Chapter

9 Transhipment and Assignment Problems

9.1 INTRODUCTION

This chapter deals with a very interesting method called the 'assignment technique', which is applicable to a class of very practical problems generally called 'assignment problems'.

The objective of assignment problems is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost or maximum profit.

9.2 TRANSHIPMENT PROBLEM

9.2.1 Definition

In a transportation problem, where shipments are allowed only between source-sink pairs, there is a possibility of existing points via which units of a goods/merchandise may be transhipped from a source to a sink. It is a strong assumption that shipments may be allowed between sources and between sinks, and also, inter-linking source-sink. Transportation models which have these additional features are called transhipment problems. Often, we see a gradual shift towards conversion from a transhipment problem to a transportation problem. This conversion procedure is of great significance as it broadens the applicability of algorithm as a solution for transportation problems. This conversion procedure can be well defined with the following example:

9.2.2 Transhipment Problem-to-Transportation Problem

An organic food company manufactures cereals in two cities, Leeds and Kent. The daily cereal production capacity at Leeds and Kent are 160 and 200 packets, respectively. Cereals are shipped by air to consumers in London and New York. The consumers in each city require 140 packets of cereals per day. However, due to the deregulation of air fares, the organic food company believes that it may be cheaper to fly some variety of cereals to Leeds or Dallas, and then do the final packaging to fly the packets of cereals to London and New York (final destinations). The table given below shows the cost of flying one packets (in £) of the cereals between these cities:

From →	*Leeds*	*Kent*	*London*	*Dallas*	*New York*
Leeds	£ 0	—	£ 9	£ 14	£ 29
Kent	—	£ 0	£ 16	£ 13	£ 26
London	—	—	£ 0	£ 7	£ 18
Dallas	—	—	£ 7	£ 0	£ 17
New York	—	—	—	—	£ 0

Now, to minimize the total incurred cost of daily shipments of the cereals to its consumers, we first need to understand terminologies, such as source and sink. Source is a city that can send products, however cannot receive any product from any other city. Whereas, sink is a city that can receive products but cannot send to any other city.

So, in this example we can say, that Leeds and Kent are source, and Leeds and Dallas are transhipment points, and finally, London and New York are sinks (each with a daily requirement of 140 packets of cereals).

So, we see a mismatch in demand and supply with the total supply equals to 156 and the total demand equals to 122.

Now, to solve this imbalance, we need to create a dummy sink, with a demand of 34. We would now have 2 sources, 2 sinks, and 2 transhipment points. As discussed before, transhipment points can act in dual roles, both as sources and sinks. As there are no transportation costs from a transhipment point to itself, the primary objective to reduce costs remain unaffected.

Therefore, we should perform a reformulation and use the transhipment points as an optimal solution for imbalanced demand-supply as well as reduce the transportation problem (costs) to ensure maximization of profits.

9.2.3 Transhipment Model

In a transhipment model, the objects are supplied from various specific sources to various specific destinations. It is also economic if the shipment passes via the transient nodes which are in between the sources and the destinations. It is different from transportation problem where the shipments are directly sent from a specific source to a specific destination, whereas in the transhipment problem the main goal is to reduce the total cost of shipments. Hence, the shipment passes via one or more intermediary nodes before it reaches its desired specific destination. Basically, there are two methods of evaluating transhipment problems as discussed below.

The following is the schematic illustration of the sources and destinations acting as transient nodes of a simple transhipment problem.

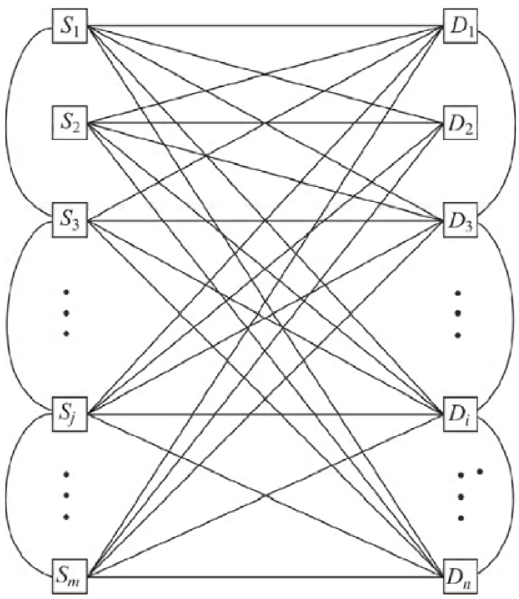


Fig. 9.1 Schematic Diagram of Simple Transhipment Model

The figure shows the shipment of objects from source S_1 to destination D_2. Shipment from source S_1 can pass via S_2 and D_1 before it reaches the desired destination D_2. Because the shipment passes via the particular transient nodes, this arrangement is named as transhipment model. The goal of the transhipment problem is to discover the optimal shipping model so that the total transportation cost is reduced.

Figure 9.2 shows a different approach where the number of first starting nodes and also the number of last ending nodes is the sum of the total number of sources and destinations of the original problem. Let B be the buffer which should be maintained at every transient source and transient destination. Considering it as a balanced problem, buffer B at the least may be equal to the sum of total supplies or the sum of total demands. Therefore, a constant B is further added to all the starting nodes and the ending nodes as shown below:

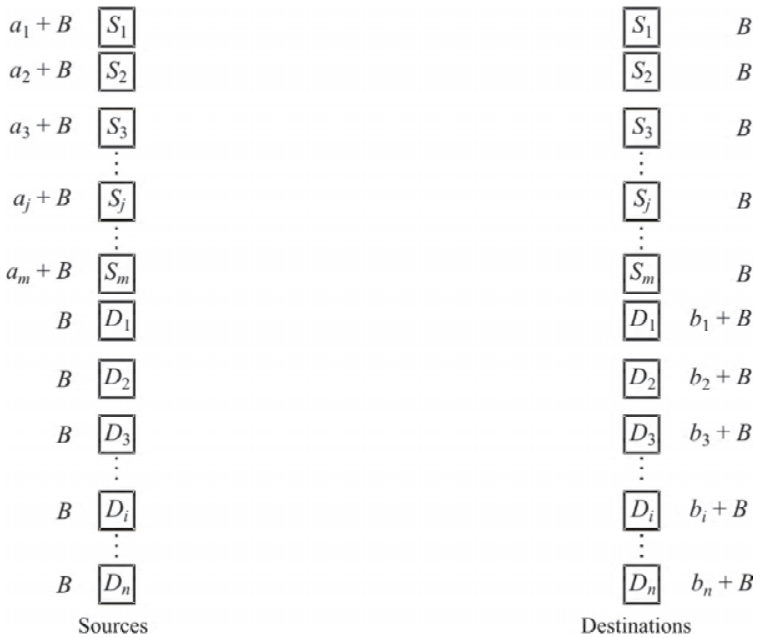


Fig. 9.2 Modified Version of Simple Transhipment Problem

Here in the modified version of simple transhipment model, the destinations $D_1, D_2, D_3, ..., D_i, ..., D_n$ are incorporated as added starting nodes which basically acts as the transient nodes. Hence, these nodes do not have the original supplies and at least the supply of every transient node must be equal to B. Therefore, every transient node is assigned B units of supply value. Also, the sources $S_1, S_2, S_3, ..., S_j ..., S_m$ are incorporated as added ending nodes which basically act as the transient nodes. These nodes too do not have the original demands but every transient node is assigned B units of demand value. To make it a balanced problem, B is further added to every starting node and to the ending nodes. Hence, the problem resembles a usual transportation problem and can be solved to obtain the optimum shipping plan.

Example 9.1 The following is the transhipment problem with 4 sources and 2 destinations. The supply values of the sources S_1, S_2, S_3 and S_4 are 100, 200, 150 and 350 units respectively. The demand values of destinations D_1 and D_2 are 350 and 450 units respectively. Transportation cost per unit between various defined sources and destinations are given in the following table. Solve the transhipment problem.

Source	Destination					
	S_1	S_2	S_3	S_4	D_1	D_2
S_1	0	4	20	5	25	12
S_2	10	0	6	10	5	20
S_3	15	20	0	8	45	7
S_4	20	25	10	0	30	6
D_1	20	18	60	15	0	10
D_2	10	25	30	23	4	0

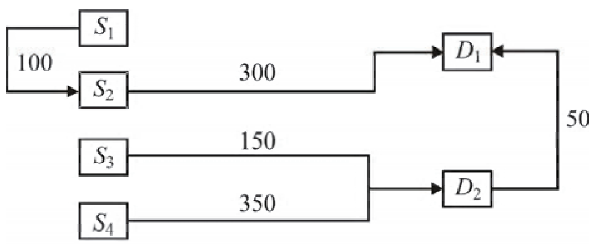
Solution: In the above table the number of sources is 4 and the number of destinations is 2. Therefore, the total number of starting nodes and the ending nodes of the transhipment problem will be 4 + 2 = 6. We also have,

$$B = \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_i$$

The following is the detailed format of the transhipment problem after including transient nodes for the sources and the destinations. Here the value of B is added to all the rows and columns.

Source	Destination						Supply
	S_1	S_2	S_3	S_4	D_1	D_2	
S_1	0	4	20	5	25	12	100 + 800 = 900
S_2	10	0	6	10	5	20	200 + 800 = 1000
S_3	15	20	0	8	45	7	150 + 800 = 950
S_4	20	25	10	0	30	6	350 + 800 = 1150
D_1	20	18	60	15	0	10	800
D_2	10	25	30	23	4	0	800
	800	800	800	800	350 + 800 = 1150	450 + 800 = 1250	

The optimal solution and the corresponding total cost transportation is ₹ 5600. The allocations defined in the main diagonal cells are ignored. The diagrammatic representation of the optimal shipping pattern of the shipments related to the off-diagonal cells is shown below:



Optimal Shipping Pattern

9.3 ASSIGNMENT PROBLEM

9.3.1 Definition

Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degrees of efficiency. Let c_{ij} be the cost if the ith person is assigned to the jth job. The problem is to find an assignment (which job should be assigned to which person, on a one-to-one basis) so that the total cost of performing all the jobs is minimum. Problems of this kind are known as assignment problems.

An assignment problem can be stated in the form of $n \times n$ cost matrix $[c_{ij}]$ of real numbers as given in the following table.

$$\begin{array}{cc} & \textbf{Jobs} \\ & \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & j & \cdots & n \end{array} \\ \textbf{Persons} \begin{array}{c} 1 \\ 2 \\ 3 \\ i \\ \vdots \\ n \end{array} & \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2j} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3j} & \cdots & c_{3n} \\ c_{i1} & c_{i2} & c_{i3} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & & & \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix} \end{array}$$

9.3.2 Mathematical Formulation of an Assignment Problem

Mathematically, an assignment problem can be stated as,

$$\text{Minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}, \text{ where, } i = 1, 2, \ldots n, \text{ and } j = 1, 2, \ldots n$$

Subject to the restrictions,

$$x_{ij} = \begin{cases} 1, \text{ if the } i\text{th person is assigned } j\text{th job} \\ 0, \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the } i\text{th person)}$$

and $$\sum_{j=1}^{n} x_{ij} = 1 \text{ (only one person should be assigned the } j\text{th job)}$$

where, x_{ij} denotes that the jth job is to be assigned to the ith person.

9.3.3 Difference between Transportation and Assignment Problems

	Transportation Problem	*Assignment Problem*
1.	Number of sources and destinations need not be equal. Hence, the cost matrix is not necessarily a square matrix.	Since assignment is done on a one-to-one basis, the number of sources and destinations are equal. Hence, the cost matrix must be a square matrix.
2.	x_{ij}, the quantity to be transported from ith origin to jth destination can take any possible positive value, and it satisfies the rim requirements.	x_{ij}, the jth job is to be assigned to the ith person and can take either the value 1 or zero.
3.	The capacity and the requirement value is equal to a_i and b_j for the ith source and jth destination (i = 1, 2, ...m; j = 1, 2, ... n).	The capacity and the requirement value is exactly one, i.e., for each source of each destination, the capacity and the requirement value is exactly one.
4.	The problem is unbalanced if the total supply and total demand are not equal.	The problem is unbalanced if the cost matrix is not a square matrix.

9.4 HUNGARIAN METHOD PROCEDURE

Solution of an assignment problem can be arrived at, by using the **Hungarian method**. The steps involved in this method are as follows.

Step 1 Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (or column) with zero cost element.

Step 2 Subtract the minimum element in each row from all the elements of the respective rows.

Step 3 Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

Step 4 Then, draw the minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be N. Now there are two possible cases.

Case I If $N = n$, where n is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.

Case II If $N < n$, then proceed to step 5.

Step 5 Determine the smallest uncovered element in the matrix (element not covered by N lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step 6 Repeat steps 3 and 4 until we get the case (*i*) of Step 4.

Step 7 (To make zero assignment) Examine the rows successively until a row-wise exactly single zero is found. Circle (O) this zero to make the assignment. Then mark a cross (×) over all zeros if lying in the column of the circled zero, showing that they cannot be considered for future assignment. Continue in this manner until all the zeros have been examined. Repeat the same procedure for columns also.

Step 8 Repeat step 6 successively until one of the following situation arises—

(*i*) If no unmarked zero is left, then the process ends or

(*ii*) If there lie more than one unmarked zero in any column or row, circle one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row or column. Repeat the process until no unmarked zero is left in the matrix.

Step 9 Thus, exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeros will give the optimal assignment.

Example 9.2 Using the following cost matrix, determine (*a*) optimal job assignment (*b*) the cost of assignments.

$$\textbf{Mechanic}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \textbf{Job} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 10 & 3 & 3 & 2 & 8 \\ 9 & 7 & 8 & 2 & 7 \\ 7 & 5 & 6 & 2 & 4 \\ 3 & 5 & 8 & 2 & 4 \\ 9 & 10 & 9 & 6 & 10 \end{bmatrix} \end{array}$$

Solution Select the smallest element in each row and subtract this smallest element from all the elements in its row.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 8 & 1 & 1 & 0 & 6 \\ B & 7 & 5 & 6 & 0 & 5 \\ C & 5 & 3 & 4 & 0 & 2 \\ D & 1 & 3 & 6 & 0 & 2 \\ E & 3 & 4 & 3 & 0 & 4 \end{array}$$

Select the minimum element from each column and subtract from all other elements in its column. With this we get the first modified matrix.

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix} \end{array}$$

In this modified matrix we draw the minimum number of lines to cover all zeros (horizontal or vertical).

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 7 & 0 & 0 & 0 & 4 \\ 6 & 4 & 5 & 0 & 3 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 2 & 3 & 2 & 0 & 2 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros is $4 = N$.

The order of matrix is $n = 5$

Hence, $N < n$.

Now we get the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding it to the element at the point of intersection of lines.

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & 0 & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & 0 & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & 0 \\ 0 & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & 0 & \cancel{0} & 2 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros $= N = 5$

The order of matrix is $n = 5$.

Hence $N = n$. Now we determine the optimum assignment.

Assignment

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 9 & \boxed{0} & \cancel{0} & 2 & 6 \\ 6 & 2 & 3 & \boxed{0} & 3 \\ 4 & \cancel{0} & 1 & \cancel{0} & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \cancel{0} & \cancel{0} \\ 2 & 1 & \boxed{0} & \cancel{0} & 2 \end{bmatrix} \end{array}$$

First row contains more than one zero. So proceed to the 2nd row. It has exactly one zero. The corresponding cell is $(B, 4)$. Circle this zero thus, making an assignment. Mark $(\times)$ for all other zeros in its column. Showing that they cannot be used for making other assignments. Now row 5 has a single zero in the cell $(E, 3)$. Make an assignment in this cell and cross the 2nd zero in the 3rd column.

Now row 1 has a single zero in the column 2, i.e., in the cell $(A, 2)$. Make an assignment in this cell and cross the other zeros in the 2nd column. This leads to a single zero in column 1 of the cell $(D, 1)$, make an assignment in this cell and cross the other zeros in the 4th row. Finally, we have a single zero left in the 3rd row, making an assignment in the cell (C, 5). Thus, we have the following assignment.

Optimal assignment and optimum cost of assignment.

Job	***Mechanic***	***Cost***
1	D	3
2	A	3
3	E	9
4	B	2
5	C	4
		₹ 21

Therefore, $1 \to D$, $2 \to A$, $3 \to E$, $4 \to B$, $5 \to C$, with minimum cost equal to ₹ 21.

Example 9.3 A company has 5 jobs to be done on five machines. Any job can be done on any machine. The cost of doing the jobs on different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

	Machines				
Jobs	***A***	***B***	***C***	***D***	***E***
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

Solution We form the first modified matrix by subtracting the minimum element from all the elements in the respective row, and the same with respective columns.

Step 1

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Since each column has the minimum element 0, we have the first modified matrix. Now we draw the minimum number of lines to cover all zeros.

Step 2

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Number of lines drawn to cover zero is $N = 4 <$ the order of matrix $n = 5$.

We find the second modified matrix by subtracting the smallest uncovered element (3) from all the uncovered elements and adding to the element that is the point of intersection of lines.

Step 3

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ 0 & 6 & 12 & 0 & 0 \\ 11 & 8 & 0 & 3 & 0 \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros = 5,
which is the order of matrix. Hence, we can form an assignment.

Assignment

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 5 & \boxed{0} & 5 & 10 & 8 \\ \cancel{0} & 6 & 12 & \cancel{0} & \boxed{0} \\ 11 & 8 & \boxed{0} & 3 & \cancel{0} \\ \boxed{0} & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & \boxed{0} & 5 \end{bmatrix} \end{array}$$

All the five jobs have been assigned to 5 different machines.

Here the optimal assignment is,

Job	*Machine*
1	B
2	E
3	C
4	A
5	D

Minimum (Total cost) = 8 + 12 + 4 + 6 + 12 = ₹ 42.

Example 9.4 Four different jobs can be done on four different machines and the take-down time costs are prohibitively high for change overs. The matrix below gives the cost in rupees for producing job *i* on the machine *j*.

	Machines			
Jobs	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized?

Solution We form a first modified matrix by subtracting the least element in the respective rows and respective columns.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix} \end{array}$$

Since the third column has no zero element, we subtract the smallest element 4 from all the elements.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now we draw minimum number of lines to cover all zeros.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Number of lines drawn to cover all zeros = 3, which is less than the order of matrix = 4.

Hence, we form the 2nd modified matrix, by subtracting the smallest uncovered element from all the uncovered elements and adding to the element that is at the point of intersection of lines.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 7 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$N = 3 < n = 4$ $\qquad$ $N = 4 = n = 4$

Hence, we can make an assignment.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} \cancel{0} & \cancel{0} & \cancel{0} & 0 \\ 5 & 0 & \cancel{0} & 2 \\ 0 & 1 & \cancel{0} & 2 \\ 8 & \cancel{0} & 0 & \cancel{0} \end{bmatrix} \end{array}$$

Since no rows and no columns have single zero, we have a different assignment (Multiple solution).

Optimal assignment

Job	***Machine***
J_1	M_4
J_2	M_2
J_3	M_1
J_4	M_3

Minimum (Total cost)

$$6 + 5 + 4 + 8 = ₹\,23.$$

Alternate Solution

$$J_1 \rightarrow M_1; \quad J_2 \rightarrow M_2; \quad J_3 \rightarrow M_3; \quad J_4 \rightarrow M_4.$$

Minimum (Total cost)

$$5 + 5 + 10 + 3 = ₹\,23.$$

Example 9.5 Solve the following assignment problem in order to minimize the total cost. The cost matrix given below gives the assignment cost when different operators are assigned to various machines.

Operators

$$\textbf{Machines}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 30 & 25 & 33 & 35 & 36 \\ 23 & 29 & 38 & 23 & 26 \\ 30 & 27 & 22 & 22 & 22 \\ 25 & 31 & 29 & 27 & 32 \\ 27 & 29 & 30 & 24 & 32 \end{bmatrix} \end{array}$$

Solution We form the first modified matrix by subtracting the least element from all the elements in the respective rows and then in the respective columns.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{bmatrix} \end{array}$$

Since each column has the minimum element 0, the first modified matrix is obtained. We draw the minimum number of lines to cover all zeros.

The number of lines drawn to cover all zeros = 4 < the order of matrix = 5. Hence, we form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element that is at the point of intersection of lines.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 8 & 0 & 8 & 13 & 11 \\ 0 & 3 & 12 & 0 & 0 \\ 11 & 5 & 0 & 3 & 0 \\ 0 & 3 & 1 & 2 & 4 \\ 3 & 2 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

N = 5, i.e., the number of lines drawn to cover all zeros = order of matrix. Hence, we can make an assignment.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \\ \cancel{0} & 6 & 12 & \cancel{0} & 0 \\ 11 & 8 & 0 & 3 & \cancel{0} \\ 0 & 6 & 1 & 2 & 4 \\ 3 & 5 & 3 & 0 & 5 \end{bmatrix} \end{array}$$

The optimum assignment is

Operators	***Machines***
I	D
II	A
III	C
IV	E
V	B

The optimum cost is given by

$$25 + 25 + 22 + 24 + 26 = ₹\ 122.$$

9.5 UNBALANCED ASSIGNMENT PROBLEM

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and columns are not equal. To make it balanced, we add a dummy row or dummy column with all the entries as zero.

Example 9.6 There are four jobs to be assigned to five machines. Only one job can be assigned to one machine. The amount of time in hours required for the jobs per machine are given in the following matrix.

	Machines				
Jobs	***A***	***B***	***C***	***D***	***E***
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

Find an optimum assignment of jobs to the machines to minimize the total processing time and also find out for which machine no job is assigned. What is the total processing time to complete all the jobs?

Solution Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zero.

Modified Matrix

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 4 & 3 & 6 & 2 & 7 \\ 10 & 12 & 11 & 14 & 16 \\ 4 & 3 & 2 & 1 & 5 \\ 8 & 7 & 6 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We subtract the smallest element from all the elements in the respective rows.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeros.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The number of lines to cover all zeros = 4 < the order of matrix. We form the 2nd modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element at the point of intersection of lines.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 2 & \boxed{0} & 3 & \cancel{0} & 4 \\ \boxed{0} & 1 & \cancel{0} & 4 & 5 \\ 3 & 1 & \cancel{0} & \boxed{0} & 3 \\ 3 & 1 & \boxed{0} & 4 & \cancel{0} \\ 1 & \cancel{0} & \cancel{0} & 1 & \boxed{0} \end{bmatrix} \end{array}$$

Here the number of lines drawn to cover all zeros = 5 = Order of matrix. Therefore, we can make the assignment.

Optimum assignment

1	B	3
2	A	10
3	D	1
4	C	6
5	E	0

For machine *D*, no job is assigned.

Optimum (minimum) cost = 3 + 10 + 1 + 6 = ₹ 20.

Example 9.7 A company has 4 machines to do 3 jobs. Each job can be assigned to only one machine. The cost of each job on each machine is given below. Determine the job assignments that will minimize the total cost.

$$\begin{array}{cc} & \textbf{Machine} \\ & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \textbf{Job} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 18 \\ 10 & 15 & 19 & 22 \end{bmatrix} \end{array}$$

Solution Since the cost matrix is not a square matrix, we add a dummy row *D* with all the elements 0.

$$\begin{array}{cc} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 18 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Subtract the minimum element in each row from all the elements in its row.

$$\begin{array}{cc} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Since each column has a minimum element 0, we draw minimum number of lines to cover all zeros.

$$\begin{array}{cc} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

∴ The number of lines drawn to cover all zeros = 2 < the order of matrix, we form a second modified matrix. By subtracting the minimum uncovered value from all other uncovered values i.e. 5 and adding 5 to the element at the point of intersection of lines →

$$\begin{array}{cc} & \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, $N = 3 < n = 4$.

Again we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

Here, $N = 4 = n$. Hence, we make an assignment.

Assignment

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ \cancel{0} & 0 & \cancel{0} & 1 \\ \cancel{0} & \cancel{0} & 0 & 3 \\ 9 & 4 & \cancel{0} & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} A \to W \\ D \to Z \\ B \to X \\ C \to Y \end{array} \qquad \text{or} \qquad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ \cancel{0} & \cancel{0} & 0 & 1 \\ \cancel{0} & 0 & \cancel{0} & 3 \\ 9 & 4 & \cancel{0} & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} A \to W \\ D \to Z \\ B \to Y \\ C \to X \end{array}$$

Since D is a dummy job, machine Z is assigned no job.

Therefore, optimum cost = 18 + 13 + 19 = ₹ 50.

9.6 MAXIMIZATION IN ASSIGNMENT PROBLEM

In this, the objective is to maximize the profit. To solve this, we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest element. For this converted loss matrix we apply the steps in Hungarian method to get the optimum assignment.

Example 9.8 The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each jobs, which are as follows:

Job

$$\textbf{Mechanic}\ \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 62 & 78 & 50 & 111 & 82 \\ 71 & 84 & 61 & 73 & 59 \\ 87 & 92 & 111 & 71 & 81 \\ 48 & 64 & 87 & 77 & 80 \end{bmatrix} \end{array}$$

By using the assignment method, find the assignment of mechanics to the job that will result in maximum profit. Which job should be declined ?