Change of Basis

Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V. Let v be a vector in V, then we can write

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Then $c_1, c_2, \ldots, c_n$ are called the co-ordinates of the vector v with respect to the B.

$$\therefore [v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \longrightarrow \text{coordinate vector}$$

$v = (2,5) \in \mathbb{R}^2$.

$\underline{\{(0,1),(1,0)\}}_B$

$\underline{\{(1,0),(0,1)\}}_B$

$$[v_B] = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$[v]_B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

$P(x) = 2 - 3x + 5x^2$, what will be the co-ordinate of the vector $[P(x)]_B$, when $B = \{1, x, x^2\}$.

$$[P(x)]_B = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$B = \{x^2, x, 1\}$.

$$[P(x)]_B = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$$

Find the co-ordinate vector $[A]_B$ of $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ with respect to the basis

$B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[A]_B = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

$$p(x) = 1 + 2x - x^2 \longrightarrow$$

$$C = \{1 + x, x + x^2, 1 + x^2\}$$

$$[p(x)]_C = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$c_1(1+x) + c_2(x + x^2) + c_3(1 + x^2)$$
$$= 1 + 2x - x^2$$

Let $B = \{v_1, v_2, \dots v_n\}$ be a bases for a vector space V. Let u and v be vectors in V and c be a scalar. Then.

a) $[u+v]_B = [u]_B + [v]_B$

b) $[cu]_B = c[u]_B$

Proof: Let $u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ ($\because$ B is a basis)

$v = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$.

$[u+v] = (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \dots + (c_n + d_n) v_n$

$cu = (cc_1) v_1 + (cc_2) v_2 + \dots + (cc_n) v_n$.

$$[u+v]_B = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$= [u]_B + [v]_B.$$

$$[cu]_B = \begin{bmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_n \end{bmatrix} = c \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c\,[u]_B.$$

co-ordinate vectors preserve linear combinations.

In $\mathbb{R}^2$, $B = \{u_1, u_2\}$

$C = \{v_1, v_2\}$

$$u_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightarrow B$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow C.$$

$$[x]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow$$ coordinate vector in the basis B.

Find $[x]_C$.

$$u_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[u_1]_C = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = -3v_1 + 2v_2. \quad \rightarrow (1)$$

$$u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[u_2]_C = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3v_1 - v_2 \quad \rightarrow (2)$$

$$[x]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= u_1 + 3u_2$$ (replace (1), (2) here)

$$= (-3\omega_1 + 2\omega_2) + 3(3\omega_1 - \omega_2)$$

$$= 6\omega_1 - \omega_2$$

$$[x]_C = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \longrightarrow$$ Laborious

Second Method

$$[x]_C = [u_1 + 3u_2]_C$$

$$= [u_1]_C + 3[u_2]_C.$$

$$= \begin{bmatrix} [u_1]_C & [u_2]_C \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

$$= \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$ it

Change of Bases Matrix.

$$= \underline{\underset{C \leftarrow B}{P}} [x]_B$$

$P_{C \leftarrow B}$

$P_{C \leftarrow B}$

Let $B = \{u_1, \ldots u_n\}$ and $C = \{v_1, v_2, \ldots, v_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are coordinate vectors $[u_1]_C, [u_2]_C, \ldots, [u_n]_C$ of the vectors in B with respect to C is denoted by $P_{C \leftarrow B}$. is called as the change of basis matrix from B to C

$$P_{C \leftarrow B} = \left[[u_1]_C \; [u_2]_C \cdots [u_r]_C \right]$$

Find the change-of-basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B = \{1, x, x^2\}$ and $C = \{1+x, x+x^2, 1+x^2\}$ of $\mathbb{P}_2$. Then find the vector of $P(x) = 1 + 2x - x^2$ with respect to C.

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Solⁿ: $P_{B \leftarrow C}$

$$P_{B \leftarrow C} = \left[[1+x]_B \; [x+x^2]_B \; [1+x^2]_B \right]$$

$$B = \{1, x, x^2\}$$

$$[1+x]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [x+x^2]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$[1+x^2]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \checkmark$$

$$P_{C \leftarrow B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \checkmark$$

$$\left(P_{B \leftarrow C}\right)^{-1} \overset{?}{=} P_{C \leftarrow B} \checkmark$$

[can you prove this in general?]

$$[P(x)]_C = P_{C \leftarrow B} [P(x)]_B$$

$$= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

Prob: In $M_{2\times 2}$ let B be the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ and let Y be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrices

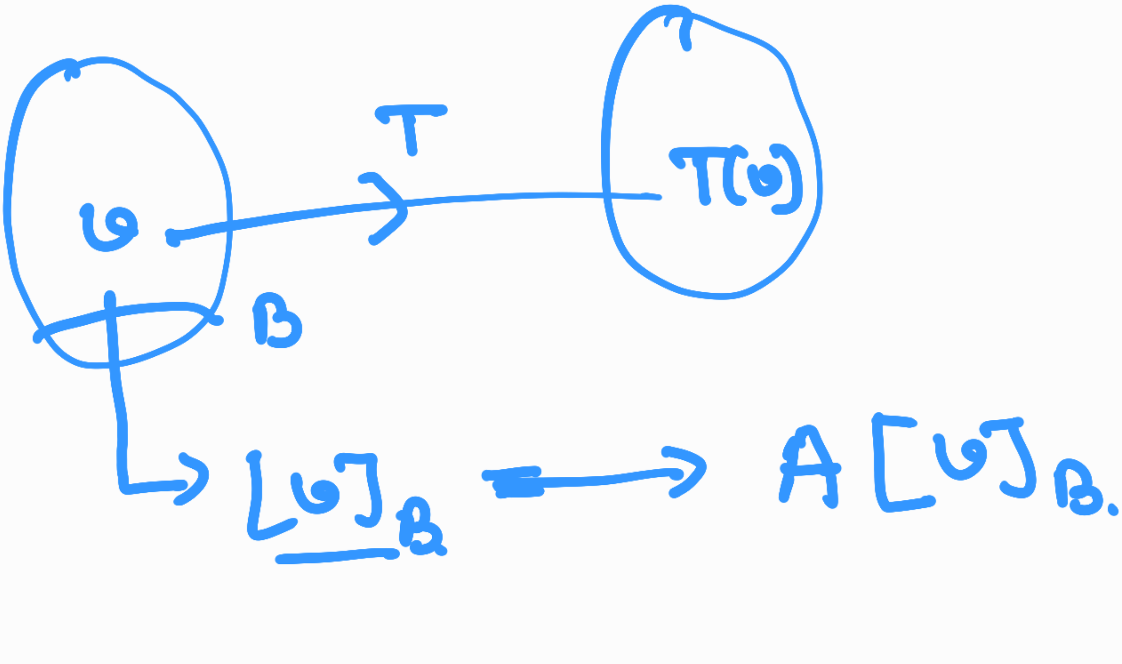
$$P_{Y \leftarrow B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change-of-basis matrix $\rightarrow [\quad]$

$B \longrightarrow C.$

(✓) (✓)

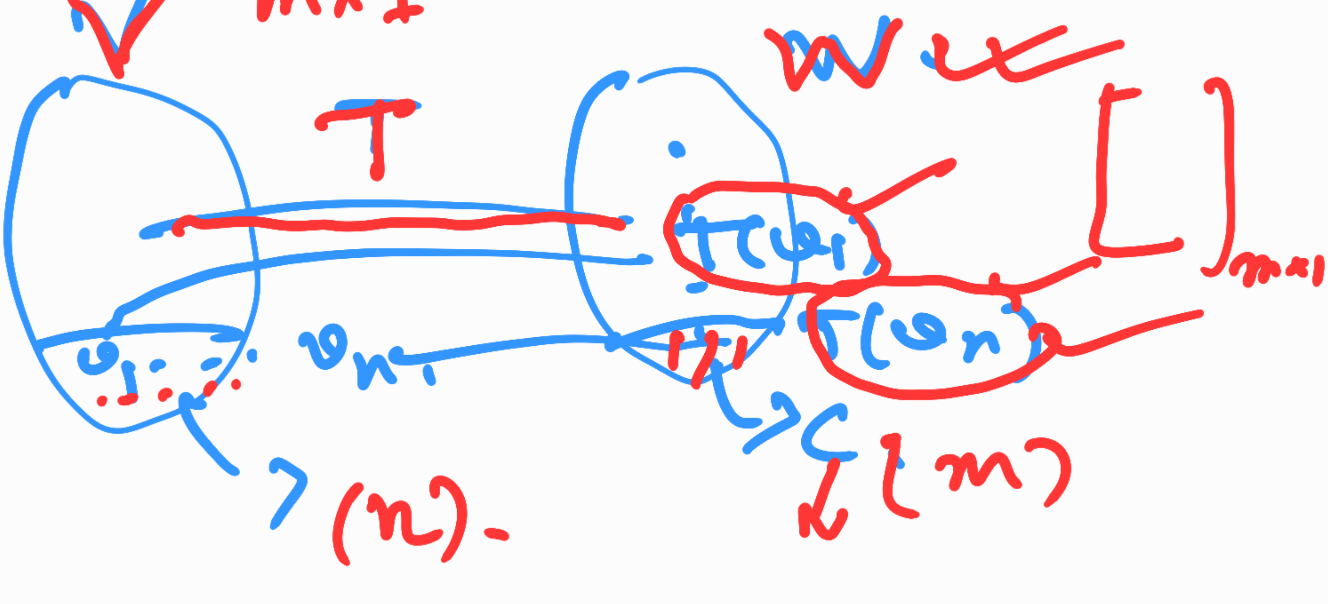
Any linear transformation can be represented as a matrix. ✓.



Let V and W be two finite-dimensional vector spaces with bases B and C, respectively where $B = \{v_1, v_2, \ldots, v_n\}$. If $T: V \to W$ be a linear transformation, then the matrix $(m \times n)$ A is defined by

$$A = \left[\underbrace{(T(v_1))_C}_{m \times 1} \quad (T(v_2))_C \quad \ldots \quad T(v_n)_C \right]$$

$m \times 1$ n



$$A = \begin{bmatrix} [T(v_1)]_{m\times 1} & [T(v_2)]_{m\times 1} & \cdots & [T(v_n)]_{m\times 1} \end{bmatrix}_{m\times n}$$

$$A[v]_B = [T[v]]_C$$

$$\begin{tikzcd} V \ni v \arrow[r, "T"] \arrow[d] & T(v) \in W \arrow[d] \\ {[v]_B} \arrow[r] & A[v]_B = [T(v)]_C \end{tikzcd}$$

$A \rightarrow m\times n$ } Linear transf. from a vector space of dimension n to a vector space of dimension m.

Let : $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

Let $B = \{e_1, e_2, e_3\}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$C = \{e_2, e_1\}$ be bases $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively

Find the matrix, map T with respect to B and C.

$$\begin{array}{ccc} e_1 , & e_2 & e_3 \\ \downarrow & | & | \\ T(e_1) & T(e_2) & T(e_3) \end{array} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \mathbb{R}^2$$

$$T(e_2) = T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

$$T(e_3) = \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix}.$$

$$\begin{bmatrix}1\\1\end{bmatrix} = 1\begin{pmatrix}0\\1\end{pmatrix} + 1\begin{pmatrix}1\\0\end{pmatrix}$$

e_1, e_2

$\underline{C = \{e_2, e_1\}}$

$$\begin{bmatrix}1\\1\end{bmatrix} = 1e_2 + 1e_1.$$

$$= 1\begin{pmatrix}0\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{bmatrix}-2\\1\end{bmatrix} = 1e_2 - 2e_1$$

$$-3\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{bmatrix}0\\-3\end{bmatrix} = -3e_2 + 0e_1.$$

$$[T(e_1)]_C = \begin{bmatrix}1\\1\end{bmatrix} \qquad [T(e_3)]_C = \begin{bmatrix}-3\\0\end{bmatrix}$$

$$[T(e_2)]_C = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$A = \left[[T(e_1)]_C \; [T(e_2)]_C \; [T(e_3)]_C \right]$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \checkmark$$

??

Prob: Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be a linear transformation defined by,

$$T(p(x)) = p(2x - 1)$$

Find the matrix representation of T with respect to

$E = \{1, x, x^2\}$.

$E = \{1, x, x^2\}$

$T(1) = 1$

$T(x) = 2x - 1$

$T(x^2) = (2x-1)^2$

$= 1 - 4x + 4x^2$

$D : P_3 \longrightarrow P_2$

$D(\underline{p(x)}) = p'(x)$.

$\left\{ \begin{array}{l} a + bx \\ + cx^2 \\ + dx^3 \end{array} \right.$

let $B = \{1, x, x^2, x^3\}$

$C = \{1, x, x^2\}$,

Find the matrix rep. of linear map D