Orthogonality in $\mathbb{R}^n$

$$\mathbb{R}^n \longrightarrow (x_1, x_2 \cdots\cdots x_n)$$

↳ $x_i \in \mathbb{R}$.

Now, $(x_1, \cdots\cdots, x_n)$

$+ (y_1, y_2 \cdots y_n)$

$= (x_1 + y_1, \cdots, x_n + y_n)$

$c(x_1, \cdots x_n)$

$= (cx_1, cx_2 \cdots cx_n)$

Is $\mathbb{R}^n$ vector space over the field $(\mathbb{R}, +, \cdot)$ ✓

↳ vector space.

Basis $= \{(1, 0 \cdots 0), (0, 1, \cdots 0), \cdots (0, 0, \cdots 1)\}$

$(x_1, \ldots x_n) \to v_i \in \mathbb{R}^n.$

Def^n: A set of vectors

$\{v_1, v_2 \ldots \cdot \, v_k\}$ in $\mathbb{R}^n$ is called orthogonal set if all pairs of distinct vectors are orthogonal — i.e

if $\boxed{v_i \cdot v_j = 0}$ whenever $i \neq j$

$\boxed{v_i \cdot v_j} \to$ [scalar] $\to 0$.

$\boxed{V \times V \to F}$

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 \cdot v_2 = 2\cdot 1 + 1\cdot 1 + (-1)\cdot 1$$

$$= 2 + \cancel{1} - \cancel{1}$$

$$= 2$$

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \rightarrow (1) \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (2)$$

$\boxed{v_1 \cdot v_2 = 0} \rightarrow v_1, v_2$ are orthogonal to each other.

$\Leftarrow$

Thm: If $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of non zero vectors in $\mathbb{R}^n$, then these set of vectors are linearly independent.

Proof: If $c_1, c_2, \ldots, c_k$ are scalars such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

then:

$$(c_1 v_1 + c_2 v_2 + \cdots + c_i v_i + \cdots + c_k v_k) \cdot v_i = 0 \cdot v_i = 0$$

$$\Rightarrow c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i) + \cdots + c_i (v_i \cdot v_i) + \cdots + c_k (v_k \cdot v_i) = 0$$

$\therefore \{v_1, \ldots, v_k\}$ is an orthogonal set of vectors.

$v_j \cdot v_i = 0$ if $i \neq j$

$\Rightarrow c_i(v_i \cdot v_i) = 0 \longrightarrow (1).$

Now since $v_i \neq 0$

$v_i \cdot v_i \neq 0$

$\Rightarrow c_i = 0.$

This is true for $i = 1, 2 \ldots \ldots k.$

$\Rightarrow c_i = 0 \ \forall \ i = 1, \ldots, k.$

$\Rightarrow \{v_1, v_2, \ldots \ldots, v_k\}$ is a linealy independent set of vectors.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$v_1 \cdot v_2$, $v_2 \cdot v_3$, $v_1 \cdot v_3$

$\downarrow \quad \downarrow \quad \downarrow$

$0 \quad 0 \quad 0.$

$\{\bar{v_1}, \bar{v_2}, \bar{v_3}\} \longrightarrow$ l.g. ✓

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$ ✓

can be expressed as l.c.

of $v_1, v_2, v_3 \longrightarrow$ ✓

$\{v_1, v_2, v_3\}$ $\longrightarrow$ check this

$\longrightarrow$ basis

Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by,

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \underline{x - y + 2z = 0} \right\}$$

$$x = y - 2z \quad \checkmark \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} \quad \text{LC}$$

$$= y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\} \begin{array}{l} \rightarrow \text{spans } W \\ \rightarrow \text{L.I.} \end{array}$$

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{r} c_1 - 2c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{array}\right\} \quad c_1 = c_2 = 0$$

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} , \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}} \right\} \longrightarrow \text{basis of } W.$$

not orthogonal.

Our task is to create an orthogonal basis from this.

$\underline{W}$

Suppose $\bar{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to $\bar{u}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\boxed{x - y + 2z = 0} \longrightarrow (1)$$

$$\boxed{x + y = 0} \longrightarrow (2)$$

$$x = -z \; , \; y = z .$$

$$\underline{\bar{w}} = z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} .$$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ of W.

$\rightarrow$ L ~~x~~

$\rightarrow$ basis or not !

$\boxed{\text{L.I.}}$ $\longrightarrow$ orthogonal basis

~~$\boxed{W}$~~

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix}$$

spans W.

$$-c_1 + c_2 = y - 2z.$$

$$c_1 + c_2 = y.$$

$$c_1 = z.$$

If $\boxed{c_1 = z, \quad c_2 = y - z.}$

Thm: Let $\{u_1, u_2, \ldots, u_k\}$ be an orthogonal basis for subspace W of $\mathbb{R}^n$ and let w be any vector in W. Then the unique scalars $c_1, c_2, \ldots, c_k$ such that

$$w = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

are given by

$$\boxed{c_i = \frac{w \cdot u_i}{u_i \cdot u_i}}$$ for $i = 1 \ldots k$.

$u_i \cdot u_j = 0$

Since $\{u_1, u_2, \ldots, u_k\}$ is a basis for W, we know that there are unique scalars $c_1, c_2, \ldots, c_k$ such that

$$w = c_1 u_1 + c_2 u_2 \cdot + c_k u_k.$$

$$w \cdot u_i = (c_1 u_1 + c_2 u_2 + \cdots + c_k u_k) \cdot u_i$$

$$= c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i)$$

$$+ c_i (v_i \cdot v_i) + c_k (v_k \cdot v_i)$$

$$w \cdot v_i = c_i (v_i \cdot v_i)$$

$\because v_i \neq 0 \quad (v_i \cdot v_i) \neq 0.$

$$\Rightarrow c_i = \frac{(w \cdot v_i)}{(v_i \cdot v_i)} \quad \text{for } i = 1, \ldots K.$$

$$c_1 = \frac{(w \cdot v_1)}{(v_1 \cdot v_1)}, \quad c_2 =$$

Ex Find the co-ordinates of

$$w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ with respect to

the orthogonal basis

$$B: \left\{ v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$c_1 = \frac{w \cdot v_1}{v_1 \cdot v_1} = \frac{2+2-3}{4+1+1}$$

$$= \frac{1}{6}.$$

$$c_2 = \frac{5}{2}, \quad c_3 = \frac{2}{3}$$

$$w = \frac{1}{6} v_1 + \frac{5}{2} v_2 + \frac{2}{3} v_3.$$

$$(w)_B = \begin{bmatrix} 1/6 \\ 5/2 \\ 2/3 \end{bmatrix}$$

A set of vectors in $\mathbb{R}^n$ is an orthonormal set if it is an orthogonal set of <u>unit</u> vectors.

$$\underline{q_i \cdot q_j = \begin{matrix} 0 \\ 1 \end{matrix}} \Big\} \begin{matrix} \text{if } i \neq j. \\ \text{if } i = j \end{matrix}$$

An orthonormal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

$$\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}_{e_1 \quad e_2 \quad e_3} \qquad \mathbb{R}^3$$

$$\left. \begin{array}{l} e_i \cdot e_i = 1 \\ e_i \cdot e_j = 0 \end{array} \right\}$$

$S = \{q_1, q_2\}$ is an orthonormal set in $\mathbb{R}^3$

$$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Let $\{q_1, q_2, \ldots, q_k\}$ be an orthonormal basis for a subspace of W of $\mathbb{R}^n$ and let w be any vector in W.

Then.

$$w = (w \cdot q_1) q_1 + (w \cdot q_2) q_2 + \cdots (w \cdot q_k) q_k.$$

$$c_i = \frac{(w \cdot q_i)}{(q_i \cdot q_i)} \quad \therefore c_i = \frac{w \cdot q_i}{1}.$$

$\therefore \{q_1, q_2 \ldots q_k\} \rightarrow$ orthonormal bases