Sudipta Halder

Roll: 2021202011

Assignment-3

1

(a) Given equations are ⇒

$$x + 3y + 5z = 14$$
$$2x - y - 3z = 3$$
$$4x + 5y - z = 7$$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$

Reducing the augmented matrix to Row-echleon form

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

This is in Row-Echleon form

So, from R_3 $-8z = -24$

$\Rightarrow \boxed{z = 3}$

from R_2 $-7y - 13z = -25$

$\Rightarrow -7y - 13 \times 3 = -25$

$\Rightarrow 7y = -14$

$\Rightarrow \boxed{y = -2}$

from R_1

$$x+3y+5z=14$$
$$\Rightarrow x+3\times(-2)+5(3)=14$$
$$\Rightarrow x-6+15=14$$
$$\Rightarrow \boxed{x=5}$$

$$\therefore \boxed{x=5,\ y=-2,\ z=3} \quad \text{(Ans)}$$

(b) Given equations are $\Rightarrow$

$$y+z=4$$
$$3x+6y-3z=3$$
$$-2x-3y+7z=10$$

The augmented matrix is: $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right]$

Reducing the matrix to Reduced-Row echelon form:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1/3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{R_3 \to R_3+2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3-R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{4}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\downarrow R_1 \longleftrightarrow R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xleftarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This is in reduced row echleon form.

Now, from R_1 ~~$x=-1$~~ $\boxed{x=-1}$

from R_2 $\boxed{y=2}$

from R_3 $\boxed{z=2}$

$\therefore$ $\boxed{x=-1,\ y=2,\ z=2}$ (Ans)

(c) Given equations are:- $\sqrt{2}x + y + 2z = 1$

$\sqrt{2}y - 3z = -\sqrt{2}$

$-y + \sqrt{2}z = 1$

The augmented matrix is $\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right]$

Reducing the matrix into reduced row-echleon form

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \xrightarrow{R_3 \to \sqrt{2}R_3 + R_2} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\downarrow \begin{array}{l} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xleftarrow[R_3 \to -R_3]{R_2 \to R_2/\sqrt{2}} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} \sqrt{2} & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1/\sqrt{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This is in reduced row-echelon form

from R_1 $\boxed{x = \sqrt{2}}$

from R_2 $\boxed{y = -1}$

from R_3 $\boxed{z = 0}$

$\therefore \boxed{x = \sqrt{2},\ y = -1,\ z = 0}$ (Ans)

$\boxed{2}$ Given equations

$$x + y - z = 1$$
$$x + y + z = 2$$
$$x - y = 3$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1(1) - 1(-1) - 1(-1-1)$$
$$= 1+1+2 = 4$$

Now, $x = \frac{D_x}{D} = \frac{1}{4} \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = \frac{1}{4}\left[1(1) - 1(-3) - 1(-2-3)\right]$

$$= \frac{1}{4}(1+3+5)$$
$$= 9/4$$

Now, $y = \frac{D_y}{D} = \frac{1}{4} \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = \frac{1}{4}\left[1(-3) - 1(-1) - 1(-3-2)\right]$

$$= \frac{1}{4}(-3+1-1)$$
$$= -3/4$$

Now, $z = \frac{D_z}{D} = \frac{1}{4}\begin{vmatrix} 1 & 1 & \boxed{1} \\ 1 & 1 & \boxed{2} \\ 1 & -1 & \boxed{3} \end{vmatrix} = \frac{1}{4}\left[1(5) - 1(1) + 1(-2)\right]$

$= \frac{1}{4}(5 - 1 - 2) = \frac{1}{2}$

$\therefore \boxed{x = 9/4,\ y = -3/4,\ z = 1/2}$ (Ans)

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3 Given equations are:

$$2x + y - 3z = 1$$
$$y + z = 1$$
$$z = 1$$

The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 2 & 1 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(1 - 0) = 2$$

Now, $x = \frac{D_x}{D} = \frac{1}{2}\begin{vmatrix} \boxed{1} & 1 & -3 \\ \boxed{1} & 1 & 1 \\ \boxed{1} & 0 & 1 \end{vmatrix}$

$= \frac{1}{2}\left[1(1) - 1(1 - 1) - 3(-1)\right]$

$= \frac{1}{2}(1 - 0 + 3) = 2$

Now, $y = \frac{D_y}{D} = \frac{1}{2}\begin{vmatrix} 2 & \boxed{1} & -3 \\ 0 & \boxed{1} & 1 \\ 0 & \boxed{1} & 1 \end{vmatrix}$

$= \frac{1}{2}\left[2(1 - 1) + 0 + 0\right] = 0$

Now, $z = \frac{D_z}{D} = \frac{1}{2}\begin{vmatrix} 2 & 1 & \boxed{1} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & \boxed{1} \end{vmatrix}$

$= \frac{1}{2}(2(1) + 0 + 0) = 1$

$\therefore \boxed{x = 2,\ y = 0,\ z = 1}$ (Ans)

4 Given equations are:- $x+y=1$

$x-y=2$

The determinant of the coefficient matrix is

$$D=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=1(-1)-1=-2$$

Now, $x=\frac{D_x}{D}=\frac{1}{-2}\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}=-\frac{1}{2}(-1-2)=3/2$

Now, $y=\frac{D_y}{D}=-\frac{1}{2}\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=-\frac{1}{2}(2-1)=-1/2$

$\therefore \boxed{x=3/2,\ y=-1/2}$ (Ans)

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5 Let $\lambda_i\ (i=1,2,...n)$ are complete set of eigen values of $A_{n\times n}$. We have to prove that $\det(A)=\prod_{i=1}^{n}\lambda_i$

Soln: Given $\lambda_1,\lambda_2,...,\lambda_n$ are eigen values of A.

We know that λ_i are roots of the characteristic equation $|A-\lambda I|=0$

We know $|A-\lambda I|=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$

Let $\lambda=0$

Then $|A-0I|=(\lambda_1-0)(\lambda_2-0)\cdots(\lambda_n-0)$

$\Rightarrow |A|=\lambda_1\lambda_2\cdots\lambda_n$

$\Rightarrow |A|=\prod_{i=1}^{n}\lambda_i$

$\therefore \boxed{\det(A)=\prod_{i=1}^{n}\lambda_i}$ (proved)

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$\boxed{6}$ We have to prove that $\det(AB) = \det(BA)$.

Soln LHS $\det(AB) = \det(A) \cdot \det(B)$ $[\because |AB| = |A| \cdot |B|]$

$= \det(B) \cdot \det(A)$ [$\because$ det is a real number and it satisfies commutative property]

$= \det(BA)$

$= R \cdot H \cdot S$

Note: The above statement holds true only when A and B are square matrices, for other matrices, it might not be always true.

$\boxed{7}$ If A is idempotent find all possible values of $\det(A)$.

Soln: If A is idempotent then $A^2 = A$.

Hence, $|A^2| = |A|$

$\Rightarrow |A \cdot A| = |A|$

$\Rightarrow |A|\,|A| = |A|$

$\Rightarrow |A| \cdot |A| - |A| = 0$

$\Rightarrow |A|\,(|A| - 1) = 0$

$\Rightarrow \boxed{|A| = 0}$ or $\boxed{|A| = 1}$

$\therefore$ The possible values ~~are~~ of A are 0 and 1 if A is idempotent

8 We need to find characteristic equation, eigen values, basis for eigen space of eigen values, algebric and geometric multiplicity of eigen values.

(a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

The characteristic eq. of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(6-\lambda)+6 = 0$$

$$\Rightarrow \boxed{\lambda^2 - 7\lambda + 12 = 0} \Rightarrow \text{characteristic equation.}$$

The eigen values are the roots of characteristic eq:-

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\Rightarrow (\lambda-4)(\lambda-3) = 0$$

$$\Rightarrow \boxed{\lambda = 4} \text{ or } \boxed{\lambda = 3}$$

eigen values.

For $\lambda = 4$: $(A-\lambda I) = (A-4I)$

$$= \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \text{ (after row operations)}$$

Thus, $-x+y = 0$

$\Rightarrow x = y$

General soln is $\begin{bmatrix} x \\ x \end{bmatrix}$

$\therefore$ Basis of eigen space for $\lambda = 4$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric multiplicity $= \dim(V_{\lambda=4}) = 1$.

• For $\lambda = 3$: $(A - \lambda I) = (A - 3I)$

$$= \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix} \text{ (after row op..)}$$

Thus, $-2x + 3y = 0$

$\Rightarrow 2x = 3y$

$\therefore$ Basis of eigen space for $\lambda = 3$ is $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric multiplicity $= \dim(V_{\lambda=3}) = 1$

(b) Given $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation for A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(\lambda+1)(\lambda-1) - 1\right] - 2\left[(\lambda-1) - 0\right] + 0 = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 2) - 2\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2 - \lambda^3 + 2\lambda - 2\lambda + 2 = 0$

$\Rightarrow \lambda^2 - \lambda^3 = 0$

$\Rightarrow \boxed{\lambda^3 - \lambda^2 = 0}$ $\longrightarrow$ characteristic eq

The eigen values are roots of the characteristic eq:

$\lambda^3 - \lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - 1) = 0$

$\Rightarrow \boxed{\lambda^2 = 0}, \boxed{\lambda = 1}$

$\boxed{\lambda = 0}, \boxed{\lambda = 0}, \boxed{\lambda = 1}$

eigen values

For $\lambda=0$

$$(A-\lambda I)=A$$

$$=\begin{bmatrix}1&2&0\\-1&-1&1\\0&1&1\end{bmatrix}\xrightarrow{R_2\to R_2+R_1}\begin{bmatrix}1&2&0\\0&1&1\\0&1&1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-R_3}\begin{bmatrix}1&2&0\\0&1&1\\0&1&1\end{bmatrix}\xrightarrow{R_2\to R_2-R_3}\begin{bmatrix}1&2&0\\0&1&1\\0&0&0\end{bmatrix}$$

Thus, $x+2y=0$ and $y+z=0$

$\Rightarrow x=-2y \qquad \Rightarrow y=-z$

$\Rightarrow x=2z$

General Solution is $\therefore \begin{bmatrix}2z\\-z\\z\end{bmatrix}$

$\therefore$ Basis of eigen space for $\lambda=0$ is $\left\{\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$

- Algebraic multiplicity $=2$
- Geometric " $=\dim(V_{\lambda=0})=1$

For $\lambda=1$

$$(A-\lambda I)=(A-I)$$

$$=\begin{bmatrix}0&2&0\\-1&-2&1\\0&1&0\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}-1&-2&1\\0&2&0\\0&1&0\end{bmatrix}$$

$$\xrightarrow[R_3\to 2R_3-R_2]{R_1\to -1*R_1}\begin{bmatrix}1&2&-1\\0&2&0\\0&0&0\end{bmatrix}\xrightarrow{R_1\to R_1-R_2}\begin{bmatrix}1&0&-1\\0&2&0\\0&0&0\end{bmatrix}$$

Thus. $x-z=0$ and $2y=0$

$\Rightarrow \boxed{x=z} \qquad \Rightarrow \boxed{y=0}$

General solⁿ is $\left\{ \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} \right\}$

$\therefore$ Basis of eigen space for $\lambda = 1$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric multiplicity $= \dim(V_{\lambda=1}) = 1$

(c) Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristic eq for A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(\lambda-1)\lambda - 1\right] - 0 + 1\left[0 - (1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - \lambda - 1\right] - 1 + \lambda = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 1 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\Rightarrow \boxed{\lambda^3 - 2\lambda^2 - \lambda + 2 = 0}$$ (characteristic eq)

The eigen values are the roots of the characteristic eq.

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1)(\lambda - 2) = 0$$

$$\Rightarrow \boxed{\lambda = 1} \text{ or } \boxed{\lambda = -1} \text{ or } \boxed{\lambda = 2}$$

eigen values

For $\lambda = 1$ $(A - \lambda I) = (A - I)$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_3 \leftarrow R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_1 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $x + y = 0$ and $z = 0$

$\Rightarrow x = -y$

$\therefore$ General soln is $\begin{bmatrix} -y \\ y \\ 0 \end{bmatrix}$

$\therefore$ Basis of eigen space for $\lambda = 1$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric " $= \dim(V_{\lambda=1}) = 1$

For $\lambda = -1$

$(A - \lambda I) = (A + I)$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 \to R_3 - R_2/2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$$

Thus, $x + z/2 = 0$ and $2y + z = 0$

$\Rightarrow \boxed{x = -z/2}$ and $\boxed{y = -z/2}$

General soln $\begin{bmatrix} -z/2 \\ -z/2 \\ z \end{bmatrix}$ or $\begin{bmatrix} -z \\ -z \\ 2z \end{bmatrix}$

$\therefore$ Basis of eigen space for $\lambda = -1$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric " $= 1$

For $\lambda = 2$

$(A - \lambda I) = (A - 2I)$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $-x + z = 0$ and $-y + z = 0$

$\Rightarrow \boxed{x = z}$ and $\boxed{y = z}$

$\therefore$ General soln is $\begin{bmatrix} z \\ z \\ z \end{bmatrix}$

$\therefore$ Basis of eigen space for $\lambda = 2$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

- Algebraic multiplicity $= 1$
- Geometric " $= 1$

9. We have to show that given vectors forms an orthogonal basis for $\mathbb{R}^3$

(a) $V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $V_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Soln If the set of vectors $\{V_1, V_2, \dots, V_n\}$ are orthogonal then it means $V_i \cdot V_j = 0 \ \forall \ i,j \in 1,2,\dots n$ and $i \neq j$

$V_1 \cdot V_2$: $1(1)+0(2)-1(1) = 1-1 = 0$

$V_1 \cdot V_3$: $1(1)+0(-1)-1(1) = 1-1 = 0$

$V_1 \cdot V_4$: $1(1)+0(1)-1(1) = 1-1=0$

$V_2 \cdot V_4$: $1(1)+2(1)+1(1) = 4 \neq 0$

$V_3 \cdot V_4$: $1(1)-1(1)+1(1) = 1 \neq 0$

The vectors are not mutually orthogonal.

$\therefore \{V_1, V_2, V_3, V_4\}$ does not form orthogonal basis.

(b) $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $V_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Soln $V_1 \cdot V_2$: $1(1)+1(-1)+1(0) = 0$

$V_1 \cdot V_3$: $1(1)+1(1)+1(-2) = 0$

$V_1 \cdot V_4$: $1(1)+1(2)+1(3) = 6 \neq 0$

$V_2 \cdot V_3$ $1(1)-1(1)+0(-2) = 0$

$V_2 \cdot V_4$ $1(1)-1(2)+0(3) = -1 \neq 0$

$V_3 \cdot V_4$ $1(1)+1(2)-2(3) = -3 \neq 0$

The vectors are not mutually orthogonal.

$\therefore \{V_1, V_2, V_3, V_4\}$ does not form orthogonal basis.

10

(a) Given $A = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

Augmenting the matrix with identity matrix $\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right]$

To find the inverse of A, we have to make the left matrix as identity matrix. Then the matrix on the right side will be the inverse.

$$\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1/-2} \left[\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 0 & 5 & 3/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2/5} \left[\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 0 & 1 & 3/10 & 1/5 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/10 & 2/5 \\ 0 & 1 & 3/10 & 1/5 \end{array}\right]$$

$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{Identity matrix}}$

$$\therefore A^{-1} = \begin{bmatrix} 1/10 & 2/5 \\ 3/10 & 1/5 \end{bmatrix}$$

(b) Given $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$

Augment the matrix with the identity matrix

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1/2$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/2 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/2 & 0 & 0 \\ 0 & -7/2 & -1 & -1/2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \to R_2 * (-2/7)$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 2/7 & 1/7 & -2/7 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_3 \to R_3 + 3R_2 \\ R_1 \to R_1 - 3/2 R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3/7 & 2/7 & 3/7 & 0 \\ 0 & 1 & 2/7 & 1/7 & -2/7 & 0 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 * (-7)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3/7 & 2/7 & 3/7 & 0 \\ 0 & 1 & 2/7 & 1/7 & -2/7 & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

$$\Big\downarrow \quad \begin{array}{l} R_1 \rightarrow R_1 + 3/7 R_3 \\ R_2 \rightarrow R_2 - 2/7 R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

$\underbrace{\qquad\qquad}$ Identity matrix

$$\therefore B^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$