Chapter 4

Simplex Method

4.1 INTRODUCTION

Simplex method is an iterative procedure for solving an LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

4.1.1 Definitions

(*i*) **Cost Vector:** Let X_B be a basic feasible solution to the LPP.

$$\text{Max } Z = CX$$

Subject to, $\quad AX = b$

and, $\quad X \geq 0$. Such that it satisfies $X_B = B^{-1}b$

where B is the basis matrix formed by the column of basic variables.

The vector $C_{Bj} = (C_{B1}, C_{B2} \cdots C_{Bm})$ where C_{Bj} are the components of C associated with the basic variables, called the *cost vector* associated with the basic feasible solution X_B.

(*ii*) **Evaluation:** Let X_B be a basic feasible solution to the LPP.

$$\text{Max } Z = CX \text{ where,}$$
$$AX = b \text{ and } X \geq 0.$$

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A_1, which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij}\, b_i$$

Then the number $Z_j = \sum\limits_{i=1}^{m} C_{Bi}\, a_{ij}$

is called the *evaluation* corresponding to a_j and the number $(Z_j - C_j)$ is called the *net evaluation* corresponding to j.

4.2 SIMPLEX ALGORITHM

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. Various steps for the computation of an optimum solution are as follows:

Step 1 Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by

$$\text{Min } Z = -\text{Max } (-Z)$$

Step 2 Check whether all b_i ($i = 1, 2 \cdots m$) are positive. If any b_i is negative then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Step 3 Express the problem in standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

Step 4 Obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table as given below:

		C_j =	C_1	C_2	C_3	..	..	0	0	 0
C_B	S_B	x_B	x_1	x_2	x_3	x_4	..... x_n	S_1	S_2	..... S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	..... a_{1n}	1	0	 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	..... a_{2n}	1	0	 0

Step 5 Compute the net evaluations $Z_j - C_j$ by using the relation $Z_j - C_j = C_B (a_j - c_j)$
Examine the sign of $Z_j - C_j$

(*i*) If all $Z_j - C_j \geq 0$, then the initial basic feasible solution x_B is an optimum basic feasible solution.
(*ii*) If at least one $Z_j - C_j < 0$, then proceed to next step as the solution is not optimal.

Step 6 (To find the entering variable, i.e., key column)
If there are more than one negative $Z_j - C_j$, choose the most negative of them. Let it be $Z_r - C_r$ for some $j = r$. This gives the entering variable x_r and is indicated by an arrow at the bottom of the r^{th} column. If there are more than one variables having the same most negative $Z_j - C_j$ then, any one of them can be selected arbitrarily as the entering variable.

(*i*) If all $a_{ir} \leq 0$ ($i = 1, 2 \ldots m$) then there is an unbounded solution to the given problem.
(*ii*) If at least one $a_{ir} > 0$ ($i = 1, 2 \ldots m$) then the corresponding vector x_r enters the basis.

Step 7 (To find the leaving variable or key row)
Compute the ratio (x_{Bi}/a_{ir}, $a_{ir} > 0$)
If the minimum of these ratios be x_{Bi}/a_{kr}, then choose the variable x_k to leave the basis called the *key row* and the element at the intersection of key row and key column is called the *key element*.

Step 8 Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using Gauss Elimination Method on the formula

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and column}}{\text{Key element}}\right]$$

Step 9 Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

Example 4.1 Use simplex method to solve the LPP.

$$\text{Max} \quad Z = 3x_1 + 2x_2$$

$$\text{Subject to,} \quad \begin{aligned} x_1 + x_2 &\le 4 \\ x_1 - x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution By introducing the slack variables S_1, S_2, convert the problem in standard form.

$$\text{Max} \quad Z = 3x_1 + 2x_2 + 0S_1 + 0S_2$$

$$\text{Subject to,} \quad \begin{aligned} x_1 + x_2 + S_1 &= 4 \\ x_1 - x_2 + S_2 &= 2 \\ x_1, x_2, S_1, S_2 &\ge 0 \end{aligned}$$

Writing in martix form $AX = 6$

$$\begin{bmatrix} x_1 & x_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by

$$x_B = B^{-1}b,$$

where, $B = I_2, x_B = (S_1\ S_2).$

i.e., $(S_1\ S_2) = I_2\,(4, 2) = (4, 2)$

Initial simplex table

$$Z_j = C_B\, a_j$$

$$Z_1 - c_1 = C_B\, a_1 - c_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3$$

$$Z_2 - c_2 = C_B\, a_2 - c_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}(1 - 1) - 2 = -2$$

$$Z_3 - c_3 = C_B\, a_3 - c_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}(1\ 0) - 0 = -0$$

$$Z_4 - c_4 = C_B\, a_4 - c_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}(0\ 1) - 0 = -0.$$

		C_j	3	2	0	0		
C_B	**Basis**	x_B	x_1	x_2	S_1	S_2	*Min* $\frac{x_B}{x_1}$	
0	S_1	4	1	1	1	0	4/1 = 4	←
←0	S_2	2	1	–1	0	1	2/1 = 2	
	Z_j	0	0	0	0	0		
	$Z_j - C_j$		–3	–2	0	0		
			↑					

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimum.

Since $Z_1 - C_1 = -3$ is the most negative, the corresponding non-basic variable x_1 enters the basis.

The column corresponding to this x_1 is called the *key column*.

$$\text{To find the ratio } = \text{Min}\left\{\frac{x_{Bi}}{x_{ir}}, x_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{2}{1}\right\} = 2, \text{ which corresponds to } S_2.$$

∴ The leaving variable is the basic variable S_2. This row is called the *key row*. Convert the leading element x_{21} to units and all other elements in its column n i.e. (x_1) to zero by using the formula or Gauss Elimination:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{The element to be zero}}{\text{Key element}} = \frac{1}{1} = 1$$

Apply this ratio for the number of elements that are converted in the key row. Multiply this ratio by key row elements as shown below.

$$\begin{array}{c} 1 \times 2 \\ 1 \times 1 \\ 1 \times -1 \\ 1 \times 0 \\ 1 \times 1 \end{array}$$

Now subtract this element from the old element. The element to be converted into zero is called the *old element row*. Finally we have,

$$\begin{aligned} 4 - (1 \times 2) &= 2 \\ 1 - (1 \times 1) &= 0 \\ 1 - (1 \times -1) &= 2 \\ 1 - (1 \times 0) &= 1 \\ 0 - (1 \times 1) &= -1 \end{aligned}$$

∴ The improved basic feasible solution is given in the following simplex table

First iteration

		C_j	3	2	0	0		
C_B	Basis	x_B	x_1	x_2	S_1	S_2	Min $\frac{x_B}{x_2}$	
←0	S_1	2	0	2	1	–1	2/2 = 1	←
0	x_1	2	1	–1	0	1	—	
	Z_j	6	3	–3	0	0		
	$Z_j - C_j$		0	–5	0	0		
				↑				

Since $Z_2 - C_2$ is most negative, x_2 enters the basis.

To find $\text{Min}\left(\frac{x_B}{x_{i2}}, x_{i2} > 0\right)$

$$\text{Min}\left(\frac{2}{2}, \frac{2}{-1}\right) = 1. \qquad (\because \text{ negative or zero value are not considered})$$

This gives the outgoing variables. Convert the leading element into one. This is done by dividing all the elements in the key row by 2. The remaining elements should be made zero using the formula as shown below.

$-\frac{1}{2}$ is the common ratio. Put this ratio 5 times and multiply each ratio by key row elements.

$$\left(-\frac{1}{2}\right) \times 2$$
$$\left(-\frac{1}{2}\right) \times 0$$
$$\left(-\frac{1}{2}\right) \times 2$$
$$\left(-\frac{1}{2}\right) \times 1$$
$$\left(-\frac{1}{2}\right) \times -1$$

Subtract this result from the old element. All the row elements that are converted into zero, are called the *old element*.

$$2 - \left(-\frac{1}{2} \times 2\right) = 3$$
$$1 - \left(-\frac{1}{2} \times 0\right) = 1$$
$$-1 - \left(-\frac{1}{2} \times 2\right) = 0$$
$$0 - \left(-\frac{1}{2} \times 1\right) = \frac{1}{2}$$
$$1 - \left(-\frac{1}{2} \times -1\right) = \frac{1}{2}$$

Second iteration

		C_j	3	2	0	0
C_B	*Basis*	x_B	x_1	x_2	S_1	S_2
2	x_2	1	0	1	1/2	–1/2
3	x_1	3	1	0	1/2	1/2
	Z_j	11	3	2	5/2	1/2
	$Z_j - C_j$		0	0	5/2	1/2

Since all $Z_j - C_j \geq 0$, the solution is optimum. The optimal solution is Max $Z = 11$, $x_1 = 3$, and $x_2 = 1$.

Example 4.2 Solve the LPP

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to,
$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$4x_1 - x_2 \le 8$$
$$x_1, x_2 \ge 0$$

Solution Convert the inequality of the constraint into an equation by adding slack variables S_1, S_2, S_3,.

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$$

Subject to,
$$4x_1 + 3x_2 + S_1 = 12$$
$$4x_1 + x_2 + S_2 = 8$$
$$4x_1 - x_2 + S_3 = 8$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

$$\begin{bmatrix} x_1 & x_2 & S_1 & S_2 & S_3 \\ 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}$$

Initial table

		C_j	3	2	0	0	0	
C_B	*Basis*	x_B	x_1	x_2	S_1	S_2	S_3	*Min* $\frac{x_B}{x_1}$
0	S_1	12	4	3	1	0	0	12/4 = 3
0	S_2	8	4	1	0	1	0	8/4 = 2
←0	S_3	8	4	–1	0	0	1	8/4 = 2
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–3	–2	0	0	0	
			↑					

$\because$ $Z_1 - C_1$ is most negative, x_1 enters the basis. And the $\min\left(\frac{x_B}{x_{il}}, x_{il} > 0\right) = \min(3, 2, 2) = 2$ gives S_3 as the leaving variable.

Convert the leading element into 1, by dividing key row element by 4 and the remaining elements into 0.

First iteration

		C_j	3	2	0	0	0	
C_B	*Basis*	x_B	x_1	x_2	S_1	S_2	S_3	*Min* $\frac{x_B}{x_2}$
0	S_1	4	0	4	1	0	–1	4/4 = 1
←0	S_2	0	0	2	0	1	–1	0/2 = 0
3	x_1	2	1	–1/4	0	0	¼	—
	Z_j	6	3	–3/4	0	0	¾	
	$Z_j - C_j$		0	–11/4	0	0	¾	
				↑				

$$8-\left(\frac{4}{4}\times 8\right)=0 \qquad 12-\left(\frac{4}{4}\times 8\right)=4$$

$$4-\left(\frac{4}{4}\times 4\right)=0 \qquad 4-\left(\frac{4}{4}\times 4\right)=0$$

$$1-\left(\frac{4}{4}\times -1\right)=2 \qquad 3-\left(\frac{4}{4}\times -1\right)=4$$

$$0-\left(\frac{4}{4}\times 0\right)=0 \qquad 1-\left(\frac{4}{4}\times 0\right)=1$$

$$1-\left(\frac{4}{4}\times 0\right)=1 \qquad 0-\left(\frac{4}{4}\times 0\right)=0$$

$$0-\left(\frac{4}{4}\times 1\right)=-1 \qquad 0-\left(\frac{4}{4}\times 1\right)=-1$$

Since $Z_2 - C_2 = -\frac{11}{4}$ is the most negative, x_2 enters the basis.

To find the outgoing variable, find $\text{Min}\left(\frac{x_B}{x_{i2}}, x_{i2} > 0\right)$

$$\text{Min}\left(\frac{4}{4}, \frac{0}{2}, -\right) = 0$$

First iteration

Therefore, S_2 leaves the basis. Convert the leading element into 1 by dividing the key row elements by 2 and make the remaining elements in that column as zero using the formula.

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

		C_j	3	2	0	0	0	
C_B	**Basis**	x_B	x_1	x_2	S_1	S_2	S_3	$Min\ \frac{x_B}{S_3}$
←0	S_1	4	0	0	1	–2	1	4/1 = 4
2	x_2	0	0	1	0	1/2	–1/2	—
3	x_1	2	1	0	0	1/8	1/8	$\frac{2}{1/18} = 16$
	Z_j	6	3	2	0	11/8	–5/8	
	$Z_j - C_j$		0	0	0	11/8	–5/8	
							↑	

Second iteration

Since $Z_5 - C_5 = -5/8$ is most negative, S_3 enters the basis and

$$\text{Min}\left(\frac{x_B}{S_{13}}, S_{i3}\right) = \text{Min}\left(\frac{4}{1}, 16\right) = 4.$$

Therefore, S_1 leaves the basis. Convert the leading element into one and remaining elements as zero.

Third iteration

		C_j	3	2	0	0	0
C_B	**Basis**	x_B	x_1	x_2	S_1	S_2	S_3
0	S_3	4	0	0	1	–2	1
2	x_2	2	0	1	1/2	–1/2	0
3	x_1	3/2	1	0	–1/8	3/8	0
	Z_j	17/2	3	2	5/8	1/8	0
	$Z_j - C_j$		0	0	5/8	1/8	0

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $x_1 = 3/2$, $x_2 = 2$ and Max $Z = 17/2$.

Example 4.3 Using simplex method solve the LPP.

$$\text{Max } Z = x_1 + x_2 + 3x_3$$

Subject to,
$$3x_1 + 2x_2 + x_3 \leq 3$$
$$2x_1 + x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution Rewrite the inequality of the constraints into an equation by adding slack variables.

$$\text{Max } Z = x_1 + x_2 + 3x_3 + 0S_1 + 0S_2$$

Subject to,
$$3x_1 + 2x_2 + x_3 + S_1 \leq 3$$
$$2x_1 + x_2 + 2x_3 + S_2 \leq 2$$

Initial basic feasible solution is,

$$x_1 = x_2 = x_3 = 0$$
$$S_1 = 3, S_2 = 2 \text{ and } Z = 0$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & S_1 & S_2 \\ 3 & 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & 0 \end{bmatrix}$$

		C_j	1	1	3	0	0	
C_B	Basis	x_B	x_1	x_2	x_3	S_1	S_2	Min $\frac{x_B}{x_3}$
0	S_1	3	3	2	1	1	0	3/1 = 3
←0	S_2	2	2	1	2	0	1	2/2 = 1
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–1	–3	0	0	
					↑			

Since $Z_3 - C_3 = -3$ is the most negative, the variable x_3 enters the basis. The column corresponding to x_3 is called the *key column*.

To determine the key row or leaving variable, find $\text{Min}\left(\frac{x_B}{x_{i3}}, x_{i3} > 0\right)$ $\text{Min}\left(\frac{3}{1}, \frac{2}{2}\right) = 1$

Therefore, the leaving variable is the basic variable S_2, the row is called the *key row* and the intersection element 2 is called the *key element*.

Convert this element into one by dividing each element in the key row by 2 and the remaining elements in that key column as zero using the formula

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

First iteration

		C_j	1	1	3	0	0
C_B	Basis	x_B	x_1	x_2	x_3	S_1	S_2
0	S_1	2	2	3/2	0	1	–1/2
3	x_3	1	1	1/2	1	0	1/2
	Z_j	3	3	3/2	3	0	3/2
	$Z_j - C_j$		2	1/2	0	0	3/2

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, Max $Z = 3$.

Example 4.4 Use simplex method to solve the LPP.

$$\text{Min } Z = x_2 - 3x_3 + 2x_5$$

Subject to,

$$3x_2 - x_3 + 2x_5 \leq 7$$
$$-2x_2 + 4x_3 \leq 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

$$x_2, x_3, x_5 \ge 0$$

Solution Since the given objective function is of minimization, we shall convert it into maximization using Min Z = –Max(–Z) = –Max Z^*

$$\text{Max } Z^* = -x_2 + 3x_3 - 2x_5$$

Subject to,

$$3x_2 - x_3 + 2x_5 \le 7$$

$$-2x_2 + 4x_3 \le 12$$

$$-4x_2 + 3x_3 + 8x_5 \le 10$$

We rewrite the inequality of the constraints into an equation by adding slack variables S_1, S_2, S_3 and the standard form of LPP becomes.

$$\text{Max } Z = -x_2 + 3x_3 - 2x_5 + 0S_1 + 0S_2 + 0S_3$$

Subject to,

$$3x_2 - x_3 + 2x_5 + S_1 = 7$$

$$-2x_2 + 4x_3 + S_2 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + S_3 = 10$$

$$x_2, x_3, x_5, S_1, S_2, S_3 \ge 0$$

$\therefore$ The initial basic feasible solution is given by $S_1 = 7$, $S_2 = 12$, $S_3 = 10$. ($x_2 = x_3 = x_5 = 0$)

Initial table

		C_j	–1	3	–2	0	0	0	
C_B	Basis	x_B	x_2	x_3	x_5	S_1	S_2	S_3	Min $\frac{x_B}{x_3}$
0	S_1	7	3	–1	2	1	0	0	—
←0	S_2	12	–2	4	0	0	1	0	12/4 = 3
0	S_3	10	–4	3	8	0	0	1	10/3 = 3.33
	Z_j	0	0	0	0	0	0		
	$Z_j - C_j$		1	–3	2	0	0	0	
				↑					

Since $Z_2 - C_2 = -3 < 0$, the solution is not optimum.

The incoming variable is x_3 (key column) and the outgoing variable (key row) is given by,

$$\text{Min}\left(\frac{x_B}{x_{i3}}\, x_{i3} > 0\right) = \text{Min}\left(-, \frac{12}{4}, \frac{10}{3}\right) = 3.$$

Hence, S_2 leaves the basis.

First iteration

		C_j	-1	3	-2	0	0	0	
C_B	B	x_B	x_2	x_3	x_5	S_1	S_2	S_3	$Min\ \frac{x_B}{x_2}$
$\leftarrow 0$	S_1	10	5/2	0	2	1	1/4	0	$\frac{10}{5/2} = 4$
3	x_3	3	–1/2	1	0	0	1/4	0	—
0	S_3	1	5/2	0	8	0	–3/4	1	2/5
	Z_j	9	–3/2	3	0	0	3/4	0	
	$Z_j - C_j$		–1/2	0	2	0	3/4	0	
			↑						

Since $Z_1 - C_1 < 0$, the solution is not optimum. Improve the solution by allowing the variable x_2 to enter into the basis and the variable S_1 to leave the basis.

Second iteration

		C_j	-1	3	-2	0	0	0
C_B	B	x_B	x_2	x_3	x_5	S_1	S_2	S_3
–1	x_2	4	1	0	4/5	2/5	1/10	0
3	x_3	5	0	1	2/5	1/5	3/10	0
0	S_3	11	0	0	10	1	– 1/2	1
	Zj	11	– 1	3	2/5	1/5	8/10	0
	$Z_j - C_j$		0	0	12/5	1/5	8/10	0

Since all $Z_j - C_j \geq 0$, the solution is optimum.

$\therefore$ The optimal solution is given by Max $Z^* = 11$

$$x_2 = 4, x_3 = 5, x_5 = 0$$

$\therefore$ $\text{Min } Z = -\text{Max } (-Z) = -11$

$\therefore$ $\text{Min } Z = -11, x_2 = 4, x_3 = 5, x_5 = 0.$

Example 4.5 Solve the following LPP using simplex method.

Max $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4$

Subject to,
$$2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$$
$$7x_1 + x_4 \leq 70$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution Rewriting the inequality of the constraint into an equation by adding slack variables S_1, S_2 and S_3, the standard form of LPP becomes.

Max $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3$

Subject to, $\quad 2x_1 + x_2 + 5x_3 + 6x_4 + S_1 = 20$

$$3x_1 + x_2 + 3x_3 + 25x_4 + S_2 = 24$$
$$7x_1 + x_4 + S_3 = 70$$
$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

The initial basic feasible solution is $S_1 = 20$, $S_2 = 24$, $S_3 = 70$ ($x_1 = x_2 = x_3 = x_4 = 0$ non-basic)

The initial simplex table is given by

		C_j	15	6	9	2	0	0	0	
C_B	**Basis**	x_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	$Min \frac{x_B}{x_1}$
0	S_1	20	2	1	5	6	1	0	0	20/2 = 10
←0	S_2	24	3	1	3	25	0	1	0	24/3 = 8←
0	S_3	70	7	0	0	1	0	0	1	70/7 = 10
	Z_j	0	0	0	0	0	0	0	0	
	$Z_j - C_j$		–15	–6	–9	–2	0	0	0	
			↑							

$\therefore$ As some of $Z_j - C_j \leq 0$ the current basic feasible solution is not optimum. $Z_1 - C_1 = -15$ is the most negative value and hence x_1 enters the basis and the variable S_2 leaves the basis.

First iteration

		C_j	15	6	9	2	0	0	0	
C_B	**Basis**	x_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3	$Min \frac{x_B}{x_2}$
←0	S_1	4	0	1/3	3	– 32/3	1	– 2/3	0	$\frac{4}{1/3} = 12$
15	x_1	8	1	1/3	1	25/3	0	1/3	0	$\frac{8}{1/3} = 24$
0	S_3	14	0	– 7/3	– 7	– 172/3	0	– 7/3	1	—
	Z_j	120	15	5	15	125	0	5	0	
	$Z_j - C_j$		0	– 1	6	123	0	5	0	
				↑						

Since $Z_2 - C_2 = -1 < 0$ the solution is not optimal therefore, x_2 enters the basis and the basic variable S_1 leaves the basis.

Second iteration

		C_j	–5	6	9	–2	0	0	0
C_B	B	x_B	x_1	x_2	x_3	x_4	S_1	S_2	S_3
6	x_2	12	0	1	9	–32	3	–2	0
15	x_1	4	1	0	–2	57/3	–1	1	0
0	S_3	42	0	0	14	–132	7	–7	1
	Z_j	132	15	6	24	93	3	3	0
	$Z_j - C_j$		0	0	15	91	3	3	0

Since all $Z_j - C_j \geq 0$, the solution is optimal and is given by,

$$\text{Max Z} = 132, x_1 = 4, x_2 = 12, x_3 = 0, x_4 = 0.$$

Example 4.6

Max $Z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 430 \\ 3x_1 + 2x_3 &\leq 260 \\ x_1 + 4x_2 &\leq 420 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution Rewrite the constraint into equation by adding slack variables S_1, S_2, S_3. The standard form of LPP becomes

Max $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to,

$$\begin{aligned} x_1 + 2x_2 + x_3 + S_1 &= 430 \\ 3x_1 + 2x_3 + S_2 &= 460 \\ x_1 + 4x_2 + S_3 &= 420 \\ x_1, x_2, x_3, S_1, S_2, S_3 &\geq 0. \end{aligned}$$

The initial basic feasible solution is,

$$S_1 = 430, S_2 = 460, S_3 = 420 \ (x_1 = x_2 = x_3 = 0)$$

Initial table

		C_j	3	2	5	0	0	0	
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	$Min \frac{x_B}{x_3}$
0	S_1	430	1	2	1	1	0	0	430/1 = 430
←0	S_2	460	3	0	2	0	1	0	460/2 = 230
0	S_3	420	1	4	0	0	0	1	
	Z_j	0	0	0	0	0	0	0	
	$Z_j - C_j$		–3	–2	–5	0	0	0	
					↑				

Since some of $Z_j - C_j \leq 0$, the current basic feasible solution is not optimum. Since $Z_3 - C_3 = -5$ is the most negative, the variable x_3 enters the basis. To find the variable leaving the basis find,

$$\text{Min}\left(\frac{x_B}{x_{i3}}, x_{i3} > 0\right) = \text{Min}\left(\frac{430}{1}, \frac{460}{2}, -\right) = 230.$$

$\therefore$ the variable S_2 leaves the basis.

First iteration

		C_j	3	2	5	0	0	0	
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	$Min\ \frac{x_B}{x_2}$
←0	S_1	200	–1/2	2	0	1	1/2	0	200/2 = 100
5	x_3	230	3/2	0	1	0	1/2	0	—
0	S_3	420	1	4	0	0	0	1	420/4 = 105
	Z_j	1150	15/2	0	5	0	5/2	0	
	$Z_j - C_j$		9/2	–2	0	0	5/2	0	
				↑					

Since $Z_2 - C_2 = -2$ is negative, the current basic feasible solution is not optimum. Therefore, the variable x_2 enters the basis and the variable S_1 leaves the basis.

Second iteration

		C_j	3	2	5	0	0	0
C_B	B	x_B	x_1	x_2	x_3	S_1	S_2	S_3
2	x_2	100	– 1/4	1	0	1/2	– 1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	S_3	20	2	0	0	– 2	1	1
	Z_j	1350	7	2	5	1	+2	0
	$Z_j - C_j$		4	0	0	1	2	0

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = 0, x_2 = 100, x_3 = 230$ and Max $Z = 1350$.

EXERCISES

1. Using the simplex method, find non-negative values of x_1, x_2 and x_3, which
   Maximize $Z = x_1 + 4x_2 + 5x_3$
   Subject to the constraints,
   $$3x_1 + 6x_2 + 3x_3 \leq 22$$
   $$x_1 + 2x_2 + 3x_3 \leq 14$$
   and, $$3x_1 + 2x_2 \leq 14$$
   [**Ans.** Max $Z = 650$, $x_1 = 0$, $x_2 = 100$, $x_3 = 50$]
2. Maximize $Z = x_1 + x_2 + 3x_3$
   Subject to, $$3x_1 + 2x_2 + x_3 \leq 2$$
   $$2x_1 + x_2 + 2x_3 \leq 2$$
   $$x_1, x_2, x_3 \geq 0$$
   [**Ans.** Max $Z = 3$, $x_1 = x_2 = 0$, $x_3 = 1$]

3. Max $Z = 10x_1 + 6x_2$

   Subject to,
   $$x_1 + x_2 \le 2$$
   $$2x_1 + x_2 \le 4$$
   $$3x_1 + 8x_2 \le 12$$
   $$x_1, x_2 \ge 0$$
   [**Ans.** Max $Z = 20$, $x_1 = 2$, $x_2 = 0$]

4. Max $Z = 30x_1 + 23x_2 + 29x_3$

   Subject to the constraints,
   $$6x_1 + 5x_2 + 3x_3 \le 52$$
   $$6x_1 + 2x_2 + 5x_3 \le 14$$
   $$x_1, x_2, x_3 \ge 0$$
   [**Ans.** Max $Z = 161$, $x_1 = 0$, $x_2 \Rightarrow x_3 = 0$]

5. Max $Z = x_1 + 2x_2 + x_3$

   Subject to,
   $$2x_1 + x_2 - x_3 \ge -2$$
   $$-2x_1 + x_2 - 5x_3 \le 6$$
   $$4x_1 + x_2 + x_3 \le 6$$
   $$x_1, x_2, x_3 \ge 6$$
   [**Ans.** Max $Z = 10$, $x_1 = 0$. $x_2 = 4$, $x_3 = 2$]

6. A manufacturer is engaged in producing 2 products *x* and *y*, the contribution margin being ₹ 15 and ₹ 45 respectively. A unit of product *x* requires 1 unit of facility *A* and 0.5 unit of facility *B*. A unit of product *y* requires 1.6 units of facility *A*, 2.0 units of facility *B* and 1 unit of raw material *C*. The availability of total facility *A*, *B* and raw material *C* during a particular time period are 240, 162 and 50 units respectively.

   Using the simplex method find out the product mix that will maximize the contribution margin.

   [**Ans.** Max $Z = 15x_1 + 45x_2$; Subject to, $x_1 + 1.6x_2 \le 240$; $0.5x_1 + 2x_2 \le 162$; $x_2 \le 50$; $x_1, x_2 \ge 0$. Also, Max $Z =$ ₹ 1815, $x_1 = 18.4$, $x_2 = 35$]

7. A firm has availability of 240, 370 and 180 kg of wood, plastic and steel respectively. The firm produces two products *A* and *B*. Each unit of *A* requires 1, 3 and 2 kg of wood, plastic and steel respectively. The corresponding requirements for each unit of *B* are 3, 4 and 1 respectively. If *A* is sold for ₹ 4, and *B* for ₹ 6, determine how many units of *A* and *B* should be produced in order to obtain the maximum gross income. Use the simplex method.

   [**Ans.** Max $Z = 4x_1 + 6x_2$; Subject to, $x_1 + 3x_2 \le 240$; $3x_1 + 4x_2 \le 370$; $2x_1 + x_2 \le 180$; $x_1, x_2 \ge 0$. Max $Z = 540$, $x_1 = 30$, $x_2 = 70$]

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