Chapter

3 Graphical Method

Simple linear programming problems with two decision variables can be easily solved by graphical method.

3.1 PROCEDURE FOR SOLVING LPP BY GRAPHICAL METHOD

The steps involved in graphical method are as follows:

Step 1 Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph, as each equation will geometrically represent a straight line.

Step 3 Mark the region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the '*feasible region*'.

Step 4 Assign an arbitrary value, say zero, to the objective function.

Step 5 Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6 Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin, passing through at least one corner of the feasible region.

Step 7 Find the co-ordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution. That is, in the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optimal solution is the corner point which gives the minimum value for the objective function.

✓ ***Example 3.1*** Solve the following LPP by graphical method.

$$\text{Minimize } Z = 20x_1 + 10x_2$$

$$\text{Subject to, } x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30$$

$$4x_1 + 3x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

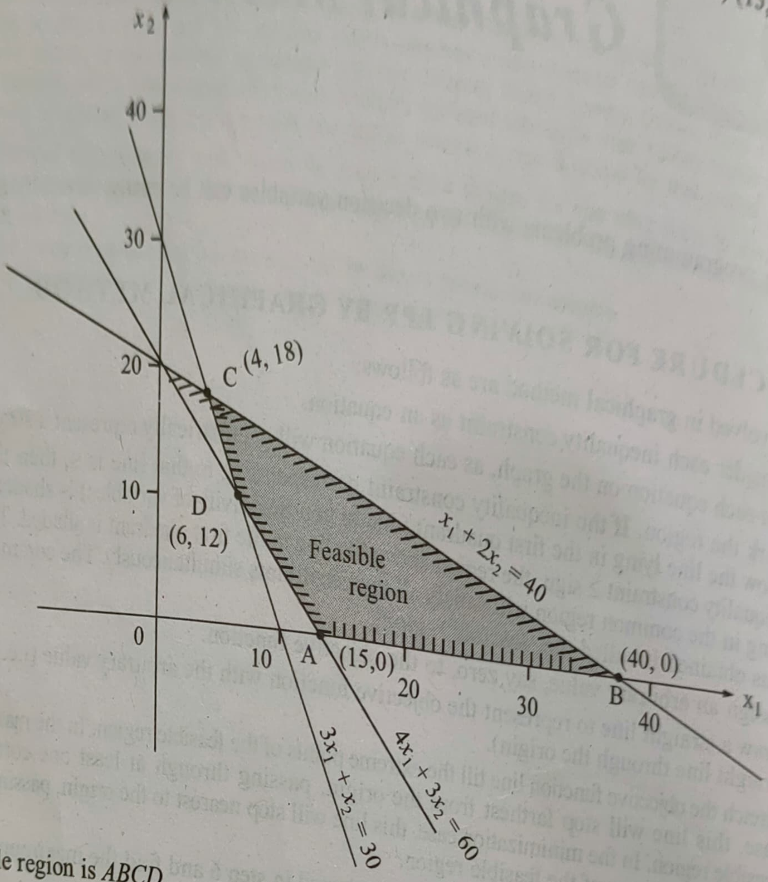
Solution Replace all the inequalities of the constraints by equation

$$x_1 + 2x_2 = 40 \text{ If } x_1 = 0 \Rightarrow x_2 = 20$$
$$\text{If } x_2 = 0 \Rightarrow x_1 = 40$$

$\therefore$

$$x_1 + 2x_2 = 40 \text{ passes through } (0, 20)\ (40, 0)$$
$$3x_1 + x_2 = 30 \text{ passes through } (0, 30)\ (10, 0)$$
$$4x_1 + 3x_2 = 60 \text{ passes through } (0, 20)\ (15, 0)$$

Plot each equation on the graph.



The feasible region is *ABCD*.

C and *D* are points of intersection of lines.

C intersect $x_1 + 2x_2 = 40,\ 3x_1 + x_2 = 30$

and, *D* intersect $4x_1 + 3x_2 = 60,\ 3x_1 + x_2 = 30$

$$C = (4, 18)$$
$$D = (6, 12)$$

Corner points	*Value of* $Z = 20x_1 + 10x_2$
$A(15, 0)$	300
$B(40, 0)$	800
$C(4, 18)$	260
$D(6, 12)$	240 (Minimum value)

$\therefore$ The minimum value of *Z* occurs at *D* (6, 12). Hence, the optimal solution is $x_1 = 6, x_2 = 12$.

Example 3.2 Find the maximum value of $Z = 5x_1 + 7x_2$

Subject to the constraints,

$$x_1 + x_2 \le 4$$
$$3x_1 + 8x_2 \le 24$$
$$10x_1 + 7x_2 \le 35$$
$$x_1, x_2 \ge 0$$

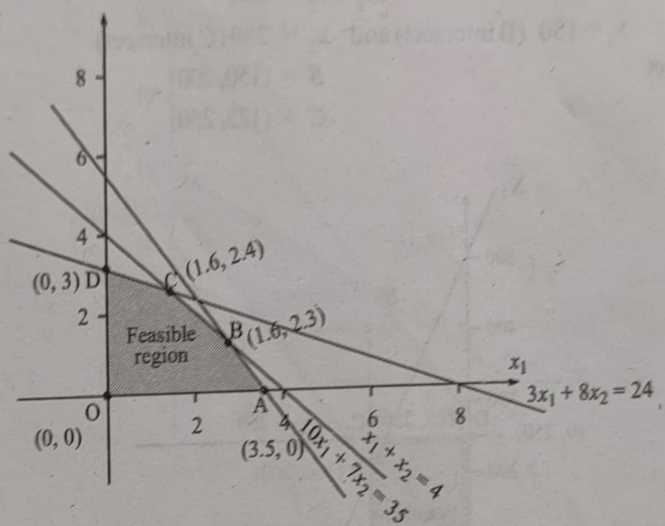
Solution Replace all the inequalities of the constraints by forming equations

$x_1 + x_2 = 4$ passes through (0, 4) (4, 0)

$3x_1 + 8x_2 = 24$ passes through (0, 3) (8, 0)

$10x_1 + 7x_2 = 35$ passes through (0, 5) (3.5, 0)

Plot these lines on the graph and mark the region below the line as the inequality of the constraint as $\le$ which is also lying in the first quadrant.



The feasible region is *OABCD*.

B and *C* are the points of intersection of lines

B intersect $x_1 + x_2 = 4, \quad 10x_1 + 7x_2 = 35$

and *C* intersect $3x_1 + 8x_2 = 24, \quad x_1 + x_2 = 4.$

On solving we get, $B = (1.6, 2.3)$

$C = (1.6, 2.4)$

Corner points	*Value of* $Z = 5x_1 + 7x_2$
$O(0,0)$	0
$A(3.5,0)$	17.5
$B(1.6,2.3)$	24.1
$C(1.6,2.4)$	24.8 (Maximum value)
$D(0,3)$	21

$\therefore$ The maximum value of *Z* occurs at *C* (1.6, 2.4) and the optimal solution is $x_1 = 1.6, x_2 = 2.4$.

Example 3.3 A company produces 2 types of hats *A* and *B*. Every hat *A* requires twice as much labour time as the second hat *B*. If the company produces only hat *B* then it can produce a total of 500 hats per day. The market limits daily sales of hat *A* and *B* to 150 and 250 respectively. The profits on hat *A* and *B* are ₹8 and ₹5 respectively. Solve graphically to get the optimal solution.

Solution Let x_1 and x_2 be the number of units of type A and type B hats respectively.

$$\text{Max } Z = 8x_1 + 5x_2$$

$$\text{Subject to, } 2x_1 + x_2 \le 500$$
$$x_1 \le 150$$
$$x_2 \le 250$$
$$x_1, x_2 \ge 0$$

First rewrite the inequality of the constraint into an equation and plot the lines on the graph.

$2x_1 + x_2 = 500$ passes through (0, 500) (250, 0)

$x_1 = 150$ passes through (150, 0)

$x_2 = 250$ passes through (0, 250)

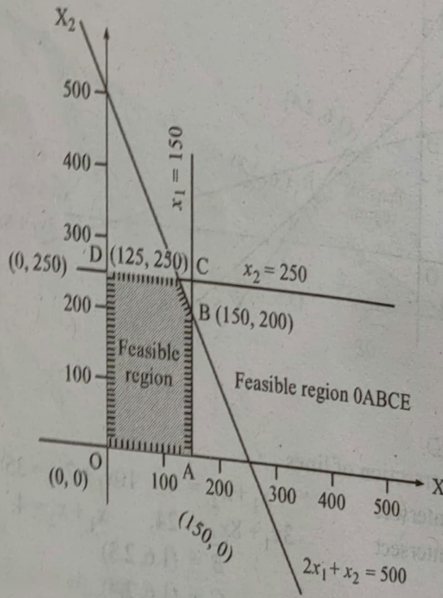
We mark the region below the lines lying in the first quadrant as the inequality of the constraints are $\le$. The feasible region is $OABCD$. B and C are the points of intersection of lines

$$2x_1 + x_2 = 500,$$

$x_1 = 150$ (B intersect) and $x_2 = 250$ (C intersect)

On solving, we get

$$B = (150, 200)$$
$$C = (125, 250)$$



Corner points	*Value of* $Z = 8x_1 + 5x_2$	
$O(0,0)$	0	
$A(150,0)$	1200	
$B(150,200)$	2200	
$C(125,250)$	2250	(Maximum Z=2250)
$D(0,250)$	1250	

The maximum value of Z is attained at $C(125, 250)$

$\therefore$ The optimal solution is $x_1 = 125, x_2 = 250$.

i.e., The company should produce 125 hats of type A and 250 hats of type B in order to get the maximum profit of ₹ 2250.

Example 3.4 By graphical method solve the following LPP.

$$\text{Max } Z = 3x_1 + 4x_2$$

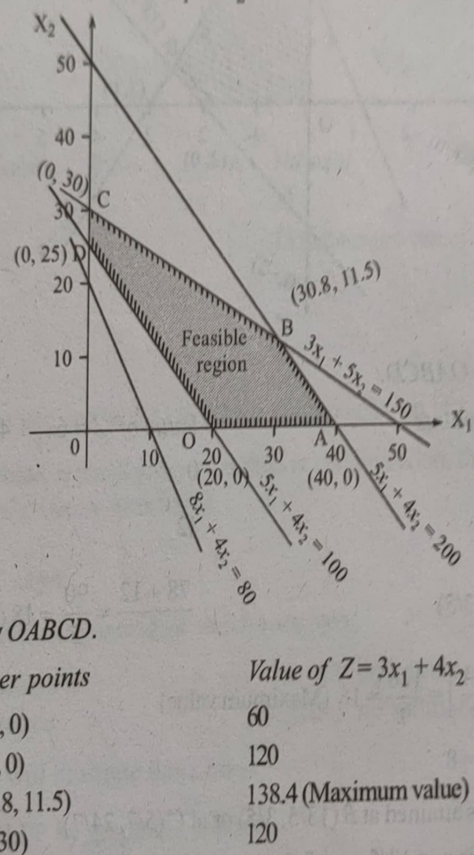
$$\text{Subject to, } 5x_1 + 4x_2 \le 200$$
$$3x_1 + 5x_2 \le 150$$
$$5x_1 + 4x_2 \ge 100$$
$$8x_1 + 4x_2 \ge 80$$

and
$$x_1, x_2 \ge 0$$

Solution

Replacing the inequality by equality

$5x_1 + 4x_2 = 200$ passes through $(0, 50), (40, 0)$
$3x_1 + 5x_2 = 150$ passes through $(0, 30), (50, 0)$
$8x_1 + 4x_2 = 80$ passes through $(0, 20), (10, 0)$
$5x_1 + 4x_2 = 100$ passes through $(0, 25), (20, 0)$



Feasible region is given by *OABCD*.

Corner points	*Value of* $Z = 3x_1 + 4x_2$
$O(20, 0)$	60
$A(40, 0)$	120
$B(30.8, 11.5)$	138.4 (Maximum value)
$C(0, 30)$	120
$D(0, 25)$	100

$\therefore$ The maximum value of Z is attained at B (30.8, 11.5)

$\therefore$ The optimal solution is $x_1 = 30.8, x_2 = 11.5$.

Example 3.5 Use graphical method to solve the LPP.

$$\text{Maximize } Z = 6x_1 + 4x_2$$

$$\text{Subject to, } -2x_1 + x_2 \le 2$$
$$x_1 - x_2 \le 2$$
$$3x_1 + 2x_2 \le 9$$
$$x_1, x_2 \ge 0$$

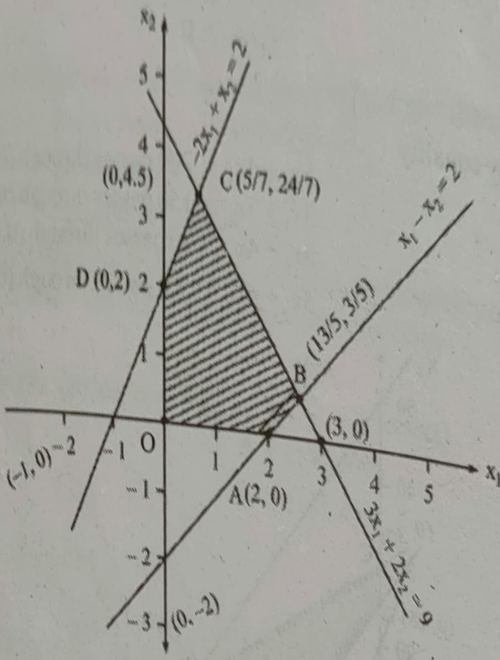
Solution

Replacing the inequality by equality

$-2x_1 + x_2 = 2$ passes through $(0, 2), (-1, 0)$

$x_1 - x_2 = 2$ passes through $(0, -2), (2, 0)$

$3x_1 + 2x_2 = 9$ passes through $(0, 4.5), (3, 0)$



Feasible region is given by *OABCD*.

Corner points	*Value of* $Z = 6x_1 + 4x_2$
$O(0,0)$	0
$A(2,0)$	12
$B(13/5, 3/5)$	$\frac{78+12}{5} = \frac{90}{5} = 18$ (Maximum value)
$C\left(\frac{5}{7}, \frac{24}{7}\right)$	$= \frac{126}{7} = 18$ (Maximum value)
$D(0,2) = 8$	

The maximum value of Z is attained at $B(13/5, 3/5)$ or at $C(5/7, 24/7)$.

$\therefore$ The optimal solution is $x_1 = 13/5, x_2 = 3/5$, or $x_1 = 5/7, x_2 = 24/7$.

Example 3.6 Use graphical method to solve the LPP.

$$\text{Maximize } Z = 3x_1 + 2x_2$$

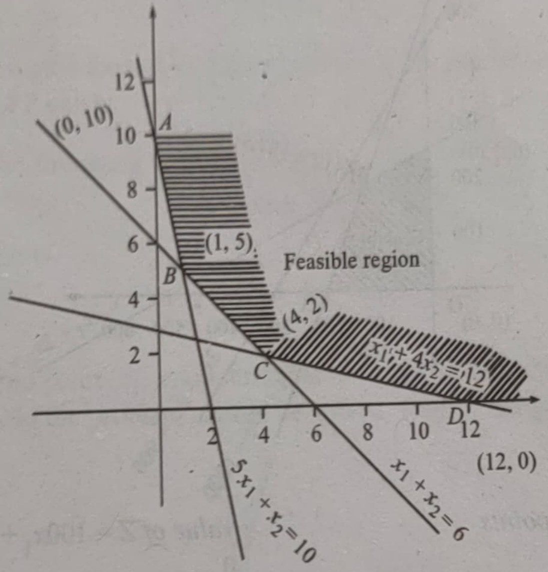
$$\text{Subject to, } 5x_1 + x_2 \geq 10$$

$$x_1 + x_2 \geq 6$$

$$x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Solution



Corner points	*Value of* $Z = 3x_1 + 2x_2$
$A(0, 10)$	20
$B(1, 5)$	13 (Minimum value)
$C(4, 2)$	16
$D(12, 0)$	36

Since the minimum value is attained at B (1, 5) the optimum solution is $x_1 = 1, x_2 = 5$.

Note: In the above problem if the objective function is maximization, then the solution is unbounded, as the maximum value of Z occurs at infinity.

3.1.1 Some More Cases

There are some linear programming problems which may have,

(*i*) a unique optimal solution
(*ii*) an infinite number of optimal solutions
(*iii*) an unbounded solution
(*iv*) no solution.

The following examples will illustrate these cases.

Example 3.7 Solve the LPP by graphical method.

$$\text{Maximize } Z = 100x_1 + 40x_2$$

$$\text{Subject to, } 5x_1 + 2x_2 \le 1{,}000$$

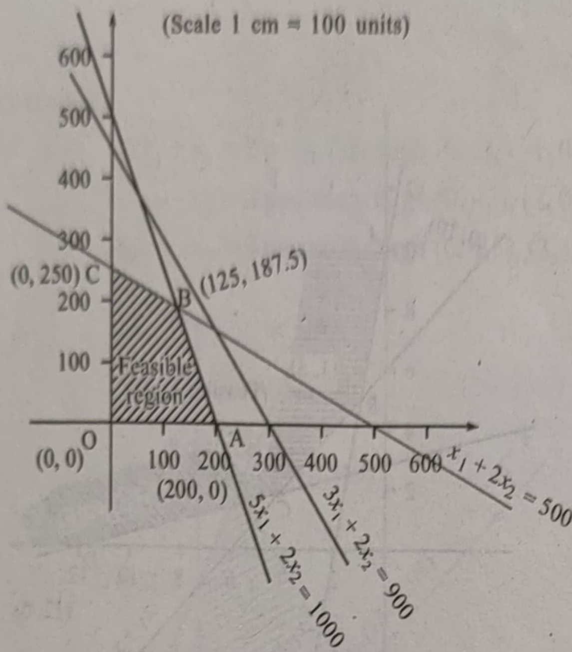
$$3x_1 + 2x_2 \le 900$$

$$x_1 + 2x_2 \le 500$$

and,
$$x_1, x_2 \ge 0$$

Solution

The solution space is given by the feasible region $OABC$.



Corner points	Value of $Z = 100x_1 + 40x_2$
$O(0, 0)$	0
$A(200, 0)$	20000 (Max value of Z)
$B(125, 187.5)$	20000 (Max value of Z)
$C(0, 250)$	10000

$\therefore$ The maximum value of Z occurs at two vertices A and B.

Since there are infinite number of points on the line joining A and B it gives the same maximum value of Z.

Thus, there are infinite number of optimal solutions for the LPP.

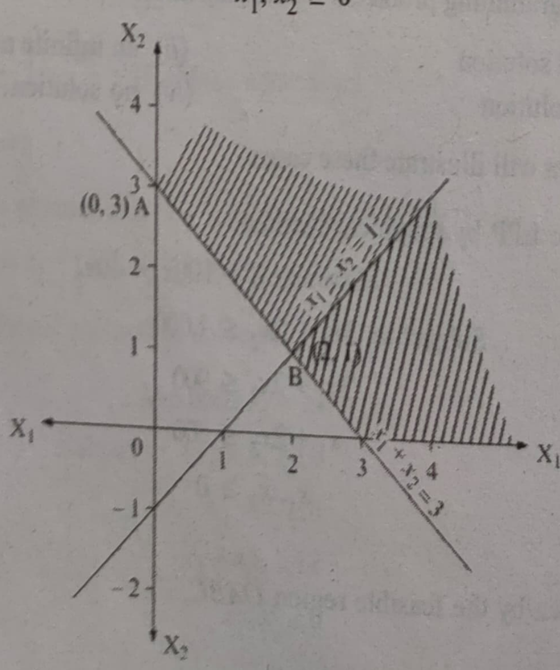
Unbounded Solution

Example 3.8 Solve the following LPP.

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to,

$$x_1 - x_2 \geq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$



Solution The solution space is unbounded. In fact, the maximum value of Z occurs at infinity. Hence, the problem has an *unbounded solution.*

No feasible solution

When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the LPP exists.

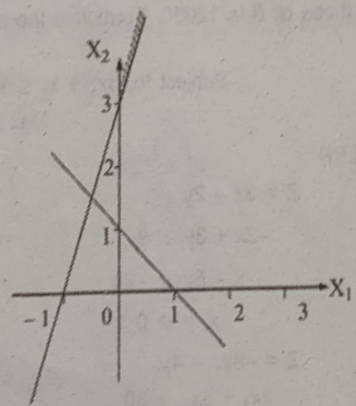
Example 3.9 Solve the following LPP.

$$\text{Max } Z = x_1 + x_2$$

Subject to the constraints,

$$x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

Solution There being no point (x_1, x_2) common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no solution.



For this problem, no feasible region in found, hence is given as infeasible solution.

EXERCISES

1. Solve the following by graphical method.

Max $$Z = x_1 - 3x_2$$
Subject to,
$$x_1 + x_2 \le 300$$
$$x_1 - 2x_2 \le 200$$
$$2x_1 + x_2 \le 100$$
$$x_2 \le 200$$
$$x_1, x_2 \ge 0$$

[**Ans.** Max $Z = 205$, $x_1 = 200$, $x_2 = 0$]

2. Max $$Z = 5x + 8y$$
Subject to,
$$3x + 2y \le 36$$
$$x + 2y \le 20$$
$$3x + 4y \le 42$$
$$x, y \ge 0$$

[**Ans.** Max $Z = 82$, $x = 2$, $y = 9$]

3. Max $$Z = x + 3y$$
Subject to,
$$x + y \le 300$$
$$x - 2y \le 200$$
$$x + y \le 100$$
$$y \ge 200$$
and, $$x, y \ge 0$$

[**Ans.** Max $Z = 700$, $x = 200$ $y = 100$]