Eigen Values and Eigen Vectors of $n \times n$ matrices

The eigen values of a square matrix A are precisely the solutions λ of the equation.

$$\det(A - \lambda I) = 0$$

$\lambda \to$ eigen value.

$$\boxed{Ax = \lambda x}$$

$\to$ eigen vectors.

eigenvalue

characteristic equation.

Find the eigen values and the eigen vectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

The characteristic equation is given by,

$$\det(A - \lambda I_3)$$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix}$$

$$\boxed{= -\lambda^3 + 4\lambda^2 - 5\lambda + 2}$$

$= 0$

algebric multiplicity 2 ← 1, 2 → 1

$$\Rightarrow \boxed{\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2}$$

Case I

Case II

$$\boxed{Ax = \lambda x}$$

<u>Case I</u>:

$$Ax = x$$

$$\boxed{\lambda = 1}$$

$$\boxed{(A - I)X = 0}$$

augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0. \end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{If } x_3 = t \\ &x_1 = t,\ x_2 = t \end{aligned}\right\}$$

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\hookrightarrow \dim E_1 \longrightarrow$ geometric mult.

$$= \underline{\text{span}} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$\lambda_3 = 2.$

$$E_2 = \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right)$$

$\hookrightarrow \dim E_2$

$\lambda \longrightarrow E_\lambda$ [the set of eigen vectors $\to$ which corresponding to λ will satisfy eq^n $Ax = \lambda x$]

$\boxed{\text{Eigen-Space}}$

vector space $\longrightarrow$

$\boxed{\dim\{E_\lambda\}}$ $\longrightarrow$ the number of vectors in the basis of E_λ.

$\rightarrow$ geometric multiplicity of λ.

Find the eigen values and the corresponding eigen spaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\lambda = \underline{\overbrace{0, 0}^{2}}, -2.$$

$$E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \rightarrow 2.$$

$$E_{-2} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right) \rightarrow 1.$$

a) The eigen values of a triangular matrix are the entries on its main diagonal

b) A square matrix A is invertible iff 0 is not an eigenvalue of A.

— x ——— x ——— x —

Let A be a square matrix with eigenvalue λ and corresponding eigenvector x.

~~(a)~~ For any positive integer n, λ^n is an.

eigen value of A^n with corresponding eigen vector x.

~~(b)~~ If A is invertible, then $1/\lambda$ is an eigen value of A^{-1} with corresponding eigen vector x.

Sol^n: $\boxed{A^n x = \lambda^n x}$.

We will prove by induction.

For $n=1$, $\boxed{Ax = \lambda x}$ → true?

Let us assume that this is true for $n=k$

$$A^k x = \lambda^k x \longrightarrow \text{holds} \longrightarrow (1)$$

we have to show that this is true for $(n=k+1)$

$$A^{k+1}x = A(A^k x)$$
$$= A(\lambda^k x)$$
$$= \lambda^k (Ax) = \lambda^k(\lambda x)$$
$$= \lambda^{k+1} x.$$

Thus $A^n x = \lambda^n x$ for arbitrary n.

(b) $Ax = \lambda x.$

Multiply both the sides by A^{-1} (A^{-1} exists, since A is invertible)

$$A^{-1}(Ax) = \lambda(A^{-1}x)$$

$$\Rightarrow (A^{-1}A)x = \lambda(A^{-1}x)$$

$$\Rightarrow Ix = \lambda(A^{-1}x)$$

$$\Rightarrow x = \lambda(A^{-1}x)$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x.$$

So $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} w.r.t. e.v x.

Compute $\longrightarrow$ $\underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}}_{A}^{10} \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{x}$

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $\boxed{x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$ $\longrightarrow A^{10}x$

$\longrightarrow$ first find out e.v. and eigenvectors of A.

$\lambda_1 = -1 \longrightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2 \longrightarrow v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

v_1 and v_2 forms a bases of $\mathbb{R}^2$.

$$c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + 2c_2 &= 0 \\ \Rightarrow c_1 = c_2 &= 0 \end{aligned}\right\} \quad \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$$

$$\begin{bmatrix}x\\y\end{bmatrix} = d_1\begin{bmatrix}1\\-1\end{bmatrix} + d_2\begin{bmatrix}1\\2\end{bmatrix}$$

$$\left\{ \overset{v_1}{\begin{bmatrix}1\\-1\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}1\\2\end{bmatrix}} \right\}$$ forms a basis of $\mathbb{R}^2$

$$x = 3v_1 + 2v_2$$

$$A^{10}x = A^{10}(\underline{3v_1 + 2v_2}).$$

$$= 3(A^{10}v_1) + 2(A^{10}v_2).$$

$$= 3(\lambda_1)^{10}v_1 + 2(\lambda_2)^{10}v_2$$

$$= 3(-1)^{10}\begin{bmatrix}1\\-1\end{bmatrix} + 2(2)^{10}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}2051\\4093\end{bmatrix}$$

<u>Thm</u>: Suppose the $n \times n$ matrix A has eigenvectors $v_1, v_2, \ldots, v_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If x is a

vector in $\mathbb{R}^n$ that can be expressed as linear combination of these eigen vector i.e

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

then for any integer k

$$A^k x = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_m \lambda_m^k v_m$$

Thm: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be $\boxed{\text{distinct}}$ eigenvalues of A

with corresponding eigenvectors $v_1, v_2, \ldots \ldots v_m$. Then $v_1, v_2, \ldots, v_m$ are linearly independent.

Proof: Contradiction!

Let us assume that the set of eigenvectors are linearly dependent

L.D

L.I

$$v_1, v_2, \ldots\ldots v_k, v_{k+1}, \ldots\ldots, v_n.$$

$$v_{k+1} = c_1 v_1 + c_2 v_2 \ldots\ldots + c_k v_k. \quad \rightarrow (1)$$

$$\boxed{\lambda_{k+1} v_{k+1}}$$

$$= c_1 \lambda_{k+1} v_1 + c_2 \lambda_{k+1} v_2 + \cdots + c_k \lambda_{k+1} v_k \longrightarrow (2)$$

$$\boxed{\lambda_{k+1} v_{k+1}} = A v_{k+1}$$

$$= A(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k)$$

$$= c_1 (A v_1) + c_2 (A v_2) + \cdots + c_k (A v_k).$$

$$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k \longrightarrow (3)$$

$-(2)+(3)$

$$0 = c_1(\lambda_1 - \lambda_{k+1}) + c_2(\lambda_2 - \lambda_{k+1}) + \cdots + c_k(\lambda_k - \lambda_{k+1})$$

c_i ?

$\lambda_i - \lambda_k$? $\neq 0$

$c_i = 0$?

$$v_{k+1} = 0v_1 + 0v_2 + \cdots + 0v_n$$

$\longrightarrow$? contradiction

$\{v_1, v_2 \cdots, v_m\}$ are L.D

$\rightarrow$ X, so $\{v_1, v_2 \cdots v_m\}$

$\longrightarrow$ L.I.

Similarity and Diagonalization

Similar Matrices:

Def^n: Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. We will then say A is similar to B. ($A \sim B$)

If $A \sim B$

$$P^{-1}AP = B. \Rightarrow (PP^{-1})AP = PB$$

$$\boxed{AP = PB.}$$

Thm: Let A, B and C are three $n \times n$ matrices

a) $A \sim A$ (reflexive)

b) If $A \sim B$, then $B \sim A$ → (Sy..

c) If $A \sim B$, $B \sim C$ then $A \sim C$. (tra

Proof: (i) $A = I^{-1}AI$ → $A \sim A$

$\sim$ is reflexive

(iii) $A \sim B$

$\Rightarrow P^{-1}AP = B$ for some invertible matrix P

$\Rightarrow (PP^{-1})A(PP^{-1}) = PBP^{-1}$

$\Rightarrow IAI = PBP^{-1}$

$\Rightarrow A = PBP^{-1}$

$= (P^{-1})^{-1}BP^{-1}$

If you call $P^{-1} = Q$.

$\rightarrow$ invertible matrix

$$A = Q^{-1} B Q$$

$\Rightarrow B \sim A.$

$\Rightarrow \sim$ is symmetric.

<u>(c)</u> $A \sim B$

$$P^{-1} A P = B$$

$B \sim C$

$Q^{-1} B Q = C.$

[$R = PQ$ is invertible]

$$\Rightarrow C = Q^{-1} B Q$$

$$= Q^{-1} (P^{-1} A P) Q$$

$$= (Q^{-1} P^{-1}) A (PQ)$$

$$= (PQ)^{-1} A (PQ)$$

$\Rightarrow \sim$ transitive.

$$= R^{-1} A R$$

Thm: Let A and B be $n \times n$ matrices with $A \sim B$, then show that

(i) $\det A = \det B$

(ii) A and B has same characteristic polynomial.

Proof:

(i)

$$A \sim B \qquad B = P^{-1} A P$$

$$\det(B) = \det(P^{-1} A P)$$

$$= \det(P^{-1}) \det(A) \det(P)$$

$$= \frac{1}{\cancel{\det(P)}} \det(A) \cancel{\det(P)}$$

$$= \det(A).$$

(ii) The characteristic polynomial of B

$$\det(B - \lambda I)$$

$$= \det(P^{-1}AP - \lambda I)$$

$$= \det(P^{-1}AP - \lambda(P^{-1}IP))$$

$$= \det(P^{-1}AP - P^{-1}(\lambda I)P)$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det P$$

$$= \frac{1}{\cancel{\det(P)}} \det(A - \lambda I) \cancel{\det(P)}$$

$$= \det(A - \lambda I)$$

Thus, $\boxed{\det(B-\lambda I) = \det(A-\lambda I)}$

Prob:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$

Then $\boxed{A \sim B.}$

$\hookrightarrow P ? = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ &

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Diagonalizable Matrix

Def^n: An $n \times n$ matrix A is $\boxed{\text{diagonalizable}}$ if there is a diagonal matrix D such that A is similar to D. — i.e if there is an invertible $n \times n$ matrix P such that

$$\boxed{P^{-1}AP = D.}$$

$$A \sim \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

diagonalizable.

Thm: Let A be $n \times n$ matrix. Then A is diagonalizable if and only if A has n - linearly independent eigenvectors

$$A \longrightarrow \begin{matrix} \lambda_1 & - & v_1 \\ \vdots & - & v_2 \\ \lambda_n & - & v_n \end{matrix}$$

$\longleftarrow$

A is diagonalization

$$\boxed{P^{-1}AP = D.}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$$

$$\longrightarrow D = \begin{bmatrix} & \\ & \end{bmatrix} ?$$