01125 Topologiske grundbegreber og metriske rum
Aflevering, uge 1

Raja Shan Zaker Mahmood

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1 Tuesday, 6

We have a topology τ defined on a set and containing the following elements.

$$\tau = \{X, \emptyset, A, B\} \tag{1}$$

A and B are distinct and non-empty subsets of X. If one of the sets A,B is a subset of the other, we have that both their intersection and union are members of τ. If on the other hand $A \cap B = \emptyset$, the union $A \cup B$, can only be X, for them to be members in the topology.To sum up, they must be a partition of X or have one be a subset of the other.

2 AM 7

Let K arbitrary and (M, d) be a metric space with $0 \le d(x, y) \le 1$ for $\forall x, y \in M$. Let $F(K, M)$ be the set of mappings $f : K \to M$. We define a mapping D, $D : F(K, M) \times F(K, M) \to \Re_0^+$ having the following property.

$$D(f, g) = \sup_{t \in K} d(f(t), g(t)) \tag{2}$$

That is, it takes two functions and returns the largest pointwise distance between the graphs, for $t \in K$. First we want to show that D is a metric. For this to be the case, it must satisfy the properties MET1,2,3, defined in the textbook.

2.1 1

2.1.1 MET1

Since the metric on M is bounded, $0 \le d(x, y) \le 1$, we have that $d(f(t), g(t)) \le 1$. This is inportant, as a supremum of for example $+\infty$, would not be defined and the metric D, would then not be defined over the entire domain $F \times F$.

$D(f, g) \ge 0$ trivially, from d.

If $D(f,g) = \sup_{t\in K} d(f(t),g(t)) = 0$, then we have that $d(f(t),g(t)) = 0$, for $\forall t \in K$.

That means $f(t) = g(t)$ over the domain, and $f = g$, confirming MET1

2.1.2 MET2

$$D(f,g) = \sup_{t\in K} d(f(t),g(t)) = \sup_{t\in K} d(g(t),f(t)) = D(g,f) \tag{3}$$

This confirms symmetry and therefore MET2

2.1.3 MET3

$d(x,y)$ is defined to be a metric, so transitivity holds for it.

$$d(f(t),g(t) \le d(f(t),h(t)) + d(h(t),g(t)) \qquad\qquad t \in K \tag{4}$$

For some $h \in M$. D is then the maximum distance.

$$D(f,g) = \sup_{t\in K} d(f(t),g(t)) \le \sup_{t\in K}(d(f(t),h(t)) + d(h(t),g(t)) \tag{5}$$

The supremum on the right side is bounded by the sum of the suprema of the terms. Taking the suprema of each metric and summing, is equal or greater than the ter m above.

$$D(f,g) \le \sup_{t\in K} d(f(t),h(t)) + d(h(t),g(t)) = D(f,h) + D(h,f) \tag{6}$$

Thus, the metric D is transitive.

2.2 2

We let $t_0 \in K$, be a point in K. We define a new mapping. $E_{v_{t_0}} : F(K,M) \to M$, by the following expression.

$$E_{v_{t_0}}(f) = f(t_0) \tag{7}$$

It evaluates a function at a point, hence an evaluation mapping. We want to show that the mapping is continuous.

$$d(E_{v_{t_0}}(f), E_{v_{t_0}}(g) = d(f(t_0), g(t_0)) \tag{8}$$

This then, must be less than or equal to the supremum on the domain.

$$d(f(t_0), g(t_0)) \le \sup_{t\in K} d(f(t),g(t)) = D(f,g) \tag{9}$$

To show continuity, we argue by the way of delta-epsilon.

For $\forall\epsilon > 0$, we choose $\delta = \epsilon$. If $D(f,g) < \delta$, we have:

$$d(E_{v_{t_0}}(f), E_{v_{t_0}}(g)) \le D(f,g) \le \delta = \epsilon \tag{10}$$

Thus proving that the map $E_{v_{t_0}}$ is continuous.