01125 Topologiske Grundbegreber og metriske rum
Aflevering, uge 3

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1 AM 52

We let $f : [a, b] \rightarrow \Re$, where f is a continuous function. The interval is a subset of the real numbers. $[a, b] \in \Re$.

1.1 1

We want to prove that the image of the interval under the function f, is a closed, bounded interval. That is, the image $f[a, b]$ is some interval $[c, d]$ of $\Re$

As $[a, b]$ is a closed, bounded interval in $\Re$ we see that it is a compact set. A well known theorem states that the image under a continuous function is itself compact and hence in this case, closed and bounded.

1.2 2

Now we will prove that a point ξ exists, so that $\xi \in [a, b]$ and

$$\int_a^b f(x)dx = f(\xi)(b-a) \tag{1}$$

As the function f is defined on the closed interval $[a, b]$ it obtains both a maximum and a minimum value. That is, numbers $x_{\text{max}}, x_{\text{min}} \in [a, b]$ exist, such that $f(x_{\text{max}}) = \max_{x\in[ab]} f(x)$ and $f(x_{\text{min}}) = \min_{x\in[ab]} f(x)$. Now the integral over the maximum and minimum of the functions must follow soft inequalities.

$$\int_a^b f(x_{\text{min}})dx \leq \int_a^b f(x)dx \leq \int_a^b f(x_{\text{max}})dx \tag{2}$$

The integrals to the right and left are over constant values, so we can easily evaluate them.

$$f(x_{\text{min}})(b-a) \leq \int_a^b f(x)dx \leq f(x_{\text{max}})(b-a) \tag{3}$$

Dividing by the length of the interval.

$$f(x_{\text{min}}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(x_{\text{max}}) \tag{4}$$

So the term in the middle is bounded by two constants. Due to the intermediate value theorem, the function will for some $\xi \in [a, b]$ attain a value between the maximum and minimum and thus we have the following relation.

$$\frac{1}{b-a}\int_a^b f(x)dx = f(\xi) \tag{5}$$

And the desired result follows.

$$\int_a^b f(x)dx = f(\xi)(b-a) \tag{6}$$

2 AM 31

We let $C([0,1],\Re)$ be a vector space of continuous real-valued functions in the interval $[0,1]$. We define a continuous function $f : [0,1] \to \Re$. Furthermore we define the following.

$$||f||_1 = \int_0^1 |f(x)|dx \tag{7}$$

2.1 1

We want to show that $||\cdot||_1$ is a norm in $C([0,1],\Re)$. For this to be the case, it must have the properties NORM 1,2,3, detailed in the textbook.

Now the absolute value of the function is non-negative by definition. The integral over a non negative is non-negative as well.

$$||f||_1 = \int_0^1 |f(x)|dx \geq 0 \qquad\qquad x \in [0,1] \tag{8}$$

For the second and third conditions, they follows from the linear properties of integrals, which we show by some simple manipulations.

$$||\alpha \cdot f||_1 = \int_0^1 |\alpha \cdot f(x)|dx = |\alpha|\int_0^1 |f(x)|dx = |\alpha| \cdot ||f||_1 \tag{9}$$

$$||f+g||_1 = \int_0^1 |f(x)+g(x)|dx \leq \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx = ||f||_1 + ||g||_1 \tag{10}$$

Thus, with NORM 1,2,3 fulfilled, we have established that $||\cdot||_1$ is a norm.

2.2 2

Now equipped with the norm $||\cdot||_1$, we define the function $C([a,b],\Re) \to \Re$, to be the following.

$$I(f) = \int_0^1 f(x)dx \tag{11}$$

We want to demonstrate the continuity and linearity of the function. We utilize the close relationship to the norm.

$$|I(f)| = |\int_0^1 f(x)dx| \leq \int_0^1 |f(x)dx| = ||f||_1 \tag{12}$$

This shows that for any ϵ with $\epsilon > 0$, we can find $\delta = \epsilon$, so that.

$$||f||_1 < \delta \to |I(f)| \leq ||f||_1 = \epsilon \tag{13}$$

Thus the function is continuous. Linearity follows again from the basic properties of integrals.

2.3 3

Now we want to determine the operator norm of I. We can bound it by our result in the previous task.

$$||I|| = \sup\{|I(f)|, ||f||_1 = 1\} \leq \sup\{||f||_1, ||f||_1 = 1\} = 1 \tag{14}$$

That is to say $||I|| \leq 1$. If on the contrary $f(x) \geq 0$.

$$|I(f)| = \int_0^1 f(x)dx = ||f||_1 \tag{15}$$

Now we have that $||I|| \geq 1$. We are therefore left with only one option and that is that the operator norm is squeezed in between these two estimates and.

$$||I|| = 1 \tag{16}$$