Chapter

8

Transportation Problem

8.1 INTRODUCTION

The transportation problem is one of the subclasses of LPPs. Here the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. To achieve this, we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

8.2 MATHEMATICAL FORMULATION

Consider a transportation problem with m origins (rows) and n destinations (columns). Let C_{ij} be the cost of transporting one unit of the product from the ith origin to jth destination. a_i be the quantity of commodity available at origin i, b_j be the quantity of commodity needed at destination j. x_{ij} is the quantity transported from ith origin to jth destination. The above transportation problem can be stated in the following tabular form.

Destinations

Origin

	1	***2***	***3***	***...*** ***n***	***Capacity***
1	C_{11} x_{11}	C_{12} x_{12}	C_{13} x_{13}	... C_{1n} ... x_{1n}	a_1
2	C_{21} x_{21}	C_{22} x_{22}	C_{23} x_{23}	... C_{2n} ... x_{2n}	a_2
3	C_{31} x_{31}	C_{32} x_{32}	C_{33} x_{33}	... C_{3n} ... x_{3n}	a_3
m	C_{m1} x_{n1}	C_{m2} x_{m2}	C_{m3} x_{m3}	... C_{mn} ... x_{mn}	a_m
Demand	b_1	b_2	b_3	... b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The Linear programming model representing the transportation problem is given by,

$$\text{Minimize Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, 2, ...n$$

(Row Sum)

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, 2, ...\ n$$

(Column Sum)

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

The given transportation problem is said to be balanced if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

i.e. if the total supply is equal to the total demand.

8.3 DEFINITIONS

Feasible Solution: Any set of non-negative allocations ($x_{ij} > 0$) which satisfies the row and column sum (rim requirement) is called a 'feasible solution'.

Basic Feasible Solution: A feasible solution is called a 'basic feasible solution' if the number of non-negative allocations is equal to $m + n - 1$, where m is the number of rows and n the number of columns in a transportation table.

Non-degenerate Basic Feasible Solution: Any feasible solution to a transportation problem containing m origins and n destinations is said to be 'non-degenerate' if it contains $m + n - 1$ occupied cells and each allocation is in an independent position.

The allocations are said to be in independent positions, if it is impossible to form a closed path.

A path which is formed by allowing horizontal and vertical lines and all the corner cells of which are occupied is called a 'closed path'.

The allocations in the following tables are not in independent positions.

	*	*
	*	*

*		*
*		*

	*	*	
	*		
	*	*	

The allocations in the following tables are in independent positions.

	*	
*	*	*
*		

*	*		
	*		*
		*	*

Degenerate Basic Feasible Solution: If a basic feasible solution contains less than $m + n - 1$ non-negative allocations, it is said to be 'degenerate'.

8.4 OPTIMAL SOLUTION

Optimal solution is a feasible solution (not necessarily basic), which minimizes the total cost.

The solution of a transportation problem can be obtained in two stages, namely initial and optimum solution.

Initial solution can be obtained by using any one of the three methods, viz.,

(*i*) North-West Corner Rule (NWCR)
(*ii*) Least Cost Method or Matrix Minima Method
(*iii*) Vogel's Approximation Method (VAM)

VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

The cells in the transportation table can be classified as occupied and unoccupied cells. The allocated cells in the transportation table are called *occupied cells* and the empty ones are called *unoccupied cells*.

The improved solution of the initial basic feasible solution is called 'optimal solution', which is the second stage of solution and can be obtained by MODI (modified distribution method).

8.4.1 North-West Corner Rule

Step 1 Starting with the cell at the upper left corner (north-west) of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, i.e., $x_{11} = \min (a_1, b_1)$.

Step 2 If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude $x_{22} = \min (a_2, b_1 - x_{11})$ in the cell (2, 1).

If $b_1 < a_1$, move right horizontally to the second column and make the second allocation of magnitude $x_{12} = \min (a_1, x_{11} - b_1)$ in the cell (1, 2).

If $b_1 = a_1$, there is a tie for the second allocation. We make the second allocations of magnitude

$x_{12} = \min (a_1 - a_1, b_1) = 0$ in cell (1, 2)

or $x_{21} = \min (a_2, b_1 - b_1) = 0$ in the cell (2, 1)

Step 3 Repeat steps 1 and 2, moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Example 8.1 Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is given below.

Origin/Destination	D_1	D_2	D_3	***Supply***
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

Solution Since $\Sigma ai = 34 = \Sigma bj$, there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The first allocation is made in the cell (1, 1), the magnitude being $x_{11} = \min (5, 7) = 5$. The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by $x_{21} = \min (8, 7 - 5) = 2$.

	D_1	D_2	D_3	***Supply***
O_1	2 ⑤	7	4	~~5~~ 0
O_2	3 ②	3 ⑥	1	~~8~~ ~~6~~ 0
O_3	5	4 ③	7 ④	~~7~~ ~~4~~ 0
O_4	1	6	2 ⑭	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	34

The third allocation is made in the cell (2, 2), the magnitude being $x_{22} = \min (8 - 2, 9) = 6$.

The magnitude of fourth allocation is made in the cell (3, 2) given by min (7, 9 – 6) = 3.

The fifth allocation is made in the cell (3, 3) with magnitude $x_{33} = \min (7 - 3, 14) = 4$.

The final allocation is made in the cell (4, 3) with magnitude $x_{43} = \min (14, 18 - 4) = 14$.

Hence we get the initial basic feasible solution to the given T.P. which is given by,

$$x_{11} = 5;\ x_{21} = 2;\ x_{22} = 6;\ x_{32} = 3;\ x_{33} = 4;\ x_{43} = 14$$

$$\text{Total cost} = (5 \times 2) + (2 \times 3) + (6 \times 3) + (3 \times 4) + (4 \times 7) + (14 \times 2)$$

$$= 10 + 6 + 18 + 12 + 28 + 28 = ₹\ 102.$$

Example 8.2 Determine an initial basic feasible solution to the following transportation problem using NWCR.

	D_1	D_2	D_3	D_4	***Supply***
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Required	6	10	15	4	35

Solution The problem is a balanced TP as the total supply is equal to the total demand. Using the steps involved in the north-west corner rule, we find the initial basic feasible solution as given in the following table.

	D_1	D_2	D_3	D_4	***Supply***
O_1	6 ⑥	4 ⑧	1	5	~~14~~ ~~8~~ 0
O_2	8	9 ②	2 ⑭	7	~~16~~ ~~14~~ 0
O_3	4	3	6 ①	2 ④	~~5~~ ~~4~~
Demand	~~6~~	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~	35

Solution is given by,

$$x_{11} = 6;\ x_{12} = 8;\ x_{22} = 2;\ x_{23} = 14;\ x_{33} = 1;\ x_{34} = 4$$

$$\text{Total cost} = (6 \times 6\ ) + (8 \times 4) + (2 \times 9) + (14 \times 2) + (1 \times 6) + (4 \times 2)$$

$$= ₹\,128.$$

8.4.2 Least Cost or Matrix Minima Method

Step 1 Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate $x_{ij} = \min\,(a_i, b_j)$ in the cell (i, j)

Step 2 If $x_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_j by a_i. Then go to step 3.

If $x_{ij} = b_j$, cross off the j^{th} column of the transportation table and decrease a_i by b_j. Go to step 3.

If $x_{ij} = a_i = b_j$, cross off either the i^{th} row or the j^{th} column but not both.

Step 3 Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Example 8.3 Obtain an initial feasible solution to the following TP using the matrix minima method.

	D_1	D_2	D_3	D_4	***Supply***
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution Since $\Sigma a_i = \Sigma b_j = 24$, there exists a feasible solution to the TP. Using the steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being $x_{31} = 4$. It satisfies the demand at the destination D_1 and we delete this column from the table as it is exhausted.

	D_1	D_2	D_3	D_4	***Supply***
O_1	1	2 ⑥	3	4	~~6~~ 0
O_2	4	3	2 ②	0 ⑥	~~8~~ ~~2~~
O_3	0 ④	2	2 ⑥	1	~~10~~ ~~6~~
Demand	~~4~~ 0	~~6~~ 0	~~8~~ ~~2~~ 0	~~6~~ 0	24

The second allocation is made in the cell (2, 4) with magnitude $x_{24} = \min(6, 8) = 6$. Since it satisfies the demand at the destination D_4, it is deleted from the table. From the reduced table the third allocation is made in the cell (3, 3) with magnitude $x_{33} = \min(8, 6) = 6$. The next allocation is made in the cell (2, 3) with magnitude x_{23} of min (2, 2) = 2. Finally the allocation is made in the cell (1, 2) with magnitude $x_{12} = \min(6, 6) = 6$. Now all the rim requirements have been satisfied and hence, initial feasible solution is obtained.

The solution is given by,

$$x_{12} = 6,\ x_{23} = 2,\ x_{24} = 6,\ x_{31} = 4,\ x_{33} = 6$$

Since the total number of occupied cells = $5 < m + n + 1$.

We get a degenerate solution.

$$\text{Total cost} = (6 \times 2) + (2 \times 2) + (6 \times 0) + (4 \times 0) + (6 \times 2)$$

$$= 12 + 4 + 12 = ₹\,28.$$

Example 8.4 Determine an initial basic feasible solution for the following TP, using least cost method.

	D_1	D_2	D_3	D_4	*Supply*
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

Solution Since $\Sigma a_i = \Sigma b_j$, there exists a basic feasible solution. Using the steps in least cost method, we make the first allocation to the cell (1, 3) with magnitude x_{13} = min (14, 15) = 14 (as it is the cell having the least cost).

This allocation exhausts the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3) which is chosen arbitrarily with magnitude x_{23} = min (1, 16) = 1, which exhausts the 3rd column destination.

From the reduced table, the next least cost cell is (3, 4) to which allocation is made with magnitude min (4, 5) = 4. This exhausts the destination D_4 requirement, deleting the fourth column from the table. The next allocation is made in the cell (3, 2) with magnitude x_{32} = Min (1, 10) = 1, which exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell (2,1) with magnitude x_{21} = min (6, 15) = 6. This exhausts the first column requirement. Hence, it is deleted from the table.

Finally the allocation is made to the cell (2, 2) with magnitude x_{22} = min (9, 9) = 9, which satisfies the rim requirement. These allocations are shown in the transportation table as follows:

(I allocation)

	D_1	D_2	D_3	D_4	*Supply*
O_1	6	4	1 ⑭	5	~~14~~
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	~~15~~ 1	4	

(II allocation)

	D_1	D_2	D_3	D_4	*Supply*
O_2	8	9	2 ①	7	~~16~~ 15
O_3	4	3	6	2	5
Demand	6	10	~~1~~ 0	4	

(III allocation)

	D_1	D_2	D_4	*Supply*
O_2	8	9	7	15
O_3	4	3	2 ④	~~5~~ 1
Demand	6	10	~~4~~ 0	

(IV allocation)

	D_1	D_2	*Supply*
O_2	8	9	15
O_3	4	3 ①	~~1~~ 0
Demand	6	~~10~~ 9	

(V, VI allocation)

	D_1	D_2	*Supply*
O_2	8 ⑥	9 ⑨	15
Demand	6	9	

The following table gives the initial basic feasible solution.

	D_1	D_2	D_3	D_4	*Supply*
O_1	6	4	1 ⑭	5	14
O_2	8 ⑥	9 ⑨	2 ①	7	16
O_3	4	3 ①	6	2 ④	5
Demand	6	10	15	4	

Solution is given by,

$$x_{13} = 14; \quad x_{21} = 6; \quad x_{22} = 9; \quad x_{23} = 1; \quad x_{32} = 1; \quad x_{34} = 4$$

Transportation cost

$$= (14 \times 1) + (6 \times 8) + (9 \times 9) + (1 \times 2) + (1 \times 3) + (4 \times 2)$$

= ₹ 156.

8.4.3 Vogel's Approximation Method (VAM)

The steps involved in this method for finding the initial solution are as follows.

Step 1 Find the penalty cost, namely the difference between the smallest and next to smallest costs in each row and column.

Step 2 Among the penalties as found in step (1), choose the maximum penalty. If this maximum penalty is more than one (i.e., if there is a tie), choose any one arbitrarily.

Step 3 In the selected row or column as by step (2), find out the cell having the least cost. Allocate to this cell as much as possible, depending on the capacity and requirements.

Step 4 Delete the row or column that is fully exhausted. Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Note: If the column is exhausted, then there is a change in row penalty, and vice versa.

Example 8.5 Find the initial basic feasible solution for the following transportation problem by VAM.

		Destination				
		D_1	***D_2***	***D_3***	***D_4***	***Supply***
Origin	O_1	11	13	17	14	250
	O_2	16	18	14	10	300
	O_3	21	24	13	10	400
	Demand	200	225	275	250	950

Solution Since $\Sigma a_i = \Sigma b_j = 950$, the problem is balanced and there exists a feasible solution to the problem.

First we find the row and column penalty P_{I} as the difference between the least and next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column choose the cell having the least cost (1, 1). Allocate to this cell with minimum magnitude (i.e., (250, 200) = 200). This exhausts the first column. Delete this column. Since the column is deleted, there is a change in row penalty P_{II} and column penalty P_{II} remains the same. Continuing in this manner we get the remaining allocations as given in the table below.

I allocation

	D_1	***D_2***	***D_3***	***D_4***	***Supply***	***P_I***
O_1	11 200	13	17	14	50 ~~250~~	2
O_2	16	18	14	10	300	4
O_3	21	24	13	10	400	3
Demand	~~200~~ 0	225	275	250		
P_{I}	5↑	5	3	0		

II allocation

	D_2	***D_3***	***D_4***	***Supply***	***P_{II}***
O_1	13 50	17	14	~~50~~	1
O_2	18	14	10	300	4
O_3	24	13	10	400	3
Demand	~~225~~ 175	275	250		
P_{II}	5↑	1	0		

III allocation

	D_2	***D_3***	***D_4***	***Supply***	***P_{III}***
O_2	18 175	14	10	~~300~~ 125	4
O_3	24	13	10	400	3
Demand	~~175~~	275	250		
P_{III}	0 6↑	1	0		

IV allocation

	D_3	***D_4***	***Supply***	***P_{IV}***
O_2	14	10 125	~~125~~ 0	4 ←
O_3	13	10	400	3
Demand	275	~~250~~ 125		
P_{IV}	1	0		

V allocation

	D_3	D_4	*Supply*	P_V
O_3	13 275	10	~~400~~ 125	3
Demand	~~275~~ 0	125		
P_V	13↑	10		

VI allocation

	D_4	*Supply*	P_{VI}
O_3	10 125	~~125~~ 0	10 ←
Demand	~~125~~ 0		
P_{VI}	10		

Finally, we arrive at the initial basic feasible solution, which is shown in the following table.

	D_1	D_2	D_3	D_4	*Supply*
O_1	11 200	13 50	17	14	250
O_2	16	18 175	14	10 125	300
O_3	21	24	13 275	10 125	400
Demand	200	225	275	250	

There are 6 positive independent allocations which are equal to $m + n - 1 = 3 + 4 - 1$. This ensures that the solution is a non-degenerate basic feasible solution.

∴ The transportation cost

$$= 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 10 + 275 \times 13 + 125 \times 10$$
$$= ₹\ 12{,}075.$$

Example 8.6 Find the initial solution to the following TP using VAM.

		Destination				
		D_1	D_2	D_3	D_4	*Supply*
Factory	F_1	3	3	4	1	100
	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
	Demand	120	80	75	25	300

Solution Since $\Sigma a_i = \Sigma b_j$, the problem is a balanced TP. So there exists a feasible solution.

	D_1	D_2	D_3	D_4	*Supply*	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
F_1	3 ㊺	3	4 ㉚	1 ㉕	100	2	2 ←	0	1 ←	4	4
F_2	4	2 (80)	4 ㊺	2	125	2	2	2 ←	0	4 ←	—
F_3	1 (75)	5	3	2	75	1	—	—	—	—	—
Demand	120	80	75	25							
P_{I}	2↑	1	1	1							
P_{II}	1	1	0	1							
P_{III}	1	1	0	—							
P_{IV}	1	—	0	—							
P_{V}	—	—	0	—							
P_{VI}	—	—	4↑	—							

Finally, we have the initial basic feasible solution as given in the following table.

	D_1	D_2	D_3	D_4	*Supply*
F_1	3 ㊺	3	4 ㉚	1 ㉕	100
F_2	4	2 (80)	4 ㊺	2	125
F_3	1 (75)	5	3	2	75
Demand	120	80	75	25	

There are 6 independent non-negative allocations equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non-degenerate basic feasible.

∴ The transportation cost

$$= 45 \times 3 + 30 \times 4 + 25 \times 1 + 80 \times 2 + 45 \times 4 + 75 \times 1$$

$$= 135 + 120 + 25 + 160 + 180 + 75 = ₹\ 695.$$

8.5 OPTIMALITY TEST

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted on any initial basic feasible solution of a TP provided such an allocation has exactly $m + n - 1$, non-negative allocations where m is the number of origins and n is the number of destinations. Also these allocations must be in independent positions.

To perform this optimality test, we shall discuss the modified distribution method (MODI). The various steps involved in MODI method for performing the optimality test are given below.

8.5.1 MODI Method

Step 1 Find the initial basic feasible solution of a TP by using any one of the three methods.

Step 2 Find out a set of numbers u_i and v_j for each row and column satisfying $u_i + v_j = c_{ij}$ for each occupied cell. To start with, we assign a number '0' to any row or column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

Step 3 For each empty (unoccupied) cell, we find the sum u_i and v_j written in the bottom left corner of that cell.

Step 4 Find out the net evaluation value $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each *empty cell*, which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

(*i*) If all $\Delta_{ij} > 0$ (i.e., all the net evaluation value), the solution is optimum and a *unique solution* exists.

(*ii*) If $\Delta_{ij} \geq 0$, then the solution is optimum, but an alternate solution exists.

(*iii*) If at least one $\Delta_{ij} < 0$, the solution is not optimum. In this case we go to the next step, to improve the total transportation cost.

Step 5 Select the empty cell having the most negative value of Δ_{ij}. From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign + and – alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

Step 6 The above step yields a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations repeat from step (2) onwards, till an optimum basic feasible solution is obtained.

Example 8.7 Solve the following transportation problem.

		Destination				
		P	***Q***	***R***	***S***	***Supply***
Source	***A***	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
	Demand	6	10	12	15	43

Origin/Dest.	***P***	***Q***	***R***	***S***	***Supply***	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
A	21	16	25	13 ⑪	11	3	—	—	—	—	—
B	17 ⑥	18	14 ③	23 ④	13	3	3	3	3 ←	—	—
C	32	17 ⑩	18 ⑨	41	19	1	1	1	1	1	17
Demand	6	10	12	15	43						
P_{I}	4	1	4	10↑							
P_{II}	15	1	4	18↑							
P_{III}	15↑	1	↑4	—							
P_{IV}	—	1	↑4	—							
P_{V}	—	17	18↑	—							
P_{VI}	—	17↑	—	—							

Solution We first find the initial basic feasible solution by using VAM. Since $\Sigma a_i = \Sigma b_j$, the given TP is a balanced one. Therefore, there exists a feasible solution.

Finally, we have the initial basic feasible solution as given in the following table.

		Destination			
		P	***Q***	***R***	***S***
Source	***A***	21	16	25	13 ⑪
	B	17 ⑥	18	14 ③	23 ④
	C	32	17 ⑩	18 ⑨	41

From this table we see that the number of non-negative independent allocation is $6 = m + n - 1 = 3 + 4 - 1$.

Hence, the solution is non-degenerate basic feasible.

$\therefore$ The initial transportation cost

$$= (11 \times 13) + (3 \times 14) + (4 \times 23) + (6 \times 17) + (10 \times 17) + (9 \times 18) = ₹\,711.$$

To find the optimal solution We apply MODI method in order to determine the optimum solution. We determine a set of numbers u_i and v_j for each row and column, with $u_i + v_j = c_{ij}$ for each occupied cell. To start with, we give $u_2 = 0$ as the 2nd row has the maximum number of allocation.

$$\begin{aligned}
c_{21} &= u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17\\
c_{23} &= u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14\\
c_{24} &= u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23\\
c_{14} &= u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = -10\\
c_{33} &= u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4\\
c_{32} &= u_3 + v_2 = 17 = 4 + v_2 = 17 \Rightarrow v_2 = 13
\end{aligned}$$

Now we find the sum u_i and v_j for each empty cell and enter it at the bottom left corner of that cell.

Next we find the net evaluation $\Delta_{ji} = C_{ij} - (u_i + v_j)$ for each unoccupied cell and enter it at the bottom right corner of that cell.

Initial table

	P	*Q*	*R*	*S*	u_i
A	21 7 14	16 3 13	25 4 21	13 ⑪	$u_i = -10$
B	17 ⑥	18 13 5	14 ③	23 ④	$u_2 = 0$
C	32 21 9	17 ⑩	18 ⑨	41 25 16	$u_3 = 4$
v_j	$v_1 = 17$	$v_2 = 13$	$v_3 = 14$	$v_4 = 23$	

Since all $\Delta_{ij} > 0$, the solution is optimal and unique. The optimum solution is given by,

$$x_{14} = 11, x_{21} = 6, x_{23} = 3, x_{24} = 4, x_{32} = 10, x_{33} = 9$$

The min. transportation cost

$$= 11 \times 13 + 6 \times 17 + 3 \times 14 + 4 \times 23 + 10 \times 17 + 9 \times 18$$

= ₹ 711.

Example 8.8 Solve the following transportation problem starting with the initial solution obtained by VAM.

	D_1	D_2	D_3	D_4	*Supply*
O_1	2	2	2	1	3
O_2	10	8	5	4	7
O_3	7	6	6	8	5
Demand	4	3	4	4	15

Solution Since $\Sigma_{ai} = \Sigma_{bj}$, the problem is a balanced TP. Therefore, there exists a feasible solution.

	D_1	D_2	D_3	D_4	*Supply*	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
O_1	2 ③	2	2	1	3	1	—	—	—	—	—
O_2	10	8	5 ③	4 ④	7	1	1	3 ←	—	—	—
O_3	7 ①	6 ③	6 ①	8	5	0	0	0	0	0	6 ←
Demand	4	3	4	4	15						
P_{I}	5↑	4	4	3							
P_{II}	3	2	1	4↑							
P_{III}	3	2	1	—							
P_{IV}	7↑	6	6	—							
P_{V}	—	6↑	6	—							
P_{VI}	—		6	—							

Finally, the initial basic feasible solution is given as below:

	D_1	D_2	D_3	D_4	*Supply*
O_1	2 ③	2	2	1	3
O_2	10	8	5 ③	4 ④	7
O_3	7 ①	6 ③	6 ①	8	5
Demand	4	3	4	4	15

Since the number of occupied cells = 6 = $m + n - 1$ and are also independent, there exists a non-degenerate basic feasible solution.

The initial transportation cost

$$= (3 \times 2) + (3 \times 5) + (4 \times 4) + (1 \times 7) + (3 \times 6) + (1 \times 6) = ₹\ 68.$$

To find the optimal solution Applying the MODI method, we determine a set of numbers u_i and v_j for each row and column, such that $u_i + v_j = c_{ij}$ for each occupied cell. Since the 3rd row has maximum number of allocations, we give number $u_3 = 0$. The remaining numbers can be obtained as given here.

$$\begin{aligned}
c_{31} &= u_3 + v_1 = 7 = 0 + v_1 = 7 \Rightarrow v_1 = 7\\
c_{32} &= u_3 + v_2 = 6 = 0 + v_2 = 6 \Rightarrow v_2 = 6\\
c_{33} &= u_3 + v_3 = 6 = 0 + v_3 = 6 \Rightarrow v_3 = 6\\
c_{23} &= u_2 + v_3 = 5 = u_2 + 6 = 5 \Rightarrow u_2 = -1\\
c_{24} &= u_2 + v_4 = 4 = -1 + v_4 = 4 \Rightarrow v_4 = 5\\
c_{11} &= u_1 + v_1 = 2 = u_1 + 7 = 2 \Rightarrow u_1 = -5
\end{aligned}$$

We find the sum of u_i and v_j for each empty cell and enter it at the bottom left corner of the cell. Next we find the net evaluation Δ_{ij} given by,

Initial table

	D_1	D_2	D_3	D_4	u_4
O_1	2 ③	2 1 1	2 1 1	1 0 1	$u_1 = -5$
O_2	10 6 4	8 5 3	5 ③	4 ④	$u_2 = -1$
O_3	7 ①	6 ③	6 ①	8 5 3	$u_3 = 0$
	$v_1 = 7$	$v_2 = 6$	$v_3 = 6$	$v_4 = 5$	

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for each empty cell and enter it at the bottom right corner of the cell.

Since all $\Delta_{ij} > 0$, the solution is optimum and unique. The solution is given by,

$$x_{11} = 3; \quad x_{23} = 3; \quad x_{24} = 4$$

$$x_{31} = 1; \quad x_{32} = 3; \quad x_{33} = 1$$

The total transportation cost

$$= (3 \times 2) + (3 \times 5) + (4 \times 4) + (1 \times 7) + (3 \times 6) + (1 \times 6) = ₹\ 68$$

Degeneracy in transportation problem In a TP, if the number of non-negative independent allocations is less than $m + n - 1$, where m is the number of origins (rows) and n is the number of destinations (columns), there exists a degeneracy. This may occur either at the initial stage or at subsequent iteration.

To resolve this degeneracy, we adopt the following steps:

1. Among the empty cells, we choose an empty cell having the least cost, which is of an independent position. If such cells are more than one, choose any one arbitrarily.
2. To the cell as chosen in step (1), we allocate a small positive quantity $\varepsilon > 0$.

The cells containing ε are treated like other occupied cells and degeneracy is removed by adding one (more) accordingly. For this modified solution, we adopt the steps involved in MODI method till an optimum solution is obtained.

Example 8.9 Solve the transportation problem for minimization.

		Destinations			
		1	*2*	*3*	*Capacity*
Sources	**1**	2	2	3	10
	2	4	1	2	15
	3	1	3	1	40
	Demand	20	15	30	65

Solution Since $\Sigma a_i = \Sigma b_j$, the problem is a balanced TP. Hence, there exists a feasible solution. We find the initial solution by north-west corner rule as given below.

	1	*2*	*3*	*Capacity*
1	2 ⑩	2	3	10
2	4 ⑩	1 ⑮	2	15
3	1	3 ⑤	1 ㉚	40
Demand	20	15	30	

Since the number of occupied cells = 5 = $m + n - 1$ and all the allocations are independent, we get an initial basic feasible solution.

The initial transportation cost

$$= 10 \times 2 + 10 \times 4 + 5 \times 1 + 10 \times 3 + 30 \times 1 = ₹\ 125.$$

To find the optimal solution (***MODI method***) We use the above table to apply MODI method. We find out a set of numbers u_i and v_j for which $u_i + v_j = c_{ij}$, only for occupied cells. To start with, as the maximum number of allocations is 2 in more than one row and column, we choose arbitrarily column 1, and assign a number 0 to this column, i.e., $v_1 = 0$. The remaining numbers can be obtained as follows.

$$\begin{aligned}
c_{11} &= u_1 + v_1 = 2 = u_1 + 0 = 2 \Rightarrow u_1 = 2 \\
c_{21} &= u_2 + v_1 = 4 \Rightarrow u_2 = 4 - 0 = 4 \\
c_{22} &= u_2 + v_2 = 1 \Rightarrow v_2 = 1 - u_2 = 1 - 4 = -3 \\
c_{32} &= u_3 + v_2 = 3 = u_3 = 3 - v_2 = 3 - (-3) = 6 \\
c_{33} &= u_3 + v_3 = 1 = v_3 = 1 - u_3 = 1 - 6 = -5
\end{aligned}$$

Initial table

	1	***2***	***3***	u_i
1	2 ⑩	2 – 1 3	3 – 3 6	$u_1 = 2$
2	4 ⑩ –	1 ⑤ +	2 –1 3	$u_2 = 4$
3	$\boxed{1}$ 6 + – 5	3 ⑩ –	1 ㉚	$u_3 = 6$
v_j	$v_1 = 0$	$v_2 = -3$	$v_3 = -5$	

We find the sum of u_i and v_j for each empty cell and write it at the bottom left corner of that cell. Find the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each empty cell and enter it at the bottom right corner of the cell. The solution is not optimum as the cell (3, 1) has a negative Δ_{ij} value. We improve the allocation by making this cell namely (3, 1) as an allocated cell. We draw a closed path from this cell and assign + and – signs alternately. From the cell having negative sign we find the min. allocation given by min (10, 10) = 10. Hence, we get two occupied cells (2, 1) (3, 2) that become empty and the cell (3, 1) is occupied, resulting in a degenerate solution. (Degeneracy in subsequent iteration).

Number of allocated cell = 4 < $m + n - 1 = 5$.

We get a degeneracy and to resolve it, we add the empty cell (1, 2) and allocate $\varepsilon > 0$. This cell namely (1, 2) is added as it satisfies the two steps for resolving the degeneracy. We assign a number 0 to the first row, namely $u_1 = 0$, we get the remaining numbers as follows.

$$\begin{aligned}
c_{11} &= u_1 + v_1 = 2 \Rightarrow v_1 = 2 - u_1 = 2 - 0 = 2 \\
c_{12} &= u_1 + v_2 = 2 \Rightarrow v_2 = 2 - u_1 = 2 - 0 = 2
\end{aligned}$$

$$c_{31} = u_3 + v_1 = 1 \Rightarrow u_3 = 1 - v_1 = 1 - 2 = -1$$
$$c_{33} = u_3 + v_3 = 1 \Rightarrow v_3 = 1 - u_3 = 1 - (-1) = 2$$
$$c_{22} = u_2 + v_2 = 1 \Rightarrow u_2 = 1 - v_2 = 1 - 2 = -1$$

Next we find the sum of u_i and v_j for the empty cell and enter it at the bottom left corner of that cell and also the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each empty cell and enter it at the bottom right corner of the cell.

Iteration table

	1	*2*	*3*	u_i
1	2 ⑩	2 (ε)	3 2 1	0
2	4 1 3	1 ⑮	2 1 1	– 1
3	1 ⑩	3 1 2	1 ㉚	– 1
v_j	2	2	2	

The modified solution is given in the following table. This solution is also optimal and unique as it satisfies the optimality condition that all $\Delta_{ij} > 0$.

	1	*2*	*3*	*Capacity*
1	2 ⑩	2 (ε)	3	10
2	4	1 ⑮	2	15
3	1 10	3	1 30	40
Demand	⑳	15	㉚	65

$$x_{11} = 10; \quad x_{22} = 15; \quad x_{33} = 30;$$
$$x_{12} = \varepsilon_j; \quad x_{31} = 10$$
$$\text{Total cost} = (10 \times 2) + (\varepsilon \times 2) + (15 \times 1) + (10 \times 1) + (30 \times 1)$$
$$= 75 + 2\,\varepsilon = ₹\,75.$$

Example 8.10 Solve the following transportation problem whose cost matrix is given below.

		Destination				
		A	***B***	***C***	***D***	***Capacity***
Origin	**1**	1	5	3	3	34
	2	3	3	1	2	15
	3	0	2	2	3	12
	4	2	7	2	4	19
	Demand	21	25	17	17	80

Solution Since $\Sigma a_i = \Sigma b_j$, the problem is a balanced transportation problem. Hence, there exists a feasible solution. We find the initial solution by north-west corner rule.

		Destination				
		A	***B***	***C***	***D***	***Capacity***
Origin	**1**	1 ㉑	5 ⑬	3	3	~~34~~ ~~13~~ 0
	2	3	3 ⑫	1 ③	2	~~15~~ ~~3~~ 0
	3	0	2	2 ⑫	3	~~12~~ 0
	4	2	7	2 ②	4 ⑰	~~19~~ ~~17~~ 0
	Demand	~~21~~ 0	~~25~~ ~~12~~	~~17~~ ~~14~~ ~~2~~	~~17~~ 0	80

We get the total number of allocated cells = 7 = 4 + 4 – 1. As all the allocations are independent, the solution is a non-degenerate solution.

Total transportation cost

$$= 21 \times 1 + 13 \times 5 + 12 \times 3 + 3 \times 1 + 12 \times 2 + 2 \times 2 + 17 \times 4$$

$$= ₹\,221.$$

***To find the optimal solution* (*MODI Method*)** We determine a set of numbers u_i and v_j for each row and each column with $u_j + v_j = c_{ij}$ for each occupied cell. To start with, we give 0 to the third column as it has the maximum number of allocations.

Initial table

	A	***B***	***C***	***D***	u_i
1	1 ㉑	5 ⑬ 3	3 0	3 5 – 2	$3 = u_1$
2	3 – 1 4	3 ⑫ –	1 + ③	2 3 – 1	$1 = u_2$
3	0 0 0	+ $\boxed{2}$ 4 – 2	2 – ⑫	3 4 – 1	$2 = u_3$
4	2 0 2	7 4 3	2 ②	4 ⑰	$2 = u_4$
v_j	$-2 = v_1$	$2 = v_2$	$0 = v_3$	$2 = v_4$	

$$
\begin{aligned}
c_{23} &= u_2 + v_3 = 1 \Rightarrow u_2 = 1 - 0 = 1\\
c_{33} &= u_3 + v_3 = 2 \Rightarrow\\
u_3 &= 2 - v_3 = 2 - 0 = 2\\
c_{43} &= u_4 + v_3 = 2 \Rightarrow\\
u_4 &= 2 - 0 = 2\\
c_{44} &= u_4 + v_4 = 4 \Rightarrow v_4 = 4 - 2 = 2\\
c_{22} &= u_2 + v_2 = 3 \Rightarrow\\
v_2 &= 3 - u_2 = 2\\
c_{12} &= u_1 + v_2 = 5 \Rightarrow\\
u_1 &= 5 - v_2 = 3\\
c_{11} &= u_1 + v_1 = 1 \Rightarrow\\
v_1 &= 1 - u_1 = 1 - 3 = -2
\end{aligned}
$$

We find the sum of u_i and v_j for each empty cell and enter it at the bottom left corner of that cell. We find the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each empty cell and enter it at the bottom right corner of that cell. The solution is not optimum as some of $\Delta_{ij} < 0$. We choose the most negative Δ_{ij}, i.e., –2. There is a tie between the cells (1, 4) and (3, 2) but we choose the cell (3, 2) as it has the least cost. From this cell we draw a closed path and assign + and – signs alternately and find the minimum allocation from the cell having – sign.

Thus we get, Min (12, 12) = 12. Hence, one empty cell (3, 2) becomes occupied and two occupied cells (2, 2) (3, 3) become empty, resulting in degeneracy (Degeneracy in subsequent iteration). By adding and subtracting this minimum allocation, we get the modified allocation as given in the table below. For these modified allocations, we repeat the steps in MODI method.

I Iteration table

	A	*B*	*C*	*D*
1	1 ㉑	5 ⑬	3	3
2	3	3	1 ⑮	2
3	0	2 ⑫	2	3
4	2	7	2 ②	4 ⑰

The number of allocation = 6 < $m + n - 1 = 7$. We add the cell (3, 3) as it is the least cost empty cell, which is of independent position. Give a small quantity ε > 0. This removes degeneracy. The modified allocation is given in the table below.

	A	*B*	*C*	*D*	u_i
1	1 ㉑	5 – ⑬	3 5 – 2	3 + ε 7 – 4	5
2	3 – 3 6	3 1 2	1 ⑮	2 3 – 1	1
3	0 – 2 2	2 + ⑫	2 – (ε)	3 4 –1	2
4	2 – 2 4	7 2 5	2 + ②	4 – ⑰	2
v_j	– 4	0	0	2	

The solution is not optimum. The next negative value of $\Delta_{ij} = -\ 4$. (the cell (1, 4)).

The minimum allocation is min. (13, ε, 17) = ε . Proceeding in the same manner we have the 2nd iteration table as given below.

II Iteration table

	A	***B***	***C***	***D***	***u_i***
1	1 21	— 5 — 13–ε	3 1 2	3 + ε	0
2	3 1 2	3 +ε 5 –2	1 – 15	2 3 –1	0
3	0 –2 2	2 12 + ε	2 –2 4	3 0 3	–3
4	2 2 0	7 6 1	+ 2 2 + ε	— 4 17 – ε	1
	1	5	1	3	

As the solution is not optimum, we improve it by using the steps involved in MODI method. The most negative value of $\Delta_{ij} = -2$. Min allocation is min. (13 – ε, 15, 17 – ε) = 13 – ε.

III Iteration table

	A	***B***	***C***	***D***	
1	1 21	5 3 2	3 1 2	3 13	0
2	3 1 2	3 13 – ε	1 2 + ε –	2 + ε 3 –1	0
3	0 0 0	2 12 + ε	2 0 2	3 2 1	0 – 1
4	2 2 0	7 4 3	2 + 15	4 – 4	1
	1	3	1	3	

Improve the solution by adding and subtracting the new allocation given by min. (2 + ε, 4) = (2 + ε)

IV Iteration table

	A	*B*	*C*	*D*	u_i
1	1 21	5 4 1	3 1 2	3 13	3
2	3 0 3	3 13 – ε	1 0 1	2 2 + ε	2
3	0 –1 1	2 12 + ε	2 –1 3	3 1 2	1
4	2 2 0	7 5 2	2 17 + ε	4 2 – ε	4
v_j	–2	1	–2	0	

Since all $\Delta_{ij} \geq 0$, the solution is optimum (alternate solution exists). The solution is given by,

$$X_{11} = 21; X_{14} = 13; X_{22} = 13 - \varepsilon = 13; X_{24} = 2 + \varepsilon = 2;$$

$$X_{32} = 12 + \varepsilon = 12; X_{43} = 17 + \varepsilon = 17; X_{44} = 2 - \varepsilon = 2$$

$$\text{Total transportation cost} = 21 \times 1 + 13 \times 3 + (13 - \varepsilon) \times 3 + (2 + \varepsilon) \times 2 + (12 + \varepsilon) \times 2$$

$$+ (17 + \varepsilon) \times 2 + (2 - \varepsilon) \times 4 = 169 - \varepsilon = ₹\ 169$$

Example 8.11 A company has three plants *A*, *B* and *C*, 3 warehouses *X*, *Y* and *Z*. The number of units available at the plants is 60, 70, 80 and the demand at *X*, *Y*, *Z* is 50, 80, 80 respectively. The unit cost of the transportation is given in the following table:

	X	*Y*	*Z*
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum.

Solution

	Warehouses				
		X	*Y*	*Z*	*Capacity*
Plants	*A*	8	7	3	60
	B	3	8	9	70
	C	11	3	5	80
	Demand	50	80	80	210

Since $\Sigma a_i = \Sigma b_j = 210$, the problem is a balanced one. Hence, there exists a feasible solution. Let us find the initial solution by least cost method.

Iteration: 1

	X	*Y*	*Z*	*Supply*
A	8	7	3	60
B	3 ㊿	8	9	70
C	11	3	5	80
Demand	50	80	80	210

Here the least cost cell is not unique, i.e., the cells (2, 1) (1, 3) and (3, 2) have the least value 3. So choose the cell arbitrarily. Let us choose the cell (2, 1) and allocate with magnitude min. (70, 50) = 50. This exhausts the first column. So delete this column. The reduced transportation table is given by,

Iteration: 2

	Y	*Z*	*Supply*
A	7	3 (60)	60
B	8	9	20
C	3	5	80
Demand	80	80	

Continuing in this manner, we finally arrive at the initial solution, which is shown in the following table:

Iteration: 3

	X	*Y*	*Z*	*Supply*
A	8	7	3 (60)	60
B	3 ㊿	8	9 ⑳	70
C	11	3 (80)	5 ⓪	80
Demand	50	80	80	

Iteration: 4

	Y	*Z*	*Supply*
B	8	9	20
C	3 (80)	5	80
Demand	80	20	

The number of allocated cells is $m + n - 1 = 5$,

This solution is non-degenerate.

The solution is given by,

$$X_{13} = 60,\ x_{21} = 50$$
$$X_{23} = 20,\ X_{32} = 80,\ x_{33} = 0.$$

The total transportation cost

$$= (60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) + (0 \times 5)$$
$$= ₹\ 750$$

To find the optimal solution We apply the steps involved in MODI method to the above table. We find a set of numbers u_i and v_j for which $u_i + v_j = c_{ij}$ is satisfied for each of the occupied cells. To start with, we assign a number 0 to the third column (i.e., $v_3 = 0$) as it has the maximum number of allocations. The remaining numbers are obtained as follows.

	X	Y	Z	u_i
A	8 –3 11	7 1 6	3 60	3
B	3 50	8 7 1	9 20	9
C	11 –1 12	3 80	5 0	5
v_j	–6	–2	0	

Since all $\Delta_{ij} > 0$, we have obtained an optimum solution.

The solution is given by, $X_{13} = 60; X_{21} = 50; X_{23} = 20; X_{32} = 80; X_{33} = 0$

Total transportation cost $= (60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) + (0 \times 5)$

$=$ ₹ 750

Unbalanced transportation problem The given TP is said to be unbalanced if $\Sigma a_i \neq \Sigma b_j$, i.e., if the total supply is not equal to the total demand.

There are two possible cases.

Case I $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

If the total supply is less than the total demand, a dummy source (row) is included in the cost matrix with zero cost, the excess demand is entered as a rim requirement for this dummy source (origin). Hence, the unbalanced transportation problem can be converted into a balanced TP.

Case II $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

If the total supply is greater than the total demand, the unbalanced TP can be converted into a balanced TP by adding a dummy destination (column) with zero cost. The excess supply is entered as a rim requirement for the dummy destination.

Example 8.12 Solve the transportation problem when the unit transportation costs, demands and supplies are as given below:

		Destination				
		D_1	D_2	D_3	D_4	***Supply***
Origins	O_1	6	1	9	3	70
	O_2	11	5	2	8	55
	O_3	10	12	4	7	70
	Demand	85	35	50	45	

Solution Since the total demand $\Sigma b_j = 215$ is greater than the total supply $\Sigma a_i = 195$, the problem is an unbalanced TP.

We convert this into a balanced TP by introducing a dummy origin O_4 with cost zero and giving supply equal to $215 - 195 = 20$ units. Hence, we have the converted problem as follows:

		Destination				
		D_1	D_2	D_3	D_4	***Supply***
	O_1	6	1	9	3	70
Origins	O_2	11	5	2	8	55
	O_3	10	12	4	7	70
	O_4	0	0	0	0	20
	Demand	85	35	50	45	215

As this problem is balanced, there exists a feasible solution to this problem. Using VAM we get the following initial solution.

	D_1	D_2	D_3	D_4	***Supply***	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}	P_{VII}
O_1	6 (65)	1 ⑤	9	3	70	2	2	2	—	—	—	—
O_2	11	5 ㉚	2 ㉕	8	55	3	3	3	3	6 ←	—	—
O_3	10	12	4 ㉕	7 ㊺	70	3	3	3	3	3	3	4 ←
O_4	0 ⑳	0	0	0	20	0	—	—	—	—	—	—
Demand	85	35	50	45								
P_{I}	6↑	1	2	3								
P_{II}	4↑	4	2	3								
P_{III}	—	4↑	2	4								
P_{IV}	—	7↑	2	1								
P_{V}	—	—	2	1								
P_{VI}	—	—	4	7↑								
P_{VII}	—	—	4	7								

The initial solution to the problem is given by,

	D_1	D_2	D_3	D_4
O_1	6 (65)	1 (5)	9	3
O_2	11	5 (30)	2 (25)	8
O_3	10	12	4 (25)	7 (45)
O_4	0 (20)	0	0	0

There are 7 independent non-negative allocations equals to $m + n - 1$. Hence, the solution is a non-degenerate one. The total transportation cost

$$= 65 \times 6 + 5 \times 1 + 30 \times 5 + 25 \times 2 + 25 \times 4 + 45 \times 7 + 20 \times 0$$

$$= ₹\ 1{,}010.$$

To find the optimal solution We apply the steps in MODI method to the above table.

Initial table

	D_1	D_2	D_3	D_4	u_i
O_1	6 (65) –	1 + (5)	9 0 9	3 3 0	0
O_2	11 10 1	– 5 (30)	2 + (25)	8 5 3	4
O_3	+ 10 12 –2	12 7 5	4 – (25)	7 (45)	6
O_4	0 (20) –5	0 5 –4	0 4 –5	0 5 –6	
v_j	6	1	–2	1	

Since all $\Delta_{ij} \geq 0$, the solution is not optimum. We introduce the cell (3, 1) as this cell has the most negative value of Δ_{ij}. We modify the solution by adding and subtracting the minimum allocation given by min (65, 30, 25). While doing this, the occupied cell (3, 3) becomes empty.

I Iteration table

	D_1	D_2	D_3	D_4	u_i
O_1	6 40	1 30	9 –2 11	3 3 0	6
O_2	11 10 1	5 5	2 50	8 7 1	10
O_3	10 25	12 5 7	4 2	7 45	10
O_4	0 20	0 –5 5	0 –8 8	0 –3 3	0
v_j	0	– 5	–8	– 3	

As the number of independent allocations are equal to $m + n - 1$, we check the optimality.

Since all $\Delta_{ij} \geq 0$, the solution is optimal and an alternate solution exists as $\Delta_{14} = 0$. Therefore, the optimum allocation is given by,

$$X_{11} = 40, X_{12} = 30, X_{22} = 5, X_{23} = 50, X_{31} = 25, X_{34} = 45, X_{41} = 20.$$

The optimum transportation cost is

$$= 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20 = ₹\,960.$$

Example 8.13 A product is produced by 4 factories F_1, F_2, F_3 and F_4. Their unit production costs are ₹ 2, 3, 1 and 5 respectively. Production capacity of the factories are 50, 70, 30 and 50 units respectively. The product is supplied to 4 stores S_1, S_2, S_3 and S_4, the requirements of which are 25, 35, 105 and 20 respectively. Unit costs of transportation are given below.

Find the transportation plan such that the total production and transportation cost is minimum.

		Destination			
		S_1	S_2	S_3	S_4
Factory	F_1	2	4	6	11
	F_2	10	8	7	5
	F_3	13	3	9	12
	F_4	4	6	8	3

Solution We form the transportation table, which consists of both production and transportation costs.

	S_1	S_2	S_3	S_4	*Capacity*
F_1	4	6	8	13	50
F_2	13	11	10	8	70
F_3	14	4	10	13	30
F_4	9	11	13	8	50
Demand	25	35	105	20	

Total capacity = 200 units
Total demand = 185 units

Therefore $\Sigma a_i > \Sigma b_j$. Hence the problem is unbalanced. We convert it into a balanced one by adding a dummy store S_5 with cost 0 and the excess supply is given as the rim requirement to this store namely (200–185) units.

	S_1	S_2	S_3	S_4	S_5	*Supply*
F_1	4 25	6 5	8 20	13	0	50
F_2	13	11	10 50	8 20	0	70
F_3	14	4 30	10	13	0	30
F_4	9	11	13 35	8	0 15	50
Demand	25	35	105	20	15	200

The initial basic feasible solution is obtained by least cost method. We get the solution containing 8 non-negative independent allocations equals to $m + n - 1$. So the solution is a non-degenerate solution.

The total transportation cost

$$= (25 \times 4) + (5 \times 6) + (20 \times 8) + (50 \times 10) + (20 \times 8) + (30 \times 4) + (35 \times 13) + (15 \times 0)$$

$$= ₹\ 1{,}525$$

To find the optimal solution We apply MODI method to the above table as it has $m + n - 1$ independent non-negative allocation.

Initial table

	S_1	S_2	S_3	S_4	S_5	u_i
F_1	4 25	6 5	8 20	13 6 7	0 –5 5	0
F_2	13 6 7	11 8 3	10 50	8 20 –	0 –3 3	2
F_3	14 2 12	4 30	10 6 4	13 4 9	0 –7 7	–2
F_4	9 9 0	11 11 0	13 35 –	8 11 –3	0 15	5
v_j	4	6	8	6	–5	

The solution is not optimum as the cell (4, 5) is having a negative net evaluation value, i.e., $\Delta_{44} = -3$. We draw a closed path from this cell and have a modified allocation by adding and subtracting the allocation min (35, 20) = 20. This modified allocation is given in the following table.

I Iteration table

	S_1	S_2	S_3	S_4	S_5	u_i
F_1	4 ㉕	6 ⑤	8 ⑳	13 3 0	0 –5 5	0
F_2	13 6 7	11 8 3	10 (70)	8 5 –3	0 –3 3	2
F_3	14 2 12	4 ㉚	10	13 1 12	0 –7 7	–2
F_4	9 9 0	11 11 0	13 ⑮	8 ⑳	0 ⑮	5
v_j	4	6	8	6	–5	

Since all the values of $\Delta_{ij} \geq 0$, the solution is optimum but an alternate solution exists.

The optimum solution or the transportation plan is given by,

X_{11} = 25 units	X_{32} = 30 units
X_{12} = 5 units	X_{43} = 15 units
X_{13} = 20 units	X_{44} = 20 units
X_{23} = 70 units	X_{45} = 15 units

This is the surplus capacity that is not transported, which is manufactured in factory F_4. The optimum production with transportation cost

$$= (25 \times 4) + (5 \times 6) + (20 \times 8) + (70 \times 10) + (30 \times 4) + (20 \times 8) + (15 \times 13) + (15 \times 0) = ₹\ 1{,}465$$

Maximization case in transportation problem Here the objective is to maximize the total profit for which the profit matrix is given. For this, first we have to convert the maximization problem into minimization by subtracting all the elements from the highest element in the given transportation table. This modified minimization problem can be solved in the usual manner.

Example 8.14 There are three factories *A*, *B* and *C*, which supply goods to four dealers D_1, D_2, D_3 and D_4. The production capacities of these factories are 1,000, 700 and 900 units per month respectively. The requirements from the dealers are 900, 800, 500 and 400 units per month respectively. The per unit return (excluding transportation cost) are ₹ 8, ₹ 7 and ₹ 9 at the three factories. The following table gives the unit transportation costs from the factories to the dealers.

	D_1	D_2	D_3	D_4
A	2	2	2	4
B	3	5	3	2
C	4	3	2	1

Determine the optimum solution to maximize the total returns.

Solution Profit = return – transportation cost. With this we form a transportation table with profit.

	D_1	D_2	D_3	D_4
A	8 – 2 = 6	8 – 2 = 6	8 – 2 = 6	8 – 4 = 4
B	7 – 3 = 4	7 – 5 = 2	7 – 3 = 4	7 – 2 = 5
C	9 – 4 = 5	9 – 3 = 6	9 – 2 = 7	9 – 1 = 8

Profit matrix

	D_1	D_2	D_3	D_4	*Capacity*
A	6	6	6	4	1,000
B	4	2	4	5	700
C	5	6	7	8	900
Requirement	900	800	500	400	2,600

The above profit matrix is converted into its equivalent loss matrix by subtracting all the elements from the highest element namely 8. Hence, we have the following loss matrix.

	D_1	D_2	D_3	D_4	*Capacity*
A	2	2	2	4	1,000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	2,600

We use VAM to get the initial basic feasible solution.

	D_1	D_2	D_3	D_4	*Capacity*	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{Vi}
A	2 200	2 800	2	4	200 1,000	0	0	0	2	2	2
B	4 700	6	4	3	0 700	1	0	2	4 ←	—	—
C	3 0	2	1 500	0 400	900	1	1 ←	1	3 ←	3	—
Demand	900	800	500	400	2,600						
P_{I}	1	0	1	3↑							
P_{II}	1	0	1	—							
P_{III}	2	4↑	—	—							
P_{IV}	2	—	—	—							
P_{V}	2	—	—	—							
P_{vi}	2↑										

The initial solution is given in the following table:

	D_1	D_2	D_3	D_4	*Capacity*
A	2 200	2 800	2	4	1,000
B	4 700	6	4	3	700
C	3 0	2	1 500	0 400	900
Demand	900	800	500	400	

Since the number of allocated cells = 5 < $m + n - 1$ = 6, the solution is non-degenerate.

This cell is the least cost cell and of independent position. The initial basic feasible solution is given as follows:

Initial table

	D_1	D_2	D_3	D_4
A	2 200	2 800	2	4
B	4 700	6	2	3
C	3 0	2	1 500	0 400

Number of allocations = 6 = $m + n - 1$ and the 6 allocations are in independent positions. Hence, we can perform the optimality test using MODI method.

	D_1	D_2	D_3	D_4	u_i
A	2 200	2 800	2 2 0	4 1 3	0
B	4 700	6 4 2	2 4 2	3 3 0	2
C	3 1 2	2 1	1 500	0 400	–1
u_i	2	2	2	1	

Since all the net evaluations Δ_{ji} are non-negative, the initial solution is optimum.

The optimum distribution is,

$A \rightarrow D_1$ = 200 units
$A \rightarrow D_2$ = 800 units
$A \rightarrow D_3$ = ε units

$$B \to D_1 = 700 \text{ units}$$
$$C \to D_3 = 500 \text{ units}$$
$$C \to D_4 = 400 \text{ units}$$

Total profit or the Max. return $= 200 \times 6 + 6 \times 800 + 4 \times 700 + 7 \times 500 + 8 \times 400 =$ ₹ 15,500.

Example 8.15 Solve the following transportation problem to maximize the profit.

		Destination				
		A	***B***	***C***	***D***	***Supply***
	1	15	51	42	33	23
Source	**2**	80	42	26	81	44
	3	90	40	66	60	33
	Demand	23	31	16	30	100

Solution Since the given problem is to maximize the profit, we convert this into loss matrix and minimize it. For converting it into minimization type, we subtract all the elements from the highest element 90. Hence, we have the following loss matrix.

		Destination				
		A	***B***	***C***	***D***	***Supply***
	1	75	39	48	57	23
Source	2	10	48	64	9	44
	3	0	50	24	30	33
	Demand	23	31	16	30	100

Since $\Sigma a_i = \Sigma b_j$, there exists a feasible solution and is obtained by VAM.

	A	***B***	***C***	***D***	***Supply***	P_{I}	P_{II}	P_{III}	P_{IV}	P_{V}	P_{VI}
1	75	39 ㉓	48	57	23	9	18	18	18	39	39
2	10 ⑥	48 ⑧	64	9 ㉚	44	1	1	1	39	48 ←	—
3	0 ⑰	50	24 ⑯	30	33	24	30 ←	—	—	—	—
Demand	23	31	16	30							
P_{I}	10	9	24↑	21							
P_{II}	10	9	—	21							
P_{III}	65↑	9	—	48							
P_{IV}	—	9	—	48↑							
P_{V}	—	9	—	—							
P_{VI}	—	39↑	—	—							

The initial basic feasible solution is given in the following table.

	A	*B*	*C*	*D*	*Capacity*
1	75	39 ㉓	48	57	23
2	10 ⑥	48 ⑧	64	9 ㉚	44
3	0 ⑰	50	24 ⑯	30	33
Demand	23	31	16	30	

As the number of independent allocated cells = 6 = $m + n - 1$, the solution is non-degenerate.

Optimality Test Using MODI Method:

	A	*B*	*C*	*D*	u_i
1	75 19 56	39 ㉓	48 23 25	57 18 39	9
2	10 ⑥	48 ⑧ 14	64 50	9 ㉚	0
3	0 ⑰	50 38 12	24 ⑯	30 –1 31	–10
v_j	10	48	14	9	

Since all the net evaluation $\Delta_{ij} > 0$, the solution is optimum and unique. The optimum solution is given by,

$$x_{12} = 23; x_{21} = 6; x_{22} = 8; x_{24} = 30; x_{31} = 17; x_{33} = 16$$

The optimum profit = (23 × 51) + (6 × 80) + (8 × 42) + (30 × 81) + (90 × 17) + (16 × 66)

= ₹ 7,005

8.6 THE STEPPING-STONE METHOD

This method is an approximation in which initial feasible solution is moved to an optimal solution.Main application of this method is to evaluate the cost effectiveness of shipping goods through routs used in transportation which are not currently being used in the solution.

Following the determination of an initial basic feasible solution to a transportation problem, we next obtain the optimum solution. An optimality test can be performed only on the feasible solution in which

(a) The number of allocations is $m + n - 1$ where m-number of rows, n-Number of columns.

(b) These $m + n - 1$ allocations should be in independent positions.