Chapter

19 Game Theory

19.1 INTRODUCTION

Competition is the watchword of modern life. We can say that a competitive solution exists, if two or more individuals make decisions in a situation that involves conflicting interests, and in which the outcome is controlled by the decision of all the concerned parties. A competitive situation is called a *game*. The term *game* represents a conflict between two or more parties. A situation is termed as game when it possesses the following properties.

(*i*) The number of competitors is finite.
(*ii*) There is a conflict in interests between the participants.
(*iii*) Each of the participants has a finite set of possible courses of action.
(*iv*) The rules governing these choices are specified and known to all players. The game begins when each player chooses a single course of action from the list of courses available to him.
(*v*) The outcome of the game is affected by the choices made by all the players.
(*vi*) The outcome for all the specific set of choices, by all the players, is known in advance and is numerically defined.

The outcome of a game consists of a particular set of courses of action undertaken by the competitors. Each outcome determines a set of payments (positive, negative or zero), one to each competitor.

19.1.1 Definition

The term '*strategy*' is defined as a complete set of plans of action specifying precisely what the player will do under every possible future contigency that might occur during the play of the game, i.e., strategy of a player is the decision rule he uses for making a choice, from his list of courses of action. Strategy can be classified as:

(*i*) Pure strategy (*ii*) Mixed strategy

A strategy is called *pure* if one knows in advance of the play that it is certain to be adopted, irrespective of the strategy the other players might choose.

The optimal strategy mixture for each player may be determined by assigning to each strategy, its probability of being chosen. The strategy so determined is called *mixed strategy* because it is a probabilistic combination of the available choices of strategy. Mixed strategy is denoted by the set, $S = \{X_1, X_2 ... X_n\}$ where, X_j is the probability of choosing the course j such that $X_j > 0, j = 1, 2...n$ and $X_1 + X_2 + ... X_n = 1$. It is evident that a pure strategy is a special case of mixed strategy.

In the case where all but one X_j is zero, a player may be able to choose only n pure strategy, but he has an infinite number of mixed strategies to choose them.

19.2 PAY-OFF

Pay-off is the outcome of playing the game. A pay-off matrix is a table showing the amount received by the player named at the left hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

If a player A has m courses of action and player B has n courses, then a pay-off matrix may be constructed using the following steps.

(*i*) Row designations for each matrix are the courses of action available to A.
(*ii*) Column designations for each matrix are the courses of action available to B.
(*iii*) With a two-person zero-sum game, the cell entries in B's pay-off matrix will be the negative of the corresponding entries in A's pay-off matrix and the matrices will be as shown below.

Player B

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & \dots & j & \dots & n \end{array} \\ \text{Player } A \quad \begin{array}{c} 1 \\ 2 \\ \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3j} & \dots & a_{3n} \\ & & & \dots & & \dots & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \end{array}$$

A's pay-off matrix.

Player B

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & \dots & j & \dots & n \end{array} \\ \text{Player } A \quad \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} & \begin{bmatrix} -a_{11} & -a_{12} & a_{13} & \cdots & -a_{1j} & \cdots & -a_{1} \\ -a_{21} & -a_{22} & a_{23} & \cdots & -a_{2j} & \cdots & a_{2} \\ \vdots & & & & & & \\ -a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{ij} & \cdots & a_{in} \\ \vdots & & & & & & \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

19.3 TYPES OF GAMES

(*i*) *Two-person games and n-person games* In two-person games, the players may have many possible choices open to them for each play of the game but the number of players remain only two. Hence, it is called a *two-person game*. In case of more than two persons, the game is generally called n-person game.

(*ii*) *Zero-sum game* A *zero-sum game* is one in which the sum of the payments to all the competitors is zero, for every possible outcome of the game if the sum of the points won, equals the sum of the points lost.

(*iii*) *Two-person zero-sum game* A game with two players, where the gain of one player equals the loss of the other, is known as a *two-person zero-sum game*. It is also called a *rectangular game* because their pay-off matrix is in the rectangular form. The characteristics of such a game are:

(*a*) Only two players participate in the game.
(*b*) Each player has a finite number of strategies to use.
(*c*) Each specific strategy results in a pay-off.
(*d*) Total pay-off to the two players at the end of each play is zero.

19.4 THE MAXIMIN-MINIMAX PRINCIPLE

This principle is used for the selection of optimal strategies by two players. Consider two players A and B. A is a player who wishes to maximize his gains, while player B wishes to minimize his losses. Since A would like to maximize his minimum gain, we obtain for player A, the value called *maximin value* and the corresponding strategy is called the *maximin strategy*.

On the other hand, since player B wishes to minimize his losses, a value called the *minimax value*, which is the minimum of the maximum losses is found. The corresponding strategy is called the *minimax strategy*. When these two are equal (maximin value = minimax value), the corresponding strategies are called *optimal strategies* and the game is said to have a *saddle point*. The value of the game is given by the saddle point.

The selection of maximin and minimax strategies by A and B is based upon the so-called maximin-minimax principle, which guarantees the best of the worst results.

Saddle point A saddle point is a position in the pay-off matrix where, the maximum of row minima coincides with the minimum of column maxima. The pay-off at the saddle point is called the *value* of the game.

We shall denote the maximin value by $\underline{\gamma}$, the minimax value of the game by $\overline{\gamma}$ and the value of the game by γ.

Notes:

(*i*) A game is said to be fair if,
maximin value = minimax value = 0, i.e., if $\overline{\gamma} = \underline{\gamma} = 0$

(*ii*) A game is said to be strictly determinable if,
maximin value = minimax value $\neq 0$. $\underline{\gamma} = \gamma = \overline{\gamma}$.

Example 19.1 Solve the game whose pay-off matrix is given by,

$$\text{Player } A \quad \overset{\text{Player } B}{\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}}$$

Solution

$$\begin{array}{cc|ccc|c} & & \multicolumn{3}{c|}{\text{Player } B} & \\ & & B_1 & B_2 & B_3 & \text{Row minima} \\ \text{Player } A & A_1 & 1 & 3 & 1 & 1 \\ & A_2 & 0 & -4 & -3 & -4 \\ & A_3 & 1 & 5 & -1 & -1 \\ \text{Column maxima} & & 1 & 5 & 1 & \end{array}$$

$$\text{Maxi (minimum)} = \text{Max}\,(1, -4, -1) = 1$$
$$\text{Mini (maximum)} = \text{Min}\,(1, 5, 1) = 1.$$

i.e., Maximin value $\underline{\gamma} = 1 =$ Minimax value $\overline{\gamma}$

$\therefore$ Saddle point exists. The value of the game is the saddle point, which is 1. The optimal strategy is the position of the saddle point and is given by, (A_1, B_1).

Example 19.2 For what value of l, is the game with the following matrix strictly determinable?

$$\text{Player } B$$
$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

Solution Ignoring the value of l, the pay-off matrix is given by,

$$\text{Player } B$$
$$\begin{array}{cc|ccc|c} & & B_1 & B_2 & B_3 & \text{Row minima} \\ & A_1 & \lambda & 6 & 2 & 2 \\ \text{Player } A & A_2 & -1 & \lambda & -7 & -7 \\ & A_3 & -2 & 4 & \lambda & -2 \\ \text{Column maxima} & & -1 & 6 & 2 & \end{array}$$

The game is strictly determinable if,

$$\underline{\gamma} = \gamma = \overline{\gamma}\text{. Hence, } \underline{\gamma} = 2, \qquad \overline{\gamma} = -1$$

$$\Rightarrow \qquad -1 \le \lambda \le 2.$$

Example 19.3 Determine which of the following two-person zero-sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.

(*i*)
$$\text{Player } B$$
$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{array}$$

(*ii*)
$$\text{Player } B$$
$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix} \end{array}$$

$$\text{Maxi (minimum)} = \underline{\gamma} = \text{Max}\,(-5, -7) = -5$$
$$\text{Mini (maximum)} = \overline{\gamma} = \text{Min}\,(-5, 2) = -5$$

Solution

(*i*)
$$\text{Player } B$$
$$\begin{array}{cc|cc|c} & & B_1 & B_2 & \text{Row minima} \\ \text{Player } A & A_1 & -5 & 2 & -5 \\ & A_2 & -7 & -4 & -7 \\ \text{Column maxima} & & -5 & 2 & \end{array}$$

Since $\underline{\gamma} = \overline{\gamma} = -5 \ne 0$, the game is strictly determinable. There exists a saddle point $= -5$. Hence, the value of the game is -5. The optimal strategy is the position of the saddle point given by, (A_1, B_1).

(*ii*)

$$\begin{array}{llcccc} & & \multicolumn{2}{c}{\text{Player } B} & \\ & & B_1 & B_2 & \text{Row minimum} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}} & \begin{array}{c} 1 \\ -3 \end{array} \\ \text{Column maximum} & & 4 & 1 & \end{array}$$

Maxi (minimum) = $\underline{\gamma}$ = Max (1, – 3) = 1.

Mini (maximum) = $\overline{\gamma}$ = Min (4, 1) = 1.

Since $\underline{\gamma} = \overline{\gamma} = 1 \neq 0$, the game is strictly determinable. Value of the game is 1. The optimal strategy is, (A_1, B_2).

Example 19.4 Solve the game whose pay-off matrix is given below.

$$\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

Solution

$$\begin{array}{llcccccc} & & B_1 & B_2 & B_3 & B_4 & B_5 & \text{Row minima} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \multicolumn{5}{c}{\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}} & \begin{array}{c} -2 \\ 1 \\ -4 \\ -6 \end{array} \\ \text{Column maxima} & & 5 & 3 & 1 & 5 & 6 & \end{array}$$

Maxi (minimum) = $\underline{\gamma}$ = Max (– 2, 1, – 4, – 6) = 1.

Mini (maximum) = $\overline{\gamma}$ = Min (5, 3, 1, 5, 6) = 1.

Since, $\underline{\gamma} = \overline{\gamma} = 1$, there exists a saddle point. Value of the game is 1. The position of the saddle point is the optimal strategy and is given by, $[A_2, B_3]$.

EXERCISES

1. For a game with the following pay-off matrix,

$$\begin{array}{cc} & \text{Player } A \\ \text{Player } B & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

determine the best strategies as well as the value of the game for players *A* and *B*. Is this game (*i*) fair, (*ii*) strictly determinable?

[**Ans.** Value of game is = 2. Game is not fair, but strictly determinable]

2. Determine the optimal minimax strategies for each player in the following game.

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} -5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{bmatrix} \end{array}$$

[**Ans.** g = 4, (A_2, B_3) is the optimum strategy]

19.5 GAMES WITHOUT SADDLE POINTS (MIXED STRATEGIES)

A game without saddle point can be solved by various solution methods.

19.5.1 2 × 2 Games without Saddle Point

Consider a 2 × 2 two-person zero-sum game without any saddle point, having the pay-off matrix for player A as,

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

where, $$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$$

$$\text{The value of the game } (\gamma) = \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}.$$

Example 19.5 Solve the following pay-off matrix. Also determine the optimal strategies and value of the game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}. \text{ Let this be,}$$

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}. \text{ The optimum mixed strategies,}$$

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

where, $$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - 3}{(5 + 4) - (1 + 3)} = \frac{1}{5}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{5} = \frac{4}{5}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - 1}{(5 + 4) - (1 + 3)} = \frac{3}{5}$$

$$q_2 = 1 - q_1 \Rightarrow q_2 = 1 - \frac{3}{5} = \frac{2}{5}$$

value of game, $$\gamma = \frac{(5 \times 4) - (1 \times 3)}{(5 + 4) - (1 + 3)} = \frac{17}{5}$$

$\therefore$ The optimum mixed strategies,

$$S_A = \left(\frac{1}{5}, \frac{4}{5}\right); S_B = \left(\frac{3}{5}, \frac{2}{5}\right)$$

$$\text{Value of game} = \frac{17}{5}.$$

Example 19.6 Solve the following game and determine its value.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

Solution It is clear that the pay-off matrix does not possess any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}, p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - (-4)}{4 + 4 - (-4 - 4)} = \frac{8}{16} = \frac{1}{2}$$

$$p_2 = 1 - p_1 \Rightarrow p_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - (-4)}{4 + 4 - (-4 - 4)} = \frac{8}{16} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

The optimum strategy is, $S_A = \left(\frac{1}{2}, \frac{1}{2}\right); S_B = \left(\frac{1}{2} \ \frac{1}{2}\right)$

The value of the game is, $$\gamma = \frac{a_{22}\, a_{11} - a_{12}\, a_{21}}{(a_{22} + a_{11}) - (a_{12} + a_{21})}$$

$$= \frac{(4 \times 4) - [-4 \times (-4)]}{(4 + 4) - [-4 + (-4)]} = 0.$$