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MCS Assignment 2

1 We need to show that the transformation $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

$T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$

<u>let's take $x_1 = 0, x_2 = 1$</u>

$\therefore T(0,1) = (0-2, 3) = (-2, 3)$

<u>let's take $x_1 = 0, x_2 = -1$</u>

$\therefore T(0,-1) = (0+2, 3) = (2, 3)$

$\therefore T(0,1) + T(0,-1) = (-2,3) + (2,3) = (0,6)$

<u>NOW,</u>

$$T((0,1)+(0,-1)) = T(0+0, 1-1)$$
$$= T(0,0)$$
$$= 4.0 - 2.0, 3.0$$
$$= (0,0)$$

$\therefore$ ~~$T(x_1, x_2) =$~~

$$\therefore \boxed{T(0,1) + T(0,-1) \neq T((0,1)+(0,-1))}$$

<u>Hence</u>, the given transformation is not linear.

$\boxed{2}$ Let $T: R^3 \to R^2$ projects each point (x_1, x_2, x_3) onto the $x_1 x_2$ plane. We have to find the matrix representation of the Linear transformation T.

The matrix representation for T will be

$$[T] = [T(\vec{e_1}), T(\vec{e_2}), T(\vec{e_3})]$$

Where $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are Standard basis vectors of R^3

$$[T] = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Such that $[T] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\{T: R^3 \to R^2\}$

$$T(\vec{e_1}) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix}$$

Similarly $T(\vec{e_2}) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$

Similarly $T(\vec{e_3}) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

Now, $T(\vec{e_1}) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$T(\vec{e_2}) = T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e_3}) = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore [T] = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

3 Find kernel and range.

i) Differential operator $D: P^3 \to P^2$ defined by $D(p(x)) = \frac{d}{dx} p(x)$, p^i is a polynomial of degree i.

Soln Since $D(a+bx+cx^2+dx^3) = b+2cx+3dx^2$, we have

$$\ker(D) = \{a+bx+cx^2+dx^3 : D(a+bx+cx^2+dx^3)=0\}$$
$$= \{a+bx+cx^2+dx^3 : b+2cx+3dx^2=0\}$$

But $b+2cx+3dx^2=0$ if and only if $b=2c=3d=0$, which implies that $b=c=d=0$.

Therefore, $\ker(D) = \{a+bx+cx^2+dx^3 : b=c=d=0\}$
$$= \{a : a \text{ in } \mathbb{R}\}$$

In other words, the kernel of D is the set of constant polynomials.

The range of D is all of P_2, since every polynomial in P_2 is the image under D (i.e. the derivative) of some polynomial in P_3. To be specific, if $a+bx+cx^2$ is in P_2, then

$$a+bx+cx^2 = D\left(ax + \left(\tfrac{b}{2}\right)x^2 + \left(\tfrac{c}{3}\right)x^3\right)$$

ii) $S: P^1 \to \mathbb{R}$, $S(p(x)) = \int_0^1 p(x)\,dx$.

Soln In detail we have $S(a+bx) = \int_0^1 (a+bx)\,dx$

$$= \left[ax + \frac{bx^2}{2}\right]_0^1$$
$$= \left(a + b/2\right) - 0$$
$$= a + b/2$$

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Therefore, $\ker(S) = \{a+bx : S(a+bx) = 0\}$

$$= \{a+bx : a + b/2 = 0\}$$

$$= \{a+bx : a = -b/2\}$$

$$= \{-\frac{b}{2} + bx\}$$

Geometrically, $\ker(S)$ consists of all those linear polynomials whose graphs have the property that the area between the line and the X-axis is equally distributed above and below the axis on the interval $[0,1]$

The range of S is R, since every real number can be obtained as the image under S of some polynomial in P_1. For example, if a is an arbitrary real number then, $\int_0^1 a\,dx = [ax]_0^1 = a - 0 = a$

So, $\boxed{a = S(a)}$

iii) $T : M_{22} \to M_{22}$, $T(A) = A^T$, M_{22} is a 2×2 matrix and T is the transpose of the matrix.

Soln We see that $\ker(T) = \{A \text{ in } M_{22} : T(A) = 0\}$

$$= \{A \text{ in } M_{22} : A^T = 0\}$$

But if $A^T = 0$, then $A = (A^T)^T = 0^T = 0$.

It follows $\ker(T) = \{0\}$

Since, for any matrix A in M_{22}, we have $A = (A^T)^T = T(A^T)$ (and A^T is in M_{22})

We deduce that $\text{range}(T) = M_{22}$.

4

Let $B = [v_1, v_2, \dots, v_n]$ be a basis of vector space V.

and let $u_1, u_2, \dots u_n$ be vectors in V.

We need to show that $[u_1, u_2 \dots u_n]$ is linearly Independent in V iff $[[u_1]_B, [u_2]_B, \dots [u_n]_B]$ is linearly independent in R^n.

Soln

Forward implication ($\rightarrow$):

Assume that $\{u_1, u_2, \dots u_n\}$ are linearly independent.

Let $\alpha_1 [u_1]_B + \dots + \alpha_n [u_n]_B = 0$ —— ①

If we can show that $\alpha_i = 0 \; \forall \; i \in (1, \dots n)$

then the implication will be proved

We know, $[u+v]_B = [u]_B + [v]_B$

$[cu]_B = c[u]_B$

So, now ① becomes

$[\alpha_1 u_1 + \dots + \alpha_n u_n]_B = 0$

Also, $[v]_B = 0 \Leftrightarrow v = 0$

i.e if coordinate vector v with respect to B is 0, then vector itself is 0.

$\Rightarrow \alpha_1 u_1 + \dots + \alpha_n u_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

(because $\{u_1, u_2, \dots u_n\}$ are linearly independent)

Hence, it is proved that $[[u_1]_B, \dots [u_n]_B]$ is linearly independent.

Backward implication ($\Leftarrow$):

Assume $[[u_1]_B, [u_2]_B, \ldots [u_n]_B]$ are linearly independent in R^n.

~~Let's~~ Consider $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ —— (1)

If we can show that $\alpha_i = 0 \; \forall \; i \in (1, \ldots, n)$ then we will prove the implication

We know, $[V]_B = 0 \Leftrightarrow V = 0$

$\therefore$ (1) becomes $[\alpha_1 u_1 + \ldots + \alpha_n u_n]_B = 0$ —— (2)

We know, $[u+v]_B = [u]_B + [v]_B$

$\therefore$ (2) becomes

$$\Rightarrow [\alpha_1 u_1]_B + \ldots + [\alpha_n u_n]_B = 0$$

$$\Rightarrow \alpha_1 [u_1]_B + \ldots + \alpha_n [u_n]_B = 0 \quad \left\{ \because [cu]_B = c[u]_B \right\}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

[because $[[u_1]_B, [u_2]_B, \ldots [u_n]_B]$ are linearly independent]

Hence, the backward implication is also proved

$\therefore$ We have proved the given statement

$\boxed{5}$ Given

$$P_{C\leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Basis $B = [x, 1+x, 1-x+x^2]$

Let $C = \{v_1, v_2, v_3\}$ be the basis.

Now, $P_{C\leftarrow B} = [[u_1]_C \, [u_2]_C \, [u_3]_C]$

$[u_1]_C, [u_2]_C, [u_3]_C$ are co-ordinate vectors of the vectors in B w.r.t C, and are also the columns of 3×3 $P_{C\leftarrow B}$ matrix.

So, $[x]_C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\Rightarrow x = v_1 - v_3$ —— ①

$[1+x]_C = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow 1+x = 2v_2 + v_3$ —— ②

$[1-x+x^2]_C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow 1-x+x^2 = v_2 + v_3$ —— ③

Now, ② − ③

$(2v_2+v_3) - (v_2+v_3) = (1+x) - (1-x+x^2)$

$\Rightarrow v_2 = \cancel{1} + x - \cancel{1} + x - x^2$

$\Rightarrow v_2 = 2x - x^2$

NOW, putting value of V_2 in ②

$$1+x = 2(2x-x^2)+V_3$$

$$\Rightarrow 1+x = 4x-2x^2+V_3$$

$$\Rightarrow V_3 = 2x^2-3x+1$$

NOW, putting value of V_3 in ①

$$x = V_1-(2x^2-3x+1)$$

$$\Rightarrow V_1 = 2x^2-2x+1$$

$$\therefore \boxed{C = \{2x^2-2x+1,\ 2x-x^2,\ 2x^2-3x+1\}}$$

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6. A linear transformation $T: V \to V$ is given. If possible find a basis C for V such that matrix $[T]_C$ of T with respect to C is diagonal.

(a) $T: R^2 \to R^2$ defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4b \\ a+5b \end{bmatrix}$

Soln First, let's try to find the matrix representation of T with respect to standard basis.

$B = \{e_1, e_2\}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then matrix $[T]_B = \left[[T(e_1)]_B \ \ [T(e_2)]_B \right]$

$$= \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}_B \begin{bmatrix} -4 \\ 5 \end{bmatrix}_B \right]$$

$$= \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix} \rightarrow$$ (Matrix representation of T w.r.t standard basis $B = \{e_1, e_2\}$)

Now, let's try to find a basis C for $\mathbb{R}^2$ such that the matrix $[T]_C$ is a diagonal matrix.

A matrix becomes diagonal when we work in an eigen basis i.e. basis made up of eigen vectors. So, we need to find the eigen vectors of $[T]_B$ and that will be the basis 'C' which we want.

$$\text{Let } A = [T]_B$$

Find eigen values: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} -\lambda & -4 \\ 1 & 5-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow -\lambda(5-\lambda) + 4 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \lambda = 1 \text{ and } \lambda = 4$$

As there are 2 eigen values, A can be diagonalizable.

Find eigen vectors:

1) $\lambda = 1$: $(A - \lambda I)X = 0$

$$\Rightarrow (A - I)X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From above $x + 4y = 0$

$$\Rightarrow x = -4y$$

Let $y = 1$ then $x = -4$

$\therefore$ Eigen vector for $\lambda = 1$ is $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

ii) $\underline{\lambda = 4}$: $(A - \lambda I)X = 0$

$$\Rightarrow (A - 4I)X = 0$$

$$\Rightarrow \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

~~So~~ From above $x + y = 0$

$$\Rightarrow x = -y$$

Now, let $y = 1$, then $x = -1$

Eigen vector for $\lambda = 4$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ The basis $C = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and

$[T]_C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ which is diagonal.

(b) $T: P_2 \to P_2$ defined by $T(P(x)) = P(3x+2)$

Soln First, let's try to find the matrix representation of T with respect to standard basis $B = \{1, x, x^2\}$

Then, $T(1) = 1$; $T(x) = 3x + 2$

$$T(x^2) = (3x+2)^2$$
$$= 9x^2 + 12x + 4$$

Then, matrix $[T]_B = [[T(1)]_B \, [T(x)]_B \, [T(x^2)]_B]$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix} \longrightarrow$$ Matrix representation of T w.r.t. standard basis $B = \{1, x, x^2\}$

Now, let's try to find a basis C such that the matrix $[T]_c$ is diagonal.

Find eigen values: $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(3-\lambda)(9-\lambda)-0] = 0$

$\Rightarrow (1-\lambda)[\lambda^2 - 12\lambda + 27] = 0$

$\Rightarrow -\lambda^3 + 12\lambda^2 - 27\lambda + \lambda^2 - 12\lambda + 27 = 0$

$\Rightarrow -\lambda^3 + 13\lambda^2 - 39\lambda + 27 = 0$

$\Rightarrow \lambda^3 - 13\lambda^2 + 39\lambda - 27 = 0$

$\Rightarrow (\lambda-1)(\lambda^2 - 12\lambda + 27) = 0$

$\Rightarrow (\lambda-1)(\lambda-3)(\lambda-9) = 0$

$\therefore \lambda = 1, \lambda = 3, \lambda = 9$

As there are 3 eigen values, A can be diagonalizable.

Find eigen vectors

1) $\lambda = 1$: $(A - \lambda I)X = 0$

$\Rightarrow (A - I)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8z = 0 \Rightarrow z = 0$

$2y + 12z = 0 \Rightarrow y = 0$

x can be any value.

Let $x = 1$, then the eigen vector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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ii) $\underline{\lambda = 3}$: $(A - \lambda I)X = 0$

$\Rightarrow (A - 3I)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 12 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 6z = 0 \Rightarrow z = 0$

y can be any value

$\therefore -2x + 2y + 4z = 0$

$\Rightarrow x = y$

Let $x = 1$, then $y = 1$

$\therefore$ Eigen vector for $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

iii) $\underline{\lambda = 9}$: $(A - \lambda I)X = 0$

$\Rightarrow (A - 9I)X = 0$

$$\Rightarrow \begin{bmatrix} -8 & 2 & 4 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -6y + 12z = 0$

$\Rightarrow 6y = 12z$

$\Rightarrow y = 2z$

Also, $-8x + 2y + 4z = 0$

$\Rightarrow -8x + 4z + 4z = 0$

$\Rightarrow 8x = 8z$

$\Rightarrow x = z$

Let $z = 1$, then $x = 1$ and $y = 2$

Eigen vector for $\lambda = 9$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\therefore \text{The basis } C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$= \{ 1, 1+x, 1+2x+x^2 \}$$

$$\text{and } [T]_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$ Which is diagonal

8 Find

w.r.

M_{2}

Soln V

T

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(a) To show that $B = \{x^2, x, 1\}$ is a basis for vector space V (here V is P_2, set of all polynomials of x of degree at most 2), we have to show that

i) B is linearly independent

ii) B spans V.

i) Suppose c_0, c_1, c_2 are scalars such that

$$c_0 x^2 + c_1 x + c_2 = 0$$

putting $x=0$, we get $c_2 = 0$

This leads to $c_0 x^2 + c_1 x = 0$

Taking derivative w.r.t. x

$$\frac{d}{dx}(c_0 x^2 + c_1 x) = 0$$

$$\Rightarrow 2c_0 x + c_1 = 0$$

putting $x=0$, we get $c_1 = 0$

Finally, we get $c_0 = c_1 = c_2 = 0$

So, $c_0 x^2 + c_1 x + c_2 = 0$

$$\Rightarrow c_0 = c_1 = c_2 = 0$$

$\therefore B = \{x^2, x, 1\}$ is linearly independent

ii) B also spans P_2, spans every polynomial in P_2

B is a linear combination of 0th, 1st, 2nd powers of x.

$\therefore$ B is a basis for P_2, more precisely, it is ~~a~~ the standard basis

(b) Showing that T is linear transformation

Given $T(x^2) = x+m$; $T : \underset{(V)}{P^2} \to \underset{(W)}{P^2}$

$T(x) = (m-1)x$

$T(1) = x^2+m$

and Basis $B = \{x^2, x, 1\}$

Assuming B is basis for both V and W.

Using the following theorem, let's prove T is linear.

Theorem: Let V, W be vector spaces over ~~R~~ F

Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for V.

and $A = \{w_1, w_2, \ldots, w_n\}$ be any subset of W

Then a transformation:

$T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \cdots + \alpha_n w_n$

is linear

Let $p(x) = a_1x^2 + b_1x + c_1 \in V$

$q(x) = a_2x^2 + b_2x + c_2 \in V$

Then $T(p(x)+q(x))$

$= T(a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2)$

$= T((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2))$

$= (a_1+a_2)(x+m) + (b_1+b_2)(m-1)(x) + (c_1+c_2)(x^2+m)$ Obtained using the above thm

$= (c_1+c_2)(x^2) + (a_1+a_2+b_1m+b_2m-b_1-b_2)(x) + (a_1m+a_2m+c_1m+c_2m)$

Now, $T(p(x)) = T(a_1x^2 + b_1x + c_1)$

$$= a_1(x+m) + b_1(m-1)x + c_1x^2 + c_1m$$
$$= a_1x + a_1m + b_1(m-1)x + c_1x^2 + c_1m$$
$$= c_1x^2 + (a_1 + b_1m - b_1)x + (a_1m + c_1m)$$

Similarly $T(d(x)) = c_2x^2 + (a_2 + b_2m - b_2)x + (a_2m + c_2m)$

Then $T(p(x)) + T(d(x))$

$$= (c_1+c_2)(x^2) + (a_1 + a_2 + b_1m - b_1 + b_2m - b_2)(x)$$
$$+ (a_1m + a_2m + c_1m + c_2m)$$

$$\therefore T(p(x) + d(x)) = T(p(x)) + T(d(x))$$

NOW, let $\alpha \in \mathbb{R}$ then

$$T(\alpha(p(x))) = T(a_1\alpha x^2 + b_1\alpha x + c_1\alpha)$$
$$= a_1\alpha(x+m) + b_1\alpha(m-1)x + c_1\alpha(x^2+m)$$
$$= c_1\alpha x^2 + (a_1\alpha + b_1\alpha m - b_1\alpha)x + (a_1\alpha m + c_1\alpha m)$$

and $\alpha T(p(x)) = \alpha[c_1x^2 + (a_1 + b_1m - b_1)x + (a_1m + c_1m)]$

$$= c_1\alpha x^2 + (a_1\alpha + b_1\alpha m - b_1\alpha)$$
$$+ (a_1\alpha m + c_1\alpha m)$$

$$\therefore \boxed{T(\alpha p(x)) = \alpha T(p(x))}$$

Hence T is a linear transformation.

(c) Find the matrix representation of T relative to the given basis B.

Soln Given, $T(x^2) = x + m$ and Basis $B = \{x^2, x, 1\}$

$$T(x) = (m-1)x$$

$$T(1) = x^2 + m$$

The matrix representation of T relative to B is

$$[T]_B = \left[[T(x^2)]_B \ [T(x)]_B \ [T(1)]_B \right]$$

$$= \left[[(x+m)]_B \ [(m-1)x]_B \ [(x^2+m)]_B \right]$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}$$

Note: $[(x+m)]_B \rightarrow a(x^2) + b(x) + c(1) = x + m$

Equating both sides,

$\Rightarrow a = 0, b = 1, c = m$

$\therefore [(x+m)]_B = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$

$[(m-1)x]_B \rightarrow a(x^2) + b(x) + c(1) = (m-1)x$

Equating both sides

$\Rightarrow a = 0, b = (m-1), c = 0$

$\therefore [(m-1)x]_B = \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$

$[(x^2+m)]_B \rightarrow a(x^2) + b(x) + c(1) = x^2 + m$

Equating both sides

$\Rightarrow a = 1, b = 0, c = m$

$\therefore [x^2+m]_B = \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$

(d) Find kernel(T) for all values of m:

Given $B=\{x^2, x, 1\}$ is basis for V then kernel is set of all vectors (polynomials) $ax^2+bx+c(1)$ such that:

$$aT(x^2)+bT(x)+cT(1)=0$$
$$\Rightarrow a(x+m)+b(m-1)(x)+c(x^2+m)=0$$
$$\Rightarrow cx^2+(bm-b+a)x+(am+cm)=0$$

Equating on both sides

$c=0$ — ①

$bm-b+a=0$ — ②

$am+cm=0$ — ③

~~Eq-③: $am+cm=0$ — ④, $bm-b+a=0$ — ②, $am+cm=0$ — ③~~

Eq-③: $am+cm=0$

$\Rightarrow am=0$ {From ①, $c=0$}

$\Rightarrow a=0$ or $m=0$

Case 1

Let $a=0$

then eq-②

$bm-b=0$

$\Rightarrow b(m-1)=0$

$b=0$ or $m-1=0$

$\Downarrow$

$m=1$

$\therefore a=0$, $b=0$, $c=0$, $m=$ any value | $a=0$, $m=1$, $b=$ any value, $c=0$

Case 2

Let $m=0$

then eq ②

$-b+a=0$

$\Rightarrow a=b$

$\therefore a=b$

$c=0$

$m=0$

$\therefore$ Kernel (T) is

if $m=0 \longrightarrow \{ax^2+bx+c \mid a=b \text{ and } c=0\}$

$= \{ax^2+ax \mid a \in \mathbb{R}\}$

$m=1 \longrightarrow \{ax^2+bx+c \mid a=0, c=0, b = \text{any value}\}$

$= \{bx \mid b \in \mathbb{R}\}$

$m = \text{other values} \longrightarrow \{ax^2+bx+c \mid a=0, b=0, c=0\}$

$= \{0\}$

(e) Find Image(T) for all values of m.

Let $ax^2+bx+c \in$ Image(T) which means there exists a polynomial $f(x) = px^2+dx+r$ such that $T(f(x)) = ax^2+bx+c$

Using matrix representation of T:

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} p \\ d \\ r \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{--- (1)}$$

Two cases possible where $|A|=0$ and $|A| \neq 0$

$\boxed{\text{case-1}}$ $|A|=0$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} = 0$$

$\Rightarrow 1(-m(m-1)) = 0$

$\Rightarrow m(m-1) = 0$

$\Rightarrow \underline{m=0}$ or $\underline{m=1}$

If $m=0$ then $\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Eq ① can be written as

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Rightarrow \boxed{r=a}; \boxed{p-q=b}; \boxed{0=c}$

Hence all the polynomials, $ax^2+bx+c \in Image(T)$

such that $a=r$ and $b=p-q$ and $c=0$

has pre-image in domain V.

$\therefore Image(T) = \{ax^2+bx+c \mid c=0\}$

$= \{ax^2+bx \mid a, b \in R\}$

If $m=1$ then $\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Eq ① can be written as

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Rightarrow r=a; \ p=b; \ p+r=c$

$\Rightarrow \cancel{c+a}\ c=a+b$

$\therefore Image(T) = \{ax^2+bx+c \mid c=a+b\}$

$= \{ax^2+bx+(a+b) \mid a, b, c \in R\}$

$\boxed{\text{case-2}}$ $|A| \neq 0$

then $\begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} \neq 0$

$\Rightarrow -m(m-1) \neq 0$

$\Rightarrow m \neq 0$ and $m \neq 1$

We can take any other value of m, and the

eq: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has

valid solution i.e has pre-image in domain

$\therefore \text{Image}(T) = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$

$= V$

Finally Image(T) is

if $m=0 \longrightarrow \{ax^2+bx \mid a,b \in \mathbb{R}\}$

$m=1 \rightarrow \{ax^2+bx+(a+b) \mid a,b \in \mathbb{R}\}$

m ~~is~~ other values $\rightarrow \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$

$= V.$

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8. Find the co-ordinate vector of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ w.r.t. Basis $B = [E_{22}, E_{21}, E_{12}, E_{11}]$ & $M_{2\times 2}$ matrices.

Soln We know that

$$E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The co-ordenate vector of $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ which satisfies

$$aE_{22} + bE_{21} + cE_{12} + dE_{11} = A$$

$$\Rightarrow a\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Hence, $a = 4, b = 3, c = 2, d = 1$

$\therefore$ Co-ordinate vector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ w.r.t. Basis $B = [E_{22}, E_{21}, E_{12}, E_{11}]$ is $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

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