\nu$  is Poisson's ratio. Young's modulus  $E$ , the shear modulus  $\mu$ , and the bulk modulus  $K$  are connected by  $2\mu = E/(1+\nu)$ ,  $K = E/3(1-2\nu)$ , and  $\lambda = 2\mu\nu/(1-2\nu)$ . The alternative expressions for (2.7) are

$$\begin{aligned}\sigma_x &= \frac{E}{1+\nu} \left\{ (\epsilon_x - \epsilon_x^*) + \frac{\nu}{1-2\nu} (\epsilon_{kk} - \epsilon_{kk}^*) \right\}, \\ \sigma_y &= \frac{E}{1+\nu} \left\{ (\epsilon_y - \epsilon_y^*) + \frac{\nu}{1-2\nu} (\epsilon_{kk} - \epsilon_{kk}^*) \right\}, \\ \sigma_z &= \frac{E}{1+\nu} \left\{ (\epsilon_z - \epsilon_z^*) + \frac{\nu}{1-2\nu} (\epsilon_{kk} - \epsilon_{kk}^*) \right\}, \\ \sigma_{xy} &= \frac{E}{1+\nu} (\epsilon_{xy} - \epsilon_{xy}^*), \\ \sigma_{yz} &= \frac{E}{1+\nu} (\epsilon_{yz} - \epsilon_{yz}^*), \\ \sigma_{zx} &= \frac{E}{1+\nu} (\epsilon_{zx} - \epsilon_{zx}^*),\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}\epsilon_x - \epsilon_x^* &= \{ \sigma_x - \nu(\sigma_y + \sigma_z) \} / E, \\ \epsilon_y - \epsilon_y^* &= \{ \sigma_y - \nu(\sigma_z + \sigma_x) \} / E, \\ \epsilon_z - \epsilon_z^* &= \{ \sigma_z - \nu(\sigma_x + \sigma_y) \} / E, \\ \epsilon_{xy} - \epsilon_{xy}^* &= \frac{1+\nu}{E} \sigma_{xy}, \\ \epsilon_{yz} - \epsilon_{yz}^* &= \frac{1+\nu}{E} \sigma_{yz}, \\ \epsilon_{zx} - \epsilon_{zx}^* &= \frac{1+\nu}{E} \sigma_{zx},\end{aligned}\tag{2.9}$$

where  $\epsilon_{kk} = \epsilon_x + \epsilon_y + \epsilon_z$  and  $\epsilon_{kk}^* = \epsilon_x^* + \epsilon_y^* + \epsilon_z^*$ . It is convenient to use (2.8) for the plane strain case where  $\epsilon_z = 0$ . Expression (2.9) is recommended for the plane stress case where  $\sigma_z = \sigma_{zx} = \sigma_{zy} = 0$ . It should be noted that solutions for the plane stress can be obtained directly from those for the plane strain by replacing  $E/(1 - \nu^2)$  with  $E$  and  $\nu/(1 - \nu)$  with  $\nu$ .

When Hooke's law (2.8) is rewritten for the two-dimensional case, we have

$$\begin{aligned}\sigma_x &= \frac{\mu}{\kappa - 1} \left\{ (\kappa + 1)(\epsilon_x - \epsilon_x^*) + (3 - \kappa)(\epsilon_y - \epsilon_y^*) \right\}, \\ \sigma_y &= \frac{\mu}{\kappa - 1} \left\{ (\kappa + 1)(\epsilon_y - \epsilon_y^*) + (3 - \kappa)(\epsilon_x - \epsilon_x^*) \right\},\end{aligned}\tag{2.9.1}$$

$$\sigma_{xy} = 2\mu(\epsilon_{xy} - \epsilon_{xy}^*),$$

$$\sigma_z = \sigma_{zx} = \sigma_{zy} = 0,$$

for the plane stress and  $\kappa = (3 - \nu)/(1 + \nu)$ . For the plane strain, we have

$$\begin{aligned}\sigma_x &= \frac{\mu}{\kappa - 1} \left\{ (\kappa + 1)(\epsilon_x - \epsilon_x^* - \nu\epsilon_z^*) + (3 - \kappa)(\epsilon_y - \epsilon_y^* - \nu\epsilon_z^*) \right\}, \\ \sigma_y &= \frac{\mu}{\kappa - 1} \left\{ (\kappa + 1)(\epsilon_y - \epsilon_y^* - \nu\epsilon_z^*) + (3 - \kappa)(\epsilon_x - \epsilon_x^* - \nu\epsilon_z^*) \right\}, \\ \sigma_{xy} &= 2\mu(\epsilon_{xy} - \epsilon_{xy}^*),\end{aligned}\tag{2.9.2}$$

$$\sigma_z = -\frac{\kappa + 1}{\kappa - 1}\mu\epsilon_z^* + \frac{3 - \kappa}{\kappa - 1}\mu(\epsilon_x + \epsilon_y - \epsilon_x^* - \epsilon_y^*),$$

$$\sigma_{zx} = \sigma_{zy} = 0,$$

where  $\kappa = 3 - 4\nu$ .

### *Equilibrium conditions*

When eigenstresses are calculated, material domain  $D$  is assumed to be free from any external force and any surface constraint. If these conditions for the free body are not satisfied, the stress field can be constructed from the superposition of the eigenstress of the free body and the solution of a proper boundary value problem.

The equations of equilibrium are

$$\sigma_{ij,j} = 0 \quad (i = 1, 2, 3).\tag{2.10}$$

The boundary conditions for free external surface forces are

$$\sigma_{ij}n_j = 0, \quad (2.11)$$

where  $n_i$  is the exterior unit normal vector on the boundary of  $D$ .

By substituting (2.4) into (2.10) and (2.11), we have

$$C_{ijkl}u_{k,lj} = C_{ijkl}\epsilon_{kl,j}^* \quad (2.12)$$

and

$$C_{ijkl}u_{k,l}n_j = C_{ijkl}\epsilon_{kl}^*n_j. \quad (2.13)$$

It can be seen that the contribution of  $\epsilon_{ij}^*$  to the equations of equilibrium is similar to that of a body force since the equations of equilibrium under body force  $X_i$  with zero  $\epsilon_{ij}^*$  are  $C_{ijkl}u_{k,lj} = -X_i$ . Similarly,  $C_{ijkl}\epsilon_{kl}^*n_j$  behaves like a surface force on the boundary. Thus, it can be said that the elastic displacement field caused by  $\epsilon_{ij}^*$  in a free body is equivalent to that caused by body force  $-C_{ijkl}\epsilon_{kl,j}^*$  and surface force  $C_{ijkl}\epsilon_{kl}^*n_j$ .

In subsequent chapters,  $D$  in most cases is considered as an infinitely extended body (infinite body), and condition (2.11) is replaced by the condition  $\sigma_{ij}(\mathbf{x}) \rightarrow 0$  for  $\mathbf{x} \rightarrow \infty$ .

### *Compatibility conditions*

The strain tensor  $\epsilon_{ij}$  has six components, while the displacement vector  $u_i$  has three components. The tensor and the vector are related to each other through the relation (2.2), which can be called the condition for the compatibility of strain  $\epsilon_{ij}$ . Generally, however, the equations of compatibility are referred to the relations which are derived from (2.2) by eliminating  $u_i$ ,

$$\epsilon_{pki}\epsilon_{qlj}\epsilon_{ij,kl} = 0, \quad (2.14)$$

where  $\epsilon_{pki}$  is the permutation tensor (see Appendix 1). Relation (2.14) will be discussed in Section 10.

The displacement differential equations of the elasticity theory are given by (2.12). In some cases, however, it is more convenient to consider (2.10), (2.3), and (2.14). Boundary conditions and various side conditions, such as singularity conditions, continuity conditions, etc., arise in problems from time to time. We can say at this point that the fundamental equations to be solved are equations (2.12).

Eigenstresses are caused by constraint from the surrounding elastic medium which prohibits the geometrically incompatible deformation of  $\epsilon_{ij}^*$ . The incompatibility of  $\epsilon_{ij}^*$  was discussed by Reissner (1931) and Neményi (1931). Dislocations due to incompatibility were studied by Weingarten (1901), Cesáro (1906), Volterra (1907), and Moriguti (1947) from the viewpoint of the elasticity theory in connection with the multiple values of displacements and rotations. Another viewpoint on dislocations, from the plasticity theory, was developed by Kondo (1955), Bilby (1960), and Kröner (1958).

In the following sections we investigate the methods of finding the associated elastic fields (displacements, strains, stresses) and the related problems for given distributions of  $\epsilon_{ij}^*$ . Particular emphasis will be placed on the case when a uniform  $\epsilon_{ij}^*$  is given in an ellipsoidal domain  $\Omega$  in an infinitely extended medium  $D$ . The results are useful for the study of the mechanical properties of solids which may contain precipitates, inclusions, voids, cracks, etc. The most fundamental contribution to this study was made by Eshelby (1951, 1956, 1957, 1959 and 1961).

### 3. General expressions of elastic fields for given eigenstrain distributions

The case where a given material is infinitely extended is of particular interest for the mathematical simplicity of the solution as well as for its practical importance. When the solution is applied to inclusion problems, it can be assumed with sufficient accuracy that the materials are infinitely extended since the size of the inclusions is relatively small compared to the size of the macroscopic material samples.

The fundamental equations to be solved for given  $\epsilon_{ij}^*$ , (2.12), are

$$C_{ijkl}u_{k,lj} = C_{ijkl}\epsilon_{kl,j}^*. \quad (3.1)$$

#### *Periodic solutions*

Suppose  $\epsilon_{ij}^*(\mathbf{x})$  is given in the form of a single wave of amplitude  $\bar{\epsilon}_{ij}^*(\xi)$ , where  $\xi$  is the wave vector corresponding to the given period of the distribution,

$$\epsilon_{ij}^*(\mathbf{x}) = \bar{\epsilon}_{ij}^*(\xi) \exp(i\xi \cdot \mathbf{x}), \quad (3.2)$$

where  $i = \sqrt{-1}$  and  $\xi \cdot \mathbf{x} = \xi_k x_k$ .

The solution of (3.1) corresponding to this distribution may also be expressed in the form of a single wave of the same period, that is,

$$u_i(\mathbf{x}) = \bar{u}_i(\xi) \exp(i\xi \cdot \mathbf{x}). \quad (3.3)$$

Substituting (3.2) and (3.3) into (3.1), we have

$$C_{ijkl}\bar{u}_k\xi_l\xi_j = -iC_{ijkl}\bar{\epsilon}_{kl}^*\xi_j \quad (3.4)$$

where in the derivation  $(i\xi \cdot \mathbf{x})_{,j} = i\xi_j$  is used. Expression (3.4) stands for three equations ( $i = 1, 2, 3$ ) for determining the three unknowns  $\bar{u}_i$  for given  $\bar{\epsilon}_{ij}^*$ .

Using the notation

$$\begin{aligned} K_{ik}(\xi) &= C_{ijkl}\xi_j\xi_l, \\ X_i &= -iC_{ijkl}\bar{\epsilon}_{kl}^*\xi_j, \end{aligned} \quad (3.5)$$

we can write (3.4) as

$$\begin{aligned} K_{11}\bar{u}_1 + K_{12}\bar{u}_2 + K_{13}\bar{u}_3 &= X_1, \\ K_{21}\bar{u}_1 + K_{22}\bar{u}_2 + K_{23}\bar{u}_3 &= X_2, \\ K_{31}\bar{u}_1 + K_{32}\bar{u}_2 + K_{33}\bar{u}_3 &= X_3. \end{aligned} \quad (3.6)$$

Then,  $\bar{u}_i$  is obtained as

$$\bar{u}_i(\xi) = X_j N_{ij}(\xi) / D(\xi), \quad (3.7)$$

where  $N_{ij}$  are cofactors of the matrix

$$\mathbf{K}(\xi) = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \quad (3.8)$$

and  $D(\xi)$  is the determinant of  $\mathbf{K}(\xi)$ . Note that  $K_{ki} = C_{kjil}\xi_j\xi_l = C_{klij}\xi_l\xi_j = C_{ijkl}\xi_l\xi_j = K_{ik}$  due to the symmetry of the elastic constants  $C_{ijkl} = C_{klij}$ , and that  $N_{ij} = N_{ji}$ . The explicit expressions for  $D(\xi)$  and  $N_{ij}(\xi)$  are

$$\begin{aligned} D(\xi) &= \epsilon_{mnl}K_{m1}K_{n2}K_{l3}, \\ N_{ij}(\xi) &= \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn}K_{km}K_{ln} \\ &= K_{im}K_{mj} - K_{mm}K_{ij} \\ &\quad + (\epsilon_{mn1}K_{m2}K_{n3} + \epsilon_{mn2}K_{m3}K_{n1} + \epsilon_{mn3}K_{m1}K_{n2})\delta_{ij}, \end{aligned} \quad (3.9)$$

where  $\epsilon_{ijk}$  is the permutation tensor.



Substituting (3.7) into (3.3), we have

$$u_i(\mathbf{x}) = -iC_{jlmn}\bar{\epsilon}_{mn}^*(\xi)\xi_l N_{ij}(\xi)D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}). \quad (3.10)$$

The corresponding strain and stress are obtained from (2.2) and (2.4) as

$$\epsilon_{ij}(\mathbf{x}) = \frac{1}{2}C_{klmn}\bar{\epsilon}_{mn}^*(\xi)\xi_l\{\xi_j N_{ik}(\xi) + \xi_i N_{jk}(\xi)\}D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}) \quad (3.11)$$

and

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}\{C_{pqmn}\bar{\epsilon}_{mn}^*(\xi)\xi_q\xi_l N_{kp}(\xi)D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}) - \epsilon_{kl}^*(\mathbf{x})\}, \quad (3.12)$$

where  $D^{-1} = 1/D$ . The above result was used by Mura (1964) for periodic distributions of dislocations and by Khachaturyan (1967) for a coherent inclusion of a new phase.

#### *Method of Fourier series and Fourier integrals*

The linear theory of elasticity allows for the superposition of solutions. If  $\epsilon_{ij}^*(\mathbf{x})$  is given in the Fourier series form,

$$\epsilon_{ij}^*(\mathbf{x}) = \sum \bar{\epsilon}_{ij}^*(\xi)\exp(i\xi\cdot\mathbf{x}), \quad (3.13)$$

the corresponding displacement, strain and stress are then obtained as superpositions of the solutions for single waves of the form (3.2), namely,

$$\begin{aligned} u_i(\mathbf{x}) &= -i\sum C_{jlmn}\bar{\epsilon}_{mn}^*(\xi)\xi_l N_{ij}(\xi)D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}), \\ \epsilon_{ij}(\mathbf{x}) &= \frac{1}{2}\sum C_{klmn}\bar{\epsilon}_{mn}^*(\xi)\xi_l\{\xi_j N_{ik}(\xi) + \xi_i N_{jk}(\xi)\}D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}), \end{aligned} \quad (3.14)$$

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}\left\{\sum C_{pqmn}\bar{\epsilon}_{mn}^*(\xi)\xi_q\xi_l N_{kp}(\xi)D^{-1}(\xi)\exp(i\xi\cdot\mathbf{x}) - \epsilon_{kl}^*(\mathbf{x})\right\},$$

where the summations in (3.13) and (3.14) are taken with respect to  $\xi$ . Similarly, if  $\epsilon_{ij}^*$  is given by the Fourier integral form (see Appendix 3),

$$\epsilon_{ij}^*(\mathbf{x}) = \int_{-\infty}^{\infty} \bar{\epsilon}_{ij}^*(\xi)\exp(i\xi\cdot\mathbf{x})d\xi, \quad (3.15)$$

where

$$\bar{\epsilon}_{ij}^*(\xi) = (2\pi)^{-3} \int_{-\infty}^{\infty} \epsilon_{ij}^*(x) \exp(-i\xi \cdot x) dx, \quad (3.16)$$

we have

$$\begin{aligned} u_i(x) &= -i \int_{-\infty}^{\infty} C_{jlmn} \bar{\epsilon}_{mn}^*(\xi) \xi_l N_{ij}(\xi) D^{-1}(\xi) \exp(i\xi \cdot x) d\xi, \\ \epsilon_{ij}(x) &= \frac{1}{2} \int_{-\infty}^{\infty} C_{klmn} \bar{\epsilon}_{mn}^*(\xi) \xi_l \{ \xi_j N_{ik}(\xi) + \xi_i N_{jk}(\xi) \} D^{-1}(\xi) \\ &\quad \times \exp(i\xi \cdot x) d\xi, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \sigma_{ij}(x) &= C_{ijkl} \left\{ \int_{-\infty}^{\infty} C_{pqmn} \bar{\epsilon}_{mn}^*(\xi) \xi_q \xi_l N_{kp}(\xi) D^{-1}(\xi) \right. \\ &\quad \left. \times \exp(i\xi \cdot x) d\xi - \epsilon_{kl}^*(x) \right\}, \end{aligned}$$

where

$$\begin{aligned} \int_{-\infty}^{\infty} d\xi &= \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3, \\ \int_{-\infty}^{\infty} dx &= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3. \end{aligned} \quad (3.18)$$

When (3.16) is substituted into (3.17), we have

$$\begin{aligned} u_i(x) &= -i(2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{jlmn} \epsilon_{mn}^*(x') \xi_l N_{ij}(\xi) D^{-1}(\xi) \\ &\quad \times \exp\{i\xi \cdot (x - x')\} d\xi dx', \\ &= -(2\pi)^{-3} \frac{\partial}{\partial x_l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{jlmn} \epsilon_{mn}^*(x') N_{ij}(\xi) D^{-1}(\xi) \\ &\quad \times \exp\{i\xi \cdot (x - x')\} d\xi dx', \end{aligned} \quad (3.19)$$

$$\begin{aligned}\epsilon_{ij}(\mathbf{x}) &= (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} C_{klmn} \epsilon_{mn}^*(\mathbf{x}') \\ &\quad \times \xi_l \{ \xi_j N_{ik}(\xi) + \xi_i N_{jk}(\xi) \} D^{-1}(\xi) \\ &\quad \times \exp\{i\xi \cdot (\mathbf{x} - \mathbf{x}')\} d\xi d\mathbf{x}',\end{aligned}\tag{3.20}$$

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &= C_{ijkl} \left\{ (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{pqmn} \epsilon_{mn}^*(\mathbf{x}') \xi_q \xi_l N_{kp}(\xi) D^{-1}(\xi) \right. \\ &\quad \left. \times \exp\{i\xi \cdot (\mathbf{x} - \mathbf{x}')\} d\xi d\mathbf{x}' - \epsilon_{kl}^*(\mathbf{x}) \right\}.\end{aligned}\tag{3.21}$$

### *Method of Green's functions*

When Green's functions  $G_{ij}(\mathbf{x} - \mathbf{x}')$  are defined as

$$G_{ij}(\mathbf{x} - \mathbf{x}') = (2\pi)^{-3} \int_{-\infty}^{\infty} N_{ij}(\xi) D^{-1}(\xi) \exp\{i\xi \cdot (\mathbf{x} - \mathbf{x}')\} d\xi,\tag{3.22}$$

(3.19) can be written as

$$u_i(\mathbf{x}) = - \int_{-\infty}^{\infty} C_{jlmn} \epsilon_{mn}^*(\mathbf{x}') G_{ij,l}(\mathbf{x} - \mathbf{x}') d\mathbf{x}',\tag{3.23}$$

where  $G_{ij,l}(\mathbf{x} - \mathbf{x}') = \partial/\partial x_l G_{ij}(\mathbf{x} - \mathbf{x}') = -\partial/\partial x'_l G_{ij}(\mathbf{x} - \mathbf{x}')$ . Sometimes Green's functions are called the fundamental solutions.

The corresponding expressions for the strain and stress become

$$\epsilon_{ij}(\mathbf{x}) = -\frac{1}{2} \int_{-\infty}^{\infty} C_{klmn} \epsilon_{mn}^*(\mathbf{x}') \{ G_{ik,lj}(\mathbf{x} - \mathbf{x}') + G_{jk,li}(\mathbf{x} - \mathbf{x}') \} d\mathbf{x}'\tag{3.24}$$

and

$$\sigma_{ij}(\mathbf{x}) = -C_{ijkl} \left\{ \int_{-\infty}^{\infty} C_{pqmn} \epsilon_{mn}^*(\mathbf{x}') G_{kp,ql}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' + \epsilon_{kl}^*(\mathbf{x}) \right\}.\tag{3.25}$$

Mura (1963) rewrote (3.25) in the form

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl} \int_{-\infty}^{\infty} \epsilon_{sth} \epsilon_{lnh} C_{pqmn} G_{kp,ql}(\mathbf{x} - \mathbf{x}') \epsilon_{sm}^*(\mathbf{x}') d\mathbf{x}',\tag{3.26}$$

which will be useful for the dislocation theory given in later sections. It is easy to prove that (3.26) is equivalent to (3.25). Since  $\epsilon_{st}h\epsilon_{lnh} = \delta_{st}\delta_{ln} - \delta_{sn}\delta_{tl}$ , (3.26) becomes

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl} \int_{-\infty}^{\infty} C_{pqmn} (G_{kp,qn} \epsilon_{ml}^* - G_{kp,ql} \epsilon_{mn}^*) d\mathbf{x}'. \quad (3.27)$$

In Section 5 it is shown that

$$C_{mnpq} G_{pk,qn}(\mathbf{x} - \mathbf{x}') = -\delta_{mk} \delta(\mathbf{x} - \mathbf{x}'), \quad (3.28)$$

where  $\delta(\mathbf{x} - \mathbf{x}')$  is Dirac's delta function having the property

$$\int_{-\infty}^{\infty} \epsilon_{ml}^*(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d\mathbf{x}' = \epsilon_{ml}^*(\mathbf{x}); \quad (3.29)$$

therefore, (3.25) follows from (3.27).

It is seen from (3.28) that Green's function  $G_{pk}(\mathbf{x} - \mathbf{x}')$  is the displacement component in the  $x_p$ -direction at point  $\mathbf{x}$  when a unit body force in the  $x_k$ -direction is applied at point  $\mathbf{x}'$  in the infinitely extended material. By this definition of Green's function we can directly derive (3.23) from (3.1). As was mentioned in Section 2, the displacement  $u_i$  in (3.1) can be considered as a displacement caused by the body force  $-C_{ilmn}\epsilon_{mn,l}^*$  applied in the  $x_i$ -direction. Since  $G_{ij}(\mathbf{x} - \mathbf{x}')$  is the solution for a unit body force applied in the  $x_j$ -direction, the solution for the present problem is the product of  $G_{ij}$  and the body force  $-C_{jlmn}\epsilon_{mn,l}^*$ , namely,

$$u_i(\mathbf{x}) = - \int_{-\infty}^{\infty} G_{ij}(\mathbf{x} - \mathbf{x}') C_{jlmn} \epsilon_{mn,l}^*(\mathbf{x}') d\mathbf{x}'. \quad (3.30)$$

Integrating by parts and assuming that the boundary terms vanish, we have

$$u_i(\mathbf{x}) = \int_{-\infty}^{\infty} C_{jlmn} \epsilon_{mn}^*(\mathbf{x}') \frac{\partial}{\partial x'_l} G_{ij}(\mathbf{x} - \mathbf{x}') d\mathbf{x}'. \quad (3.31)$$

For an infinite body it holds that  $(\partial/\partial x'_l)G_{ij}(\mathbf{x} - \mathbf{x}') = -(\partial/\partial x_l)G_{ij}(\mathbf{x} - \mathbf{x}')$ ; (3.23) is thereby obtained.

Expression (3.31) or (3.23) is preferable to expression (3.30). When  $\epsilon_{mn}^*$  is constant in  $\Omega$  and is zero in  $D-\Omega$ , it can be seen that the integrand in (3.30) vanishes except on the boundary of  $\Omega$ .

As will be seen in (5.9),  $G_{ij}(\mathbf{x} - \mathbf{x}')$  has a singularity at  $\mathbf{x} = \mathbf{x}'$  with the order of  $|\mathbf{x} - \mathbf{x}'|^{-1}$ . Thus, the integrals in (3.24) and (3.25) do not exist in the

sense of Riemann integrals. This difficulty can be avoided by writing (3.25) in the form

$$\sigma_{ij}(\mathbf{x}) = -C_{ijkl} \left\{ \frac{\partial}{\partial x_l} \int_{-\infty}^{\infty} C_{pqmn} \epsilon_{mn}^*(\mathbf{x}') G_{kp,q}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' + \epsilon_{kl}^*(\mathbf{x}) \right\}. \quad (3.32)$$

Expressions (3.25) or (3.17) are permissible in the context of generalized functions (Lighthill 1964).

Expressions (3.23), (3.25) and their equivalents were developed by Fredholm (1900). In connection with the solution for dislocations, many papers have discussed these expressions more extensively: Weingarten (1901), Volterra (1907), Somigliana (1914), Burgers (1939), Leibfried (1953), Eshelby (1961), Kröner (1958), Steketee (1958), de Wit (1960), Indenbom (1966), Kunin (1964), Kosevich (1965), Bacon, Barnett, and Scattergood (1978), Hirth and Lothe (1982), Teodosiu (1982), and Steeds and Willis (1979), among others.

As will be seen in Section 5, Green's functions have been obtained explicitly only for isotropic and transversely isotropic materials. Therefore, for practical calculations the usage of Green's functions as seen in (3.23) ~ (3.25) is limited, and the use of Fourier integral expressions (3.17) is much more convenient. For this reason the integrands appearing in (3.17) are written down in detail.

### Isotropic materials

$$\begin{aligned} D(\xi) &= \mu^2(\lambda + 2\mu)\xi^6, \\ N_{ij}(\xi) &= \mu\xi^2 \{ (\lambda + 2\mu)\delta_{ij}\xi^2 - (\lambda + \mu)\xi_i\xi_j \}, \end{aligned} \quad (3.33)$$

where  $\xi^2 = \xi_k\xi_k$ ,

$$\begin{aligned} C_{jlmn}\xi_l N_{ij}(\xi) D^{-1}(\xi) &= (\lambda + 2\mu)^{-1} \xi^{-4} \{ \lambda \delta_{mn} \xi_i \xi^2 + (\lambda + 2\mu) \delta_{im} \xi_n \xi^2 \\ &\quad + (\lambda + 2\mu) \delta_{in} \xi_m \xi^2 - 2(\lambda + \mu) \xi_i \xi_m \xi_n \}, \end{aligned} \quad (3.34)$$

$$\begin{aligned} C_{ijkl} C_{pqmn} \xi_q \xi_l N_{kp}(\xi) D^{-1}(\xi) \\ = (\lambda + 2\mu)^{-1} \xi^{-4} \{ \lambda^2 \delta_{ij} \delta_{mn} \xi^4 + 2\lambda \mu \delta_{mn} \xi_i \xi_j \xi^2 + 2\lambda \mu \delta_{ij} \xi_m \xi_n \xi^2 \\ + \mu(\lambda + 2\mu) (\delta_{im} \xi_j \xi_n + \delta_{jm} \xi_i \xi_n + \delta_{in} \xi_j \xi_m + \delta_{jn} \xi_i \xi_m) \xi^2 \\ - 4\mu(\lambda + \mu) \xi_i \xi_j \xi_m \xi_n \}. \end{aligned}$$

*Cubic crystals*

$$D(\xi) = \mu^2(\lambda + 2\mu + \mu')\xi^6 + \mu\mu'(2\lambda + 2\mu + \mu')\xi^2(\xi_1^2\xi_2^2 + \xi_2^2\xi_3^2 + \xi_3^2\xi_1^2) \\ + \mu'^2(3\lambda + 3\mu + \mu')\xi_1^2\xi_2^2\xi_3^2, \quad (3.35)$$

$$N_{11}(\xi) = \mu^2\xi^4 + \beta\xi^2(\xi_2^2 + \xi_3^2) + \gamma\xi_2^2\xi_3^2, \quad (3.36)$$

$$N_{12}(\xi) = -(\lambda + \mu)\xi_1\xi_2(\mu\xi^2 + \mu'\xi_3^2),$$

and the other components are obtained by the cyclic permutation of 1, 2, 3, where

$$\xi^2 = \xi_i\xi_i,$$

$$\beta = \mu(\lambda + \mu + \mu'),$$

$$\gamma = \mu'(2\lambda + 2\mu + \mu'),$$

$$\lambda = C_{12}, \quad (3.37)$$

$$\mu = C_{44},$$

$$\mu' = C_{11} - C_{12} - 2C_{44}.$$

*Hexagonal crystals (transversely isotropic)*

$$D(\xi) = (\alpha'\eta^2 + \gamma\xi_3^2)\{\alpha\gamma\eta^4 + (\alpha\beta + \gamma^2 - \gamma'^2)\eta^2\xi_3^2 + \beta\gamma\xi_3^4\} \\ = (\alpha'\eta^2 + \gamma\xi_3^2)\{(\gamma\eta^2 + \beta\xi_3^2)(\alpha\eta^2 + \gamma\xi_3^2) - \gamma'^2\eta^2\xi_3^2\}, \quad (3.38)$$

$$N_{11}(\xi) = (\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2)(\gamma\eta^2 + \beta\xi_3^2) - \gamma'^2\xi_2^2\xi_3^2,$$

$$N_{12}(\xi) = \gamma'^2\xi_1\xi_2\xi_3^2 - (\alpha - \alpha')\xi_1\xi_2(\gamma\eta^2 + \beta\xi_3^2),$$

$$N_{13}(\xi) = (\alpha - \alpha')\gamma'\xi_1\xi_2^2\xi_3 - \gamma'\xi_1\xi_3(\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2),$$

$$N_{22}(\xi) = (\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2)(\gamma\eta^2 + \beta\xi_3^2) - \gamma'^2\xi_1^2\xi_3^2, \quad (3.39)$$

$$N_{23}(\xi) = (\alpha - \alpha')\gamma'\xi_1^2\xi_2\xi_3 - \gamma'\xi_2\xi_3(\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2),$$

$$N_{33}(\xi) = (\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2)(\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2) - (\alpha - \alpha')^2\xi_1^2\xi_2^2,$$

where

$$\begin{aligned}
 \alpha &= C_{11} = C_{22}, & \alpha' &= C_{66} = \frac{1}{2}(C_{11} - C_{12}), \\
 \beta &= C_{33}, & \gamma' - \gamma &= C_{13} = C_{23}, \\
 \gamma &= C_{44} = C_{55}, & \eta^2 &= \xi_1^2 + \xi_2^2.
 \end{aligned} \tag{3.40}$$

For isotropic materials,  $\alpha = \beta = \lambda + 2\mu$ ,  $\gamma' = \lambda + \mu$ , and  $\gamma = \alpha' = \mu$ .

#### 4. Exercises of general formulae

General formulae are given in Section 3 for the elastic fields associated with prescribed eigenstrains. This section provides exercises in the usage of these formulae: the best way to understand general statements is to work out specific examples.

##### *A straight screw dislocation*

Let us consider the case where  $\Omega$  in Fig. 1.1 is the half plane ( $x_2 = 0, x_1 < 0$ ) (see Fig. 4.1), and  $\epsilon_{23}^*$  is prescribed on  $\Omega$ . Other components of  $\epsilon_{ij}^*$  are zero.

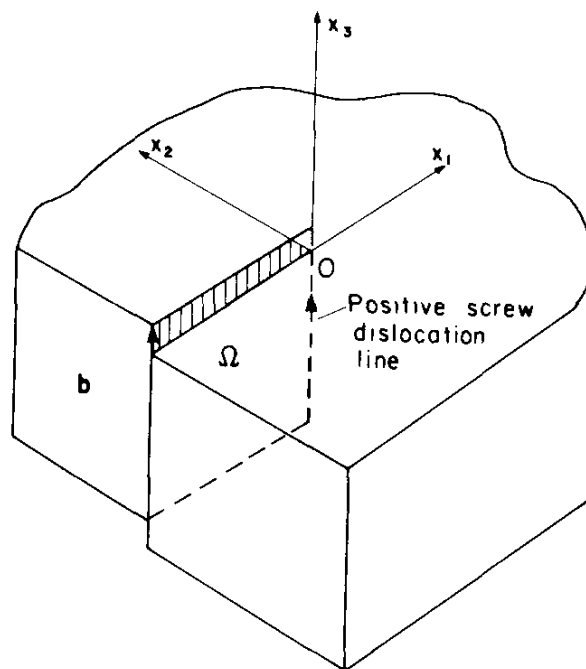


Fig 4.1 A screw dislocation