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(Q1, Q3, Q5, Q7, Q9)

(Q5) Let u, v, w be distinct vectors of a vector space V.

We have to show that if $[u, v, w]$ is a basis for V then $[u+v+w, v+w, w]$ is also a basis for V.

Soln

Let $u, v, w \in V$ a vector space over a field F such that $u \neq v \neq w$. Let $\{u, v, w\}$ be a basis for V.

Because $\{u, v, w\}$ is a basis, then u, v, w are linearly independent and $\langle \{u, v, w\} \rangle = V$.

Let $x \in V$ be an arbitrary vector then x can be uniquely expressed as a linear combination of $\{u, v, w\}$. Let's suppose

$x = au + bv + cw$ for some $a, b, c \in F$.

On the other hand, let's consider $\{u+v+w, v+w, w\} \subseteq V$.

Then,

$$\langle \{u+v+w, v+w, w\} \rangle = \{d(u+v+w) + e(v+w) + f(w) \mid d, e, f \in F\}$$
$$= \{du + (d+e)v + (d+e+f)w \mid d, e, f \in F\}$$

If $x \in V$, then $x = du + (d+e)v + (d+e+f)w$ is another unique representation of $x \in V$.

Then for arbitrary $x \in V$, we have $d = a$, $d+e = b$ and ~~d+e~~

$d+e+f = c \in F$

Because $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ ~~that~~ must also be a basis for V.

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(91)

(97)

Let $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$

$|A| = 0$ and $|B| = 0$

$\therefore A \in M$ and $B \in M$

Now, let's consider $C = A + B$

$\therefore C = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

Now, $|C| = i \times i - 0 \times 0 = i^2 = -1$

$\therefore C \notin M$

$\Rightarrow A + B \notin M$

which means, M is not closed under addition.

$\therefore$ M is not a subspace of $C^{2\times2}$

(*)

b)

Q7 (a) $T(x) = x+1$

we need to show

For linear transformation, $T(x+y) = T(x)+T(y)$ — ①

$T(cx) = cT(x)$ — ②

Now, $T(x) = x+1$

$T(y) = y+1$

$\therefore T(x)+T(y) = x+1+y+1 = x+y+2$

and, $T(x+y) = T(x)+T(y)$

$\therefore T(x+y) \neq T(x)+T(y)$

~~So, T: R→R defined by T(x) = x+1~~

So, since the first property itself is failed, so we don't need to check the 2nd property.

Hence, $T: R \to R$, $T(x) = x+1$ is not linear transformable

Required for proof

(*) A linear transformation (or a linear map) is a function $T: R^n \to R^m$ that satisfies the following properties.

1. $T(x+y) = T(x)+T(y)$
2. $T(ax) = aT(x)$

} for any vectors $x, y \in R^n$ and any scalar $a \in R$

b) $T(x) = 2^x$

$T(y) = 2^y$

Now, $T(x+y) = 2^{x+y}$

and, $T(x)+T(y) = 2^x + 2^y$

So, $\boxed{T(x+y) \neq T(x)+T(y)}$

Hence, $T: R \to R$ $T(x) = 2^x$ is not linear transformable.

We can see that,

$$\ker(T) = \{A \text{ in } M_{22} : T(A) = 0\}$$
$$= \{A \text{ in } M_{22} : A^T = 0\}$$

But, If $A^T = 0$, then $A = (A^T)^T = 0^T = 0$

It follows that $\text{Kernel}(T) = \{0\}$

Since, for any matrix A is M_{22}, we have $A = (A^T)^T$
$= T(A^T)$
(and A^T is in M_{22})

We deduce that $\text{range}(T) = M_{22}$.

In all ~~the~~ of the above examples, the kernel and range of a linear transformation are sub-spaces of the domain and codomain, respectively, of the transformation. Since we are generalizing the null space and column space of a matrix, this is perhaps not suprising. Nevertheless, we should not take anything for granted, so we need to prove that it is not a coincidence.

(9)

The range of T is the set of all vectors in W that are of the form T(v), where v is in V. Let T(v) be in range of T. Since B spans V, there are scalars $c_1, c_2, ---, c_n$ such that.

$$v = c_1 v_1 + \cdots + c_n v_n$$

Applying T and using the fact that it is a linear transformation we can see that,

$$T(v) = T(c_1 v_1 + \cdots + c_n v_n)$$
$$= c_1 T(v_1) + \cdots + c_n T(v_n)$$

In other words, T(v) is in span (T(B)), as required.