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(3.1) We have to show that, for a discrete non-negative random variable: $E(X) = \sum_{x \geq 0} Pr(X > x)$

We know that,

$$Pr(X>x) = Pr(X=x+1) + Pr(X=x+2) + Pr(X=x+3) + \cdots \infty \quad \text{——— (1)}$$

Also, we know that,

$$E(X) = \sum_{\substack{\text{all possible} \\ x}} x \, Pr(X=x)$$

here $x \geq 0$ $\therefore E(X) = \sum_{x \geq 0} x \, Pr(X=x)$ ——— (2)

Now,

putting $x=0$ in eq (1): $Pr(X>0) = Pr(X=1) + Pr(X=2) + Pr(X=3) + \cdots$

putting $x=1$ in eq (1): $Pr(X>1) = \qquad\qquad Pr(X=2) + Pr(X=3) + \cdots$

putting $x=2$ in eq (1): $Pr(X>2) = \qquad\qquad\qquad\qquad Pr(X=3) + \cdots$

$\vdots$

putting $x=K$ in eq (1): $Pr(X>K) = \qquad\qquad\qquad\qquad \cdots Pr(X=K+1) + \cdots$

$\vdots$

$+ \qquad\qquad\qquad +$

$$Pr(X>0) + Pr(X>1) + Pr(X>2) + \cdots = 1 \cdot Pr(X=1) + 2 \cdot Pr(X=2) + 3 \cdot Pr(X=3) + \cdots + K \cdot Pr(X=K) + (K+1) Pr(X=K+1) + \cdots$$

$$\Rightarrow \sum_{x \geq 0} Pr(X>x) = \sum_{x \geq 1} x \cdot Pr(X=x) + 0 \cdot Pr(X=0)$$

$$= \sum_{x \geq 0} x \cdot Pr(X=x) = E(X) \quad \text{[from (2)]}$$

$$\therefore \boxed{E(X) = \sum_{x \geq 0} Pr(X>x)} \quad \text{(proved)}$$

(3.2)

Given a permutation π of $\{1,2,3,\ldots,n\}$, $i \in \{1,2,3,\ldots,n\}$ is said to be a fixed point of π if $\pi(i)=i$. Let σ be a random permutation of $\{1,2,3,\ldots,n\}$

X is a random variable corresponding to the number of fixed points in σ.

(a) Let's consider X_i a random variable such that

$$X_i = \begin{cases} 1 & \text{if } \pi(i)=i \text{ (ith point is fixed)} \\ 0 & \text{if } \pi(i)\neq i \text{ (ith point is not fixed)} \end{cases}$$

$i \in \{1,2,3,\ldots,n\}$

point i is fixed so, the number of ways rest $(n-1)$ points can be arranged.

Then, Probability of $X_i = P(X_i) = \begin{cases} \dfrac{(n-1)!}{n!} = \dfrac{1}{n} & ; \text{ if } X_i = 1 \\ 1-\dfrac{1}{n} & ; \text{ if } X_i = 0 \end{cases}$

The number of ways all n points can be arranged.

$\therefore$ Expectation of $X_i = E(X_i) = \sum X_i P(X_i)$

$$= 1\cdot\frac{1}{n} + 0\left(1-\frac{1}{n}\right) = \frac{1}{n}.$$

Now, we can definitely say that

$$X = X_1 + X_2 + X_3 + X_4 + \cdots + X_n.$$

nth point fixed

Hence, from linearity of expectation we can say that,

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

$$= n \cdot \frac{1}{n} = 1$$

total no of points

$\left[\because E(X_i) = \frac{1}{n}\right]$

$\therefore \boxed{E(X) = 1}$ (Ans)

(b) We need to find PMF of x.

Let's first look at the formula for derangement.

A permutation of the elements of a set where no element remains at the same place is called derangement. It is denoted by D_n.

The number of derangements of a set with n objects is given by the formula
$$\boxed{D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}} \quad \text{--- (1)}$$

Now, Probability that no. of fixed points in σ is K

$= P(X = K)$

$$= \frac{\underbrace{{}^nC_K}_{\substack{\text{Choose K points} \\ \text{among n points} \\ \text{which will be} \\ \text{considerd as fixed} \\ \text{points}}} \cdot \underbrace{(1)}_{\substack{\text{Those K fixed} \\ \text{points have only} \\ \text{1 option to} \\ \text{arrange themselves}}} \cdot (\text{Derangement of } (n-K) \text{ points})}{\underbrace{n!}_{\substack{\text{all possible arrangement} \\ \text{of n points.}}}}$$

Rest $(n-K)$ points should be arranged in such a way that, no one should be at their correct pos. i.e. $\pi(i) \neq i$ to maintain the fact that exactly K points are fixed

$$= \frac{\frac{n!}{K!(n-K)!}(D_{n-K})}{n!}$$

$$= \frac{\frac{\cancel{n!}}{K!\cancel{(n-K)!}}\left[\cancel{(n-K)!}\sum_{i=0}^{n-K}\frac{(-1)^i}{i!}\right]}{\cancel{n!}} \quad \left[\text{putting } n = n-K \text{ in eq (1)}\right]$$

$$= \frac{1}{K!}\sum_{i=0}^{n-K}(-1)^i \frac{1}{i!}$$

$$\boxed{\therefore \text{PMF}(X) = P(X = K) = \frac{1}{K!}\sum_{i=0}^{n-K}\frac{(-1)^i}{i!}} \quad (\text{Ans})$$

(c) We have to find expected number of swaps in a uniformly random permutation σ.

Let's define a random variable Y_{ij} in such a way that

$$Y_{ij} = \begin{cases} 1 & \text{if } \pi(i)=j \text{ and } \pi(j)=i \text{ i.e. } i \text{ is in } j\text{th position and } j \text{ is in } i\text{th position.} \\ 0 & \text{otherwise} \end{cases}$$

i, j these two points are in place j, i. Rest (n-2) points can be arranged (n-2)! ways

$$\therefore P(Y_{ij}) = \begin{cases} \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)} & ; \text{if } Y_{ij}=1 \\ 1-\dfrac{1}{n(n-1)} & ; \text{if } Y_{ij}=0 \end{cases}$$

all possible arrangement of n points

$$\therefore E(Y_{ij}) = 1\cdot\frac{1}{n(n-1)} + 0\cdot\left(1-\frac{1}{n(n-1)}\right)$$
$$= \frac{1}{n(n-1)}$$

Now, we have to find out how many Y_{ij} are possible.

Since, here ordering does not matter i.e swap of (1, 2) and (2, 1) are same thing.

$\therefore$ It will be equal to number of ways we can choose 2 points from n points. $= {}^{n}C_2 = \dfrac{n(n-1)}{2}$

Now, let Y be a random variable which denotes expected no. of swaps.

Hence we can say that $Y = \sum\limits_{\substack{i<j \\ 1\le i,j\le n}} Y_{ij}$

$\therefore$ From linearity of expectation, we can say,

$$E(Y) = E\left(\sum_{\substack{i<j \\ 1\le i,j\le n}} Y_{ij}\right)$$

$$= \frac{n(n-1)}{2} \cdot E(Y_{ij})$$

total no. of such ~~points~~ (i,j) pairs.

$$= \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)}$$

$$= \frac{1}{2}$$

$$\boxed{\therefore E(Y) = \frac{1}{2}}$$ (Ans)

(d) We have to show that $P(X > 10) \le \frac{1}{10}$

We have already proved that, $E(X) = 1$ [where X is a r.v. corresponding to the no of fixed points in σ]

From Markov's inequality, we know that.

$$P(X \geq K) \le \frac{E(X)}{K}$$

$$\Rightarrow P(X \geq 10) \le \frac{1}{10} \quad \left[\text{putting } K = 10\right]$$

Now, $n(X > 10) \subset n(X \geq 10)$

We know that if A is a subset of B i.e. $A \subset B$

Then $P(A) \le P(B)$.

$$\therefore P(X > 10) \le P(X \geq 10)$$

$$\therefore \boxed{P(X > 10) \le \frac{1}{10}}$$ (proved)

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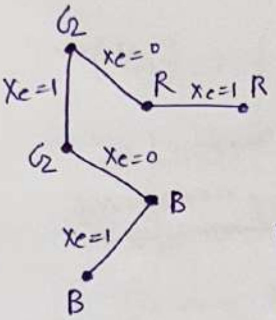
① For any edge $e \in E$. Let X_e be random variable which is 1 when e is monochromatic and 0 otherwise.

We have to show that set of random variable $\{X_e\}_{e \in E}$ are pairwise independent. Show that they are not independent.



$$X_e = \begin{cases} 1 & \text{if } e \text{ is monochromatic} \\ 0 & \text{if } e \text{ is non-chromatic} \end{cases}$$

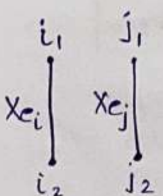
Let's define a sample space which denotes the two endpoints colour of any edge

$$S = \{ \underbrace{RR, BB, GG}_{\substack{X_e = 1 \\ \text{monochromatic}}}, \underbrace{RB, BR, GB, BG, RG, GR}_{\substack{X_e = 0 \\ \text{non-monochromatic}}} \}$$

$$\therefore P(X_e) = \begin{cases} 3/9 = 1/3 & \text{if } X_e = 1 \quad \text{—— ①} \\ 6/9 = 2/3 & \text{if } X_e = 0 \quad \text{—— ②} \end{cases}$$

$\boxed{\text{Proving pairwise independence}}$

$\boxed{\text{case 1}}$ (Two edges are not connected)



We need to show, $P(X_{e_i} = 1, X_{e_j} = 1) = P(X_{e_i} = 1)\, P(X_{e_j} = 1)$

LHS

$$P(X_{e_i} = 1, X_{e_j} = 1) = \frac{\text{no of ways s.t. } (\text{colour}(i_1) = \text{colour}(i_2)) * \text{no of ways s.t. } (\text{colour}(j_1) = \text{colour}(j_2))}{\text{no of all possible colourings of } \{i_1, i_2, j_1, j_2\}}$$

$$= \frac{3 \times 3}{3^4}$$

$$= \frac{1}{3^2} = \frac{1}{9}$$

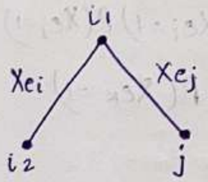
<u>RHS</u> $P(X_{e_i}=1) = \frac{1}{3}$ (from ①)

$P(X_{e_j}=1) = \frac{1}{3}$ (from ②)

$\therefore P(X_{e_i}=1) \cdot P(X_{e_j}=1) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$\therefore \boxed{P(X_{e_i}=1, X_{e_j}=1) = P(X_{e_i}=1) \cdot P(X_{e_j}=1)}$ (proved) (case 1)

Case-2 (Two edges are connected)



<u>We need to show</u> $P(X_{e_i}=1, X_{e_j}=1) = P(X_{e_i}=1)\, P(X_{e_j}=1)$

<u>LHS</u>

Let's consider X_{e_i} mono-chromatic and

let colour (i_1) = colour (i_2) = R

<u>Hence</u> if X_{e_j} has to be mono-chromatic then colour (j) has to R

$\therefore$ colour (i_1) = colour (i_2) = colour (j) = $\{R, G, B\}$

$$\therefore P(X_{e_i}=1, X_{e_j}=1) = \frac{\text{no of ways } (\text{colour}(i_1) = \text{colour}(i_2) = \text{colour}(j))}{\text{no of ways of colouring of } (i_1, i_2, j)}$$

$$= \frac{3}{3^3} = \frac{1}{3^2} = 1/9$$

<u>RHS</u> $P(X_{e_i}=1) = \frac{1}{3}$ $\therefore P(X_{e_i}=1) \cdot P(X_{e_j}=1) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

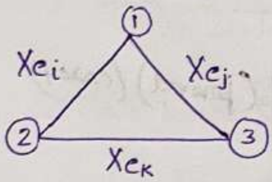
$P(X_{e_j}=1) = 1/3$

$\therefore P(X_{e_i}=1, X_{e_j}=1) = P(X_{e_i}=1) \cdot P(X_{e_j}=1)$

$\therefore$ Pairwise independence is proved

Proving non-independence

Consider cycle C_3



<u>To ~~d~~ prove non-independence</u>

We need to prove at least one of these.

i) $P(X_{ei}=1, X_{ej}=1) \neq P(X_{ei}=1)\,P(X_{ej}=1)$

ii) $P(X_{ei}=1, X_{ek}=1) \neq P(X_{ei}=1)\,P(X_{ek}=1)$

iii) $P(X_{ej}=1, X_{ek}=1) \neq P(X_{ej}=1)\,P(X_{ek}=1)$

iv) $P(X_{ei}=1, X_{ej}=1, X_{ek}=1) \neq P(X_{ei}=1)\,P(X_{ej}=1)\,P(X_{ek}=1)$

<u>Let's try to prove iv)</u>

~~$P(X_{ei}=1, X_{ej}=1, X_{ek}=1)$~~

Let's consider X_{ei} mono-chromatic. Hence colour(1) = colour(2) = R (suppose)

So, ~~to~~ if X_{ej} and X_{ek} wants to be mono-chromatic

then colour(1) = colour(2) = colour(3) = {B, G, R}

<u>LHS</u>

$$\therefore P(X_{ei}=1, X_{ej}=1, X_{ek}=1) = \frac{\text{no of ways (colour(1) = colour(2) = colour(3))}}{\text{no of ways of colouring } \{1,2,3\}}$$

$$= \frac{3}{3^3} = \frac{1}{3^2} = \frac{1}{9}$$

<u>RHS</u>

$P(X_{ei}=1) = \frac{1}{3}$

$P(X_{ej}=1) = \frac{1}{3}$ $\quad \therefore P(X_{ei}=1)\cdot P(X_{ej}=1)\,P(X_{ek}=1) = \frac{1}{27}$

$P(X_{ek}=1) = \frac{1}{3}$

$$\boxed{\therefore P(X_{ei}=1, X_{ej}=1, X_{ek}=1) \neq P(X_{ei}=1)\,P(X_{ej}=1)\,P(X_{ek}=1)}$$

(proved)

$\therefore$ Non-independence proved

② Y is the random variable corresponding to the number of non-monochromatic edges.

Let's define random variable Y_i such that

$$Y_i = \begin{cases} 1 & \text{if the edge is non-monochromatic} \\ 0 & \text{if the edg is monochromatic.} \end{cases}$$

$$\therefore P(Y_i) = \begin{cases} 6/9 = 2/3 & \text{if } Y_i = 1 \\ 3/9 = 1/3 & \text{if } Y_i = 0. \end{cases}$$

$$\left\{ \underbrace{RR, BB, GG}_{Y_i = 0}, \underbrace{RB, BR, RG, GR, BG, GB}_{Y_i = 1} \right\}$$

$$\therefore E(Y_i) = \sum Y_i P(Y_i)$$
$$= 1 \times 2/3 + 0 \cdot 1/3 = 2/3$$

Now, we can say that $Y = \sum Y_i$

From linearity of expectation we can write

$$E(Y) = E\left(\sum Y_i\right)$$
$$= \sum E(Y_i)$$
$$= (\text{total no of edges}) \times E(Y_i)$$
$$= |E| \times 2/3 \quad \left[\begin{array}{l}\text{where} \\ |E| = \text{total no of edges in Graph}\end{array}\right]$$
$$= \frac{2|E|}{3}$$

(3) We have to show that there can't be any graph for which all 3-colour assignments make $< \frac{2|E|}{3}$ edges non-chromatic.

Let's try to prove it by contradiction

Assumption: Let's suppose that there can be some graph for which all 3-colour assignments make $< \frac{2|E|}{3}$ edges non-monochromatic

If that's true then $E(Y) < \frac{2|E|}{3}$ (Since all possibilities make $< \frac{2|E|}{3}$ edges non-~~chr~~ monochromatic)

(If all numbers are less than x, then the ~~avg~~ average of those numbers can't be $\geq x$; it should be $< x$)

But, in 4.2 we have proved that $E(Y) = \frac{2|E|}{3}$

Hence, our assumption was wrong.

$\therefore$ There can't be any graph for which all 3-colour assignments make $< \frac{2|E|}{3}$ edges non-monochromatic

(proved)

④ We have to show that $P\left(Y \geq \frac{|E|}{2}\right) \geq \frac{1}{3}$

We know that Y is the random variable corresponding to the number of ~~m~~ non-monochromatic edges.

We have already proved in 4.2 that $E(Y) = \frac{2|E|}{3}$

[where $|E|$ = total no of edges]

<u>NOW,</u> Let X be the random variable corresponding to the number of monochromatic edges.

Let also X_e be the r.v. such that $X_e = \begin{cases} 1 & \text{if edge is monochromatic} \\ 0 & \text{if " is nonmonochromatic.} \end{cases}$

$$P(X_e) = \begin{cases} 3/9 = 1/3 & \text{if } X_e = 1 \\ 6/9 = 2/3 & \text{if } X_e = 0 \end{cases}$$

$$\therefore E(X_e) = 1 \times 1/3 + 0 \times 2/3 = 1/3$$

$$\therefore X = \sum X_e$$

$$\therefore E(X) = E\left(\sum X_e\right) = |E| \cdot E(X_e) = |E| \cdot 1/3 = \frac{|E|}{3}$$

[$|E|$ = total no of edges]

<u>NOW,</u> $P\left(Y \geq \frac{|E|}{2}\right)$

$= 1 - P\left(X \geq \frac{|E|}{2}\right)$ [$\because$ X, Y are complimentary as an edge is either mono-chromatic or non mono-chromatic]

$= 1 - \left(\leq \frac{E(X)}{\frac{|E|}{2}}\right)$ [From Markov's inequality $E(X \geq K) \leq \frac{E(X)}{K}$]

$= 1 - \left(\leq \frac{\frac{|E|}{3}}{\frac{|E|}{2}}\right)$ [$\because E(X) = \frac{|E|}{3}$]

$= 1 - \left(\leq 2/3\right) \geq \frac{1}{3}$ (<u>proved</u>)

(5) We have to devise a method that can find an assignment for which the number of non-monochromatic edges is at least $|E|/2$ with probability at least $99/100$.

Let's try to devise the algorithm first.

Algorithm (input: any Graph X)

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<u>Step 1:</u> Let $X = \{x_1, x_2, x_3, \dots, x_k\}$ be a set of K randomly selected assignments.

<u>Step 2:</u> Let x_i = ith assignment of graph

<u>Step 3:</u> IF the assignment x_i contains number of non-monochromatic edges $\geq \frac{|E|}{2}$ return the assignment x_i

ELSE check for next assignment x_{i+1}

}

<u>NOW</u> let's try to find how many assignments we need to check to achieve probability at least $99/100$.

Let G_i be a random variable such that

$$G_i = \begin{cases} 1 & ; \text{if ith assignment } x_i \text{ has } \geq \frac{|E|}{2} \text{ mono-chromatic edges} \\ 0 & ; \text{else} \end{cases}$$

<u>Now</u> from result of 4.4

$$P(G_i = 1) \leq 2/3$$

$$\therefore E(G_i) \leq 1 \times 2/3 \leq 2/3$$

Now let's define a random variable G such that

$$G = \frac{\sum_{i=1}^{K} G_i}{K} = \frac{G_1 + G_2 + G_3 + \dots + G_K}{K}$$

From linearity of expectation,

$$E(G) = \frac{E(G_1 + G_2 + \dots + G_K)}{K}$$

$$\Rightarrow E(G) = \frac{E(G_1) + E(G_2) + \dots + E(G_K)}{K}$$

$$\Rightarrow E(G) = \frac{\cancel{K} \cdot E(G_i)}{\cancel{K}}$$

$$\Rightarrow E(G) \le 2/3 \quad \left[\because E(G_i) \le 2/3\right]$$

Now $G = 1 \Rightarrow G_1, G_2, \dots, G_K$ all are 1 individually

i.e all assignments $\{G_1, G_2, \dots, G_K\}$ have mono-chromatic edges $\geq \frac{|E|}{2}$

which implies all " " have non-monochromatic edges $< \frac{|E|}{2}$

which implies our above stated algorithm failed to find any assignment G_i with non-monochromatic edges $\geq \frac{|E|}{2}$

Now, from Q. $P(G=1) \le \left(1 - \frac{99}{100}\right)$

$$\le (1 - 0.99)$$

$$\le 0.01$$

Now G can hold max. value $= 1$

$$\therefore P(G >= 1) = P(G = 1)$$

$$\therefore P(G >= 1) \le 0.01$$

$$\Rightarrow P(G >= 1) = P(G - E(G) >= 1 - E(G))$$

$$\Rightarrow P(|G - E(G)| >= 1 - E(G)) >= P(G - E(G) >= 1 - E(G))$$

$$\Rightarrow P(G - E(G) >= 1 - E(G)) \le P(|G - E(G)| >= 1 - E(G))$$

Now, from chebyshev we know

$$P(|X-\mu| \geq K) \leq \frac{Var(X)}{K^2}$$

$$\therefore P(|G-E(G)| >= 1-E(G)) \leq \frac{Var(G)}{(1-E(G))^2}$$

$$\leq \frac{1}{4K(1-E(G))^2}$$

$$\leq \frac{1}{4K(1-2/3)^2}$$

$$\leq \frac{1}{4K(1/3)^2}$$

$$\leq \frac{1}{4K \cdot \frac{1}{9}}$$

$$\leq \frac{9}{4K}.$$

Now, $P(G >= 1) \leq 0.01$

$$\Rightarrow P(|G-E(G)| >= 1-E(G)) \leq 0.01$$

$$\Rightarrow \frac{9}{4K} \leq 0.01$$

$$\Rightarrow \frac{4K}{9} \geq \frac{1}{0.01}$$

$$\Rightarrow \frac{\cancel{4}K}{9} \geq \cancel{100}\,25$$

$$\Rightarrow K \geq 25 \times 9$$

$$\Rightarrow \boxed{K \geq 225}$$

(which means we need to atleast check 225 assignments of graph to achieve probability $\geq \frac{99}{100}$)