Assignment 2

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A set of equations give an approximation to a gravitational problem.

$$\dot{\theta} = \sin(\theta)\, e + \theta e^2$$
$$\dot{e} = -e + \theta^2 - \theta^2 e$$

With θ describing the phase and e is the orbital parameter. We will sometimes describe the system by the following shorthand.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

With $\mathbf{x} = (\theta, e)^T$ and $\mathbf{f}$ being a vector of the functions.

a)

The equilibrium point is found by setting the solving for θ and e in the following expression, setting the temporal derivatives equal to 0.

$$\sin(\theta)\, e + \theta e^2 = 0$$
$$-e + \theta^2 - \theta^2 e = 0$$

$(\theta, e) = (0, 0)$ is an equilibrium point for the system. We investigate the properties of the solutions near the point by linearisation. This is done by taking the Jacobian of the system.

$$Df(\theta, e) = \begin{pmatrix} -\cos(\theta)e + e^2 & \sin(\theta) + 2\theta e \\ 2\theta - 2\theta e & -1 - \theta^2 \end{pmatrix}$$

So for the equilibrium point, we get the following Jacobian.

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The Jacobian in the point has an eigenvalue with zero real part and an eigenvalue with real part negative. According to the Center Manifold Theorem (Perko p. 155, Theorem 1), this shows that we have a stable manifold tangent to the e-axis in the origin, as well as a center manifold tangent to the θ-axis in the origin. The center manifold can be found by applying the Center Manifold theorem. We wish to compute it up to and excluding cubic terms. As the stable manifold is tangent to the θ-axis, the stable manifold can be constructed as a graph over θ.

$$e = h(\theta) = a\theta^2 + \mathcal{O}(\theta^3) \tag{1}$$

By the chain rule, we get the following condition on the graph defining the center manifold. This is also stated in (Perko, p. 155, Theorem 1).

$$\dot{e} = Dh(\theta)\dot{\theta}$$

We insert the derivatives as they are defined in the initial system, with $e = h(\theta)$.

$$Dh(\theta)(\sin(\theta)h(\theta) + \theta h(\theta)^2 = -h(\theta) + \theta^2 - \theta^2 h(\theta) \tag{2}$$

Inserting the power series in equation 1 into equation 2 gives us a relation that the center manifold must obey.

$$D\left(a\theta^2\right)\left(\sin(\theta)(a\theta^2) + \theta(a\theta^2)^2\right) = -a\theta^2 + \theta^2 - \theta^2 a\theta^2$$
$$\leftrightarrow$$
$$D\left(a\theta^2\right)\left(\sin(\theta)(a\theta^2) + \theta(a\theta^2)^2\right) + a\theta^2 - \theta^2 + \theta^2 a\theta^2 = 0$$

This puts a condition on the constant a, as each power of θ must vanish due to linear independence of polynomials. We ignore cubic terms and higher, leaving very few terms and fixing the value of a.

$$a = 1$$

And the center manifold then takes the following form.

$$h(\theta) = \theta^2 + \mathcal{O}(\theta^3)$$

A comment on notation. In the entire assignment, we discard cubic or higher terms. This is done as soon as they appear, and they are treated as zero. We will mostly not write them in the equations, unless we want to point something out. When we later deal with two variables, cubic terms in both will be discarded as well as cross terms with quadratic term in one variable and linear in the second as the term is cubic.

b)

The dynamics on the center manifold can be found from the equation for the temporal evolution of the tangent line of the center manifold. Ignoring cubic terms.

$$\dot{\theta} = \sin(\theta)h(\theta) + \theta h(\theta)^2 = \sin(\theta)\theta^2 + \mathcal{O}(\theta^5)$$

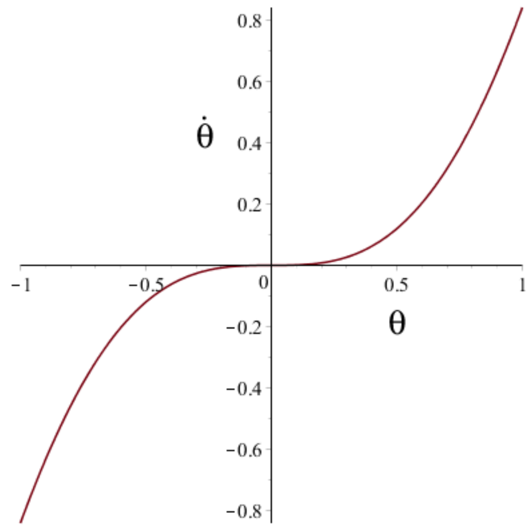


Figure 1: Stability of the equilibrium at the origin. We see that it is unstable, as any perturbation will lead an orbit away from the origin.

c)

The stability of the equilibrium is straightforward to state, as the center manifold defines a 1-dimensional manifold. We therefore see that the time evolution of the phase is negative for negative phase and positive for positive phase. This makes the equilibrium unstable, as small perturbations away from the point will evolve towards the center manifold, away from the equilibrium. This can be illustrated by use of pplane8 in figure 2

We see that the solution curves decay towards the center manifold.

d)

We now change the system to include a small parameter μ.

$$\begin{aligned} \dot{\theta} &= \sin(\theta)\, e + \theta e^2 \\ \dot{e} &= -e + \theta^2 - \theta^2 e + \mu \\ \dot{\mu} &= 0 \end{aligned}$$

We now want to find the center manifold of this system, near the equilibrium at $(\theta, e, \mu) = (0, 0, 0)$. The only change in the linearisation is, that we know have a 2-dimensional center manifold and so, we must adjust the method accordingly. The equation is no longer in normal form, due to the addition of the parameter μ. This can be seen by the linear part of the system no longer being in Jordan form. We can rectify this by performing a change of coordinates. The procedure for how to do this, is exemplified in (Perko. p. 159, example 3). Our end goal is to have a system of the following form, where A is a diagonal matrix.

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x})$$

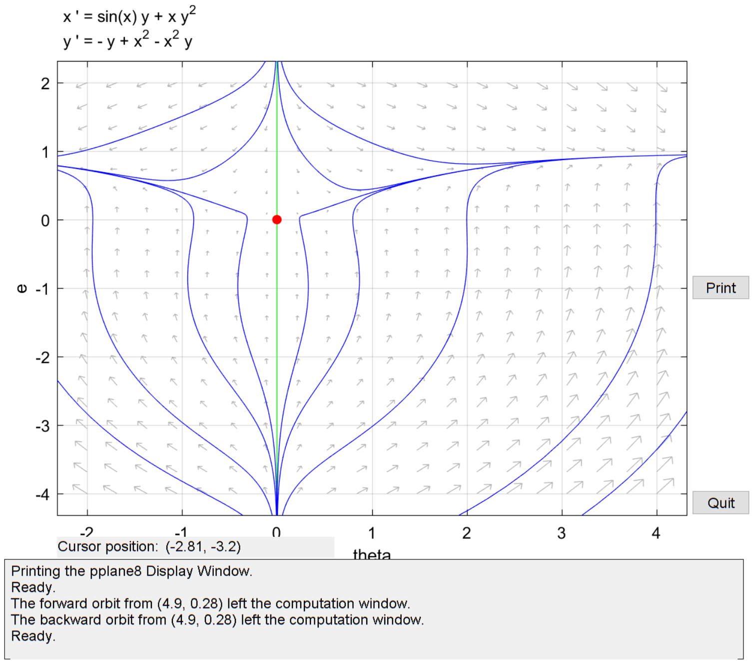


Figure 2: We see that the e-axis is a stable manifold. We also see that the stable manifold attracts the orbits and lead them away from the equilibrium as we calculated. We also see that close to the origin, the center manifold looks like a quadratic function.

As it stands now, the linear part of the stable manifold contains μ as well. So we begin with the linear part of the system, expressed as a matrix.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We find the eigenvalues.

$$\det(A - \lambda I) = -\lambda^2(1+\lambda)$$

This gives the eigenvalues $\lambda = -1$ and $\lambda = 0$. We find the associated eigenvectors. For the eigenvalue $\lambda = -1$.

$$(A - (-1)I)\mathbf{v} = 0$$

We get a possible eigenvector $\mathbf{v}_1$.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For the eigenvalue $\lambda = 0$.

$$(A))\,\mathbf{v} = 0$$

We get $v_2 = v_3$, giving a 2-dimensional eigenvector space.

$$\mathbf{v}_2 = \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

From this we can construct a matrix for changing coordinates. It is composed of the eigenvectors of A as its columns.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We also find the inverse, as $\det(P) \neq 0$.

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

The system can now be expressed in the new basis. The linear part is diagonalized as we are used to, while the non-linear part is transformed, followed by a reverse transform on the non-linear functions.

$$\dot{\mathbf{x}} = P^{-1}AP\mathbf{x} + P^{-1}f(P\mathbf{x})$$

The transformed linear system matrix is then.

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \theta \\ e \\ \mu \end{pmatrix} = \begin{pmatrix} \theta \\ e+\mu \\ \mu \end{pmatrix}$$

$$P^{-1}f(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sin(\theta)e + \theta e^2 \\ \theta^2 - \theta^2 e \\ 0 \end{pmatrix} = \begin{pmatrix} \sin(\theta)e + \theta e^2 \\ -\theta^2 + \theta^2 e \\ 0 \end{pmatrix}$$

We can now state the transformed system.

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \theta \\ e \\ \mu \end{pmatrix} + \begin{pmatrix} \sin(\theta)(e+\mu) + \theta(e+\mu)^2 \\ -\theta^2 + \theta^2(e+\mu) \\ 0 \end{pmatrix}$$

Written explicitly an an ODE system.

$$\begin{aligned}\dot{\theta} &= \sin(\theta)(e+\mu)+\theta(e+\mu)^2\\ \dot{e} &= -e-\theta^2+\theta^2(e+\mu)\\ \dot{\mu} &= 0\end{aligned}$$

After the transformation, we find the manifold as in the previous example, now as a function of θ and μ. Remembering to use a 2-dimensional power series as trial function. The center manifold is tangent to the e and μ axis at the origin, so we can define the center manifold as a graph over these.

$$\theta = h(\theta,\mu) = a\theta^2 + b\theta\mu + c\mu^2$$

The equation for finding h.

$$\dot{e} = D_\theta h(\theta,\mu)\dot{\theta} + D_\mu h(\theta,\mu)\dot{\mu}$$

As $\dot{\mu}=0$, we only have to worry about one of the terms on the right. Substituting what we found for the system.

$$\left(2a\theta+b\mu\right)\left(\sin(\theta)\left(h(\theta,\mu)+\mu\right)+\theta\left(h(\theta,\mu)+\mu\right)^2\right)-(-h(\theta,\mu))-\left(\theta^2-\theta^2h(\theta,\mu)\right)=0$$

We solve for $h(e,\mu)$ in this equation and due to linear independence, this imposes restrictions upon the constants a,b,c. We find that $a=1$, $b=c=0$.

This again gives the following power series representation for the center manifold.

$$h(\theta,\mu)=\theta^2+\mathcal{O}(\theta^3,\mu^3)$$

e)

The reduced system is found by inserting the equation for the center manifold into the original system. The dynamics we get from the μ equation are trivial and so.

$$\dot{\theta}=\sin(\theta)\left(\theta^2+\mu\right)+\theta\left(\theta^2+\mu\right)^2$$

We simplify and discard cubic and higher terms.

$$\dot{\theta}=\sin(\theta)\theta^2+\sin(\theta)\mu$$

As the parameter μ passes through 0, the reduced system undergoes a qualitative change of dynamics. This can be shown with a bifurcation diagram, with μ taking the role of independent variable and the fixed points for θ being the dependent variable. The form of the bifurcation is that of a pitchfork. We also see that it is of the subcritical variety.

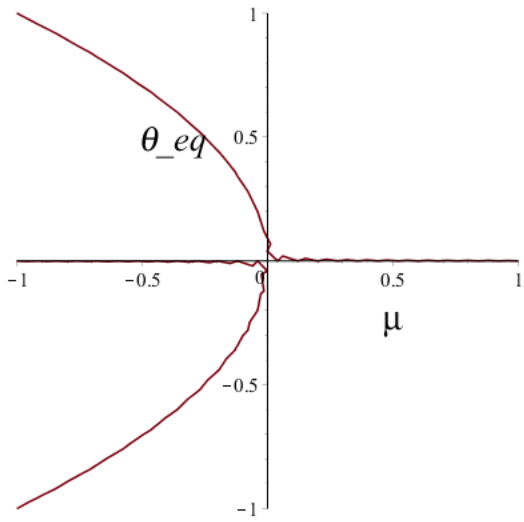


Figure 3: Bifurcation diagram for the reduced dynamics on the center manifold. For μ>0, we have that the equilibrium is unstable. For $\mu < 0$, the origin is stable and the prongs of the pitchforks are saddles and thus unstable. The point of bifurcation is thus $\mu = 0$.

To make it explicit.

$$\sin(\theta)\theta^2 + \sin(\theta)\mu = 0$$
$$\leftrightarrow$$
$$\theta^2 = -\mu$$

So for $\mu > 0$, we only have the equilibrium $\theta = 0$, while for $\mu < 0$, we have $\theta = 0, \sqrt{\mu}, -\sqrt{\mu}$.

f)

We have an expression for the center manifold, as well as the dynamics upon it. This together with the bifurcation discovered in the previous part, allows us to sketch the phase portrait for the system. We take the linearisation in the point $(\theta, e) = (0, 0)$, by use of the Jacobian.

$$Df(0,0) = \begin{pmatrix} \mu + \mu^2 & 0 \\ 0 & -1 \end{pmatrix}$$

For μ small and negative, the equilibrium at the origin is a nodal sink, as both the eigenvalues of the Jacobian are negative. For $\mu = 0$, the origin is non-hyperbolic, due to one of the eigenvalues of the Jacobian having zero real part. For μ small and positive, the origin is a saddle, as it has a positive (θ-direction) and negative (e-direction) eigenvalue. For $\mu = 0$, we are reduced to the old system, before μ was introduced.

The stability in the other equilibria can be found by fixing μ in the desired interval and computing the Jacobian. A small note in that this new system has it's e-axis inverted from the coordinate transformation. So the center manifold is concave downwards, when it was upwards for the original system. The sketches can be seen in figures 4, 5 and 6.

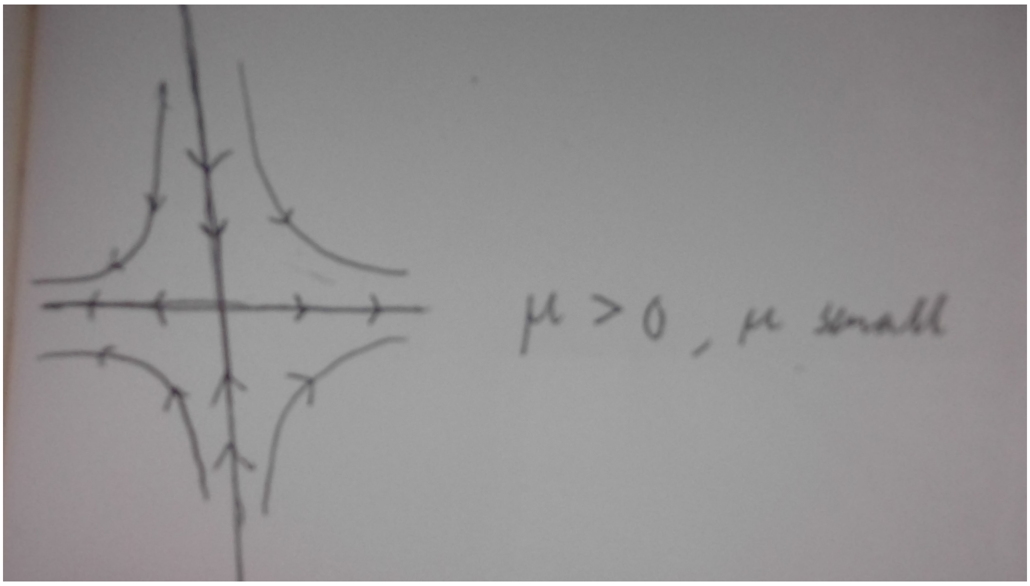


Figure 4:

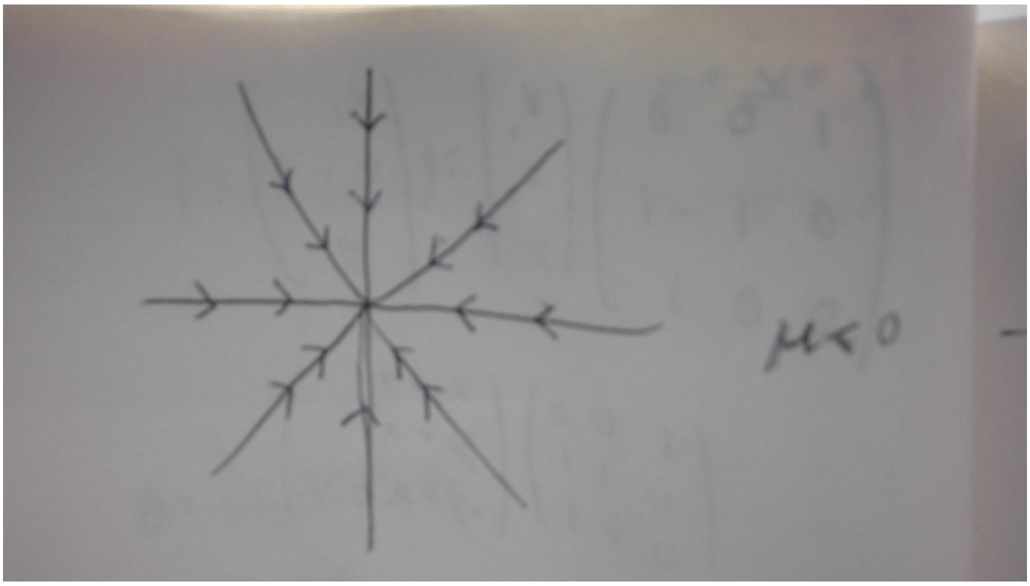


Figure 5:

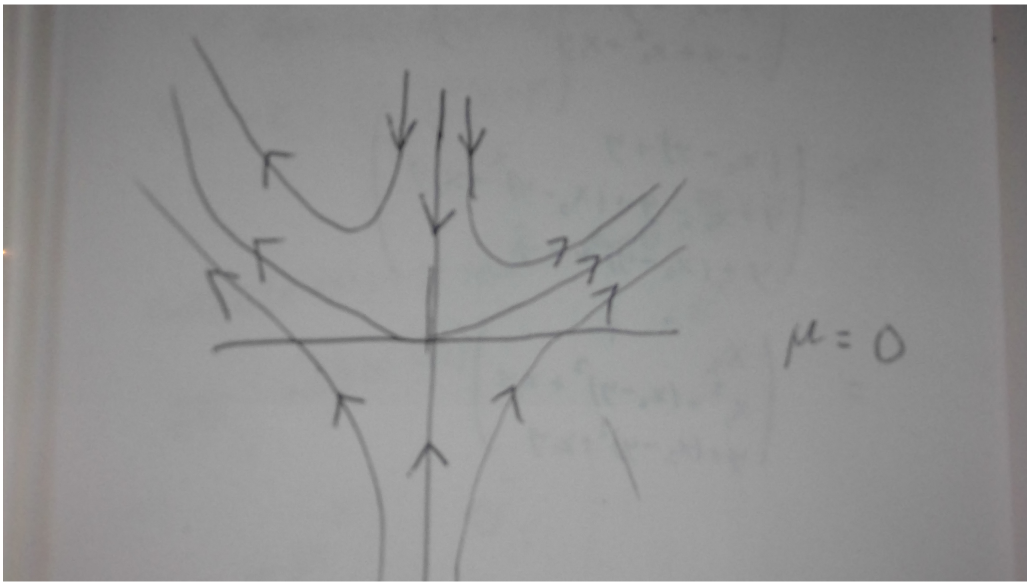


Figure 6:

Remark that μ is supposed small in all the cases and we are looking at the local system, far from other, eventual equilibria.

2

In a model of biological autocatalysis, we have two variables, x and y. Their coupled evolution is determined by the following ODE system.

$$\dot{x} = -\alpha x + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - \beta y$$

Where $\alpha, \beta > 0$.

a)

We want to begin by locating the nullclines for the system. They are found by identifying the curves at which the derivatives of the system are 0. On the x-nullcline, the derivative of x is zero and the same for y. We therefore isolate for y in both equations after setting them equal to zero and thereby obtain expression for the nullclines.

$$y_{null-x} = \alpha x$$
$$y_{null-y} = \beta^{-1} \frac{x^2}{1+x^2}$$

They have relatively simple forms, so we can sketch them as it is done in figure 7.

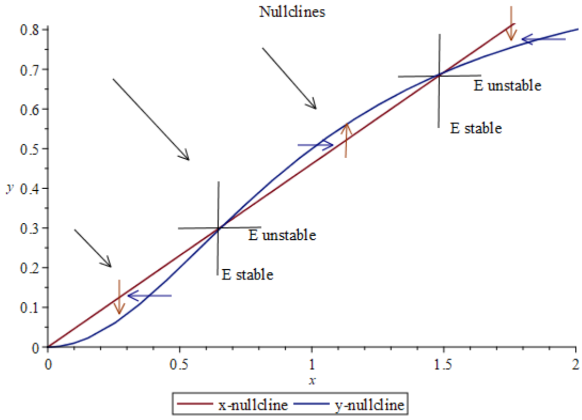


Figure 7:

b)

We now show that for β fixed and small α, we find three stationary points. For each of the points, we determine the stability and sketch the location of the linear subspaces E^u, E^s and E^c.

The nullclines intersect in the equilibrium points, in other words where.

$$y_{null-x} = y_{null-y}$$
$$\leftrightarrow$$
$$y_{null-x} - y_{null-y} = 0$$

Since for fixed β and sufficiently small α this gives a third degree polynomial with three solutions for x, we get three equilibrium points.

$$\alpha x - \frac{x^2}{\beta\left(1+x^2\right)} = \beta\alpha x^3 - x^2 + \beta\alpha x = 0$$

To verify this, we can examine the factorization and see the first equilibrium point at x = 0.

$$x\left(\beta\alpha x^2 - x + \beta\alpha\right) = 0$$

Additionally, the quadratic polynomial has two real solutions when $\alpha \leq 0.5$, since the discriminant in the quadratic root is then positive (giving both a positive and negative square root and using $\beta = 1$ for simplicity).

$$x_{eq} = \frac{1 \pm \sqrt{1-4\alpha^2}}{2\alpha}$$

$d = 1 - 4\alpha^2 > 0$ for $\alpha < \frac{1}{2}$. Let's use $(\alpha, \beta) = (.25, 1)$, so we get.

$$y_{null-x} = 0.25x$$
$$y_{null-y} = \frac{x^2}{1+x^2}$$

These lines intersect in the origin and in two more points.

$$x_{eq} = \left[x : \frac{x^2}{4} - x + \frac{1}{4} = 0\right] = \frac{1 \pm \sqrt{1-4(\frac{1}{4})^2}}{2(\frac{1}{4})}$$
$$\leftrightarrow$$
$$x_{eq} = 2 \pm 2\sqrt{3/4} = 2 \pm \sqrt{3}$$

There are thus three equilibrium points in this given settings. The y-values for the points are found from the y-values using either nullcline:

$$(0,0)$$
$$\left(2-\sqrt{3}, 1/2-(1/4)\sqrt{3}\right)$$
$$\left(2+\sqrt{3}, 1/2+(1/4)\sqrt{3}\right)$$

The stability of an equilibrium point can be found by examining the eigenvalues of the linearised system evaluated at the given equilibrium point. The linearised system is the Jacobian of the original system.

$$\begin{pmatrix} \dot{x}_{lin} \\ \dot{y}_{lin} \end{pmatrix} = D\begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} -\alpha & 1 \\ \frac{2x}{x^2+1} - \frac{2x^3}{(x^2+1)^2} & -\beta \end{pmatrix} = \begin{pmatrix} -\alpha & 1 \\ \frac{2x}{(x^2+1)^2} & -\beta \end{pmatrix} = A(x,y)$$

The origin is easily found to be a stable equilibrium point since the eigenvalues of the linearised system evaluated at the origin are both negative (remember that the eigenvalues of an upper-triangular matrix are the diagonal).

$$A(0,0) = \begin{pmatrix} -\frac{1}{4} & 1 \\ 0 & -1 \end{pmatrix}$$

Since both dimensions are stable, we have.

$$E^s = \{(a,b)|a,b \in \mathbb{R}^2\}, E^u = E^c = \emptyset$$

The two other equilibrium points can be evaluated in a similar fashion, but also geometrically. First a quick algebraic approach in which we will use an expression for the equilibrium x in the Jacobian and use what we know about the determinant and the trace to determine the stability. We saw above that.

$$x\left(\beta\alpha x^2 - x + \beta\alpha\right) = 0$$

Divide x out once and move the negative x to the other side to get.

$$x = \beta\alpha\left(1 + x^2\right)$$

Insert this in the numerator of the complicated entry in the Jacobian and we have.

$$\frac{2x}{\left(1+x^2\right)^2} = \frac{2\beta\alpha\left(1+x^2\right)}{\left(1+x^2\right)^2} = \frac{2\beta\alpha}{1+x^2}$$

Since the trace is always negative, all the equilibria must be either stable or saddles (Perko p. 25). The determinant of the Jacobian at the equilibria is therefore.

$$\delta = \det\left(\begin{pmatrix} -\alpha & 1 \\ \frac{2}{\left(x^2+1\right)^2} & -\beta \end{pmatrix}\right) = \alpha\beta - \frac{2\beta\alpha}{1+x^2}$$

When $\frac{2\beta\alpha}{1+x^2} > \alpha\beta$, the determinant is negative thus giving a saddle.

$$\begin{aligned} \frac{2\beta\alpha}{1+x^2} &> \alpha\beta \\ &\rightarrow \\ \frac{2}{1+x^2} &> 1 \\ &\rightarrow \\ 2 &> 1+x^2 \\ &\rightarrow \\ x^2 &< 1 \\ &\rightarrow \\ x &< 1 \end{aligned}$$

In other words, the equilibrium where $x < 1$ is a saddle because the determinant is negative (ibid.). The determinant is positive for the equilibrium where $x > 1$ and is therefore either a stable node (or a stable focus) (ibid.). We can determine that the stable equilibrium is a node since Perko says this is the case when $\tau - 4\delta > 0$.

$$A\left(2-\sqrt{3}, \frac{1}{2} - \frac{1}{4}\sqrt{3}\right) = \begin{pmatrix} \frac{-1}{4} & 1 \\ \frac{2(2-\sqrt{3})}{1+(2-\sqrt{3})^2} - \frac{2(2-\sqrt{3})^3}{(1+(2-\sqrt{3})^2)^2} & -1 \end{pmatrix}$$

The eigenvalues for this matrix are numerically evaluated. In the x-direction, it is equal to 0.15 to two significant digits and −1.40 in the y-direction. The eigenvalues reveal that the x dimension is unstable whereas the y dimension is stable for this equilibrium.

$$E^s = \text{span}\{(0,1)\}, E^u = \text{span}\{(1,0)\}, E^c = \emptyset$$

$$A\left(2+\sqrt{3}, \frac{1}{2}+\frac{1}{4}\sqrt{3}\right) = \begin{pmatrix} -\frac{1}{4} & 1 \\ \frac{2(2+\sqrt{3})}{1+(2+\sqrt{3})^2} - \frac{2(2+\sqrt{3}^3}{(1+(2+\sqrt{3})^2} & -1 \end{pmatrix}$$

THe eigenvalues for this matrix are evaluated to -0.21 in the x-direction and -1.04 in the y-direction. The eigenvalues reveal that both dimensions are stable for this second equilibrium:

$$E^s = \text{span}\{(0,1),(1,0)\}, E^u, E^c = \emptyset$$

The geometric approach will identify a perhaps more intuitive and enlightening observation: The nullclines intersect each other from opposite directions. In the hand-drawing of the vector field, follow the y-nullcline (blue) from left to right and notice that the horizontal- (x-)direction of the flow points away from the equilibrium point both before and after their intersection. This is the geometric meaning of an unstable dimension. Similarly, follow the y-nullcline (red) from left to right around the second equilibrium point and notice that the horizontal flow is directed towards the equilibrium point on both sides. This is the geometric meaning of a stable dimension. Using the same approach for the y-dimension yields the complete picture of stable and unstable sets, and is of course identical to the algebraic approach.

c)

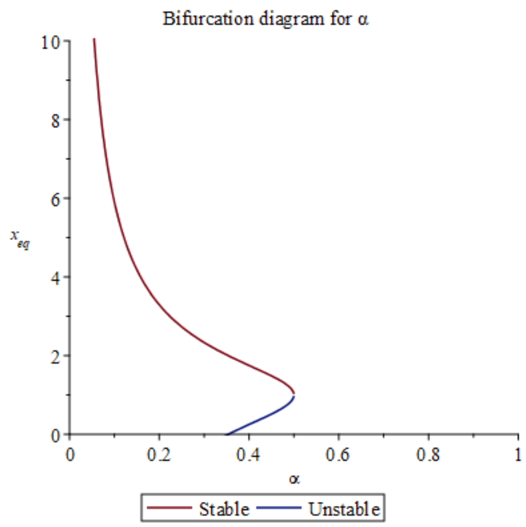
Now, still for fixed β, we argue that increasing the value of α causes a saddle-node bifurcation to occur. Again we can use an algebraic and a geometric approach. Let's start with the geometric approach: As α increases, the two intersections of the nullclines in the diagram becomes a single point when the x-nullcline is tangent to the y-nullcline and then they vanish all together. The value of alpha; when the two are tangent is α_c and marks the bifurcation between two equilibria on one side and zero equilibria on the other side.

We saw algebraically that the equilibrium points were the solutions to a quadratic polynomial in which the discriminant is a function of α . As α increases, this discriminant goes from positive to negative. This annihilates the two real valued solutions and the two nullclines no longer intersect each other for points on the real axis.

$$x_{eq} = \frac{1 \pm \sqrt{1-4\alpha^2}}{2\alpha}$$

3

For a simple model of the time evolution of two regulating hormones is given by the following coupled dynamical system.



$$\dot{F} = F(t)\left(2\sqrt{F(t)^2+G(t)^2} - \left(F(t)^2+G(t)^2\right)^{3/2}\right) - G(t)\left(2 + \frac{G(t)}{\sqrt{F(t)^2+G(t)^2}}\right)$$
$$\dot{G} = G(t)\left(2\sqrt{F(t)^2+G(t)^2} - \left(F(t)^2+G(t)^2\right)^{3/2}\right) + F(t)\left(2 + \frac{G(t)}{\sqrt{F(t)^2+G(t)^2}}\right)$$

In addition, $(F,G) \neq (0,0)$.

a)

We begin by establishing the coordinate transformation "dictionary". By treating the F and G axes as planar, cartesian coordinates, we can define polar coordinates in standard form. For $(F,G) \rightarrow (r,\theta)$.

$$F = r\cos(\theta)$$
$$G = r\sin(\theta)$$

And for $(r,\theta) \rightarrow (F,G)$.

$$r = \sqrt{F^2+G^2}$$
$$\theta = \tan^{-1}\left(\frac{G}{F}\right)$$

We then have.

$$\begin{pmatrix}\dot{F}\\ \dot{G}\end{pmatrix} = \begin{pmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{pmatrix}\begin{pmatrix}\dot{r}\\ \dot{\theta}\end{pmatrix}$$

The determinant of the coordinate matrix can be calculated by using the trigonometric identity and it is $r = \sqrt{F^2 + G^2} \neq 0$, by the assumption of non-zero coordinates. This means the matrix has an inverse. This can be used to find the reverse coordinate transformation.

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}^{-1} \begin{pmatrix} \dot{F} \\ \dot{G} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{F} \\ \dot{G} \end{pmatrix}$$

We handle the substitution of the coordinates piece by piece.

$$\begin{aligned} F(t)^2 + G(t)^2 &= r^2 \\ \sqrt{F(t)^2 + G(t)^2} &= \sqrt{r^2} = r \\ \left(F(t)^2 + G(t)^2\right)^{3/2} &= r^3 \end{aligned}$$

For the system, we therefore have.

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} r\cos(\theta)\left(2r - r^3\right) - r\sin(\theta)\left(2 + \frac{r\sin(\theta)}{r}\right) \\ r\sin(\theta)\left(2r - r^3\right) + r\cos(\theta)\left(2 + \frac{r\sin(\theta)}{r}\right) \end{pmatrix}$$

Gathering and simplifying the trigonometric terms, we arrive at a system of uncoupled equations.

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -r\left(r^2 - 2\right) \\ 2 + \sin(\theta) \end{pmatrix}$$

We look at the stability of the r-axis, by plotting the temporal derivative of r against r. The result is seen in figure 8.

b)

The period of oscillation can be found be a change of coordinates.

$$T = \int_0^T dt = \int_0^{2\pi} \dot{\theta}^{-1} d\theta$$

We insert the expression found for $\dot{\theta}$ in the uncoupled system.

$$T = \int_0^{\pi} \frac{1}{2 + \sin(\theta)} d\theta$$

This evaluates to the following.

$$T = \frac{2}{3}\pi\sqrt{3} \approx 3.63$$

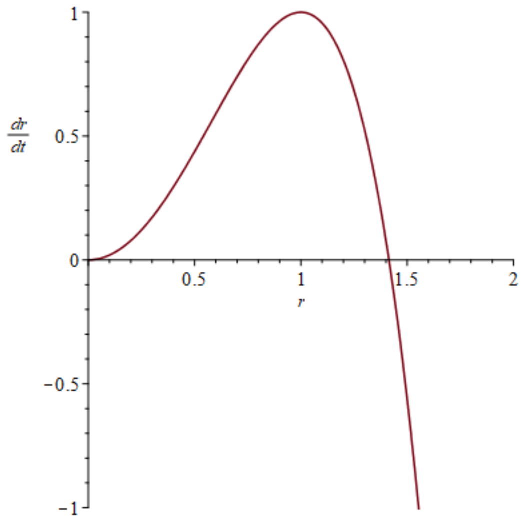


Figure 8: Illustration that we can find a stable value of r. We see that there is an equilibrium as the derivative crosses 0. To the left, r is increasing and to the right, r is decreasing. Solutions will thus tend toward the stable value of r found at the intersection.