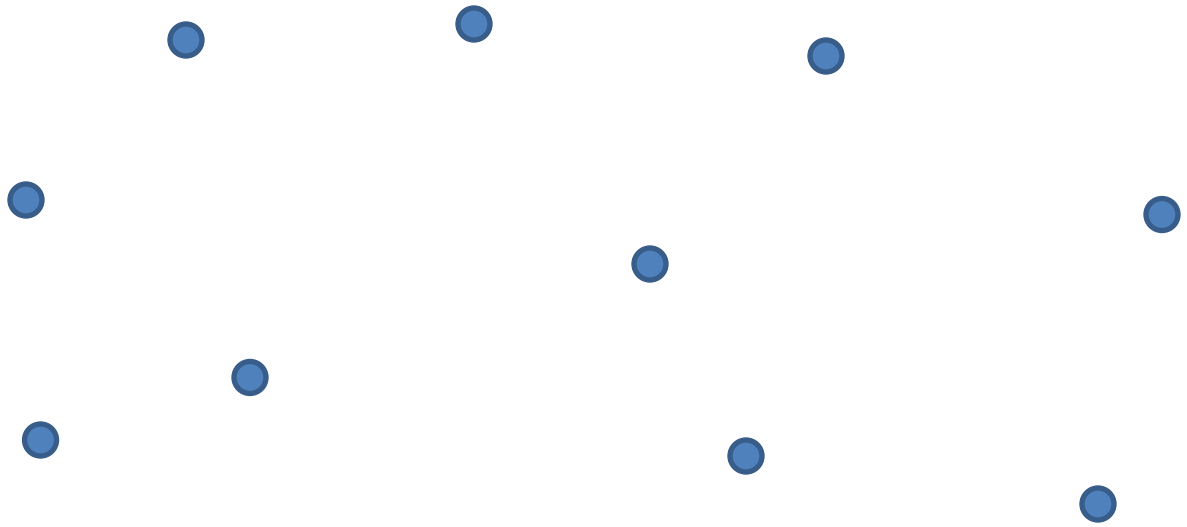
Module – 5 Geometric Algorithms
Voronoi Diagram and Delaunay Triangulation

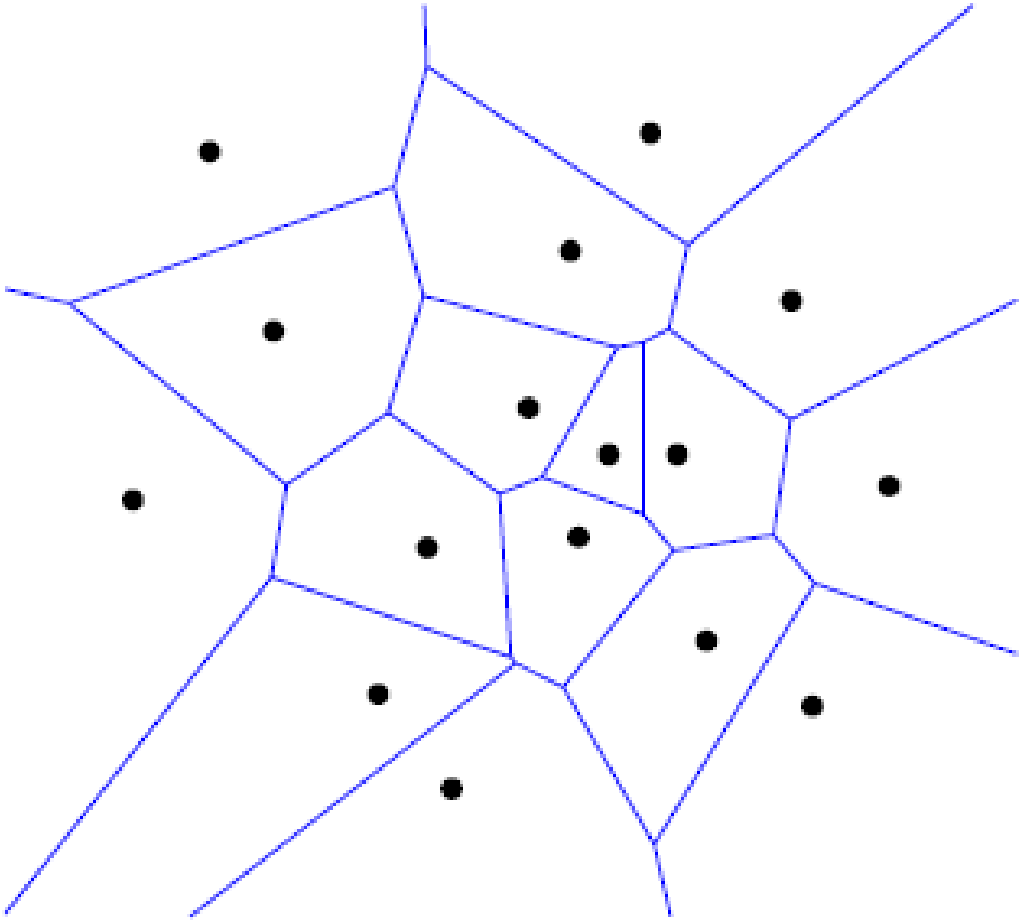
Definition

- Voronoi Diagrams encode ***proximity information*** that help answer questions like:
  - ***"Which object is closest to point p?"***
  - ***"where is the nearest hospital from q?"***
- For a given set of objects,
  - called sites, each Voronoi cell defines the set of points in the plane
  - (or in any higher dimension) that are closer to each site than to any other site.
  - The outline of all ***voronoi cells*** form the ***Voronoi Diagram.***

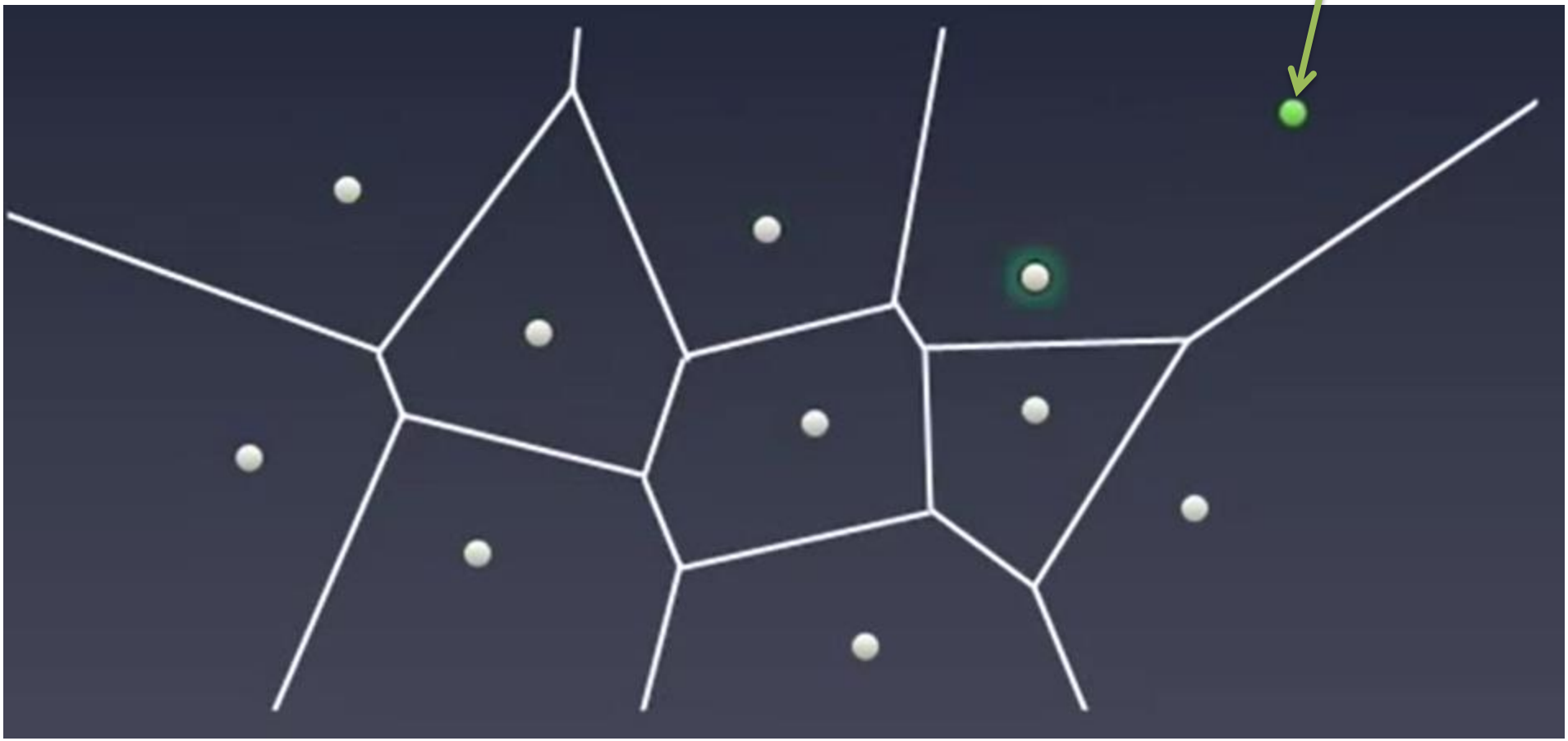
Example: Set of sites



Voronoi Diagram of Point set

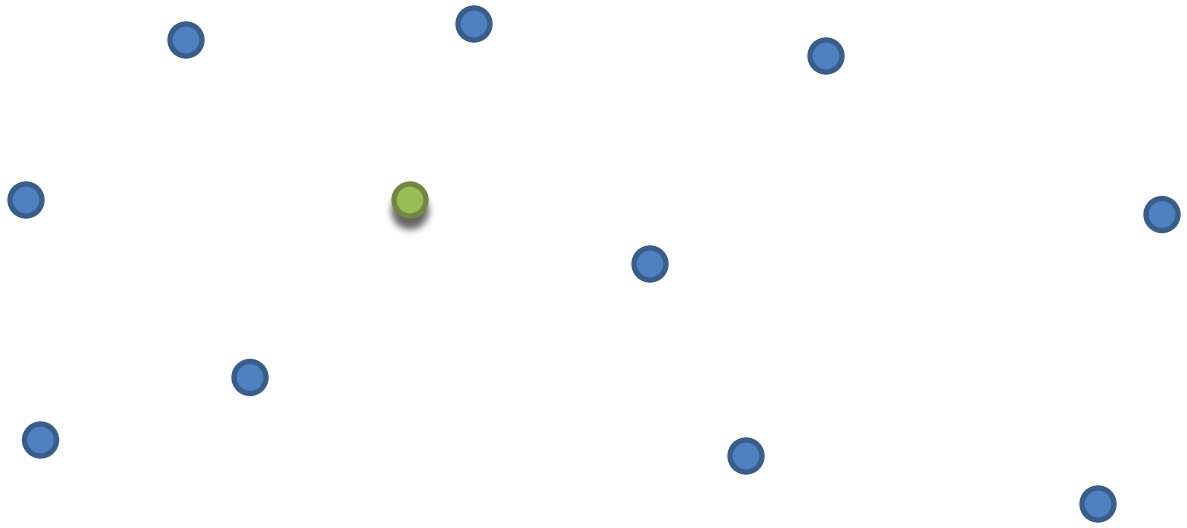


How to find the query point q?



Without Voronoi Diagram

- The increase in difficulty without a voronoi diagram.
- **O(n)** distance queries will be computed every frame



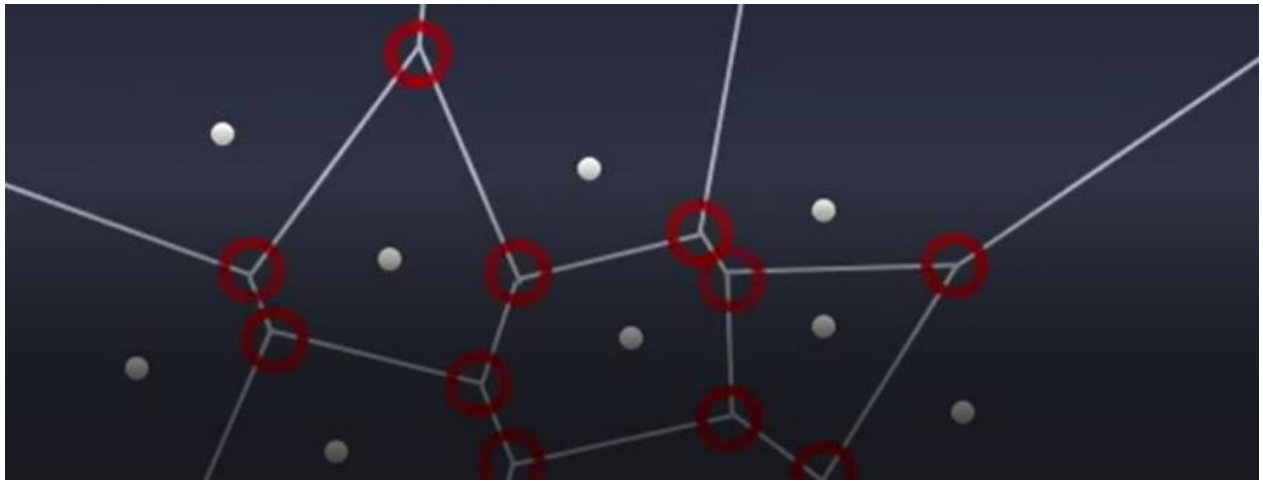
With Voronoi Diagram

Given a voronoi diagram,

a point location query on the cell containing q can now be done in only ***O(log n) times.***

Property #1

- Voronoi diagram is a ***planar graph where every vertex is of degree 3*** (assuming general position, where no sites are cocircular)

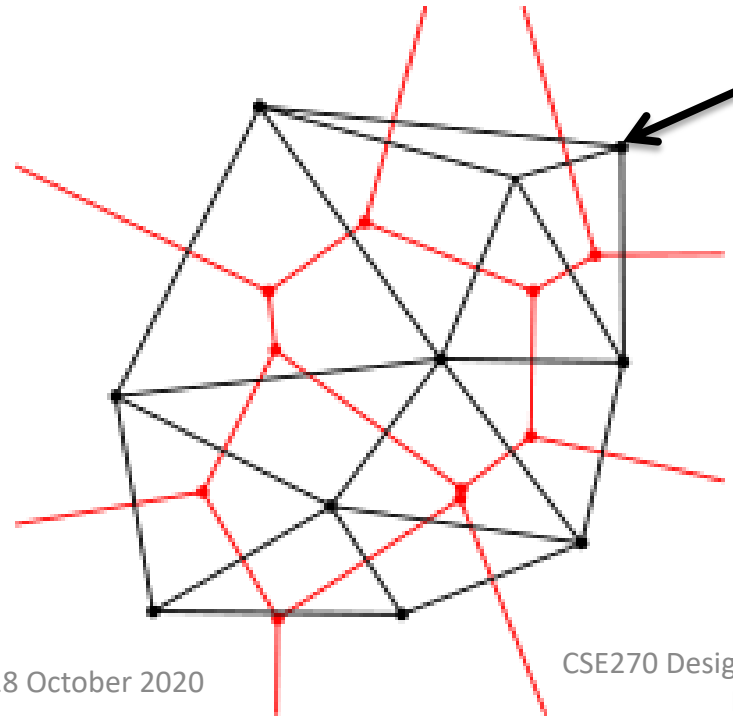


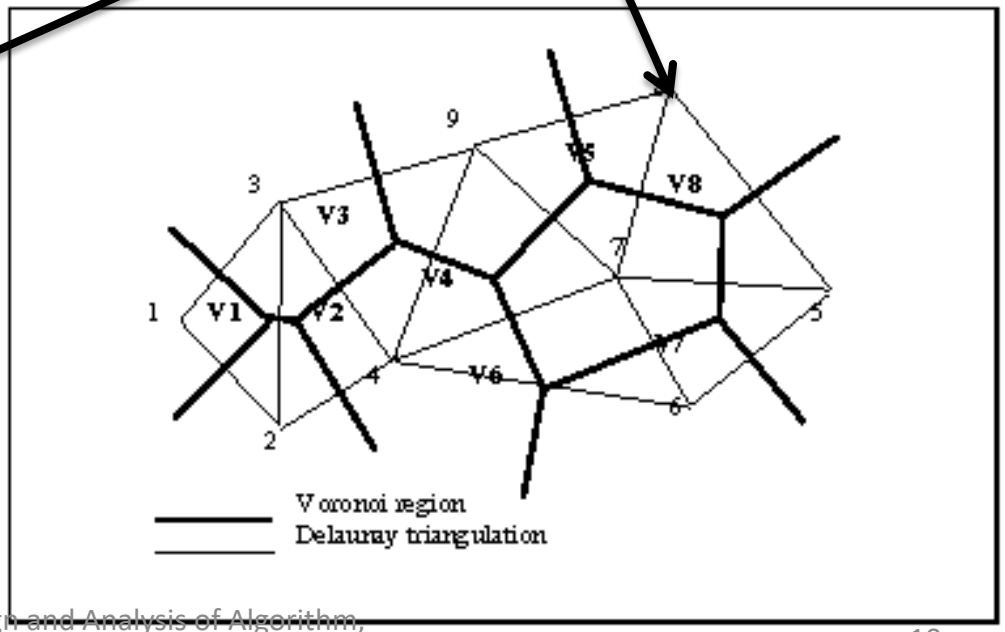
Property #2

- The complexity of Voronoi diagram is ***linear***.
  - For n sites, there are ***n faces***, at most ***2n-5 vertices*** and at most ***3n-6 edges.***

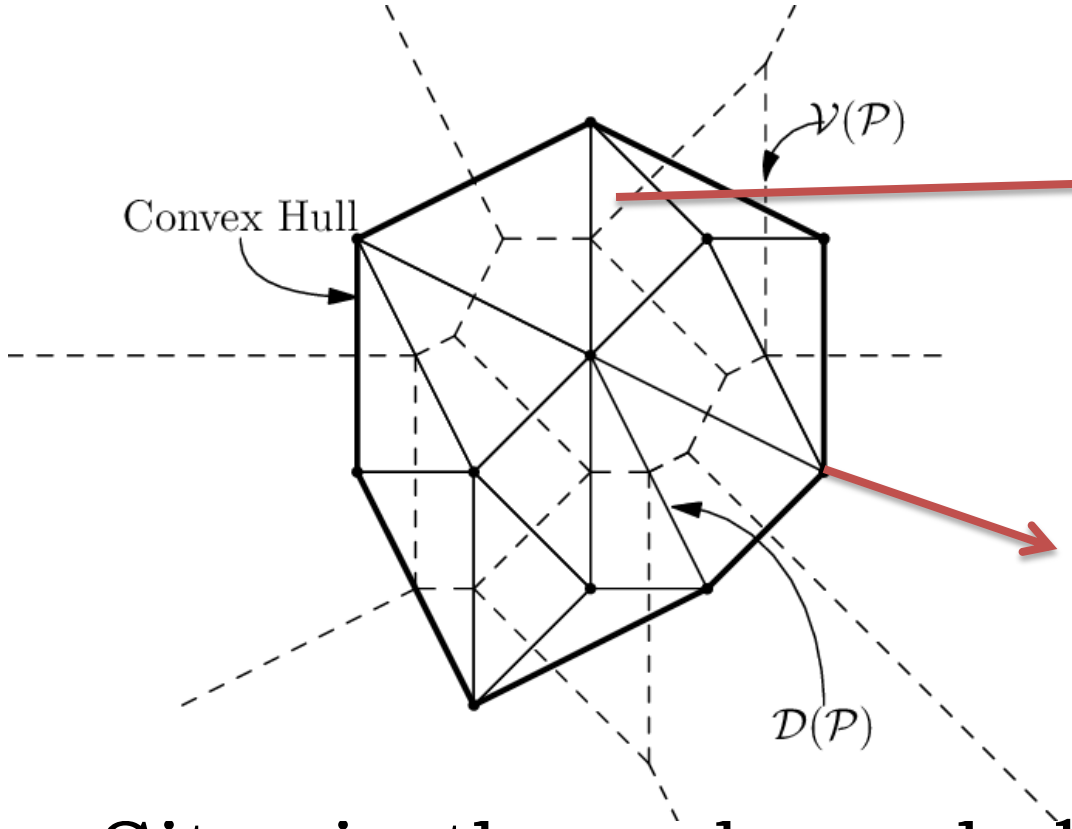
Property #3

- The dual graph forms a special kind of triangualtion called the ***Delaunay Triangulation.***





Property #4



Voronoi Diagram

Delaunay Triangulation

- Sites in the unbounded voronoi cells correspond to the vertices of the ***convex hull***

General Properties

- If the points P are collinear, the voronoi diagram consists of n-1 parallel lines forming n cells.
- If the points P are not collinear, the edges of the voronoi diagram are segments, or half lines(which may terminate at useful border).
- The number of vertices in a voronoi diagram of a set of n points is at most 2n-5 and the number of edges are at most 3n-6.
- Voronoi cells or voronoi partitions are convex.
- The voronoi cell of P_i is unbounded if P_i lies on the convex hull of P.
- "Largest Empty Circle of q wrt P" is the largest circle whose center is q and which contains no point of P in its interior.
- A point q is a vertex of a voronoi diagram if its largest empty circle has three or more points of P on its circumference.
- This circle is the circumcircle of the corresponding Delaunay Triangle.
- The bisector between P_i and P_j defines an edge of the voronoi diagram of P iff there is a point q in the plane such that its largest empty circle contains both P_i and P_j on its circumference and no other member of P.

PROPERTIES (without proofs)

- *If the points P are collinear, the* ***voronoi diagram*** *consists of* ***n-1 parallel lines forming n cells****.*
- *If the points P are not collinear, the edges of the* ***voronoi diagram are segments****, or* ***half lines****(which may terminate at useful border).*
- *The number of vertices in a voronoi diagram of a set of n points is at most* ***2n-5*** *and the number of edges are* ***at most 3n-6****.*
- *Voronoi cells or voronoi partitions are* ***convex.***
- ***The voronoi cell of P_i is unbounded*** *if P_i lies on the convex hull of P.*
- ***"Largest Empty Circle of q wrt P"*** *is the largest circle whose center is q and which contains no point of P in its interior.*
- ***A point q*** *is a vertex of a voronoi diagram if its largest empty circle has* ***three or more points of P*** *on its circumference.*
- *This circle is the* ***circumcircle of the corresponding Delaunay Triangle****.*
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Applications

- ***Nearest neighbor search:*** For a query point ,
  - finding its nearest neighbor from a fixed set of points is simply a matter of determining which cell in the Voronoi diagram of contains .
- ***Facility location:*** Suppose McDonald's wanted to open another restaurant.
  - To minimize interference with existing McDonald's, it should be located as far away from the closest restaurant as possible.
  - This location is always at a vertex of the Voronoi diagram, and it can be found in a linear-time search through all the Voronoi vertices.
- ***Largest empty circle:*** Suppose you needed to obtain a large,
  - contiguous, undeveloped piece of land on which to build a factory.
  - The same condition used for picking McDonald's locations is appropriate for other undesirable facilities, namely that it be as far as possible from any relevant sites of interest.
  - A Voronoi vertex defines the center of the largest empty circle among the points.
- ***Path planning:*** If the sites of are the centers of obstacles we seek to avoid,
  - the edges of the Voronoi diagram define the possible channels that maximize the distance to the obstacles.
  - Thus in planning paths among the sites, it will be safest to stick to the edges of the Voronoi diagram.