Chapter 3

Graphical Method

Simple linear programming problems with two decision variables can be easily solved by graphical method.

3.1 PROCEDURE FOR SOLVING LPP BY GRAPHICAL METHOD

The steps involved in graphical method are as follows:

Step 1 Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph, as each equation will geometrically represent a straight line.

Step 3 Mark the region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the '*feasible region*'.

Step 4 Assign an arbitrary value, say zero, to the objective function.

Step 5 Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6 Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin, passing through at least one corner of the feasible region.

Step 7 Find the co-ordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution. That is, in the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optimal solution is the corner point which gives the minimum value for the objective function.

Example 3.1 Solve the following LPP by graphical method.

$$\begin{aligned} \text{Minimize } Z &= 20x_1 + 10x_2 \\ \text{Subject to, } x_1 + 2x_2 &\leq 40 \\ 3x_1 + x_2 &\geq 30 \\ 4x_1 + 3x_2 &\geq 60 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution Replace all the inequalities of the constraints by equation

$$x_1 + 2x_2 = 40 \text{ If } x_1 = 0 \Rightarrow x_2 = 20$$
$$\text{If } x_2 = 0 \Rightarrow x_1 = 40$$

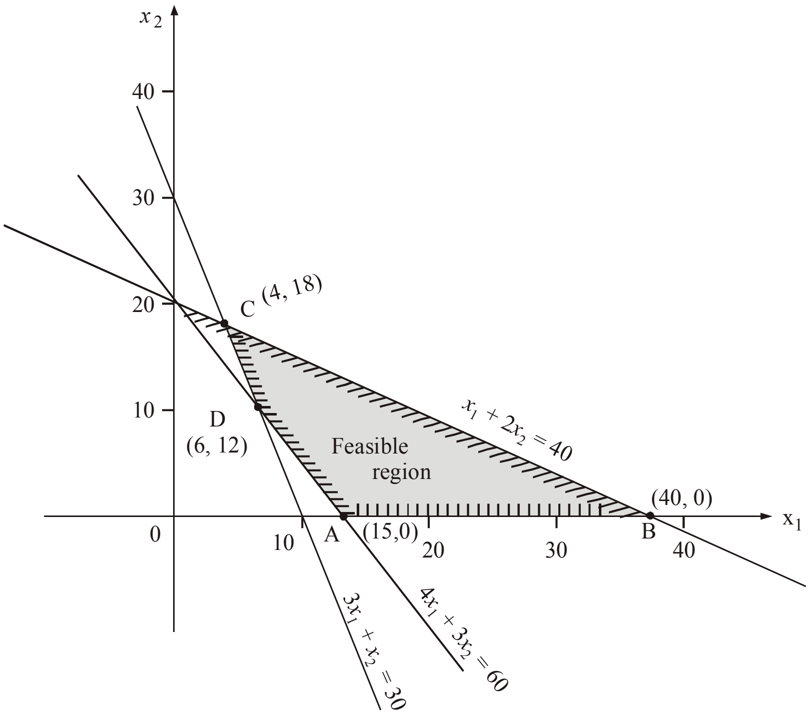
$\therefore$

$x_1 + 2x_2 = 40$ passes through (0, 20) (40, 0)

$3x_1 + x_2 = 30$ passes through (0, 30) (10, 0)

$4x_1 + 3x_2 = 60$ passes through (0, 20) (15, 0)

Plot each equation on the graph.



The feasible region is *ABCD*.

C and *D* are points of intersection of lines.

	C intersect	$x_1 + 2x_2 = 40, 3x_1 + x_2 = 30$
and,	*D* intersect	$4x_1 + 3x_2 = 60, 3x_1 + x_2 = 30$

$$C = (4, 18)$$
$$D = (6, 12)$$

Corner points	*Value of* $Z = 20x_1 + 10x_2$
$A(15, 0)$	300
$B(40, 0)$	800
$C(4, 18)$	260
$D(6, 12)$	240 (Minimum value)

$\therefore$ The minimum value of *Z* occurs at *D* (6, 12). Hence, the optimal solution is $x_1 = 6, x_2 = 12$.

Example 3.2 Find the maximum value of $Z = 5x_1 + 7x_2$

Subject to the constraints,

$$x_1 + x_2 \le 4$$
$$3x_1 + 8x_2 \le 24$$
$$10x_1 + 7x_2 \le 35$$
$$x_1, x_2 \ge 0$$

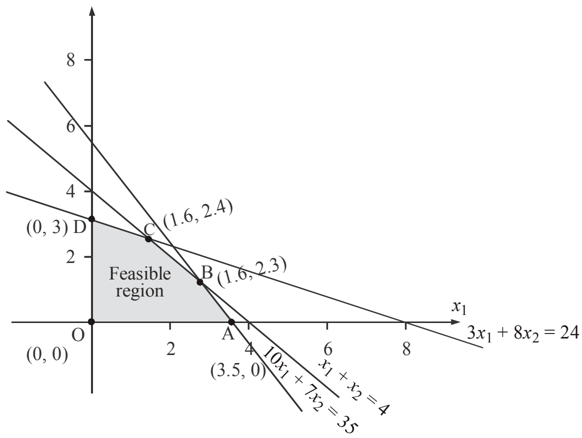
Solution Replace all the inequalities of the constraints by forming equations

$x_1 + x_2 = 4$ passes through (0, 4) (4, 0)

$3x_1 + 8x_2 = 24$ passes through (0, 3) (8, 0)

$10x_1 + 7x_2 = 35$ passes through (0, 5) (3.5, 0)

Plot these lines on the graph and mark the region below the line as the inequality of the constraint as ≤ which is also lying in the first quadrant.



The feasible region is *OABCD*.

B and *C* are the points of intersection of lines

B intersect $x_1 + x_2 = 4, \quad 10x_1 + 7x_2 = 35$

and *C* intersect $3x_1 + 8x_2 = 24, \quad x_1 + x_2 = 4.$

On solving we get, $B = (1.6, 2.3)$

$C = (1.6, 2.4)$

Corner points	*Value of* $Z = 5x_1 + 7x_2$
$O(0, 0)$	0
$A(3.5, 0)$	17.5
$B(1.6, 2.3)$	24.1
$C(1.6, 2.4)$	24.8 (Maximum value)
$D(0, 3)$	21

∴ The maximum value of Z occurs at C (1.6, 2.4) and the optimal solution is $x_1 = 1.6, x_2 = 2.4$.

Example 3.3 A company produces 2 types of hats *A* and *B*. Every hat *A* requires twice as much labour time as the second hat *B*. If the company produces only hat *B* then it can produce a total of 500 hats per day. The market limits daily sales of hat *A* and *B* to 150 and 250 respectively. The profits on hat *A* and *B* are ₹ 8 and ₹ 5 respectively. Solve graphically to get the optimal solution.

Solution Let x_1 and x_2 be the number of units of type A and type B hats respectively.

$$\text{Max } Z = 8x_1 + 5x_2$$

$$\begin{aligned} \text{Subject to, } 2x_1 + x_2 &\le 500 \\ x_1 &\le 150 \\ x_2 &\le 250 \\ x_1, x_2 &\ge 0 \end{aligned}$$

First rewrite the inequality of the constraint into an equation and plot the lines on the graph.

$2x_1 + x_2 = 500$ passes through (0, 500) (250, 0)

$x_1 = 150$ passes through (150, 0)

$x_2 = 250$ passes through (0, 250)

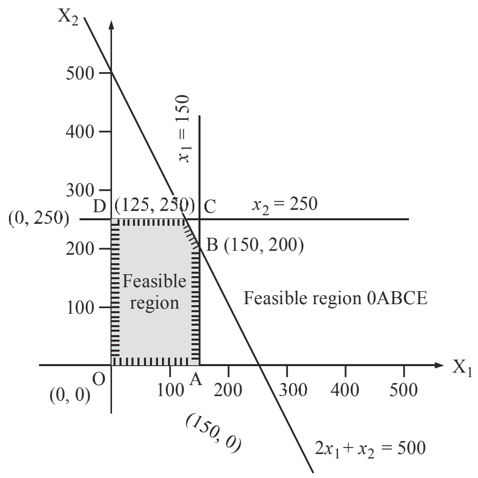
We mark the region below the lines lying in the first quadrant as the inequality of the constraints are ≤. The feasible region is *OABCD*. *B* and *C* are the points of intersection of lines

$$2x_1 + x_2 = 500,$$

$x_1 = 150$ (B intersect) and $x_2 = 250$ (C intersect)

On solving, we get $B = (150, 200)$

$C = (125, 250)$



Corner points	*Value of* $Z = 8x_1 + 5x_2$	
$O(0, 0)$	0	
$A(150, 0)$	1200	
$B(150, 200)$	2200	
$C(125, 250)$	2250	(Maximum $Z = 2250$)
$D(0, 250)$	1250	

The maximum value of Z is attained at $C(125, 250)$

∴ The optimal solution is $x_1 = 125, x_2 = 250$.

i.e., The company should produce 125 hats of type A and 250 hats of type B in order to get the maximum profit of ₹ 2250.

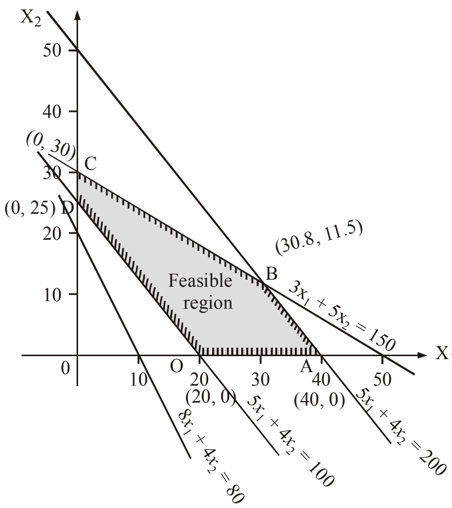
Example 3.4 By graphical method solve the following LPP.

$$\text{Max } Z = 3x_1 + 4x_2$$
$$\text{Subject to, } 5x_1 + 4x_2 \leq 200$$
$$3x_1 + 5x_2 \leq 150$$
$$5x_1 + 4x_2 \geq 100$$
$$8x_1 + 4x_2 \geq 80$$
and
$$x_1, x_2 \geq 0$$

Solution

Replacing the inequality by equality

$$5x_1 + 4x_2 = 200 \text{ passes through } (0, 50), (40, 0)$$
$$3x_1 + 5x_2 = 150 \text{ passes through } (0, 30), (50, 0)$$
$$8x_1 + 4x_2 = 80 \text{ passes through } (0, 20), (10, 0)$$
$$5x_1 + 4x_2 = 100 \text{ passes through } (0, 25), (20, 0)$$



Feasible region is given by *OABCD*.

Corner points	*Value of* $Z = 3x_1 + 4x_2$
$O(20, 0)$	60
$A(40, 0)$	120
$B(30.8, 11.5)$	138.4 (Maximum value)
$C(0, 30)$	120
$D(0, 25)$	100

$\therefore$ The maximum value of Z is attained at B (30.8, 11.5)

$\therefore$ The optimal solution is $x_1 = 30.8, x_2 = 11.5$.

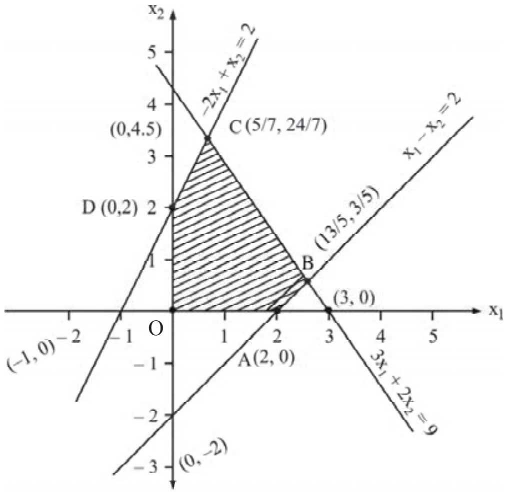
Example 3.5 Use graphical method to solve the LPP.

$$\text{Maximize } Z = 6x_1 + 4x_2$$
$$\text{Subject to, } -2x_1 + x_2 \leq 2$$
$$x_1 - x_2 \leq 2$$
$$3x_1 + 2x_2 \leq 9$$
$$x_1, x_2 \geq 0$$

Solution

Replacing the inequality by equality

$$-2x_1 + x_2 = 2 \text{ passes through } (0, 2), (-1, 0)$$
$$x_1 - x_2 = 2 \text{ passes through } (0, -2), (2, 0)$$
$$3x_1 + 2x_2 = 9 \text{ passes through } (0, 4.5), (3, 0)$$



Feasible region is given by *OABCD*.

Corner points	*Value of* $Z = 6x_1 + 4x_2$
$O(0, 0)$	0
$A(2, 0)$	12
$B(13/5, 3/5)$	$\frac{78+12}{5} = \frac{90}{5} = 18$ (Maximum value)

$C\left(\frac{5}{7}, \frac{24}{7}\right) = \frac{126}{7} = 18$ (Maximum value)

$D(0, 2) = 8$

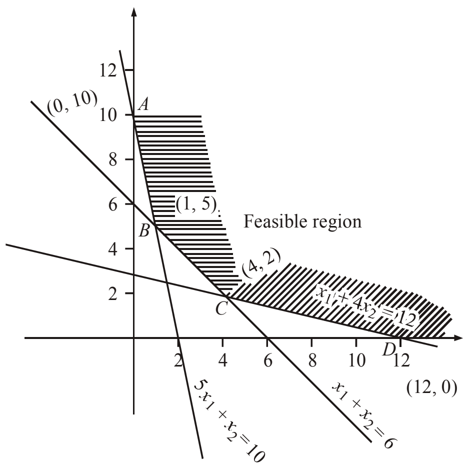
The maximum value of Z is attained at B (13/5, 3/5) or at C (5/7, 24/7).

$\therefore$ The optimal solution is $x_1 = 13/5, x_2 = 3/5$, or $x_1 = 5/7, x_2 = 24/7$.

Example 3.6 Use graphical method to solve the LPP.

$$\text{Maximize } Z = 3x_1 + 2x_2$$
$$\text{Subject to, } 5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \geq 6$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution



Corner points	*Value of* $Z = 3x_1 + 2x_2$
A (0, 10)	20
B (1, 5)	13 (Minimum value)
C (4, 2)	16
D (12, 0)	36

Since the minimum value is attained at B (1, 5) the optimum solution is $x_1 = 1$, $x_2 = 5$.

Note: In the above problem if the objective function is maximization, then the solution is unbounded, as the maximum value of Z occurs at infinity.

3.1.1 Some More Cases

There are some linear programming problems which may have,

(*i*) a unique optimal solution
(*ii*) an infinite number of optimal solutions
(*iii*) an unbounded solution
(*iv*) no solution.

The following examples will illustrate these cases.

Example 3.7 Solve the LPP by graphical method.

$$\text{Maximize } Z = 100x_1 + 40x_2$$

$$\text{Subject to, } 5x_1 + 2x_2 \leq 1{,}000$$

$$3x_1 + 2x_2 \leq 900$$

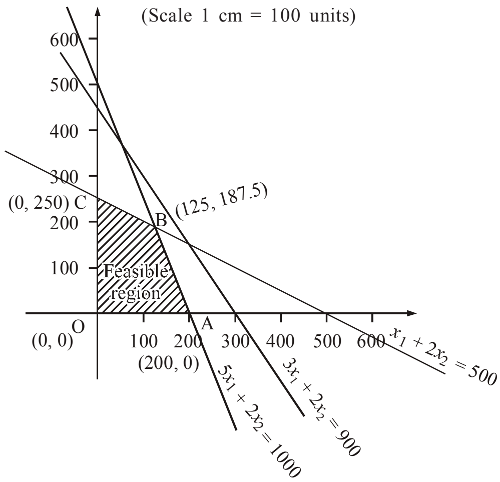
$$x_1 + 2x_2 \leq 500$$

and,

$$x_1, x_2 \geq 0$$

Solution

The solution space is given by the feasible region *OABC*.



Corner points	*Value of* $Z = 100x_1 + 40x_2$
O (0, 0)	0
A (200, 0)	20000 (Max value of Z)
B (125, 187.5)	20000 (Max value of Z)
C (0, 250)	10000

$\therefore$ The maximum value of Z occurs at two vertices A and B.

Since there are infinite number of points on the line joining A and B it gives the same maximum value of Z.

Thus, there are infinite number of optimal solutions for the LPP.

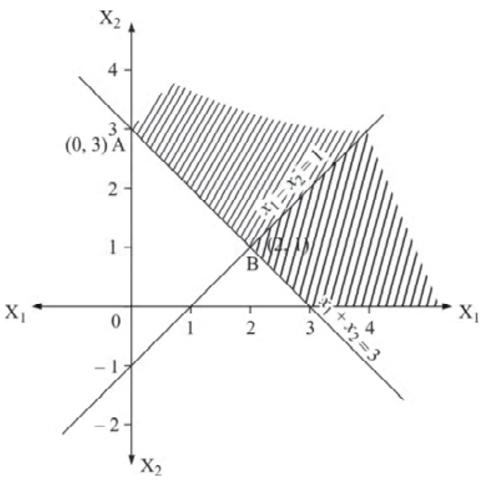
Unbounded Solution

Example 3.8 Solve the following LPP.

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to,

$$x_1 - x_2 \geq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$



Solution The solution space is unbounded. In fact, the maximum value of Z occurs at infinity. Hence, the problem has an *unbounded solution*.

No feasible solution

When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the LPP exists.

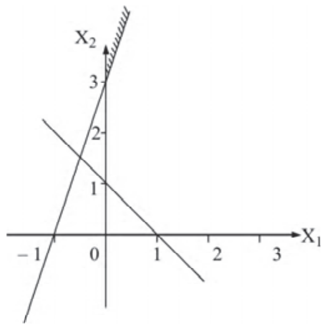
Example 3.9 Solve the following LPP.

$$\text{Max } Z = x_1 + x_2$$

Subject to the constraints,

$$x_1 + x_2 \leq 1$$
$$-3x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution There being no point (x_1, x_2) common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no solution.



For this problem, no feasible region in found, hence is given as infeasible solution.

EXERCISES

1. Solve the following by graphical method.

   Max $Z = x_1 - 3x_2$

   Subject to, $x_1 + x_2 \leq 300$

   $x_1 - 2x_2 \leq 200$

   $2x_1 + x_2 \leq 100$

   $x_2 \leq 200$

   $x_1, x_2 \geq 0$

   [**Ans.** Max $Z = 205$, $x_1 = 200$, $x_2 = 0$]

2. Max $Z = 5x + 8y$

   Subject to, $3x + 2y \leq 36$

   $x + 2y \leq 20$

   $3x + 4y \leq 42$

   $x, y \geq 0$ [**Ans.** Max $Z = 82$, $x = 2$, $y = 9$]

3. Max $Z = x + 3y$

   Subject to, $x + y \leq 300$

   $x - 2y \leq 200$

   $x + y \leq 100$

   $y \geq 200$

   and, $x, y \geq 0$ [**Ans.** Max $Z = 700$, $x = 200$ $y = 100$]

4. An egg contains 6 units of vitamin A and 7 units of vitamin B per gram and costs 12 paise per gram. Milk contains 8 units of vitamin A and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix. [**Ans.** Min Z = 205, x_1 = 15, x_2 = 1.25]

5. Solve graphically the following LPP.

$$\begin{array}{ll} \text{Max} & Z = 20x_1 + 10x_2 \\ \text{Subject to,} & x_1 + 2x_2 \le 40 \\ & 3x_1 + x_2 \ge 30 \\ & 4x_1 + 3x_2 \ge 60 \\ \text{and,} & x_1, x_2 \ge 0 \end{array}$$

[**Ans.** Max Z = 240, x_1 = 6, x_2 = 12]

6. A company produces two different products, A and B and makes a profit of ₹ 40 and ₹ 30 per unit respectively. The production process has a capacity of 30000 man-hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market survey indicates that the maximum number of units of product A that can be sold is 8000 and those of B is 12000. Formulate the problem and solve it by graphical method to get maximum profit. [**Ans.** Max Z = $40x_1 + 30x_2$
Subject to, $3x_1 + x_2 \le 30000$; $x_1 \le 8000$; $x_2 \le 12000$; $x_1, x_2 \ge 0$
Max Z = 600000, x_1 = 6000 and x_2 = 12000]

7. Solve graphically the following LPP

$$\begin{array}{ll} \text{Min} & Z = 3x - 2y \\ \text{Subject to,} & -2x + 3y \le 9 \\ & x - 5y \ge -20 \\ & x, y \ge 0 \end{array}$$

8. $$\begin{array}{ll} \text{Min} & Z = -6x_1 - 4y_2 \\ \text{Subject to,} & 2x_1 + 3x_2 \ge 30 \\ & 3x_1 + 2x_2 \le 24 \\ & x_1 + x_2 \ge 3 \\ & x_1, x_2 \ge 0 \end{array}$$

[**Ans.** Infinite number of solutions with min Z = –48
(*i*) x_1 = 81, x_2 = 0; (*ii*) $x_1 = \frac{12}{5}$, $x_2 = \frac{42}{5}$, etc.]

9. $$\begin{array}{ll} \text{Max} & Z = 3x_1 - 2x_2 \\ \text{Subject to,} & x_1 + x_2 \le 1 \\ & 2x_1 + 2x_2 \ge 4 \\ & x_1, x_2 \ge 0 \end{array}$$

[**Ans.** No feasible region]

10. $$\begin{array}{ll} \text{Max} & Z = -x_1 + x_2 \\ \text{Subject to,} & x_1 - x_2 \ge 0 \\ & -x_1 + x_2 \ge 3 \\ & x_1, x_2 \ge 0 \end{array}$$

[**Ans.** No feasible region so no solultion]

3.2 GENERAL FORMULATION OF LPP

The general formulation of the LPP can be stated as follows:

Maximize or Minimize

$$Z = C_1x_1 + C_2x_2 + ... + C_nxs_n \quad ... \quad (1)$$

Subject to m constraints

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n & (\le\ =\ \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n & (\le\ =\ \ge)\ b_2 \\ \vdots & \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{1j}x_j + \ldots + a_{in}x_n & (\le\ =\ \ge)\ b_i \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n & (\le\ =\ \ge)\ b_m \end{cases}\right\} \tag{2}$$

In order to find the values of n decision variables $X_1\ X_2 \ldots X_n$ to maximize or minimize the objective function and the non-negativity restrictions

$$x_1 \ge 0, x_2 \ge 0 \ldots x_n \ge 0 \ldots \tag{3}$$

3.3 MATRIX FORM OF LPP

The linear programming problem can be expressed in the matrix form as follows:

Maximize or Minimize $Z = CX$

Subject to
$$AX \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$$

$$X \ge 0.$$

where, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, $C = (C_1\ C_2 \ldots C_n)$

and,
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

SOME IMPORTANT DEFINITIONS

1. A set of values $x_1, x_2 \ldots x_n$ that satisfies the constraints (2) of the LPP is called its *solution*.
2. Any solution to a LPP, which satisfies the non-negativity restrictions (3) of the LPP is called its *feasible solution*.
3. Any feasible solution, which optimizes (minimizes or maximizes) the objective function (1) of the LPP is called its *optimum solution*.
4. Given a system of m linear equations with n variables ($m < n$), any solution that is obtained by solving for m variables keeping the remaining n–m variables zero is called a *basic solution*. Such m variables are called *basic variables* and the remaining are called *non-basic variables*.

   The number of basic solutions $= \le \dfrac{n!}{m!(n-m)!}$

5. A *basic feasible* solution is a basic solution which also satisfies (3), that is all basic variables are non-negative. Basic feasible solutions are of two types:
   (*a*) *Non-degenerate*: A non-degenerate basic feasible solution is the basic feasible solution that has exactly m positive x_i ($i = 1, 2 \ldots m$) i.e., None of the basic variables are zero.
   (*b*) *Degenerate*: A basic feasible solution is said to degenerate if one or more *basic variables* are zero.
6. If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called *unbounded solutions.*

3.4 CANONICAL OR STANDARD FORMS OF LPP

The general LPP can be classified as canonical or standard forms.

In *standard form*, irrespective of the objective function, namely, maximize or minimize, all the constraints are expressed as equations. Moreover RHS of each constraint and all variables are non-negative.

3.4.1 Characteristics of the Standard Form

Following are the characteristics of Standard form of LPP.

(*i*) The objective function is of maximization type.
(*ii*) All constraints are expressed as equations.
(*iii*) Right hand side of each constraint is non-negative.
(*iv*) All variables are non-negative.

In *canonical form,* if the objective function is of maximization type, all the constraints other than non-negative conditions are '≤' type. If the objective function is of minimization type, all the constraints other than non-negative condition are '≥' type.

3.4.2 Characteristics of the Canonical Form

Following are the characteristics of Canonical form of LPP.

(*i*) The objective function is of maximization type.
(*ii*) All constraints are of (≤) type.
(*iii*) All variables x_i are non-negative.

Notes:

(*i*) Minimization of a function Z is equivalent to maximization of the negative expression of this function, i.e., Min Z = – Max (– Z)

(*ii*) An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).

(*iii*) Suppose we have the constraint equation,

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$

This equation can be replaced by two weak inequalities in opposite directions,

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$$

and, $$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1$$

(*iv*) If a variable is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, i.e., if x_1 is unrestricted in sign, then $x_1 = x_1' - x_1''$, where x_1', x_1'' are ≥ 0.

(*v*) In standard form, all the constraints are expressed in equation, which is possible by introducing some additional variables called 'slack variables' and 'surplus variables' so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that $b_i \geq 0$.

DEFINITIONS

(*i*) **Slack Variables:** If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i = 1, 2 \ldots m).$$

Then the non-negative variables S_i, which are introduced to convert the inequalities (≤) to the equalities,

$\sum_{j=1}^{n} a_{ij}x_j + S_i = b_i$ $(i = 1, 2 \ldots m)$ are called 'slack variables'.

Slack variables are also defined as the non-negative variables that are added in the LHS of the constraint to convert the inequality '≤' into an equation.

(*ii*) **Surplus Variables:** If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i \quad (i = 1, 2 \ldots m).$$

Then the non-negative variables S_i, which are introduced to convert the inequalities ≥ to the equalities

$\sum_{j=1}^{n} a_{ij}x_j - S_i = b_i$ $(i = 1, 2 \ldots m)$ are called surplus variables.

Surplus variables are defined as the non-negative variables that are removed from the LHS of the constraint to convert the inequality (≥) into an equation.

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