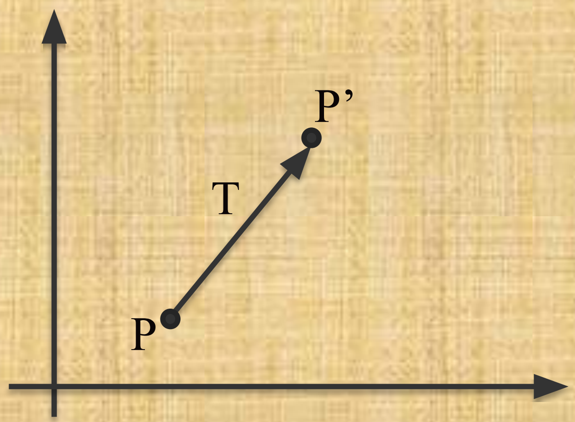
# 2-D Transformations

# Geometric Transformations

- Sometimes also called modeling transformations
  - Geometric transformations: Changing an object's position (translation), orientation (rotation) or size (scaling)
  - Modeling transformations: Constructing a scene or hierarchical description of a complex object
- Others transformations: reflection and shearing operations

# Basic 2D Geometric Transformations

- 2D Translation




  - $x' = x + t_x \, , \, y' = y + t_y$

$$P = \begin{bmatrix} x \\ y \end{bmatrix}, P' = \begin{bmatrix} x' \\ y' \end{bmatrix}, T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

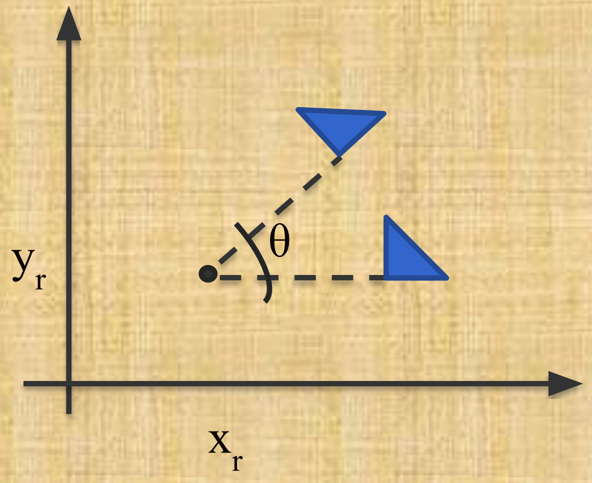
  - P'=P+T
  - Translation moves the object without deformation (rigid-body transformation)

# Basic 2D Geometric Transformations (cont.)

- 2D Translation
  - To move a line segment, apply the transformation equation to each of the two line endpoints and redraw the line between new endpoints
  - To move a polygon, apply the transformation equation to coordinates of each vertex and regenerate the polygon using the new set of vertex coordinates

# Basic 2D Geometric Transformations (cont.)

- 2D Rotation
  - Rotation axis
  - Rotation angle
  - rotation point or pivot point $(x_r,y_r)$



# Basic 2D Geometric Transformations (cont.)

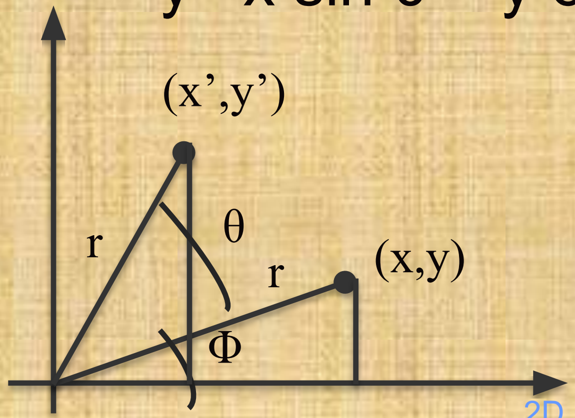
- 2D Rotation
  - If θ is positive □ counterclockwise rotation
  - If θ is negative □ clockwise rotation
  - Remember:
    - $\cos(a + b) = \cos a \cos b - \sin a \sin b$
    - $\cos(a - b) = \cos a \sin b + \sin a \cos b$

# Basic 2D Geometric Transformations (cont.)

- 2D Rotation
  - At first, suppose the pivot point is at the origin
  - $x' = r\cos(\theta+\Phi) = r\cos\theta\cos\Phi - r\sin\theta\sin\Phi$

    $y' = r\sin(\theta+\Phi) = r\cos\theta\sin\Phi + r\sin\theta\cos\Phi$
  - $x = r\cos\Phi,\ y = r\sin\Phi$
  - $x' = x\cos\theta - y\sin\theta$

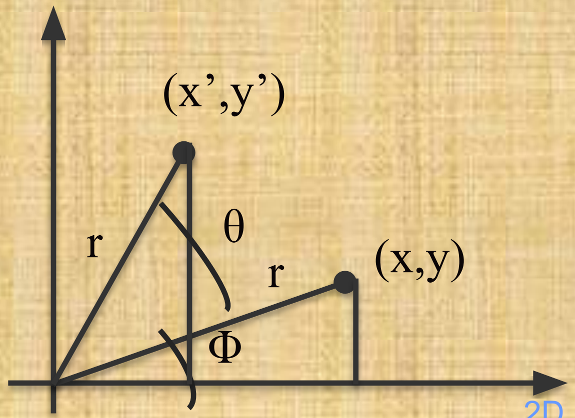
    $y' = x\sin\theta + y\cos\theta$



# Basic 2D Geometric Transformations

- 2D Rotation
  - P’=R·P

$$R = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$$



# Basic 2D Geometric Transformations (cont.)

- 2D Rotation
  - Rotation of a point about any specified position $(x_r, y_r)$
    $x' = x_r + (x - x_r) \cos\theta - (y - y_r) \sin\theta$
    $y' = y_r + (x - x_r) \sin\theta + (y - y_r) \cos\theta$
  - Rotations also move objects without deformation
  - A line is rotated by applying the rotation formula to each of the endpoints and redrawing the line between the new end points
  - A polygon is rotated by applying the rotation formula to each of the vertices and redrawing the polygon using new vertex coordinates

# Basic 2D Geometric Transformations (cont.)

- 2D Scaling
  - Scaling is used to alter the size of an object
  - Simple 2D scaling is performed by multiplying object positions (x, y) by scaling factors $s_x$ and $s_y$

    $x' = x \cdot s_x$

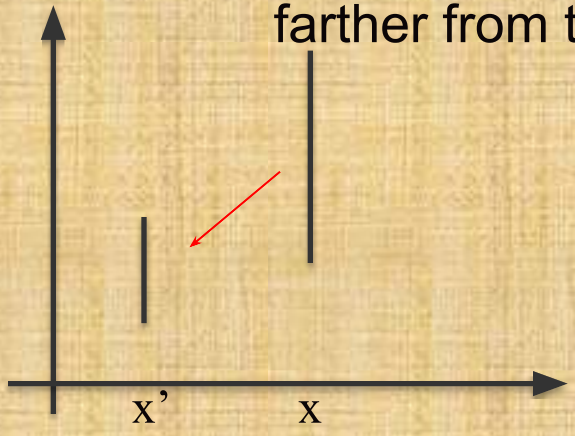
    $y' = y \cdot s_x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

or $P' = S \cdot P$

# Basic 2D Geometric Transformations (cont.)

- 2D Scaling
  - Any positive value can be used as scaling factor
    - Values less than 1 reduce the size of the object
    - Values greater than 1 enlarge the object
    - If scaling factor is 1 then the object stays unchanged
    - If $s_x = s_y$ , we call it <u>uniform scaling</u>
    - If scaling factor <1, then the object moves closer to the origin and If scaling factor >1, then the object moves farther from the origin



# Basic 2D Geometric Transformations (cont.)

- 2D Scaling
  - Why does scaling also reposition object?
  - Answer: See the matrix (multiplication)
  - Still no clue?
  - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x * s_x + y * 0 \\ x * 0 + y * s_y \end{bmatrix}$

# Basic 2D Geometric Transformations (cont.)

- 2D Scaling
  - We can control the location of the scaled object by choosing a position called the **fixed point ($x_f$,$y_f$)**

    $x' - x_f = (x - x_f)\, s_x \quad y' - y_f = (y - y_f)\, s_y$

    $x' = x \cdot s_x + x_f (1 - s_x)$

    $y' = y \cdot s_y + y_f (1 - s_y)$
  - Polygons are scaled by applying the above formula to each vertex, then regenerating the polygon using the transformed vertices

# Matrix Representations and Homogeneous Coordinates

- Many graphics applications involve sequences of geometric transformations
  - Animations
  - Design and picture construction applications
- We will now consider matrix representations of these operations
  - Sequences of transformations can be efficiently processed using matrices

# Matrix Representations and Homogeneous Coordinates (cont.)

- $P' = M_1 \cdot P + M_2$
  - P and P' are column vectors
  - $M_1$ is a 2 by 2 array containing multiplicative factors
  - $M_2$ is a 2 element column matrix containing translational terms
  - For translation $M_1$ is the identity matrix
  - For rotation or scaling, $M_2$ contains the translational terms associated with the pivot point or scaling fixed point

# Matrix Representations and Homogeneous Coordinates (cont.)

- To produce a sequence of operations, such as scaling followed by rotation then translation, we could calculate the transformed coordinates one step at a time
- A more efficient approach is to combine transformations, without calculating intermediate coordinate values

# Matrix Representations and Homogeneous Coordinates (cont.)

- Multiplicative and translational terms for a 2D geometric transformation can be combined into a single matrix if we expand the representations to 3 by 3 matrices
  - We can use the third column for translation terms, and all transformation equations can be expressed as matrix multiplications

# Matrix Representations and Homogeneous Coordinates (cont.)

- Expand each 2D coordinate *(x,y)* to three element representation $(x_h, y_h, h)$ called **homogeneous coordinates**
- h is the **homogeneous parameter** such that
  $x = x_h/h, \quad y = y_h/h,$
- □ infinite homogeneous representations for a point
- A convenient choice is to choose h = 1

# Matrix Representations and Homogeneous Coordinates (cont.)

- 2D Translation Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or, $\mathbf{P}' = \mathbf{T}(t_x, t_y) \cdot \mathbf{P}$

# Matrix Representations and Homogeneous Coordinates (cont.)

- 2D Rotation Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or, **P’** = **R**(θ)·**P**

# Matrix Representations and Homogeneous Coordinates (cont.)

- 2D Scaling Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or, $\boldsymbol{P'} = \boldsymbol{S}(s_x, s_y) \cdot \boldsymbol{P}$

# Inverse Transformations

- 2D Inverse Translation Matrix

$$T^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- By the way:

$$T^{-1} * T = I$$

# Inverse Transformations (cont.)

- 2D Inverse Rotation Matrix

$$R^{-1} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- And also:

$$R^{-1} * R = I$$

# Inverse Transformations (cont.)

- 2D Inverse Rotation Matrix:
  - If θ is negative □ clockwise
  - In

$$R^{-1} * R = I$$

  - Only sine function is affected
  - Therefore we can say

$$R^{-1} = R^{T}$$

  - Is that true?
  - Proof: It's up to you ☺

# Inverse Transformations (cont.)

- 2D Inverse Scaling Matrix

$$S^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Of course:

$$S^{-1} * S = I$$

# 2D Composite Transformations

- We can setup a sequence of transformations as a **composite transformation matrix** by calculating the product of the individual transformations
- $P' = M_2 \cdot M_1 \cdot P$
  $= M \cdot P$

# 2D Composite Transformations (cont.)

- Composite 2D Translations
  - If two successive translation are applied to a point P, then the final transformed location P' is calculated as

$$\mathbf{P}' = \mathbf{T}(t_{x_2}, t_{y_2}) \cdot \mathbf{T}(t_{x_1}, t_{y_1}) \cdot \mathbf{P} = \mathbf{T}(t_{x_1} + t_{x_2}, t_{y_1} + t_{y_2}) \cdot \mathbf{P}$$

$$\begin{bmatrix} 1 & 0 & t_{2x} \\ 0 & 1 & t_{2y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{1x} \\ 0 & 1 & t_{1y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{1x} + t_{2x} \\ 0 & 1 & t_{1y} + t_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

# 2D Composite Transformations (cont.)

- Composite 2D Rotations

$$\mathbf{P}' = \mathbf{R}(\theta_1 + \theta_2) \cdot \mathbf{P}$$

$$\begin{bmatrix} \cos\Theta_2 & -\sin\Theta_2 & 0 \\ \sin\Theta_2 & \cos\Theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_1 & -\sin\Theta_1 & 0 \\ \sin\Theta_1 & \cos\Theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\Theta_1 + \Theta_2) & -\sin(\Theta_1 + \Theta_2) & 0 \\ \sin(\Theta_1 + \Theta_2) & \cos(\Theta_1 + \Theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# 2D Composite Transformations (cont.)

- Composite 2D Scaling

$$\mathbf{S}\left(s_{x_2}, s_{y_2}\right) \cdot \mathbf{S}\left(s_{x_1}, s_{y_1}\right) = \mathbf{S}\left(s_{x_1} \cdot s_{x_2},\ s_{y_1} \cdot s_{y_2}\right)$$

$$\begin{bmatrix} s_{2x} & 0 & 0 \\ 0 & s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{1x} & 0 & 0 \\ 0 & s_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{1x} \cdot s_{2x} & 0 & 0 \\ 0 & s_{1y} \cdot s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

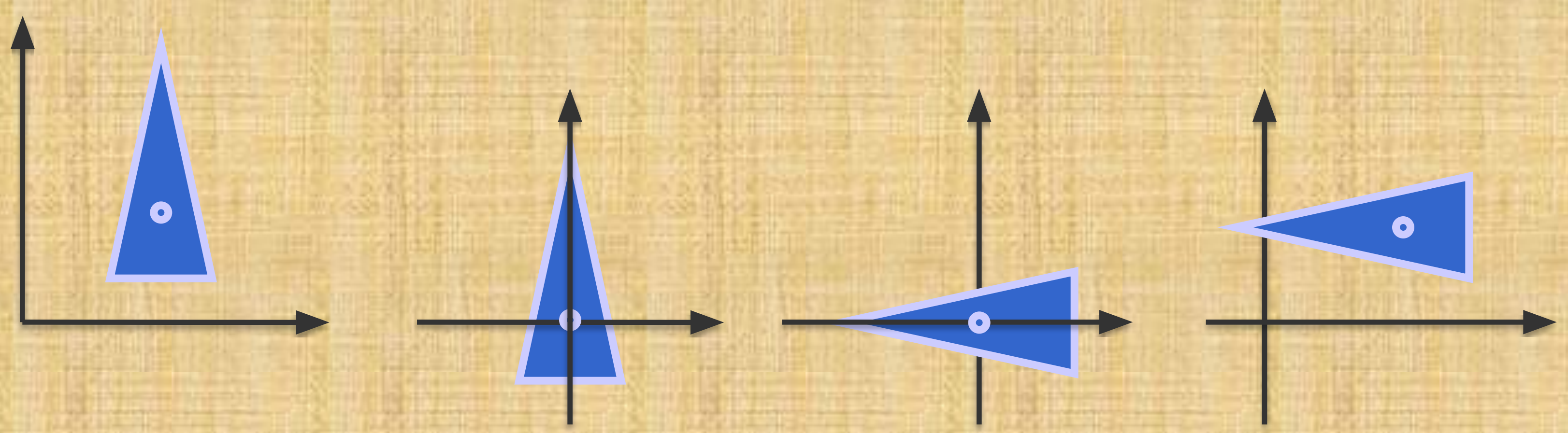
# 2D Composite Transformations (cont.)

- Don’t forget:
- Successive translations are additive
- Successive scalings are multiplicative
  - For example: If we triple the size of an object twice, the final size is nine (9) times the original
  - 9 times?
  - Why?
  - Proof: Again up to you ☺

# General Pivot Point Rotation

- Steps:
  1. Translate the object so that the pivot point is moved to the coordinate origin.
  2. Rotate the object about the origin.
  3. Translate the object so that the pivot point is returned to its original position.

# General Pivot Point Rotation



# 2D Composite Transformations (cont.)

- General 2D Pivot-Point Rotation

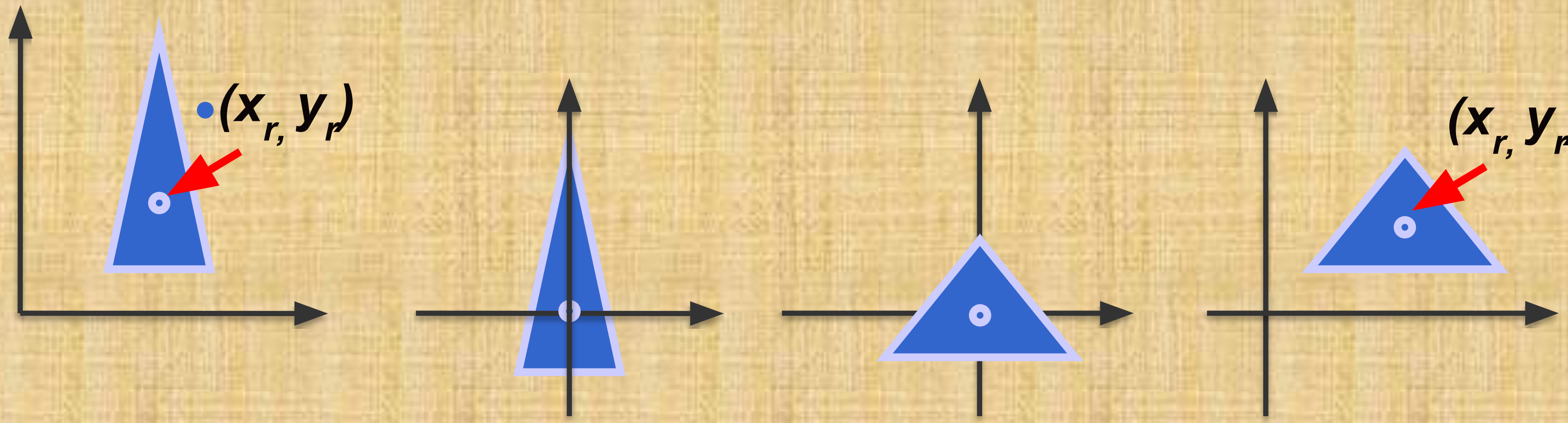
$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Theta & -\sin\Theta & x_r(1-\cos\Theta) + y_r \sin\Theta \\ \sin\Theta & \cos\Theta & y_r(1-\cos\Theta) - x_r \sin\Theta \\ 0 & 0 & 1 \end{bmatrix}$$

# General Fixed Point Scaling

- Steps:
  1. Translate the object so that the fixed point coincides with the coordinate origin.
  2. Scale the object about the origin.
  3. Translate the object so that the pivot point is returned to its original position.

# General Fixed Point Scaling (cont.)



# General Fixed Point Scaling (cont.)

- General 2D Fixed-Point Scaling:

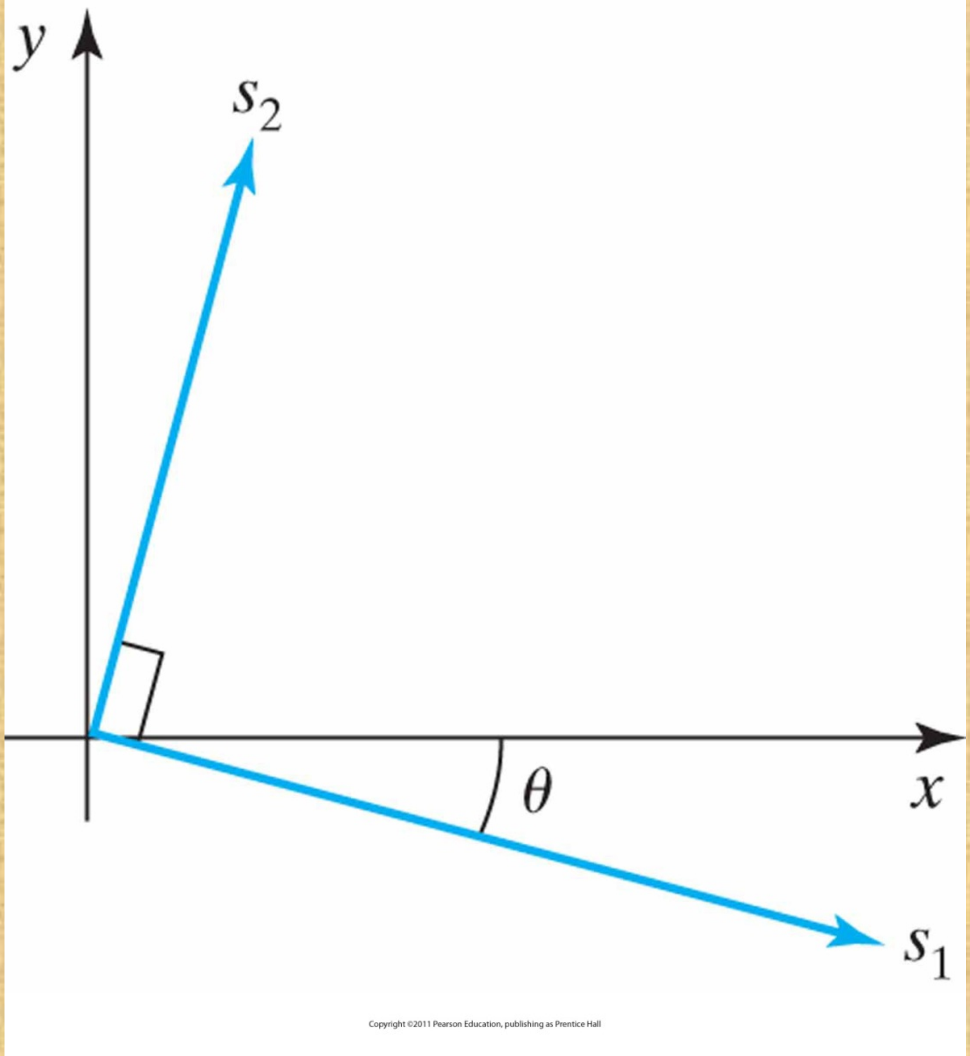
$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(x_f, y_f) \cdot \mathbf{S}(s_x, s_y) \cdot \mathbf{T}(-x_f, -y_f) = \mathbf{S}(x_f, y_f, s_x, s_y)$$

# 2D Composite Transformations (cont.)

- General 2D scaling directions:
  - Above: scaling parameters were along $x$ and $y$ directions
  - What about arbitrary directions?
  - Answer: See next slides

# General 2D Scaling Directions




Scaling parameters $s_1$ and $s_2$ along orthogonal directions defined by the angular displacement $\theta$.

# General 2D Scaling Directions (cont.)

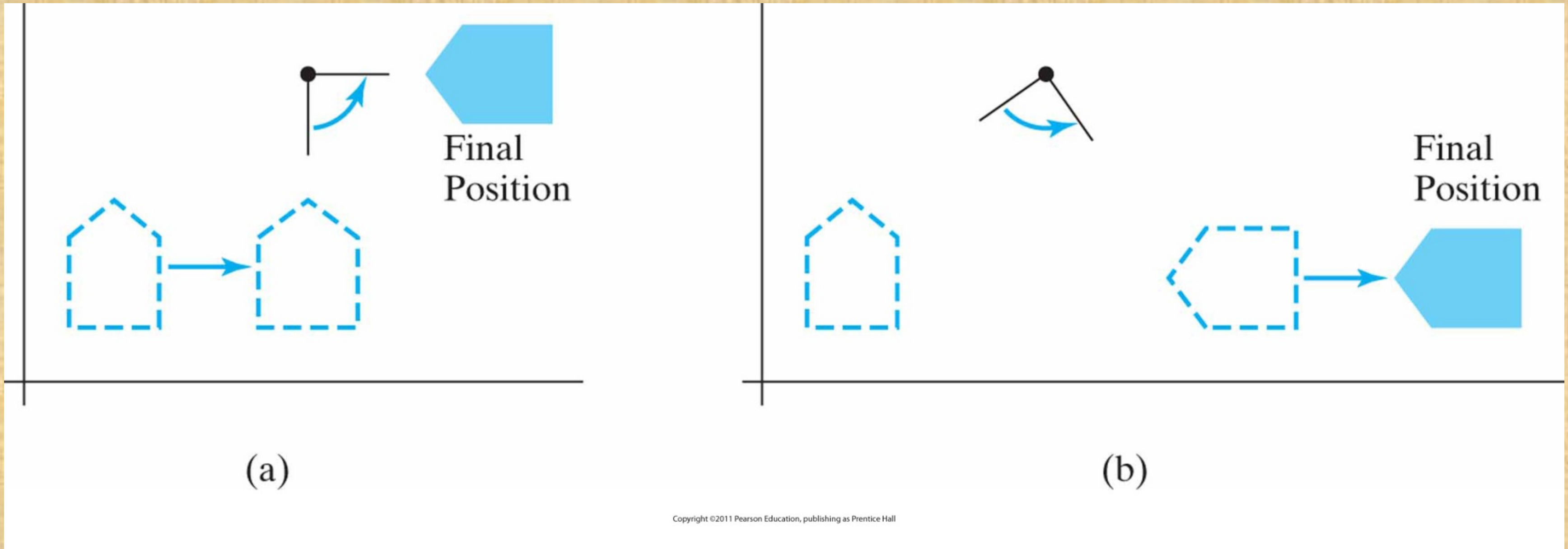
- General procedure:
  1. Rotate so that directions coincides with *x* and *y* axes
  2. Apply scaling transformation $S(s_1, s_2)$
  3. Rotate back
- The composite matrix:

$$R^{-1}(\Theta) * S(s_1, s_2) * R(\Theta) = \begin{bmatrix} s_1 \cos^2\Theta + s_2 \sin^2\Theta & (s_2 - s_1)\cos\Theta\sin\Theta & 0 \\ (s_2 - s_1)\cos\Theta\sin\Theta & s_1 \sin^2\Theta + s_2 \cos^2\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# 2D Composite Transformations (cont.)

- Matrix Concatenation Properties:
  - Matrix multiplication is associative !
    - $M_3 \cdot M_2 \cdot M_1 = (M_3 \cdot M_2) \cdot M_1 = M_3 \cdot (M_2 \cdot M_1)$
    - A composite matrix can be created by multiplicating left-to-right (premultiplication) or right-to-left (postmultiplication)
  - Matrix multiplication is ***not*** commutative !
    - $M_2 \cdot M_1 \neq M_1 \cdot M_2$

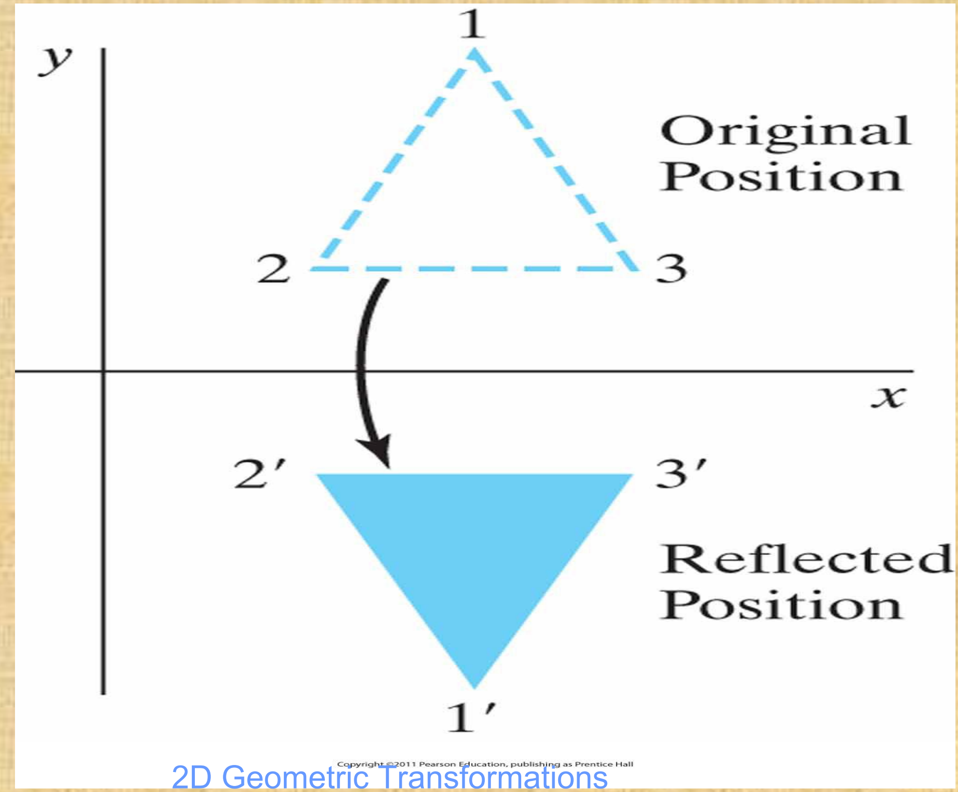
# Reversing the order




in which a sequence of transformations is performed may affect the transformed position of an object.
In (a), an object is first translated in the x direction, then rotated counterclockwise through an angle of 45°.
In (b), the object is first rotated 45° counterclockwise, then translated in the *x* direction

# Other 2D Transformations

- Reflection
  - Transformation that produces a mirror image of an object



# Other 2D Transformations (cont.)

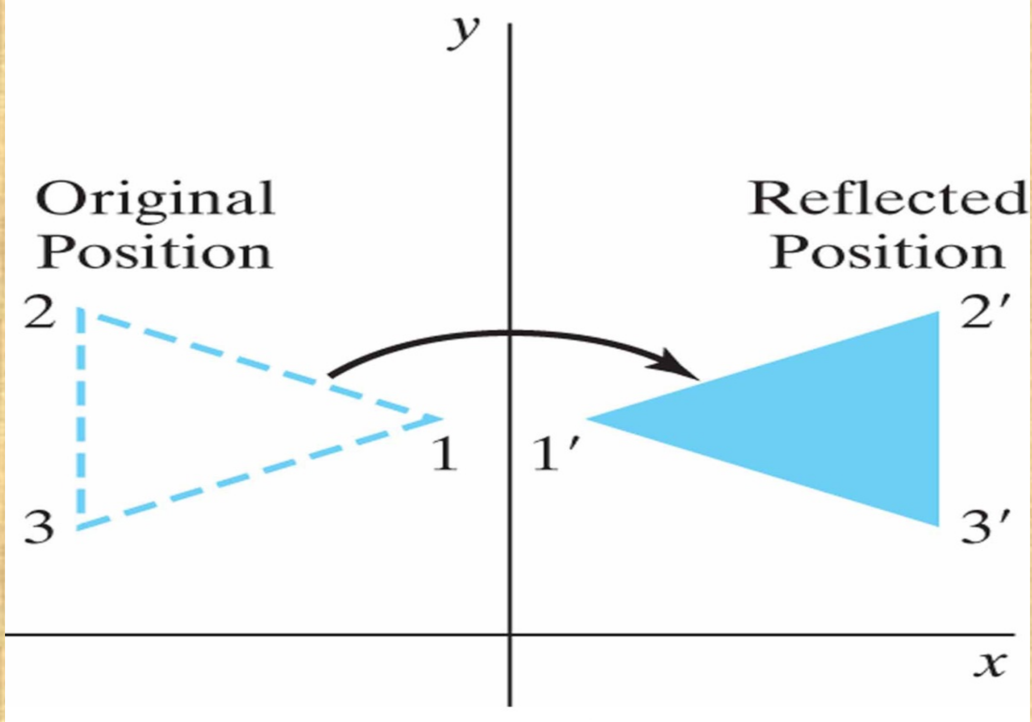
- Reflection
  - Image is generated relative to an axis of reflection by rotating the object 180° about the reflection axis
  - Reflection about the line y=0 (the x axis) (previous slide)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Other 2D Transformations (cont.)

- Reflection
  - Reflection about the line x=0 (the y axis)

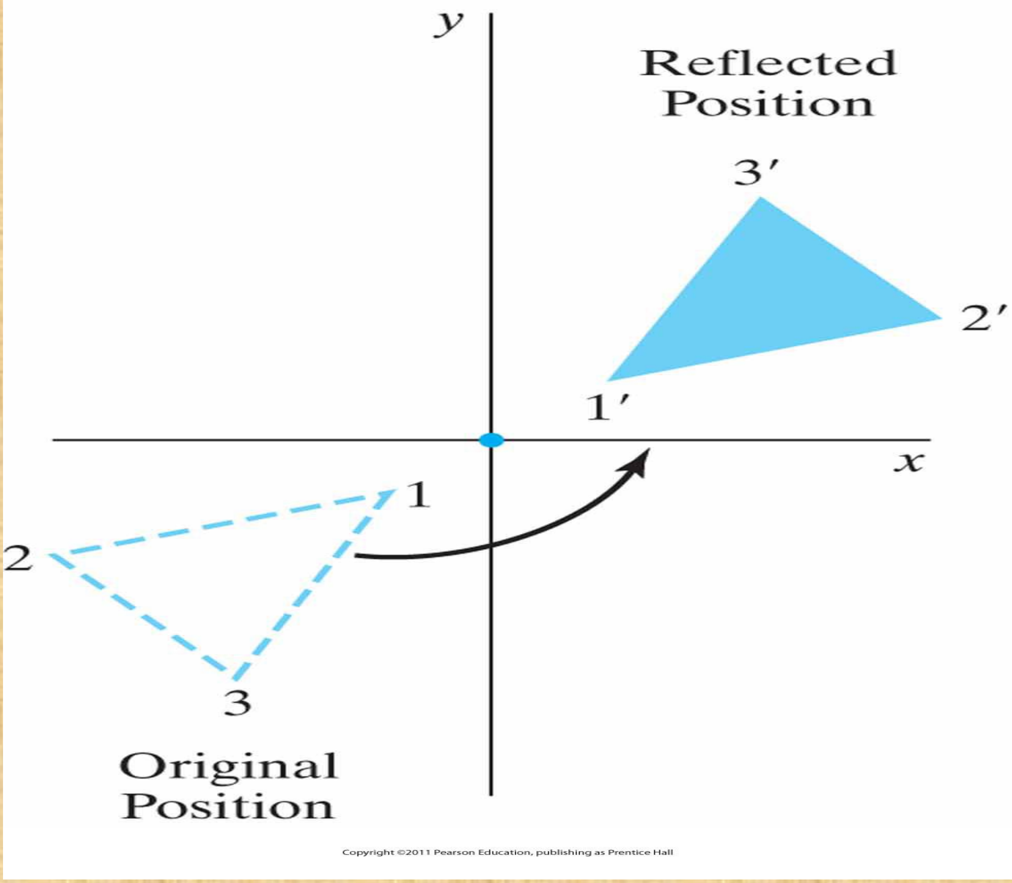
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Other 2D Transformations (cont.)

- Reflection about the origin

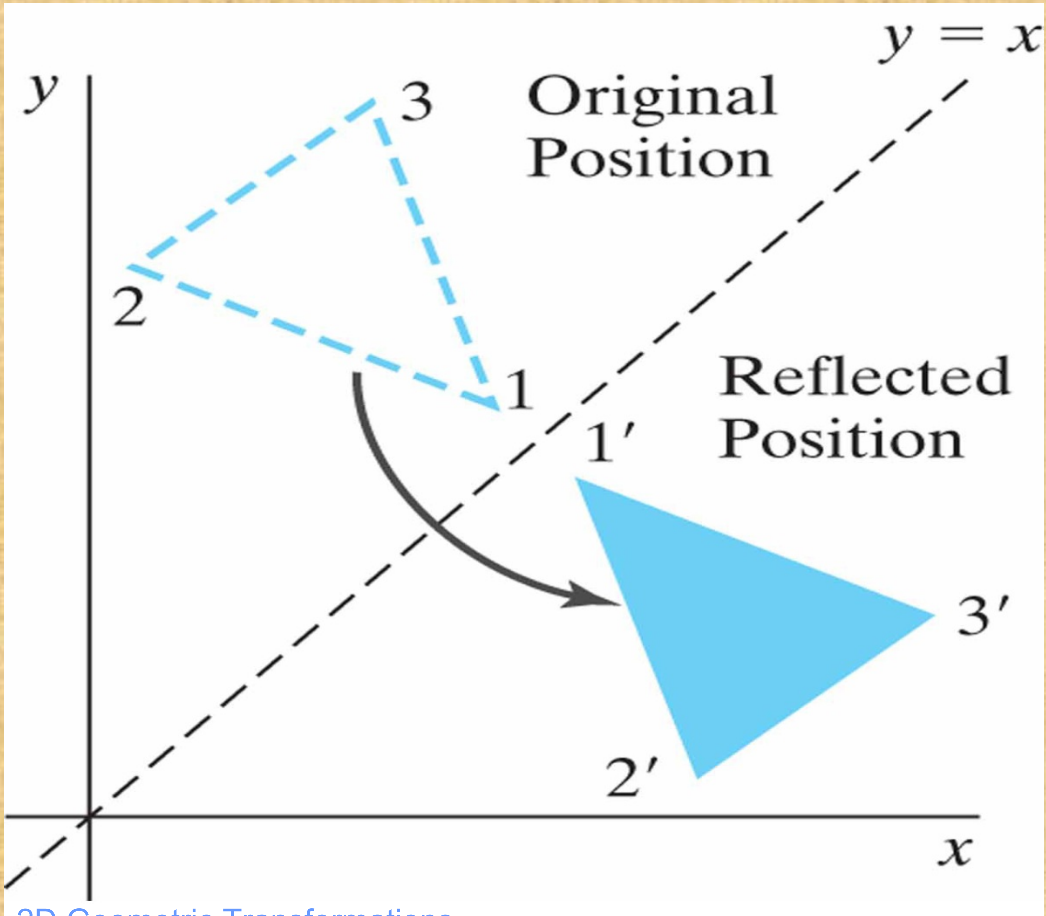
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Other 2D Transformations (cont.)

- Reflection about the line *y=x*

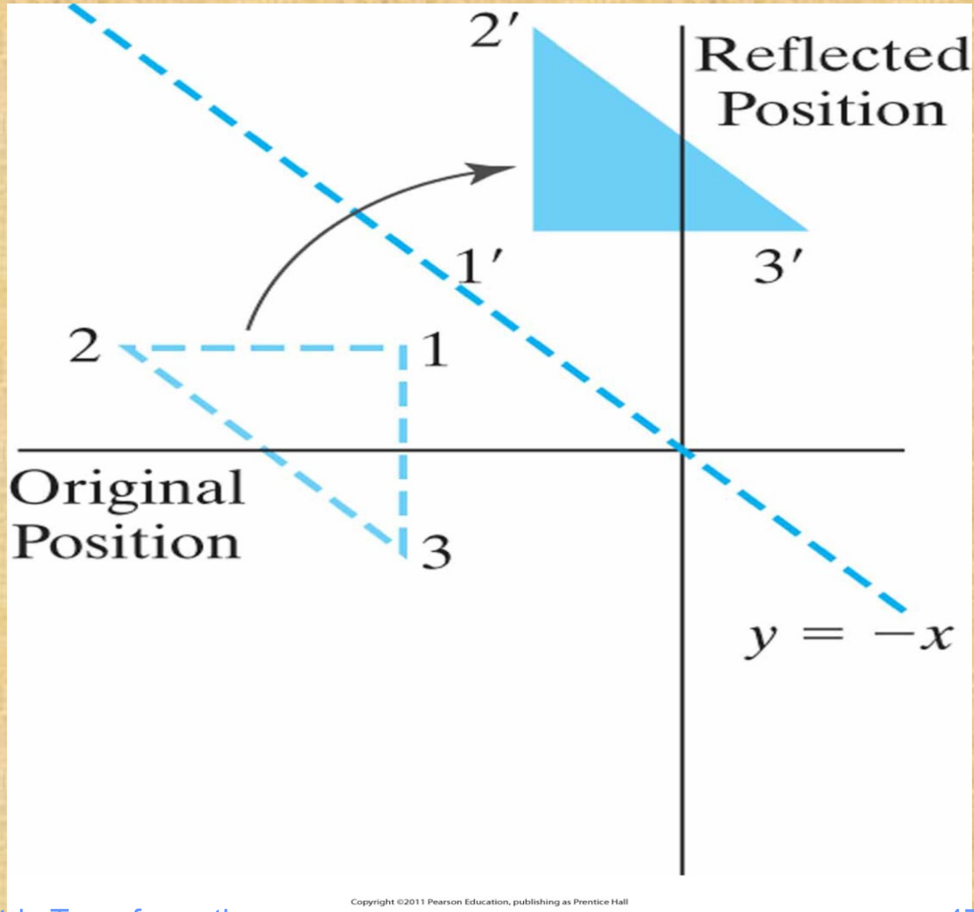
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Other 2D Transformations (cont.)

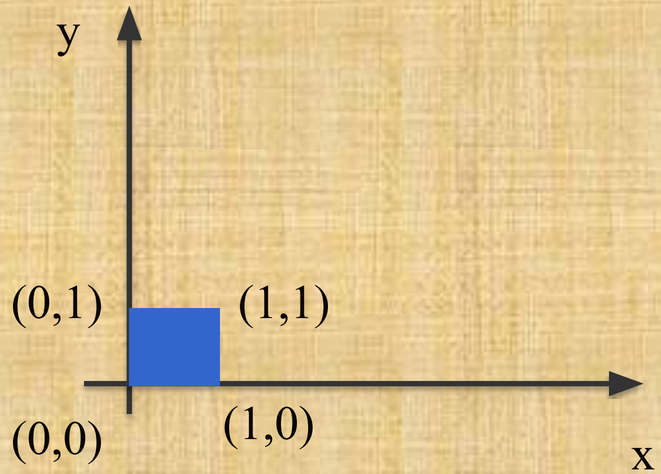
- Reflection about the line *y=-x*

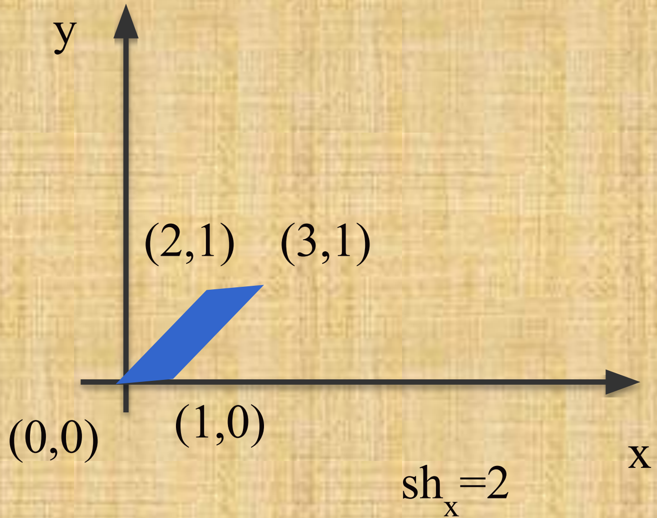
$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Other 2D Transformations (cont.)

- Shear
  - Transformation that distorts the shape of an object such that the transformed shape appears as the object was composed of internal layers that had been caused to slide over each other



# Other 2D Transformations (cont.)

- Shear
  - An x-direction shear relative to the x axis

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x' &= x + sh_x \cdot y \\ y' &= y \end{aligned}$$

  - An y-direction shear relative to the y axis

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Other 2D Transformations (cont.)

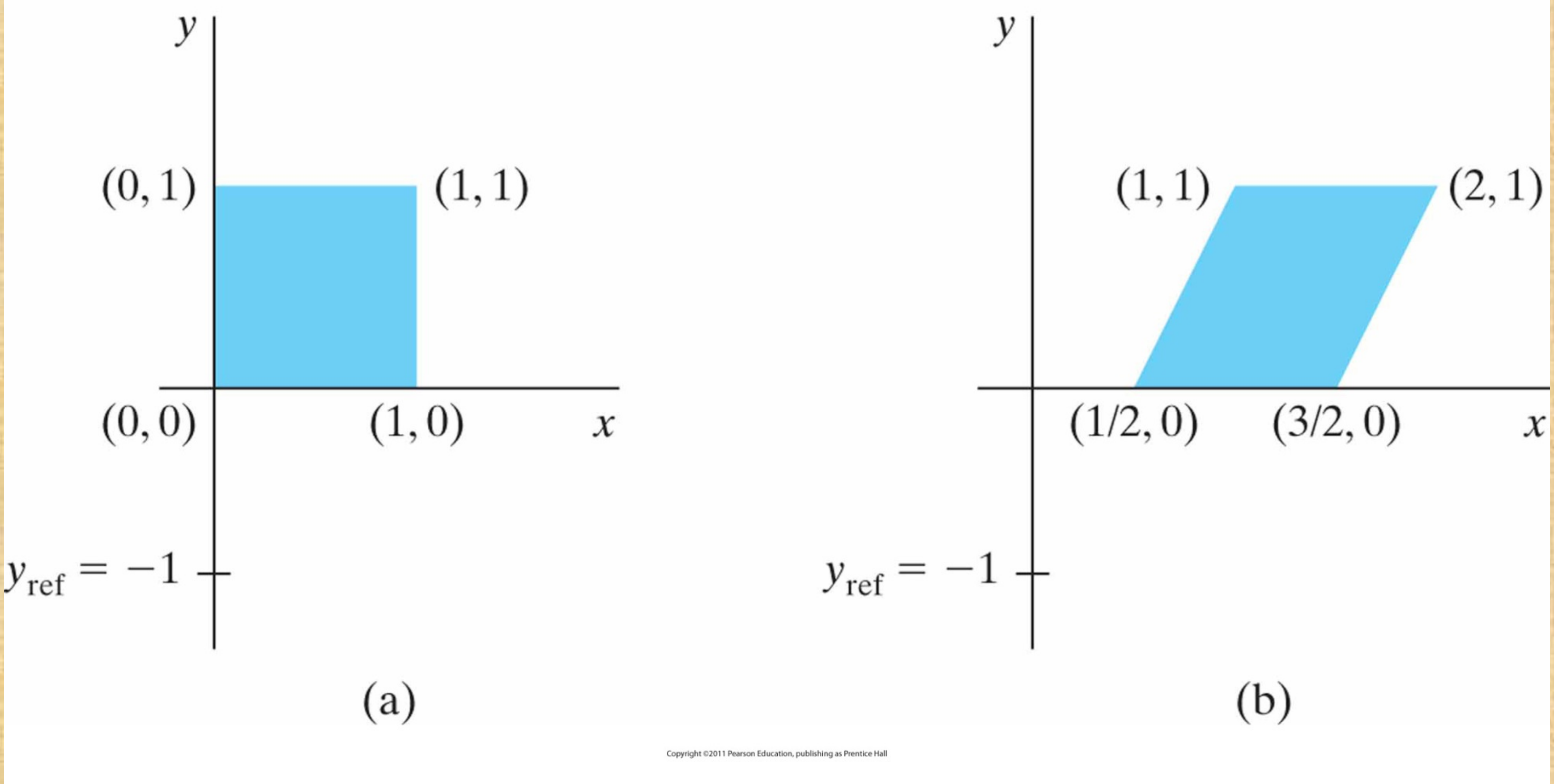
- Shear
  - x-direction shear relative to other reference lines

$$\begin{bmatrix} 1 & sh_x & -sh_x * y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x + sh_x * \left(y - y_{ref}\right)$$

$$y' = y$$

# Example




A unit square (a) is transformed to a shifted parallelogram (b) with $sh_x = 0.5$ and $y_{ref} = -1$ in the shear matrix from Slide 56

# Other 2D Transformations (cont.)

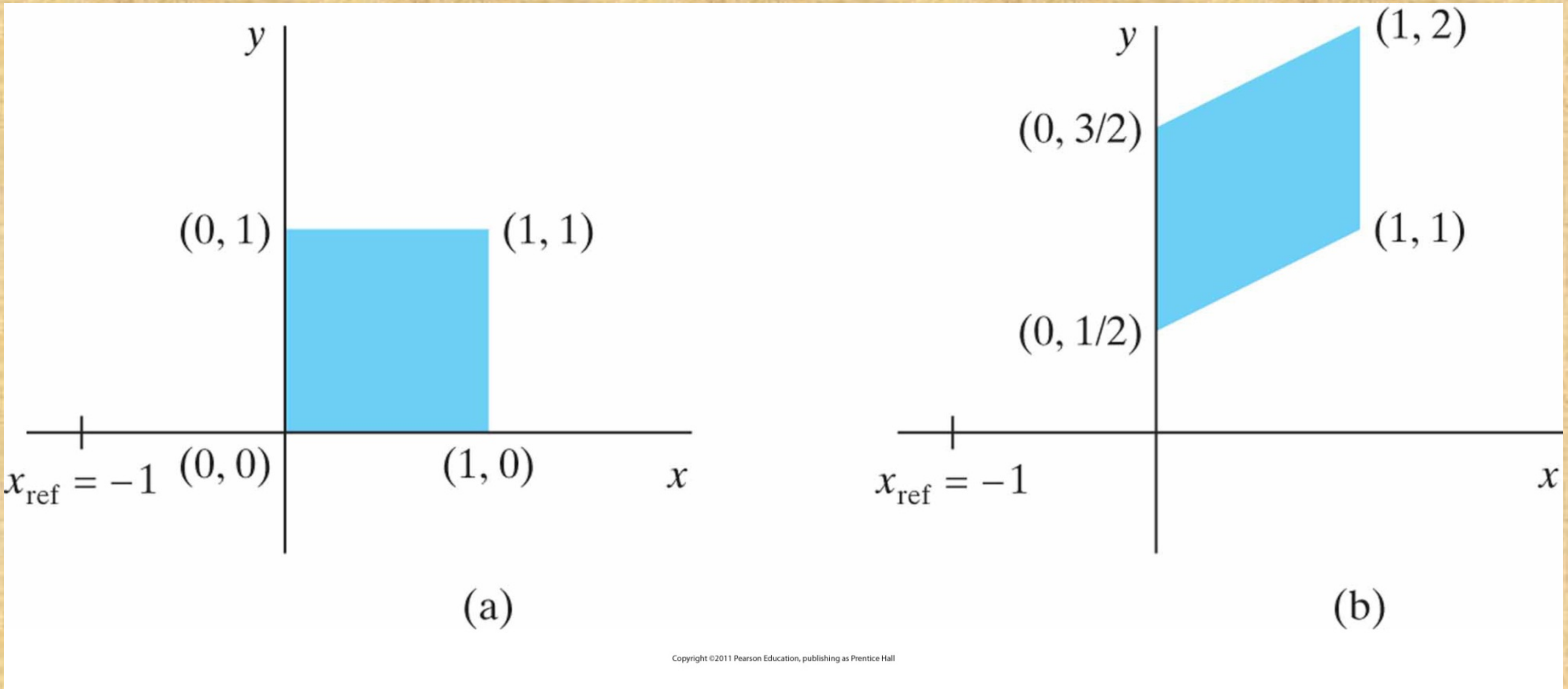
- Shear
  - y-direction shear relative to the line $x = x_{ref}$

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y * x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = x$$

$$y' = x + sh_y * \left(x - x_{ref}\right)$$

# Example




A unit square (a) is turned into a shifted parallelogram (b) with parameter values $sh_y = 0.5$ and $x_{ref} = -1$ in the $y$ -direction shearing transformation from Slide 58

# Transformation Between Coordinate Systems

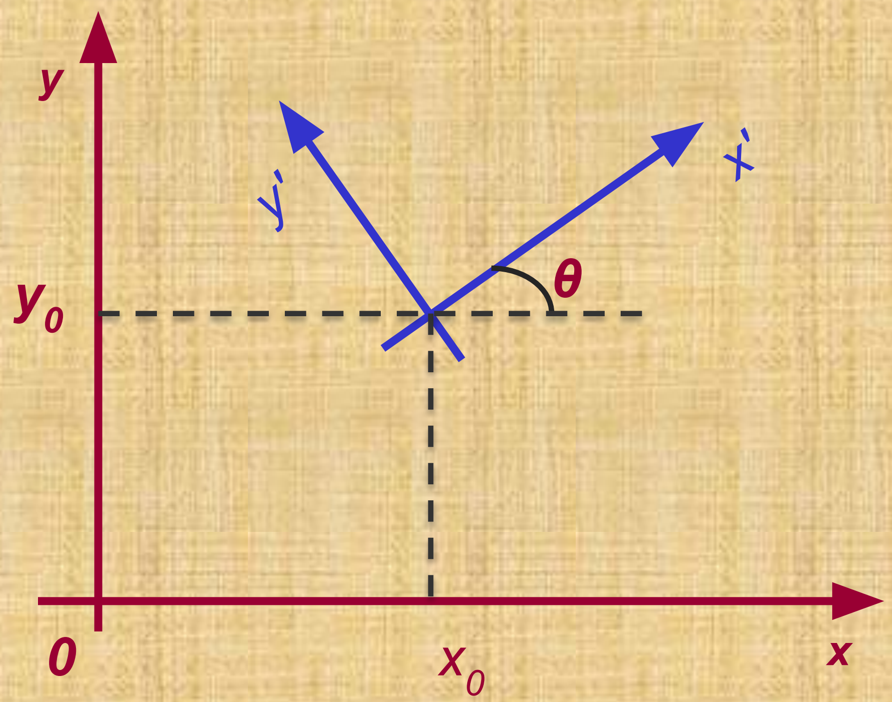
- Individual objects may be defined in their local cartesian reference system.
- The local coordinates must be transformed to position the objects within the scene coordinate system.

# Transformation Between Coordinate Systems

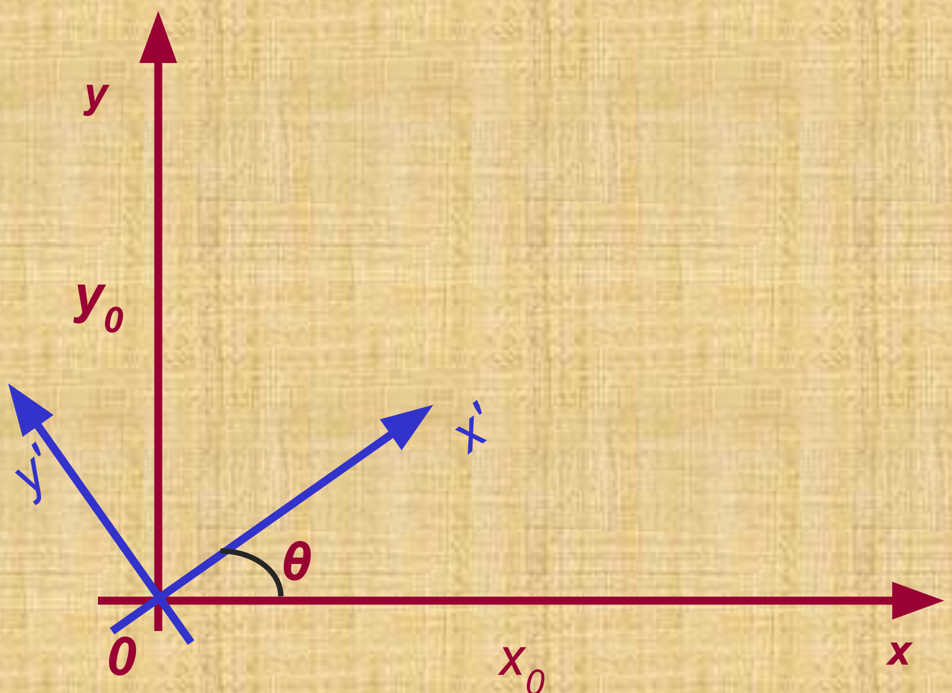
**Steps for coordinate transformation**

1. Translate so that the origin ($x_0$, $y_0$ ) of the $x'$-$y'$ system is moved to the origin of the $x$-$y$ system.
2. Rotate the $x'$ axis on to the axis $x$.

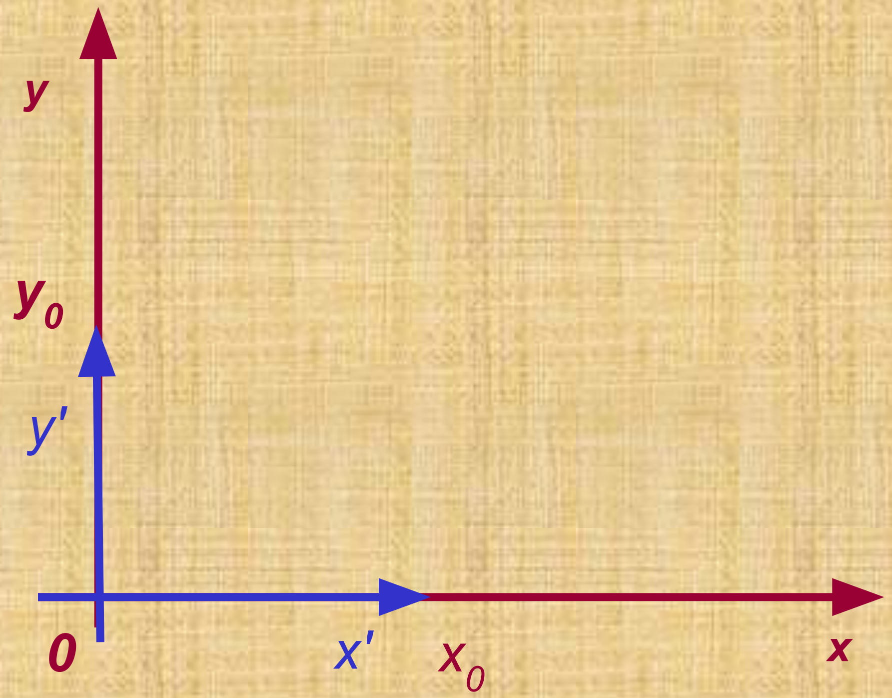
# Transformation Between Coordinate Systems (cont.)



# Transformation Between Coordinate Systems (cont.)



# Transformation Between Coordinate Systems (cont.)



# Transformation Between Coordinate Systems (cont.)

$$\mathbf{T}(-x_0, -y_0) = \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(-\theta) = \begin{bmatrix} Cos\theta & Sin\theta & 0 \\ -Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{xy,x'y'} = \mathbf{R}(-\theta) \cdot \mathbf{T}(-x_0, -y_0)$$

# Transformation Between Coordinate Systems (cont.)

An alternative method:

-Specify a vector **V** that indicates the direction for the positive y′ axis. Let

$$\mathbf{v} = \frac{\mathbf{V}}{|\mathbf{V}|} = (v_x, v_y)$$

-Obtain the unit vector **u**=($u_x$ ,$u_y$) along the x′ axis by rotating **v** $90^0$ clockwise.

# Transformation Between Coordinate Systems (cont.)

- Elements of any rotation matrix can be expressed as elements of orthogonal unit vectors. That is, the rotation matrix can be written as

$$\mathbf{R} = \begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Transformation Between Coordinate Systems (cont.)