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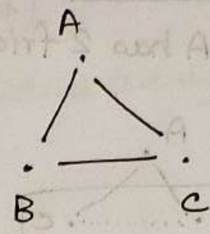
1.1

Case 1

Let's say, there are 3 persons A, B, C.



Now, Probability (A, B, C are mutual friend)

$= \text{Probability}$ (AB, BC, AC all are friends)

$= P(\text{A and B frnd}) \times P(\text{B and C are frnd}) \times P(\text{A, C are frnd})$

$= 1/2 \times 1/2 \times 1/2$

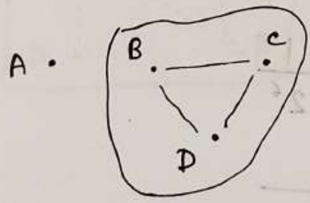
$= 1/8 = 1/2^3$

Case 2

Now, Let's say, there are 4 persons A, B, C, D.

Case 2.1

A has 0 friend. Probability (at least 3 mutual friend)



~~= Probability (B, C, D are mutual friend)~~

$= 3C_0 \times \underbrace{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}\times$ Probability (B, C, D are mutual friend)

Selecting 0 edges from {AB, AC, AD}

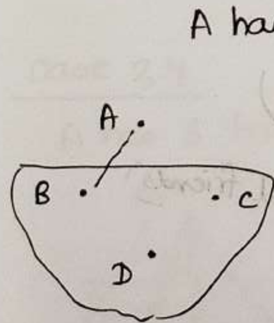
Probability of {AB, AC, AD} edge not present

$= 1 \times \frac{1}{2^3} \times \frac{1}{2^3} = \frac{1}{2^6}$

Case 2.2

A has 1 friend.

Probability (at least 3 mutual friend)



$= 3C_1 \times \underbrace{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}\times$ Probability (B, C, D are mutual friend)

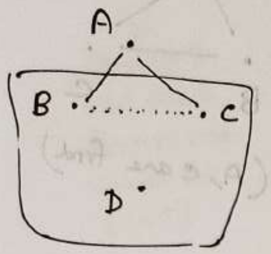
Selecting any 1 edge from {AB, AC, AD}

Probability of {AB present, AC not ", AD not "}

$= \frac{3}{2^3} \times \frac{1}{2^3} = \frac{3}{2^6}$

Case 2.3

A has 2 friend



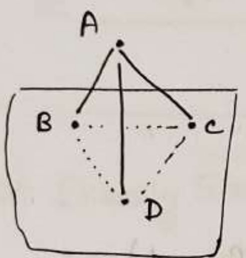
Probability (at least 3 mutual friend)

$$= {}^3C_2 \times \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times \text{Prob}(B, C \text{ are friend})$$

$$= \frac{3}{2^3} \times \frac{1}{2} = \frac{3}{2^4} = \frac{12}{2^6}$$

case 2.4

A has 3 friend



Probability (at least 3 mutual friend)

$$= {}^3C_3 \times \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times \text{Prob}(\text{either } B, C \text{ or } C, D \text{ or } B, D \text{ are mutual friend})$$

$$= \frac{1}{2^3} \times \left(1 - \text{Prob}(\text{none of } B, C, D \text{ are mutual friend})\right)$$

$$= \frac{1}{2^3} \times \left(1 - \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)$$

$$= \frac{1}{2^3} \times \left(1 - \frac{1}{2^3}\right) = \frac{1}{2^3} - \frac{1}{2^6}$$

$$= \frac{7}{2^6}$$

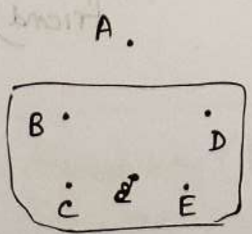
$\therefore$ For four persons, Prob(at least 3 mutual friend)

$$= \frac{1}{2^6} + \frac{3}{2^6} + \frac{12}{2^6} + \frac{7}{2^6} = \frac{23}{2^6}$$

Case 3 (The final answer)

case 3.1

A has 0 friend



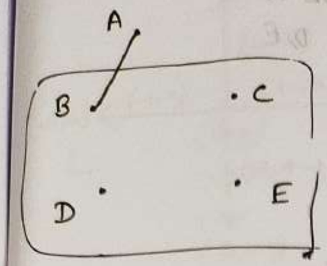
Probability (at least 3 mutual friend)

$$= {}^4C_0 \times \frac{1}{2^4} \times \text{Prob}(B, C, D, E \text{ are mutual friends})$$

$$= \frac{1}{2^4} \times \frac{23}{2^6} = \frac{23}{2^{10}} \quad \text{(1)}$$

case 3.2

A has 1 friend



Probability (at least 3 mutual friend)

$$= {}^4C_1 \times \frac{1}{2^4} \times \text{Prob}(B, C, D, E \text{ are mutual friends})$$

$\underbrace{{}^4C_1}_{\text{choose any edge from } \{AB, AC, AD, AE\}}$, $\underbrace{\frac{1}{2^4}}_{\text{Prob of } \{AB \text{ present, } AC \text{ not, } AD \text{ not, } AE \text{ not}\}}$

$$= \frac{4}{2^4} \times \frac{23}{2^6} = \frac{92}{2^{10}}$$

(2)

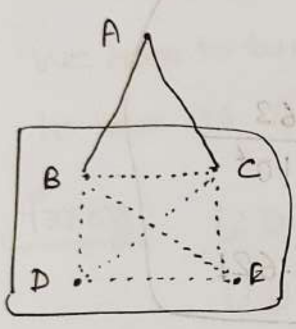
$\frac{1}{2^4} = \frac{1}{2^{10}}$

$= \frac{2^6 - 1}{2^{10}}$

$=$

case 3.3

A has 2 friends



Probability (at least 3 mutal friend)

$$= {}^4C_2 \times \frac{1}{2^4} \times \Big[\text{Prob}(B, C \text{ mutual friend}) + \underbrace{\tfrac{1}{2}}_{(B, C \text{ not frnd})}\text{Prob}(B, D, E \text{ or } C, D, E \text{ mutual frnd})\Big]$$

$$= \frac{6}{2^4}\left[\frac{1}{2} + \frac{1}{2}P(\Delta BDE \cup \Delta CDE)\right]$$

$$= \frac{6}{2^4}\left[\frac{1}{2} + \frac{1}{2}P(\Delta BDE) + \frac{1}{2}P(\Delta CDE) - \frac{1}{2}P(\Delta BDE \cap \Delta CDE)\right]$$

$$= \frac{6}{2^4}\left[\frac{1}{2} + \frac{1}{2}\frac{1}{2 \times 2^3} + \frac{1}{2}\frac{1}{2 \times 2^3} - \frac{1}{2}\frac{1}{2 \times 2^5}\right]$$

$$= \frac{6}{2^5} + \frac{6}{2^8} + \frac{6}{2^8} - \frac{6}{2^{10}} = \frac{6 \times 2^5 + 6 \times 2^2 + 6 \times 2^2 - 6}{2^{10}}$$

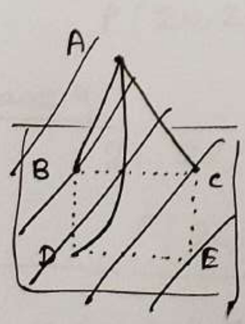
$$= \frac{6 \times 2^5 + 6 \times 2^3 - 6}{2^{10}}$$

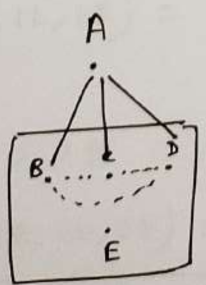
$$= \frac{234}{2^{10}}$$

(3)

case 3.4

A has 3 friends





Probability (at least 3 mutual friends)

$$= {}^4C_3 \times \frac{1}{2^4} \times \text{Prob}(B, C \text{ or } C, D \text{ or } B, D \text{ are mutual friends})$$

$\underbrace{{}^4C_3}_{\text{choose 3 edges from } \{AB, AC, AD, AE\}}$

$$= \frac{4}{2^4} \times \left[1 - \frac{1}{2^3}\right]$$

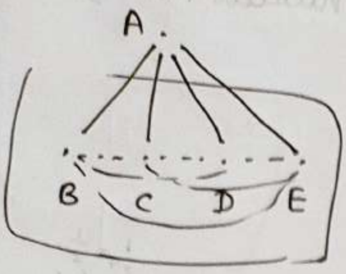
$$= \frac{4}{2^4} - \frac{4}{2^7} = \frac{8 \times 4 - 4}{2^7} = \frac{28}{2^7} = \frac{224}{2^{10}}$$

(4)

$P \times 28 = 224$

Case 3.5

A has 4 friends. Probability (at least 3 mutual friends)



$$= {}^4C_4 \times \frac{1}{2^4} \times \text{Prob}\left(\begin{matrix} B,C \text{ or } B,D \text{ or } B,E \text{ or} \\ C,D \text{ or } C,E \text{ or } D,E \\ \text{m.f} \end{matrix}\right).$$

$$= \frac{1}{2^4} \times \left(1 - \frac{1}{2^6}\right)$$

$$= \frac{1}{2^4} - \frac{1}{2^{10}} = \frac{2^6-1}{2^{10}} = \frac{63}{2^{10}}$$ (5)

∴ Probability (at least 3 mutual friends among 5 persons)

$$= (1) + (2) + (3) + (4) + (5)$$

$$= \frac{23}{2^{10}} + \frac{92}{2^{10}} + \frac{234}{2^{10}} + \frac{224}{2^{10}} + \frac{63}{2^{10}}$$

$$= \frac{636}{2^{10}} = \frac{636}{1024}$$ (Ans) $= 0.621$

Sudipta Halder

Roll: 2021202011

1.2 10 Red cards, 10 blue cards.

Shuffled and numbered from 1 to 20.

$\therefore$ Sample space $= 20!$

ⓐ All red cards are assigned numbers ≤ 15

10R, 5B | 15 | 5B | 20 upto

We have to place all 10 red cards ~~before~~ upto 15.

$\therefore$ we need to place $(15-10) = 5$ blue cards ~~before~~ upto 15

$\therefore$ Favourable events

$$= \left(\underbrace{{}^{10}C_{10}}_{\text{Choosing 10 red cards from 10 red cards for [1,15]}} \times \underbrace{{}^{10}C_5}_{\text{choose 5 blue cards from 10 blue cards for [1,15]}} \times \underbrace{15!}_{\text{Arrange first 15 cards [1,15]}} \times \underbrace{{}^5C_5}_{\text{Choosing 5 blue cards from 5 blue cards for [16,20]}} \times \underbrace{5!}_{\text{Arranging Last 5 cards [16,20]}} \right)$$

$\therefore$ Probability (All red cards are assigned numbers ≤ 15)

$$= \frac{\text{Favourable events}}{\text{Sample space}} = \frac{{}^{10}C_{10} \times {}^{10}C_5 \times 15! \times {}^5C_5 \times 5!}{20!} = \frac{21}{1292} = 0.0162 \text{ (Ans)}$$

ⓑ Exactly 8 red cards are assigned numbers ≤ 15

8R, 7B | 15 | 2R, 3B | 20

We have to place 8 red cards ~~b~~ upto 15

$\therefore$ We need to place $(15-8) = 7$ blue cards upto 15.

As a result, we need to place 2 red cards and 3 blue cards in ~~from~~ [16, 20]

∴ Favourable events

$$= \left({}^{10}C_8 \times {}^{10}C_7 \times 15! \times {}^{2}C_2 \times {}^{3}C_3 \times 5! \right)$$

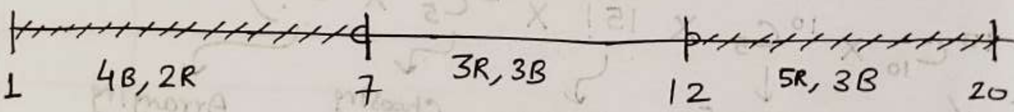
- Choosing 8 red cards from 10 red cards for [1,15]
- Choose 7 ~~red~~ blue cards from 10 blue cards for [1,15]
- Arranging first 15 cards [1,15]
- Choosing 2 red cards from 2 red cards for [16,20]
- Choosing 3 blue cards from 3 blue cards for [16,20]
- Arranging last 5 cards [16,20]

∴ Probability (Exactly 8 red cards ≤ 15)

$$= \frac{{}^{10}C_8 \times {}^{10}C_7 \times 15! \times {}^{2}C_2 \times {}^{3}C_3 \times 5!}{20!} = \frac{225}{646} \text{ (Ans)}$$

$$= 0.3482$$

(c) Exactly 5 red cards are assigned numbers > 12 and exactly 4 blue cards " " " < 7



We have place 4 blue cards and 2 red cards in [1,6]

" " " 3 blue cards and 3 red " " [7,12]

" " " 3 blue " and 5 red " " [13,20]

∴ Favourable events

$$= \left({}^{10}C_2 \times {}^{10}C_4 \times 6! \times {}^{8}C_5 \times {}^{6}C_3 \times 8! \times {}^{3}C_3 \times {}^{3}C_3 \times 6! \right)$$

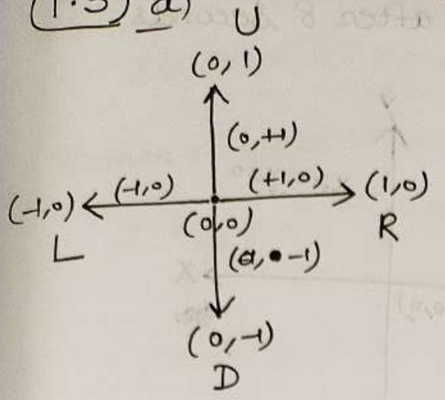
- Choose 2 red from 10 red for [1,6]
- Chose 4 blue from 10 blue for [1,6]
- Arranging first 6 cards [1,6]
- Choose 5 red cards from 8 red cards for [13,20]
- Choose 3 blue cards from 6 blue ca for [13,20]
- Arranging cards in [13,20]
- Arranging cards in [7,12]

∴ Probability (Exactly 5 red cards > 12 and Exactly 4 blue cards < 7)

$$= \frac{{}^{10}C_2 \times {}^{10}C_4 \times 6! \times {}^{8}C_5 \times {}^{6}C_3 \times 8! \times {}^{3}C_3 \times {}^{3}C_3 \times 6!}{20!}$$

$$= \frac{4200}{46189} \text{ (Ans)} = 0.09093$$

(1.3) a)



Four moves are possible

(0,1) (1,0) (-1,0) (0,-1)
U R L D

In 6 sec, we can take 6 moves and for each move we have 4 choices

$\therefore$ Sample Space $= 4^6$

$\begin{Bmatrix} U \leftarrow D \\ L \leftarrow R \end{Bmatrix}$

~~In order to take 6 steps and what~~

In order to get back to origin after 6 seconds, we have to take 6 steps and whatever step we take, we have to take its opposite step same number of times.

for eg   U D U D L R

U U U D D D

case 1

3U, 3D

$$P(3U,3D) = \frac{\frac{6!}{3!\times3!}}{4^6} = \frac{20}{4^6}$$

case 2

3L, 3R

$$P(3L,3R) = \frac{\frac{6!}{3!\times3!}}{4^6} = \frac{20}{4^6}$$

case 3

2U, 2D, 1L, 1R

$$P(2U,2D,1L,1R) = \frac{\frac{6!}{2!\times2!}}{4^6} = \frac{180}{4^6}$$

case 4

2L, 2R, 1U, 1D

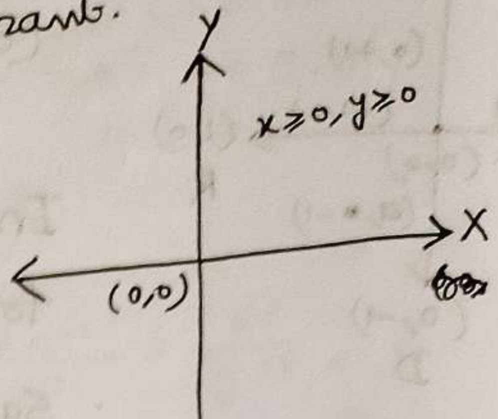
$$P(2L,2R,1U,1D) = \frac{\frac{6!}{2!\times2!}}{4^6} = \frac{180}{4^6}$$

$$\therefore P(\text{at } t=6\ x=0, y=0) = P(3U,3D) + P(3L,3R) + P(2U,2D,1L,1R) + P(2L,2R,1U,1D)$$

$$= \frac{20}{4^6} + \frac{20}{4^6} + \frac{180}{4^6} + \frac{180}{4^6} = \frac{400}{4^6}$$

$$= 0.0976 = \frac{400}{4096} \text{ (Ans)}$$

(1.3 b) The particle comes back to the origin after 8 seconds by remaining only in first quadrant.



Here also, there are 4 moves possible.

(0,1), (0,−1), (1,0), (−1,0)
U D R L.

To move back in origin after 8 seconds, we have to take 8 steps and whatever step we take, we have to take opposite step same number of times.

But, here the constraint is → The particle ~~can't~~ can remain only in 1st quadrant.

We will take the help of catalan numbers to solve this problem

Total Sample Space $= 4^8$ | Catalan number$(C_n) = \frac{1}{n+1}\left({}^{2n}C_n\right)$

4 65536

Here, C_n = String of length 2n consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.

$$= \frac{1}{n+1}\binom{2n}{n}$$

Case 1:

4U, 4D _ _ _ _ _ _ _ _

Number of ways in which particle comes back to $(x=0, y=0)$ at $t=8$ being in $x \geq 0, y \geq 0$

$$= \underbrace{{}^{8}C_{8}}_{\text{Choosing(4U,4D) total 8 positions from 8 positions}} \times \underbrace{C_4}_{\text{String of length 8 consisting of 4U and 4D such that no initial segment of the string has more D's than U's}} = \frac{1}{5}\left({}^{8}C_{4}\right) \text{ ——— } (1)$$

Case 2:

4L, 4R _ _ _ _ _ _ _ _

$$\text{Number of ways} = \underbrace{{}^{8}C_{8}}_{\text{Choosing 4L,4R Total 8 positions from 8 positions}} \times \underbrace{C_4}_{\text{String of length 8 consisting of 4L and 4R such that no initial segment of the string has more L's than R's}} = \frac{1}{5}\left({}^{8}C_{4}\right) \text{ ——— } (2)$$

Case 3:

3U, 3D, 1L, 1R — — — — — — — —

Number of ways = $^8C_6 \times C_3 \times {}^2C_2 \times C_1$

Choosing 6 positions from 8 positions for (3U, 3D)

String of length 6 consisting of 3U and 3D such that no initial segment of the string has more D's than U's

Choosing 2 positions from 2 pos. for (1L, 1R)

String of length 2 consisting of 1L and 1R s.t. no initial segment of the string has more L's than R's

$= {}^8C_6 \times \frac{1}{4}({}^6C_3) \times 1 \times \frac{1}{2}({}^2C_1)$

$= {}^8C_6 \times {}^6C_3 \times \frac{1}{4}$ ——— (3)

Case 4:

3L, 3R, 1U, 1D — — — — — — — —

Number of ways = $^8C_6 \times C_3 \times {}^2C_2 \times C_1$

Choosing 6 positions from 8 positions for (3L, 3R)

String of length 6 consisting of 3L and 3R such that no initial segment of the string has more L's than R's

Choosing 2 positions from 2 pos for (1U, 1D)

String of length 2 consisting of 1U, 1D s.t. no initial segment of the string has more D's than U's

$= {}^8C_6 \times \frac{1}{4}({}^6C_3) \times 1 \times \frac{1}{2}({}^2C_1)$

$= {}^8C_6 \times {}^6C_3 \times \frac{1}{4}$ ——— (4)

Case 5:

2U, 2D, 2L, 2R - - - - - - - - - -

Number of ways $= {}^8C_4 \times C_2 \times {}^4C_4 \times C_2$

Choosing 4 positions from 8 positions for (2U, 2D)

String of length 4 consisting of 2U and 2D such that no initial segment of string has more D's than U's

Choosing 4 positions from 4 positions for (2L, 2R)

String of length 4 consisting of 2L and 2R such that no initial segment of string has more L's than R's

$$= {}^8C_4 \times \frac{1}{3}({}^4C_2) \times 1 \times \frac{1}{3}({}^4C_2)$$

$$= {}^8C_4 \times {}^4C_2 \times {}^4C_2 \times 1/9 \quad \text{——— } (5)$$

$\therefore$ Total Number of favourable cases =

$(1) + (2) + (3) + (4) + (5)$

$$= \frac{1}{5} \times ({}^8C_4) + \frac{1}{5} \times ({}^8C_4) + \frac{1}{4} \times {}^8C_6 \times {}^6C_3 + \frac{1}{4} \times {}^8C_6 \times {}^6C_3 + {}^8C_4 \times {}^4C_2 \times {}^4C_2 \times \frac{1}{9}$$

$= 14 + 14 + 140 + 140 + 280$

$= 588$

$\therefore$ Probability (Number of ways in which particle comes back to $(x=0, y=0)$ at $t=8$ remaining in $x \geq 0, y \geq 0$)

$$= \frac{588}{4^8}$$

$$= \frac{588}{65536} = 0.0089 \text{ (Ans)}$$