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PART - 1

1) (a) $f(x) = \frac{1}{\sqrt{|x| - x}}$

As $f(x)$ is real, $\frac{1}{\sqrt{|x| - x}}$ ~~with~~ is a real
the value of x must be such that
quantity.

$\therefore$ $|x| - x > 0 \Rightarrow |x| > x$

This inequality is satisfied for all values of $x < 0$.

$\therefore$ the domain of defination of $f(x)$ is -

$(-\infty, 0)$ or $-\infty < x < 0$.

(b) $f : [0, \infty) \rightarrow R$, $f(x) = \frac{2x+1}{3x+5}$

Let, x_1 and x_2 be any two points in $[0, \infty)$ such that $x_1 < x_2$.

we have to prove that $f(x_1) < f(x_2)$

Suppose, $f(x_1) \geq f(x_2)$

$$\Rightarrow \frac{2x_1+1}{3x_1+5} \geq \frac{2x_2+1}{3x_2+5}$$

$\Rightarrow (2x_1+1)(3x_2+5) \geq (2x_2+1)(3x_1+5)$

$\Rightarrow \cancel{6x_1x_2} + 10x_1 + 3x_2 + \cancel{5} \geq \cancel{6x_1x_2} + 10x_2 + 3x_1 + \cancel{5}$

$\Rightarrow 7x_1 \geq 7x_2$

$\Rightarrow x_1 \geq x_2$

So, it implise that if $f(x_1) \geq f(x_2)$, then $x_1 \geq x_2$. so, we can say that,

if $x_1 < x_2$, then, $f(x_1) < f(x_2)$.

so, $f(x)$ is a strictly monotonically increasing function.

3) (a) Let $f(x)$ be a function defined on $[a, b]$.

if,

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is derivable that means $f'(x)$ exist on (a, b)

(iii) $f(a) = f(b)$

~~There~~ Then, there must exist atleast one point c $(a<c<b)$ such that, $f'(c) = 0$

This is Rolle's theorem.

Let take an ~~intv~~ interval $[a, b]$.

$f(x) = 2x^3 - x^2 - 6x + 3$

$f'(x) = 6x^2 - 2x - 6$

(i) $f(x)$ is continious on $[a, b]$ as it is a Polynomial function.

(ii) $f'(x)$ Exist on (a, b)

(iii) $f(a) = \cancel{6a^2}\ 2a^3 - a^2 - 6a + 3$

$f(b) = 2b^3 - b^2 - 6b + 3$

$\therefore f(a) = f(b)$

$\therefore f(x)$ satisfies all three conditions, ~~at~~ so ~~Rolle's th~~ we should have,

$f'(c) = 0 \qquad [a < c < b]$

$\Rightarrow 6c^2 - 2c - 6 = 0$

$\Rightarrow 3c^2 - c - 3 = 0$

$c = \dfrac{1 \pm \sqrt{1 - 4 \cdot 3 \cdot (-3)}}{2 \cdot 3}$

$= \dfrac{1 \pm \sqrt{1 + 36}}{6}$

$= \dfrac{1 \pm \sqrt{37}}{6} \in (a, b)$

So, Rolle's theorem is verified.

(b) $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ $\forall$ $x \in [a, b]$.

with $0 < a < b$.

According to Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

~~Let $b = a + \theta h$ $0 < \theta < 1$~~

~~$\therefore$ $f(a + \theta h) - f(a)$~~

(i) $f(x)$ and $g(x)$ both are continuous on $[a, b]$,

(ii) both are derivable on (a, b) and $g'(x) \neq 0$ for any value of $x \in (a, b)$

So

$f'(x) = -\frac{2}{x^3}$ $g'(x) = -\frac{1}{x^2}$

$\therefore$ we should have,

$$\frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}} = \frac{2 \times \cancel{c^2}}{\cancel{c^3}\, c}$$

$$\Rightarrow \frac{a^2 - b^2}{\cancel{a^2}\cancel{b^2}} \times \frac{\cancel{ab}}{(a-b)} = \frac{2}{c}$$

$$\Rightarrow \frac{a+b}{ab} = \frac{2}{c}$$

$$\Rightarrow c = \frac{2ab}{a+b}$$

c is the harmonic mean.

1) (c) LHS $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

~~Let~~ $A = \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^{1/h^2}$

$$\Rightarrow \log A = \lim_{h \to 0} \frac{\log\left(\frac{\sin h}{h}\right)}{h^2}$$

$$= \lim_{h \to 0} \frac{\left[\frac{h}{\sin h} \cdot \frac{h\cos h - \sin h}{h^2}\right]}{2h}$$

[By L'Hospital rule]

$$= \lim_{h \to 0} \left[\frac{h}{\sin h} \cdot \frac{h\cos h - \sin h}{2h^3}\right]$$

$$= \lim_{h \to 0} \left[\frac{h\cos h - \sin h}{2h^2 \sin h}\right]$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\frac{-h\sin h}{2h\sin h + h^2\cos h}\right]$$

$$= \frac{1}{2} \lim_{h \to 0} \left(\frac{-\sin h}{2\sin h + h\cos h}\right)$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{-\cos h}{3\cos h - h\sin h}$$

$$= \frac{1}{2} \times \frac{(-1)}{3}$$

$\therefore \log A = \frac{-1}{6}$

$\therefore A = e^{-\frac{1}{6}}$ RHS (proved) $\therefore \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^{\frac{1}{h^2}} = e^{-\frac{1}{6}}$ (proved)

2) Continuous function - Let f(x) be a function. f(x) is called to be continuous on at x = a,

if $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$

$f(x) = \cos x$

Let $\epsilon > 0$ is given.

$\therefore |\cos x - \cos a| < \epsilon$

$\Rightarrow |2 \sin(\frac{x+a}{2}) \sin(\frac{x-a}{2})| < \epsilon$

~~$\Rightarrow |\sin(\frac{x+a}{2}) \sin(\frac{x-a}{2})| < \frac{\epsilon}{2}$~~

~~$\Rightarrow |x - a| < \delta$ [where, $\frac{\epsilon}{2} = \delta$]~~

~~$\therefore |\cos x - \cos a| < \epsilon$ whenever, $|x-a| < \delta$.~~
~~So f(x) is continuous.~~

$\Rightarrow |2| \, |\sin(\frac{x+a}{2})| \, |\sin(\frac{x-a}{2})| < \epsilon$

we know that $|\sin x| \leq 1 \quad \therefore \sin(\frac{x+a}{2}) \leq 1$

Now, $|\cos x - \cos a| \leq \cancel{2} \cdot 1 \cdot |\frac{x-a}{\cancel{2}}|$

$\Rightarrow |\cos x - \cos a| \leq x - a$

But $|x-a|<\delta$ and $\delta=\epsilon$

So, $|\cos x - \cos a| \le |x-a| < \delta$

$\therefore$ $|\cos x - \cos a| < \epsilon$ whenever $|x-a|<\delta$.

$\therefore$ $f(x) = \cos x$ is continuous function.

b) ~~Let~~ $y = \frac{\sin x}{\sqrt{1-x^2}}$

$\cancel{\Rightarrow (1-x^2)y^2 = \sin^2 x}$

$\cancel{\Rightarrow (1-x^2)}$

$\Rightarrow y_1 = \frac{\sqrt{1-x^2}\cos x - \sin x \cdot \frac{(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)}$

$\Rightarrow y_1 = \frac{\sqrt{1-x^2}\cos x - \frac{x\sin x}{\sqrt{1-x^2}}}{(1-x^2)}$

$\Rightarrow (1-x^2)y_1 = \sqrt{1-x^2}\cos x - xy$

$\Rightarrow (1-x^2)y_1 = \sqrt{1-x^2}\cdot\sqrt{1-\sin^2 x} - xy$

$\Rightarrow (1-x^2)y_1 = \sqrt{1-x^2}\cdot\sqrt{1-y^2(1-x^2)} - xy$

LHS

$$y = \frac{\sin n}{\sqrt{1-n^2}}$$

$$y_1 = \frac{1}{1-n^2} - \frac{1}{2}\cdot\frac{(-2n)\sin^2 n}{(1-n^2)^{3/2}}$$

$\Rightarrow (1-n^2)y_1 = 1+xy$

$\Rightarrow (1-n^2)y_2 - 2xy_1 = ny_1 - y$ [differentiating both sides w.r.t n]

$\Rightarrow (1-n^2)y_2 - 3xy_1 + y = 0$

Taking ~~n~~ the derivative of n-th order in both sides

$$y_{n+2}(1-n^2) - 2x\cdot n\cdot y_{n+1} - n(n-1)\cdot y_n - 3\{y_{n+1}\cdot x + ny_n\} + y_n = 0$$

$\Rightarrow y_{n+2}(1-n^2) - (2n+3)ny_{n+1} - (n-1)^2 y_n = 0$

RHS (Proved)

4) (a) $f(x,y) = xy$

LHS

$D_{\cancel{(a_1,a_2)}} \cancel{f(x,y)}_{1,2} =$

$$D_{(1,2)} f(a_1,a_2) = \lim_{t \to 0} \frac{f(a_1+t, a_2+2t) - f(a_1,a_2)}{t}$$

$$= \lim_{t \to 0} \frac{(a_1+t) \cdot (a_2+2t) - a_1 a_2}{t}$$

$$= \lim_{t \to 0} \frac{a_1 \cancel{a_2} + 2a_1 t + a_2 t + 2t^2 - \cancel{a_1 a_2}}{t}$$

$$= \lim_{t \to 0} \frac{t(2a_1 + a_2 + 2t)}{t}$$

$= a_2 + 2a_1$ RHS (proved)

(b) $f(x) = \begin{cases} \frac{x^2 y}{x^4+y^2} & , \text{if } (x,y) \neq 0 \\ 0 & , \text{if } (x,y) = 0 \end{cases}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4+y^2}$$

Put,
$y = mx^2$

$\therefore \lim_{x \to 0} \frac{x^2 \cdot mx^2}{x^4 + m^2 x^4}$

$= \lim_{x \to 0} \frac{m \cdot x^4}{x^4(1+m^2)}$

$= \lim_{x \to 0} \frac{m}{1+m^2}$

$= \frac{m}{1+m^2}$

this gives us different values ~~of~~ at different value of m. So, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4 + y^2}$ does n't exist.

But, $f(0,0) = 0$

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) \neq f(0,0)$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$ (Proved).

$$D_{(v_1,v_2)} f(0,0) = \lim_{t\to 0} \frac{f(v_1 t, v_2 t) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{v_1^2 t^2 \cdot v_2 t - 0}{(v_1^4 t^4 + v_2^2 t^2)\cdot t}$$

$$= \lim_{t\to 0} \frac{v_1^2 v_2 \cdot \cancel{t^3}}{\cancel{t^2}(v_1^4 t^2 + v_2^2)\cdot \cancel{t}}$$

$$= \lim_{t\to 0} \frac{v_1^2 v_2}{v_1^4 t^2 + v_2^2}$$

$$= \frac{v_1^2 \cancel{v_2}}{\cancel{v_2^2}\, v_2}$$

$$= \frac{v_1^2}{v_2}$$

$\therefore$ ~~f(x,y)~~ $f(x,y)$ has a directional derivative at $(0,0)$ in any direction (v_1, v_2). (Proved)

c) Let,

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{where } x^2+y^2 \neq 0 \\ 0 & \text{where } x^2+y^2 = 0 \end{cases}$$

$$f_x(x,y) = \frac{(x^2+y^2)y - xy \cdot 2x}{(x^2+y^2)^2}$$

$f_x(0,0) = 0$ [$\because$ when, $x^2+y^2=0$, then $f(x,y)=0$]

$$f_y(x,y) = \frac{(x^2+y^2)x - xy \cdot 2y}{(x^2+y^2)^2}$$

$f_y(0,0) = 0$ [$\because$ when, $x^2+y^2=0$, then $f(x,y)=0$]

$\therefore$ $f_x(0,0)$ and $f_y(0,0)$ exist (Proved).

$$D_{(1,1)} f(0,0) = \lim_{t \to 0} \frac{f(0+t, 0+t) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\frac{t \cdot t}{t^2+t^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{t^2}}{2\cancel{t^2} \cdot t}$$

$\lim_{t \to 0} \frac{1}{2t}$ doesn't exist.

$\therefore$ The directional derivative doesn't exist at $(0,0)$ in the direction $(1,1)$.

PART-B

5) (a) Let, $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x+\cos x} dx$ —(i)

$I = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)\, dx}{1+\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin x+\cos x}$ —(ii)

By adding (i) and (ii), we get-

$2I = \int_0^{\frac{\pi}{2}} \frac{(1+\sin x+\cos x)-1}{1+\sin x+\cos x} dx$

$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$

$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{(1+\tan^2\frac{x}{2})}{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}} dx$

$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{1+\tan\frac{x}{2}} dx$

$$= \int_0^{\frac{\pi}{2}} dx - \int_0^1 \frac{dz}{1+z}$$

put,
$\tan\frac{x}{2} = z$
$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dz$

x	0	$\frac{\pi}{2}$
z	0	1

$$= [x]_0^{\frac{\pi}{2}} - [\log(1+z)]_0^1$$

$$= \frac{\pi}{2} - \log 2$$

$$\therefore I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x+\cos x}dx = \frac{\pi}{4} - \frac{1}{2}\log 2$$

(b) Let, $I = \int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}dx$ — (i)

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)dx}{\sin x+\cos x} \quad \text{— (ii)}$$

adding (i) and (ii) we get,

$$2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+\cos x}$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{(1+\tan^2\frac{x}{2})}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$= \frac{\Delta\pi}{4}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$$

put
$\tan\frac{x}{2} = z$
$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

x	0	$\frac{\pi}{2}$
z	0	1

$$= \pi\int_0^1 \frac{dz}{2z+1-z^2}$$

$$= \pi\int_0^1 \frac{dz}{(\sqrt{2})^2-(z-1)^2}$$

$-\{z^2-2z-1\}$
$= -\{z^2-2\cdot z\cdot 1+(1)^2-2\}$
$= (\sqrt{2})^2-(z-1)^2$

$$= \pi\cdot\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right|\right]_0^1$$

$$= \frac{\pi}{2\sqrt{2}}\left[-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$

$$= \frac{\pi}{\sqrt{2}}\log\sqrt{\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}}$$

$$= \frac{\pi}{\sqrt{2}}\log\sqrt{\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1}}$$

$$= \frac{\pi}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\therefore I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

$\therefore \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$

7) LHS

$\int_a^b (x-a)^m (b-x)^n dx$

$= \int_0^{b-a} z^m (b-a-z)^n dz$

put, $x-a = z$

$\Rightarrow dx = dz$

x	a	b
z	0	$b-a$

$= \int_0^{b-a} z^m \{(b-a)-z\}^n dz$

$= (b-a)^n \int_0^{b-a} z^m \left\{1 - \frac{z}{b-a}\right\}^n dz$

~~$= (b-a)^n \int_0^1 z^m (1-t)^n \cdot b-$~~

put, $\frac{z}{b-a} = t$

$\Rightarrow dz = (b-a) dt$

$= (b-a)^n \int_0^1 (b-a)^m \cdot t^m (1-t)^n (b-a) dt$

z	0	$b-a$
t	0	1

$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt$

$= (b-a)^{m+n+1} \int_0^1 t^{\{(m+1)-1\}} (1-t)^{\{(n+1)-1\}} dt$

$= (b-a)^{m+n+1} B(m+1, n+1)$ —(1)

from 1 we can say,

$$\int_a^b (x-a)^3 (b-x)^2 \, dx$$

$$= (b-a)^{3+2+1} \cdot B(4,3)$$

$$= (b-a)^6 \cdot \frac{\Gamma(4) \cdot \Gamma(3)}{\Gamma(7)}$$

$$= (b-a)^6 \cdot \frac{3! \times 2!}{6!}$$

$$= (b-a)^6 \frac{\cancel{3} \times \cancel{2} \times 2}{\cancel{6}_3 \times 5 \times 4 \times \cancel{3} \times \cancel{2}}$$

$$= \frac{(b-a)^6}{60} \text{ RHS (proved)}$$

(b) LHS $B(n,m) = \dfrac{(n-1)!}{m(m+1)(m+2) \cdots (m+n-1)}$

$$= \frac{(n-1)!\,(m-1)!}{m(m+1)(m+2) \cdots (m+n-1)(m-1)!}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1) \cdot (m+n-2) \cdots (m+2)(m+1) \cdot m \cdot (m-1)!}$$

$= \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$ R.H.S (m+n)

(c) $B(\frac{5}{6}, 6) = B(6, \frac{5}{6})$ [from, $B(m,n) = B(n,m)$]

$= \dfrac{(6-1)!}{\frac{5}{6}(\frac{5}{6}+1)(\frac{5}{6}+2)(\frac{5}{6}+3)(\frac{5}{6}+4)(\frac{5}{6}+5)}$ $\cancel{(\frac{5}{6}+6}$

$= \dfrac{5! \times 6^6}{5 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 35}$

6) (b) $\int_1^2 \frac{x}{\sqrt{2-x}}\,dx$

this improper integral has a singularity point and that is $= 2$

$\therefore \int_1^2 \frac{x}{\sqrt{2-x}}\,dx$

$= \lim_{t \to 0^+} \int_1^{2-t} \frac{x}{\sqrt{2-x}}\,dx$

$= \lim_{t \to 0^+} \int_1^{\sqrt{t}} \frac{(2-z^2)(-2\cancel{z})}{\cancel{z}}\,dz$ put $2 - x = z^2$
$\Rightarrow -dx = 2z\,dz$

x	1	2-t
z	1	$\sqrt{t}$

$$= -2 \lim_{t \to 0^+} \int_1^{\sqrt{t}} (2 - z^2)\, dz$$

$$= -2 \lim_{t \to 0^+} \left[2z - \frac{z^3}{3} \right]_1^{\sqrt{t}}$$

$$= -2 \lim_{t \to 0^+} \left[\left(2\sqrt{t} - \frac{t\sqrt{t}}{3} \right) - \left(2 - \frac{1}{3} \right) \right]$$

$$= -2 \lim_{t \to 0^+} \left[\left(2\sqrt{t} - \frac{t\sqrt{t}}{3} \right) - \frac{5}{3} \right]$$

$$= -2 \times \left(-\frac{5}{3}\right) \quad \left[\because \lim_{t \to 0^+} \left(2\sqrt{t} - \frac{t\sqrt{t}}{3}\right) = 0 \right]$$

$$= \frac{10}{3}$$

$\therefore$ ~~the value of the~~ this integral is convergent and its value $= \frac{10}{3}$

(a) $\int_0^{\pi/2} \log(\sin x)\, dx$

This integral has a singularity point at end that is $= 0$

$\therefore \int_0^{\pi/2} \log(\sin x)\, dx$

$$= \lim_{t \to 0^+} \int_{0+t}^{\pi/2} \log(\sin x)\, dx$$

$$= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\frac{\pi}{2}} \log(\sin x)\, dx$$

$$= \lim_{\epsilon \to 0^+} \left[\left\{ x \log(\sin x) \right\}_{\epsilon}^{\frac{\pi}{2}} - \int_{\epsilon}^{\frac{\pi}{2}} \frac{x \cdot \cos x}{\sin x}\, dx \right]$$

$$= \lim_{\epsilon \to 0^+} \left[\left\{ x \log(\sin x) \right\}_{\epsilon}^{\frac{\pi}{2}} \right] \left[\because \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\frac{\pi}{2}} \frac{x \cos x}{\sin x}\, dx = 0 \right]$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{\pi}{2} \cancel{\log} \cdot 0 - \epsilon \log(\sin \epsilon) \right]$$

$\lim_{\epsilon \to 0^+} [-\epsilon \log(\sin \epsilon)]$ ~~is~~ does n't exist. so this integral is not convergent.

PART-C

8) (a) Gibbs Phenomemon - It States that the peculiar manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity. The uniform limit of continuous function is continuous, and so the Fourier series of a function cannot converge uniformly where the function is discontinuous. Gibbes phenomenon is ~~usa~~ usually demonstrated with examples that have a single discontinuity ~~and~~ at the end of their period, such as a square wave or a saw tooth wave.

E.g -

Gibbes formula - $\frac{f(x_0+) + f(x_0-)}{2}$

(b) Consider the function,

$$f(x) = \begin{cases} \cos x & , 0 < x < \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

~~Th~~ we extand this function by defining $f(x+\frac{\pi}{2}) = f(x)$. This is a periodic function of period $\frac{\pi}{2}$. The Fourier coefficients are-

$$a_0 = \frac{1}{\left(\frac{\pi}{4}\right)} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\,dx$$

$$= \cancel{\frac{4}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [\,/\,}$$

$$= \frac{1}{\left(\frac{\pi}{4}\right)} \cdot \int_0^{\frac{\pi}{4}} f(x)\,dx \quad [\because f(x) = 0, \text{ when } \frac{\pi}{4} < x < \frac{\pi}{2}]$$

$$= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

$$= \frac{4}{\pi \times 2} [\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \cancel{\frac{\pi}{8}(1+0)} = \frac{2}{\pi} \times 1$$

$$= \cancel{\frac{\pi}{8}} \qquad = \frac{2}{\pi}$$

$$a_n = \frac{1}{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x) \cos nx\,dx$$

$$= \frac{4}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \cos nx\,dx$$

$= \frac{4}{\pi} \cdot 2 \int_0^{\frac{\pi}{4}} \cos 2x \cos nx \, dx$ [$\because \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \cos nx \, dx$ is an even function]

$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} 2 \cos 2x \cos nx \, dx$

$= \cancel{\frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos\left(\frac{2x+nx}{2}\right) \cos\left(\frac{2x-nx}{2}\right) dx}$

$= \cancel{\frac{4}{\pi} \left[\sin\left(\frac{nx+2x}{2}\right) \right.}$

$= \cancel{\frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos\left(\frac{n+2}{2}\right)x \cos}$

$= \cancel{\frac{4}{\pi} \cdot \sin \frac{(n+2)x}{2}}$

$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \{\cos(n+2)x + \cos(n-2)x\} \, dx$

$= \frac{4}{\pi} \left[\frac{\sin(n+2)x}{n+2} + \frac{\sin(n-2)x}{n-2} \right]_0^{\frac{\pi}{4}}$

$= \frac{4}{\pi} \left[\frac{\sin(n+2)\frac{\pi}{4}}{n+2} + \frac{\sin(n-2)\frac{\pi}{4}}{n-2} \right]$

$b_n = \frac{4}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \sin nx \, dx = 0$ [$\because \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x \sin nx$ is an odd function]

$\therefore$ the Fourier series is-

$$\frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi} \left[\frac{\sin(n+2)\frac{\pi}{4}}{n+2} + \frac{\sin(n-2)\frac{\pi}{4}}{n-2} \right] \cos nx$$

9)(a) Consider the function,

$$f(x) = x + x^2$$

we extend this function by defining $f(x+2\pi) = f(x)$. This is a periodic function of period 2π.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2\,dx$$

$$= \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} x + (x)^2}\ \int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_{0}^{\pi} x^2\,dx\right] \left[\because \int_{-\pi}^{\pi} x\,dx \text{ is an odd and } \int_{-\pi}^{\pi} x^2\,dx \text{ is an even function}\right]$$

$$= \frac{2}{\pi}\cdot\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\cancel{\pi}}\cdot \cancel{\pi^3}\pi^2$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_{0}^{\pi} x^2\cos nx\,dx\right]$$ $\left[\because \int_{-\pi}^{\pi} x\cos nx\right.$ is an odd and $\int_{-\pi}^{\pi} x^2\cos nx$ is an even function$\left.\right]$

$$= \frac{2}{\pi}\left[\left\{x^2 \cdot \frac{\sin nx}{n}\right\}_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{n}\left\{-\frac{x\cos nx}{n}\right\}_0^{\pi} + \frac{1}{n^2}\left\{\sin nx\right\}_0^{\pi}\right]$$

$$= -\frac{4}{n\pi}\left(-\frac{\pi\cos n\pi}{n}\right)$$

$$= \frac{4\cos n\pi}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,dx + \int_{-\pi}^{\pi} x^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx + 0\right]$$ $\left[\because \int_{-\pi}^{\pi} x\sin nx\right.$ is an even and $\int_{-\pi}^{\pi} x^2\sin nx$ is an odd function$\left.\right]$

$$= \frac{2}{\pi}\left[\left\{-\frac{x\cos nx}{n}\right\}_0^{\pi} + \frac{1}{n^2}\left\{\sin nx\right\}_0^{\pi}\right]$$

$$= -\frac{2}{\pi}\cdot\frac{\pi\cos n\pi}{n} \;;\; = -\frac{2\cos n\pi}{n}$$

∴ the fourier series is —

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\Rightarrow \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi \cos nx}{n^2} - \frac{2 \cos n\pi \sin nx}{n}$$

its also satisfy Dirichlet's condition, It is define on $[-\pi, \pi]$.

$$\therefore x^2 + x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi \cos nx}{n^2} - \frac{2\cos n\pi \sin nx}{n}$$

putting $x=0$ in both side

$$\Rightarrow 0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi \cdot 1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \cos n\pi = 0$$

$$\Rightarrow \frac{\pi^2}{3} + {}^{2}\cancel{4} \cdot \frac{\pi}{{}_{3}\cancel{6}} \sum_{n=1}^{\infty} \cos n\pi = 0 \quad \left[\because \text{given } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}\right]$$

$$\Rightarrow \frac{\pi^2}{3} + \frac{2\pi}{3} \sum_{n=1}^{\infty} \cos n\pi = 0$$

(b) $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\oint_S \vec{F}\cdot\hat{n}\,ds = \int_V \nabla\vec{F}\,\tau$$

This is Gauss divergence theorem

10) (b) $f(x,y,z) = x^2 - y^2 + 2z^2$

$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

at $P(1,2,3)$ ∇f,

$\Rightarrow \nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$\vec{RS} = (-\hat{i} + 3\hat{j} + 7\hat{k}) - (3\hat{i} - 4\hat{j} + 5\hat{k})$

$= -4\hat{i} + 10\hat{j} + 2\hat{k}$

Let $\vec{a} = -4\hat{i} + 10\hat{j} + 2\hat{k}$

$\hat{a} = \dfrac{-4\hat{i} + 10\hat{j} + 2\hat{k}}{\sqrt{16+100+4}}$

$= \dfrac{-4\hat{i} + 10\hat{j} + 2\hat{k}}{\sqrt{120}}$

$40 \times 3 = 0$
$= 2\sqrt{30}$

$= \dfrac{-4\hat{i} + 10\hat{j} + 2\hat{k}}{2\sqrt{30}}$

$= \dfrac{-2\hat{i} + 5\hat{j} + \hat{k}}{\sqrt{30}}$

$\therefore$ the directional derivative of f in the direction $\hat{a}$ is $= \cancel{\nabla f}$, $\nabla f \cdot \hat{a} = \dfrac{(2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot (-2\hat{i} + 5\hat{j} + \hat{k})}{\sqrt{30}}$

$= \dfrac{-4 - 20 + 12}{\sqrt{30}}$

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$= -\frac{12}{\sqrt{30}}$

$\therefore$ the directional derivative of the function is

$= -\frac{12}{\sqrt{30}}$

the directional derivative of the function is the greatest rate of increase of $f(x,y,z)$ at R.

a) LHS A $\text{curl } \vec{A}\times\vec{B} = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\nabla\cdot\vec{B})$

$\text{curl}(\vec{A}\times\vec{B}) = \nabla\times(\vec{A}\times\vec{B})$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\hat{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})\right\}$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\hat{i}\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}\vec{B}\right)\right\}$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right\}$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A} - (\hat{i}\cdot\vec{A})\frac{\partial\vec{B}}{\partial x}\right\}$

$+ \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{(\hat{i}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}\right\}$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\vec{A}\right)\right\} - \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

$\left\{(\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}\right\} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).$

$\left\{(\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x}\right\} - \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

$\left\{\left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\}$

$= (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B}$

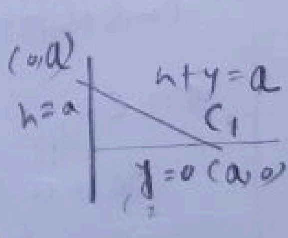
R.H.S (proved)

11)(b) Green's theorem =

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint M\,dx + N\,dy$

$\vec{F} = (3x^2 - 8y^2)\hat{i} + (2y - 3xy)\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$



$\vec{F} \cdot d\vec{r} = (3x^2 - 8y^2)\,dx + (2y - 3xy)\,dy$

$M\,dx + N\,dy = (3x^2 - 8y^2)\,dx + (2y - 3xy)\,dy$

$\int_0^a \int_0^{a-y} (-3y + 16y)\,dx\,dy$

$= \int_0^a \int_0^{a-y} 13y\,dx\,dy$

$$= 13\int_0^a \int_0^{a-x} y\, dx\, dy$$

$$= \frac{13}{2}\int_0^a \left[y^2\right]_0^{a-x} dx$$

$$= \frac{13}{2}\int_0^a \left[(a-x)^2\right] dx$$

$$= \frac{13}{2}\int_0^a (a^2-2ax+x^2)\, dx$$

$$= \frac{13}{2}\left[a^2x - \frac{\cancel{2}ax^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^a$$

$$= \frac{13}{2}\left[\cancel{a^3} - \cancel{a^3} + \frac{a^3}{3}\right]$$

$$= \frac{13a^3}{6}$$

Let us evaluate the line integral $\oint M dx + N dy$ on closed C. C is the piecewise smooth curve consisting, C_1, ~~C_2, C_3~~.

On C_1,

$\cancel{x = a}$ $\cancel{dx = 0}$

$\cancel{\therefore \int_{C_1} M dx + N dy = (3x^2 - 8y^2}$

$y = a - x$, $dy = -dx$

$$\int_C M dx + N dy = \{38x^2 - 8(a-x)^2\} dx - \{2(a-x) - 3x(a-x)\} dx$$

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$$\int_C M dx + N dy = \{3x^2 - 8(a^2 - 2ax + x^2)\} dx -$$

$$\{2a - 2x - 3(ax - x^2)\} dx$$

$$= \{3x^2 - 8a^2 + 16ax - 8x^2\} dx -$$

$$\{2a - 2x - 3ax + 3x^2\} dx$$

$$= \{-5x^2 - 8a^2 + 16ax - 2a + 2x + 3ax - 3x^2\} dx$$

$$= \{-8x^2 - 8a^2 + 19ax - 2a + 2x\} dx$$

C_1 varies from 0 to a

$$\therefore \int_0^a (-8x^2 - 8a^2 + 19ax - 2a + 2x) dx$$

$$= \left[-\frac{8x^3}{3} - 8a^2 x + \frac{19ax^2}{2} - 2ax + \cancel{2}a\frac{x^2}{\cancel{2}}\right]_0^a$$

$$= \left[-\frac{8a^3}{3} - 8a^3 + \frac{19a^3}{2} - 2a^3\right]$$

$$= \cancel{13a^3} \; \frac{13a^3}{6}$$

$$\therefore \oint_C M dx + N dy = \int_C M dx + N dy$$

$$= \frac{13a^3}{6}$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy = \frac{13a^3}{6} \oint M dx + N dy$$

$\therefore$ Hence Green's theorem is verified.

(a) To verify the Stoke's theorem,

i.e $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \int_C \vec{F} \cdot d\vec{r}$

Now $\int_C \vec{F} \cdot dr$ RHS

$= \int_C \{(z^2 - y)\hat{i} + (x - 2yz)\hat{j} + (2xz - y^2)\hat{k}]$

$\{dx\hat{i} + dy\hat{j} + dz\hat{k}\}$

Here, $z = 0$, and $dz = 0$

$= \int_C (-y\,dx + x\,dy)$

Now, Converting into Polar coordinates, we get,

$x = 3\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta \leq 2\pi$

$= \int_0^{2\pi} (9\sin^2\theta + 9\cos^2\theta)\, d\theta$

$= 9 \int_0^{2\pi} d\theta$

$= 9 \times 2\pi$

$= 18\pi$

Now, $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ LHS

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y & x - 2xyz & 2xz - y^2 \end{vmatrix}$$

$$= (-2y + 2y)\hat{i} + (2z - 2z)\hat{j} + (1+1)\hat{k}$$

$$= 2\hat{k}$$

Now $(\nabla \times F) \cdot \hat{n} = (2\hat{k}) \cdot \hat{k}$

Now, $\iint_S (\nabla \times \vec{F}) \hat{n}\, dS = \iint 2\, dx\, dy$

$$= 2 \iint dx\, dy$$

$$= 2 \times 3^2 \times \pi$$

$$= 18\pi$$

Thus Stokes's theorem is verified.

$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$ R.H.S (proved)

(c) $B(\frac{5}{6}, 6) = B(6, \frac{5}{6})$ [from, $B(m,n) = B(n,m)$]

$= \frac{(6-1)!}{\frac{5}{6}(\frac{5}{6}+1)(\frac{5}{6}+2)(\frac{5}{6}+3)(\frac{5}{6}+4)(\frac{5}{6}+5)}$ ~~$(\frac{5}{6}+6$~~

$= \frac{5! \times 6^6}{5 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 35}$

The value is $= \frac{5! \times 6^6}{5 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 35}$

6) (b) $\int_1^2 \frac{x}{\sqrt{2-x}}\,dx$

this improper integral has a singularity point and that is $= 2$

$\therefore \int_1^2 \frac{x}{\sqrt{2-x}}\,dx$

$= \lim_{t \to 0^+} \int_1^{2-t} \frac{x}{\sqrt{2-x}}\,dx$

$= \lim_{t \to 0^+} \int_1^{\sqrt{t}} \frac{(2-z^2)(-2z)}{z}\,dz$

put $2 - x = z^2$

$\Rightarrow -dx = 2z\,dz$