The elastic field outside an ellipsoidal inclusion

By J. D. Eshelby

Department of Metallurgy, University of Birmingham

(*Communicated by R. E. Peierls, F.R.S.—Received* 5 *June* 1959)

The results of an earlier paper are extended. The elastic field outside an inclusion or inhomogeneity is treated in greater detail. For a general inclusion the harmonic potential of a certain surface distribution may be used in place of the biharmonic potential used previously. The elastic field outside an ellipsoidal inclusion or inhomogeneity may be expressed entirely in terms of the harmonic potential of a solid ellipsoid. The solution gives incidentally the velocity field about an ellipsoid which is deforming homogeneously in a viscous fluid. An expression given previously for the strain energy of an ellipsoidal region which has undergone a shear transformation is generalized to the case where the region has elastic constants different from those of its surroundings. The Appendix outlines a general method of calculating biharmonic potentials.

1. Introduction

In a previous paper (Eshelby 1957; to be referred to as I) a method was given for finding the stresses set up in an elastic solid when a region within it (the 'inclusion') undergoes a change of form which would be a uniform homogeneous deformation if the surrounding material were absent. It was also shown that the results for the inclusion can be used to find how a uniform stress is disturbed by the presence of an ellipsoidal cavity, or more generally an ellipsoidal region whose elastic constants differ from those of the remaining material. It was emphasized in I that the elastic field inside the inclusion can be calculated without having to find the field outside the inclusion, and that a good deal of information can be derived from a knowledge of the internal field alone. Consequently the question of determining the field outside the inclusion or inhomogeneity was only briefly touched on. In the present paper the external problem is considered in more detail.

We first show (§ 2) that the biharmonic potential introduced in I in discussing the general inclusion may be replaced by the harmonic potential of a certain surface distribution. In § 3 the displacement due to an ellipsoidal inclusion is given in a form which involves only the harmonic potential of a solid ellipsoid, and it is reduced to a form suitable for numerical calculation of the stress. The Appendix describes a general method for calculating biharmonic potentials; an explicit expression is obtained for the ellipsoid.

It will be convenient to note here some errors in I. A shortcoming in the notation on p. 379 is corrected in § 2 of the present paper. On p. 380, last line, *for* p_{jk} *read* p^T_{jk}. The arithmetic (though not the principle) is incorrect in the second paragraph of p. 392. On the same page, sixth line from the bottom, *for* $V\gamma$ *read* V/γ.

2. The general inclusion

In I the following problem was solved. An inclusion bounded by a surface S in an infinite homogeneous and isotropic elastic medium undergoes a change of form which in the absence of the surrounding material would be an arbitrary homogeneous strain e^T_{ij}: to find the elastic state inside and outside S.

It was shown that the resulting displacement u^C_i is the same as that produced by a layer of body-force spread over S of amount $p^T_{ij}n_j\,\mathrm{d}S$ on each surface element $\mathrm{d}S$, where p^T_{ij} is the stress derived from e^T_{ij} by Hooke's law. The displacement produced at $\mathbf{r}$ by a point-force F_i at $\mathbf{r}'$ is

$$U_i = U_{ij}F_j,$$

where
$$U_{ij} = \frac{1}{4\pi\mu}\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{16\pi\mu(1-\sigma)}\frac{\partial^2}{\partial x_i \partial x_j}|\mathbf{r}-\mathbf{r}'|. \tag{2·1}$$

Hence
$$u^C_i(\mathbf{r}) = \int_S \mathrm{d}S\, p^T_{jk} n_k U_{ij}(\mathbf{r}-\mathbf{r}'). \tag{2·2}$$

(This corrects a deficiency of notation in (I, 2·5) and (I, 2·6).) On substituting (2·1) in (2·2) and converting the surface integrals to volume integrals we have

$$u^C_i = \frac{1}{16\pi\mu(1-\sigma)} p^T_{jk}\psi_{,ijk} - \frac{1}{4\pi\mu} p^T_{ik}\phi_{,k} \tag{2·3}$$

with
$$\phi = \int_V \frac{\mathrm{d}v}{|\mathbf{r}-\mathbf{r}'|} \quad \text{and} \quad \psi = \int_V |\mathbf{r}-\mathbf{r}'|\,\mathrm{d}v, \tag{2·4}$$

where the integrals are taken over the volume V enclosed by S.

The harmonic potential ϕ and the biharmonic potential ψ have the following properties:

$$\nabla^2\psi = 2\phi; \quad \nabla^4\psi = 2\nabla^2\phi = \begin{cases} -8\pi & \text{inside } S, \\ 0 & \text{outside } S. \end{cases} \tag{2·5}$$

The quantities
$$\phi,\ \phi_{,i},\ \psi,\ \psi_{,i},\ \psi_{,ij},\ \psi_{,ijk} \tag{2·6}$$

are continuous across S, while $\phi_{,ij}$ and $\psi_{,ijkl}$ have the discontinuities

$$\left.\begin{aligned} \phi_{,ij}(\text{out}) - \phi_{,ij}(\text{in}) &= 4\pi n_i n_j, \\ \psi_{,ijkl}(\text{out}) - \psi_{,ijkl}(\text{in}) &= 8\pi n_i n_j n_k n_l, \end{aligned}\right\} \tag{2·7}$$

where n_i is the outward normal to S.

The Appendix gives one method of reducing the determination of ψ to a problem in ordinary potential theory. Alternatively, u^C_i can be expressed in terms of ϕ and an additional harmonic function in the following way.

Equation (2·3) can be rewritten in terms of a harmonic vector B_i and a harmonic scalar β (Papkovich–Neuber):

$$u^C_i = B_i - \frac{1}{4(1-\sigma)}(x_m B_m + \beta)_{,i} \tag{2·8}$$

with and
$$\left.\begin{aligned} 4\pi\mu B_i &= -p^T_{ik}\phi_{,k} \\ 4\pi\mu\beta &= p^T_{jk}(x_j\phi_{,k} - \psi_{,jk}). \end{aligned}\right\} \tag{2·9}$$

In verifying that β is harmonic, and elsewhere, the relation

$$\nabla^2(pq) = p\nabla^2 q + q\nabla^2 p + 2p_{,m}q_{,m}$$

is useful.

The determination of β can be reduced to a standard problem in the theory of the ordinary harmonic potential. The normal derivative of β in the neighbourhood of the interface S is

$$\partial\beta/\partial n = \beta_{,m}n_m = (\delta_{jm}\phi_{,k}+x_j\phi_{,km}-\psi_{,jkm})\,n_m p^T_{jk}/4\pi\mu.$$

On crossing S, $\phi_{,km}$ increases abruptly by $4\pi n_k n_m$; $\phi_{,k}$ and $\psi_{,jkm}$ on the other hand are both continuous. Thus the corresponding increase in $\partial\beta/\partial n$ is

$$(\partial\beta/\partial n)\,(\text{out})-(\partial\beta/\partial n)\,(\text{in}) = p^T_{jk}x_j n_k/\mu. \tag{2·10}$$

Since β falls to zero far from the inclusion (2·9) implies that β is the potential of a layer of attracting matter distributed with surface density $-p^T_{jk}x_j n_k/4\pi\mu$ over the interface.

3. The ellipsoidal inclusion

When the inclusion is bounded by the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2 = 1$$

the harmonic potential is

$$\phi = \pi abc\int_\lambda^\infty \frac{U\,\mathrm{d}u}{\Delta}, \tag{3·1}$$

where
$$U(u) = 1-x^2/(a^2+u)-y^2/(b^2+u)-z^2/(c^2+u) \tag{3·2}$$

and
$$\Delta = (a^2+u)^{\frac{1}{2}}\,(b^2+u)^{\frac{1}{2}}\,(c^2+u)^{\frac{1}{2}}. \tag{3·3}$$

For an external point λ in the greatest root of $U(u) = 0$ and for an internal point $\lambda = 0$.

A similar expression for ψ is derived in the Appendix (equation (A 1)). This completes the formal solution of the inclusion problem. However, to reach a point where numerical calculation is possible it is more convenient to find the derivatives of ψ directly in terms of ϕ. Consider the function

$$f = x_1\phi_{,2}-\psi_{,12}$$

which figures as the coefficient of p^T_{12} in (2·9). It is harmonic inside and outside the inclusion, falls to zero at infinity and according to (2·7) there is a discontinuity

$$(\partial f/\partial n)\,(\text{out})-(\partial f/\partial n)\,(\text{in}) = 4\pi x_1 n_2$$

in its normal derivative across the interface. The function

$$g = x_1\phi_{,2}-x_2\phi_{,1},$$

familiar in the hydrodynamic theory of the rotating ellipsoid, is harmonic and falls to zero at infinity. The discontinuity in its normal derivative across S is

$$(\partial g/\partial n)\,(\text{out})-(\partial g/\partial n)\,(\text{in}) = 4\pi(x_1n_2-x_2n_1) = 4\pi x_1 n_2(1-b^2/a^2)$$

since the normal to the ellipsoid at x_i has components (I, 3·23)

$$n_1 = x_1/a^2h,\quad n_2 = x_2/b^2h,\quad n_3 = x_3/c^2h$$

and so $b^2x_1n_2 = a^2x_2n_1$. Hence the harmonic functions f and $ga^2/(a^2-b^2)$ are identical, since they behave similarly at infinity and have the same discontinuity in their normal derivative. Thus

$$f = \frac{a^2}{a^2-b^2}(x_1\phi_{,2}-x_2\phi_{,1})$$

and so
$$\psi_{,12} = \frac{a^2}{a^2-b^2}x_2\phi_{,1}+\frac{b^2}{b^2-a^2}x_1\phi_{,2}.$$

Similarly
$$\psi_{,23} = \frac{b^2}{b^2-c^2}x_3\phi_{,2}+\frac{c^2}{c^2-b^2}x_2\phi_{,3},$$

$$\psi_{,31} = \frac{c^2}{c^2-a^2}x_1\phi_{,3}+\frac{a^2}{a^2-c^2}x_3\phi_{,1}.$$

The requirement that $(\psi_{,ij})_{,k}$ be equal to $(\psi_{,ik})_{,j}$ leads to the identity

$$(b^2-c^2)x_1\phi_{,23}+(c^2-a^2)x_2\phi_{,31}+(a^2-b^2)x_3\phi_{,12} = 0$$

which may be verified independently.

If we were to try to find $\psi_{,11}$ by treating the function $x_1\phi_1-\psi_{,11}$ in the same way as f we should have to construct a harmonic function whose normal derivative has a discontinuity $4\pi x_1n_1$, and this is rather laborious (cf. Dyson 1891). But in fact there is no need to determine $\psi_{,11}$, $\psi_{,22}$ and $\psi_{,33}$ since the third derivatives which appear in (2·3) can all be made to depend on the four quantities ϕ, $\psi_{,12}$, $\psi_{,23}$, $\psi_{,31}$. If i, j, k are not all equal we have, trivially, $\psi_{,112} = (\psi_{,12})_{,1}$ and so forth, while if i, j, k are all equal we can use

$$\psi_{,111} = 2\phi_{,1}-(\psi_{,12})_{,2}-(\psi_{,13})_{,3}$$

and the two similar relations which are obtained by differentiating $\nabla^2\psi = 2\phi$.

In this way we find from (2·8)

$$8\pi(1-\sigma)u_1^C = \frac{e_{22}^T-e_{11}^T}{a^2-b^2}\frac{\partial}{\partial x_2}(a^2x_2\phi_{,1}-b^2x_1\phi_{,2})+\frac{e_{33}^T-e_{11}^T}{c^2-a^2}\frac{\partial}{\partial x_3}(c^2x_1\phi_{,3}-a^2x_3\phi_{,1})$$
$$-2\{(1-\sigma)e_{11}^T+\sigma(e_{22}^T+e_{33}^T)\}\phi_{,1}-4(1-\sigma)(e_{12}^T\phi_{,2}+e_{13}^T\phi_{,3})+\frac{\partial}{\partial x_1}\bar{\beta}, \qquad (3·4)$$

where
$$\bar{\beta} = \frac{2e_{12}^T}{a^2-b^2}(a^2x_2\phi_{,1}-b^2x_1\phi_{,2})+\frac{2e_{23}^T}{b^2-c^2}(b^2x_3\phi_{,2}-c^2x_2\phi_{,3})+\frac{2e_{31}^T}{c^2-a^2}(c^2x_1\phi_{,3}-a^2x_3\phi_{,1}).$$

The expressions for u_2^C and u_3^C follow by cyclic permutation of $(1,2,3)$ and (a,b,c).

The harmonic potential can be written in the form

$$\phi = \frac{2\pi abc}{l^3}\left\{\left[l^2-\frac{x^2}{k^2}+\frac{y^2}{k^2}\right]F(\theta,k)+\left[\frac{x^2}{k^2}-\frac{y^2}{k^2k'^2}+\frac{z^2}{k'^2}\right]E(\theta,k)+\frac{l}{k'^2}\left[\frac{C}{AB}y^2-\frac{B}{AC}z^2\right]\right\},$$

where
$$A = (a^2+\lambda)^{\frac{1}{2}},\quad B = (b^2+\lambda)^{\frac{1}{2}},\quad C = (c^2+\lambda)^{\frac{1}{2}},\quad a>b>c,$$

$$l = (a^2-c^2)^{\frac{1}{2}},\quad k = (a^2-b^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}},\quad k' = (b^2-c^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}},$$

and F, E are elliptic integrals of modulus k and argument $\theta = \sin^{-1}(l/A)$. The differentiations required to find u_i^C, e_{ij}^C or p_{ij}^C can be carried out with the help of the relations

$$\partial F/\partial\lambda = -\tfrac{1}{2}l/ABC, \quad \partial E/\partial\lambda = -\tfrac{1}{2}lB/A^3C,$$

$$\partial\lambda/\partial x = 2x/Ah, \quad \partial\lambda/\partial y = 2y/Bh, \quad \partial\lambda/\partial z = 2z/Ch,$$

where
$$h^2 = x^2/A^4 + y^2/B^4 + z^2/C^4.$$

It is evident from (3·1) that λ may be treated as a constant in forming the *first* derivatives of ϕ. Finally λ has to be put equal to the greatest (in fact positive) root of

$$\lambda^3 - L\lambda^2 + M\lambda - N = 0, \tag{3·5}$$

where
$$L = r^2 - R^2,$$

$$M = a^2x^2 + b^2y^2 + c^2z^2 - a^2b^2 - b^2c^2 - c^2a^2 + r^2R^2,$$

$$N = a^2b^2c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

with
$$R^2 = a^2 + b^2 + c^2, \quad r^2 = x^2 + y^2 + z^2.$$

4. Discussion

The field outside a homogeneous transformed inclusion is found by substituting the appropriate e_{ij}^T in (3·4). The external perturbing field of an inhomogeneous ellipsoid is found by inserting the e_{ij}^T of the 'equivalent inclusion', calculated from the unperturbed field e_{ij}^A by solving (I, 4·3, 4·4). It is possible to treat in the same way the case where the inhomogeneous ellipsoid has elastic constants, c_{ijkl}^*, say, which are anisotropic. It is only necessary to replace (I, 4·3, 4·4) by the six equations

$$c_{ijkl}^*(e_{ij}^C + e_{ij}^A) = \lambda(e^C + e^A - e^T)\,\delta_{ij} + 2\mu(e_{ij}^C + e_{ij}^A - e_{ij}^T) \tag{4·1}$$

and solve them for e_{ij}^T. It may seem pointless to solve the problem for an anisotropic inhomogeneity when we cannot deal with an anisotropic matrix. However, the result has been used by Kröner (1958) in discussing the elastic constants of anisotropic aggregates (cf. also Hershey 1954).

There is another problem which can be solved in terms of an equivalent inclusion. This is the case of an ellipsoidal region which undergoes a transformation strain, e_{ij}^{T*} say, and which in addition has elastic constants (which need not be isotropic) different from those of the matrix. From the present point of view it is irrelevant whether the difference in elastic constants existed originally or developed during or after the transformation. For brevity we shall refer to this inclusion as E^* and use E to denote our standard inclusion which has the same elastic constants as the matrix and has undergone a transformation strain e_{ij}^T. We suppose that before transformation E and E^* are of identical form. After transformation E can be replaced by E^*, with continuity of displacement and stress, provided that E^* when constrained to have the same final form as E develops the same stresses as E. This requires that the conditions

$$p_{ij}^I \equiv \lambda(e^C - e^T)\,\delta_{ij} + 2\mu(e_{ij}^C - e_{ij}^T) = c_{ijkl}^*(e_{kl}^C - e_{kl}^{T*}) \tag{4·2}$$

be satisfied, or, if the inclusion is isotropic,

$$p^I \equiv \kappa(e^C - e^T) = \kappa^*(e^C - e^{T*}), \quad 'p^I_{ij} \equiv 2\mu('e^C_{ij} - 'e^T_{ij}) = 2\mu^*('e^C_{ij} - 'e^{T*}_{ij}) \tag{4·3}$$

with the notation of I, § 2. Here e^C_{ij} specifies the final form of E or E^*, and e^T_{ij} or e^{T*}_{ij} is the part of e^C_{ij} which produces no stress. If we replace e^C_{kl} by $S_{klmn}e^T_{mn}$ (cf. I, 3·6) (4·1) becomes a set of equations from which e^T_{ij} can be determined when e^{T*}_{ij} and c^*_{ijkl} are prescribed. The elastic field outside the inclusion is given by (3·4) with these values of e^T_{ij}; the stress inside is, of course, given by (4·2) or (4·3) directly. The elastic energy in the matrix is given by (I, 2·20), while the energy in the inclusion is given by (I, 2·19) with e^T_{ij} replaced by e^{T*}_{ij}. Consequently the total elastic energy is

$$E_{\text{el.}} = -\tfrac{1}{2}Vp^I_{ij}e^{T*}_{ij}.$$

When only the non-diagonal components of e^{T*}_{ij} are involved the solution of (4·3) for e^C_{ij} or e^T_{ij} is simple. We may, for example, generalize a result in I (p. 391) which has been used in discussing the formation of martensite (Christian 1958, 1959; Kaufman 1959). Let an ellipsoidal region with elastic constants μ^*, κ^* in a matrix with constants μ, κ undergo a shear transformation in which e^{T*}_{13} is the only non-vanishing component. Then we have

$$E_{\text{el.}} = 2\gamma^*\mu V(e^{T*}_{13})^2 \tag{4·4}$$

and

$$\frac{\text{energy in matrix}}{\text{energy in inclusion}} = \frac{\mu^*}{\mu}\frac{1-\gamma}{\gamma}, \tag{4·5}$$

where

$$\gamma^* = \frac{\gamma\mu^*}{\gamma\mu + (1-\gamma)\mu^*}$$

and γ is the accommodation coefficient defined by (I, 5·2). For good accommodation (small γ) $E_{\text{el.}}$ is fairly insensitive to the relative values of μ and μ^*, though (4·5) is not. In fact if μ^*/μ increases from $\frac{1}{2}$ to 2 the ratio (4·5) increases by a factor 4, but (4·4) only by a factor $(1+\gamma)/(1-\frac{1}{2}\gamma)$. The value of σ to be used in calculating γ is that for the matrix. The expressions (4·4) and (4·5) are independent of κ^*, as they must be, since the inclusion is in pure shear. To deal with the case where e^{T*}_{11}, e^{T*}_{22}, e^{T*}_{33} are not zero we should have to solve a set of simultaneous equations. Robinson (1951) has given an extensive discussion of the case where e^{T*}_{ij} is a pure dilatation.

If the inclusion is entirely rigid ($c^*_{ijkl} \to \infty$) we must have, according to (4·1),

$$e^C_{ij} = S_{ijkl}e^T_{kl} = e^{T*}_{ij}; \tag{4·6}$$

that is, the constrained strain e^C_{ij} is equal to the transformation strain e^T_{ij}, or, in engineering language, there is no spring-back. In other words, the problem becomes that of finding the elastic field outside an ellipsoidal surface on which the displacement is required to take the value

$$u_i = (e^C_{ij} + \varpi^C_{ij})x_j. \tag{4·7}$$

This is not a completely arbitrary linear function of the x_i, since if e^C_{ij} is prescribed, e^T_{ij} may be found from (4·6) and ϖ^C_{ij} is then fixed by (I, 3·19). The origin of this connexion lies in the fact that there is no net couple acting on the rigid inclusion E^*, since there was none on the inclusion E which it replaced. To obtain an arbitrary ϖ_{ij}^C we should have somehow to apply a suitable external couple to the rigid inclusion. Edwardes (1893) has discussed the elastic field about an embedded ellipsoid which is given an arbitrary small rotation in this way without change of form. Daniele (1911) has solved the combined problem of determining the elastic field outside an ellipsoidal surface which is subject to the linear displacement (4·7) with independent e_{ij}^C and ϖ_{ij}^C.

It was pointed out in I (p. 387) that the elastic solution of the problem of a rigid ellipsoidal inclusion perturbing a uniform stress could be adapted to give the solution of a related problem in slow viscous motion. We can use the results for a rigid transformed inclusion in the same way to solve the following problem. An ellipsoid in a fluid of viscosity μ is undergoing a steady homogeneous change of form specified by the rate-of-strain tensor e_{ij}^{T*}, that is, its semi-axes are changing length at the rate $\dot{a} = e_{11}^{T*}$, ... and the angles between them are altering at the rate $\dot{\theta}_{ab} = 2e_{12}^{T*}, \ldots$; to find the velocity of the fluid. Solve (4·6) for e_{ij}^T, putting $\sigma = \frac{1}{2}$ in the relations (I, 3·7) which define the S_{ijkl}. Insert these values of e_{ij}^T in (3·4) and put $\sigma = \frac{1}{2}$. Then u_i^C is the velocity required.

Several authors (see Robinson (1951) and the references he gives) have solved problems concerning ellipsoidal inclusions and inhomogeneities by using ellipsoidal co-ordinates. It is perhaps worth comparing this approach with the method outlined in I and the present paper. It is clear that in order to set up the problem and solve it formally there is no need to introduce ellipsoidal co-ordinates. If our object is merely to find the stresses immediately outside the ellipsoid we calculate the S_{ijkl} from b/a and c/a by consulting a table of elliptic integrals or Osborn's (1945) curves. The e_{ij}^T are either given or have to be calculated by solving the set of simultaneous equations (4·1) or (4·2) when e_{ij}^A or e_{ij}^{T*} are the initial data. The e_{ij}^C for an internal point are then calculated from the e_{ij}^T and S_{ijkl} and the stress or strain immediately outside the ellipsoid follows from (I, 2·13). The elastic energy, interaction energy and the remote field also follow from (I, 2·21, 3·22, 4·10, 2·18) without trouble. If the calculation is to stop at this point the use of ellipsoidal co-ordinates seems to offer no advantages.

In order to find the elastic field at an arbitrary external point we have to insert the appropriate e_{ij}^T in (3·4) and carry out several tedious but straightforward differentiations. Finally (3·5) must be solved for λ. This equation reduces to a quadratic if the point of interest lies on one of the co-ordinate planes. (It is also a quadratic for arbitrary x, y, z if the ellipsoid is a spheroid.) Again, if we are only interested in points on, say, the x-axis we have simply $\lambda = x^2 - a^2$. If the field at a general point is really required we may use the tables given by Emde (1940) or by Jahnke & Emde (1943) where, in effect, λ/L is tabulated as a function of M/L^2 and N/L^3. Alternatively, if it is merely required to survey the stresses around the ellipsoid it may be sufficient to pick a triplet of values for x, y, λ, calculate the corresponding z from $U(\lambda) = 0$ (equation (3·2)) and repeat the process for other choices of x, y, λ.

It has to be admitted that, except in the simplest cases, a calculation of the external field is laborious. The real value of having a solution for the ellipsoid with three unequal axes lies rather in the fact that it furnishes a number of useful quantities (for example, elastic and interaction energies, the interior and remote fields, the stresses at the inner surface of the matrix) without much effort even in the general case, and that from it can be obtained the solutions of a number of more specialized problems.

The use of ellipsoidal co-ordinates does not really simplify the calculation of the external field. If we have already available a three-dimensional ellipsoidal co-ordinate network (constructed for the appropriate value of k) we may locate points relative to the ellipsoid without having to solve a cubic equation. If not, it is necessary to carry out the equivalent of finding all three roots of (3·5) in order to obtain the ellipsoidal co-ordinates of a point whose Cartesian co-ordinates are specified. The stresses are somewhat complicated expressions involving elliptic functions of the ellipsoidal co-ordinates. They appear in the analysis referred to local rectangular axes parallel to the curvilinear co-ordinate lines at the point under consideration. There does not seem to be any special advantage in this orientation of axes except perhaps at the surface of the ellipsoid, and there, as we have seen, a much simpler treatment is adequate.

Appendix. A method of calculating biharmonic potentials

If into the integrand of the expression for ψ (equation (2·4)) we introduce a factor unity in the form

$$1 = \frac{(x_i - x_i')(x_i - x_i')}{|\mathbf{r}-\mathbf{r}'|^2} = \frac{r^2 - 2x_i x_i' + r'^2}{|\mathbf{r}-\mathbf{r}'|^2}$$

we obtain

$$\psi = r^2\phi - 2x_i \int_V \frac{x_i' \mathrm{d}v}{|\mathbf{r}-\mathbf{r}'|} + \int_V \frac{r'^2 \mathrm{d}v}{|\mathbf{r}-\mathbf{r}'|}.$$

The integrals are the harmonic potentials of solids bounded by S and having variable densities $\rho(\mathbf{r}) = x_i$, $\rho(\mathbf{r}) = r^2$.

Ferrers (1877) and Dyson (1891) have shown how to calculate the harmonic potential of a solid ellipsoid whose density is proportional to a polynominal in x, y, z. (There is a factor 2 missing from Ferrers's expression (p. 10) for the potential of a density distribution proportional to x). From their results we obtain

$$\psi = \pi abc \int_\lambda^\infty \left\{ \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{U^2 u^2}{\Delta}\right) - \tfrac{1}{4}\frac{U^2 u}{\Delta} \right\} \mathrm{d}u. \qquad \text{(A 1)}$$

At first sight it seems as if the first term could be removed by a trivial integration. Unfortunately, although the contribution from the lower limit is zero, that from the upper limit is infinite. In fact the two terms in the integrand of (A 1) behave for large u like $\frac{1}{4}u^{-\frac{1}{2}}$ and $-\frac{1}{4}u^{-\frac{1}{2}}$ respectively, so that they must be taken together to secure convergence. We can, however, treat the integral arising from the first term as an infinite constant independent of x, y, z in the following sense. Let us put formally

$$\psi + \text{const.} = -\tfrac{1}{4}\pi abc \int_\lambda^\infty \frac{U^2 u}{\Delta} \mathrm{d}u \qquad \text{(A 2)}$$

and differentiate with respect to x_i, ignoring the undefined contribution from the variation of the upper limit. (The lower limit gives no contribution, since $U = 0$ there.) The result is the correct expression for $\psi_{,i}$. Dirichlet (1839) has given an expression for the potential of a solid ellipsoid when the law of attraction is the p^{-1}th power, valid only for $2 \leqslant p < 3$. If we put $p = 0$ (the biharmonic case) we obtain the divergent integral (A 2). Hadamard (1923) has given a method for extracting a 'finite part' from a certain type of divergent integral. Applied to the integral on the right of (A 2) it gives precisely (A 1).

References

Christian, J. W. 1958 *Acta Metall.* **6**, 379.
Christian, J. W. 1959 *Acta Metall.* **7**, 218.
Daniele, E. 1911 *Nuovo Cim.* [6], **1**, 211.
Dirichlet, G. L. 1839 *Verh. K. Preuss Akad. Wiss.* p. 18 (*Werke*, **1**, 383).
Dyson, F. W. 1891 *Quart. J. Pure Appl. Math.* **25**, 259.
Edwardes, D. 1893 *Quart. J. Pure Appl. Math.* **26**, 270.
Emde, F. 1940 *Tables of elementary functions.* Leipzig: Teubner.
Eshelby, J. D. 1957 *Proc. Roy. Soc.* A, **241**, 376.
Ferrers, N. M. 1877 *Quart. J. Pure Appl. Math.* **14**, 1.
Hadamard, J. 1923 *Lectures on Cauchy's problem*, p. 133. New Haven: Yale University Press.
Hershey, A. V. 1954 *J. Appl. Mech.* **21**, 1.
Jahnke, E. & Emde, F. 1943 *Tables of functions.* New York: Dover Publications.
Kaufman, L. 1959 *Acta Metall.* **7**, 216.
Kröner, E. 1958 *Z. Phys.* **151**, 504.
Osborn, J. A. 1945 *Phys. Rev.* **67**, 351.
Robinson, K. 1951 *J. Appl. Phys.* **22**, 1045.