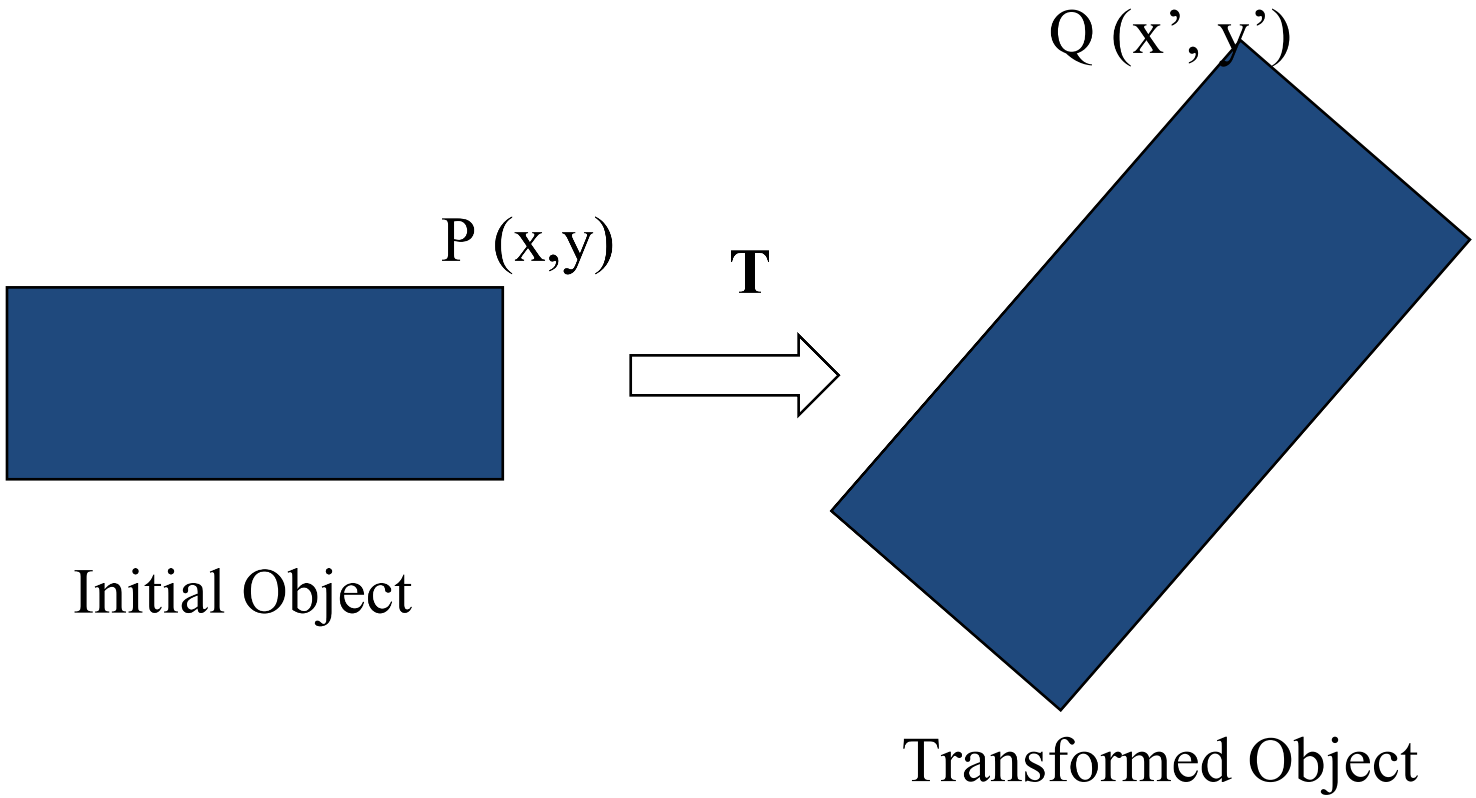
TWO DIMENSIONAL TRANSFORMATION

CONTENT

TRANSFORMATION PRINCIPLES
CONCATENATION
MATRIX REPRESENTATIONS

Transformation

Transform every point on an object according to certain rule.

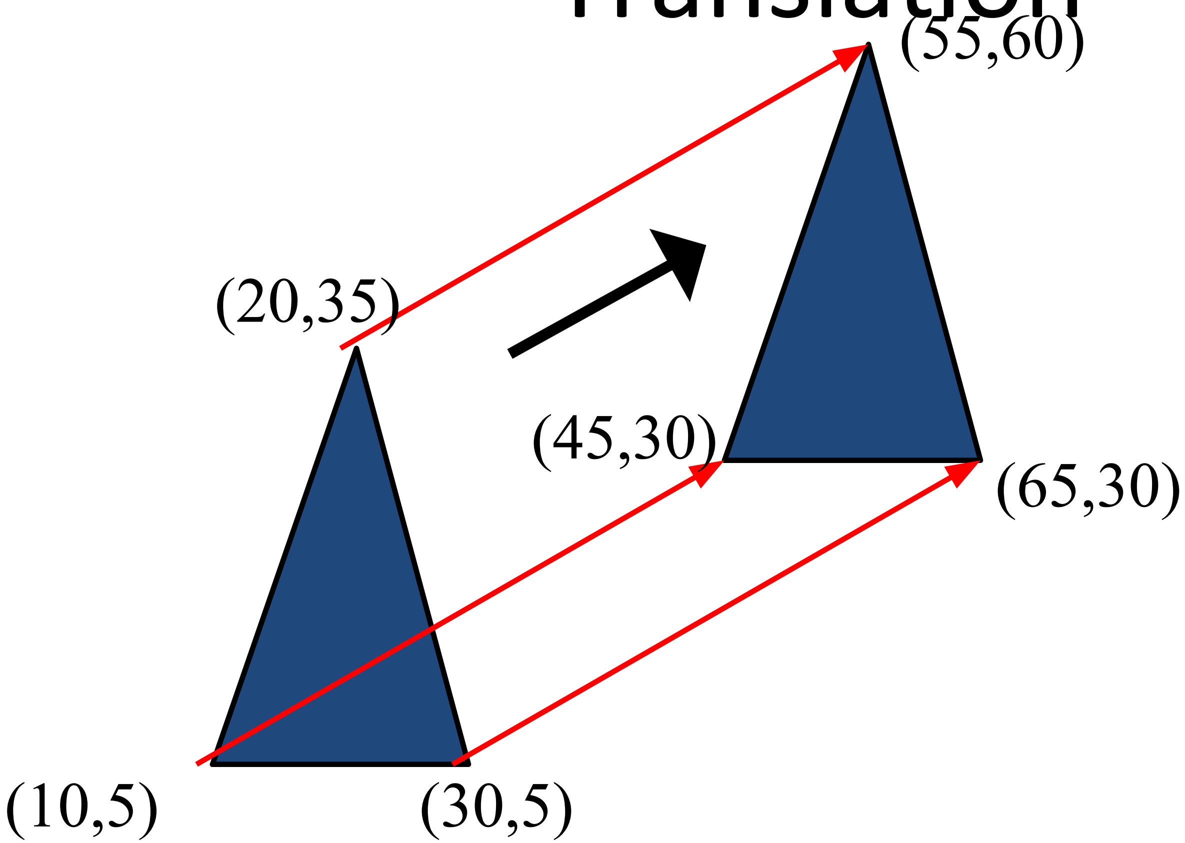


$x \rightarrow x'$

$y \rightarrow y'$

The point Q is the image of P under the transformation T.

Translation



$$x' = x + t_x$$

$$y' = y + t_y$$

The vector (t_x, t_y) is called the *offset vector*.

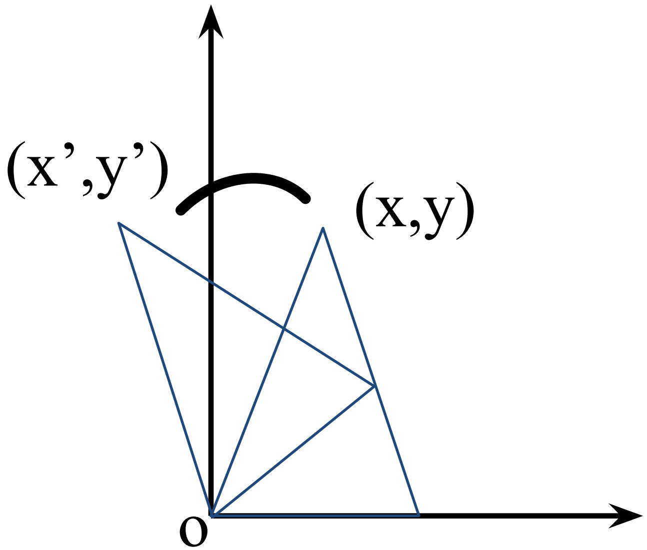
Translation (OpenGL)

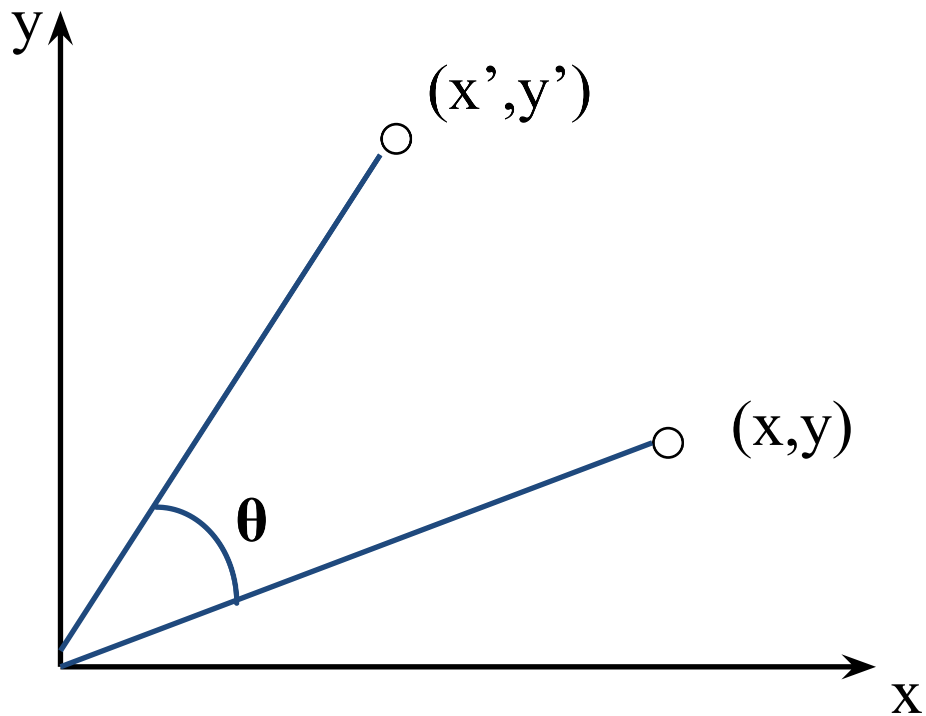
Specifying a 2D-Translation:

```
glTranslatef(tx, ty, 0.0);
```

(The z component is set to 0 for 2D translation).

Rotation About the Origin





$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

The above 2D rotation is actually a rotation about the z-axis (0,0,1) by an angle θ.

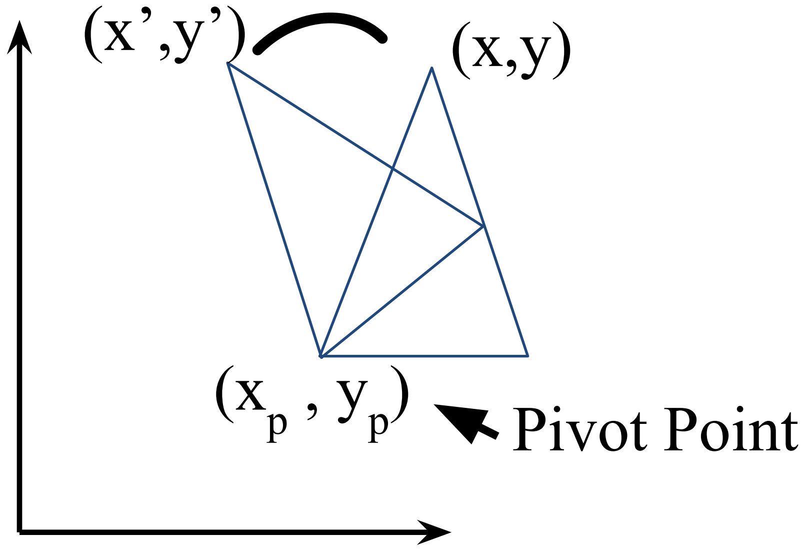
Rotation About the Origin

Specifying a 2D-Rotation about the origin:

```
glRotatef(theta, 0.0, 0.0, 1.0);
```

`theta`: Angle of rotation in degrees.
The above function defines a rotation about the z-axis (0,0,1).

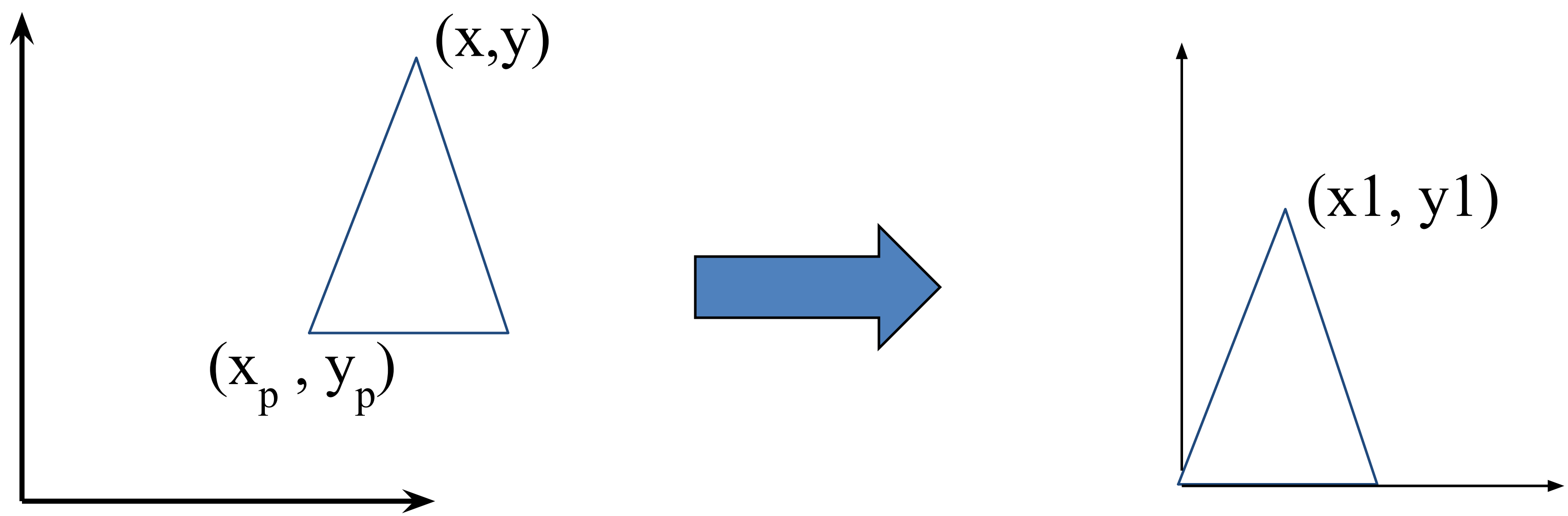
Rotation About a Pivot Point



- Pivot point is the point of rotation
- Pivot point need not necessarily be on the object

Rotation About a Pivot Point

STEP-1: Translate the pivot point to the origin

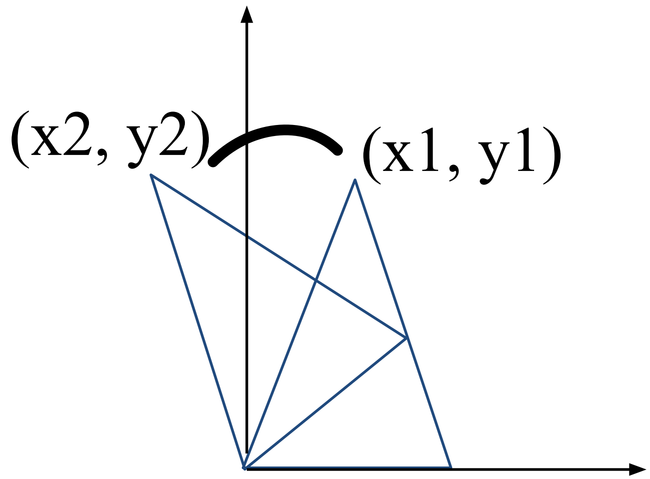


$$x1 = x - x_p$$

$$y1 = y - y_p$$

Rotation About a Pivot Point

STEP-2: **Rotate about the origin**

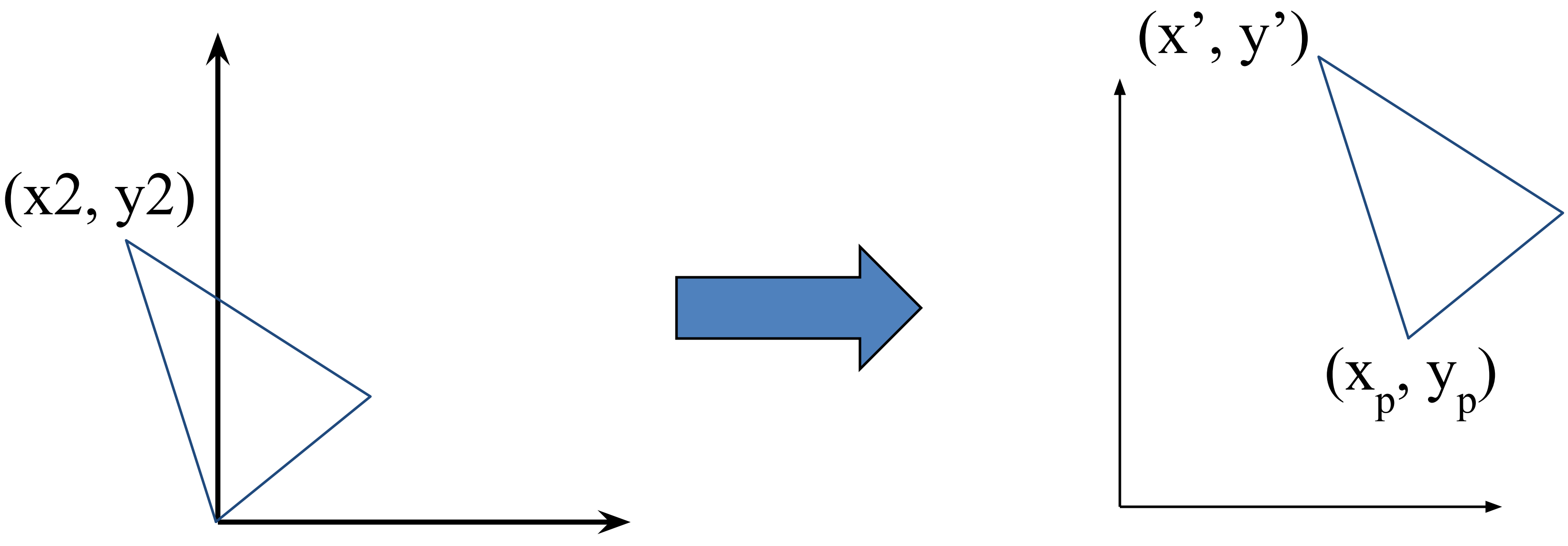


$$x2 = x1\cos\theta - y1\sin\theta$$

$$y2 = x1\sin\theta + y1\cos\theta$$

Rotation About a Pivot Point

STEP-3: **Translate the pivot point to original position**



$$x' = x2 + x_p$$

$$y' = y2 + y_p$$

Rotation About a Pivot Point

$$x' = (x - x_p)\cos\theta - (y - y_p)\sin\theta + x_p$$

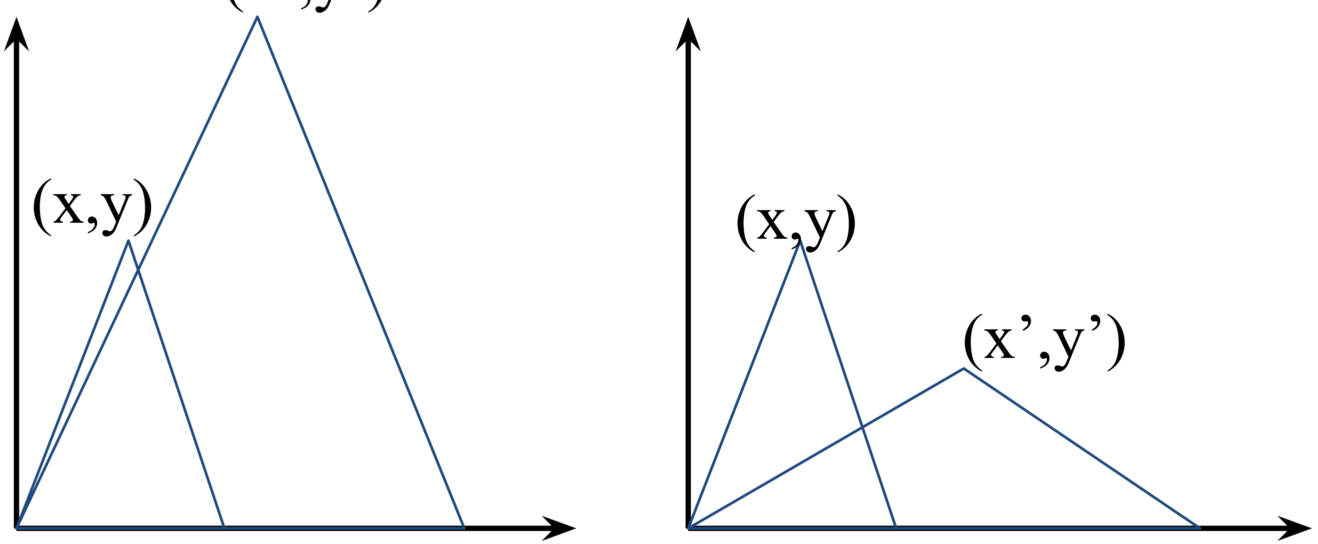
$$y' = (x - x_p)\sin\theta + (y - y_p)\cos\theta + y_p$$

Specifying a 2D-Rotation about a pivot point (xp,yp):

```
glTranslatef(xp, yp, 0);
glRotatef(theta, 0, 0, 1.0);
glTranslatef(-xp, -yp, 0);
```

Note the OpenGL specification of the sequence of transformations in the reverse order !

Scaling About the Origin



Uniform

Non-Uniform

$$x' = x.s_x$$

$$y' = y.s_y$$

$$(s_x, s_y > 0)$$

The parameters s_x, s_y are called *scale factors*.

$s_x = s_y$ $\qquad$ $s_x \neq s_y$

Scaling About the Origin

Specifying a 2D-Scaling with respect to the origin:

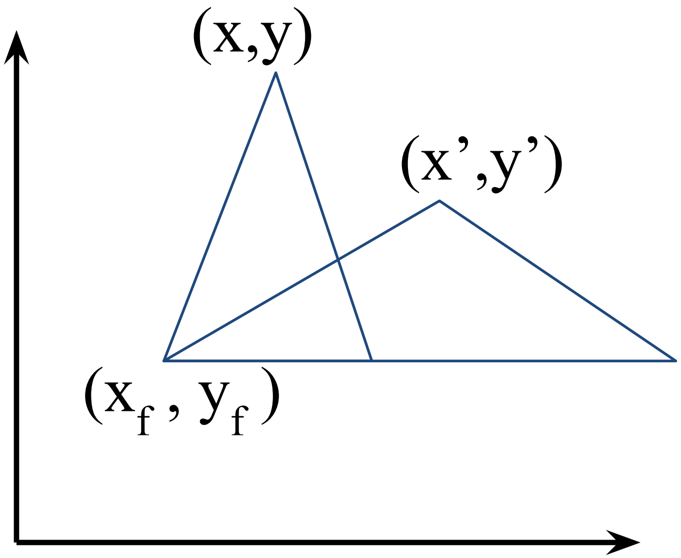
```
glScalef(sx, sy, 1.0);
```

`sx, sy`: Scale factors along x, y.

For proper scaling `sx, sy` must be positive.

For 2D scaling, the third scale factor must be set to 1.0.

Scaling About a Fixed Point



- Translate the fixed point to origin
- Scale with respect to the origin
- Translate the fixed point to its original position.

$$x' = (x - x_f).s_x + x_f$$

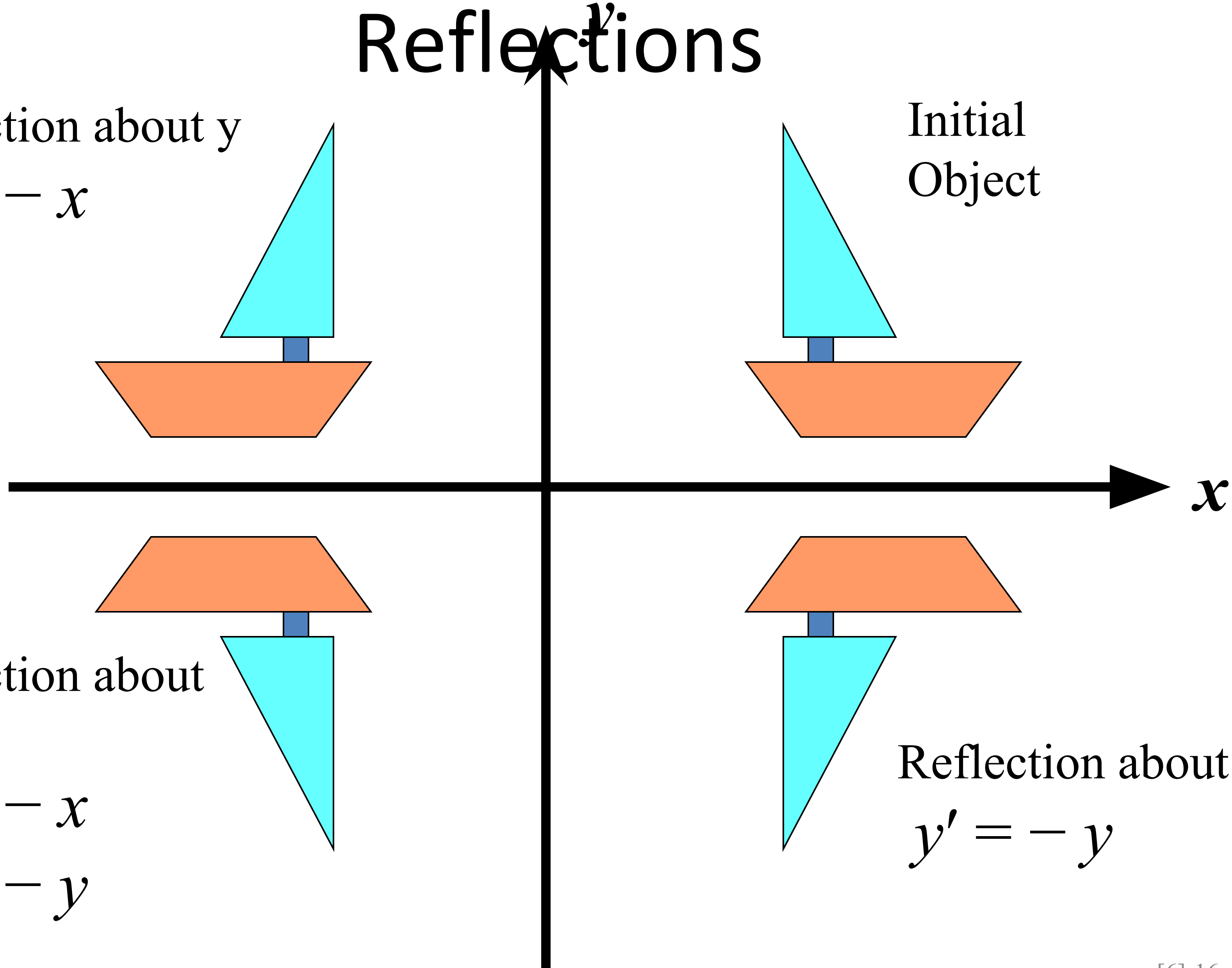
$$y' = (y - y_f).s_y + y_f$$

Reflections

Reflection about y

$$x' = -x$$

Initial Object



Reflection about origin

$$x' = -x$$

$$y' = -y$$

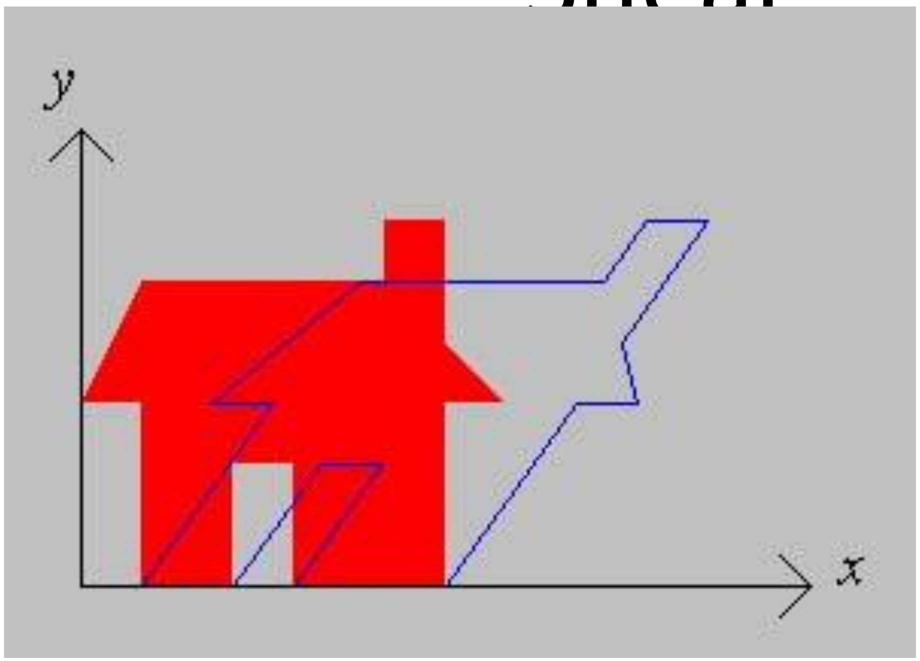
Reflection about x

$$y' = -y$$

Reflections

Reflection about x:	`glScalef(1, -1, 1);`
Reflection about y:	`glScalef(-1, 1, 1);`
Reflection about origin:	`glScalef(-1, -1, 1);`

Shear



$$x' = x + h_x y$$
$$y' = y$$

- A shear transformation in the x-direction (along x) shifts the points in the x-direction proportional to the y-coordinate.
- The y-coordinate of each point is unaffected.

Matrix Representations

Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Rotation [Origin]

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling [Origin]

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Representations

Reflection about x

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection about y

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection about the Origin

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Representations

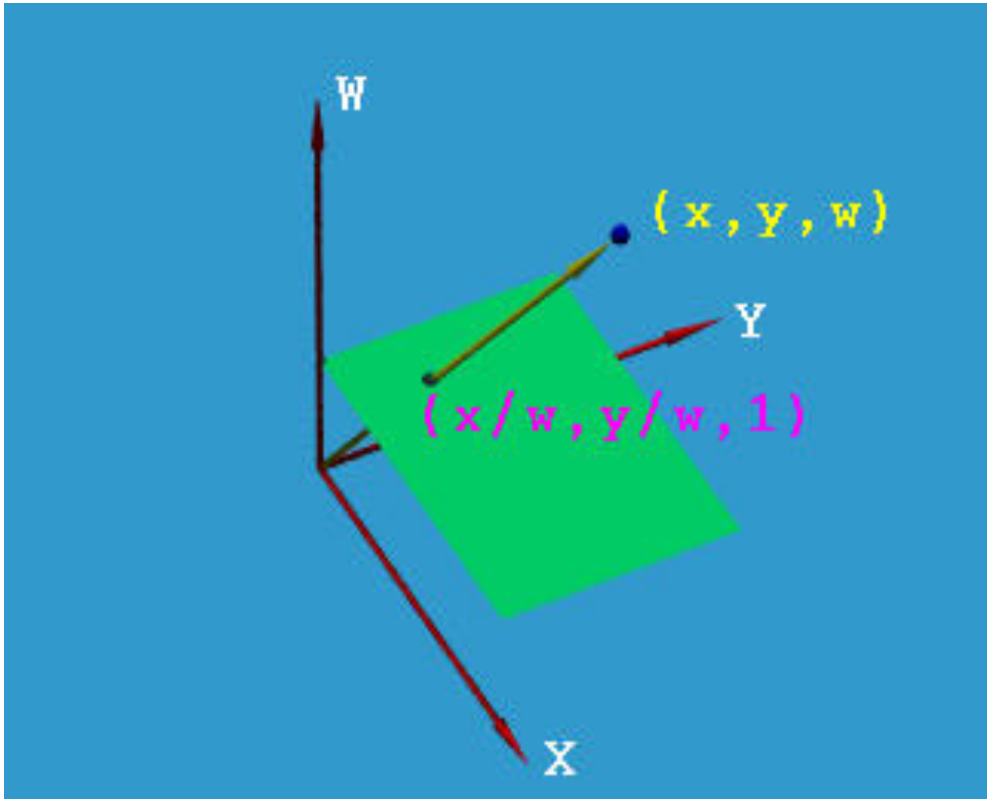
Shear along x $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear along y $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Coordinates

To obtain square matrices an additional row was added to the matrix and an additional coordinate, the *w*-coordinate, was added to the vector for a point. In this way a point in 2D space is expressed in three-dimensional homogeneous coordinates.

This technique of representing a point in a space whose dimension is one greater than that of the point is called homogeneous representation. It provides a consistent, uniform way of handling affine transformations.



Homogeneous Coordinates

- If we use homogeneous coordinates, the geometric transformations given above can be represented using only a matrix pre-multiplication.
- A composite transformation can then be represented by a product of the corresponding matrices.

<u>Cartesian</u> <u>Homogeneous</u>

$$(x, y) \longrightarrow (xh, yh, h), h \neq 0$$

$$\left(\frac{a}{c}, \frac{b}{c}\right) \longleftarrow (a, b, c), c \neq 0$$

Examples: (5, 8) (15, 24, 3)
(x, y) → (x, y, 1)

Homogeneous Coordinates

Basic Transformations

Translation

P'=TP

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation [O]

P'=RP

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling [O]

P'=SP

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverse of Transformations

If, $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [T] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ then, $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [T]^{-1} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$

Examples:

$$T^{-1}(t_x, t_y) = T(-t_x, -t_y)$$

$$R^{-1}(\theta) = R(-\theta)$$

$$S^{-1}(s_x, s_y) = S\left(\frac{1}{s_x}, \frac{1}{s_y}\right)$$

$$H_x^{-1}(h) = H_x(-h)$$

Transformation Matrices

Additional Properties:

$$T(t_x, t_y)T(u_x, u_y) = T(t_x + u_x,\ t_y + u_y)$$

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

$$\left|R(\theta)\right| = 1$$

$$S(s_x, s_y)S(w_x, w_y) = S(s_x w_x,\ s_y w_y)$$

Composite Transformations

Transformation T followed by Transformation Q followed by Transformation R:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [R][Q][T]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Example: (Scaling with respect to a fixed point)

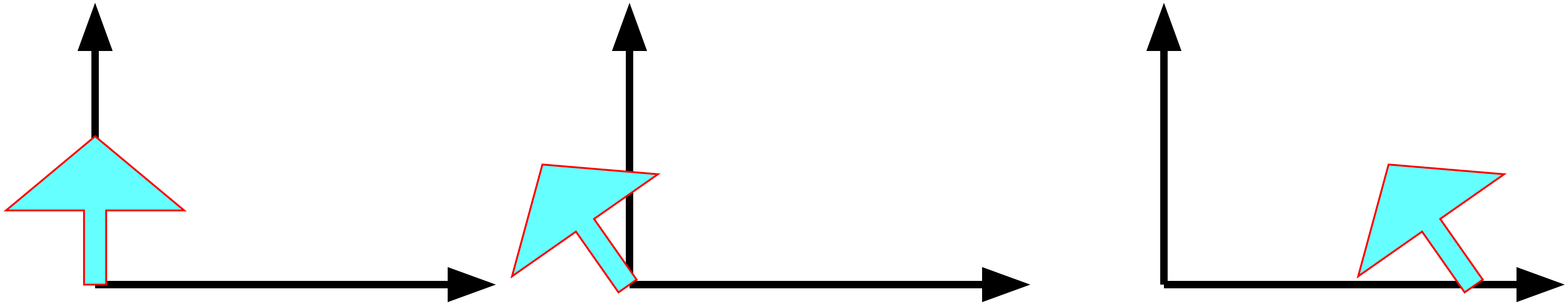
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Order of Transformations

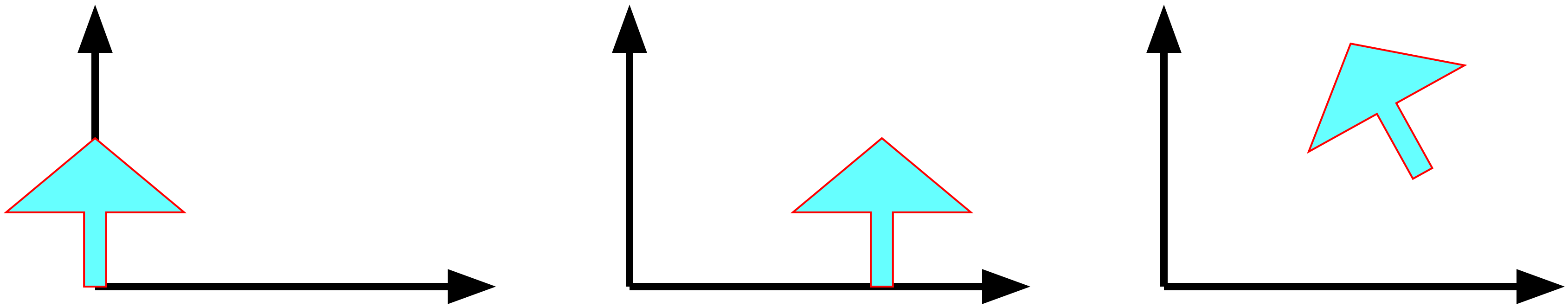
Order of Transformations

In composite transformations, the order of transformations is very important.

Rotation followed by Translation:



Translation followed by Rotation:



Order of Transformations (OpenGL)

OpenGL ***postmultiplies*** the current matrix with the new transformation matrix

	Current Matrix
`glMatrixMode(GL_MODELVIEW);`	
`glLoadIdentity();`	**[I]**
`glTranslatef(tx, ty, 0);`	**[T]**
`glRotatef(theta, 0, 0, 1.0);`	**[T] [R]**
`glVertex2f(x,y);`	**[T] [R] P**

Rotation followed by Translation !!

General Properties

Preserved Attributes

	Line	Angle	Distance	Area
Translation	Yes	Yes	Yes	Yes
Rotation	Yes	Yes	Yes	Yes
Scaling	Yes	No	No	No
Reflection	Yes	Yes	Yes	Yes
Shear	Yes	No	No	Yes

Affine Transformation

A general <u>invertible</u>, <u>linear</u>, transformation.

$$x' = a_1 x + b_1 y + c_1$$

$$y' = a_2 x + b_2 y + c_2$$

$$(a_1 b_2 - a_2 b_1 \neq 0)$$

Transformation Matrix:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenation.

- We perform 2 translations on the same

$$P' = T(d_{x1}, d_{y1}) \cdot P$$

$$P'' = T(d_{x2}, d_{y2}) \cdot P'$$

$$P'' = T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2}) \cdot P = T(d_{x1} + d_{x2}, d_{y1} + d_{y2}) \cdot P$$

So we expect :

$$T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2}) = T(d_{x1} + d_{x2}, d_{y1} + d_{y2})$$

Concatenation.

The matrix product $T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2})$ is :

$$\begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} = ?$$

Concatenation.

The matrix product $T(d_{x1}, d_{y1}) \cdot T(d_{x2}, d_{y2})$ is:

$$\begin{bmatrix} 1 & 0 & d_{x1} \\ 0 & 1 & d_{y1} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & d_{x2} \\ 0 & 1 & d_{y2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_{x1} + d_{x2} \\ 0 & 1 & d_{y1} + d_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of translations.

1. $T(0,0) = I$
2. $T(s_x, s_y) \cdot T(t_x, t_y) = T(s_x + t_x, s_y + t_y)$
3. $T(s_x, s_y) \cdot T(t_x, t_y) = T(t_x, t_y) \cdot T(s_x, s_y)$
4. $\mathrm{T}^{-1}(s_x, s_y) = T(-s_x, -s_y)$

Homogeneous form of scale.

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$