Assignment 1

Raja Shan Zaker Mahmood, s144102
Peter Alexander Rousing Bork, s145779

October 2, 2017

Intro

All references to Perko refer to the 3rd edition from 2000 by Springer.

Problem 1

We will be looking at the following linear IVP.

$$\dot{\mathbf{z}} = \mathbf{B}\mathbf{z},$$
$$\mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{R}^3$$

Where the matrix $\mathbf{B}$ is given by.

$$\mathbf{B} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a)

We begin by solving the IVP. From this, we will find the stable and unstable space, E^s and E^u, respectively. From the fundamental theorem for linear systems (Perko, p. 17), we define the solution by way of the matrix exponential. As the matrix $\mathbf{B}$ is upper triangular, we can read of the eigenvalues of the matrix directly from the diagonal. We therefore have an eigenvalue $\lambda_1 = -1$, with algebraic multiplicity (abbr. AM) 2 and an eigenvalue $\lambda_2 = 1$ with AM=1. We find the associated eigenspace for each eigenvalue. In the following, I denotes the identity matrix, of dimension 3 throughout. $\mathbf{v}$ denotes the eigenvector for the associated eigenvalue. First for λ_1

$$
\begin{aligned}
(\mathbf{B}-\lambda_1 I)\,\mathbf{v} &= 0\\
&\rightarrow\\
\left(\begin{pmatrix}-1 & -1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}-(-1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= 0\\
&\rightarrow\\
\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= 0\\
&\rightarrow\\
v_2 &= 0\\
2v_3 &= 0
\end{aligned}
$$

So v_2 and v_3 must be 0 and we have no restrictions on v_1. We can thus choose $\mathbf{v}=(1,0,0)^T$. This eigenvector has geometric multiplicity (abbr GM) 1, less than the AM for the associated eigenvalue. Now for λ_2

$$
\begin{aligned}
(\mathbf{B}-\lambda_2 I)\,\mathbf{v} &= 0\\
&\rightarrow\\
\left(\begin{pmatrix}-1 & -1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}-(1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= 0\\
&\rightarrow\\
\begin{pmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} &= 0\\
&\rightarrow\\
-2v_1+v_2 &= 0\\
-2v_2 &= 0
\end{aligned}
$$

So from the relations for the components, we can choose $\mathbf{v}=(0,0,1)$. As we had $GM < AM$ for λ_1, we must find a generalized eigenvector to fill out the space.

$$
\begin{aligned}
(\mathbf{B}-\lambda_1 I)^2\mathbf{v} &= 0 \\
&\rightarrow \\
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}^2 \mathbf{v} &= 0 \\
&\rightarrow \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{v} &= 0 \\
4v3 &= 0
\end{aligned}
$$

We can thus choose eigenvectors of the general form $\mathbf{v}(a,b,0)^T$, $a,b \in \mathbb{R}$. We want to obtain a basis for a 2 dimensional space, so we can choose the orthonormal set $\{(1,0,0)^T,(0,1,0)^T\}$. Now we can classify the stable and unstable spaces for the linear problem. They are the eigenvector-spaces belonging to the negative and positive eigenvalues respectively.

$$
E^s = \text{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)
$$

$$
E^u = \text{span}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)
$$

If our coordinate system is interpreted as conventional, cartesian 3D-space, we can thus say that the x,y-plane is stable and the z-axis is unstable. Continuing with the solution, we have now found our matrix P. The calculation is greatly simplified by P being the identity matrix. It's inverse is thus the identity matrix itself.

$$
P = (v_1\, v_2\, v_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I = P^{-1}
$$

So the diagonal matrix of the eigenvalues is not changed under the transform defined by the P matrices. The diagonal matrix, will be called Λ. From this we easily find the S and N matrix.

$$
S = P\Lambda P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
N = \mathbf{B} - S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

N is nilpotent of order 2, as $N^2 = 0$. This gives us a simple expression for the matrix exponential. From theorem 1 (Perko, p. 33).

$$e^{\mathbf{B}t} = Pe^{\Lambda t}P^{-1}e^{Nt}$$
$$\rightarrow$$
$$e^{\mathbf{B}t} = P\begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} P^{-1}(I + Nt)$$
$$\rightarrow$$
$$e^{\mathbf{B}t} = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} + \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}\begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\rightarrow$$
$$e^{\mathbf{B}t} = \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}$$

We can thus state that the solution to the problem is the following.

$$\mathbf{z}(t) = \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}\mathbf{z}_0$$

We can check the IC, by inserting $t = 0$ into the equation.

$$\mathbf{z}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{z}_0 = \mathbf{z}_0$$

As desired.

b)

We consider the subspace S of $\mathbb{R}^3$. We assume $\mathbf{z}_0 \in S$. From this we want to show that $\mathbf{z}(t) \in S$ for all $t \in \mathbb{R}$. So $\mathbf{z}_0 \in S$ means:

$$\mathbf{z}_0 \in \operatorname{span}\begin{pmatrix} k_1 \\ 0 \\ k_2 \end{pmatrix}, k_{1,2} \in \mathbb{R}$$

We can then write for the time dependent solution, using the matrix exponential from section 1(a).

$$\mathbf{z}(t) = e^{\mathbf{B}t}\mathbf{z}_0$$
$$\rightarrow$$
$$\mathbf{z}(t) = \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}\begin{pmatrix} k_1 \\ 0 \\ k_2 \end{pmatrix} = \begin{pmatrix} k_1 e^{-t} \\ 0 \\ k_2 e^{t} \end{pmatrix}$$

We thus see that if the solution starts in S, it will stay there for all time, making the set invariant with respect to the flow of the differential system investigated. In other words $\mathbf{z}_0 \in S \to \mathbf{z}(t) \subset S$ for all $t \in \mathbb{R}$.

c)

Now we want to show that if $\mathbf{w} \in S$, then $\mathbf{Bw} \subset S$. We assume that S is the set defined in exercise $1(b)$ and that $\mathbf{B}$ is the matrix defined in exercise $1(a)$. We explicitly do the calculation of the element $\mathbf{Bw}$.

$$\mathbf{Bw} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ 0 \\ k_2 \end{pmatrix} = \begin{pmatrix} -k_1 \\ 0 \\ k_2 \end{pmatrix} \in S$$

And this is the desired result.

d)

We now look at a general subspace of $\mathbb{R}^n$, which we call E. With A a $n \times n$ matrix, we state that for all $\mathbf{v} \in E$.

$$A\mathbf{v} \in E$$

We assume $\mathbf{x}(t) \in \mathbb{R}^n$ is the solution to the following IVP:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

With $\mathbf{x}(0) = \mathbf{x}_0$. As this is a linear system, we can solve it explicitly. The solution to the system can be written in terms of the matrix exponential of the matrix A, as stated in the fundamental theorem for linear systems (Perko, p. 17).

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

Using the definition of the matrix exponential, this can be rewritten as a series of matrix powers (Perko p. 12).

$$\mathbf{x}(t) = \sum_{k=0}^{n} \frac{(At)^k}{k!}\mathbf{x}_0$$

For $k = 0$, the above expression belongs to E, as $\mathbf{x}_0 \in E$. The series is absolutely convergent (Perko, p. 12). As the space E is closed, the limit of any convergent sequence contains its limit point. As the series converges towards e^{At}, we have to show that each partial sum belongs to E. We can do this recursively, by seeing that at each summation, we are multiplying an element in E by a matrix raised to some power k, together with a scaling by the factorial of k. This corresponds to successive matrix multiplications and so, the elements still belongs to E. For $k = [0, n] \in \mathbb{N}$.

$$A^k v = A^{k-1}(A\mathbf{v}) \in E$$

The scaling does not change membership of E. So the sequence is in E and the sequence is convergent. That the set E is complete, means that the limit point of the sequence is in the set. THe limit point of the sequence is exactly the solution $\mathbf{x}(t)$.

Problem 2

Now we consider a planar system.

$$\dot{u} = u^2 \tag{1}$$

$$\dot{v} = -v \tag{2}$$

a)

We want to check whether $(u, v)^T = (0, 0)^T$ is a hyperbolic point in the phase space. First we see whether it is a critical point for the system and it trivially is, as the right hand sides of the system are both 0 in the point. To find out whether this critical point is hyperbolic, we must investigate the linearisation of the system, as stated in Definition 1 (Perko, p. 102). We state the system matrix and call it $f(u, v)$.

$$f(u,v) = \begin{pmatrix} u^2 \\ -v \end{pmatrix} = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}$$

The linearisation is found by the Jacobian.

$$Df(u,v) = \begin{pmatrix} \frac{\partial f_1(u,v)}{\partial u} & \frac{\partial f_1(u,v)}{\partial v} \\ \frac{\partial f_2(u,v)}{\partial u} & \frac{\partial f_2(u,v)}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 0 \\ 0 & -1 \end{pmatrix}$$

And in the critical point.

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrix is diagonal and so, we can read the eigenvalues of the diagonal. One of the eigenvalues is identically 0 and so, the critical point is not hyperbolic.

b)

We now define a function $h : \mathbb{R} \to \mathbb{R}$, that is smooth and is given by.

$$h(u) = \begin{cases} \exp(u^{-1}), & u < 0 \\ 0, & u \geq 0 \end{cases}$$

We now want to find the flow for the system, satisfying $(u(t), v(t)) = \varphi_t(u_0, v_0)$. Even though the system is nonlinear, we can make headway by using that the equations are decoupled and we can use separation of variables. Using the IC $(u(0), v(0)) = (u_0, v_0)$, we solve the equations, first for u.

$$\begin{aligned}
\dot{u} &= u^2 \\
&\rightarrow \\
u^{-2} du &= dt \\
&\rightarrow \\
\int_{u_0}^{u} u^{-2} du &= \int_0^t dt \\
&\rightarrow \\
-\frac{1}{u} + \frac{1}{u_0} &= t \\
&\rightarrow \\
u(t) &= -\frac{u_0}{tu_0 - 1}
\end{aligned}$$

As a consistency check, we insert the IC. Now for the equation for v.

$$\begin{aligned}
\dot{v} &= -v \\
&\rightarrow \\
-v^{-1} dv &= dt \\
&\rightarrow \\
\int_{v_0}^{v} v^{-1} dv &= \int_0^t dt \\
&\rightarrow \\
\ln(v) - \ln(v_0) &= t \\
&\rightarrow \\
v(t) &= v_0 e^t
\end{aligned}$$

So we can state the flow of the system as.

$$\varphi_t(u_0, v_0) = \varphi(t, u_0, v_0) = \begin{bmatrix} -\frac{u_0}{tu_0 - 1} \\ \\ v_0 e^t \end{bmatrix}$$

Now we want to find out whether the following set is invariant with respect to the found flow.

$$U = \{(u, v) \in \mathbb{R} | v = h(u)\}$$

To investigate whether the set is in fact invariant, we set the start condition to be a member of the set, $v_0 = h(u_0) = e^{u_0^{-1}}$. The time evolution will then take the form.

$$\varphi_t(u_0, h(u_0)) = \begin{bmatrix} -\frac{u_0}{tu_0-1} \\ \\ e^{u_0^{-1}+t} \end{bmatrix} = \begin{bmatrix} -\frac{u_0}{tu_0-1} \\ \\ e^{\frac{1+tu_0}{u_0}} \end{bmatrix}$$

We can from this see that for all negative t, the set U is invariant with respect to the flow. For positive as well as the exponential term is 0 instead on both sides of the second equality above.

c)

We now want to show that the relation $v = h(u)$ satisfies the following relation holds.

$$\dot{v} = h'(u)\dot{u}$$

As u depends on t, we can use the chain rule to obtain a derivative of $h(u(t))$. The chain rule states that the derivative of the composition of functions is equal to the product of their respective derivatives.

$$\frac{d}{dt}(h(u(t))) = \frac{d}{du}h(u) \cdot \frac{d}{dt}u(t) = h'(u)\dot{u}$$

Substituting in $\dot{u} = u^2$ and $\dot{v} = -v$, as well as $h(u) = e^{-u}$ for $t < 0$.

$$-v = \left(e^{u^{-1}}\right)' u^2$$
$$\rightarrow$$
$$-v = -\frac{e^{u^{-1}}}{u^2} u^2$$
$$\rightarrow$$
$$v = e^{u^{-1}} = h(u)$$

For $t > 0$, $h(u) = v = 0$ and so the relation is trivially fulfilled for all u.

d)

We now try to generalize the previous assertions by investigating a general, planar system.

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Here $(x, y) \in \mathbb{R}$, and both f and g are in C^1. In addition the solution to the problem is assumed to exist for all t. We then make the assumption that for a function $m : \mathbb{R} \rightarrow \mathbb{R}$, that is smooth, the following holds for $y = m(x)$.

$$\dot{y} = m'(x)\dot{x}$$

Using the definition of the nonlinear system, this can be restated, namely that for all x belonging to the real numbers, the following holds.

$$g(x, m(x)) = m'(x) f(x, m(x))$$

The assertion we are trying to show is that the following set is invariant, given the above assumptions. (Assumption A in the problem text, together with the problem statement).

$$X = \{(x, y) \in \mathbb{R} | y = m(x)\}$$

In these preliminaries, we had no dependence on time in the assumptions. We try to construct a solution to the problem, by postulating a function that does depend on time: $\tilde{x}(t)$. To prove invariance, we will have to check that the solution that starts in $(x, m(x))$ will stay there for all t.

$$\begin{aligned} x &= \tilde{x}(t) \\ y = m(x) &= m(\tilde{x}(t)) \end{aligned}$$

Taking the time derivatives and calling the resulting function h. We can use the existence and uniqueness theorem (Perko, p. 74) to show that if the function h is C^1, the function $\tilde{x}(t)$ exists and is unique on an interval around a point $\tilde{x}(0) = x_0$.

$$\begin{aligned} \dot{x} &= \tilde{x}'(t) = h(x) \\ \dot{y} = m'(\tilde{x}(t)) &= m'(x)h(x) \end{aligned}$$

By choosing $h(x) = f(x, m(x)) \in C^1$, we see from assumption A, that the function lies on the graph $y = m(x)$.

$$\begin{aligned} \dot{x} &= f(x, m(x)) \\ \dot{y} = m'(x) f(x, m(x)) &= g(x, m(x)) \end{aligned}$$

As this is the case for all x, the set $(x, m(x)) \in \mathbb{R}$ is invariant.

e)

The assumptions in part 2(d) can be interpreted geometrically. Basically they assert that the slope of the vector field in a point $(x, m(x))$ is equal to the slope of the function m in the point x.

$$m'(x) = \frac{g(x, m(x))}{f(x, m(x))} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$$

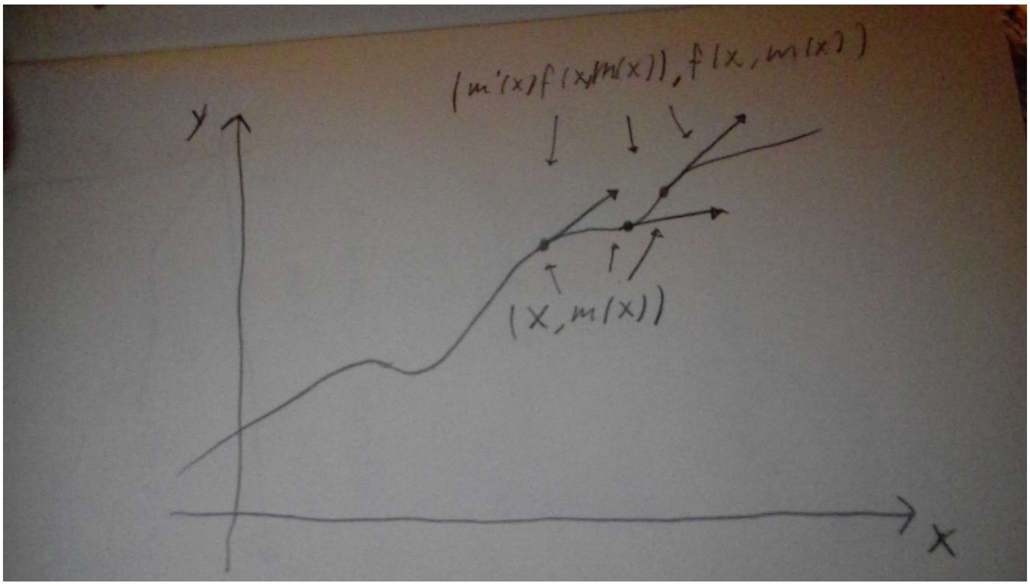
h 

Figure 1: The phase plane and the graph $y = m(x)$. Assumption A dictates that a starting on the curve, follows the lines defined by the tangent vectors in each point.

This would then mean that if we start on the function, the slope would define a trajectory tangent to the graph of the function. The vector field in the phase plane is composed of vectors in each point of the form $(g(x, m(x)), f(x, m(x)))^T$. From the assumption, we can write $(m'(x)f(x, m(x)), f(x, m(x)))^T$ instead. This is a vector that is tangent to the function m. It might be easier to explain with a picture and this is done in figure 1

That in turn explains why the set X from exercise 2(d) is invariant. A solution will follow the vector field from the point it starts to trace out the flow of the solution,. If the vector field obeys assumption A, the vectors of the field will be tangent to the graph of the function m.

f)

If the assumption is violated, i.e that $g(x, m(x)) \neq m'(x)f(x, m(x))$, we see geometrically that the change in a point in the phase plane is not necessarily equal to the tangent of the function. The set of points on the function is hence no longer invariant on the function m. Let us assume the hypothesis that the assumption is violated and look at a point on the graph. The solution curve now follows a trajectory that is not necessarily dictated by the tangent vectors of the function m, taking the solution outside the graph and hereby the set X from 2(d). This situation is shown in figure 2. The flow of a point $(x_0, m(x_0))$ starting on the curve, can go outside the curve.

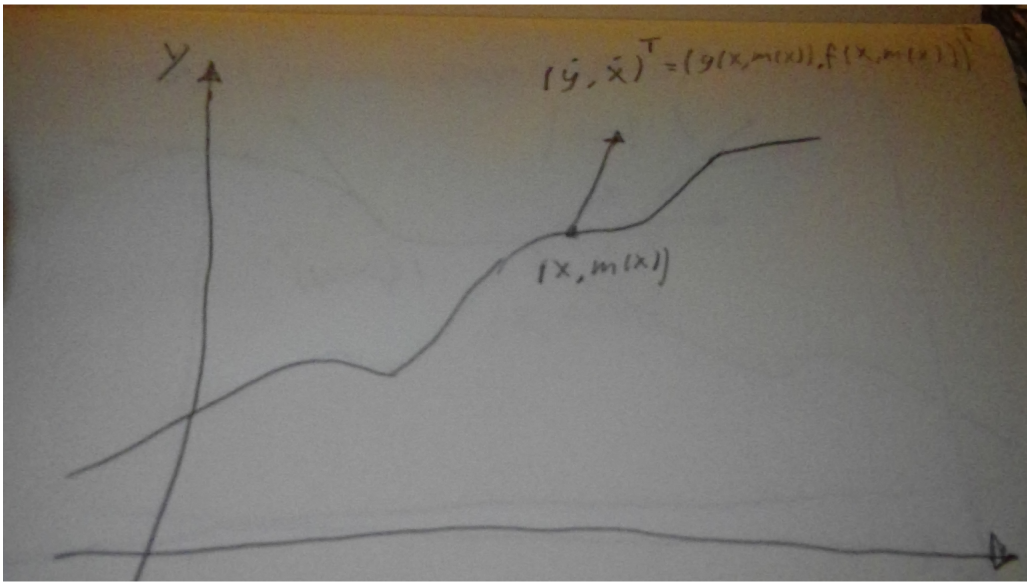
h 

Figure 2: The phase plane and the graph M(x), with $g(x, m(x) \neq m'(x) f(x, m(x))$. We see that the vector in the point does not necessarily point along the tangent.