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Q1.

Assuming field F is to be the set of real numbers R. R is our set of scalars.

(a) Let Q be the set of rational numbers.

i) Addition of 2 rational numbers result in a rational number. If $a, b \in Q$. then $a+b \in Q$.

$\therefore$ closure property under addition satisfied

ii) For $a, b \in Q$. $a+b = b+a$

$\therefore$ commutativity under ~~under~~ addition satisfied

iii) For any $a, b, c \in Q$ $a+(b+c) = (a+b)+c$.

$\therefore$ Associativity of addition satisfied

iv) For all $a \in Q$ $a+0 = a$. and $0 \in Q$.

$\therefore$ Additive identity exists

v) For all $a \in Q$, $\exists\, b \in Q$ such that $a+b=0$ where $b=-a$

$\therefore$ Unique additive inverse exists. $\forall a \in Q$

vi) Consider $\sqrt{2} \in R$. and $1 \in Q$. $\sqrt{2} \cdot 1 = \sqrt{2} \notin Q$

$\therefore Q$ is not closed under scalar multiplication

vii) Let $c \in F$ and $a, b \in Q$ then $c(a+b) = ca+cb$.

$\therefore$ Scalar multiplication is distributive w.r.t. addition of Q.

viii) Let $a, b \in F$. be two scalars and let $c \in Q$ be a rational number.

$\therefore (a+b)c = ac+bc$

$\therefore$ Scalar multiplication is distributive w.r.t field addition

ix) For 2 scalars $a, b \in \mathbb{R}$ and $c \in \mathbb{Q}$: $(ab)c = a(bc)$

$\therefore$ Associative multiplication property is satisfied

x) $\forall a \in \mathbb{Q}$. $1.a = a$ where $1 \in \mathbb{R}$.

$\therefore$ Scalar identity exists.

Result: The set of rational numbers with usual addition and multiplication is not a vector space.

[Violated axioms: (vi) closure under scalar multiplication]

(b) A is a skew symmetric matrix if $a_{ij} = -a_{ji}$ $[i \neq j]$

i) Let $X = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ $Y = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$ be two skew-symmetric matrix.

$$X+Y = \begin{bmatrix} a+d & b+e \\ -b-e & c+f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ -(b+e) & (c+f) \end{bmatrix}$$

$\therefore$ $X+Y$ is also a skew-symmetric matrix.

This argument can be extended to all $n \times n$ skew symmetric matrix.

$\therefore$ closure under addition is satisfied

ii) Let $X = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ $Y = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$

Now $X+Y = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix}$ $Y+X = \begin{bmatrix} d+a & e+b \\ -e-b & f+c \end{bmatrix}$

$$= \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix}$$

$= X+Y$

$\therefore$ Addition is commutative.

iii) for any X, Y, Z ($n\times n$ skew symmetric matrices)

$(X+Y)+Z = X+(Y+Z)$

$\therefore$ Addition is associative

iv) For any skew symmetric matrix X, $X+0=X$, where 0 is $n\times n$ zero matrix.

$\therefore$ Additive identity exists.

v) For any skew symmetric matrix, $X+(-X)=0$ where 0 is zero matrix.

$\therefore$ Unique additive inverse exists for any $(n\times n)$ skew symmetric matrix.

vi) Let any scalar $c\in\mathbb{R}$ and a skew-symmetric matrix

$$X = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$$

$$\therefore cX = \begin{bmatrix} ca & cb \\ -cb & cc \end{bmatrix}$$ is ~~also a sk~~

$\therefore$ cX is also a skew symmetric matrix. and this arguement can be extended to all $n\times n$ skew symmetric matrix.

$\therefore$ Set of skew symmetric matrices is closed under scalar multiplication.

vii) Let $c\in\mathbb{R}$ be any scalar and X and Y be two skew-symmetric matrices. Then $c(X+Y) = cX+cY$.

$\therefore$ Scalar multiplication is distributive w.r.t. addition of skew-symmetric matrices.

viii) Let $c, d\in\mathbb{R}$ be two scalars and X be a skew-symmetric matrix.

$(c+d)X = cX+dX$.

$\therefore$ Scalar multiplication is distributive w.r.t field addition

ix) Let ~~a, b~~ $c, d \in R$ be two scalars and X be a skew-symmetric matrix.

$\therefore (cd)X = c(dX)$

Proof

Let $X = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$

$\therefore (cd)X = \begin{bmatrix} cda & cdb \\ -cdb & cdc \end{bmatrix} = c(dX)$

$\therefore$ Associative multiplication is satisfied

X) For all skew symmetric matrices X, $1.X = X$. [where $1 \in R$]

$\therefore$ Scalar identity exists.

Result: The set of all skew symmetric matrices is a vector space bcoz all axioms are satisfied

(c)

i) Let $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $Y = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ be two upper triangular matrix [UTM]

$X+Y = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$ which is also (UTM)

$\therefore$ closure under addition is satisfied

ii) $X+Y = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$

$Y+X = \begin{bmatrix} d+a & e+b \\ 0 & f+c \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} = X+Y$

$\therefore$ Addition is commutative.

iii) For any X, Y, Z upper triangular matrices.

$X+(Y+Z) = (X+Y)+Z$.

$\therefore$ Addition is associative

iv) For any $n \times n$ U.T.M. X, $O + X = X + O = X$

where O is $n \times n$ zero matrix.

$\therefore$ Identity element exists

v) For any U.T.M. $X + (-X) = O$ where O is zero matrix.

$\therefore$ Unique additive inverse exists for every U.T.M.

vi) Consider a scalar $x \in R$ and UTM $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

$$\therefore xX = \begin{bmatrix} xa & xb \\ 0 & xc \end{bmatrix} \text{ is also U.T.M}$$

$\therefore$ Set of U.T.M. is closed under scalar multiplication

vii) Let $x \in R$ be a scalar and $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $Y = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ two U.T.M. (upper triangular matrix)

$$x(X+Y) = x\begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} = \begin{bmatrix} x(a+d) & x(b+e) \\ 0 & x(c+f) \end{bmatrix}$$

$$xX + xY = \begin{bmatrix} xa & xb \\ 0 & xc \end{bmatrix} + \begin{bmatrix} xd & xe \\ 0 & xf \end{bmatrix} = \begin{bmatrix} x(a+d) & x(b+e) \\ 0 & x(c+f) \end{bmatrix}$$

$$\therefore x(X+Y) = xX + xY.$$

$\therefore$ Scalar multiplication is distributive w.r.t. addition of U.T.M

viii) Let $x, y \in R$ be two scalars and $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ be an U.T.M.

$$(x+y)X = \begin{bmatrix} (x+y)a & (x+y)b \\ 0 & (x+y)c \end{bmatrix}$$

$$xX + yX = \begin{bmatrix} xa & xb \\ 0 & xc \end{bmatrix} + \begin{bmatrix} ya & yb \\ 0 & yc \end{bmatrix} = \begin{bmatrix} (x+y)a & (x+y)b \\ 0 & (x+y)c \end{bmatrix}$$

$$\therefore (x+y)X = xX + yX.$$

$\therefore$ Scalar multiplication is distributive wrt field addition

(ix) Let $x, y \in \mathbb{R}$ be two Scalars and $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ be U.T.M.

$$(xy)X = \begin{bmatrix} xya & xyb \\ 0 & xyc \end{bmatrix}$$

$$x(yX) = x\begin{bmatrix} ya & yb \\ 0 & yc \end{bmatrix} = \begin{bmatrix} xya & xyb \\ 0 & xyc \end{bmatrix}$$

$$\therefore (xy)X = x(yX).$$

$\therefore$ Associative multiplication is satisfied

x) $\forall$ U.T.M. X, $1 \cdot X = X$ where $1 \in \mathbb{R}$ and X is an U.T.M.

$\therefore$ Scalar identity exists.

Result: Set of all U.T.M. is a vector space since it satisfies all axioms

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$\boxed{Q2}$ We have to show that $R[0,1]$ forms a vector space over $\mathbb{R}$ $R[0,1] \left\{ \text{all functions } f:[0,1] \to R \text{, } f \text{ is continuous} \right\}$

i) closure under addition.

Suppose $f, g \in R[0,1]$.

The defn of addition is $(f+g)(x) = f(x) + g(x) \; \forall \; x \in [0,1]$

By the algebra of continuous functions, $f+g$ is continuous on $[0,1]$. Since real numbers are closed under addition, $(f+g)(x)$ would also map to a real number

$\therefore$ Axiom is satisfied

ii) Commutative property

For above defn. of f, g and x

$(g+f)(x) = g(x)+f(x) = f(x)+g(x)$ [commutativity of real nos]

and $f(x)+g(x) = (f+g)(x)$ by definition

$\therefore (f+g)(x) = (g+f)(x)$.

Thus this axiom is satisfied

iii) Associative property

for $f, g, h \in R[0,1]$ and $x \in [0,1]$

using the definition of vector addition of associativity of real numbers we get, $((f+g)+h)(x) = (f+g)x + h(x)$ (from definition)

$= (f(x)+g(x))+h(x)$ (")

$= f(x)+(g(x)+h(x))$ (associativity)

$= f(x) + (g+h)(x)$

$= (f+(g+h))(x)$

It holds for any $x \in [0,1]$ $\therefore (f+g)+h = f+(g+h)$.

$\therefore$ Axiom is satisfied

iv) Additive identity

Let's define a function $f_0 \in R[0,1]$ that maps everything to 0.

This is a constant fn, hence it is continuous.

Now, for any $f \in R[0,1]$ and $x \in [0,1]$

$(f_0+f)(x) = f(x) + f(x) = 0 + f(x) = f(x)$

$\therefore f_0$ is additive identity

V) Additive inverse

Let's define a function g that takes a number $x \in [0,1]$, calculates $f(x)$ for $f \in R[0,1]$, multiplies the ans by -1 and outputs the result. It would be continuous over $[0,1]$ as f is continuous over $[0,1]$ and for any $\lambda \in R$ (-1 in this case) so is λf (by definition of algebra of continuous functions) Hence $g \in R[0,1]$.

~~VI~~ For any $x \in [0,1]$, $g(x) = (-f(x))$

$\therefore \forall x \in [0,1]\ f(x) + g(x) = 0$

i.e. $f + g = f_0$ (f_0 is additive identity)

Hence axiom is satisfied

VI) Scalar multiplication

Suppose $f \in R[0,1]$ and $\lambda \in \mathbb{R}$. Scalar multiplication is defined as. $(\lambda f)(x) = \lambda f(x)$.

By algebra of continuous fn, λf is cts on $[0,1]$

Hence axiom is satisfied

VII) Distributive property i)

Let $f, g \in R[0,1]$ and $\lambda \in \mathbb{R}$ be a scalar then,

$$\lambda(f+g)(x) = \lambda(f(x) + g(x))$$

$$= \lambda(f(x) + \lambda(g(x)) \qquad \text{[distributive property of real no's]}$$

$$= \lambda f + \lambda g$$

$\therefore$ Axiom is satisfied

viii) Distributive property ii)

Let $f \in R[0,1]$ and $\lambda_1, \lambda_2 \in R$ be 2 scalars

$$(\lambda_1 + \lambda_2) f(x) = \lambda_1 f(x) + \lambda_2 f(x) \qquad \text{[distributive property of real nos]}$$

$$= \lambda_1 f + \lambda_2 f$$

$$\Rightarrow (\lambda_1 + \lambda_2) f = \lambda_1 f + \lambda_2 f$$

$\therefore$ Axiom is satisfied

(ix) Associative property

Let $f \in R[0,1]$ and scalars $c, d \in \mathbb{R}$

$c(df) = c(df(x)) = (cd)f(x)$ [Associativity of real nos]

$= (cd)f$

$\Rightarrow c(df) = (cd)f$

$\therefore$ Axiom satisfied

x) Scalar identity

Let $f \in R[0,1]$

$1 \times f = 1 \times f(x) = f(x)$ $\because$ 1 is multiplicative identity for real nos.

Hence scalar identity exists.

Axiom is satisfied

$\therefore R[0,1]$ forms a vector space over $\mathbb{R}$.

$\boxed{Q3}$ $f(x) = x$, $g(x) = e^x$, $h(x) = e^{-x}$; $x \in [0,1]$

f, g, h ~~be~~ are linearly independent iff

$\cancel{c_1 f + c}$ $c_1 f + c_2 g + c_3 h = 0$

Should be satisfied only when $c_1 = c_2 = c_3 = 0$

$\Rightarrow$ To find c_1, c_2, c_3 we need 3 equations.

$c_1 x + c_2 e^x + c_3 e^{-x} = 0$ ——— 1

Now if the original equation is true, then if we differentiate everything, the new equation must also be true.

$\therefore f, g, h$ are differentiable in the entire dimension of real no's.

$c_1 + c_2 e^x - c_3 e^{-x} = 0$ ——— 2

Again f', g', h' are all differentiable in the entire domain of real nos

$c_2 e^x + c_3 e^x = 0$ ——— 3

Now. 3 − 1

$c_1 x = 0$ $[\forall x \in [0,1]]$

$\Rightarrow c_1 = 0$

Putting value of c_1 from 4 in 2 and then adding 2 and 3

we get, $2c_2 e^x = 0 \quad \forall x \in [0,1]$

$\Rightarrow c_2 = 0$ —— 5

Putting value of c_2 from 5 in 3 we get

$c_3 e^{-x} = 0 \quad \forall x \in [0,1]$

$\Rightarrow c_3 = 0$

Hence, $c_1 = c_2 = c_3 = 0. \Rightarrow f, g, h$ are linearly independent.

Q4

a) Set of all invertible matrix A.

Let I be the invertible matrix

Then $-I$ would also be invertible

$\therefore I+(-I)=0$ (apply additive property)

But '0' matrix is not invertible

$\therefore$ Set of all invertible matrix is not subspace as not closed under addition

b) Set of all non-invertible matrix A.

Let take two non-invertible matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

but their addition $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

lead to invertible matrix.

$\therefore$ Set of all non-invertible matrix will not be a subspace as not closed under addition

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c) All A such that $AB = BA$ for a fixed matrix B in V.

Fixing matrix B, we have to find one A s.t. $AB = BA$.

Let A_1, A_2 both commutes with B

$\therefore A_1B = BA_1$

$\therefore A_2B = BA_2$

Hence (A_1+A_2) will also commute due to closure of your set under vector addition

$\therefore (A_1+A_2)B = B(A_1+A_2)$

~~& Let B be any scalar then (BA_1)~~

& Let c be any scalar then (cA_1) also commutes ~~under~~ with B due to closure of your set under scalar multiplication.

$(cA_1)B = B(cA_1)$.

$\therefore$ The set will be a subspace

d) Set of idempotent matrix subspace?

A will be idempotent if $A^2 = A$.

I will be a subspace & Let A be also a subspace

So, $I+A$ would also be subspace (additive property)

So, $(I+A)^2 = (I+A)$ [This should hold true]

$I^2 + A^2 + 2A = I + A$

$\Rightarrow I + 3A = I + A$

$\boxed{A = 0}$

So, we can write $(A = I)$ that makes $(I+I)^2 = I+I$

$4I = 2I$

Which is not true

So, set of all idempotent matrices won't be subspace.

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Q5 V is a vector space of all function $f: R \to R$.

a) All continuous function

As V is the vector space of all functions from $R \to R$.

Let W be the subset of all continuous functions in V.

So, we need to prove that W is non-empty & it's closed under addition & scalar multiplication

It will always be non-empty, as it contains zero function

For scalar multiplication

$f: R \to R$ is continuous & $c \in R$ then

$c \cdot f : x \to cf(x)$ will also be continuous.

For vector addition

f and g be continuous function belonging to w then

$f+g : x \to f(x)+g(x)$

So, linear combination of continuous fn. is a continuous fn & will be the subspace of V.

b) f such that $f(x^2) = f(x)^2$

It will not be a subspace of V as let's take

$f(x) = \{ 1$ for $\forall x$

then $f \in S$.

but $2 * f = \{ 2 \ \forall x$ which is $\notin S$

So, it is not a subspace.

c)

c) $f(3) = 1 + f(-5)$

It will be not be a Subspace of V.

$$f(x) = \begin{cases} 1 & ; x = 3 \\ 0 & ; \text{otherwise} \end{cases}$$

$$2 \times f = \begin{cases} 2 & ; x = 3 \\ 0 & ; \text{otherwise} \end{cases}$$ which is $\notin S$

$\therefore$ It is not a Subspace of V.

Q6 V and W are vector spaces over field F

$$Z = \{ (v, w) ; v \in V \text{ and } w \in W \}$$

We need to prove Z is a vector space over F with the operations :-

$\rightarrow (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$

$\rightarrow c(v_1, w_1) = (cv_1, cw_1)$

i) closure prop.

Let $Z_1 = (v_1, w_1)$ and $Z_2 = (v_2, w_2) \in Z$

Now $Z_1 + Z_2 = (v_1, w_1) + (v_2, w_2)$

$= (v_1 + v_2, w_1 + w_2)$

Now $v_1 + v_2 \in V$, $w_1 + w_2 \in W$ (bcoz V, W are vector spaces)

Then $Z_1 + Z_2 = (a, b)$ where $a = v_1 + v_2 \in V$
$b = w_1 + w_2 \in W$

$\therefore Z_1 + Z_2 \in Z$.

$\therefore$ ~~Axiom~~ closure under vector addition satisfied

ii) Identity element

As both V and W are vector spaces there will be an identity element 'e' in V and W s.t.

$e + v = v \in V$ — ①

$e + w = w \in W$ — ②

Now, let $Z = (v, w) \in Z$ and the identity element (e, e) such that $(e, e) + (v, w) = (e + v, e + w) = (v, w)$ [from ①, ②]

$\therefore$ Identity element (e, e) exists

iii) Inverse

As both V,W are vector spaces

$\forall\, v \in V$, $\exists\, v^{-1} \in V$ s.t. $v + v^{-1} = e \in V$

$\forall\, w \in W$, $\exists\, w^{-1} \in W$ s.t. $w + w^{-1} = e \in W$

Now $z = (v, w) \in Z$ then

$$\Rightarrow \underbrace{(\underbrace{v^{-1}, w^{-1}}_{\in V\ \in W})}_{\in Z} + \underbrace{(v, w)}_{\in Z} = (v^{-1}+v,\ w^{-1}+w) = (e, e) \in Z$$

$\therefore$ ~~Inver~~ Inverse for all elements exists

iv) Associativity

Let $z_1 = (v_1, w_1)$, $z_2 = (v_2, w_2)$, $z_3 = (v_3, w_3)$

where $z_1, z_2, z_3 \in Z$.

Now, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ [This has to hold]

LHS $\quad z_1 + (z_2 + z_3) = (v_1, w_1) + [(v_2, w_2) + (v_3, w_3)]$

$= (v_1, w_1) + [(v_2 + v_3), (w_2 + w_3)]$

$= (v_1 + (v_2 + v_3),\ w_1 + (w_2 + w_3))$

RHS $\quad (z_1 + z_2) + z_3 = [(v_1, w_1) + (v_2, w_2)] + (v_3, w_3)$

$= [(v_1 + v_2,\ w_1 + w_2)] + (v_3, w_3)$

$= ((v_1 + v_2) + v_3,\ (w_1 + w_2) + w_3)$

Now, $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
$w_1 + (w_2 + w_3) = (w_1 + w_2) + w_3$] as V, W are vector spaces

$\therefore$ LHS = RHS

Associativity under vector addition satisfies.

v) Commutativity

As, V, W are vector spaces over field F.

~~v_1+w_2~~ $v_1+v_2 = v_2+v_1 \ \forall \ v_1, v_2 \in V$ — ①

$w_1+w_2 = w_2+w_1 \ \forall \ w_1, w_2 \in W$ — ②

Now, let $z_1 = (v_1, w_1)$ and $z_2 = (v_2, w_2)$

where $z_1, z_2 \in Z$

$\therefore z_1 + z_2 = z_2 + z_1$ [This should hold]

LHS $z_1+z_2 = (v_1,w_1)+(v_2,w_2)$

$= (v_1+v_2, w_1+w_2)$

RHS $z_2+z_1 = (v_2,w_2)+(v_1,w_1)$

$= (v_2+v_1, w_2+w_1)$

from ① & ② $v_1+v_2 = v_2+v_1$ & $w_1+w_2 = w_2+w_1$

$\therefore$ LHS = RHS

$\therefore$ commutativity under vector addition is satisfied

vi) closure under scalar multiplication

As V, W are vector spaces over field F

$\forall \ v \in V, \forall \ \alpha \in F \Rightarrow \alpha v \in V$ — ①

$\forall \ w \in W, \forall \ \alpha \in F \Rightarrow \alpha w \in W$ — ②

Now let $z = (v, w) \in Z$ and $\alpha \in F$

then $\alpha z = \alpha(v, w)$

$= (\alpha v, \alpha w)$ from ①, ② ~~$\alpha \in V$~~

$\alpha v \in V$

$\alpha w \in W$

$\underbrace{\alpha v}_{\in V}, \underbrace{\alpha w}_{\in W}$ $\in Z$

Thus $(\alpha v, \alpha w) \in Z$

$\therefore$ closure under scalar multiplication is satisfied

vii) <u>Identity element in field F</u>

As V, W are vector spaces.

$1.v = v \in V$, where $1 \in F$ and $v \in V$ —— ①

$1.w = w \in W$ where $1 \in F$ and $w \in W$ —— ②

Now let $z = (v, w)$ then

$\Rightarrow 1.(v, w) = (1.v, 1.w)$

$= (v, w)$

$= (v, w) \in Z$

$\therefore 1.(v, w) = (v, w)$

There exists identity element 1 is fied F s.t. $1.z = z \ \forall \ z \in Z$

viii) <u>Associativity</u>

Let $\alpha, \beta \in F$ and $z = (v, w) \in Z$

Then $(\alpha\beta)z = \alpha(\beta z)$ [This should hold]

LHS $(\alpha\beta)z = (\alpha\beta)(v, w)$

$= (\alpha\beta v, \alpha\beta w)$

RHS $\alpha(\beta z) = \alpha(\beta(v, w))$

$= \alpha(\beta v, \beta w)$

$= (\alpha\beta v, \alpha\beta w)$

$\therefore$ LHS = RHS

$\therefore$ Associativity under scalar multiplication satisfies

ix) <u>Distributivity (one scalar, two vectors)</u>

Let $\alpha \in F$ and $z_1, z_2 \in Z$ then

$\alpha(z_1 + z_2) = (\alpha z_1 + \alpha z_2)$ [This should hold]

LHS $\alpha(z_1 + z_2) = \alpha((v_1, w_1) + (v_2, w_2))$

$= \alpha(v_1 + v_2, w_1 + w_2)$

$= (\alpha(v_1 + v_2), \alpha(w_1 + w_2))$

RHS $\alpha z_1 + \alpha z_2 = \alpha(v_1, w_1) + \alpha(v_2, w_2)$

$= (\alpha v_1, \alpha w_1) + (\alpha v_2, \alpha w_2)$

$= (\alpha v_1 + \alpha v_2, \alpha w_1 + \alpha w_2)$

As V, W are vector spaces $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ and $\alpha(w_1 + w_2) = \alpha w_1 + \alpha w_2$: $\forall \ \alpha \in F$

$\therefore$ LHS = RHS (Axiom satisfied) $\forall \ v_1, v_2 \in V$ $\forall \ w_1, w_2 \in W$

X) Distributivity (Two Scalar, one vector)

Let $\alpha, \beta \in F$ and $z \in Z$ then

$$(\alpha+\beta)z = \alpha z + \beta z.$$

LHS $(\alpha+\beta)z = (\alpha+\beta)(v,w) = ((\alpha+\beta)v, (\alpha+\beta)w)$

RHS $\alpha z + \beta z = (\alpha v, \alpha w) + (\beta v, \beta w)$

$= (\alpha v + \beta v, \alpha w, \beta w)$

As v, w are vector spaces $(\alpha+\beta)v = \alpha v + \beta v$

$(\alpha+\beta)w = \alpha w + \beta w$

$\therefore$ LHS = RHS

Distributivity satisfies

$\therefore Z = \{(v,w), v \in V, w \in W\}$ over a field F with given operations is a vector space

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Q7 V is a vector space and $S_1 \subseteq S_2 \subseteq V$.

Given that S_2 is linearly independent, we have to prove S_1 is linearly independent.

Ans Let vectors $v_1, v_2, \ldots, v_n \in S_1$ are pairwise distinct and $a_1, a_2, \ldots, a_n \in F$ s.t.

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \quad (1)$$

Now, we need to prove that the above equation holds true ~~iff~~ only if $a_i = 0 \ \forall a_i \in F$

As $S_1 \subseteq S_2$ is given, then all the vectors $v_1, v_2, \ldots v_n \in S_2$ also

and since S_2 is linearly independent eq (1) holds true only when $a_i = 0$ for $\forall i$ and no other possibility

Thus, all $a_i = 0$ $(\forall i)$ which means that is satisfies eq (1)

$\therefore$ S_1 is also linearly independent because

$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ only when $a_i = 0 \ \forall \ i$

but not for any other a_i values

Q8 We need to show that subset W of vector space V is a subspace of V iff Span (W) = W

Ans:- We need to prove:-

W is a subspace of V $\Leftrightarrow$ Span (W) = W where $W \subseteq V$.

$\boxed{\Rightarrow}$: If W is a subspace of V, then Span (W) = W

Let $u \in \text{Span}(W)$ where

$u = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

for vectors $v_1, v_2, \ldots, v_n \in W$

and $a_1, a_2, \ldots, a_n \in$ Field F

Since, W is a subspace of V and also $v_1, v_2, \ldots, v_n \in W$

Then $u = a_1v_1 + a_2v_2 + \cdots + a_nv_n \in W$ (closure property)

Thus, for $\forall$ $u \in \text{Span}(W)$, then $u \in W$.

Hence, Span (W) = W if W is subspace of V.

$\boxed{\Leftarrow}$ If Span (W) = W then W is a subspace of V.

Let u and $v \in W$ [= Span (W)]

We need to prove the closure satisfies in set W.

As $u \in \text{Span}(W)$, it can be written as:-

$u = \sum_{i=1}^{n} a_iu_i$ where $a_i \in F$ and $u_i \in W$.

Similarly, $v = \sum_{j=1}^{n} b_jv_j$ where $b_j \in F$ and $v_j \in W$.

Then closure property

$$au + bv = a\left[\sum_{i=1}^{n} a_i u_i\right] + b\left[\sum_{j=1}^{n} b_j v_j\right]$$

$$= \sum_{i=1}^{n} \underbrace{(a a_i)}_{\text{Scalar}} u_i + \sum_{j=1}^{n} \underbrace{(b b_j)}_{\text{Scalar}} v_j$$

Thus. $au+bv$ can be written as linear combination of vectors in W i.e. $au+bv \in Span(W)$

And as $Span(W) = W$; it means that $au+bv \in W$

$\therefore$ closure property satisfied

Proving W is non-empty

If $W = \emptyset$ (empty) then $Span(W) = \{0\}$ —— ①

[Span of the empty set is the empty set containing only zero vector]

And we know that $Span(W) = W$ but from ①

$Span(W) \neq W$ Which is contradicting

$\therefore$ W is not empty set.

Proving that W consists of zero vector

As $Span(W)$ always consists of zero vector bcoz

Let $u = a_1v_1 + a_2v_2 + \cdots + a_nv_n \in Span(W)$

where $v_1, v_2, \ldots, v_n \in W$

$a_1, a_2, \ldots a_n \in F$

If all the scalars are zero then $u=0$ (zero vector)

And as $u \in Span(W)$, the zero vector belongs to the $Span(W)$.

We know that, $Span(W) = W$, which means W also consists of zero vector.

∴ W is also a subspace of V if Span (W) = W

as it satisfies the following :-

1) Not an empty set

2) contains zero vector

3) Satisfied closure under addition and scalar multiplication

∴ Subset W of vector space V is a subspace of V iff Span(W) = W.