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Mod: 2 $\boxed{\text{Final Exam}}$

1. $\boxed{\text{Section-}\cancel{B}A}$

$\boxed{2.}$ (a) Given equations

$$x+y-z=1$$
$$x+y+z=2$$
$$x-y=3$$

$$A=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & 0\end{bmatrix} \qquad B=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

NOW,

$$D=Det(A)=|A|=\begin{vmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & 0\end{vmatrix}$$

$$=1(0+1)-1(0-1)-1(-1-1)$$

$$=1+1+2=4$$

NOW,

$$D_x=\begin{vmatrix}\boxed{\begin{matrix}1\\2\\3\end{matrix}} & \begin{matrix}1\\1\\-1\end{matrix} & \begin{matrix}-1\\1\\0\end{matrix}\end{vmatrix}=1(0+1)-1(0-3)-1(-2-3)$$

$$=1+3+5=9$$

$$D_y=\begin{vmatrix}1 & 1 & -1\\ 1 & 2 & 1\\ 1 & 3 & 0\end{vmatrix}=1(0-3)-1(0-1)-1(3-2)$$

$$=-3+1\overset{-1}{\cancel{+5}}=\cancel{0}\,-3$$

$$D_z=\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 2\\ 1 & -1 & 3\end{vmatrix}=1(5)-1(1)+1(-2)$$

$$=5-1-2=2.$$

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$\therefore x = \frac{D_x}{D} = \frac{1}{4} \times 9 = \frac{9}{4}$

$\therefore y = \frac{D_y}{D} = \frac{1}{4} \times (-3) = -3/4$

$\therefore z = \frac{D_z}{D} = \frac{1}{4}(2) = \frac{1}{2}$

$\therefore \boxed{x = \frac{9}{4},\ y = -3/4,\ z = 1/2}$ (Ans)

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \end{bmatrix}$$

The augmented matrix is $= [A : I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & 0 & 0 & 0 & 1 \end{array}\right]$

To find the inverse matrix, augment it with the identity matrix and perform row operations trying to make the identity matrix to the left. Then to the right will be the inverse matrix.

So, augmented matrix $= [A : I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & 0 & 0 & 0 & 1 \end{array}\right]$

$\downarrow R_4 = R_4 - aR_1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & b & c & d & -a & 0 & 0 & 1 \end{array}\right]$$

$\xleftarrow{R_3 = R_3 - R_2}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & b & c & d & -a & 0 & 0 & 1 \end{array}\right]$$

Since, row 3 consists solely of zeros, the determinant of the matrix equals 0.

Thus, the matrix is not invertible

$\therefore$ Inverse does not exist (Ans)

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(a) $3x+2y+z=-1$

$2x-y+4z=3$

$$[A:B] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1 = R_1/3$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & -1/3 \\ 2 & -1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_2 = R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & -1/3 \\ 0 & -7/3 & 10/3 & 11/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_2 = -3R_2/7$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & -10/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1 = R_1 - 2R_2/3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9/7 & 5/7 \\ 0 & 1 & -10/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From row ① $x + \frac{9z}{7} = 5/7 \Rightarrow x = \frac{(5-9z)}{7}$

$y - 10z/7 = -11/7 \Rightarrow y = \frac{(10z-11)}{7}$

$\therefore$ Soln is $\boxed{x = \frac{(5-9z)}{7},\ y = \frac{10z-11}{7},\ z = z}$

(b) $a+b+c+d=4$

$a+2b+3c+4d=10$

$a+3b+6c+10d=20$

$a+4b+10c+20d=35$

The augmented matrix is $=[A \vdots B]$

$$=\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right]$$

$R_2 - 1R_1 \to R_2$
$R_3 - 1R_1 \to R_3$
$R_4 - 1R_1 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 2 & 5 & 9 & 16 \\ 0 & 3 & 9 & 19 & 31 \end{array}\right]$$

$R_1 - 1R_2 \to R_1$
$R_3 - 2R_2 \to R_3$
$R_4 - 2R_3 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 10 & 13 \end{array}\right]$$

$R_1 + R_3 \to R_1$, $R_4 - 3R_3 \to R_4$
$R_2 - 2R_3 \to R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 - R_4 \to R_1$
$R_2 + 3R_4 \to R_2$
$R_3 - 3R_4 \to R_3$

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]}_{I}$$

$\therefore$ $\boxed{a=1,\ b=1,\ c=1,\ d=1}$ (Ans)

4.

Section-D

2

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the complete set of eigen values of the nxn matrix.

(i) P.T. $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$$\det(A - \lambda I) = p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$
$$= (-1)(\lambda - \lambda_1)(-1)(\lambda - \lambda_2) \cdots (-1)(\lambda - \lambda_n)$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

The first equality follows from the factorization of a polynomial given its roots; the leading (highest degree) coefficient $(-1)^n$ can be obtained by expanding the determinant along the diagonal.

Now, by setting λ to zero (simply because it is a variable) we get on the left side $\det(A)$, and on the right side $\lambda_1 \lambda_2 \cdots \lambda_n$, that is, we indeed obtain the desired result.

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

So, the determinant of the matrix is equal to the product of its eigen values. (proved)

(ii) PT $\mathrm{Trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

$$p(t) = \det(A - tI) = (-1)^n (t^n - (\mathrm{tr}A) t^{n-1} + \cdots + (-1)^n \det(A))$$

On the other hand, $p(t) = (-1)^n (t - \lambda_1) \cdots (t - \lambda_n)$, where the λ_j are the eigenvalues of A.

So, comparing coefficients, we have $\boxed{\mathrm{Trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n}$

(proved)

3 A, B are ~~of~~ orthogonally diagonalizable.

(2) A, B orthogonally diagonalizable

$\therefore AB = BA$.

The Spectral theorem States that a $n \times n$ matrix A is orthogonally diagonalizable iff it is symmetric, and by defn a matrix is symmetric if $A^T = A$.

So, now we have that $A^T = A$ and $B^T = B$.

$$(AB)^T = B^T A^T$$

Now, its easy to show that $(AB)^T = B^T A^T = BA = AB$

So, using the spectral theorem, we have that AB is orthogonally diagonalizable (proved)

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(1) If A is symmetric then $A^T = A$

If B is — — — $B^T = B$

NOW $(A+B)^T = A^T + B^T = A + B$

$\therefore$ we have proved that $(A+B)^T = A+B$

$\therefore A+B$ is orthogonally diagonalizable (proved)