Operation Research

20/02/2024

Suggested Books :-

1. HA Taha, operation Research, Pearson
2. Ghosh and chakraborty, Linear Programming & Theory of Games, Central Book Agency.
3. S Kalarathy, operation Research, Viteach Publishers.

What is Operation Research

Operation Research is a scientific methodology - analytical, experimental and quantitative which by assessing the overall implication of various alternative courses of action in a management system to provide an improved basis for management decisions.

The term operations research was first wined by Mc closky and Trefthen in 1940 in a small town called Bowdsey in UK.

Characteristics of OR (Imp 5 marks)

i) <u>Inter-disciplinary team approach</u> :- The optimum solution is obtained by a team of scientists selected from various disciplines.

ii) <u>Who listic approach to the system</u> :- OR takes into account all significant factors and finds the best optimum solution to the total organisation.

iii) <u>Imperfectness of solution</u> :- OR improves the quality of solution.

iv) <u>Use of Scientific Research</u> :- OR uses scientific research to reach optimum solution.

v) <u>To optimize the total output</u> :- OR tries to optimize by maximizing the profit and minimizing the loss.

Application of OR :- Operation research can be applied to a variety of use cases, including :-

- Scheduling and time management.
- Urban and agricultural planning.
- Enterprise resource planning (ERP) and supply chain management (SCM).
- Inventory management.
- Network Organization and engineering.
- Packet routing optimization.
- Risk Management.

Phases of OR :-

Step I : Formulation of problem

This is the most important process. It is generally lengthy and time consuming. The activities that constitute this step are visits, observations research etc. With the help of such activities, the O.R scientist gets sufficient information and support to proceed and is better prepared to formulate the problem.

This process starts with understanding of the organizational climate, its objectives and expectations. Further, the alternative courses of action are discovered in this step.

Step II :- Construction of Mathematical Model

This phase is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form ~~by~~ for this purpose is to construct a mathematical model representing the system under study. Model must include 3 important factors -

a) Decision Variable

b) Constraints or Restrictions

c) Objective function.

Step III :- Deriving the solutions from the model

After formulating the mathematical model for the problem. Under consideration, the next phase is to ~~desire~~ derive a solution from this model. Here in OR we are always in search for an optimal solution. Optimal solution is the one which maximizes the objective function.

Step IV :- Provide a solution and test its reasonableness

This step is to get a solution with the help ~~to~~ of model and input data. This solution is not implemented immediately, instead the solution is used to test the model and to find there is any limitations. Suppose, if the solution is not reasonable or the behaviour of the model is not proper, the model is updated and modified at this stage. The output of this stage is the solution(s) that supports the current organizational objectives.

step-V :- Controlling the solutions.

The model requires immediate modification as soon as the controlled variables (one or more) changes significantly, otherwise the model goes out of control. As the conditions are ~~currently~~ constantly changing in the world and the solution may not remain valid for a long time.

Step-VI : Implement the solution

At this step the solution obtained from the previous step is implemented. The implementation of the solution involves many behavioural issues. Therefore, before implementation, the implementation authority has to resolve the issues. A properly implemented solution results in quality of work and gains the support from the management.

Linear Programming :-

A linear form is mean a mathematical expression of type $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_1, a_2, \ldots, a_n$ are constants and $x_1, x_2, \ldots, x_n$ are variables. The term programming refers to the process of determining a particular program or plan of action.

Linear Programming (LP) is one of the most important optimization (maximization, minimization) techniques developed in the field of operation Research.

$$\boxed{z_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n}$$

Subject to restriction

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq \text{ or } \geq)\ b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq \text{ or } \geq)\ b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq \text{ or } \geq)\ b_m$$

and $x_i \geq 0$, $i = 1, 2, \ldots n$

Example: A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields of profit of Rs 45. Product B requires 20 quintals of raw materials, 15 man hours and yields a profit of Rs 80. Formulate the LPP.

Solⁿ:- x_1 be the number of units of product A

x_2 be the number of units of product B

	Product A	Product B	Availability
Raw Material	5	20	400
Man hours	10	15	450
Profit	Rs 45	Rs. 80	

Hence, Maximize $Z = 45x_1 + 80x_2$

Subject to

$$5x_1 + 20x_2 \leq 400$$

$$10x_1 + 15x_2 \leq 450$$

Example : A firm manufactures two types of products A and B and sells them at a profit of Rs 2 on type A and 3 on type B. Each product is processed on two machines G and H. Type A requires 1 minute of processing time on G and 2 minutes on H, type B requires 1 minutes on G and 3 minutes on H. The machine G is available for not more than 6 hrs 40 minutes. while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem solution.

Solⁿ:– Let x_1 be the number of products of type A

x_2 be the number of products of type B.

After understanding the problem, the given information can be arranged in the form of following table

Types of Products (minutes)

Machine	Type A (x_1 unit)	Type B (x_2 unit)	Available time (min)
G	1	1	400
H	2	3	600
Profit / Unit	Rs 2	Rs 3	

Total profit on selling x_1 units of type A and x_2 units of type B is given by

maximize $z = 2x_1 + 3x_2$ (objective function).

Since Machine G takes 1 minute time on type A and 1 minute on type B.

So the total number of minutes required or machine G is given by $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is $2x_1 + 3x_2$.

But machine G is available for not more than 6 hours 40 minutes i.e 400 minutes.

Therefore,

$$x_1 + x_2 \leq 400 \text{ (first constraint)}$$

Also, the machine H is available for 10 hours i.e 600 minutes only.

Therefore,

$$2x_1 + 3x_2 \leq 600 \text{ (Second constraint)}$$

Since it is not possible to produce negative quantities.

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ (non-negative restrictions)}$$

Hence,

Maximize $z = 2x_1 + 3x_2$

Subject to restrictions,

$$x_1 + x_2 \leq 400$$

$$2x_1 + 3x_2 \leq 600$$

and non-negative restrictions

$$x_1 \geq 0, \quad x_2 \geq 0$$

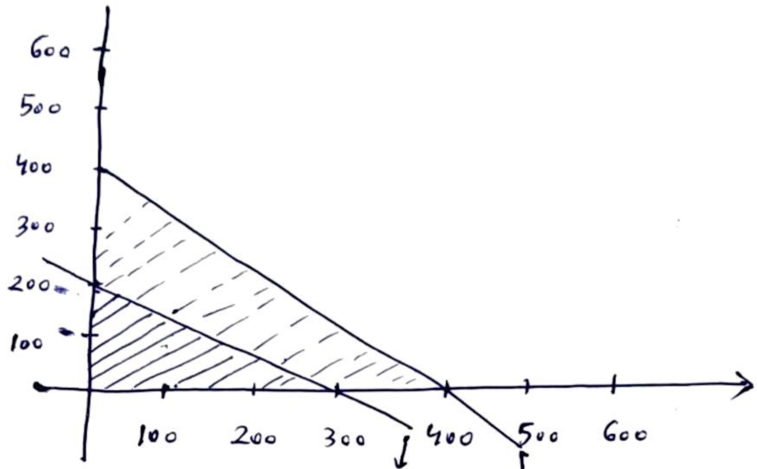
Graphical Method :-

Solving the LPP by graphical method :-

step-1 :- Consider each inequality constraint as an equation.

step-2 :- Put each equation on the ~~region~~ graph, as equation will geometrically represent a straight line.

step-3 :- Mark the region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line lying the first quadrant (due to non-negativity of variables) is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.



$$x_1 + x_2 = 400$$

$$\frac{x_1}{\underset{a}{400}} + \frac{x_2}{\underset{b}{400}} = 1$$

$a = (400, 0)$ $b = (0, 400)$

$$2x_1 + 3x_2 = 600$$

$$\Rightarrow \frac{x_1}{300} + \frac{x_2}{300} = 1$$

$a = (300, 0)$, $b = (0, 200)$

There are four cases :-

a) a unique optimal solution.

b) An infinite no of optimal solution.

c) An unbounded solution.

d) No solution.

Example :- Solve the LPP by Graphical Method

Maximize $Z = 20x_1 + 10x_2$

Subject to, $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$

Solution :- Replace all the inequalities of the constraints by equation we get,

$x_1 + 2x_2 = 40$ — (i)

$3x_1 + x_2 = 30$ — (ii)

$4x_1 + 3x_2 = 60$ — (iii)

Dividing eqn (1) by 40, we get

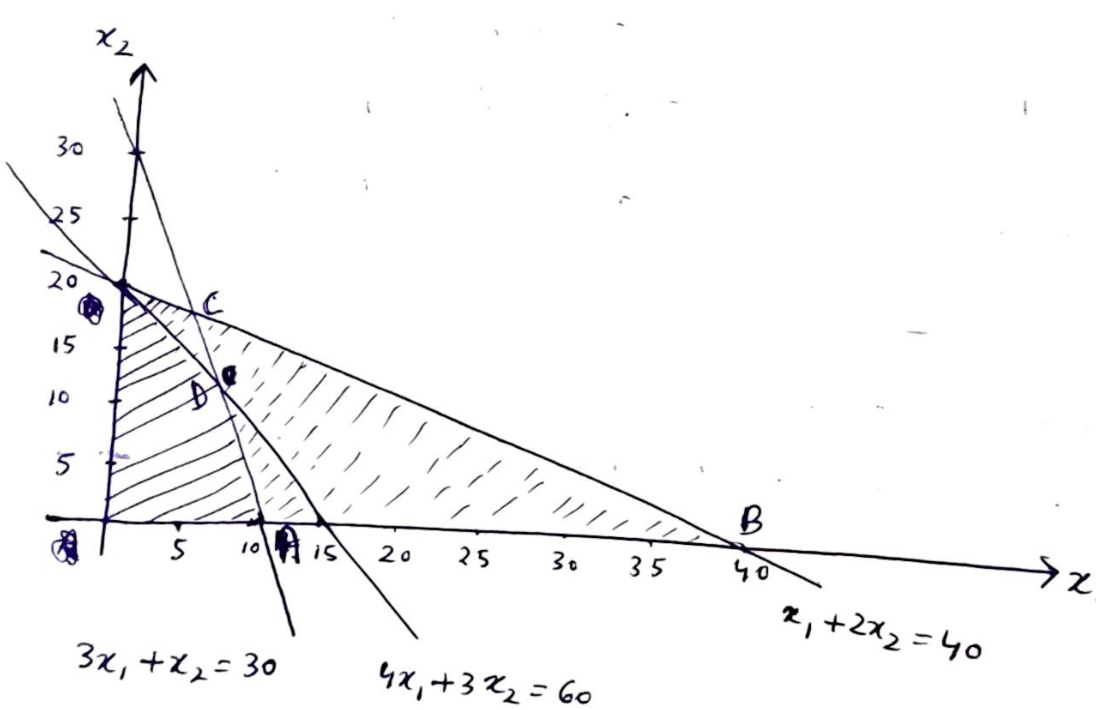
$$\frac{x_1}{40} + \frac{x_2}{20} = 1$$

Dividing eqn (2) by 30, we get

$$\frac{x_1}{10} + \frac{x_2}{30} = 1$$

Dividing eqn (3) by 60, we get

$$\frac{x_1}{15} + \frac{x_2}{20} = 1$$



The feasible region is ABCD.

The coordinates of A is (15, 0) and B is (40, 0)

C and D are points of intersection of lines

C intersect $x_1 + 2x_2 = 40$, $3x_1 + x_2 = 30$

D intersect $3x_1 + x_2 = 30$, $4x_1 + 3x_2 = 60$

① × 3, we get,

$$3x_1 + 6x_2 = 120 \quad \text{——(iv)}$$

(iv) − (iii), we get.

$$3x_1 + 6x_2 = 120$$
$$3x_1 + x_2 = 30$$
$$\overline{\quad 5x_2 = 90 \quad}$$
$$\Rightarrow x_2 = 18$$

Now putting $x_2 = 18$ in eqn ① we get

$$x_1 = 40 - 2 \times 18 = 40 - 36 = 4$$

so, the coordinates of C is $(4, 18)$

[3 × (ii)] − (iii)

$$9x_1 + 3x_2 = 90$$
$$4x_1 + 3x_2 = 60$$
$$\overline{\quad 5x_1 = 30 \quad}$$
$$\Rightarrow x_1 = 6$$

Putting $x_1 = 12$ at eqn (ii)

$$x_2 = 30 - 3 \times 6 = 12$$

Corner Points	Values of $Z = 20x_1 + 10x_2$
A (15, 0)	300
B (40, 0)	800 (Maximize)
C (4, 18)	260
D (6, 12)	240

∴ The maximize value of Z occurs at B (40, 0)

Hence the optimal solution is $x_1 = 40$, $x_2 = 0$.

27.02.24

Example Solve the following LPP by Graphical Method

Max $Z = 5x_1 + 7x_2$

Subject to extracts $x_1 + x_2 \le 4$

$3x_1 + 8x_2 \le 24$

$10x_1 + 7x_2 \le 35$

$x_1, x_2 \ge 0$

Sol^n:–

Replace all the inequalities of the constraints by forming equation

$$x_1 + x_2 = 4 \quad \text{—— (i)}$$

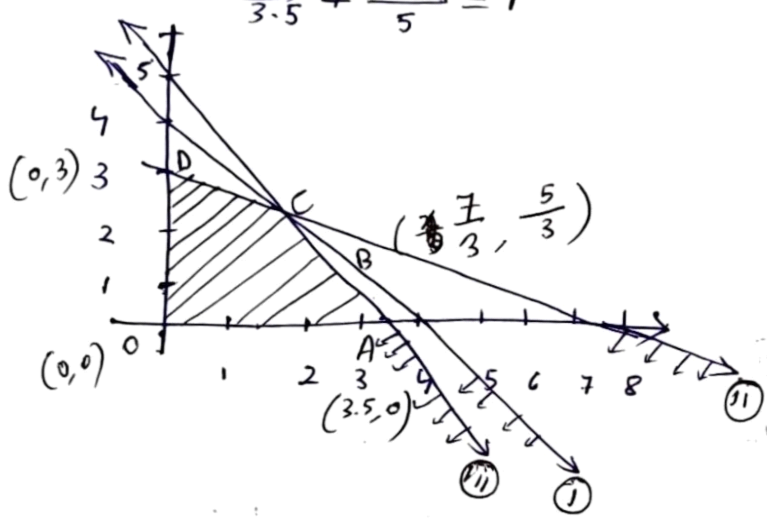
$$3x_1 + 8x_2 = 24 \quad \text{—— (ii)}$$

$$10x_1 + 7x_2 = 35 \quad \text{—— (iii)}$$

Dividing (i) $\div 4 \Rightarrow \frac{x_1}{4} + \frac{x_2}{4} = 1$

" (ii) $\div 24 \Rightarrow \frac{x_1}{8} + \frac{x_2}{3} = 1$

" (iii) $\div 35 \Rightarrow \frac{x_1}{3.5} + \frac{x_2}{5} = 1$



C

$x_1 + x_2 = 4$

$3x_1 + 8x_2 = 24$

B

$x_1 + x_2 = 4$

$10x_1 + 7x_2 = 35$

C

$8x_1 + 8x_2 = 32$

$3x_1 + 8x_2 = 24$

$5x_1 = 8$

$x_1 = \frac{8}{5}$

$x_2 = 4 - \frac{8}{5} = \frac{12}{5}$

$8 + \frac{84}{5}$

$8 + 16.8$

(i) $\times 10 \Rightarrow 10x_1 + 10x_2 = 40$ — (iv)

(iv) − (iii) $\Rightarrow 3x_2 = 5$

$x_2 = \frac{5}{3}$

From (i) $\Rightarrow x_1 = 4 - x_2 = 4 - \frac{5}{3} = \frac{7}{3}$

The feasible region is OABCD.

B and C are the points of intersection of lines.

B intersect $x_1 + x_2 = 4$ and $10x_1 + 7x_2 = 35$ and

C intersect $x_1 + x_2 = 4$ and $3x_1 + 8x_2 = 24$.

On solving we get, $B = (\frac{7}{3}, \frac{5}{3})$, $C = (\frac{8}{5}, \frac{12}{5})$

(i) $\times 3$ − (ii) $\Rightarrow$

$3x_1 + 3x_2 = 12$

$3x_1 + 8x_2 = 24$

$5x_2 = 12 \Rightarrow x_2 = \frac{12}{5}$

From (i) $\Rightarrow x_1 = 4 - \frac{12}{5} = \frac{8}{5}$

Corner Points	Value of $Z = 5x_1 + 7x_2$
O (0,0)	0
A (3.5, 0)	17.5
B $(\frac{7}{3}, \frac{5}{3})$	23.33
✓ C $(\frac{8}{5}, \frac{12}{5})$	~~56.8~~ 24.8 Max value
D (0, 3)	21

The max value of Z occurs at $C(\frac{8}{5}, \frac{12}{5})$ and the optimal sol^n is $x_1 = \frac{8}{5}$ and $x_2 = \frac{12}{5}$.

2) Example solve the LPP by Graphical method Max $Z = 100x_1 + 40x_2$

Subject to, $5x_1 + 2x_2 \le 1000$

$3x_1 + 2x_2 \le 900$

$x_1 + 2x_2 \le 500$

and $x_1, x_2 \ge 0$

Solⁿ:– Replace all the inequalities of the constraints by forming eqⁿ

$5x_1 + 2x_2 = 1000$ — (i)

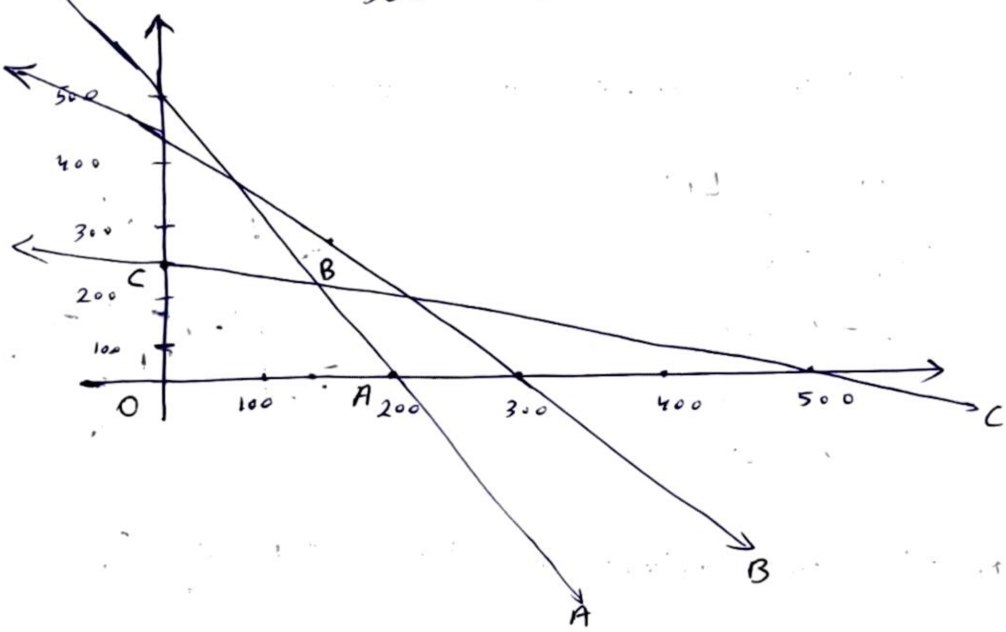
$3x_1 + 2x_2 = 900$ — (ii)

$x_1 + 2x_2 = 500$ — (iii)

Dividing (i) ÷ 1000 ⇒ $\frac{x_1}{200} + \frac{x_2}{500} = 1$

(ii) ÷ 900 ⇒ $\frac{x_1}{300} + \frac{x_2}{450} = 1$

(iii) ÷ 500 ⇒ $\frac{x_1}{500} + \frac{x_2}{250} = 1$



From (i) & (iii)

$5x_1 + 2x_2 = 1000$

$x_1 + 2x_2 = 500$

$4x_1 = 500$

$x_1 = 125$

$x_2 = \frac{500 - 125}{2} = \frac{375}{2} = 187.5$

$x_1, x_2 = (125, 187.5)$

The feasible region is OABC

B is the points of intersection of lines

B intersect $5x_1 + 2x_2 = 1000$ and $x_1 + 2x_2 = 500$

On solving we get, $B = (125, \frac{375}{2})$

Corner Points	Value of $Z = 100x_1 + 40x_2$
$O(0,0)$	0
$A(200, 0)$	20000
$B(125, \frac{375}{2})$	20000
$C(0, 300)$	12000

The maximum value of Z occurs at two vertices A and B. Since there are infinite number of points on the line joining A & B. it gives the same maximum value of Z.

Thus, there are infinite number of optimal solution for this LPP.

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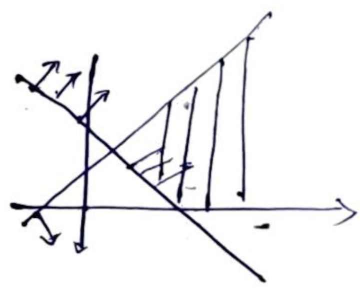
A unique solution :- an infinite number of solution.

Example :- Solve the following LPP

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to

$$x_1 - x_2 \geq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$



Solution :- Replace all the inequalities of the constraints by equation

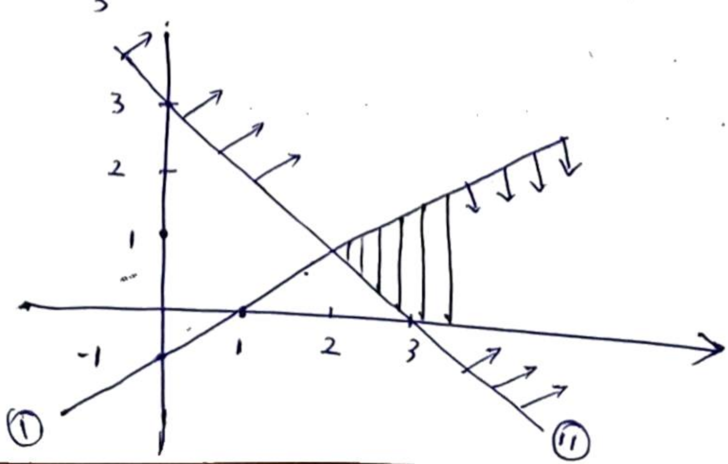
$$x_1 - x_2 = 1 \quad \text{--- (i)}$$
$$x_1 + x_2 = 3 \quad \text{--- (ii)}$$

Dividing eq^n (i) by 1 we get

$$\frac{x_1}{1} + \frac{x_2}{-1} = 1$$

Dividing eq^n (ii) by 3 we get

$$\frac{x_1}{3} + \frac{x_2}{3} = 1$$



The solution space is unbounded. Infact, the maximum value of Z occurs at infinity.

Hence the problem has an unbounded solution.

2) **Example** :- Solve the following LPP

$$\text{Max } Z = 3x_1 + 4x_2$$

subject to

$$-3x_1 + 2x_2 \le 6$$
$$-x_1 + 3x_2 \le 18$$
$$x_1, x_2 \ge 0$$

Solⁿ :- Replace all the inequalities of the constraints by equation

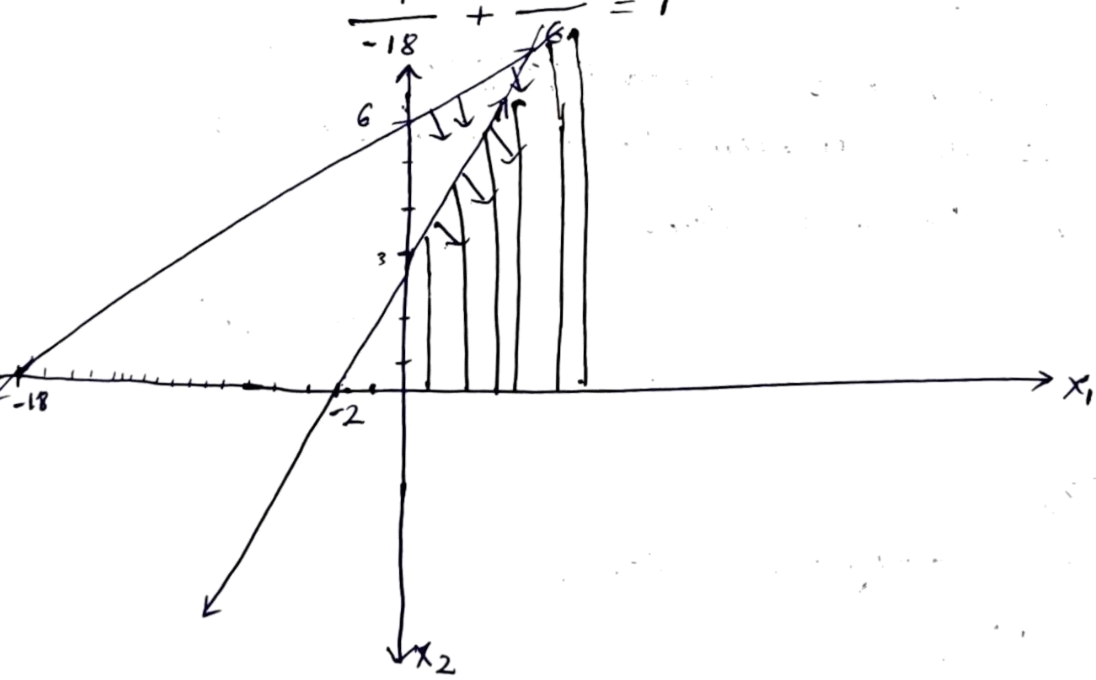
$$-3x_1 + 2x_2 = 6 \quad \text{— (i)}$$
$$-x_1 + 3x_2 = 18 \quad \text{— (ii)}$$

Dividing eqⁿ (i) by 6 we get

$$\frac{x_1}{-2} + \frac{x_2}{3} = 1$$

Dividing eqⁿ (ii) by 18 we get

$$\frac{x_1}{-18} + \frac{x_2}{6} = 1$$

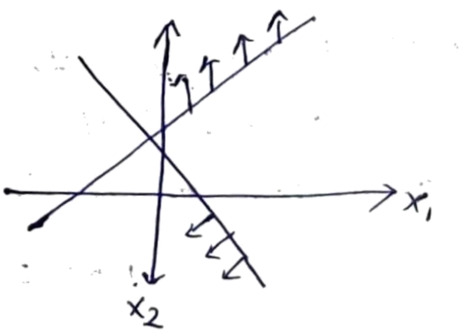


The solution space is unbounded. Infact the maximum value of Z occurs at infinity.

Hence the problem has an unbounded solution.

No Feasible solution :-

When there is no feasible region formed by the constant in conjunction with non-negativity continuous, then no solution to the LPP exists.



Example :- Solve the following LPP

$$\text{Max} \quad z = x_1 + x_2$$

Subject to

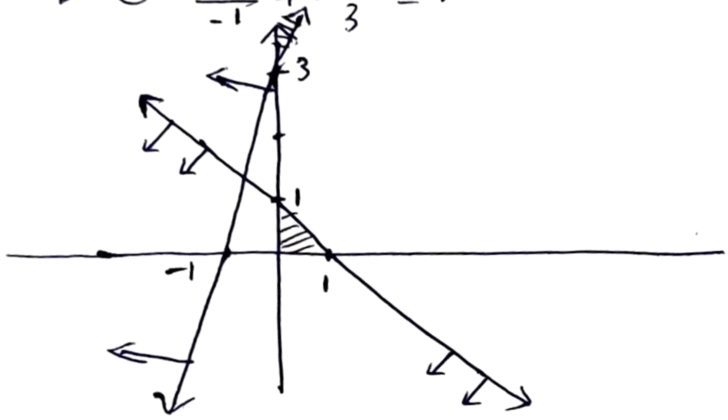
$$x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

Solⁿ:- Replace all the inequalities of the constraints by equation.

$$x_1 + x_2 = 1 \quad \text{——(i)}$$
$$-3x_1 + x_2 = 3 \quad \text{——(ii)}$$

From eqⁿ (i), $\frac{x_1}{1} + \frac{x_2}{1} = 1$

From eqⁿ (ii) $\frac{x_1}{-1} + \frac{x_2}{3} = 1$



There is no point common to both the shaded region, we can not find a feasible region for this problem. So the problem cannot be solved. Hence the problem has no solution.

5-03-24

D Simplex Method

$$\text{Max} \quad Z = 2x_1 + 3x_2$$

Graphical method involves exactly two variables

$$\text{Max} \quad Z = 2x_1 + 3x_2 + 5x_3$$

Subject to

Graphical method can not solve optimization problem involving three or more decision variable

To overcome problem, we use Simplex Method

D General Form into Standard form

(i) Check whether the objective function to LPP is maximized or minimized. If it is to be minimized then we connect it into a problem of maximization by

$$\text{Max} \quad Z' = -\text{Min } Z$$

Example

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to $x_1 + x_2 \le 4$

$x_1 - x_2 \le 2$

$x_1, x_2 \ge 0$

standard LPP is given by

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to $x_1 + x_2 \le 4$

$x_1 - x_2 \le 2$

$x_1, x_2 \ge 0$

Example

$$\text{Min } Z = 3x_1 + 2x_2 + 5x_3$$

Subject to $x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_3 \le 260$

$x_1 + 4x_2 + x_3 \le 420$

$x_1, x_2, x_3, x_4 \ge 0$

standard LPP is given by

$$\text{Max } z' = -\text{Min } Z = -3x_1 - 2x_2 - 5x_3$$

Subject to $x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_3 \le 260$

$x_1 + 4x_2 + x_3 \le 420$

$x_1, x_2, x_3, x_4 \ge 0$

ii) Check all the decision variable are ≥ 0. If any decision variables are unristricted rewrite to $2x_1 + x_2 \le 4$, $x_1 \ge 0$ and x_2 is unristricted

$$2x_1 + (x_2' - x_2'') \le 4, \quad x_1, x_2', x_2'' \ge 0$$

iii) Express the problem into standard form by introducing slack or surplus variables to convert the inequality constraints into equations.

Example

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to $x_1 + x_2 \le 4$

$x_1 - x_2 \le 2$

$x_1, x_2 \ge 0$

Standard LPP is given by

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to $x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

s_1 and s_2 are slack variables with cost

Example

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3$$

Subject to $x_1 + x_2 + x_3 \le 430$

$3x_1 + 2x_3 \ge 260$

$x_1 + 4x_2 + x_3 \le 420$

Standard LPP is given by

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3$$

$x_1 + x_2 + x_3 + s_1 = 430$

$3x_1 + 2x_3 - s_2 = 260$

$x_1 + 4x_2 + x_3 + s_3 = 420$

s_1, s_3 are ~~are~~ slack variables and s_2 is variable.

6.03.24

Example Use simplex method to solve the LPP

$\le$ slack ~~stack~~

$\ge$ surplus

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3$$

Subject to
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 \quad + 2x_3 \le 460$$
$$x_1 + 4x_2 \quad \le 420$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n:- By introducing stack variables s_1, s_2, s_3 convert the Problem in standard form.

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0.s_1 + 0.s_2 + 0.s_3$$
$$x_1 + 2x_2 + x_3 + s_1 = 430$$
$$3x_1 \quad + 2x_3 \quad + s_2 = 460$$
$$x_1 + 4x_2 \quad + s_3 = 420$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

An initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0, \quad s_1 = 430, \quad s_2 = 460, \quad s_3 = 420.$$

Writing in matrix form, $AX = B$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix}$$

An initial simplex table

pivot element

Entering incoming vector

Departing vector

			C_j	3	2	5	0	0	0	Min ratio
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	$B/x_3, x_3>0$
R_1	0	s_1	430	1	2	1	1	0	0	$430/1 = 430$
R_2	0	s_2	460	3	0	$\boxed{2}$	0	1	0	$460/2 = 230$
R_3	0	s_3	420	1	4	0	0	0	1	—
	$Z_j - C_j$			-3	-2	$-5\uparrow$	0	0	0	

Calculation for initial ~~sus~~ simplex table

$$Z_j - c_j = C_B X_j - c_j \qquad Z_1 - c_1 = C_B X_1 - c_1 = -3$$
$$Z_2 = C_B X_2 - C_2 = -2$$

$$R_2' \leftarrow R_2/2$$
$$R_1' \leftarrow R_1 - R_2'$$
$$R_3' \leftarrow R_3 \ \cancel{-R_2}$$

First iteration

		c_j	3	2	5	0	0	0	Min value
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	$B/x_2,\ x_2>0$
R_1'	0	s_1	200	$-\frac{1}{2}$	$\boxed{2}$	0	1	$-\frac{1}{2}$	0	$\frac{200}{2} = 100$
R_2'	5	x_3	230	3/2	0	1	0	1/2	0	–
R_3'	0	s_3	420	1	4	0	0	0	1	$\frac{420}{4} = 105$
	$z_j - c_j$			9/2	-2	0	0	5/2	1	

pivot element

Entering Incoming vector

departing vector

≥ 0

Calculation for the above simplex table

$$z_j - c_j = c_B x_j - c_j$$

$$z_1 - c_1 = c_B x_1 - c_1 = (0 \times (-\tfrac{1}{2}) + 5 \times \tfrac{3}{2} + 0 \times 1) - 3 = \frac{15-6}{2} = \frac{9}{2}$$

$$z_2 - c_2 =$$

$$z_3 - c_3 =$$

Second iteration

$R_1'' \leftarrow R_1'/2$ $R_2'' \leftarrow R_2'$ $R_3'' \leftarrow R_3' - 4R_1''$

			c_j	3	2	5	0	0	0
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3
R_1''	2	x_2	100	$-\frac{1}{4}$	1	0	1/2	$-\frac{1}{4}$	0
R_2''	5	x_3	230	3/2	0	1	0	$\frac{1}{2}$	0
R_3''	0	s_3	~~10~~ 20	2	0	0	-2	1	1
	$z_j - c_j$			4	0	0	1	2	0

Calculation for the above simplex table

$$z_j - c_j = c_B x_j - c_j$$

$$z_1 - c_1 = c_B x_1 - c_1 =$$

Since $z_j - c_j \geq 0$, for all j, the solution is optimum and is given by $x_1 = 0$, $x_2 = 100$ $x_3 = 230$.

Max $z = 3x_1 + 2x_2 + 5x_3$

The optimal solution is given by Max $z = c_B x_j$

$= 3 \times 0 + 2 \times 100 + 5 \times 230$

$= 0 + 200 + 1150$

$= 1350$

Example : Use simplex method to solve the following LPP

$$\text{Min } Z = x_1 - 3x_2 + 2x_3$$

subject to
$$3x_1 - x_2 + 2x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution :- By introducing stack variables s_1, s_2, s_3 convert the problem in the standard form.

$$\text{Max } z' = x_1 + 3x_2 - 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

subject to,
$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 + 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

An initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0, \quad s_1 = 7, \quad s_2 = 12, \quad s_3 = 10$$

Writing the constraint equations in matrix form $AX = B$

$$\begin{bmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

An initial simplex table, Pivot element

Entering vector

		c_j	-1	3	-2	0	0	0	Min rate
C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	B/x_2, $x_2 > 0$
0	s_1	7	3	-1	2	1	0	0	–
0	s_2	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4} = 3$
0	s_3	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.33$
$z_j - c_j$			1	$-3 \uparrow$	2	0	0	0	

Departing vector

Calculation for initial simplex table

$$z_j - c_j = C_B x_j - c_j$$

$$z_1 - c_1 = C_B x_1 - c_1 = 0 \times 3 + 0 \times (-2) + 0 \times (-4) - (-1) = 1$$

$$z_2 - c_2 = C_B x_2 - c_2 = 3$$

$$z_3 - c_3 = C_B x_3 - c_3 = 2$$

$$z_4 - c_4 = 0$$

$$z_5 - c_5 = 0$$

$$z_6 - c_6 = 0$$

$(S_1 \quad S_2 \quad S_3) \rightarrow$ in $\qquad (S_1 \quad x_2 \quad S_3)$ Pivot element

First Iteration

$R_2' \rightarrow \frac{R_2}{4}$ $\qquad$ Pivot element $\qquad R_1' \rightarrow R_1 + R_2'$ $\qquad R_3' \rightarrow R_3 - 3R_2'$

Entering vector

Departing vector

			C_j	-1	3	-2	0	0	0	Min Ratio
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	$B/x_1,\ x_1 > 0$
R_1'	0	s_1	10	$\boxed{5/2}$	0	2	1	$\frac{1}{4}$	0	$10/\frac{5}{2} = 4$
R_2'	-2	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	-
R_3'	0	s_3	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	-
	$Z_j - C_j$			$-\frac{1}{2}\uparrow$	0	2	0	$\frac{3}{4}$	0	

$\frac{5}{2} \cdot \frac{1}{2}$

Second Iteration

$R_1'' \rightarrow R_1' \times \frac{2}{5}$ $\qquad R_2'' \rightarrow R_2' + \frac{R_1''}{2}$

$R_3'' \rightarrow R_3' + \frac{5}{2} R_1''$

			C_j	-1	3	-2	0	0	0
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3
R_1''	-1	x_1	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
R_2''	3	x_2	5	0	1	$2/5$	$\frac{1}{5}$	$\frac{3}{10}$	0
R_3''	0	x_3	11	0	0	10	1	$-\frac{1}{2}$	0
	$Z_j - C_j$			0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

$x_1 \quad x_2 \quad S_3$

Since all $Z_j - C_j \geq 0$, the solution is optimum and given by

$x_1 = 4, \quad x_2 = 5, \quad x_3 = 0$

The optimum solution is given by

$$\text{Max } Z' = C_B \cdot B = (-1 \times 4 + 3 \times 5 + 0 \times 11) = -4 + 15 = 11$$

$$\text{Min } Z = -\text{Max}(Z') = -11$$

13.03.24

Degeneracy in Simplex Method tie for minimum ratio.

Example :- Use simplex Method to solve the LPP

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

subject to,

$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

Solution :- By introducing slack and surplus variables s_1, s_2, s_3 convert the problem in standard form

$$\text{Max } Z = x_1 + 2x_2 + x_3 + 0.s_1 + 0.s_2 + 0.s_3$$

Subject to,

$$2x_1 + x_2 - x_3 + s_1 = 2 \quad \text{—— (i)}$$
$$-2x_1 + x_2 - 5x_3 - s_2 = -6$$
$$\Rightarrow 2x_1 - x_2 + 5x_3 + s_2 = 6 \quad \text{—— (ii)}$$
$$4x_1 + x_2 + x_3 + s_3 = 6 \quad \text{—— (iii)}$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

An initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0, \quad s_1 = 2, \quad s_2 = 6, \quad s_3 = 6$$

Writing in matrix form $AX = B$

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}$$

An initial simplex table

Entering vector

			C_j	1	2	1	0	0	0	Min ratio
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	$B/x_2,\ x_2 > 0$
R_1	0	s_1	2	2	$\boxed{1}$	-1	1	0	0	$2/1 = 2$
Departing vector R_2	0	s_2	6	2	-1	5	0	1	0	—
R_3	0	s_3	6	4	1	1	0	0	1	$6/1 = 6$
	$Z_j - C_j$			-1	-2 ↑	-1	0	0	0	

First Iteration

$$R_1' \rightarrow R_1$$
$$R_2' \rightarrow R_2 + R_1'$$
$$R_3' \rightarrow R_3 - R_1'$$

		C_j	1	2	1	0	0	0	Min Ratio		
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3	B/x_3	x_2/x_3	s_2/x_3
R_1'	2	x_2	2	2	1	-1	1	0	0	—	—	—
R_2'	0	s_2	8	4	0	4	1	1	0	$8/4=2$	$0/4$	$1/4$
R_3'	0	s_3	4	2	0	$\boxed{2}$	-1	0	1	$4/2=2$	$0/2$	0
	$Z_j - C_j$			3	0	$-3 \uparrow$	2	0	0			

Entering vector

Departing vector

Second Iteration

$R_1'' \rightarrow R_1' + R_3''$ $\quad$ $R_2'' \rightarrow R_2' - 4R_3''$ $\quad$ $R_3'' \rightarrow R_3'/2$

			C_j	1	2	1	0	0	0
	C_B	X_B	B	x_1	x_2	x_3	s_1	s_2	s_3
R_1''	2	x_2	4	3	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
R_2''	0	s_2	0	0	0	0	3	1	-2
R_3''	1	x_3	2	1	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$
	$Z_j - C_j$			6	0	0	$\frac{1}{2}$	0	$\frac{3}{2}$

Calculation $Z_j - C_j = C_B X_j - C_j$

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by

$x_1 = 0,\ x_2 = 4$ and $x_3 = 2$

The optimum solution is given by

$\text{Max } Z = C_B B = 2 \times 4 + 0 \times 0 + 1 \times 2 = 10$

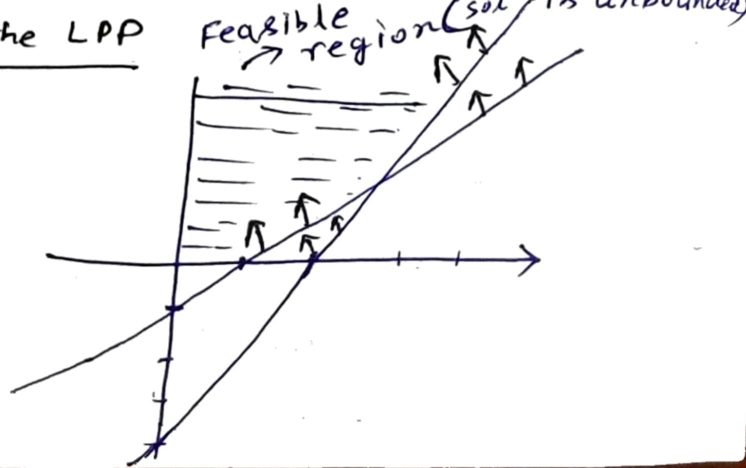
19.03.24

Unbounded Solution in Simplex Method :-

i) Unbounded feasible region and unbounded optimal solution

ii) Unbounded feasible region but bounded optimal solution.

Use Graphical method to solve the LPP



$\text{Max } Z = 2x_1 + x_2$

Subject to, $x_1 - x_2 \leq 10$

$2x_1 - x_2 \leq 40$

$x_1, x_2 \geq 0$

Solⁿ:- $x_1 - x_2 = 10$ $\qquad$ $2x_1 - x_2 = 40$

$\Rightarrow \frac{x_1}{10} + \frac{x_2}{-10} = 1$ $\qquad$ $\Rightarrow \frac{x_1}{20} + \frac{x_2}{-40} = 1$

Use simplex method to solve the LPP. $\text{Max } Z = 2x_1 + x_2$

subject to, $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

$x_1, x_2 \ge 0$

Sol^n:– By introducing slack variables S_1 and S_2 convert the problem into the standard form

$$\text{Max } Z = 2x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2$$

subject to,
$$x_1 - x_2 + S_1 = 10$$
$$2x_1 - x_2 + S_2 = 40 \qquad x_1, x_2, S_1, S_2 \ge 0$$

An initial basic feasible solution is given by

$$x_1 = x_2 = 0, \quad S_1 = 10, \quad S_2 = 40$$

Writing in Matrix form, $AX = B$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

An initial simplex table for the LPP

Entering vector   Pivot element

		C_j	2	1	0	0	Min Ratio
C_B	X_B	B	x_1	x_2	S_1	S_2	B/x_1, $x_1 > 0$
0	S_1	10	$\boxed{1}$	−1	1	0	$10/1 = 10$
0	S_2	40	2	−1	0	1	$40/2 = 20$
	$Z_j - C_j$		−2 ↑	−1	0	0	

Departing vector

$$Z_j - C_j = C_B X_j - C_j \qquad Z_1 - C_1 = C_B X_1 - C_1 = 0 \times 1 + 0 \times 2 - 2 = -2$$

First Iteration :– $R_1' \to \frac{R_1}{2}$, $R_2' \to R_2 - 2R_1'$

Entering vector

			C_j	2	1	0	0	Min Ratio
	C_B	X_B	B	x_1	x_2	S_1	S_2	B/x_2, $x_2 > 0$
R_1'	2	x_1	10	1	−1	1	0	–
R_2'	0	S_2	20	0	$\boxed{1}$	−2	1	$20/1 = 20$
		$Z_j - C_j$		0	−3 ↑	2	0	

Departing vector

Second Iteration :– $R_2'' \to \frac{R_2}{1}$, $R_1'' \to R_1' + R_2'$

			C_j	2	1	0	0	Min Ratio
	C_B	X_B	B	x_1	x_2	S_1	S_2	
R_1''	2	x_1	30	1	0	−1	1	–
R_2''	1	x_2	20	0	1	−2	1	–
		$Z_j - C_j$		0	0	−4 ↑	3	

$Z_j - C_j = C_B X_j - C_j$ $Z_3 - C_3 = -4 < 0$, the solution is not optimum.

But all values in the key columns are negative, which indicates the solution is not bounded i.e the sol^n is unbo[illegible]

Example :- Unbounded feasible region, but bounded optimum solⁿ

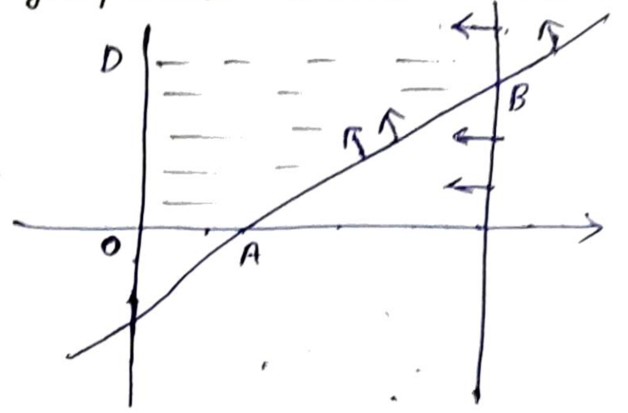
$$\text{Max } z = 6x_1 - 2x_2$$

Subject to, $2x_1 - x_2 \le 2$

$x_1 \le 4$

$x_1, x_2 \ge 0$

use graphical Method to solve the LPP



Sol^n:- $\text{Max } Z = 6x_1 - 2x_2$

Subject to, $2x_1 - x_2 = 2$

$$\Rightarrow \frac{x_1}{1} + \frac{x_2}{-2} = 1$$

$x_1 = 4$

Putting $x_1 = 4$ in eq^n ① $2x_1 - x_2 = 2 \Rightarrow 2\times 4 - x_2 = 2 \Rightarrow x_2 = 6$

Point	$Z = 6x_1 - 2x_2$
O(0,0)	0
A(1,0)	6
B(4,6)	12 → Max value
C(4,8)	8
D(0,8)	-16

Simplex Method :- By introducing slack variables s_1, s_2. Convert the problem into the standard form

$$\text{Max } Z = 6x_1 - 2x_2 + 0.s_1 + 0.s_2$$

subject to, $2x_1 - x_2 + s_1 = 2$

$x_1 + s_2 = 4$ $\quad x_1, x_2, s_1, s_2 \ge 0$

An initial basic feasible solⁿ is given by

$x_1 = x_2 = 0, \; s_1 = 2, \; s_2 = 4$

writing in matrix form $AX = B$

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The initial simplex table for the LPP

Entering vector, Pivot

		c_j	6	-2	0	0	Min Ratio
C_B	X_B	B	x_1	x_2	s_1	s_2	$B/x_1,\ x_1>0$
0	s_1	2	$\boxed{2}$	-1	1	0	$\frac{2}{2} = 1$
0	s_2	4	1	0	0	1	$4/1 = 4$
	$z_j - c_j$		-6 ↑	2	0	0	

Departing vector

$z_j - c_j = C_B X_j - c_j$

First iteration :- $R_1' \to R_1/2$, $R_2' \to R_2 - R_1'$

Departing vector

Entering vector

			c_j	6	-2	0	0	Min Ratio
	C_B	X_B	B	x_1	x_2	s_1	s_2	$B/x_2,\ x_2>0$
R_1'	6	x_1	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	—
R_2'	0	s_2	3	0	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	1	$3/\frac{1}{2} = 6$
		$z_j - c_j$	[illegible]	0	-1 ↑	3	0	

Second Iteration

$R_2'' \to \frac{R_2'}{\frac{1}{2}}$ $R_1'' \to R_1' + \frac{1}{2} R_2''$

		C_j	6	-2	0	0
	C_B X_B	B	x_1	x_2	s_1	s_2
R_1''	6 x_1	4	1	0	0	1
R_2''	-2 x_2	6	0	1	-1	2
	$Z_j - C_j$		0	0	2	2

Since all $Z_j - C_j \geq 0$, the sol^n is optimum and given by $x_1 = 4$ and $x_2 = 6$.
The optimum sol^n is given by Max $Z = 6 \times 4 + (-2) \times 6$
$= 24 - 12 = 12$

3.04.24

Charne's Big M Method of Penalty

In Simplex Method to convert general to standard form

≤ 0 add a stack

≥ 0 subtract a surplus variable

$=$ Nothing.

Example Max $z = x_1 - x_2 + 3x_3$

Subject to $x_1 + x_2 \leq 20$

$x_1 + x_3 = 5$

$x_2 + x_3 \geq 10$

By introducing stack variable s_1 and surplus variable s_2 the given LPP cannot be converted into standard form

Max $Z = x_1 - x_2 + 3x_3$

subject $x_1 + x_2 + s_1 = 20$

$x_1 + x_3 = 5$

$x_2 + x_3 - s_2 = 10$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

initial have feasible sol^n given by

$x_1 = x_2 = x_3 = 0$, $s_1 = 20$, $s_2 = -10$

In charne's Big M Method

In charne's Big M method to convert general to standard form

$\leq$ add a stack variable

$\geq$ subtract a surplus variable ~~and add~~

$=$ add an artificial variable.

5.04.24

Example :- Use Charne's Big M method to solve the LPP

$$\text{Max } Z = x_1 - x_2 + 3x_3$$

Subject to,
$$x_1 + x_2 \leq 20$$
$$x_1 + x_3 = 5$$
$$x_2 + x_3 \geq 10$$

Sol^n:- By introducing slack variable S_1, surplus variable S_2 and artificial variable A_1, A_2, the given LPP can be reformulated as follow

$$\text{Max } Z = x_1 - x_2 + 3x_3 + 0.S_1 + 0.S_2 - M.A_1 - M.A_2$$

Subject to,
$$x_1 + x_2 + S_1 = 20$$
$$x_1 + x_3 + A_1 = 5$$
$$x_2 + x_3 - S_2 + A_2 = 10$$

$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$$

An initial basic feasible solution is given by

$x_1 = x_2 = x_3 = S_2 = 0$, $S_1 = 20$, $A_1 = 5$, $A_2 = 10$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 10 \end{bmatrix}$$

Initial Table :-

Pivot element   Entering vector

	C_B	X_B	C_j / B	1 / x_1	-1 / x_2	3 / x_3	0 / S_1	0 / S_2	-M / A_1	-M / A_2	Min Ratio B/x_3, $x_3 > 0$
R_1	0	S_1	20	1	1	0	1	0	0	0	–
R_2	-M	A_1	5	1	0	$\boxed{1}$	0	0	1	0	$5/1 = 5$
R_3	-M	A_2	10	0	1	1	0	-1	0	1	$10/1 = 10$
		$Z_j - C_j$		-M-1	-M+1	-2M-3 ↑	0	M	0	0	

Departing vector

Since some of $Z_j - C_j \leq 0$, the current feasible solution is not optimum. Choose the most negative ~~are~~ one $Z_j - C_j = -2M-3$

The variable x_3 enter the basis and the artificial variable A_1 leaves the basis.

First Iteration :- $R_2' \to R_2$, $R_2' \to R_1$, $R_3' \to R_3 - R_2'$

pivot element

Entering vector

		C_j	1	-1	3	0	0	-M	-M	Min Ratio
	C_B X_B	B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	B/x_2, $x_2 > 0$
R_1'	0 S_1	20	1	1	0	1	0	0	0	$\frac{20}{1}$, $x_2 > 0$
R_2'	3 x_3	5	1	0	1	0	0	1	0	-
R_3'	-M A_2	5	-1	$\boxed{1}$	0	0	-1	-1	1	$\frac{5}{1} = 5$
	$Z_j - C_j$		M+2	-M+1 ↑	0	0	M	3+2M	0	

Departing vector

Since some of $Z_j - C_j \leq 0$, the current feasible solution is not optimum.

Choose the most negative are $Z_j - C_j = -M+1$

The variable x_2 enters the basis and the artificial variable A_2 leaves the basic in new table.

Second Iteration $R_3'' \to R_3'$, $R_2'' \to R_2'$, $R_1' \to R_1' - R_2''$

			C_j	1	-1	3	0	0	-M	-M
	C_B	X_B	B	x_1	x_2	x_3	S_1	S_2	A_1	A_2
R_1''	0	S_1	15	2	0	0	1	1	1	-1
R_2''	3	x_3	5	1	0	1	0	0	1	0
R_3''	-1	x_2	5	-1	1	0	0	-1	-1	1
	$Z_j - C_j$			3	0	0	0	1	4+M	-1+M

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 0$, $x_2 = 5$, $x_3 = 5$

Max $Z = C_B \times B$

$= 0 \times 15 + 3 \times 5 + (-1) \times 5$

$= 0 + 15 - 5$

$= 10$

pg-260

Imp for exam