ALIAH UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
Odd Semester End Examination, 2021
COURSE: ENGINEERING MATHEMATICS   COURSE CODE: MATUGBS01
Total marks : 8x10=80   **Time allowed : 3 hours**

Candidates are requested to read the following instructions carefully before answering the questions:

Instructions: Answer any **EIGHT** questions. The marks in each questions are equally distributed.

1. (a) Find the values of a and b such that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$$

(b) Find the value of k, for which given function

$$f(x)=\begin{cases}\left\{\tan(\frac{\pi}{4}+x)\right\}^{\frac{1}{x}}, & \text{when } x\neq 0,\\ k, & \text{when } x=0.\end{cases}$$

is continuous at $x=0$.

2. (a) If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$$

(b) If

$$V=\log(x^3+y^3+z^3-3xyz),$$

prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2V=-\frac{9}{(x+y+z)^2}.$$

3. (a) State Euler's theorem for homogeneous functions. If $u=\log\left(\frac{x^4+y^4}{x+y}\right)$, show by Euler's theorem that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

(b) Let

$$f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } (x,y)\neq(0,0),\\ 0, & \text{when } (x,y)=(0,0).\end{cases}$$

show that $f_{xy}(0,0)\neq f_{yx}(0,0)$.

4. (a) Starting from $\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$, obtain

$$\frac{d^2x}{dy^2}=-\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}\quad\text{and}\quad\frac{d^3x}{dy^3}=-\frac{\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^5}.$$

(b) Find the condition for the line $x\cos\theta + y\sin\theta = p$ to touch the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

5. (a) Find the Fourier cosine series of $f(x) = x^2$ on the interval $[-\pi, \pi]$.

   (b) Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -5 < x < 0, \\ 3, & 0 < x < 5. \end{cases}$$

6. (a) Evaluate:

$$\int \frac{\sin x \cos x}{a^2\cos^2 x + b^2 \sin^2 x}\, dx.$$

   (b) Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(1+\sqrt{2}).$$

7. (a) Evaluate the limit:

$$\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{n+3n}\right]$$

   as an integral.

   (b) Examine the convergence of

      i.
$$\int_0^2 \frac{dx}{(2x-x^2)},$$

      ii.
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

8. (a) Prove that

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx,$$

   where B denotes the Beta function.

   (b) Write the relation between Beta and Gamma functions. Using the definition of Beta function find the value of the integral:

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^7\theta d\theta.$$

9. (a) State the Cauchy Mean value theorem. If in the Cauchy Mean value theorem $f(x) = e^x$ and $g(x) = e^{-x}$, show that c is arithmetic mean between a and b,

   (b) Evaluate:

$$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$$

.

10. (a) If $u = \tan^{-1}\left(\frac{x^{\frac{5}{7}}+y^{\frac{5}{7}}}{\sqrt{x}-\sqrt{y}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

(b) If

$$v = z\sin\frac{y}{x} \quad \text{where} \quad x = 3r^2 + 2s,\ y = 4r - 2s,\ z = 2r^2 - 3s^2$$

then find $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial s}$.

11. (a) Check whether the limits exist or not at the origin, if exist then find limit(/s) for the functions given below:

i. $f(x,y) = \begin{cases} \frac{x^4+y^4}{x-y}, & \text{when } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$

ii. $f(x,y) = \begin{cases} \frac{x^3y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0), \\ 37, & \text{if } (x,y) = (0,0). \end{cases}$

.

(b) If $ax + by = 1$ is normal to the parabola $y^2 = 4\lambda x$, then prove that $\lambda a^3 + 2a\lambda b^2 = b^2$.

12. (a) State Lagrange's Mean value theorem. If $f(x) = (x-1)(x-2)(x-3)$, verify Lagrange's theorem in the interval $[0,4]$.

(b) If $y = \cos(m\sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

Hence prove that

$$\begin{aligned}(y_n)_0 &= 0,\ \textit{if } n \textit{ be odd}, \\ &= -m^2(2^2-m^2)(4^2-m^2)\ldots\{(n-2)^2-m^2\},\ \textit{if } n \textit{ be even.}\end{aligned}$$