01125 Topologiske Grundbegreber og metriske rum
Aflevering, uge 2

Raja Shan Zaker Mahmood, s144102

January 13, 2017

1 Tuesday, 3

We let (M, d) denote a compact metric space. Now $F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$ is a decreasing sequence of non-empty and closed subsets. We wish to prove that their union is non empty, i.e

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset \tag{1}$$

If it was not the case, we would find that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. This in turn shows that since the intervals are nested, and have no points in common, the union of their complements span the metric space. We have assumed that M is compact and as such, any infinite cover has a finite subcover due to the theorem of Heine-Borel. These open subsets are nested in the opposite order, $F_1^c \subset F_2^c \subset F_3^c$.... Since the cover is finite, we must have, for some finite n, that $M \subseteq F_n^c$. This in turn must mean that in M, F_n is empty, contradicting that the closed subsets are non-empty. Therefore, the union of closed, non empty subsets of the compact set M, is non empty.

2 AM 17

We define the set S.

$$S = \{x \in \Re | 0 \leq x < 1\} \tag{2}$$

We consider the family of subset τ in S, which consists of the empty set $\emptyset$ and every subset of S, U, of the form.

$$U = \{x \in \Re | 0 \leq x < k\}, \text{ for } \qquad 0 < k \leq 1 \tag{3}$$

We want to show three properties, here being presented in their own subsections. The set seems arcane at first glance, but it is simply the interval $[0, 1[$, partitioned into smaller intervals.

1

τ is a topology on S. For this to be the case, the family of subsets must satisfy the three poperties TOP1,2,3 of the texbook and we illustrate each one in turn. We label the subsets U with an index $i \in I$, so that we have $U_i = \{x \in \Re | 0 \leq x < k_i\}$.

1. An arbitrary union of subsets U_i, $\bigcup_{i \in I} U_i = \{x \in \Re | 0 \leq x < k_i\}, i \in I$. The highest number that k_i can attain, is $\sup\{k_i | i \in I\} \in]0, 1]$. This means that the union is also a member of the topology. $\bigcup_{i \in I} U_i = \{x \in \Re | 0 \leq x < \sup_{i \in I} k_i\} \in \tau$
2. An intersection of U_i, belonging to a finite index $I = [1, 2, 3...]$, has a minimum value of k, in analogy to the previous task. $\min_{k \in I} k_i \in]0, 1]$. To reiterate. $\bigcap_{i=1}^{n} U_i = \{x \in \Re | 0 \leq x < \min_{k \in I} k_i\} \in \tau$
3. By definition $\emptyset \in \tau$ and we can see that S itself is a member of the topology, for k=1.

The important part was that k contained its supremum 1, but the topology did not.

2

Now that we have confirmed that (S, τ) is a topological space, we want to show that the sequence x_n, will have every point in S as a limit point, where:

$$x_n = \frac{1}{n+1} \tag{4}$$

On the set S, we have that $0 \leq x < 1$, thus we can define an open neighbourhood of each x, with $x < k$

$$U_x = \{y \in \Re | 0 \leq y < k\} \tag{5}$$

Now we have that $x < k$. so for every x, equation 5 refers to an open neighbourhood around it. For $n > \frac{1}{k} - 1 = n_0$

$$x_n = \frac{1}{1+n} < k \tag{6}$$

It is then a consequence that $x_n \to x$, when $n \to \infty$, for $\forall x \in S$ in the topology. Thus, every point in S, is a limit point for the sequence x_n.

3

If the topology τ was stemming from a metric on S, we would not have that every point could be a limit point of the same sequence. This is due to the property that every metric space is also a Hausdorff space (Theorem 2.7.5). In a Hausdorff space, every sequence can have at most one limit point. The topology is hence not stemming from a metric.