Contents

General theory of eigenstrains

The definition of eigenstrains is given first. Then the associated general solutions for elastic fields for given eigenstrains are expressed by Fourier integrals and Green's functions. Some details of calculations for Green's functions are described for static and dynamic cases.

As fundamental formulae for the subsequent chapters, general expressions of elastic fields are given for inclusions, dislocations, and disclinations. The stress discontinuity on boundaries of inclusions and the incompatibility of eigenstrains are discussed as general theories.

Throughout this work, a fixed rectangular Cartesian coordinate system with coordinate axes x_i, $i = 1, 2, 3$, is used.

1. Definition of eigenstrains

'Eigenstrain' is a generic name given by the author to such nonelastic strains as thermal expansion, phase transformation, initial strains, plastic strains, and misfit strains. 'Eigenstress' is a generic name given to self-equilibrated internal stresses caused by one or several of these eigenstrains in bodies which are free from any other external force and surface constraint. The eigenstress fields are created by the incompatibility of the eigenstrains.

This new English terminology was adapted from the German 'Eigenspannungen und Eigenspannungsquellen,' which is the title of H. Reissner's paper (1931) on residual stresses. Eshelby (1957) referred to eigenstrains as stress-free transformation strains in his celebrated paper which has stimulated the present author to work on inclusion and dislocation problems. The term 'elastic polarization' was used by Kröner (1958) for eigenstrains in a slightly different context—when the nonhomogeneity of polycrystal deformation is under consideration.

Engineers have used the term 'residual stresses' for the self-equilibrated internal stresses when they remain in materials after fabrication or plastic deformation. Eigenstresses are called thermal stresses when thermal expansion

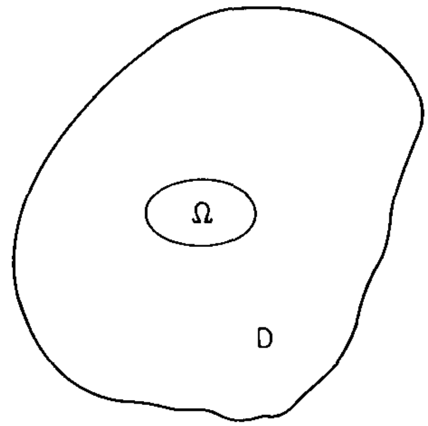

Fig 1 1 Inclusion Ω

is a cause of the corresponding elastic fields. For example, when a part Ω of a material (Fig. 1.1) has its temperature raised by T, thermal stress σ_{ij} is induced in the material D by the constraint from the part which surrounds Ω. The thermal expansion αT, where α is the linear thermal expansion coefficient, constitutes the thermal expansion strain,

$$\epsilon^*_{ij} = \delta_{ij}\alpha T, \tag{1.1}$$

where δ_{ij} is the Kronecker delta (see Appendix 1). The thermal expansion strain is the strain caused when Ω can be expanded freely with the removal of the constraint from the surrounding part.

The actual strain is then the sum of the thermal and elastic strains. The elastic strain is related to the thermal stress by Hooke's law. The thermal expansion strain (1.1) is a typical example of an eigenstrain. In the elastic theory of eigenstrains and eigenstresses, however, it is not necessary to attribute ϵ^*_{ij} to any specific source. The source could be phase transformation, precipitation, plastic deformation or a fictitious source necessary for the equivalent inclusion method (to be discussed in Section 22).

When an eigenstrain ϵ^*_{ij} is prescribed in a finite subdomain Ω in a homogeneous material D (see Fig. 1.1) and it is zero in the matrix D-Ω, then Ω is called an inclusion. The elastic moduli of the material are assumed to be homogeneous when inclusions are under consideration.

If a subdomain Ω in a material D has elastic moduli different from those of the matrix, then Ω is called an inhomogeneity. Applied stresses will be disturbed by the existence of the inhomogeneity. This disturbed stress field will be simulated by an eigenstress field by considering a fictitious eigenstrain ϵ^*_{ij} in Ω in a *homogeneous* material.

When Ω in Fig. 1.1 is a plane embedded in a three-dimensional material D and ϵ^*_{ij} is given on Ω as a plastic strain caused by a finite slip $\boldsymbol{b}$, the boundary of Ω is called a dislocation loop. If ϵ^*_{ij} is created by a rigid rotation of plane Ω by ω, the boundary of Ω is called a disclination loop.

2. Fundamental equations of elasticity

In this section the field equations for the elasticity theory will be reviewed with particular reference to solving eigenstrain problems. These problems consist of finding displacement u_i, strain ϵ_{ij}, and stress σ_{ij} at an arbitrary point $x(x_1, x_2, x_3)$ when a free body D is subjected to a given distribution of eigenstrain ϵ^*_{ij}. A free body is one which is free from any external surface or body force.

Hooke's law

For infinitesimal deformations considered in this book, the total strain ϵ_{ij} is regarded as the sum of elastic strain e_{ij} and eigenstrain ϵ^*_{ij},

$$\epsilon_{ij} = e_{ij} + \epsilon^*_{ij}. \tag{2.1}$$

The total strain must be compatible,

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2}$$

where $u_{i,j} = \partial u_i/\partial x_j$.

The elastic strain is related to stress σ_{ij} by Hooke's law;

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}(\epsilon_{kl} - \epsilon^*_{kl}) \tag{2.3}$$

or

$$\sigma_{ij} = C_{ijkl}(u_{k,l} - \epsilon^*_{kl}), \tag{2.4}$$

where C_{ijkl} are the elastic moduli (constants) (see Appendix 2), and the summation convention for the repeated indices is employed (see Appendix 1). Since C_{ijkl} is symmetric ($C_{ijlk} = C_{ijkl}$), we have $C_{ijkl}u_{l,k} = C_{ijkl}u_{k,l}$. In the domain where $\epsilon^*_{ij} = 0$, (2.4) becomes

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = C_{ijkl}u_{k,l}. \tag{2.5}$$

The inverse expression of (2.3) is

$$\epsilon_{ij} - \epsilon^*_{ij} = C^{-1}_{ijkl}\sigma_{kl}, \tag{2.6}$$

where C^{-1}_{ijkl} is the elastic compliance.

For isotropic materials, (2.3) and (2.6) can be written as

$$
\begin{aligned}
&\sigma_{ij} = 2\mu(\epsilon_{ij} - \epsilon^*_{ij}) + \lambda\delta_{ij}(\epsilon_{kk} - \epsilon^*_{kk}),\\
&\epsilon_{ij} - \epsilon^*_{ij} = \left\{\sigma_{ij} - \delta_{ij}\sigma_{kk}\nu/(1+\nu)\right\}/2\mu,
\end{aligned}
\tag{2.7}
$$

where λ and μ are the Lamé constants, and ν is Poisson's ratio. Young's modulus E, the shear modulus μ, and the bulk modulus K are connected by $2\mu = E/(1+\nu)$, $K = E/3(1-2\nu)$, and $\lambda = 2\mu\nu/(1-2\nu)$. The alternative expressions for (2.7) are

$$
\begin{aligned}
&\sigma_x = \frac{E}{1+\nu}\left\{(\epsilon_x - \epsilon^*_x) + \frac{\nu}{1-2\nu}(\epsilon_{kk} - \epsilon^*_{kk})\right\},\\
&\sigma_y = \frac{E}{1+\nu}\left\{(\epsilon_y - \epsilon^*_y) + \frac{\nu}{1-2\nu}(\epsilon_{kk} - \epsilon^*_{kk})\right\},\\
&\sigma_z = \frac{E}{1+\nu}\left\{(\epsilon_z - \epsilon^*_z) + \frac{\nu}{1-2\nu}(\epsilon_{kk} - \epsilon^*_{kk})\right\},\\
&\sigma_{xy} = \frac{E}{1+\nu}(\epsilon_{xy} - \epsilon^*_{xy}),\\
&\sigma_{yz} = \frac{E}{1+\nu}(\epsilon_{yz} - \epsilon^*_{yz}),\\
&\sigma_{zx} = \frac{E}{1+\nu}(\epsilon_{zx} - \epsilon^*_{zx}),
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&\epsilon_x - \epsilon^*_x = \left\{\sigma_x - \nu(\sigma_y + \sigma_z)\right\}/E,\\
&\epsilon_y - \epsilon^*_y = \left\{\sigma_y - \nu(\sigma_z + \sigma_x)\right\}/E,\\
&\epsilon_z - \epsilon^*_z = \left\{\sigma_z - \nu(\sigma_x + \sigma_y)\right\}/E,\\
&\epsilon_{xy} - \epsilon^*_{xy} = \frac{1+\nu}{E}\sigma_{xy},\\
&\epsilon_{yz} - \epsilon^*_{yz} = \frac{1+\nu}{E}\sigma_{yz},\\
&\epsilon_{zx} - \epsilon^*_{zx} = \frac{1+\nu}{E}\sigma_{zx},
\end{aligned}
\tag{2.9}
$$

where $\epsilon_{kk} = \epsilon_x + \epsilon_y + \epsilon_z$ and $\epsilon^*_{kk} = \epsilon^*_x + \epsilon^*_y + \epsilon^*_z$. It is convenient to use (2.8) for the plane strain case where $\epsilon_z = 0$. Expression (2.9) is recommended for the plane stress case where $\sigma_z = \sigma_{zx} = \sigma_{zy} = 0$. It should be noted that solutions for the plane stress can be obtained directly from those for the plane strain by replacing $E/(1-\nu^2)$ with E and $\nu/(1-\nu)$ with ν.

When Hooke's law (2.8) is rewritten for the two-dimensional case, we have

$$\begin{aligned}
\sigma_x &= \frac{\mu}{\kappa - 1}\left\{(\kappa+1)(\epsilon_x - \epsilon^*_x) + (3-\kappa)(\epsilon_y - \epsilon^*_y)\right\},\\
\sigma_y &= \frac{\mu}{\kappa - 1}\left\{(\kappa+1)(\epsilon_y - \epsilon^*_y) + (3-\kappa)(\epsilon_x - \epsilon^*_x)\right\},\\
\sigma_{xy} &= 2\mu\left(\epsilon_{xy} - \epsilon^*_{xy}\right),\\
\sigma_z &= \sigma_{zx} = \sigma_{zy} = 0,
\end{aligned} \tag{2.9.1}$$

for the plane stress and $\kappa = (3-\nu)/(1+\nu)$. For the plane strain, we have

$$\begin{aligned}
\sigma_x &= \frac{\mu}{\kappa - 1}\left\{(\kappa+1)(\epsilon_x - \epsilon^*_x - \nu\epsilon^*_z) + (3-\kappa)(\epsilon_y - \epsilon^*_y - \nu\epsilon^*_z)\right\},\\
\sigma_y &= \frac{\mu}{\kappa - 1}\left\{(\kappa+1)(\epsilon_y - \epsilon^*_y - \nu\epsilon^*_z) + (3-\kappa)(\epsilon_x - \epsilon^*_x - \nu\epsilon^*_z)\right\},\\
\sigma_{xy} &= 2\mu\left(\epsilon_{xy} - \epsilon^*_{xy}\right),\\
\sigma_z &= -\frac{\kappa+1}{\kappa-1}\mu\epsilon^*_z + \frac{3-\kappa}{\kappa-1}\mu\left(\epsilon_x + \epsilon_y - \epsilon^*_x - \epsilon^*_y\right),\\
\sigma_{zx} &= \sigma_{zy} = 0,
\end{aligned} \tag{2.9.2}$$

where $\kappa = 3 - 4\nu$.

Equilibrium conditions

When eigenstresses are calculated, material domain D is assumed to be free from any external force and any surface constraint. If these conditions for the free body are not satisfied, the stress field can be constructed from the superposition of the eigenstress of the free body and the solution of a proper boundary value problem.

The equations of equilibrium are

$$\sigma_{ij,j} = 0 \quad (i = 1, 2, 3). \tag{2.10}$$

The boundary conditions for free external surface forces are

$$\sigma_{ij} n_j = 0, \tag{2.11}$$

where n_i is the exterior unit normal vector on the boundary of D.

By substituting (2.4) into (2.10) and (2.11), we have

$$C_{ijkl} u_{k,lj} = C_{ijkl} \epsilon^*_{kl,j} \tag{2.12}$$

and

$$C_{ijkl} u_{k,l} n_j = C_{ijkl} \epsilon^*_{kl} n_j. \tag{2.13}$$

It can be seen that the contribution of ϵ^*_{ij} to the equations of equilibrium is similar to that of a body force since the equations of equilibrium under body force X_i with zero ϵ^*_{ij} are $C_{ijkl} u_{k,lj} = -X_i$. Similarly, $C_{ijkl} \epsilon^*_{kl} n_j$ behaves like a surface force on the boundary. Thus, it can be said that the elastic displacement field caused by ϵ^*_{ij} in a free body is equivalent to that caused by body force $-C_{ijkl} \epsilon^*_{kl,j}$ and surface force $C_{ijkl} \epsilon^*_{kl} n_j$.

In subsequent chapters, D in most cases is considered as an infinitely extended body (infinite body), and condition (2.11) is replaced by the condition $\sigma_{ij}(\boldsymbol{x}) \to 0$ for $\boldsymbol{x} \to \infty$.

Compatibility conditions

The strain tensor ϵ_{ij} has six components, while the displacement vector u_i has three components. The tensor and the vector are related to each other through the relation (2.2), which can be called the condition for the compatibility of strain ϵ_{ij}. Generally, however, the equations of compatibility are referred to the relations which are derived from (2.2) by eliminating u_i,

$$\epsilon_{pki} \epsilon_{qlj} \epsilon_{ij,kl} = 0, \tag{2.14}$$

where ϵ_{pki} is the permutation tensor (see Appendix 1). Relation (2.14) will be discussed in Section 10.

The displacement differential equations of the elasticity theory are given by (2.12). In some cases, however, it is more convenient to consider (2.10), (2.3), and (2.14). Boundary conditions and various side conditions, such as singularity conditions, continuity conditions, etc., arise in problems from time to time. We can say at this point that the fundamental equations to be solved are equations (2.12).

Eigenstresses are caused by constraint from the surrounding elastic medium which prohibits the geometrically incompatible deformation of ϵ^*_{ij}. The incompatibility of ϵ^*_{ij} was discussed by Reissner (1931) and Neményi (1931). Dislocations due to incompatibility were studied by Weingarten (1901), Cesáro (1906), Volterra (1907), and Moriguti (1947) from the viewpoint of the elasticity theory in connection with the multiple values of displacements and rotations. Another viewpoint on dislocations, from the plasticity theory, was developed by Kondo (1955), Bilby (1960), and Kröner (1958).

In the following sections we investigate the methods of finding the associated elastic fields (displacements, strains, stresses) and the related problems for given distributions of ϵ^*_{ij}. Particular emphasis will be placed on the case when a uniform ϵ^*_{ij} is given in an ellipsoidal domain Ω in an infinitely extended medium D. The results are useful for the study of the mechanical properties of solids which may contain precipitates, inclusions, voids, cracks, etc. The most fundamental contribution to this study was made by Eshelby (1951, 1956, 1957, 1959 and 1961).

3. General expressions of elastic fields for given eigenstrain distributions

The case where a given material is infinitely extended is of particular interest for the mathematical simplicity of the solution as well as for its practical importance. When the solution is applied to inclusion problems, it can be assumed with sufficient accuracy that the materials are infinitely extended since the size of the inclusions is relatively small compared to the size of the macroscopic material samples.

The fundamental equations to be solved for given ϵ^*_{ij}, (2.12), are

$$C_{ijkl}u_{k,lj} = C_{ijkl}\epsilon^*_{kl,j}. \tag{3.1}$$

Periodic solutions

Suppose $\epsilon^*_{ij}(x)$ is given in the form of a single wave of amplitude $\bar{\epsilon}^*_{ij}(\xi)$, where ξ is the wave vector corresponding to the given period of the distribution,

$$\epsilon^*_{ij}(x) = \bar{\epsilon}^*_{ij}(\xi)\exp(i\xi\cdot x), \tag{3.2}$$

where $i=\sqrt{-1}$ and $\xi\cdot x = \xi_k x_k$.

The solution of (3.1) corresponding to this distribution may also be expressed in the form of a single wave of the same period, that is,

$$u_i(x) = \bar{u}_i(\xi)\exp(i\xi\cdot x). \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$C_{ijkl}\bar{u}_k\xi_l\xi_j = -iC_{ijkl}\bar{\epsilon}^*_{kl}\xi_j \tag{3.4}$$

where in the derivation $(i\boldsymbol{\xi}\cdot\boldsymbol{x})_{,l} = i\xi_l$ is used. Expression (3.4) stands for three equations ($i = 1, 2, 3$) for determining the three unknowns $\bar{u}_i$ for given $\bar{\epsilon}^*_{ij}$.

Using the notation

$$\begin{aligned} K_{ik}(\boldsymbol{\xi}) &= C_{ijkl}\xi_j\xi_l, \\ X_i &= -iC_{ijkl}\bar{\epsilon}^*_{kl}\xi_j, \end{aligned} \tag{3.5}$$

we can write (3.4) as

$$\begin{aligned} K_{11}\bar{u}_1 + K_{12}\bar{u}_2 + K_{13}\bar{u}_3 &= X_1, \\ K_{21}\bar{u}_1 + K_{22}\bar{u}_2 + K_{23}\bar{u}_3 &= X_2, \\ K_{31}\bar{u}_1 + K_{32}\bar{u}_2 + K_{33}\bar{u}_3 &= X_3. \end{aligned} \tag{3.6}$$

Then, $\bar{u}_i$ is obtained as

$$\bar{u}_i(\boldsymbol{\xi}) = X_j N_{ij}(\boldsymbol{\xi})/D(\boldsymbol{\xi}), \tag{3.7}$$

where N_{ij} are cofactors of the matrix

$$\boldsymbol{K}(\boldsymbol{\xi}) = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \tag{3.8}$$

and $D(\boldsymbol{\xi})$ is the determinant of $\boldsymbol{K}(\boldsymbol{\xi})$. Note that $K_{ki} = C_{kjil}\xi_j\xi_l = C_{klij}\xi_l\xi_j = C_{ijkl}\xi_l\xi_j = K_{ik}$ due to the symmetry of the elastic constants $C_{ijkl} = C_{klij}$, and that $N_{ij} = N_{ji}$. The explicit expressions for $D(\boldsymbol{\xi})$ and $N_{ij}(\boldsymbol{\xi})$ are

$$\begin{aligned} D(\boldsymbol{\xi}) &= \epsilon_{mnl}K_{m1}K_{n2}K_{l3}, \\ N_{ij}(\boldsymbol{\xi}) &= \tfrac{1}{2}\epsilon_{ikl}\epsilon_{jmn}K_{km}K_{ln} \\ &= K_{im}K_{mj} - K_{mm}K_{ij} \\ &\quad + (\epsilon_{mn1}K_{m2}K_{n3} + \epsilon_{mn2}K_{m3}K_{n1} + \epsilon_{mn3}K_{m1}K_{n2})\delta_{ij}, \end{aligned} \tag{3.9}$$

where ϵ_{ijk} is the permutation tensor.

Substituting (3.7) into (3.3), we have

$$u_i(x) = -iC_{jlmn}\bar{\epsilon}^*_{mn}(\xi)\xi_l N_{ij}(\xi)D^{-1}(\xi)\exp(i\xi\cdot x). \tag{3.10}$$

The corresponding strain and stress are obtained from (2.2) and (2.4) as

$$\epsilon_{ij}(x) = \tfrac{1}{2}C_{klmn}\bar{\epsilon}^*_{mn}(\xi)\xi_l\{\xi_j N_{ik}(\xi)+\xi_i N_{jk}(\xi)\}D^{-1}(\xi)\exp(i\xi\cdot x) \tag{3.11}$$

and

$$\sigma_{ij}(x) = C_{ijkl}\{C_{pqmn}\bar{\epsilon}^*_{mn}(\xi)\xi_q\xi_l N_{kp}(\xi)D^{-1}(\xi)\exp(i\xi\cdot x)-\epsilon^*_{kl}(x)\}, \tag{3.12}$$

where $D^{-1}=1/D$. The above result was used by Mura (1964) for periodic distributions of dislocations and by Khachaturyan (1967) for a coherent inclusion of a new phase.

Method of Fourier series and Fourier integrals

The linear theory of elasticity allows for the superposition of solutions. If $\epsilon^*_{ij}(x)$ is given in the Fourier series form,

$$\epsilon^*_{ij}(x) = \sum \bar{\epsilon}^*_{ij}(\xi)\exp(i\xi\cdot x), \tag{3.13}$$

the corresponding displacement, strain and stress are then obtained as superpositions of the solutions for single waves of the form (3.2), namely,

$$u_i(x) = -i\sum C_{jlmn}\bar{\epsilon}^*_{mn}(\xi)\xi_l N_{ij}(\xi)D^{-1}(\xi)\exp(i\xi\cdot x),$$

$$\epsilon_{ij}(x) = \tfrac{1}{2}\sum C_{klmn}\bar{\epsilon}^*_{mn}(\xi)\xi_l\{\xi_j N_{ik}(\xi)+\xi_i N_{jk}(\xi)\}D^{-1}(\xi)\exp(i\xi\cdot x), \tag{3.14}$$

$$\sigma_{ij}(x) = C_{ijkl}\{\sum C_{pqmn}\bar{\epsilon}^*_{mn}(\xi)\xi_q\xi_l N_{kp}(\xi)D^{-1}(\xi)\exp(i\xi\cdot x)-\epsilon^*_{kl}(x)\},$$

where the summations in (3.13) and (3.14) are taken with respect to ξ. Similarly, if ϵ^*_{ij} is given by the Fourier integral form (see Appendix 3),

$$\epsilon^*_{ij}(x) = \int_{-\infty}^{\infty}\bar{\epsilon}^*_{ij}(\xi)\exp(i\xi\cdot x)\,d\xi, \tag{3.15}$$

where

$$\bar{\epsilon}^*_{ij}(\boldsymbol{\xi}) = (2\pi)^{-3}\int_{-\infty}^{\infty} \epsilon^*_{ij}(\boldsymbol{x}) \exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \tag{3.16}$$

we have

$$u_i(\boldsymbol{x}) = -i\int_{-\infty}^{\infty} C_{jlmn}\bar{\epsilon}^*_{mn}(\boldsymbol{\xi})\xi_l N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi}) \exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi},$$

$$\begin{aligned}\epsilon_{ij}(\boldsymbol{x}) = \tfrac{1}{2}\int_{-\infty}^{\infty} C_{klmn}\bar{\epsilon}^*_{mn}(\boldsymbol{\xi})\xi_l\{\xi_j N_{ik}(\boldsymbol{\xi}) + \xi_i N_{jk}(\boldsymbol{\xi})\}D^{-1}(\boldsymbol{\xi})\\ \times\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\sigma_{ij}(\boldsymbol{x}) = C_{ijkl}\Big\{\int_{-\infty}^{\infty} C_{pqmn}\bar{\epsilon}^*_{mn}(\boldsymbol{\xi})\xi_q\xi_l N_{kp}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\\ \times\exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi} - \epsilon^*_{kl}(\boldsymbol{x})\Big\},\end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\xi} &= \int_{-\infty}^{\infty} \mathrm{d}\xi_1\int_{-\infty}^{\infty} \mathrm{d}\xi_2\int_{-\infty}^{\infty} \mathrm{d}\xi_3,\\ \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x} &= \int_{-\infty}^{\infty} \mathrm{d}x_1\int_{-\infty}^{\infty} \mathrm{d}x_2\int_{-\infty}^{\infty} \mathrm{d}x_3.\end{aligned} \tag{3.18}$$

When (3.16) is substituted into (3.17), we have

$$\begin{aligned}u_i(\boldsymbol{x}) &= -i(2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{jlmn}\epsilon^*_{mn}(\boldsymbol{x}')\xi_l N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\\ &\qquad\times\exp\{i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{x}')\}\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{x}'\\ &= -(2\pi)^{-3}\frac{\partial}{\partial x_l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{jlmn}\epsilon^*_{mn}(\boldsymbol{x}')N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\\ &\qquad\times\exp\{i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{x}')\}\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{x}',\end{aligned} \tag{3.19}$$

$$\epsilon_{ij}(x) = (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2} C_{klmn}\epsilon^*_{mn}(x')$$
$$\times \xi_l\{\xi_j N_{ik}(\xi) + \xi_i N_{jk}(\xi)\} D^{-1}(\xi)$$
$$\times \exp\{i\xi\cdot(x - x')\}\, d\xi\, dx', \tag{3.20}$$

$$\sigma_{ij}(x) = C_{ijkl}\Big\{(2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{pqmn}\epsilon^*_{mn}(x')\xi_q\xi_l N_{kp}(\xi) D^{-1}(\xi)$$
$$\times \exp\{i\xi\cdot(x - x')\}\, d\xi\, dx' - \epsilon^*_{kl}(x)\Big\}. \tag{3.21}$$

Method of Green's functions

When Green's functions $G_{ij}(x - x')$ are defined as

$$G_{ij}(x - x') = (2\pi)^{-3}\int_{-\infty}^{\infty} N_{ij}(\xi) D^{-1}(\xi)\exp\{i\xi\cdot(x - x')\}\, d\xi, \tag{3.22}$$

(3.19) can be written as

$$u_i(x) = -\int_{-\infty}^{\infty} C_{jlmn}\epsilon^*_{mn}(x') G_{ij,l}(x - x')\, dx', \tag{3.23}$$

where $G_{ij,l}(x - x') = \partial/\partial x_l G_{ij}(x - x') = -\partial/\partial x'_l G_{ij}(x - x')$. Sometimes Green's functions are called the fundamental solutions.

The corresponding expressions for the strain and stress become

$$\epsilon_{ij}(x) = -\tfrac{1}{2}\int_{-\infty}^{\infty} C_{klmn}\epsilon^*_{mn}(x')\{G_{ik,lj}(x - x') + G_{jk,li}(x - x')\}\, dx' \tag{3.24}$$

and

$$\sigma_{ij}(x) = -C_{ijkl}\Big\{\int_{-\infty}^{\infty} C_{pqmn}\epsilon^*_{mn}(x') G_{kp,ql}(x - x')\, dx' + \epsilon^*_{kl}(x)\Big\}. \tag{3.25}$$

Mura (1963) rewrote (3.25) in the form

$$\sigma_{ij}(x) = C_{ijkl}\int_{-\infty}^{\infty} \epsilon_{sth}\epsilon_{lnh} C_{pqmn} G_{kp,qt}(x - x')\epsilon^*_{sm}(x')\, dx', \tag{3.26}$$

which will be useful for the dislocation theory given in later sections. It is easy to prove that (3.26) is equivalent to (3.25). Since $\epsilon_{sth}\epsilon_{lnh} = \delta_{sl}\delta_{tn} - \delta_{sn}\delta_{tl}$, (3.26) becomes

$$\sigma_{ij}(x) = C_{ijkl}\int_{-\infty}^{\infty} C_{pqmn}\left(G_{kp,qn}\epsilon^*_{ml} - G_{kp,ql}\epsilon^*_{mn}\right)\,\mathrm{d}x'. \tag{3.27}$$

In Section 5 it is shown that

$$C_{mnpq}G_{pk,qn}(x-x') = -\delta_{mk}\delta(x-x'), \tag{3.28}$$

where $\delta(x-x')$ is Dirac's delta function having the property

$$\int_{-\infty}^{\infty} \epsilon^*_{ml}(x')\delta(x-x')\,\mathrm{d}x' = \epsilon^*_{ml}(x); \tag{3.29}$$

therefore, (3.25) follows from (3.27).

It is seen from (3.28) that Green's function $G_{pk}(x-x')$ is the displacement component in the x_p-direction at point x when a unit body force in the x_k-direction is applied at point x' in the infinitely extended material. By this definition of Green's function we can directly derive (3.23) from (3.1). As was mentioned in Section 2, the displacement u_i in (3.1) can be considered as a displacement caused by the body force $-C_{ilmn}\epsilon^*_{mn,l}$ applied in the x_i-direction. Since $G_{ij}(x-x')$ is the solution for a unit body force applied in the x_j-direction, the solution for the present problem is the product of G_{ij} and the body force $-C_{jlmn}\epsilon^*_{mn,l}$, namely,

$$u_i(x) = -\int_{-\infty}^{\infty} G_{ij}(x-x')C_{jlmn}\epsilon^*_{mn,l}(x')\,\mathrm{d}x'. \tag{3.30}$$

Integrating by parts and assuming that the boundary terms vanish, we have

$$u_i(x) = \int_{-\infty}^{\infty} C_{jlmn}\epsilon^*_{mn}(x')\frac{\partial}{\partial x'_l}G_{ij}(x-x')\,\mathrm{d}x'. \tag{3.31}$$

For an infinite body it holds that $(\partial/\partial x'_l)G_{ij}(x-x') = -(\partial/\partial x_l)G_{ij}(x-x')$; (3.23) is thereby obtained.

Expression (3.31) or (3.23) is preferable to expression (3.30). When ϵ^*_{mn} is constant in Ω and is zero in D-Ω, it can be seen that the integrand in (3.30) vanishes except on the boundary of Ω.

As will be seen in (5.9), $G_{ij}(x-x')$ has a singularity at $x=x'$ with the order of $|x-x'|^{-1}$. Thus, the integrals in (3.24) and (3.25) do not exist in the sense of Riemann integrals. This difficulty can be avoided by writing (3.25) in the form

$$\sigma_{ij}(\boldsymbol{x}) = -C_{ijkl}\left\{\frac{\partial}{\partial x_l}\int_{-\infty}^{\infty} C_{pqmn}\epsilon^*_{mn}(\boldsymbol{x}')G_{kp,q}(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' + \epsilon^*_{kl}(\boldsymbol{x})\right\}. \quad (3.32)$$

Expressions (3.25) or (3.17) are permissible in the context of generalized functions (Lighthill 1964).

Expressions (3.23), (3.25) and their equivalents were developed by Fredholm (1900). In connection with the solution for dislocations, many papers have discussed these expressions more extensively: Weingarten (1901), Volterra (1907), Somigliana (1914), Burgers (1939), Leibfried (1953), Eshelby (1961), Kröner (1958), Steketee (1958), de Wit (1960), Indenbom (1966), Kunin (1964), Kosevich (1965), Bacon, Barnett, and Scattergood (1978), Hirth and Lothe (1982), Teodosiu (1982), and Steeds and Willis (1979), among others.

As will be seen in Section 5, Green's functions have been obtained explicitly only for isotropic and transversely isotropic materials. Therefore, for practical calculations the usage of Green's functions as seen in (3.23) ~ (3.25) is limited, and the use of Fourier integral expressions (3.17) is much more convenient. For this reason the integrands appearing in (3.17) are written down in detail.

Isotropic materials

$$\begin{aligned} D(\boldsymbol{\xi}) &= \mu^2(\lambda+2\mu)\xi^6, \\ N_{ij}(\boldsymbol{\xi}) &= \mu\xi^2\{(\lambda+2\mu)\delta_{ij}\xi^2 - (\lambda+\mu)\xi_i\xi_j\}, \end{aligned} \quad (3.33)$$

where $\xi^2 = \xi_k\xi_k$,

$$\begin{aligned} C_{jlmn}\xi_l N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi}) = (\lambda+2\mu)^{-1}\xi^{-4}\{&\lambda\delta_{mn}\xi_i\xi^2 + (\lambda+2\mu)\delta_{im}\xi_n\xi^2 \\ &+(\lambda+2\mu)\delta_{in}\xi_m\xi^2 - 2(\lambda+\mu)\xi_i\xi_m\xi_n\}, \end{aligned} \quad (3.34)$$

$$\begin{aligned} &C_{ijkl}C_{pqmn}\xi_q\xi_l N_{kp}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi}) \\ &= (\lambda+2\mu)^{-1}\xi^{-4}\{\lambda^2\delta_{ij}\delta_{mn}\xi^4 + 2\lambda\mu\delta_{mn}\xi_i\xi_j\xi^2 + 2\lambda\mu\delta_{ij}\xi_m\xi_n\xi^2 \\ &\qquad +\mu(\lambda+2\mu)(\delta_{im}\xi_j\xi_n + \delta_{jm}\xi_i\xi_n + \delta_{in}\xi_j\xi_m + \delta_{jn}\xi_i\xi_m)\xi^2 \\ &\qquad -4\mu(\lambda+\mu)\xi_i\xi_j\xi_m\xi_n\}. \end{aligned}$$

Cubic crystals

$$D(\xi) = \mu^2(\lambda + 2\mu + \mu')\xi^6 + \mu\mu'(2\lambda + 2\mu + \mu')\xi^2(\xi_1^2\xi_2^2 + \xi_2^2\xi_3^2 + \xi_3^2\xi_1^2) + \mu'^2(3\lambda + 3\mu + \mu')\xi_1^2\xi_2^2\xi_3^2, \tag{3.35}$$

$$\begin{aligned} N_{11}(\xi) &= \mu^2\xi^4 + \beta\xi^2(\xi_2^2 + \xi_3^2) + \gamma\xi_2^2\xi_3^2, \\ N_{12}(\xi) &= -(\lambda + \mu)\xi_1\xi_2(\mu\xi^2 + \mu'\xi_3^2), \end{aligned} \tag{3.36}$$

and the other components are obtained by the cyclic permutation of 1, 2, 3, where

$$\begin{aligned} \xi^2 &= \xi_i\xi_i, \\ \beta &= \mu(\lambda + \mu + \mu'), \\ \gamma &= \mu'(2\lambda + 2\mu + \mu'), \\ \lambda &= C_{12}, \\ \mu &= C_{44}, \\ \mu' &= C_{11} - C_{12} - 2C_{44}. \end{aligned} \tag{3.37}$$

Hexagonal crystals (transversely isotropic)

$$\begin{aligned} D(\xi) &= (\alpha'\eta^2 + \gamma\xi_3^2)\{\alpha\gamma\eta^4 + (\alpha\beta + \gamma^2 - \gamma'^2)\eta^2\xi_3^2 + \beta\gamma\xi_3^4\} \\ &= (\alpha'\eta^2 + \gamma\xi_3^2)\{(\gamma\eta^2 + \beta\xi_3^2)(\alpha\eta^2 + \gamma\xi_3^2) - \gamma'^2\eta^2\xi_3^2\}, \end{aligned} \tag{3.38}$$

$$\begin{aligned} N_{11}(\xi) &= (\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2)(\gamma\eta^2 + \beta\xi_3^2) - \gamma'^2\xi_2^2\xi_3^2, \\ N_{12}(\xi) &= \gamma'^2\xi_1\xi_2\xi_3^2 - (\alpha - \alpha')\xi_1\xi_2(\gamma\eta^2 + \beta\xi_3^2), \\ N_{13}(\xi) &= (\alpha - \alpha')\gamma'\xi_1\xi_2^2\xi_3 - \gamma'\xi_1\xi_3(\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2), \\ N_{22}(\xi) &= (\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2)(\gamma\eta^2 + \beta\xi_3^2) - \gamma'^2\xi_1^2\xi_3^2, \\ N_{23}(\xi) &= (\alpha - \alpha')\gamma'\xi_1^2\xi_2\xi_3 - \gamma'\xi_2\xi_3(\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2), \\ N_{33}(\xi) &= (\alpha\xi_1^2 + \alpha'\xi_2^2 + \gamma\xi_3^2)(\alpha'\xi_1^2 + \alpha\xi_2^2 + \gamma\xi_3^2) - (\alpha - \alpha')^2\xi_1^2\xi_2^2, \end{aligned} \tag{3.39}$$

where

$$\begin{aligned}
&\alpha = C_{11} = C_{22}, \quad && \alpha' = C_{66} = \tfrac{1}{2}(C_{11} - C_{12}), \\
&\beta = C_{33}, && \gamma' - \gamma = C_{13} = C_{23}, \\
&\gamma = C_{44} = C_{55}, && \eta^2 = \xi_1^2 + \xi_2^2.
\end{aligned} \tag{3.40}$$

For isotropic materials, $\alpha = \beta = \lambda + 2\mu$, $\gamma' = \lambda + \mu$, and $\gamma = \alpha' = \mu$.

4. Exercises of general formulae

General formulae are given in Section 3 for the elastic fields associated with prescribed eigenstrains. This section provides exercises in the usage of these formulae: the best way to understand general statements is to work out specific examples.

A straight screw dislocation

Let us consider the case where Ω in Fig. 1.1 is the half plane ($x_2 = 0$, $x_1 < 0$) (see Fig. 4.1), and ϵ_{23}^* is prescribed on Ω. Other components of ϵ_{ij}^* are zero.

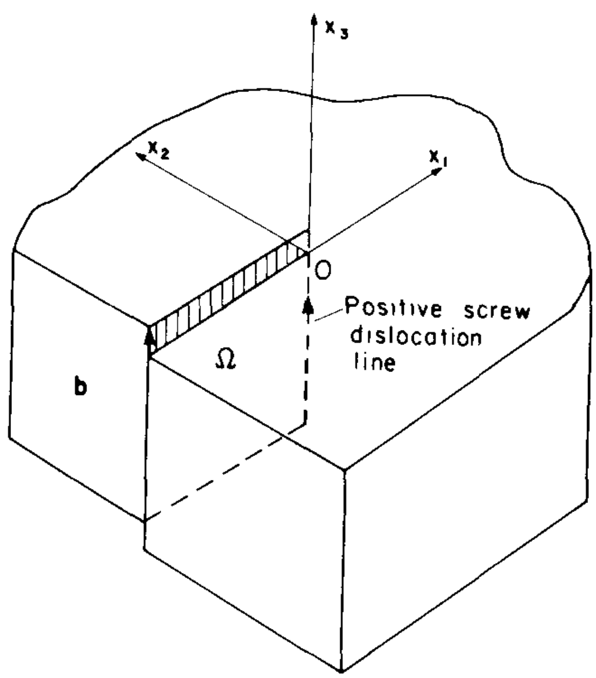

Fig 4 1 A screw dislocation