Data Structures & Algorithms for Problem Solving (CS1.304)

Sorting & Parallelism

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Introduction

- Sorting is a fundamental concept in Computer Science.
  - several applications and a lot of literature.
  - We shall see two algorithms for sorting today
  - Try to introduce parallelism in sorting

QuickSort

- The quick sort algorithm designed by Tony Hoare is a simple yet highly efficient algorithm.
- It works as follows:
  - Start with the given array A of n elements.
  - Consider a pivot, say A[n].
  - Now, partition the elements of A into two arrays A_L and A_R such that:
    - the elements in A_L are less than A[n]
    - the elements in A_R are greater than A[n].
  - Sort A_L and A_R, recursively.

QuickSort

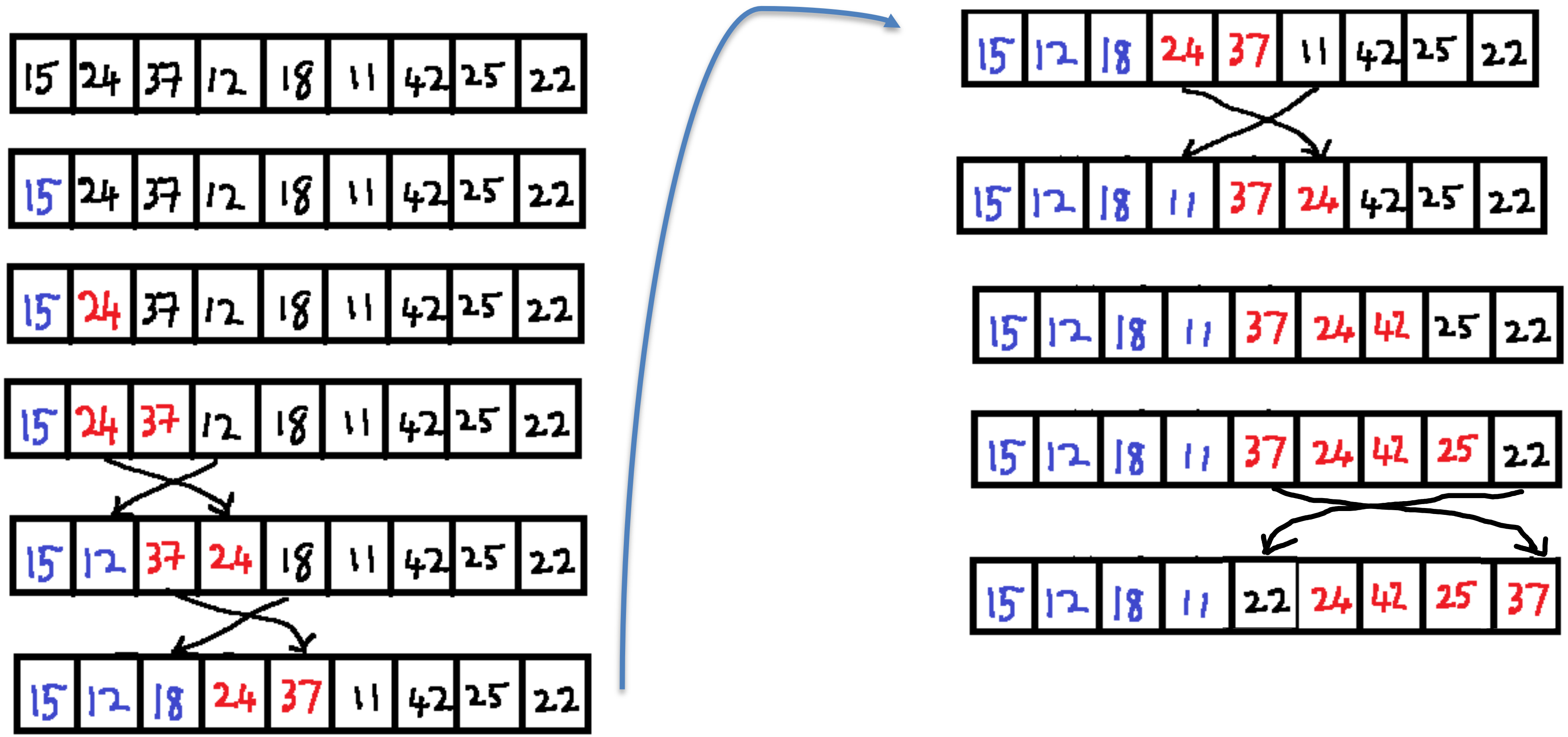
- How to partition ?
  - Suppose we take each element, compare it with A[n] and then move it to A_L or A_R accordingly.
  - Works in O(n) time.
  - Can write the program easily.
  - But, recall that space is also an resource. The above approach requires extra space for the arrays A_L and A_R
  - A better approach exists.

QuickSort

```
Procedure Partition(A,n)
begin
  pivot = A[n];
  less = 0; more = 1;
  for more = 1 to n-1 do
     if A[more] < pivot then
        less++;
        swap(A[more], A[less]);
     end
   end
   swap (A[less+1], A[n]);
end
```

15	24	37	12	18	11	42	25	22

QuickSort



15 24 37 12 18 11 42 25 22
15 24 37 12 18 11 42 25 22
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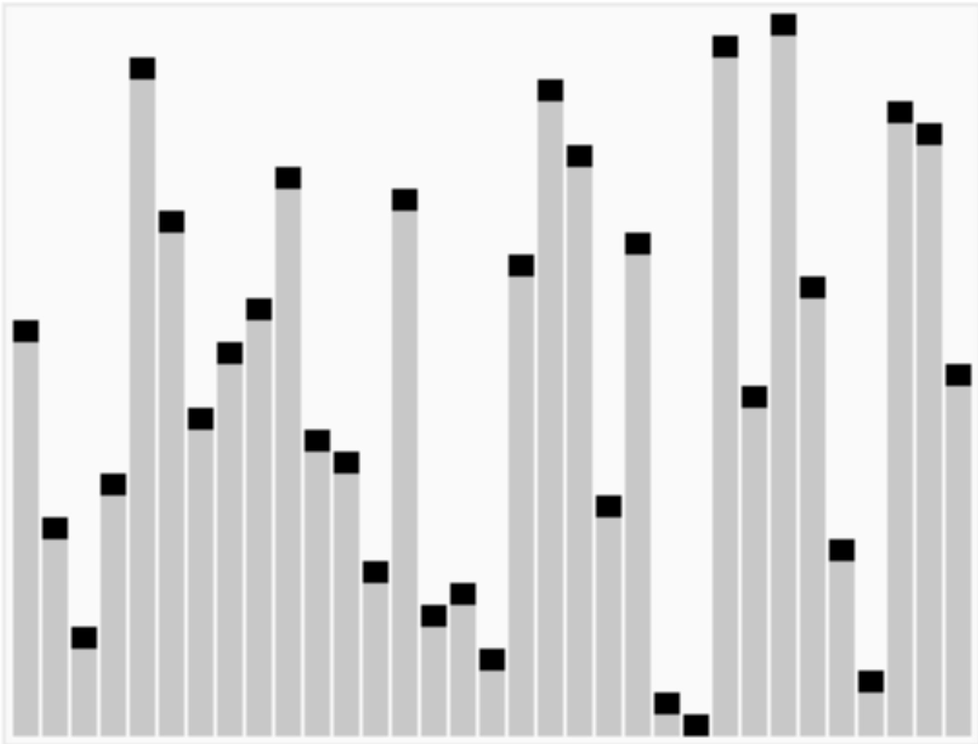
QuickSort

- Graphical Visualization of recursive partitioning,



Graphics Courtsey: Wikipidea

Analyzing Quick Sort

- Suppose we run quick sort with A[n] as the pivot.
- Let A_L and A_R be the two subarrays obtained after partitioning.
- What is the time taken by quicksort?
- As a recurrence relation, $T(n) = T(|A_L|) + T(|A_R|) + O(n)$.
- To be able to solve this recurrence relation, need to know the sizes of arrays A_L and A_R.

Analyzing Quick Sort

- We know that $|A_L| + |A_R| = n-1$.
- But, if the pivot is such that all elements are smaller (or larger) than the pivot, then $|A_L|$ (or $|A_R|$) $= n-1$.
- The recurrence relation in that case is

$$T(n) = T(n-1) + O(n).$$

- Suppose the same situation happens over every recursive call. So, the above recurrence relation holds during every recursive call.
- When will this happen ?

Analyzing Quick Sort

$$\underbrace{12 \quad 15 \quad 17 \quad 24 \quad 29 \quad 36}_{A_L} \mid 42$$

$$\underbrace{12 \quad 15 \quad 17 \quad 24 \quad 29}_{A_L} \mid 36$$

$$\underbrace{12 \quad 15 \quad 17 \quad 24}_{A_L} \mid 29$$

$$\underbrace{12 \quad 15 \quad 17}_{A_L} \mid 24$$

$$\underbrace{12 \quad 15}_{A_L} \mid 17$$

Analyzing Quick Sort

- Is it always that bad?
- What if the pivot is such that each recursive iteration, the sizes of $|A_L|$ and $|A_R|$ is exactly the same?
- The recurrence relation then stands as:

  $T(n) = 2T(n/2) + O(n).$

- Which element as the pivot ensures that the sizes of $|A_L|$ and $|A_R|$ are exactly the same?
- Can that happen in every run when you pick a pivot to be the last element? Or the first element? Or even uniformly at random?

Analyzing Quick Sort

- In general, if the sizes of $|A_L|$ and $|A_R|$ are such that they are a constant away from each other, then the recurrence relation is:

  $$T(n) = T(an) + T((1-a)n) + O(n)$$

  where a is a constant < 1.

- In practice, it turns out that most often the partitions are not too skewed.
- So, quick sort runs in $O(n \log n)$ time almost always.

Merge Sort

- Another sorting technique.
- Based on the divide and conquer principle.
- We will first explain the principle and then apply it to merge sort.

Divide and Conquer

- Divide the problem P into $k \geq 2$ sub-problems P_1, P_2, …, P_k.
- Solve the sub-problems P_1, P_2, …, P_k.
- Combine the solutions of the sub-problems to arrive at a solution to P.

Divide and Conquer

- A useful paradigm with several applications.
- Examples include merge sort, convex hull, median finding, matrix multiplication, and others.
- Typically, the sub-problems are solved recursively.
  - Recurrence relation

$$T(n) = \underbrace{D(n)}_{\text{Divide time}} + \underbrace{\sum_i T(n_i)}_{\text{Recursive cost}} + \underbrace{C(n)}_{\text{Combine time}}$$

Divide and Conquer

- Combination Procedure: Merge

$\longrightarrow$ 8 10 12 27 $\qquad\qquad$ $\longrightarrow$ 15 17 24 32

8

8 10

8 10 12

8 10 12 15

8 10 12 15 17

8 10 12 15 17 24

8 10 12 15 17 24 27

8 10 12 15 17 24 27 32

Algorithm Merge

```
Algorithm Merge(L, R)
// L and R are two sorted arrays of size n each.
// The output is written to an array A of size 2n.
int i=1, j=1;
L[n+1] = R[n+1] = MAXINT; // so that index does not
                          // fall over
for k = 1 to 2n do
        if L[i] < R[j] then
                A[k] = L[i]; i++;
        else
                A[k] = R[j]; j++;
end-for
```

Time complexity is O(n).

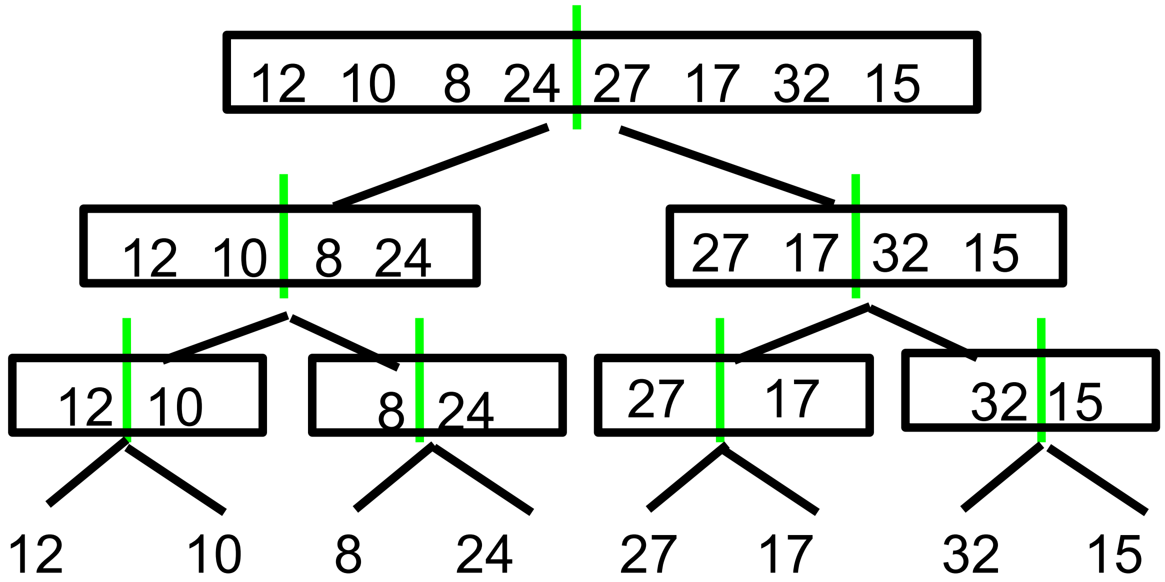
From Merging to Sorting

- How to use merging to finally sort?
- Using the divide and conquer principle
  - Divide the input array into two halves.
  - Sort each of them.
  - Merge the two sub-arrays. This is indeed procedure Merge.
- The algorithm can now be given as follows.

Algorithm Merge Sort

```
Algorithm MergeSort(A)
begin
        mid = n/2; //divide step
        L = MergeSort(A[1..mid]);
        R = MergeSort(A[mid+1..n]);
        Merge(L, R); //combine step
end-Algorithm
```

Divide & Conquer Merge Sort



- Example via merge sort: 1) Divide is split into two parts

  2)Recursively solve each subproblem

Analyzing Merge Sort

- Recurrence relation for merge sort as:

$$T(n) = 2T(n/2)+O(n).$$

  - This can be explained by the O(n) time for merge and
  - The two subproblems obtained during the divide step each take T(n/2) time.
  - Now use the general format for divide and conquer based algorithms.
- Solving this recurrence relation is done using say the substitution method giving us T(n) = O(n log n).
  - Look at previous examples.

Introducing Parallelism in Computing

- My laptop has 8 cores.
- What does it mean?
  - In principle, each core can run some instructions on its own independently.
  - So, while one core is possibly running the browser, the one can run an editor, the third can run a PDF reader, and the fourth can run a mail client, etc.
  - Is that helpful in all situations?

Introducing Parallelism in Computing

- There are two reasons why the previous model is not enough.
- There are applications that are time critical and can use ALL the cores for themselves.
  - Examples include weather forecasting, cyber-security, and the like.
- Secondly, while the number of cores is increasing, per core performance is dropping.

Introducing Parallelism in Computing

- So, how can MY program use ALL the cores simultaneously?
- Can the architecture help?
- Can the OS help?
- Can the compiler help?
- Can I help?
  - Very much. That is what parallel computing is all about.
- How can I help?
  - Algorithm Designer
  - Programmer
  - Computer Science student,
  - All of the above.

Maxima of n Numbers

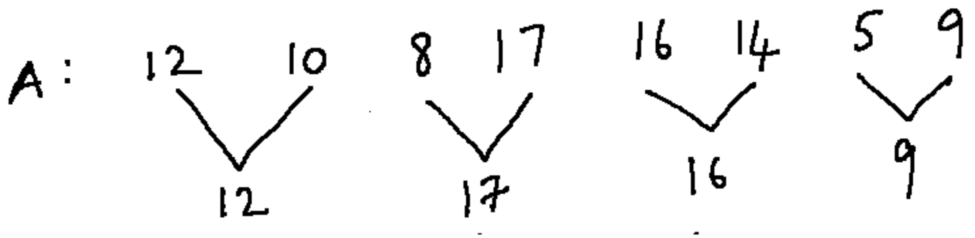
- A sequential program resembles the code below.

```
Program Maxima(A)
Begin
    int max = A[1];
    for i = 1 to n do
        if max < A[i]
            max = A[i]
        End-if
    End-for
End
```

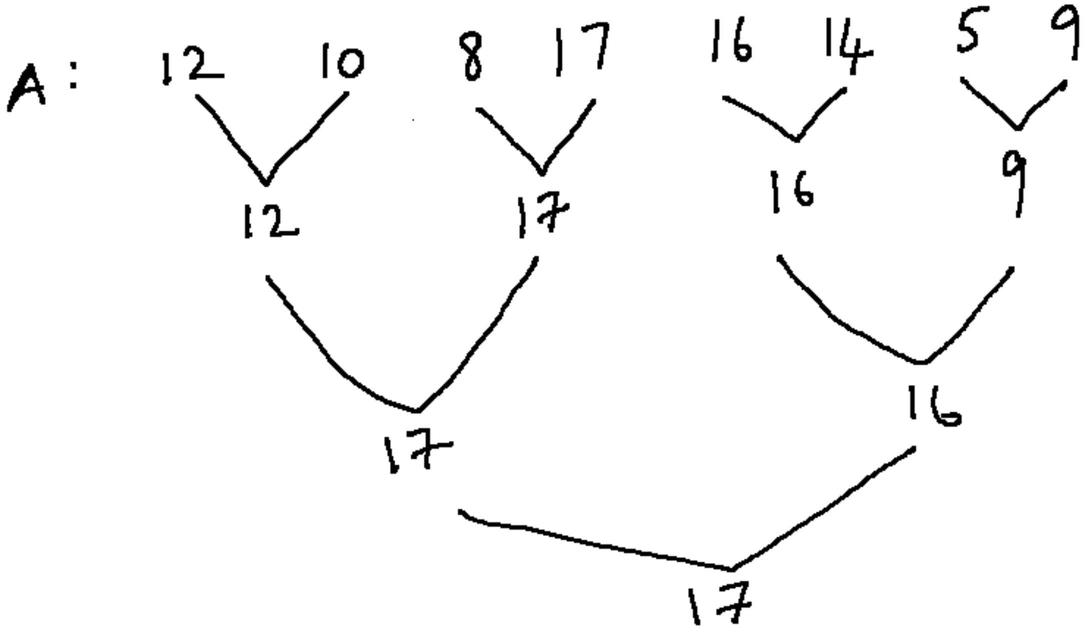
- How do we run this program in parallel?

Maxima of n Numbers

- Suppose we compare pairs of numbers and record the maximum.
- The overall maximum is one of these local maxima.
- So, we could compare pairs of local maxima, to get more local maxima.
- Continue until only one pair remains with the property that the maxima of the input is the maximum of the pair that remains.



Maxima of n Numbers



Maxima of n Numbers

- Can you write the above idea as a parallel algorithm?
- Some hints. Let n be the size of the array.
  - How many pairs are compared in the first iteration?
  - In the second iteration?
  - In the ith iteration?
  - How many iterations have to be executed to get the final maxima?
  - Which parts of the algorithm are parallel?
    - Within iterations or across iterations?

Parallelism in Computing

- Think of the sequential computer as a machine that executes jobs or instructions.
- With more than one processor, can execute more than one job (instruction) at the same time.
  - Cannot however execute instructions that are dependent on each other.
- This opens up a new world where computations have to specified in parallel.
- Sometimes have to rethink on known sequential approaches.

Parallelism in Computing

- How many independent operations can be done at a time?
  - Depends on the number of processors available.
- Assume that as many as n, or n/2, processors are available.
- Most often, our analysis suggests that the computation takes only O(log n) time, but we need n processors for this.
- Cannot ensure that the number of processors also grow with the input size.
- In practice, the number of processors on a machine does not change!

Parallelism in Computing

- The idea of the parallel algorithm is to show the extent of parallelism available in the computation.
- Plus, if there are fewer processors than what is required, can always simulate more processors.
- For instance, if there are p processors and n processors are required, then each of the p processors simulates the actions of n/p processors.

Parallel Merge Sort

- An algorithm is a sequence of tasks T1, T2, ...
- These tasks may have inter-dependecies,
  - Such as task Ti should be completed before task Tj for some i,j.
- However, it is often the case that there are several algorithms where many tasks are independent of each other.
  - In some cases, the algorithm or the computation has to be expressed in that independent-task fashion.

Parallel Merge Sort

- In such a setting, one can imagine that tasks that are independent of each other can be done simultaneously, or in parallel.
- Let us think of arriving at a parallel merge sort algorithm.

Parallel Merge Sort

- What are the independent tasks in merge sort?
  - Sorting the two parts of the array.
  - This further breaks down to sorting four parts of the array, etc.
  - Eventually, every element of the array is a sorted sub-array.
  - So the main work is in merge itself.

Parallel Merge Sort

- So, we just have to figure out a way to merge in parallel.

- Recall the merge algorithm as we developed it earlier.
  - Too many dependent tasks.
  - Not feasible in a parallel model.

```
for k = 1 to 2n do
        if L[i] < R[j] then
                A[k] = L[i]; i++;
        else A[k] = R[j]; j++;
end-for
```

Parallel Merge Sort

- Need to rethink on a parallel merge algorithm
- Start from the beginning.
  - We have two sorted arrays L and R.
  - Need to merge them into a single sorted array A.

Parallel Merge Sort

- Need to rethink on a parallel merge algorithm
- Start from the beginning.
  - We have two sorted arrays L and R.
  - Need to merge them into a single sorted array A.
- Define the rank of an element x in a sorted array A as the number of elements of A that are smaller than x.
- To merge L and R, need to know the rank of every element from L and R in the merged array L U R.

Parallel Merge Sort

- Importantly, for any x in L or R,

  Rank(x, L U R) = Rank(x, L) + Rank(x, R).

- So, merging is equivalent to finding the two ranks on the right hand side.

Parallel Merge Sort

- Now, consider an element x in L at index k.
- How many elements of L are smaller than x?
  - k-1.
- How many elements of R are smaller than x?
  - No easy answer, but
  - can do binary search for x in R and get the answer.
  - Say k' elements in R are smaller than x.

Parallel Merge Sort

- How many elements in LUR are smaller than x?
  - Precisely k + k‘ - 1.
- So, in the merged output, what index should x be placed in?
  - precisely at k+k'.
- Can this be done for every x in L?
  - Yes, it is an independent operation.
- Can this be done for every x in R also?
  - Yes, replace the roles of L and R.
- All these operations are independent.

Parallel Merge Sort

L = [8 10 12 27]

R = [15 17 24 32]

Element	8	10	12	27	15	17	24	32
Rank in L	0	1	2	3	3	3	3	4
Rank in R	0	0	0	3	0	1	2	3
Rank in L U R	0	1	2	6	3	4	5	7

L U R = [8 10 12 15 17 24 27 32]

Parallel Merge Sort

```
Algorithm ParallelMergeSort(A)
Begin
        mid = n/2; //divide step
        L = MergeSort(A[1..mid]);
        R = MergeSort(A[mid+1..n]);
        ParallelMerge(L, R); //combine step
  end-Algorithm
```

Thank You