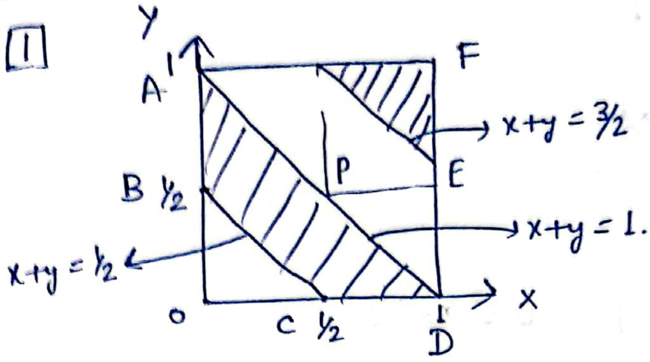
Name: Sudipta Halder
Roll : 2021202011

5.1

[1]



Since $f_{X,Y}(x,y) = \begin{cases} c & , \text{ in shaded region} \\ 0 & ; \text{ otherwise} \end{cases}$

we know Area under curve = 1 $\left[\because \iint f_{X,Y}(x,y) = 1\right]$

$$\Rightarrow c = \frac{1}{\text{Area of shaded region}}$$

$$\Rightarrow c = \frac{1}{\text{Area}(ABCD) + \text{Area}(EFG)}$$

$$\Rightarrow c = \frac{1}{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 4}$$

$$\Rightarrow \boxed{c = 2}$$

[2] Marginal PDF of X = Integration of $f_{X,Y}(x,y)$ w.r.t to y

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$\Rightarrow$ when $0 \le x < 1/2$, Area of shaded region, is given by the area b/w curve $x+y = 1/2$ & $x+y = 1$

So, $\left| \int_{y=\frac{1}{2}-x}^{1-x} 2\,dy \right|$ when $0 < x < 1/2$

$$= 2\left[1 - x - 1/2 + x\right] = 1$$

When $\frac{1}{2} < x < 1$, Area of shaded region is given by the sum of area b/w curve $x+y=1$ & $y=0$ and $x+y=3/2$ & $y=1$

So, $\left|\int_0^{1-x} 2dy\right| + \left|\int_{3/2-x}^{1} 2dy\right| \quad \frac{1}{2} \le x < 1$

$= 2(1-x) + 2(1 - 3/2 + x)$

$= 2 - 2x + 2 - 3 + 2x = 1$

So $f_x(x) = \begin{cases} \int_{1/2-x}^{1-x} 2dy = 1 & 0 \le x \le 1/2 \\ \int_0^{1-x} 2dy + \int_{3/2-x}^{1} 2dy = 1 & 1/2 < x \le 1 \end{cases}$

Similarly marginal PDF of y = Integration of $f_{x,y}(x,y)$ wrt to x

When $0 \le y < 1/2$ Area of shaded region is given by the area b/w the curve $x+y = 1/2$ & $x+y=1$

So, $\left|\int_{1/2-y}^{1-y} 2dx\right|$ when $0 \le y < 1/2$

$= 2[1 - y - 1/2 + y] = 1$

& when $\frac{1}{2} \le y < 1$ Area of shaded region is given by the sum of area b/w curve $x+y=1$ & $x=0$ and $x+y=3/2$ and $x=1$

So, $\left|\int_0^{1-y} 2dx + \right| + \left|\int_{3/2-y}^{1} 2dx\right| = 1$ when $1/2 \le y \le 1$

$= 2(1-y) + 2(1 - 3/2 + y) = 1$

So, $f(x,y) = \begin{cases} \int_{\frac{1}{2}-y}^{1-y} 2dx = 1 & 0 \le y < \frac{1}{2} \\ \int_{0}^{1-y} 2dx + \int_{\frac{3}{2}-y}^{1} 2dx = 1 & \frac{1}{2} \le y \le 1 \end{cases}$

$= 1$

3

$$E(X/Y=\tfrac{1}{4}) = \int_{-\infty}^{\infty} x\, f_{x,y}(x,y)\,dx$$

Now $f_{x,y}(x/y) = \dfrac{f_{xy}(x,y)}{f_y(y)} = 1.$

(from Graph) $= \begin{cases} 2 & \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{else} \end{cases}$

when $y = \frac{1}{4}$
$x+y = \frac{1}{2}$
$\Rightarrow x = \frac{1}{4}$
$x+y = 1$
$\Rightarrow x = \frac{3}{4}$

So, $E(X/Y=\frac{1}{4}) = \int_{1/4}^{3/4} x.2dx = \frac{9}{16} - \frac{1}{16} = \frac{8}{16} = \frac{1}{2}$

$\therefore E(X/Y = \frac{1}{4}) = \frac{1}{2}$

Now $\text{var}(X/Y=\frac{1}{4}) = \int_{-\infty}^{\infty} x^2 f_{x,y}(x/y)\,dx$

$= \int_{1/4}^{3/4} x^2 \cdot 2dx = \frac{2}{3}\left[\frac{27}{64} - \frac{1}{64}\right] = \frac{13}{48}$

~~4~~

$\therefore \boxed{\text{var}(X/Y = \frac{1}{4}) = \frac{13}{48}}$

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4

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y=\tfrac{3}{4}) = \frac{f_{XY}(x,y)}{1}$$

When $y = \frac{3}{4}$ $f_{XY}(x,y)$ is in shaded region

When $x+y=1 \Rightarrow x = 1-3/4 = \frac{1}{4}$

$x+y = 3/2 \Rightarrow x = 3/2 - 3/4 = 3/4$

So, $f_{X|Y}(x|y=3/4) = \begin{cases} 2, & 0 \le x \le 1/4 \ \& \ 3/4 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\boxed{f_{X|Y}(x|y=3/4) = \begin{cases} 2, & 0 \le x \le \frac{1}{4} \ \& \ \frac{3}{4} \le x \le 1 \\ 0, & \text{else} \end{cases}}$$

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5.2

[1] $T_i - T_{i-1}$ = Random variable which is the number of shuffle that was done after $i-1$ cards went below n for next card to go below n.

Then, $P(T_i - T_{i-1} = j)$

When exactly $i-1$ cards are under the original bottom card n. The chance that the current top card is inserted below n is $\frac{(i-1)+1}{n} = i/n$

For, current top card to go below n in the j th step, $(j-1)$ failure (if i goes to above bottom card n). Should occur and at j th step success (it goes below bottom card n)

$$P(T_i - T_{i-1} = j) = \frac{i}{n}\left(\frac{n-i}{n}\right)^{(j-1)}$$

$$= \frac{i}{n}\left(1 - i/n\right)^{(j-1)}$$

So it is geometric distribution with $p = i/n$

Here T_{i-1} depends on it. This is bcoz at a time the distribution of T_i depends on the immediate past. Where the next card would go depends only on the current card position. not on where other cards have been placed.

2 Here card no. n is fixed. At beginning there is no card below n. Now, we observe that whenever we put a card below n, we put in a completely random position with respect to the rest of the card below n.

So, we can say card below n are in uniformly random permutation.

Here T_{n-1} = first time that n becomes the top card of the deck as $(n-1)$ cards get below it.

At this points we have a uniformly random permutation of all other cards $(n-1)$.

This is because bottom $(n-1)$ cards can take any 1 of possible $(n-1)!$ arrangement.

So, after $T_{n-1}+1$ shuffles all possible $n!$ arrangements are equally likely.

3 Let T be the first time that 'n' becomes the top card of the deck.

Then let $T = T_{n-1}+1$

$$\Rightarrow T = (T_{n-1} - T_{n-2}) + (T_{n-2} - T_{n-3}) + \cdots + (T_2 - T_1) + (T_1 - T_0)$$

$$E(T) = E(T_{n-1} - T_{n-2}) + \cdots + E(T_1 - T_0)$$

$$= \frac{n}{n-1} + \frac{n}{n-2} + \cdots + 1$$

($\because T_i - T_{i-1}$ has geometric distribution with $p = i/n$ so expected value in geometric " $= 1/p = n/i$ in card i)

$$E(T) = n(1/1 + 1/2 + \cdots + 1/(n-2) + 1/(n-1) + 1/n)$$

$$E(T) \approx n \cdot H_n \quad [H_n = n\text{th harmonic no}]$$

$\boxed{4}$ Let T be the first time that bottom card 'n' becomes the top card of the deck.

Then the deck will be completely random. (proved in $\boxed{2}$)

Also, $T = T_{n-1} + 1$

& $E(T) = n.H_n \approx n.\log n$ [$\boxed{3}$]

Then if we shuffle K-times (say) so that with 99%. hence the deck will be completely random or there is an $\frac{1}{100}$ chance for it not to be completely shuffled

So, $P(T \geq K) \leq \frac{1}{100}$ —— (1)

It means we need to shuffle more than K times with only 1%. chance to not to be randomized shuffle

Using markov inequality,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$\Rightarrow P(T \geq K) \leq \frac{E(X)}{K}$$

$$\Rightarrow P(T \geq K) \leq \frac{n \log n}{K} \quad \text{—— (2)}$$

Comparing (1) & (2)

$$\frac{n \log n}{K} \leq \frac{1}{100}$$

$$\Rightarrow K \geq 100 \, n \log n$$

So, if we shuffle 100 n logn times, with $\frac{99}{100}$ chance, the deck will be completely random

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(5.3)

[1] The transition matrix P for this Markov chain is as follows.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

[2] The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any n even $P^{(n)}_{6,1}$ will be '0' and for any n odd $P^{(n)}_{6,5}$ will also be zero. This means that there is no power of P that would have all its entries strictly positive.

[3] for finding out the stationary distribution, let's consider the stationary state $S = [a_1\ a_2\ a_3\ a_4\ a_5\ a_6]$

Now, we know that $\underbrace{S}_{\text{Stationary state}} \cdot \underbrace{P}_{\text{transition matrix}} = \underbrace{S}_{\text{Stationary state}}$

$$\Rightarrow [a_1\ a_2\ a_3\ a_4\ a_5\ a_6] \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= [a_1\ a_2\ a_3\ a_4\ a_5\ a_6]$$

Also, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1$

$$\Rightarrow \left[\left(\tfrac{1}{4}a_3\right)\left(\tfrac{1}{4}a_3\right)\left(a_1+a_2+\tfrac{1}{2}a_4+\tfrac{1}{2}a_5\right)\left(\tfrac{1}{4}a_3+\tfrac{1}{2}a_6\right)\left(\tfrac{1}{4}a_3+\tfrac{1}{2}a_6\right)\left(\tfrac{1}{2}a_4+\tfrac{1}{2}a_5\right)\right]$$

$$= [a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6]$$

$\therefore \frac{1}{4}a_3 = a_1$ $\quad \therefore \frac{1}{4}a_3 = a_2$ $\quad \boxed{a_1 + a_2 + \frac{1}{2}a_4 + \frac{1}{2}a_5 = a_3}$ — 3

$\Rightarrow \boxed{a_3 = 4a_1}$ — 1 $\quad \therefore \boxed{a_3 = 4a_2}$ — 2

$a_5 = \frac{1}{4}a_3 + \frac{1}{2}a_6$ $\quad a_6 = \frac{1}{2}a_4 + \frac{1}{2}a_5$

$\Rightarrow \boxed{4a_5 = a_3 + 2a_6}$ — 4 $\quad \boxed{2a_6 = a_4 + a_5}$ — 5

$\boxed{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1}$ — 6

<u>Solving</u> 1, 2, 3, 4, 5, 6 we get.

$a_1 = 1/12$

$a_2 = 1/12$

$a_3 = 4/12$

$a_4 = 2/12$

$a_5 = 2/12$

$a_6 = 2/12$

$\therefore$ Stationary dist $(S) = [1/12 \ 1/12 \ 4/12 \ 2/12 \ 2/12 \ 2/12]$

4

Let t_i be the expected number of steps until the chain hits state 5 for the first time given that $X_0 = 1$.

So, by total probability law,

$$t_1 = 1 + t_3$$

$$t_3 = 1 + t_1/4 + t_2/4 + t_4/4 + \cancel{t_5/4}$$

$$t_2 = 1 + t_3$$

$$t_4 = 1 + t_6/2 + t_3/2$$

$$t_6 = 1 + t_4/2 + \cancel{t_5/2}$$

$$\Rightarrow t_4 = 1 + \tfrac{1}{2}(1 + t_4/2) + t_3/2 = 3/2 + t_4/4 + t_3/2$$

$$\therefore t_3 = 1 + \tfrac{1}{4}(1 + t_3) + \tfrac{1}{4}(1 + t_3) + t_4/4$$

$$\Rightarrow \frac{3t_3}{4} = 3/2 + t_3/2$$

$$\Rightarrow t_3 = 3/2 + t_3/2 + t_4/4$$

$$\Rightarrow 3t_4 = 6 + 2t_3$$

$$t_3 = 3 + t_4/2$$

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$\Rightarrow 3t_4 = 6+6+t_4$

$\Rightarrow 2t_4 = 12$

$\Rightarrow t_4 = 6$

$\Rightarrow t_3 = 3 + 6/2 = 6$

$\Rightarrow \boxed{t_1 = 1+6 = 7}$

5

Let r_ℓ be the mean return time to State ℓ.

Then $r_l = 1 + \sum_k t_k p_{lk}$

When t_k is the expected time until the chain hits State l. given $X_0 = k$

$$r_1 = 1 + t_3 \cdot 1$$
$$= 1 + t_3$$

Here t_3 is the expected time until chain hits State 1 given $X_0 = 3$

Again by law of total probability

we have, $t_1 = 0$

$$t_2 = 1 + t_3$$
$$t_3 = 1 + t_2/4 + t_4/4 + t_5/4$$
$$t_4 = 1 + t_3/2 + t_6/2$$
$$t_5 = 1 + t_3/2 = t_6/2$$
$$t_6 = 1 + t_4/2 + t_5/2$$

$\Rightarrow t_4 = t_5$

$\Rightarrow t_6 = 1 + t_4$

$\Rightarrow t_3 = 1 + t_2/4 + t_4/4$

$\Rightarrow t_3 = 1 + \frac{1+t_3}{4} + \frac{1}{2} + t_3/4 + t_2/4$

$\Rightarrow 4t_3 = 4 + 1 + t_3 + 2 + t_3 + t_2$

$\Rightarrow 2t_3 = 7 + t_2$

$= 7 + (1 + t_1)$

$= 7 + (1 + 2t_3 - 2 - t_2/2)$

$= 7 + \left(1 + 2t_3 - 2 - \left(\frac{1+t_3}{2}\right)\right)$

$\Rightarrow \frac{t_3}{2} = 11/2$

$\Rightarrow t_3 = 11$

<u>So</u> $r_1 = 1 + t_3 = 12$

<u>Aliter</u>

r_l = Mean time to return to state l

$r_l = \frac{1}{\pi_4} = \frac{1}{1/12} = 12$ Ans [from $\boxed{3}$]

$\boxed{\therefore r_l = 12}$