22/3/22

Matrix (m×n) :

Defination - of A matrix is a rectangular of mn quantites a_{ij} $\begin{pmatrix} i=1 \dots m \\ j=1 \dots, n \end{pmatrix}$ in m rows and n coulmns

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Row matrix - $[a_{11} \quad a_{12} \quad a_{13}]$

Coulmn matrix - $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$ upper triangular matrix

lower triangular matrix

Addition of a matrix -

$A+B =$

$A = \begin{bmatrix} 0 & 2 \\ 5 & 6 \\ 8 & -1 \end{bmatrix}$ $B \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 4 \\ 9 & 7 \\ 8 & 0 \end{bmatrix}$

multiplication of matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix}$

AB not possible

BA possible

matrix multiplication is not commutative in general

Sol.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Find AB and BA and show $(AB \neq BA)$

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+4+3 & 1+2+6 & 1+6+9 \\ 4+10+6 & 4+5+12 & 4+15+18 \\ 7+16+9 & 7+8+18 & 7+24+27 \end{bmatrix}$

$= \begin{bmatrix} 8 & 9 & 16 \\ 20 & 21 & 37 \\ 32 & 33 & 58 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$= \begin{bmatrix} 1+8+7 & 2+10+8 & 3+12+9 \\ 2+4+21 & 4+5+24 & 6+6+27 \\ 1+8+21 & 2+10+24 & 3+12+27 \end{bmatrix}$

$= \begin{bmatrix} 10 & 75 & 78 \\ 27 & 33 & 39 \\ 30 & 36 & 42 \end{bmatrix}$

So, $AB \neq BA$ (Proved)

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 2 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 6I$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ → Transpose

$A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix}$

Symmetric matrix — $A = [a_{ij}]$, $a_{ij} = a_{ji}$

A^T $A = A^T$

Skew symmetric — $A = -A^T$

For diagonal matrix → $a_{kk} = -a_{kk}$

$2a_{kk} = 0$, $a_{kk} = 0$

Sm.

(2) Select the symmetric matrix, skew-symmetric matrix from the following —

$\begin{bmatrix} 0 & -2 & 4 \\ -3 & 0 & -4 \\ 4 & & \end{bmatrix}$, $\begin{bmatrix} 0 & -2 & 8 \\ 2 & 0 & \end{bmatrix}$, $\begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$

skew symmetric symmetric

(3) give the ex of a matrix which is neither symmetric nor skew symmetric matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Thm
Result I: Product of any matrix with is transpose is any matrix.

Sol: Let A be an $m \times n$ matrix and $X = AA^T$, hence A^T is a $n \times m$ matrix

Now, $X^T = (AA^T)^T = (A^T)^T A = AA^T$. Thus X is symmetric.

Result - II, Any square matrix can be expressed as the sum of a symmetric matrix and skew symmetric matrix:

Proof:- Let A be a square matrix and

$X = \frac{1}{2}(A + A^T)$

$X^T = \frac{1}{2}[(A + A^T)^T] = \frac{1}{2}(A^T + A) = X$

Thus X is symmetric

Let $Y = \frac{1}{2}(A - A^T)$

$Y^T = \frac{1}{2}[(A - A^T)^T] = \frac{1}{2}[A^T - A]$

$= -\frac{1}{2}[A - A^T] = -Y$

Thus Y is skew symmetric and $A = X + Y$

(Proved)

Qn

(1) Express $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrix

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

$= \begin{bmatrix} 2 & 5 & 8 \\ 5 & 8 & 11 \\ 8 & 11 & 14 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

the ans is—

$$\therefore \frac{1}{2}\begin{bmatrix} 2 & 0 & 8 \\ 5 & 8 & 11 \\ 8 & 11 & 14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 & -2 \\ 10 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Qn-2 Express $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrix

Let, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A^T$

$$A + A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$A - A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$= 0$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Adjoint matrix—

Let $A = [a_{ij}]$ is a square matrix and B is matrix whose elements are cofactors of the corresponding element in $|A|$, then the transpose of B is called the adjoint or adjugate matrix of A and we got as $[A_{ij}]$

(1) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ Find A^{-1}.

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$= 1(16-15) + 1(9-8)$

$= 1 + 1$

$= 2 \neq 0 \quad \therefore A^{-1}$ exists

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (16-10) & -(12-10) & (2-3) \\ -(0-3) & (4-2) & -(3-0) \\ (0-4) & -(5-3) & (4-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{|A|}{adj A} = \frac{1}{2}\begin{bmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

Differential equation -

1st order and first degree equation -

1) exact equation
2) method of separation of variables
3) Homogeneous eqn
4) Linear eqn

1) Exact eqn - The general form of 1st order and first degree eqn is $M_{(x,y)}dx + N_{(x,y)}dy = 0$

$Mdx + Ndy = 0$

the equation $Mdx + Ndy = 0$ is called exact, if there exist a function $u(x,y)$ such that $du = Mdx + Ndy$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Mdx + Ndy$$

$$M = \frac{\partial u}{\partial x} \text{ -(1)} \quad N = \frac{\partial u}{\partial y} \text{ -(2)}$$

Diff (1) partialy w.r.t y and (2) w.r.t x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}} \qquad \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

→ necessary and sufficient condition for an eqn to be exact

① Evaluate $\int m\,dx$ (take y as constant)

$\int n\,dy$ (take x as constant)

Solⁿ (Common terms of both the integrals) + the terms not in common = Constant

Ex① solve $(4x^3y^3-2xy)dx+(3x^4y^2-x^2)dy=0$

$m\,dx+n\,dy=0$

Here, $m=4x^3y^3-2xy$, $n=3x^4y^2-x^2$

$\frac{\partial m}{\partial y}=12x^3y^2-2x \qquad \frac{\partial n}{\partial x}=12x^3y^2-2x$

$\boxed{\frac{\partial m}{\partial y}=\frac{\partial n}{\partial x}}$ so the given eqn is exact.

$\int m\,dx=\int(4x^3y^3-2xy)dx$

$=4\frac{x^4}{4}y^3-2\frac{x^2}{2}y$

$=x^4y^3-x^2y$

$\int n\,dy=$ ~~$\int(12x^3y^2-2x)d$~~

$\int(3x^4y^2-x^2)dy$

$=3\frac{x^4y^3}{3}-x^2y$

$=x^4y^3-x^2y$

The Solution is $x^4y^3-x^2y=$ Constant $=c$

② Solve: $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

$m\,dx+n\,dy$

$m=y^2e^{xy^2}+4x^3$, $n=2xye^{xy^2}-3y^2$

$\frac{\partial m}{\partial y}=2ye^{xy^2}+y^2e^{xy^2}\cdot x2y$

$=2ye^{xy^2}+2xy^3e^{xy^2}=e^{xy^2}(2y+2xy^3)$

$\frac{\partial n}{\partial x}=2y\{(e^{xy^2}\cdot x)+e^{xy^2}[(y^2)]\}=2ye^{xy^2}+2xy^3e^{xy^2}+0$

$=e^{xy^2}(2y+2xy^3)$

$\boxed{\frac{\partial m}{\partial y}=\frac{\partial n}{\partial x}}$ So the given eqn is constant

$\int m\,dx=\int(y^2e^{xy^2}+4x^3)dx$

$=y^2\frac{e^{xy^2}}{y^2}+4\frac{x^4}{4}$

$=y^2e^{xy^2}+x^4$

$\int n\,dy=\int(2xye^{xy^2}-3y^2)dy$ put $xy^2=t$

$\Rightarrow x\cdot 2y\,dy+y^2\cdot 0=dt$

$=\int e^t\,dt-\int 3y^2dy$ $2xy\,dy$

$=e^t-y^3$

$=e^{xy^2}-y^3$

$\therefore$ Sol is $=e^{xy^2}+x^4-y^3=c$

③ (i) solve $(3xy^2 - x^2)dx + (1+6y^2+3x^2y)dy = 0$

$Mdx + Ndy$

$M = 3xy^2 - x^2$, $N = 1+6y^2+3x^2y$

$\Rightarrow \frac{\partial M}{\partial y} = 6xy$ $\frac{\partial N}{\partial x} = 6xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given eqn is exact.

$\int M dx = \int (3xy^2 - x^2) dx$

$= 3\frac{x^2}{2} \cdot y^2 - \frac{x^3}{3}$

$= \frac{3x^2y^2}{2} - \frac{x^3}{3}$

$\int N dy = \int (1+6y^2+3x^2y) dy$

$= y + \cancel{6}^2\frac{y^3}{\cancel{3}} + 3x^2\frac{y^2}{2}$

$= \frac{3x^2y^2}{2} + 2y^3 + y$

$\therefore$ Soln is $= \frac{3x^2y^2}{2} - \frac{x^3}{3} + 2y^3 + y = c$

(ii) show that the equation $(x^3 - 3x^2y + 2xy^2)dx$

$- (x^3 - 2x^2y + y^3) dy = 0$

is exact and find the solution if $y = 1, x = 1$

$M = x^3 - 3x^2y + 2xy^2$ $N = -x^3 + 2x^2y - y^3$

$\frac{\partial M}{\partial y} = -3x^2 + 4xy$ $\frac{\partial N}{\partial x} = -3x^2 + 4x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the eqn is exact

$\int M dx = \int (x^3 - 3x^2y + 2xy^2) dx$

$= \frac{x^4}{4} - \cancel{3}\frac{x^3}{\cancel{3}} \cdot y + \cancel{2}\frac{x^2}{\cancel{2}}y^2$

$= \frac{x^4}{4} - x^3y + x^2y^2$

$\int N dy = \int (-x^3 + 2x^2y - y^3) dy$

$= -x^3y + \cancel{2}x^2\frac{y^2}{\cancel{2}} - \frac{y^4}{4}$

$= -x^3y + x^2y^2 - \frac{y^4}{4}$

The solution is $= (-x^3y + x^2y^2 + \frac{x^4}{4} - \frac{y^4}{4}) = c$

if $x = y = 1$

$-1\cdot1 + 1\cdot1 + \frac{1}{4} - \frac{1}{4} = c$

$\therefore c = 0$

Soln is $-x^3y + x^2y^2 + \frac{x^4}{4} - \frac{y^4}{4} = 0$

(2) method of seperation of variables-

$M(x,y)dx + N(x,y)dy = 0$

$\Rightarrow \int f_1(x)dx + \int f_2(y)dy = 0$

$\Rightarrow \phi_1(x) + \phi_2(y) = c$

Ex - (1) solve $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow \int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$

$\Rightarrow \log|\tan x| = -\log|\tan y| + \log c$

$\Rightarrow \log|\tan x \tan y| = \log c$

$\Rightarrow |\tan x \tan y| = c$

(2) $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

$\Rightarrow 3e^x \tan y \, dx = -(1-e^x)\sec^2 y \, dy$

$\Rightarrow -\int \frac{3e^x \, dx}{-(1-e^x)} = -\int \frac{\sec^2 y \, dy}{\tan y}$

$\Rightarrow -3\log|1-e^x| = -\log|\tan y| + \log c$

$\Rightarrow \log|\tan y| - \log|1-e^x|^3 = \log c$

$\Rightarrow \frac{\tan y}{|1-e^x|^3} = c$

Homogeneous equation - A function $f(x,y)$ is called a homogeneous function of degree n, if $f(x,y)$ can be written as $x^n \phi(y/x)$ or $y^n \phi(x/y)$

$M dx + N dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$

If both $M(x,y)$ and $N(x,y)$ are homogeneous of same degree then the equation $Mdx + Ndy = 0$ is called homogeneous.

$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = \frac{-x^n \phi_1(y/x)}{x^n \phi_2(y/x)}$

$= -f(y/x)$

Put $y = vx$, where v is a function of x

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = -f(v)$

$\Rightarrow x\frac{dv}{dx} = -f(v)$

$\Rightarrow \int \frac{dv}{v+f(v)} = -\int \frac{1}{x} dx$

$\Rightarrow \int \frac{dv}{v+f(v)} + \int \frac{1}{x} dx = 0$

Ex-① : Solve $\Rightarrow (y^4 - 2x^3y)\,dx + (x^4 - 2xy^3)\,dy = 0$

$$(y^4 - 2x^3y)\,dx + (x^4 - 2xy^3)\,dy = 0$$

$$\Rightarrow (x^4 - 2xy^3)\,dy = -(y^4 - 2x^3y)\,dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^4 - 2x^3y}{x^4 - 2xy^3}$$

$y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{v^4x^4 - 2x^3\cdot vx}{x^4 - 2x\cdot v^3x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{v^4x^4 - 2x^4v}{x^4 - 2x^4v^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{\cancel{x^4}(v^4 - 2v)}{\cancel{x^4}(1 - 2v^3)}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{(v^4 - 2v)}{1 - 2v^3} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(v^4 - 2v) - v(1 - 2v^3)}{1 - 2v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4 + 2v - v + 2v^4}{1 - 2v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^4}{1 - 2v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^4 + v}{1 - 2v^3}$$

$$\Rightarrow \int\frac{1 - 2v^3}{v(1 + v^3)}\,dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{1 + v^3 - 3v^3}{v(1 + v^3)}\,dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{1 - 2v^3}{v^4}\,dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow \int\left(\frac{1}{v^4} - \frac{2}{v}\right)dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{3v^3} - 2\log v = \log x + c$$

$$\Rightarrow -\frac{x^3}{3y^3} - 2\log\left(\frac{y}{x}\right) - \log x = c$$

$$\Rightarrow \frac{x^3}{3y^3} + \log\left(\frac{y}{x}\right)^2 + \log x = c$$

$$\Rightarrow \frac{x^3}{3y^3} + \log\left(\frac{y^2}{x^2}\cdot x\right) = c$$

$$\Rightarrow \frac{x^3}{3y^3} + \log\frac{y^2}{x} = c$$

$$\Rightarrow \int\left(\frac{1}{v} - \frac{3v^2}{1 + v^3}\right)dv = \int\frac{1}{x}\,dx$$

$$\Rightarrow \log v - \log(1 + v^3) = \log x + \log c$$

$$\Rightarrow \log\frac{y}{x} - \log x - \log\left|\frac{x^3 + y^3}{x^3}\right| = \log c$$

$$\Rightarrow \log\left[\frac{y\cdot x^3}{x\cdot x(x^3 + y^3)}\right] = \log c$$

$$\Rightarrow \frac{xy}{x^3 + y^3} = c$$

② Solve $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$

$$\Rightarrow e^{x/y}(1 - x/y)\,dy = (1 + e^{x/y})\,dx$$

$x = vy$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{e^{x/y}(1 - \frac{x}{y})}{1 + e^{x/y}}$$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{-e^v(1 - v)}{1 + e^v}$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v(1 - v) - v + 2v(1 + e^v)}{1 + e}$$

$\Rightarrow y\frac{dv}{dy} = \frac{-e^v(1-v) - v(1+e^v)}{1+e^v}$

$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$

$\Rightarrow \int \frac{1+e^v}{v+e^v}\,dv = -\int \frac{1}{y}\,dy$

$\Rightarrow \log(v+e^v) + \log y = \log c$

$\Rightarrow \log|y\cdot(v+e^v)| = \log c$

$\Rightarrow y\cdot\left(\frac{x}{y} + e^{\frac{x}{y}}\right) = c$

$\Rightarrow x + ye^{\frac{x}{y}} = c$

③ solve $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

~~$\Rightarrow y\,dy + x\,dy +$~~

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{vx - x + 1}{v}$

put

$x = X + h,\ y = Y + k$

i.e. $\frac{dy}{dX} = \frac{Y+k-X-h+1}{\ }$

i.e. $\frac{dY}{dX} = \frac{Y+k-X-h+1}{Y+k+X+h+5}$

we choose h, k such that

$k - h + 1 = 0,\quad k + h + 1 = 0$

$2k + 6 = 0 \qquad -3 + h + 1 = 0$

$\Rightarrow k = -3 \qquad \Rightarrow h = 2$

$\frac{dY}{dX} = \frac{Y-X}{Y+X}$

$\Rightarrow v + X\frac{dv}{dX} = \frac{v-1}{v+1}$ [put $Y = vX$]

$\Rightarrow X\frac{dv}{dX} = \frac{v-1}{v+1} - v$

$\Rightarrow X\frac{dv}{dX} = \frac{v - 1 - v^2 - v}{v+1}$

$\Rightarrow \int \frac{v+1}{1+v^2}\,dv = -\int \frac{1}{X}\,dX$

$\Rightarrow \frac{1}{2}\int \frac{2v}{1+v^2}\,dv + \int \frac{dv}{1+v^2} = -\int \frac{1}{X}\,dX$

$\Rightarrow \frac{1}{2}\log|1+v^2| + \tan^{-1}v = -\log X + \log c$

$\Rightarrow \frac{1}{2}\log\left|1+\frac{Y^2}{X^2}\right| + \tan^{-1}\frac{Y}{X} + \log X = \log c$

$\Rightarrow \log\sqrt{\frac{X^2+Y^2}{X^2}} + \tan^{-1}\frac{Y}{X} + \log X = \log c$

$\Rightarrow \log\sqrt{X^2+Y^2} + \tan^{-1}\frac{Y}{X} = \log c$ $\quad \left[\begin{array}{l} x-2 = X \\ y+3 = Y \end{array}\right.$

$\Rightarrow \tan^{-1}\frac{y-3}{x+2} + \frac{1}{2}\log\left[1 + \frac{(y-3)^2}{(x+2)^2}\right] + \log(x+2) = \log c$

Ans $\log\sqrt{(x-2)^2 + (y+3)^2} + \tan^{-1}\frac{y+3}{x-2} = \log c$

(4) Solve, $\frac{dy}{dx} = \frac{3x-4y-2}{6x-8y-5}$

$$\frac{dy}{dx} = \frac{3x-4y-2}{6x-8y-5}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3x-4y)-2}{2(3x-4y)-5} \quad -(1)$$

Put,

$3x-4y = z$

$\Rightarrow 3 - 4\frac{dy}{dx} = \frac{dz}{dx} \quad -(2) \quad \Rightarrow 4\frac{dy}{dx} = 3 - \frac{dz}{dx}$

from (1)

$\Rightarrow \frac{dy}{dx} = \frac{3}{4} - \frac{1}{4}\frac{dz}{dx} \quad -(4)$

$$\frac{3}{4} - \frac{1}{4}\frac{dz}{dx} = \frac{z-2}{2z-5}$$

$$\Rightarrow -\frac{1}{4}\frac{dz}{dx} = \frac{z-2}{2z-5} - \frac{3}{4}$$

$$\Rightarrow -\frac{1}{4}\frac{dz}{dx} = \frac{4(z-2)-3(2z-5)}{4(2z-5)}$$

$$\Rightarrow -\frac{1}{4}\frac{dz}{dx} = \frac{4z-8-6z+15}{4(2z-5)}$$

$$\Rightarrow -\frac{dz}{dx} = \frac{-2z+7}{2z-5}$$

$$\Rightarrow \frac{dz}{dx} = \frac{2z-7}{2z-5}$$

$$\Rightarrow \int \frac{2z-5}{2z-7}dz = \int dx$$

$$\Rightarrow \int \frac{2z-7+7}{2z-7}dz - \frac{5}{2}\int \frac{2dz}{2z-7} = \int dx$$

$$\Rightarrow \int dz + \frac{7}{2}\int \frac{2dz}{2z-7} - \frac{5}{2}\int \frac{2dz}{2z-7} = \int dx$$

$$\Rightarrow \int dz + \int \frac{2dz}{2z-7} = \int dx$$

$$\Rightarrow z + \log|2z-7| = x + C$$

$$\Rightarrow 3x - 4y + \log|6x-8y-7| = x + C$$

$$\Rightarrow 2x - 4y + \log|6x-8y-7| = C$$

Linear eqn — An eqn of the form $\frac{dy}{dx}+Py=Q$ where P, Q are function of x is called a linear eqn.

Integrating factor (IF) $Mdx+Ndy=0$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$ then, $e^{\int f(x)dx}$ is IF.

$\frac{dy}{dx}=(Q-Py)$

$(Q-Py)\,dx-dy=0$

$\Rightarrow M\,dx+N\,dy=0$

$M=Q-Py$, $N=-1$, $\frac{1}{-1}(P-0)=P$

$\frac{dy}{dx}+Py=Q$

$I.F=e^{\int P dx}$

$\Rightarrow \frac{dy}{dx}e^{\int P dx}+Py\,e^{\int Pdx}=Qe^{\int Pdx}$

$\Rightarrow \frac{d}{dx}(ye^{\int Pdx})=Qe^{\int Pdx}$

$\Rightarrow ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$

Bernoulli's eqn —

$\frac{dy}{dx}+Py=Qy^n$

$\Rightarrow \frac{1}{y^n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q$ Put, $\frac{1}{y^{n-1}}=z$

$\Rightarrow y^{1-n}=z$

$\Rightarrow (1-n)y^{-n}\cdot\frac{dy}{dx}=\frac{dz}{dx}$

$\Rightarrow \frac{1}{y^n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx}$

$\frac{1}{1-n}\frac{dz}{dx}+Pz=Q$

$\Rightarrow \frac{dz}{dx}+P(1-n)z=Q(1-n)$

Ex — ① $(x^2y^3+2xy)dy=dx$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x^2y^3+2xy}$

$\Rightarrow \frac{dx}{dy}=x^2y^3+2xy$

$\Rightarrow \frac{dx}{dy}-2xy=x^2y^3$

$\Rightarrow \frac{1}{x^2}\frac{dx}{dy}-\frac{2y}{x}=y^3$

Put $\Rightarrow -\frac{2y}{x}=z$

$\Rightarrow \frac{-2}{x^2}\frac{dx}{dy}=\frac{dz}{dy}$

$\Rightarrow \frac{1}{2}\frac{dz}{dy}+zy=y^3$

$\Rightarrow \frac{dz}{dy}+2zy=2y^3$ $IF=e^{\int 2y\,dy}=e^{y^2}$

multiplying both sides by e^{y^2} and then integrating

$ze^{y^2}=\int 2y^3e^{y^2}dy$

$\Rightarrow ze^{y^2}=\int 2y^3e^{y^2}dy$ Put $y^2=t$

$\Rightarrow 2y\frac{dy}{dt}=dt$

$\Rightarrow ze^{y^2}=2\int t\,e^t\frac{dt}{2}$

$\Rightarrow ze^{y^2}=\frac{2}{2}[(t-1)e^t]+C$

$\frac{y}{?} - \frac{2}{x} e^{y^2} = \ldots [y^2 - 1] e^{y^2}] + c$

$\Rightarrow \frac{?}{2}(y^2-1)e^{y^2}+c$

$\frac{-2}{x} e^{y^2} =$

(2)

$(x^2y^3 + 2xy)\, dy = dx$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2y^3+2xy}$

$\Rightarrow \frac{dx}{dy} = x^2y^3 + 2xy$

$\Rightarrow \frac{dx}{dy} - 2xy = x^2y^3$

$\Rightarrow \frac{dx}{dy} + Px = Q$, $P = -2y$, $Q = x^2y^3$

$e^{\int P dy} = e^{\int -2y\, dy} = e^{-y^2}$

multiplying e^{-y^2} in the each side of eqn -

$\Rightarrow e^{-y^2}\frac{dx}{dy} - 2yxe^{-y^2} = x^2y^3\, e^{-y^2}$

$\Rightarrow \frac{d}{dy}(xe^{-y^2}) = x^2y^3\, e^{-y^2}$

$\Rightarrow \int d(xe^{-y^2}) = \int (x^2y^3 e^{-y^2})\, dy$

$y^2 = z$

$\Rightarrow 2y\,dy = dz$

$\Rightarrow dy = \frac{dz}{2y}$

$\Rightarrow xe^{-y^2} = x^2y^3\int e^{-y^2}dy - e^{-y^2}x^2\frac{y^4}{4}$

$= x^2y^3\frac{1}{2}\int z^{-\frac{1}{2}}e$

~~$\Rightarrow xe^{-y^2} = x^2\left[y^3\int e^{-y^2}dy - e^{-y^2}\frac{y^4}{4}\right] + c$~~

$\Rightarrow xe^{-y^2} = \frac{x^2}{2}\int y^2\, 2ye^{-y^2}dy \ldots$

20/3/23

Ex-1

without expanding show that

$$\begin{vmatrix} 5 & 2 & 3 \\ 7 & 3 & 4 \\ 7 & 4 & 5 \end{vmatrix} = 0$$

L.H.S. Δ

$$= \begin{vmatrix} 5 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

$R_2 = R_2 - R_1$

$R_3 = R_3 - R_1$

$\Delta = 0$ $[\because R_2 = R_3]$

(2)

$$\begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y + \sin y \\ 1 & \cos z - \sin z & \cos z + \sin z \end{vmatrix} = 4 \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \sin y \\ 1 & \cos z & \sin z \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y + \sin y \\ 1 & \cos z - \sin z & \cos z + \sin z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2\cos x & 2\sin x \\ 1 & 2\cos y & 2\sin y \\ 1 & 2\cos z & 2\sin z \end{vmatrix}$$

$[\because C_2' = C_2 + C_3$

$C_3' = C_3 - C_2]$

$$= 4\begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \sin y \\ 1 & \cos z & \sin z \end{vmatrix}$$ (Proved)

y^3

HW ② $[\because R_2 = R_3]$

$\begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y \ldots \end{vmatrix}$

$\int y e^{y^2} dy$

$z e^{y^2} = \int 2y^3 e^{y^2} dy$

$= \int y^2 e^{y^2} 2y\, dy \qquad y^2 = t$

$\qquad \Rightarrow 2y\, dy = dt$

$= \int t e^t dt$

$= e^t(t-1) + c$

$z e^{y^2} = e^t(t-1) + c$

$\qquad = e^{y^2}(y^2-1) + c \quad ; \Rightarrow -\frac{2}{x} = y^2 - 1 + c e^{-y^2}$

$\Rightarrow z e^{y^2} = e^{y^2}(y^2-1) + c$

$2 \sin z$

$= 4 \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \ldots \end{vmatrix}$

$\cos x \ \sin$
$\cos y \ \sin$
$\cos z \ \sin$

③ Show that

$$\begin{vmatrix} \beta+\gamma & \alpha & 1 \\ \gamma+\alpha & \beta & 1 \\ \alpha+\beta & \gamma & 1 \end{vmatrix} = 0$$

$C_1 = C_1 + C_2$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \alpha & 1 \\ \alpha+\beta+\gamma & \beta & 1 \\ \alpha+\beta+\gamma & \gamma & 1 \end{vmatrix}$$

$$= \alpha+\beta+\gamma \begin{vmatrix} 1 & \alpha & 1 \\ 1 & \beta & 1 \\ 1 & \gamma & 1 \end{vmatrix}$$

$= 0$

④

$$\begin{vmatrix} a-c & a-c & a-b \\ 2b & 2a & 0 \\ 2c & 0 & 2a \end{vmatrix} = 8abc$$

$[R_1 \to R_1 + R_2 + R_3]$

$$= \begin{vmatrix} \cancel{-b-c} & \cancel{-a-c} & -a-b \\ \cancel{2b} & \cancel{2a} & \cancel{0} \\ \cancel{2c} & 0 & 2a \end{vmatrix}$$

$[C_1 = C_1 + C_2 + C_3]$

$[C_2 = C_1 + C_2 + C_3$

$C_2 = C_1 + C_2 + C_3]$

$$= \begin{vmatrix} 2a & 2a \\ 2b+2a & 2b+2a \\ 2c+2a & 2c+2a \end{vmatrix}$$

$= (b+c)(4a-0) - (a-c)$

$(4ab-0) + (a-b)$

$(0-4ac)$

$= 4ab + 4ac + 4a^2b + 4abc$

$- 4a^2c + 4abc$

$$\begin{vmatrix} b+c & a-c & a-b \\ 2b & 2a & 0 \\ 2c & 0 & 2a \end{vmatrix} = 2 \times 2 \begin{vmatrix} b+c & a-c & a-b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$$

Substraction the sum of the corresponding elements of the 2nd and 3rd row we gets

$$= 4\begin{vmatrix} 0 & -c & -b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} \qquad [R_1 = R_1 - (R_2 + R_3)]$$

$= c(a \cdot b) - b -$

$$= 4\begin{vmatrix} 0 & -c & -b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} = 4\left[c(ab) + b(ac)\right] = 8abc \text{ (Proved)}$$

S.T.

⑤

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \qquad [R_1 = R_1 + R_2 + R_3]$$

$= (a+b+c)\{(a+b+c)^2 - 4bc\} - 2a\{2b(a+b+c) - 4bc\}$

$+ 2a\{4bc - 2c(a+b+c)\}$

$= (a+b+c)^3 - 4bc(a+b+c) - 4ab(a+b+c) + 8abc$

$+ 8abc - 4ac(a+b+c)$

$= (a+b+c)^3 - 4abc - 4b^2c - 4bc^2 - 4a^2b - 4ab^2$

$- 4abc + 8abc +$

$8abc - 4a^2c - 4abc - 4ac^2$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \qquad \begin{matrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \end{matrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -b-c-a \end{vmatrix} = (a+b+c)^3$$

Ex-⑥ Find the value of $\begin{vmatrix} b+c & a-c & a-b \\ 2b & 2a & 0 \\ 2c & 0 & 2a \end{vmatrix}$

(Ans = 8abc)

(Same in line ⑨)

- **Elementary operation** – An elementary operation on a matrix A is an operation of the following type –

i) Interchange of two rows (or column) of A

ii) multiplication of two rows (or column) of A by a constant (to)

iii) Addition of a scalar multiple of two one row (or column) to another row.

R_{ij} or C_{ij} | $R_i(c)$ $C_j(t)$

$R_i + R_j(t)$ or $C_i + C_j(t)$

write –
$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 7 \\ 4 & 9 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{2C_3} \begin{bmatrix} 2 & 1 & 6 \\ 4 & 5 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$$

__Row reduced__ An $m \times n$ matrix A is called row reduced if

a) the first non-zero element in each non-zero row is called leading 1 and

b) Each column containing the leading 1 of some row has all its other elements 0

Ex –
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow$$ This one is not row reduced

Ex –
$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \longrightarrow$$ This is row reduced

Ex – 1
$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 4 & 2 & 6 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 8 \\ 0 & 1 & 0 & \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[This one is row reduced]

__Imp Defination__

A $m \times n$ matrix A is said to be a row reduced echelon matrix (or a row echelon matrix) if

(a) A is row – reduced

(b) there is an integer r $(0 \le r \le m)$ such that the first r rows of A are non-zero rows and the remain row (if they exist) are all zero rows.

c) If the leading element of the non-zero rows occurs in the kth column of it then $k_1 < k_2 < k_r$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Ex-(1)

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ This is non reduced
→ This is not echelon matrix

6.4.22

math (sin)

The differential eqn of 1st order but of higher degree the form of this type of eqn are –

$$\boxed{\left(\frac{dy}{dx}\right)^n + A_1\left(\frac{dy}{dx}\right)^{n-1} + \dots + A_n = 0 \quad (1)}$$

whose $A_1, A_2, \dots A_n$ as functions of x, y if we write $\frac{dy}{dx} = p$ then (1) becomes

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0 \quad (ii)$$

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ are roots of (ii) then (ii) can be written as

$$(p-\alpha_1)(p-\alpha_2)\dots(p-\alpha_n) = 0$$

Let $p - \alpha_n = 0$, $r = 1, \dots, n$

$$\frac{dy}{dx} = \alpha_n$$

$$\phi_n(x, y, c) = 0$$

$$\phi_1(x, y, c) \cdot \phi_2(x, y, c) \dots \times \phi_n(x, y, c) = 0$$

Ex-1

$$p^2 - 9p + 18 = 0$$

$$\left(\frac{dy}{dx}\right)^2 - 9\left(\frac{dy}{dx}\right) + 18 = 0$$

$\Rightarrow p^2 - 9p + 18 = 0$

$\Rightarrow p^2 - (3+6)p + 18 = 0$

$\Rightarrow p^2 - 3p - 6p + 18 = 0$

$\Rightarrow p(p-3) - 6(p-3) = 0$

$\Rightarrow (p-3)(p-6) = 0$

$\therefore p - 3 = 0$	$p - 6 = 0$
$\Rightarrow \frac{dy}{dx} - 3 = 0$	$\Rightarrow \frac{dy}{dx} - 6 = 0$
$\Rightarrow \int dy - 3\int dx = c$	$\Rightarrow \int dy - 6\int dx = c$
$\Rightarrow y - 3x = c$	$\Rightarrow y - 6x = c$

The general sol^n –

$$(y - 3x - c)(y - 6x - c) = 0$$

Ex-2

$p^2 + 2px - 3x^2 = 0$

$\Rightarrow p^2 + (3-1)px - 3x^2 = 0$

$\Rightarrow p^2 + 3px - px - 3x^2 = 0$

$\Rightarrow p(p + 3x) - x(p + 3x) = 0$

$\Rightarrow (p + 3x)(p - x) = 0$

$p + 3x$	$p - x = 0$
$\Rightarrow \frac{dy}{dx} + 3x = 0$	$\Rightarrow \frac{dy}{dx} - x = 0$
$\Rightarrow \int dy + 3\int x\,dx = c$	$\Rightarrow \int dy - \int x\,dx = c$
$\Rightarrow y + \frac{3x^2}{2} = c$	$\Rightarrow y - \frac{x^2}{2} = c$

The general solution is –

$$\left(y + \frac{3}{2}x^2 - c\right)\left(y - \frac{x^2}{2} - c\right) = 0$$

There will be no 'p' in general sol^n and singular sol^n for any case. i.e. p have to be eliminated at any cost.

c will present in general sol. But no 'p' will present in singular sol^n. If p is not in condition to reduce then we have to declare that by eliminating p we will get general sol^n.

Ex $p^3 - p(x^2 + xy + y^2) + xy(x+y) = 0$

$\Rightarrow p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$

$\Rightarrow p(p^2 - x^2) - xy(p - x) - y^2(p - x) = 0$

$\Rightarrow (p-x)\{p(p+x) - xy - y^2\} = 0$

$\Rightarrow (p-x)\{p^2 + px - xy - y^2\} = 0$

$\Rightarrow (p-x)\{(p^2 - y^2) + x(p - y)\} = 0$

$\Rightarrow (p-x)(p-y)(p+y+x) = 0$

$\therefore p - x = 0$

$\Rightarrow \frac{dy}{dx} - x = 0$

$\Rightarrow dy - x\,dx = 0$

$\Rightarrow \int dy - \int x\,dx = 0$

Integrated $y - \frac{x^2}{2} = c$

$p - y = 0$

$\Rightarrow \frac{dy}{dx} - y = 0$

$\Rightarrow \int \frac{dy}{y} = \int dx$

$\Rightarrow \log|y| = x + c$

Integrate

$p + y + x = 0$

$\Rightarrow \frac{dy}{dx} + y = -x$ —(1)

$\therefore I.F = e^{\int 1\,dx} = e^x$

multiply (1) by IF and then integrating

$e^x \frac{dy}{dx} + e^x y = -xe^x$

$\Rightarrow \int \frac{d}{dx}(e^x y) = \int -xe^x\,dx$

$\Rightarrow ye^x = \int -xe^x\,dx = e^x(1-x) + c$

$y - c(1+x) - ce^{-x} = 0$

The general soln is –

$(y - \frac{x^2}{2} - c)(1+xy - x - c)[y - c(1+x) - ce^{-x}] = 0$

Ex

(1) $x + yp^2 = p(1+xy)$, $p = \frac{dy}{dx}$

(2) $x^2p^2 + 3xyp + 2y^2 = 0$, $p = \frac{dy}{dx}$

Solvable for x

The general form of the eqn is

$x = f(y, p)$, $p = \frac{dy}{dx}$ —(1)

$\Rightarrow$ Diff (1) w.r.t y and get

$\frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}$

$\Rightarrow \frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}$

$\Rightarrow g(y, p, \frac{dp}{dy}) = 0$

$\psi(y, p, c) = 0$ —(2)

Ex Solve $p^2y + 2px = y$ —(1)

$2px = y - p^2y$

$\Rightarrow x = \frac{y(1-p^2)}{2p}$

Diff w.r.t y and get

$\frac{dx}{dy} = \frac{1-p^2}{2p} + y\frac{(-2p)}{2p}\frac{dp}{dy} - \frac{y(1-p^2)}{2p^2}\frac{dp}{dy}$

$\Rightarrow \frac{dx}{dy} = \frac{1-p^2}{2p} + \frac{dp}{dy}\left(-y - y\frac{(1-p^2)}{2p^2}\right)$

$\Rightarrow \frac{dx}{dy} = \frac{1-p^2}{2p} - y\frac{dp}{dy}\left(1 + \frac{1-p^2}{2p^2}\right)$

$\Rightarrow \frac{dx}{dy} = \frac{1-p^2}{2p} - y\frac{dp}{dy}\frac{1+p^2}{2p^2}$

$\Rightarrow \frac{1}{p} - \frac{1-p^2}{2p} = -y\frac{(1+p^2)}{2p^2}\frac{dp}{dy}$

$\Rightarrow \frac{2-(1-p^2)}{2p} = -\frac{y(1+p^2)}{2p^2}\frac{dp}{dy}$

$\Rightarrow 1 = -\frac{y}{p}\frac{dp}{dy}$

$\Rightarrow ydp + pdy = 0$

$\Rightarrow d(yp) = 0$

$\Rightarrow \int d(yp) = c$

$\Rightarrow yp = c$

$\Rightarrow p = \frac{c}{y}$ —(2)

elementing 'p' from (1) and (2) ~~according to~~ we get the general soln –

$\frac{c^2}{y^2}y + \frac{2cx}{y} = y$

$\Rightarrow c^2 + 2cx = y^2$

Ex-2 Solve $x = py - p^2$

$x = py - p^2$ —(1)

Diff w.r.to y and we get

$$\frac{dx}{dy} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{dx}{dy} - p = \frac{dp}{dy}(y - 2p)$$

$$\Rightarrow \frac{1}{p} - p = \frac{dp}{dy}(y - 2p)$$

$$\Rightarrow \frac{dy}{dp}\left(\frac{1}{p} - p\right) = y - 2p$$

$$\Rightarrow \frac{dy}{dp} = \frac{y - 2p}{\frac{1}{p} - p}$$

$$\Rightarrow \frac{dy}{dp} - \frac{y}{\frac{1}{p} - p} = \frac{-2p}{\frac{1}{p} - p}$$

$$\Rightarrow \frac{dy}{dp} - \frac{p}{1-p^2}\,y = -\frac{2p^2}{1-p^2}$$

$$I.F. = e^{\int \frac{-2p}{1-p^2}dp}$$

$$= e^{\frac{1}{2}\log|1-p^2|}$$

$$= \sqrt{1-p^2}$$

multiplying by I.F and then integrating

$$y\cdot\sqrt{1-p^2} = -\int \frac{2p^2\sqrt{1-p^2}}{1-p^2}\,dp$$

$$= -\int \frac{2p^2}{\sqrt{1-p^2}}\,dp$$

$$= 2\int \frac{1-p^2-1}{\sqrt{1-p^2}}\,dp$$

$$= 2\left[\int\sqrt{1-p^2} - \int\frac{dp}{\sqrt{1-p^2}}\right]$$

$$= 2\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p - \sin^{-1}p\right] + c$$

$$= 2\left[\frac{p}{2}\sqrt{1-p^2} - \frac{1}{2}\sin^{-1}p\right] + c$$

$$= p\sqrt{1-p^2} - \sin^{-1}p + c$$

$$y = p - \frac{1}{\sqrt{1-p^2}}\sin^{-1}p + c \quad \text{—(ii)}$$

From (i) (elemenating p from (1) and (2) we get the general sol^n) [If we can't elemenate p Proprly then we have to write the above statement]

Solvable form

The eqn $y = f(x, p)$ —(1) where $p = \frac{dy}{dx}$

Diff w.r.t x and get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx}$$

$$\Rightarrow p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\cdot\frac{dp}{dx} = 0$$

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0$$

Ex-1

$y = px + p^2 x, \quad \frac{dy}{dx} = p$

$y = px + p^2 x \quad \text{---(1)}$

Diff. w.r.t. x and get

$\frac{dy}{dx} = p + x\frac{dp}{dx} + p^2 + x \cdot 2p\frac{dp}{dx}$

$\Rightarrow \frac{dy}{dx} - p = \frac{dp}{dx}(x + \cancel{p^2} + 2xp) + p^2$

$\Rightarrow \cancel{p} - \cancel{p} = \frac{dp}{dx}(x + \cancel{p^2} + 2xp) + p^2$

$\Rightarrow -p^2 \frac{dx}{dp} = x + 2xp = x(1+2p)$

$\Rightarrow \frac{dx}{x} = -\frac{(1+2p)}{p^2}dp$

$\Rightarrow \int \frac{dx}{x} + \int\left(\frac{1+2p}{p^2}\right)dp = c$

$\Rightarrow \log x + \left(-\frac{1}{p} + 2\log p\right) = c$

$\Rightarrow \log x + 2\log p - \frac{1}{p} = c \quad \text{---(ii)}$

now elemenating p from (i) and (ii) we get the general soln.

Ex-2 Solve $y = (1+p)x + ap^2, \; p = \frac{dy}{dx}$ ---(1)

Diff w.r.t. x and we get

$\frac{dy}{dx} = (1+p) + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$

$\Rightarrow \cancel{p} - 1 - \cancel{p} = x\frac{dp}{dx} + 2ap\frac{dp}{dx}$

$\Rightarrow \cancel{\int -\frac{1}{x}dx + \int dp = c}$

$\Rightarrow \cancel{\log x + p = c}$

$\Rightarrow \cancel{p = c - \log x} \quad \text{---(ii)}$

$\Rightarrow \cancel{p} = 1 + p + \cancel{p} + x\frac{dp}{dx} + 2ap\frac{dp}{dx}$

$\Rightarrow (x + 2ap)\frac{dp}{dx} + 1 = 0$

$\Rightarrow x + 2ap + \frac{dx}{dp} = 0$

$\Rightarrow \frac{dx}{dp} + x = -2ap$

which is a linear eqn of x and p

$I.F. = e^{\int dp} = e^p$

multiplying both side by I.F and then integrating we get

$xe^p = -\int 2ap\,e^p\,dp$

$= -2a[pe^p - e^p] + c$

$= -2a(p-1)e^p + c$

$\Rightarrow x = -2a(p-1) + ce^{-p} \quad \text{---(ii)}$

elemenating p from (1) and (ii) we can get the general soln.

Ex-(3) H.W. Solve : (1) $p^3 x + p^2 y - 1 = 0 \quad \left(p = \frac{dy}{dx}\right)$

(2) $y = 2px + p^2$

(3) $y = 2px + \tan^{-1}(xp^2)$

(4) $x = y + p^2$

(5) $2px = y - y^2 p^3$

Clairaut's equation

The general from of clairaut's eqn is

$$\boxed{y = px + f(p)}, \; p = \frac{dy}{dx} \quad \text{---(i)}$$

Diff w.r.t x and get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow [x + f'(p)]\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c$ —(ii)

or $x + f'(p) = 0$ —(iii)

Elimenating p from (i) and (ii) we get

$y = cx + f(c)$

which is the complete primitive of the general soln.

If we eliminate p from (i) and (iii) we will get a relation which satisfy the given diff eqn (1) but that soln can not be deduced from the general soln and this soln does not contain any arbitrary constant.

This solution is called a singular solution.

Ex-1 $y = px + a\sqrt{1+p^2}$, $p = \frac{dy}{dx}$, a is a constant

Diff w.r.t x and get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + a\cdot\frac{1}{2}(1+p^2)^{-\frac{1}{2}} \cdot 2p\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + \frac{ap}{\sqrt{1+p^2}}\frac{dp}{dx}$$

$$\Rightarrow \left(x + \frac{ap}{\sqrt{1+p^2}}\right)\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c$ —(ii)

or $x + \frac{ap}{\sqrt{1+p^2}} = 0$

eliminating p from (i) and (ii) we get

$y = cx + a\sqrt{1+c^2}$ which is a general soln.

Also from (i) and (iii) we get

$$y = p\left(\frac{-ap}{\sqrt{1+p^2}}\right) + a\sqrt{1+p^2}$$

$$\Rightarrow y = \frac{-ap^2 + a(1+p^2)}{\sqrt{1+p^2}} \quad \text{---(iv)}$$

from (iii), $x = \frac{-ap}{\sqrt{1+p^2}}$ —(v)

Squaring (iv) and (v) and adding we get

$$x^2 + y^2 = \frac{a^2p^2}{1+p^2} + \frac{a^2}{1+p^2} = \frac{a^2(1+p^2)}{(1+p^2)} = a^2$$

$\Rightarrow x^2+y^2=a^2$ which is the singular soln.

Ex-2 Find the general Soln and the singular soln of

for $y = px+\sqrt{a^2p^2+b^2}$ —(i), $p=\frac{dy}{dx}$

Diff w.r.t x

$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2}\frac{2a^2p}{\sqrt{a^2p^2+b^2}}\frac{dp}{dx}$

$\Rightarrow \left[x+\frac{a^2 p}{\sqrt{a^2p^2+b^2}}\right]\frac{dp}{dx}=0$

either $\frac{dp}{dx}=0 \Rightarrow p=c$ —(ii)

or $x = \frac{-a^2p}{\sqrt{a^2p^2+b^2}}$ —(iv)

Eliminating p from (i) and (ii), we get

$y = cx+\sqrt{a^2c^2+b^2}$ which is general soln.

Also from (i) and (iii) we get

$y = p\left(\frac{-a^2p}{\sqrt{a^2p^2+b^2}}\right)+\sqrt{a^2p^2+b^2}$

$= \frac{-a^2p^2+a^2p^2+b^2}{\sqrt{a^2p^2+b^2}}$

$\Rightarrow \frac{y}{b} = \frac{b}{\sqrt{a^2p^2+b^2}}$ —(v)

from (iv) $\frac{x}{a} = \frac{-ap}{\sqrt{a^2p^2+b^2}}$ —(5)

Squaring (iv) and (v) and adding we get

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{a^2p^2}{a^2p^2+b^2}+\frac{b^2}{a^2p^2+b^2} = \frac{a^2p^2+b^2}{a^2p^2+b^2}=1$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is the singular soln.

Find the general soln and the singular soln.

$y = px+\sqrt{a^2p^2-b^2}$ —(i) $p=\frac{dy}{dx}$

Diff w.r.t x

$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx}+\frac{1}{2}(a^2p^2-b^2)^{-\frac{1}{2}}a^2 \cdot 2p\frac{dp}{dx}$

$\Rightarrow \cancel{p} = \cancel{p}+x\frac{dp}{dx}+\frac{\cancel{2}a^2p}{\cancel{2}\sqrt{a^2p^2-b^2}}\frac{dp}{dx}$

$\Rightarrow \left[x+\frac{a^2p}{\sqrt{a^2p^2-b^2}}\right]\frac{dp}{dx}=0$

either $\frac{dp}{dx}=0 \Rightarrow p=c$ —(ii)

(or) $x = -\frac{a^2p}{\sqrt{a^2p^2-b^2}}$ —(iii)

Eliminating p from (i) and (ii) we get

$y = cx+\sqrt{a^2c^2-c^2}$ —(v) which is general soln.

Also from (i) and (iii) we get

$$y = p\left(\frac{-a^2p}{\sqrt{a^2p^2-b^2}}\right) + \sqrt{a^2p^2-b^2}$$

$$= \frac{-a^2p^2 + a^2p^2 - b^2}{\sqrt{(a^2p^2-b^2)}}$$

$$\Rightarrow \frac{y}{b} = \frac{-b}{\sqrt{a^2p^2-b^2}} \quad \text{---(iv)}$$

from (iii) we get, $\frac{x}{a} = \frac{-ap}{\sqrt{a^2p^2-b^2}}$

Squaring and substracting (iii) and (iv)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{-b^2}{a^2p^2-b^2} + \frac{a^2p^2}{a^2p^2-b^2}$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ This is a singular soln.

HW

1) $x + yp^2 = p(1+xy)$, $p = \frac{dy}{dx}$

$x + yp^2 = p(1+xy)$

$\Rightarrow x + yp^2 = p + pxy$

$\Rightarrow x - pxy + yp^2 - p = 0$

$\Rightarrow x(1-py) + p(py-1) = 0$

$\Rightarrow x(1-py) - p(1-py) = 0$

$\Rightarrow (1-py)(x-p) = 0$

$1 - py = 0$

$\Rightarrow 1 - y\frac{dy}{dx} = 0$

$\Rightarrow y\frac{dy}{dx} = 1$

$\Rightarrow \int y\,dy - \int dx = 0$

$\Rightarrow \frac{y^2}{2} - x = c$

$x - p = 0$

$\Rightarrow x - \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = x$

$\Rightarrow \int dy - x\int dx = c$

$\Rightarrow y - \frac{x^2}{2} = c$

$\therefore$ The general soln $= \left(\frac{y^2}{2} - x - c\right)\left(y - \frac{x^2}{2} - c\right)$

2) $x^2p^2 + 3xyp + 2y^2 = 0$, $p = \frac{dy}{dx}$

$\Rightarrow x^2p^2 + (2+1)xyp + 2y^2 = 0$

$\Rightarrow x^2p^2 + 2xyp + xyp + 2y^2 = 0$

$\Rightarrow xp(xp+2y) + y(xp+2y) = 0$

$\Rightarrow (xp+y)(xp+2y) = 0$

$xp + y = 0$

$\Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow x\frac{dy}{dx} = -y$

$\Rightarrow \frac{dy}{y} = -\frac{1}{x}dx$

$\Rightarrow \int \frac{dy}{y} + \int \frac{1}{x}dx = \log c$

$\Rightarrow \log|y| + \log|x| = \log c$

$\Rightarrow xy = c$

$xp + 2y = 0$

$\Rightarrow x\frac{dy}{dx} + 2y = 0$

$\Rightarrow x\frac{dy}{dx} = -2y$

$\Rightarrow \frac{dy}{2y} = -\frac{1}{x}dx$

$\Rightarrow \int \frac{dy}{2y} + \int \frac{dx}{x} = \log c$

$\Rightarrow \frac{1}{2}\log y + \log x = \log c$

$\Rightarrow \log|x\sqrt{y}| = \log c$

$\Rightarrow x\sqrt{y} = c$

The general sol^n = $(xy - c)(x\sqrt{y} - c)$

3) $p^3x - p^2y - 1 = 0 \quad (1), \; p = \frac{dy}{dx}$

$\Rightarrow p^2y = p^3x - 1$

$\Rightarrow y = \frac{p^3x - 1}{p^2}$

$\Rightarrow \frac{dy}{dx} = \frac{p^2(p^3 + x \cdot 3p^2\frac{dp}{dx}) - (p^3x - 1) \cdot 2p\frac{dp}{dx}}{p^4}$

$\Rightarrow \frac{dy}{dx} = p^5 + 3p^4x\frac{dp}{dx} - 2p^4x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$\Rightarrow \frac{dy}{dx} = p^5 + p^4x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$\Rightarrow p - p^5 = p(p^3x + 2)\frac{dp}{dx}$

$\Rightarrow p(1 - p^4) = p(p^3x + 2)\frac{dp}{dx}$

~~$1 - p^4 = p^3x\frac{dp}{dx} + x^2$~~

$\Rightarrow \frac{p^3x + 2}{1 - p^4}dp = dx$

$\Rightarrow p^3x$

$p^3x = 1 + p^2y \quad (1)$

$\Rightarrow p = \frac{1 + p^2y}{p^3}$

$\Rightarrow x = \frac{1}{p^3} + \frac{y}{p}$

$\Rightarrow \frac{dx}{dy} = -\frac{3}{p^4}\cdot\frac{dp}{dy} + \left(-\frac{1}{p^2}\frac{dp}{dy}y + \frac{1}{p}\right)$

$\Rightarrow \frac{1}{p} - \frac{1}{p} = -\frac{3}{p^4}\frac{dp}{dy} - \frac{y}{p^2}\frac{dp}{dy}$

$\Rightarrow \frac{dp}{dy}\frac{1}{p^2}\left(\frac{3}{p^2} + y\right) = 0$

$\Rightarrow d$

$\Rightarrow p^3x - p^2y - 1 = 0$

$\Rightarrow p^2y = p^3x - 1$

$\Rightarrow y = \frac{p^3x - 1}{p^2}$

$\Rightarrow y = px - \frac{1}{p^2}$

$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{2}{p^3}\frac{dp}{dx}$

$\Rightarrow \left(x + \frac{2}{p^3}\right)\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0, \Rightarrow p = c$ —(II)

$x = -\frac{2}{p^3}$ —(III)

Eliminating (II) and (I) we get,

$c^3x - c^2y - 1 = 0$ → This is general soln.

Also from (I) and (III) we get,

$p^3\left(-\frac{2}{p^3}\right) - p^2y - 1 = 0$

$\Rightarrow -p^2y - 3 = 0$

$\Rightarrow p^2y + 3 = 0$

$\Rightarrow y = -\frac{3}{p^2}$

$\Rightarrow \frac{y}{3} = -\frac{1}{p^2}$ —(IV) ; $\Rightarrow \left(\frac{y}{3}\right)^3 = -\frac{1}{p^6}$ —(IV)

from (III), $\frac{x}{2} = -\frac{1}{p^3}$

$\Rightarrow \left(\frac{x}{2}\right)^2 = \frac{1}{p^6}$ —(V)

now, (IV) ÷ (V) we get

$$\frac{\left(\frac{y}{3}\right)^3}{\left(\frac{x}{2}\right)^2} = -1$$

$\Rightarrow \frac{y^3}{27} + \frac{x^2}{4} = 0$ → This is the singular soln.

(4) $y = 2px + p^2$ —(I)

$\Rightarrow \frac{dy}{dx} = 2\left(p + x\frac{dp}{dx}\right) + 2p\frac{dp}{dx}$

$\Rightarrow \frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$

$\Rightarrow p - 2p = 2(x+p)\frac{dp}{dx}$

$\Rightarrow -p = 2(x+p)\frac{dp}{dx}$

$\Rightarrow \frac{-p}{2(x+p)} = \frac{dp}{dx}$

$\Rightarrow \frac{dx}{dp} = \frac{-2(x+p)}{p}$

$\Rightarrow \frac{dx}{dp} + \frac{2x}{p} = -2$

$e^{\int \frac{2}{p}dp}$
$= e^{2\log p}$
$= e^{\log p^2}$
$= p^2$

$p^2\frac{dx}{dp} + \frac{2x}{p}\cdot p^2 = -2p^2$

$\Rightarrow \int d(p^2x) = -2\int p^2 dp$

$\Rightarrow p^2x = -\frac{2p^3}{3} + c$ —(II)

now eliminating p from (I) and (II) we get the general soln.

6) $y = 2px + \tan^{-1}(xp^2)$ --(1)

$$\frac{dy}{dx} = 2\left(p + x\frac{dp}{dx}\right) + \frac{1}{1+x^2p^4}\left(2xp\frac{dp}{dx} + p^2\right)$$

$$\Rightarrow p = 2p + 2x\frac{dp}{dx} + \frac{2xp}{1+x^2p^4}\frac{dp}{dx} + \frac{p^2}{1+x^2p^4}$$

$$\Rightarrow p - 2p - \frac{p^2}{1+x^2p^4} = \left(2x + \frac{2xp}{1+x^2p^4}\right)\frac{dp}{dx}$$

$$\Rightarrow -p - \frac{p^2}{1+x^2p^4} = 2x\left(1 + \frac{p}{1+x^2p^4}\right)\frac{dp}{dx}$$

$$\Rightarrow -p\left(1 + \frac{p}{1+x^2p^4}\right) = 2x\left(1 + \frac{p}{1+x^2p^4}\right)\frac{dp}{dx}$$

$$\Rightarrow -p = 2x\frac{dp}{dx}$$

$$\Rightarrow \int\frac{dp}{p} = -\int\frac{1}{2x}dx$$

$$\Rightarrow \log p = -\frac{1}{2}\log x + \log c$$

$$\Rightarrow \log p + \log\sqrt{x} = \log c$$

$$\Rightarrow p\sqrt{x} = c$$

$$\Rightarrow p = \frac{c}{\sqrt{x}} \quad \text{--(11)}$$

Eliminating p from (1) we get the general Soln.

$$y = \frac{2c}{\sqrt{x}}x + \tan^{-1}\left(x \cdot \frac{c^2}{x}\right)$$

$$\Rightarrow y = 2c\sqrt{x} + \tan^{-1}(c^2)$$

6) $x = y + p^2$ --(1)

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} - 1 = 2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1-p}{p} = 2p\frac{dp}{dy}$$

$$\Rightarrow \int\frac{2p^2\,dp}{1-p} - \int dy = c$$

$$\Rightarrow 2\int\frac{1-(1-p^2)}{1-p}dp - \int dy = c$$

$$\Rightarrow 2\left[\int\frac{dp}{1-p} - \int(1+p)\,dp\right] - \int dy = c$$

$$\Rightarrow 2\left[-\int\frac{-dp}{1-p} - \int(1+p)\,dp\right] - \int dy = c$$

$$\Rightarrow 2\left[-\log|1-p| - p - \frac{p^2}{2}\right] - \frac{y}{2} = c$$

$$\Rightarrow 2\log(1-p) + 2p + p^2 + \frac{y}{2} = c \quad \text{--(11)}$$

From (11) (eliminating p from (1) and (11)) we get the general Soln.

(5) $2px = y - y^2p^3$ —(i)

⇒ $x = \dfrac{y - y^2p^3}{2p}$

⇒ $x = \dfrac{y}{2p} - \dfrac{y^2p^3}{2p}$

⇒ $x = \dfrac{y}{2p} - \dfrac{y^2p^2}{2}$

⇒ $\dfrac{dx}{dy} = \dfrac{1}{2}\left(\dfrac{p - y\frac{dp}{dy}}{p^2}\right) - \dfrac{1}{2}\left(y^2 \cdot 2p\dfrac{dp}{dy} + p^2 \cdot 2y\right)$

⇒ $\dfrac{dx}{dy} = \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy}\right) - \left(y^2p\dfrac{dp}{dy} + p^2y\right)$

⇒ $\dfrac{1}{p} - \dfrac{1}{2p} + p^2y = \left(-\dfrac{y}{2p^2} - y^2p\right)\dfrac{dp}{dy}$

⇒ $\dfrac{2-1}{2p} + p^2y = \left(-\dfrac{y}{2p^2} - py^2\right)\dfrac{dp}{dy}$

⇒ $\dfrac{1}{2p} + p^2y = \left(-\dfrac{y}{2p^2} - py^2\right)\dfrac{dp}{dy}$

⇒ $\dfrac{1+2p^3y}{2p} = \left(\dfrac{-y-2p^3y^2}{2p^2\,p}\right)\dfrac{dp}{dy}$

⇒ $\dfrac{p+2p^4y}{-y-2p^3y^2} = \dfrac{dp}{dy}$

let $y = vp$

⇒ $\dfrac{dy}{dp} = v + p\dfrac{dv}{dp}$

$p = vy$

⇒ $\dfrac{dp}{dy} = v + y\dfrac{dv}{dy}$

$v + y\dfrac{dv}{dy} = \dfrac{vy + 2v^4y^4 \cdot y}{-y - 2v^3y^3y^2}$

⇒ $v + y\dfrac{dv}{dy} = \dfrac{vy + 2v^4y^5}{-y - 2v^3y^5}$

⇒ $v + y\dfrac{dv}{dy} = \dfrac{y(v+2v^4y^4)}{-y(1+2v^3y^4)}$

⇒ $\left(\dfrac{1}{2p} + p^2y\right) = -\dfrac{y}{p}\left(\dfrac{1}{2p} + p^2y\right)\dfrac{dp}{dy}$

⇒ $-\dfrac{y}{p}\dfrac{dp}{dy} = 1$

⇒ $\int \dfrac{1}{p}\,dp = -\int \dfrac{1}{y}\,dy$

⇒ $\log p + \log y = \log c$

⇒ $py = c$

⇒ $p = \dfrac{c}{y}$ —(ii)

Eliminating p from (i) & (ii) we get the general solⁿ —

$2\dfrac{c}{y}\cdot x = y - y^2\cdot\dfrac{c^3}{y^3}$

⇒ $2cx = y^2 - c^3$

7/4/22

Imp

Obtain a row reduced echelon matrix which is row equivalent to

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

Soln: Let us apply elementary row operation on the matrix

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

$$\Big\downarrow R_3 - 2R_1,\ R_4 - 3R_1$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2}$$
$R_3 + 2R_2$
$R_4 + 5R_2$

$$= \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_{34}}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3}$$

$$= \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3}$$
$R_1 - 2R_3$

$$= \begin{bmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required echelon form.

Definition

Characteristic equation: Let A be a $n \times n$ matrix over a field F where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

then det.
$$\begin{vmatrix} a_{11}-x & a_{12} & a_{1n} \\ a_{21} & a_{22}-x & a_{2n} \\ \\ a_{n1} & a_{n2} & a_{nn}-x \end{vmatrix}$$

is called Characteristic polynomial and is denoted by $\psi_A(x)$. The equation $\psi_A(x) = 0$ is said to be characteristic equation of A.

Ex- $\psi_A(x) = \begin{vmatrix} 2-x & 3 \\ 5 & 4-x \end{vmatrix}$

$= (2-x)(4-x) - 15$

$= 8 - 6x + x^2 - 15$

$= x^2 - 6x - 7$

Rajashree

Ex(1)- Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find the characteristic eqn of A.

Soln The characteristic eqn of A

$$\psi_A(x) = \begin{vmatrix} 0-x & -1 \\ 1 & 0-x \end{vmatrix} = 0$$

$\Rightarrow x^2 + 1 = 0$

$\therefore x = i$ and $-i$. This are the eigen values of A.

~~Roots of this equation~~ Roots of $x^2+1=0$ are i and $-i$. Therefore i and $-i$ are the eigen values of A.

(2) Find Characteristic equation of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

$$\psi_A(x) = \begin{vmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{vmatrix}$$

$\Rightarrow (1-x)\left[-\{(2-x)(2+x)\} + 2\right] + 1(-2-x+3) = 0$

$\Rightarrow (1-x)\left[-(4-x^2) + 2\right] + (1-x) = 0$

$\Rightarrow 2(1-x) - (1-x)(4-x^2) + 1 - x = 0$

$\Rightarrow 2-2\lambda-(4-\lambda^2-4\lambda+\lambda^3)+1-\lambda=0$

$\Rightarrow 2-2\lambda-4+\lambda^2+4\lambda-\lambda^3+1-\lambda=0$

$\Rightarrow -\lambda^3+\lambda^2+\lambda-1=0$

$\Rightarrow \lambda^3-\lambda^2-\lambda+1=0$ $\quad [\lambda = 1, 1, -1]$

$\Rightarrow \lambda^2(\lambda-1)-1(\lambda-1)=0$

$\Rightarrow (\lambda-1)(\lambda^2-1)=0$

$\Rightarrow (\lambda-1)^2(\lambda+1)=0$

$\lambda = 1, 1, -1$ ∴ The roots of $(\lambda^2-1)(\lambda-1)=0$ are 1, 1, −1. Therefore, 1, 1, −1 are the eigen values of A.

Eigen value and eigen vector

A root of the characteristic equation of a square matrix A is said to be an eigen value of A.

Eigen vector – Let A be a $n \times n$ matrix. A non-null vector x is said to be an eigen vector of A, if we can find a number λ such that $Ax = \lambda x$ holds.

[x = eigen value
λ = eigen vector]

Ex-1 Cayley Hamilton theorem: Every square matrix satisfies its own characteristic equation.

Ex-1 verification Cayley Hamilton with the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The characteristic equation of A is–

$$\psi_A(\lambda) = \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 1 = 0$

$A^2 + I_2$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2 + I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \text{ (Null-matrix)}$$

$\therefore A$ is satisfy this equation.

② $x = 1, 1, -1$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+0 & -1-2+0 & 0+1+0 \\ 1+2-3 & -1+4-2 & 0-2+2 \\ 3+2-6 & -3+4-4 & 0-2+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0-1 & 0-3-3 & 0+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0-2 & 3-3+6 & -1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -6 & 2 \\ 0 & 1 & 0 \\ -2 & 6 & 3 \end{bmatrix}$$

In correct

$$-A^3 + A^2 + A = \begin{bmatrix} 1 & 6 & -2 \\ 0 & -1 & 0 \\ 2 & -6 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

Incorrect

Qn verify cayley Hamilton theorem with the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

characteristic eqn.

$$\psi_A(x) = \begin{vmatrix} 1-x & 1 & 1 \\ -1 & -1-x & -1 \\ 0 & 0 & 1-x \end{vmatrix} = 0$$

$\Rightarrow (1-x)\left[-\{(1-x)(1+x)\}\right] - 1\left[-1+x\right] + 1\left[0-0\right] = 0$

$\Rightarrow (1-x)\left[-(1-x^2)\right] + 1 - x + 0 = 0$

$\Rightarrow -(1-x)(1-x^2) + 1 - x = 0$

$\Rightarrow -(1 - x^2 - x + x^3) + 1 - x = 0$

$\Rightarrow -\cancel{1} + x^2 + \cancel{x} - x^3 + \cancel{1} - \cancel{x} = 0$

$\Rightarrow -x^3 + x^2 = 0$

$\Rightarrow x^3 - x^2 = 0$

$\Rightarrow x^2(x-1) = 0$

$\therefore x = 0, 0, 1$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+0 & -1-2+0 & 0+1+0 \\ 1+2-3 & -1+4-2 & 0-2+2 \\ 3+2-6 & -3+4-4 & 0-2+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+3 & 0-6+2 & 0+3-2 \\ 0+1+0 & 0+2+0 & 0-1+0 \\ -1-3+6 & 1-6+4 & 0+3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+0 & 1-1+0 & 1-1+1 \\ -1+1+0 & -1+1+0 & -1+1-1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^2 + I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+1 \\ 0+0+0 & 0+0+0 & 0+0-1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$\boxed{A^3 + I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}$$

Thus,

$A^3 - A^2 = 0 \quad [\because A^3 = A^2]$

11/4/22

(S.N.)

Ex-① Reduce the differential equation:

$(px - y)(x - py) = 2p$, (where $p = \frac{dy}{dx}$)

to clairaut's form by the substitution $x^2 = u$, $y^2 = v$, and then find the general solⁿ and singular solⁿ

Solⁿ: The given equation is $(px-y)(x-py) = 2p$ — (i), $p = \frac{dy}{dx}$

Put $u = x^2$, $v = y^2$

$\Rightarrow du = 2x\,dx$, & $dv = 2y\,dy$

$$\frac{2y\,dy}{2x\,dx} = \frac{dv}{du}$$

$$\Rightarrow \frac{y}{x}p = \frac{dv}{du}$$

Put $\frac{dv}{du} = q$

$\therefore p = \frac{x}{y}q$ — (ii)

Substituting in (i) we get

$$\left(\frac{x}{y}q \cdot x - y\right)\left(x - \frac{x}{y}q\,y\right) = 2\frac{x}{y}q$$

$$\Rightarrow \frac{(x^2q - y^2)}{y} \cdot x\left(\frac{y - qy}{y}\right) = \frac{2xq}{y}$$

$$\Rightarrow \left(\frac{x^2q - y^2}{y}\right) \cdot x(1-q) = \frac{2xq}{y}$$

$\Rightarrow (uq - v)(1-q) = 2q$

$\Rightarrow (uq - v) = \frac{2q}{1-q}$

$\Rightarrow v = uq - \frac{2q}{1-q}$

$v = uq + f(q)$

The general solution is $uc - \frac{2c}{1-c}$

$y^2 = x^2 c - \frac{2c}{1-c}$

$\Rightarrow (1-c)y^2 = x^2(1-c)c - 2c$

$\Rightarrow y^2 - cy^2 = x^2 c - x^2c^2 - 2c$

$\Rightarrow x^2c^2 - cy^2 - x^2 c + 2c + y^2 = 0$

$\Rightarrow x^2c^2 - (x^2 + y^2 - 2)c + y^2 = 0$

c - discriminant (from (ii))

$(x^2 + y^2 - 2)^2 - 4 \cdot x^2 \cdot y^2 = 0$

which is the singular soln.

② Reduce the differential equation $x^2p^2 + y(2x+y)p + y^2 = 0$ (where, $p = \frac{dy}{dx}$) to clairaut's form by the substitution $y = u$, $xy = v$ and then find the general soln and singular soln.

Soln The general eqn is -

$$x^2p^2 + y(2x+y)p + y^2 = 0, \; p = \frac{dy}{dx} \quad \text{-(i)}$$

$y = u$, $\quad xy = v$

$\Rightarrow dy = du \quad \Rightarrow x\,dy + y\,dx = dv$

$x + y\frac{dx}{dy} = \frac{dv}{du}$

$\Rightarrow x + \frac{y}{p} = q$ (where, $\frac{dv}{du} = q$)

$\Rightarrow \frac{y}{p} = q - x$

$\Rightarrow \frac{1}{p} = \frac{q-x}{y}$, $\Rightarrow p = \frac{y}{q-x}$ — (ii)

Substituting in (i) we get -

$x^2\frac{(q-x)^2}{y^2} + (2x+y)\frac{(q-x)}{y} + y^2 = 0$

$x^2\frac{y^2}{(q-x)^2} + y(2x+y)\frac{y}{q-x} + y^2 = 0$

$\Rightarrow x^2y^2 + y^2(2x+y)(q-x) + y^2(q-x)^2 = 0$

$\Rightarrow x^2 + y(2x+y)(q-x) + (q-x)^2 = 0$

$\Rightarrow \cancel{x^2} - 2x^2 - xy + 2qx + qy - \cancel{2qx} + \cancel{x^2} = 0$

$\Rightarrow -v + qu + q^2 = 0$

$\Rightarrow v = qu + q^2$ —(iii)

which is the Clairaut's equation.
The general solⁿ is

$$v = uc + c^2$$
$$xy = yc + c^2 \quad \text{---(iv)}$$

which is the general solⁿ

now $c^2 + yc - xy = 0$

c - discriminant is

$$y^2 - 4(-xy) = 0$$
$$\Rightarrow y^2 + 4xy = 0$$
$$\Rightarrow y + 4x = 0$$ which is singular solⁿ.

H.W

(1) Reduce the following differential equation to Clairaut's form by suitable transformation (mentioned in the bracket) and then find the general and singular solⁿ

a) $\left(p = \frac{dy}{dx}\right)$

1) $x^2(y - px) = p^2 y, \; [x^2 = u, \; y^2 = v]$
2) $(x^2 + y^2 - 1)p = xy(1 + p^2) \; [x^2 = u, \; y^2 = v]$
3) $yp^2 - 2xp + y = 0 \; [x^2 = u, \; y^2 = v]$

2) Find the general solⁿ and the singular solⁿ (if any) of the following equation -

1) $y = px + p - p^2$
2) $y = p(x - p) + a/p$
3) $y = px + \sin^{-1} p$
4) $(y - px)(p - 1) = p$

• Higher order linear equations with constant coefficient -

The general form of higher order linear equation with constant coefficients is -

$$\boxed{\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q} \quad \text{--- (1)}$$

Where $P_1, P_2, \ldots P_n$ where constants and Q is function of x.

When, $n = 2$

$$\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q \quad \text{(2nd order eqn)}$$

when $n = 3$

$$\frac{d^3 y}{dx^3} + P_1 \frac{d^2 y}{dx^2} + P_2 \frac{dy}{dx} + P_3 y = Q \quad \text{(3rd order eqn)}$$

if $Q = 0$, then

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0$$ is called reduced eqn and the solution of this equation is called a complementary function.

Let us consider the 2nd order linear diff equation with constant coefficients

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \quad \text{---(1)}$$

Let $y = e^{mx}$ be a trial solution of (1),

then $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$

Substituting in (1) we get

$m^2e^{mx} + P_1 me^{mx} + P_2 e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 + P_1m + P_2) = 0$

$\Rightarrow m^2 + P_1m + P_2 = 0$ ($\because e^{mx} \neq 0$)

which is called auxiliary eqn.

Three cases may arise,

Let the roots of the A.E (Auxiliary eqn)

case-1 $m^2 + P_1m + P_2 = 0$ are real and distinct.

say $m_1 \neq m_2$

$y_1 = e^{m_1x}$, $y_2 = e^{m_2x}$

$\boxed{y = c_1e^{m_1x} + c_2e^{m_2x}}$ the is general soln

case-2

Let the roots of the A.E $m^2 + P_1m + P_2 = 0$ are imaginary

say $\alpha \pm i\beta$

$y = c_1e^{(\alpha + i\beta)x} + c_2e^{(\alpha - i\beta)x}$

$y = c_1e^{\alpha x} \cdot e^{i\beta x} + c_2e^{\alpha x} \cdot e^{-i\beta x}$ [De Moivre's theorem]

$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$

$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x]$

$= \boxed{e^{\alpha x}[A\cos\beta x + B\sin\beta x]}$ where $A = c_1 + c_2$, $B = (c_1 - c_2)i$

Put $A = R\cos\epsilon$

$B = -R\sin\epsilon$

$= e^{\alpha x}[R\cos\epsilon\cos\beta x - R\sin\epsilon\sin\beta x]$

$\boxed{y = Re^{\alpha x}\cos(\beta x + \epsilon)}$ $\boxed{e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)}$

case-III If the roots of A.E. are equal

Say $m_1 = m_2 = \lambda$

Then the soln is

$[m_1 = m_2 = \lambda]$ $\boxed{y = (c_1 + c_2x)e^{\lambda x}}$

For 3rd degree (1st case) $\boxed{y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}}$

$[m_1, \alpha \pm i\beta]$ (2nd case) $\boxed{y = c_1e^{m_1x} + e^{\alpha x}[c_2\cos\beta x + c_3\sin\beta x]}$

$[m_1 = m_2 = m_3 = \lambda]$ (3rd case) $\boxed{y = (c_1 + c_2x + c_3x^2)e^{\lambda x}}$

if 2 equal 1 unequal, then

$\boxed{y = c_1e^{m_1x} + (c_2 + c_3x)e^{\lambda x}}$

Ex-① $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Let $y = e^{mx}$ be actual solⁿ, then the A.E is $m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow m^2(m-1) - 5m(m-1) + 6(m-1) = 0$

$\Rightarrow (m-1)(m^2 - 5m + 6) = 0$

$\Rightarrow (m-1)(m-3)(m-2) = 0$

$\Rightarrow m = 1, 2, 3$

The solⁿ is $y = c_1 e^{1x} + c_2 e^{2x} + c_3 e^{3x}$

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

where c_1, c_2, c_3 are arbitrary constant

② $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = Q(x)$ —(i)

Let $Q(x) = 0$ then (i) ~~reading~~ reduces to $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0$ —(ii)

This eqⁿ is called reduced eqⁿ.

The solution of (ii) is called a complementary function (C.F)

(i) can be written as

$\left(\frac{d^2}{dx^2} + P_1\frac{d}{dx} + P_2\right)y = Q(x)$

If $\frac{d}{dx} = D$

$(D^2 + P_1 D + P_2)y = Q(x)$

$\Rightarrow F(D)y = Q(x)$

F(D) is called linear differential operator

$[F(D)]^{-1} F(D) y = [F(D)]^{-1} Q(x)$

$\boxed{\Rightarrow y = \frac{1}{F(D)} Q(x)}$

Particular integral (P.I)

The general solⁿ is $y = y_c$ (complementary function) + P.I

$P.I. = \frac{1}{F(D)} Q(x)$ [General methode]

$= \frac{1}{D^2 + P_1 D + P_2} Q(x)$

$= \frac{1}{(D-\alpha)(D-\beta)} Q(x)$

Let $u = \frac{1}{D-\beta} Q(x)$

$\Rightarrow (D-\beta)u = Q(x)$

$\Rightarrow \frac{du}{dx} - \beta u = Q(x)$, $u = \phi_1(x, y)$

$y = \frac{1}{(D-\alpha)} u$

$Dy - \alpha y = u$

[shortcut method]

i) $P.I = \frac{1}{F(D)} e^{ax}$

if $\phi(x) = e^{ax}$

$P.I = \frac{1}{F(D)} e^{ax}$

$= \frac{1}{F(a)} e^{ax}$ (Provided $F(a) \neq 0$)

ii) Let $\phi(x) = x^n$

$P.I = \frac{1}{F(D)} (x^n)$

$= \frac{1}{D^2 + P_1 D + P_2} (x^n)$

$= \frac{1}{P_2\left(1 + \frac{P_1}{P_2} D + \frac{D^2}{P_2}\right)} x^n$

$= \frac{1}{P_2} (1 + A_1 D + A_2 D^2)^{-1} (x^n)$

$= \frac{1}{P_2} (1 + B_1 D + B_2 D^2 + B_3 D^3 + \cdots B_n D^n \cdots) (x^n)$

$= \frac{1}{P_2} (x^n + B_1 n x^{n-1} + \cdots + B_n n!)$

iii) $\phi(x) = \sin ax / \cos ax$

$P.I = \frac{1}{F(D)} \sin ax$

$= \frac{1}{\phi(D^2)} \sin(ax)$

$= \frac{1}{\phi(-a^2)} \sin(ax)$ [$\phi(-a^2) \neq 0$]

Ex ① solve $\frac{d^2y}{dx^2} + 4y = \sin 2x$

Let $y = e^{mx}$ be a trial soln of the sheduled eqn $\frac{d^2y}{dx^2} + 4y = 0$

$\therefore$ A.F $= m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$P.I = \frac{1}{D^2+4} \sin(2x)$

Let $Z = \frac{1}{D^2+4} \cos 2x$

$Z + iy = \frac{1}{D^2+4} (\cos 2x + i \sin 2x)$

$= \frac{1}{D^2+4} (e^{2ix})$

$= e^{2ix} \frac{1}{(D+2i)^2 + 4}$ (1)

$= e^{2ix} \cdot \frac{1}{4iD\left(1 + \frac{D}{4i}\right)} (1)$

$= e^{2ix} \cdot \frac{1}{4iD} \left(1 + \frac{D}{4i}\right)^{-1} (1)$

$= e^{2ix} \frac{1}{4iD} \left(1 - \frac{D}{4i} + \cdots\right) (1)$

$= e^{2ix} \cdot \frac{1}{4iD} (1)$ $= e^{2ix} \frac{1}{4i} x$

$\boxed{\frac{1}{4iD}} (1)$ → integration (opposite of derivative)

$= x\left(\frac{\cos 2x + i\sin 2x}{4i}\right)$

$= \frac{x \sin 2x}{4} - \frac{ix\cos 2x}{4}$

Equating the imaginary part

$y = -\frac{x\cos 2x}{4}$

general soln

$y = c_1 \cos 2x + c_2 \sin 2x - \frac{x\cos 2x}{4}$

1)

$x^2(y - px) = p^2 y$ — (1) $[x^2 = u,\ y^2 = v]$

$x^2 = u$; $y^2 = v$

$\Rightarrow 2x\,dx = du$; $\Rightarrow 2y\,dy = dv$

$\frac{dv}{du} = \frac{2y\,dy}{2x\,dx}$ $\left[\text{where } \frac{dv}{du} = q\right]$

$\Rightarrow q = \frac{y}{x} \cdot p$

$\Rightarrow p = \frac{qx}{y}$ — (ii)

substituting in (1) we get,

$x^2\left(y - \frac{qx}{y} \cdot x\right) = \frac{q^2 x^2}{y^2 y} y$

$\Rightarrow x^2\left(\frac{y^2 - qx^2}{y}\right) = \frac{q^2 x^2}{y}$

$\Rightarrow u(v - qu) = q^2 u$

$\Rightarrow v - qu = q^2$

$\Rightarrow v = q^2 + qu$ — (iii)

which is the clairaut's eqn.

The general soln is

$v = c^2 + cu$

The general soln is $cu + c^2$

now, $y^2 = c^2 + cx^2$

$\Rightarrow c^2 + cx^2 - y^2 = 0$

discriminant is

$x^4 - 4 \cdot 1 \cdot (-y^2) = 0$

$\Rightarrow x^4 + 4y^2 = 0$, which is singular soln.

2) $(x^2 + y^2 - 1)p = xy(1 + p^2)$ $[x^2 = u,\ y^2 = v]$ — (1)

$x^2 = u$; $y^2 = v$

$\Rightarrow 2x\,dx = du$; $\Rightarrow 2y\,dy = dv$

$\frac{dv}{du} = \frac{y\,dy}{x\,dx}$

$\Rightarrow q = \frac{y}{x} p$ $\left[\text{where } \frac{dv}{du} = q\right]$

$\Rightarrow p = \frac{qx}{y}$ — (ii)

substituting in (1) we get

$$(x^2+y^2-1)\frac{qx}{y} = xy\left(1+\frac{q^2x^2}{y^2}\right)$$

$$\Rightarrow (x^2+y^2-1)\frac{qx}{y} = xy\left(\frac{y^2+q^2x^2}{y^2}\right)$$

$$\Rightarrow (x^2+y^2-1)q = y^2+q^2x^2$$

$$\Rightarrow (u+v-1)q = v+q^2u$$

$$\Rightarrow uq+vq-q = v+q^2u$$

$$\Rightarrow vq-v = q^2u-uq+q$$

$$\Rightarrow v(q-1) = q^2u-qu+q$$

$$\Rightarrow v = \frac{q^2u-qu+q}{q-1} \quad \text{---(iii)}$$

which is the clairaut's eqn.

The general soln is-

$$v = \frac{c^2u-cu+c}{c-1}$$

The general soln is - $\frac{c^2u-cu+c}{c-1}$

Now, $y^2 = \frac{c^2x^2-cx^2+c}{c-1}$

$$\Rightarrow y^2c-y^2-c^2x^2+cx^2-c=0$$

$$\Rightarrow -c^2x^2+c(x^2+y^2-1)-y^2=0$$

$$\Rightarrow c^2x^2-c(x^2+y^2-1)+y^2=0$$

c- discriminant is-

$$(x^2+y^2-1)^2-4\cdot x^2\cdot y^2=0$$

$\Rightarrow (x^2+y^2-1)^2-4x^2y^2=0$, which is singular soln.

3) $yp^2-2xp+y=0$ $[x^2=u,\ y^2=v]$ →(1)

$x^2=u$, $y^2=v$

$\Rightarrow 2x\,dx=du$ $\quad \Rightarrow 2y\,dy=dv$

$$\Rightarrow \frac{dv}{du} = \frac{2y\,dy}{2x\,dx}$$

$$\Rightarrow q = \frac{y}{x}p \quad \left[\text{where } \frac{dv}{du}=q\right] \quad \text{---(ii)}$$

$$\Rightarrow p = \frac{qx}{y}$$

substituting in (1) we get,

$$\frac{yq^2x^2}{y^2}-2x\cdot\frac{qx}{y}+y=0$$

$$\Rightarrow q^2x^2-2qx^2+y^2=0$$

$$\Rightarrow q^2u-2qu+v=0$$

$$\Rightarrow v = 2qu-q^2u \quad \text{---(iii)}$$

which is the clairaut's eqn.

The general soln is-

$$v = 2cu-c^2u$$

The general soln is- $2cu-c^2u$

Now, $y^2 = 2cx^2 - c^2x^2$

$\Rightarrow c^2x^2 - 2cx^2 + y^2 = 0$

c-discriminant is—

$4x^4 - 4x^2y^2 = 0$

$\Rightarrow x^4 - x^2y^2 = 0$

$\Rightarrow x^2 - y^2 = 0$, which is singular soln.

4) $y = px + p - p^2$ —(i)

$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$

$\Rightarrow p = p + (x + 1 - 2p)\frac{dp}{dx}$

$\Rightarrow (x+1-2p)\frac{dp}{dx} = 0$

$\Rightarrow y - p + p^2 = px$

$\Rightarrow x = \frac{y}{p} - 1 + p$

$\Rightarrow x = \frac{y}{p} + p - 1$

$\Rightarrow \frac{dx}{dy} = -\frac{y}{p^2}\frac{dp}{dy} + \frac{dp}{dy} + \frac{1}{p}$

$\Rightarrow \left(-\frac{y}{p^2} + 1\right)\frac{dp}{dy} = 0$

$\Rightarrow (-y + p^2)dp = 0$ [Let the y as constant]

$\Rightarrow \int(-y + p^2)dp = c$

$\Rightarrow -yp + \frac{p^3}{3} = c$ —(ii)

Now eliminating p from (i) and (ii) we get the general soln.

$y = px + p - p^2$ —(i)

$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$

$\Rightarrow (x - 2p + 1)\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0$ $\Rightarrow p = c$ —(ii)

$x - 2p + 1 = 0 \Rightarrow x = 2p - 1$ —(iii)

eliminating (i) and (ii) we get

$y = cx + c - c^2$ → this is general soln.

Also from (i) and (iii)

$y = p(2p-1) + p - p^2$
$= 2p^2 - p + p - p^2$
$= p^2$ —(iv)

from (iii) $\frac{x+1}{2} = p$ —(v)

substituting (v) in (iv)

$y = \frac{(x+1)^2}{4}$; $(x+1)^2 = 4y$ → This is singular soln.

5) $y = p(x-p) + \frac{a}{p}$ —(i)

$\frac{dy}{dx} = p\left(1 - \frac{dp}{dx}\right) + (x-p)\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$

$\Rightarrow p = p - \frac{p\,dp}{dx} + (x-p)\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$

$\Rightarrow \left(-p + x - p - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$

$\Rightarrow \left(x - 2p - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$

$\Rightarrow \int(p^2x - 2p^3 - a)dp = c$

$\Rightarrow \frac{xp^3}{3} - \frac{2p^4}{4} - ap = c$ [Let the x as constant]

$\Rightarrow \frac{xp^3}{3} - \frac{p^4}{2} - ap = c$ —(ii)

Now eliminating p from (i) and (ii) we get the general soln.

$\Rightarrow \left(x - 2p - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0$, $\Rightarrow p = c$ —(ii)

$x - 2p - \frac{a}{p^2} = 0 \Rightarrow x = 2p + \frac{a}{p^2}$ —(iii)

substituting (i) and (ii) we get,

$y = c(x - c) + \frac{a}{c}$ → This is general soln.

Also from (i) and (iii) we get,

$y = p\left(2p + \frac{a}{p^2} - p\right) + \frac{a}{p}$

$= p\left(p + \frac{a}{p^2}\right) + \frac{a}{p}$

$= p\left(\frac{p^3 + a}{p^2}\right) + \frac{a}{p}$

$\Rightarrow y = \frac{p^3 + 2a}{p}$ —(iv)

Now eliminating p from (iii) and (iv) we get the singular soln.

3) $y = px + \sin^{-1} p$ —(i)

$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

$\Rightarrow \left(x + \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow \int\left(x\sqrt{1-p^2} + 1\right)dp = c$

$\Rightarrow x\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1} p\right] + p = c$ —(ii)

now eliminating p from (i) and (ii) we get the general solⁿ.

$\frac{dp}{dx} = 0 \; ; \Rightarrow p = c$ —(ii)

$x + \frac{1}{\sqrt{1-p^2}} = 0$

$\Rightarrow x = -\frac{1}{\sqrt{1-p^2}}$ —(iii)

Eleminating (i) and (ii) we get,

$y = cx + \sin^{-1} c$ → This is the general solⁿ.

Also from (i) and (iii) we get,

$y = p\left(-\frac{1}{\sqrt{1-p^2}}\right) + \sin^{-1} p$

$= \frac{-p}{\sqrt{1-p^2}} + \sin^{-1} p$ —(iv)

∴ by eleminating p from (i) and (iii) we also get the singular solⁿ.

4) $(y - px)(p-1) = p$ → $\boxed{\text{Incorrect}}$

$\left(y - px\right)\frac{dp}{dx} + (p-1)\left(\frac{dy}{dx}\right.$ → $p = 0$

$\Rightarrow py - y - p^2x + px = p$

$\Rightarrow y(p-1) = p + p^2x - px$

$\Rightarrow y = \frac{p + p^2x - px}{p-1}$

$\Rightarrow \cancel{\frac{dy}{dx} = \frac{(p-1)\left[\frac{dp}{dx}\right.}{(p-1)^2}}$

$\Rightarrow y = \frac{p-1+1}{p-1} + \frac{p^2x}{p-1} - \frac{px}{p-1}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{(p-1)^2}\frac{dp}{dx} + \frac{p^2}{p-1} + x\left\{\frac{p^2}{(p-1)^2}\frac{dp}{dx} + \frac{2p}{(p-1)}\frac{dp}{dx}\right\}$

$\Rightarrow p = -\frac{1}{(p-1)^2}\frac{dp}{dx} + \frac{p^2}{p-1} - \frac{xp^2}{(p-1)^2}\frac{dp}{dx} + \frac{2px}{(p-1)}\frac{dp}{dx}$

$\Rightarrow p - \frac{p^2}{(p-1)} = \frac{dp}{dx}\left\{-\frac{1}{(p-1)^2} - \frac{xp^2}{(p-1)^2} + \frac{2px}{(p-1)}\right\}\frac{dp}{dx}$

$\Rightarrow \frac{p^2 - p - p^2}{(p-1)} = \left\{\frac{-1 - xp^2 + 2px(p-1)}{(p-1)^2}\right\}\frac{dp}{dx}$

$\Rightarrow p(p-1) = \left(1 + xp^2 - 2p^2x - 2px\right)\frac{dp}{dx}$

$\Rightarrow p(p-1) = \{-1 + x(p^2 - 2p^2 - 2p)\}\frac{dp}{dx}$

$\Rightarrow p(p-1) = \{1 + x(-p^2 - 2p)\}\frac{dp}{dx}$

$\Rightarrow \frac{p(p-1)}{1 + x(-p^2-2p)} = \frac{dp}{dx}$

$\Rightarrow \frac{dx}{dp} = \frac{1 + x(-p^2-2p)}{p(p-1)}$

$\Rightarrow \frac{dx}{dp} - x\left\{\frac{-p^2-2p}{p(p-1)}\right\} = \frac{1}{p(p-1)}$

$\Rightarrow \frac{dx}{dp} + x\left\{\frac{\cancel{p}(p+2)}{\cancel{p}(p-1)}\right\} = \frac{1}{p(p-1)}$

$I.F = e^{\int \frac{p+2}{p-1} dp}$

$= e^{\int \frac{p-1+3}{p-1} dp}$

$= e^{\int \left(1 + \frac{3}{p-1}\right) dp}$

$= e^{[p + 3\log(p-1)]}$

$\frac{dx}{dp} e^{p+3\log(p-1)} + \frac{x(p+2)}{(p-1)} e^{p+3\log(p-1)} = \frac{1}{p(p-1)} e^{p+3\log(p-1)}$

$\Rightarrow \frac{d}{dp}\left(e^{p+3\log(p-1)} x\right) = \frac{e^{p+3\log(p-1)}}{p(p-1)}$

$\Rightarrow \int d\{e^{p+3\log(p-1)}\} = \int \frac{e^{p+3\log(p-1)}}{p(p-1)} dp$

$\Rightarrow x e^{p+3\log(p-1)} = \int \frac{e^{\log e^p + \log(p-1)^3}}{p(p-1)} dp$

$\Rightarrow x e^{\log e^p + \log(p-1)^3} = \int \frac{e^{\log\{e^p (p-1)^3\}}}{p(p-1)} dp$

$\Rightarrow x e^p (p-1)^3 = \int \frac{e^p \cancel{(p-1)}^{3} (p-1)^2}{p\cancel{(p-1)}} dp$

$\Rightarrow x e^p (p-1)^3 = \int e^p \frac{(p^2-2p+1)}{p} dp$

$\Rightarrow x e^p (p-1)^3 = \int e^p \left(p - 2 + \frac{1}{p}\right) dp$

LC

$(y - px)(p-1) = p$ — (i)

$\Rightarrow y - px = \frac{(p-1)+1}{p-1}$

$\Rightarrow y = px + 1 + \frac{1}{p-1}$

$\Rightarrow \frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} - \frac{1}{(p-1)^2}\frac{dp}{dx}$

$\Rightarrow \left\{x - \frac{1}{(p-1)^2}\right\}\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0, \Rightarrow p = c$ — (ii)

$x = \frac{1}{(p-1)^2}$ — (iii)

Eleminating (i) and (ii) we get, $(y - cx)(c-1) = c \rightarrow$ This is general soln.

Also from (i) and (iii) $\left(y - \frac{p}{(p-1)^2}\right)(p-1) = p$ — (iv)

Now eleminating p from (iii) and (iv) we will get the singular solⁿ.

15/4/22

Class

Ex-1 Solve by Cramer's rule—

$x + y - z = 6$

$3x - 2y + z = -5$

$x + 3y - 2z = 14$

$$\left[\begin{matrix} 1 & 1 & -1 & 1 & 1 \\ 3 & -2 & 1 & 3 & -2 \\ 1 & 3 & -2 & 1 & 3 \end{matrix}\right]$$

This is Cramer's rule (calculation of matrix)

$= (4 + 1 - 9) - (2 + 3 - 6)$

$= -4 + 1$

$= -3$ $D = -3$

Here coefficient matrix $D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}$

$D_x = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix}$

$= 6(4 - 3) - 1(10 - 14) - 1(-15 + 28)$

$= 6 + 4 - 13$

$= -3$

$D_y = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix}$

$\boxed{= 1(-10 - 14) - 6(6 - 1) - 1(42 + 5) \\ = -24 - 30 - 47 \\ =}$

$= 1(10 - 14) - 6(-6 - 1) - 1(42 + 5)$

$= -9$

$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$

$= 1(-28 + 15) - 1(42 + 5) + 6(9 + 2)$

$= -13 - 47 + 66$

$= -6$

As $D \neq 0$, unique solution for x, y, z exists

$x = \frac{D_x}{D} = \frac{-3}{-3} = 1$

$y = \frac{D_y}{D} = \frac{-9}{-3} = 3$

$z = \frac{D_z}{D} = \frac{-6}{-3} = 2$

③ Solve by Cramer's rule:—

$x + 2y - 3z = 1$

$2x - y + z = 4$

$x + 3y = 5$

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$$

$= 1(0-3) - 2(0-1) - 3(6+1)$

$= -3 + 2 - 21$

$= -22$

$$D_x = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 5 & 3 & 0 \end{vmatrix}$$

$= 1(0-3) - 2(0-5) - 3(12+5)$

$= -3 + 10 - 51$

$= -44$

$$D_y = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & 1 \\ 1 & 5 & 0 \end{vmatrix}$$

$= 1(0-5) - 1(0-1) - 3(10-4)$

$= -5 + 1 - 18$

$= -22$

$$D_z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 3 & 5 \end{vmatrix}$$

$= 1(-5-12) - 2(10-4) + 1(6+1)$

$= -17 - 12 + 7$

$= -22$

$x = \frac{D_x}{D} = \frac{-44}{-22} = 2$

$y = \frac{D_y}{D} = \frac{-22}{-22} = 1$

$z = \frac{D_z}{D} = \frac{-22}{-22} = 1$

$x = 2$

$y = 1$

$z = 1$

H.W.

① Solve the equation by cramer's rule:- [Ans: $x = \frac{13}{10}, y = \frac{7}{14}, z = \frac{1}{20}$]

$x + 2y + 3z = 1$

$3x - 2y + z = 2$

$4x + 2y + z = 3$

② Apply cramer's rule to show that $x = 3, y = 1, z = 2$ is a solution of

$2x + 3y + z = 11$

$x + y + z = 6$

$5x - y + 10z = 34$

① $D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$

$= 1(-2-2) - 2(3-4) + 3(6+8)$

$= -4 + 2 + 42$

$= 40$

$D_x = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix}$

$= 1(-2-2) - 2(2-3) + 3(4+6)$

$= -4 + 2 + 30$

$= 28$

$D_y = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$

$= 1(2-3) - 1(3-4) + 3(9-8)$

$= -1 + 1 + 3$

$= 3$

$D_z = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -2 & 2 \\ 4 & 2 & 3 \end{vmatrix}$

$= 1(-6-4) - 2(9-8) + 1(6+8)$

$= -10 - 2 + 14$

$= 2$

$x = \frac{D_x}{D} = \frac{7}{10}$

$y = \frac{D_y}{D} = \frac{3}{40}$

$z = \frac{D_z}{D} = \frac{1}{20}$

(3) $D = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 10 \end{vmatrix}$

$= 2(10+1) - 3(10-5) + 1(-1-5)$

$= 22 - 15 - 6$

$= 1$

$D_x = \begin{vmatrix} 11 & 3 & 1 \\ 6 & 1 & 1 \\ 34 & -1 & 10 \end{vmatrix}$

$= 11(10+1) - 3(60-34) + 1(-6-34)$

$= 121 - 78 - 40$

$= 3$

$D_y = \begin{vmatrix} 2 & 11 & 1 \\ 1 & 6 & 1 \\ 5 & 34 & 10 \end{vmatrix}$

$= 2(60-34) - 11(10-5) + 1(34-30)$

$= 52 - 55 + 4$

$= 1$

$D_z = \begin{vmatrix} 2 & 3 & 11 \\ 1 & 1 & 6 \\ 5 & -1 & 34 \end{vmatrix}$

$= 2(34+6) - 3(34-30) + 11(-1-5)$

$= 80 - 12 - 66$

$= 2$

$\therefore x = 3, y = 1, z = 2$ (Ans)

③ Solve by matrix method :-

$x + z = 0$

$3x + 4y + 5z = 2$

$2x + 3y + 4z = 1$

Soln

Here co-efficient matrix is $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

So we can write

$AX = B$

$X = A^{-1}B$

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$= 1(16-15) + 1(9-8)$

$= 1+1$

$= 2$

$$A^{-1} = \text{adj}A = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (16-15) & -(12-10) & (9-8) \\ -(0-3) & (4-2) & -(3-0) \\ (0-4) & -(5-3) & (4-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$X = \frac{1}{2} \begin{bmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0+6-4 \\ 0+4-2 \\ 0-6+4 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$x = 1$, $y = 1$, $z = -1$

Ex-(2) Solve the equations by matrix method

$2x + 3y + z = 11$
$x + y + z = 6$
$5x - y + 10z = 34$

$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 11 \\ 6 \\ 34 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$AX = B$, $X = A^{-1}B$

$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & 10 \end{vmatrix}$

$= 2(10+1) - 3(10-5) + 1(-1-5)$

$= \cancel{20}\ 22 - 15 - 6$

$= 1 \neq 0$, A^{-1} exist, uniquely the system exists.

$adj A = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ -1 & 10 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 10 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ -1 & 10 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 5 & 10 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$

$= \begin{bmatrix} (10+1) & -(10-5) & (-1-5) \\ -(30+1) & (20-5) & -(-2-15) \\ (3-1) & -(2-1) & (2-3) \end{bmatrix}^T$

$= \begin{bmatrix} 11 & -5 & -6 \\ -31 & 15 & 17 \\ 2 & -1 & -1 \end{bmatrix}^T$

$= \begin{bmatrix} 11 & -31 & 2 \\ -5 & 15 & -1 \\ -6 & 17 & -1 \end{bmatrix}$

$X = \begin{bmatrix} 11 & -31 & 2 \\ -5 & 15 & -1 \\ -6 & 17 & -1 \end{bmatrix} \begin{bmatrix} 11 \\ 6 \\ 34 \end{bmatrix}$

$= \begin{bmatrix} 121 - 186 + 68 \\ -55 + 90 - 34 \\ -66 + 102 - 34 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$x = 3$, $y = 1$, $z = 2$

19/4/20

• vector space (mam)

Definition – A non-empty set V is said to form a vector space over a field F if –

(i) there is a binary composition $+$ on V, called addition, satisfying the conditions

(V1) $\alpha+\beta \in V$, for all $\alpha, \beta \in V$

(V2) $\alpha+\beta = \beta+\alpha$, for all $\alpha, \beta \in V$

(V3) $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha, \beta, \gamma \in V$

(V4) there exists an element θ in V that $\alpha+\theta=\alpha$ for all α in V.

(V5) for each α in V there exists an element $-\alpha$ in V such that $\alpha+(-\alpha)=\theta$

and (ii) there is an external composition of F with V, satisfying (V6) $c\alpha \in V$, for all $c \in F$, all $\alpha \in V$.

(V7) $c(d\alpha) = (cd)\alpha$ for all $c, d \in F$, all $\alpha \in V$

(V8) $c(\alpha+\beta) = c\alpha + c\beta$, for all $c \in F$, all $\alpha, \beta \in V$

(V9) $(c+d)\alpha = c\alpha + d\alpha$ for all $c, d \in F$, all $\alpha \in V$

(V10) $1\alpha = \alpha$, 1 being the identity in F.

Theorem – In a vector space V over a field F

(i) $0\alpha = \theta$ for all $\alpha \in V$

(ii) $c\theta = \theta$ for all $c \in F$

(iii) $(-1)\alpha = -\alpha$ for all $\alpha \in V$, 1 being the identity element in F

(iv) $c\alpha = \theta$ implies either $c = 0$, or $\alpha = \theta$

Proof

(i) 0 is the zero element

$0+0=0 \Rightarrow (0+0)\alpha = 0\alpha$ in V

$\Rightarrow 0\alpha + 0\alpha = 0\alpha + \theta$

$\Rightarrow 0\alpha + 0\alpha + (-0\alpha) = 0\alpha + \theta - 0\alpha$

$\Rightarrow 0\alpha + (0\alpha + (-0\alpha)) = \theta$

$\Rightarrow 0\alpha = \theta$

Ex-1 R^3 is a vector space over R (R = real no.)

$R^3 = \{(a_1, a_2, a_3) / a_i \in R\}$ — a_1, a_2, a_3 = 3 ordered

$(-2, 5, 0) \in R^3$

$(1, 0, 0) \in R^3$

external operation

$3 \times (5, 6, 8)$
$= (15, 18, 24)$
$(2,3) = 2 \times 3$.

If we take the element from F and multiply it with V, ultimately the product will be V, we will get V ultimately. second type of element $= V$, scalar elements from F

Proof

(i) $c\theta = \theta$

$F \ni c$, $\alpha \in V \in R^3$

Ex – $7 \times (-5, 0, 2) \in V = R^3$
$= (-35, 0, 14) \in V = R^3$

c is the identity vector

- (ii) $c\theta = \theta$

Proof – $\theta + \theta = \theta$ — ①

Now $c(\theta + \theta) = c\theta$

$\Rightarrow c\theta + c\theta = c\theta$

$\Rightarrow -c\theta \in V$ add $-c\theta$ to ①, we have

$-c\theta + [c\theta + c\theta] = -c\theta + c\theta$

$\Rightarrow \theta + c\theta = \theta$

$\Rightarrow c\theta = \theta$

- **Defination – subspace**

Let V be a vector space over a field F with respect to $+$ and multiplication by elements of F. Then W is said to be a subspace of V if W is a non-empty subset of V and W is a vector space over F.

Ex-①

Examine if the set vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly independent in R^3.

Soln – Let $\alpha_1 = (2,1,1)$

$\alpha_2 = (1,2,2)$

$\alpha_3 = (1,1,1)$

If $\alpha_1, \alpha_2, \alpha_3$ vectors are linearly independent then we have $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (0,0,0)$

then $a_1 = 0, a_2 = 0, a_3 = 0$.

Consider $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0,0)$

Now $c_1(2,1,1) + c_2(1,2,2) + c_3(1,1,1) = (0,0,0)$

$\Rightarrow (2c_1, c_1, c_1) + (c_2, 2c_2, 2c_2) + (c_3, c_3, c_3) = (0,0,0)$

$\Rightarrow (2c_1 + c_2 + c_3,\ c_1 + 2c_2 + c_3,\ c_1 + 2c_2 + c_3) = (0,0,0)$

Therefore,

$2c_1 + c_2 + c_3 = 0$ — (i)

$c_1 + 2c_2 + c_3 = 0$ — (ii)

$c_1 + 2c_2 + c_3 = 0$ — (iii)

$\cancel{A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}}$ $\cancel{B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

From first two equation –

$$\frac{c_1}{1-2} = \frac{c_2}{1-2} = \frac{c_3}{4-1} = k \text{ (say)}$$

$$\Rightarrow \frac{c_1}{-1} = \frac{c_2}{-1} = \frac{c_3}{3} = k \text{ (say)}$$

$\Rightarrow c_1 = -k$

$c_2 = -k$

$c_3 = -3k$

The last equation is satisfied by these values. Since k is any number

c_1, c_2, c_3 are not all zero

$\therefore$ this system is not linearly independent.

$c_1(2,1,1) + c_2(1,2,2) + c_3(1,1,1) = (0,0,0)$

$\Rightarrow 1(2,1,1) + 1(1,1,2) + -3(1,1,1)$

$= (2,1,1) + (1,2,2) + (-3,-3,-3)$

$= (0,0,0)$ satisfies $\boxed{\text{Rough}}$

2. $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a linearly independent set in R^3.

Let $\alpha_1 = 1,0,0$

$\alpha_2 = 0,1,0$

$\alpha_3 = 0,0,1$

Consider $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0,0)$

$\Rightarrow c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$\Rightarrow (c_1,0,0) + (0,c_2,0) + (0,0,c_3) = (0,0,0)$

$\Rightarrow (c_1, c_2, c_3) = (0,0,0)$

$\therefore c_1 = 0,\ c_2 = 0,\ c_3 = 0$

$\therefore$ This is linearly independent equation.

20/4/22

math (2nd)

Ex-1 Solve: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x}$

Let $y = e^{mx}$ be a trial solution of the reduced eqn. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

The axilary eqn

$m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$m = 0, m = -1, -1$

$\therefore$ The Complementary function is $y_c = c_1 e^{0x} + (c_2 + c_3 x)e^{-1x}$ (C.F)

$= c_1 + (c_2 + c_3 x)e^{-x}$

where c_1, c_2, c_3 all are arbitrary constants.

P.I $(y_p) = \frac{1}{D(D+1)^2}(e^{2x}) = \frac{1}{2(2+1)^2} e^{2x} = \frac{1}{18} e^{2x}$

The soln is $y = y_c + y_p = c_1 + (c_2 + c_3)e^{-x} + \frac{1}{18}e^{2x}$

2. Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = xe^x + e^x$

Let $y = e^{mx}$ be a trial solution of the reduced eqn.

$m^3 - 3m^2 + 3m - 1 = 0$

$\Rightarrow (m-1)^3 = 0 \Rightarrow m = 1, 1, 1$

$\therefore$ The C.F. $(y_c) = (c_1 + c_2 x + c_3 x^2) e^{+1 \cdot x}$

where c_1, c_2, c_3 are arbitrary contents.

P.I. $(y_p) = \frac{1}{(D-1)^3}(xe^x + e^x)$

$= \frac{1}{(D-1)^3}(xe^x) + \frac{1}{(D-1)^3} e^x$

$= e^x \frac{1}{(D+1-1)^3}(x) + e^x \cdot \frac{1}{(D+1-1)^3}(1)$

E: when there is a product in x and, then there is an exponential then the 'D' is replaced by $(D+1)$' i.e. $\frac{1}{(D-1)^3}(xe^x) = e^x \cdot \frac{1}{(D+1-1)^3}(x)$

$= e^x \cdot \frac{1}{D^3} x + e^x \cdot \frac{1}{D^3}$

$\frac{x^4}{24} e^x + \frac{x^3}{6} e^x$

$\boxed{\begin{array}{l} D^3(x) = \text{diff} \\ \frac{1}{D^3}(x) = \text{integration} \\ \text{3 times integration} \end{array}}$

$= \frac{e^x x^3}{6}\left(1 + \frac{x}{4}\right)$

Soln is

$y = (c_1 + c_2 x + c_3 x^2) e^x + \frac{x^3 e^x}{6}\left(1 + \frac{x}{4}\right)$

(3) Solve $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 5\cos x$

Let $y = e^{mx}$ be a trial solution of the reduced eqn

$3m^2 + 2m - 8 = 0$

$\Rightarrow 3m^2 + (6-4)m - 8 = 0$

$\Rightarrow 3m^2 + 6m - 4m - 8 = 0$

$\Rightarrow 3m(m+2) - 4(m+2) = 0$

$\Rightarrow (m+2)(3m-4) = 0$

$\Rightarrow m = -2, \frac{4}{3}$

The C.F., $y_c = c_1 e^{-2x} + c_2 e^{\frac{4x}{3}}$

where c_1, c_2 are arbitrary Contants.

P.I. $(y_p) = \frac{1}{(3D^2 + 2D - 8)}(5\cos x)$

$= 5 \frac{1}{3D^2 + 2D - 8}(\cos x)$

$= 5 \cdot \frac{1}{3(-1) + 2D - 8}(\cos x) = 5 \cdot \frac{1}{(2D - 11)}(\cos x)$

$= 5 \cdot \frac{(2D+11)}{4D^2 - 121}(\cos x)$

$= \frac{5(2D+11)(\cos x)}{4D^2 - 121}$

$= -\frac{5}{125}[-2\sin x + 11\cos x]$

$= \frac{1}{25}(2\sin x \pm 11\cos x)$

The solⁿ is =

$y = c_1 e^{-2x} + c_2 e^{\frac{4x}{3}} + \frac{1}{25}(2\sin x \mp 11\cos x)$

(4) Solve: $(D^4 - 4D^3 + 6D^2 - 4D + 1)y = e^{2x}\cos x$

Let $y = e^{mx}$ be a trial solution of the reduced sol eqn.

The A.E is $m^4 - 4m^3 + 6m^2 - 4m + 1 = 0$

$\Rightarrow (m-1)^4 = 0 \quad m = 1, 1, 1, 1$

$m^4 - m_{c_1} m^3 + m_{c_2} m^2 - m_{c_3} m \cdot 1 = 14$

The CF $(y_c) = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{x}$

The (P.F) $(y_p) = \frac{1}{(D-1)^4}(e^{2x}\cos x)$

$= \cancel{\frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 1}(e^{2x}\cos x)}$

$= e^{2x}\frac{1}{(D+2-1)^4}(\cos x)$

$= e^{2x} \cdot \frac{1}{(D+1)^4}(\cos x)$

$= e^{2x} \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1}(\cos x)$

$= e^{2x} \cdot \frac{1}{(-1)^2 + 4(-1)D + 6(-1) + 4D + 1}(\cos x)$

$= e^{2x} \cdot \frac{1}{1 - 4D - 6 + 4D + 1}(\cos x)$

$= e^{2x} \cdot \frac{1}{-4}(\cos x)$

$= \cancel{\frac{1}{-4}} e^{2x} = -\frac{1}{4}e^{2x}\cos x$

The solⁿ is -

$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{x} - \frac{1}{4}e^{2x}\cos x$

(4) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 5\cos x$, given that $y = \frac{dy}{dx} = 0$, $x = 0$.

$m^2 - 2m + 2 = 0$

$\cancel{m^3 - (2+1)m + 2 = 0}$

$\cancel{\Rightarrow m^3 - 2m - m + 2 = 0}$

$\cancel{\Rightarrow m(m-2) - 1(m-2) = 0}$

$\cancel{\Rightarrow (m-2)(m-1) = 0}$

$\cancel{\Rightarrow m = 2, 1}$

$m = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

~~$y_c = C.F = c_1 e^{2x} + c_2 e^{x}$~~

$y_c = C.F = e^{1+x}(c_1 \cos x + c_2 \sin x)$

$P.I = (y_p) = \frac{1}{D^2-2D+2}(5\cos x)$

$= 5 \frac{1}{D^2-2D+2}(\cos x)$

$= 5 \cdot \frac{1}{(-1)-2D+2}(\cos x)$

$= 5 \cdot \frac{1}{1-2D}(\cos x)$

$= 5 \cdot \frac{1+2D}{1-4D^2}(\cos x)$

$= 5 \cdot \frac{(1+2D)(\cos x)}{1+4}$

$= (\cos x - 2\sin x)$

$\therefore$ The solⁿ is -

$y = e^x(c_1 \cos x + c_2 \sin x) + (\cos x - 2\sin x)$ — (I)

$\frac{dy}{dx} = e^x(-c_1 \sin x + c_2 \cos x) + (c_1 \cos x + c_2 \sin x)e^x$

$- \sin x - 2\cos x$ — (II)

$= e^x[\cos x(c_1 + c_2) + \sin x(-c_1 + c_2)]$

$- \sin x - 2\cos x$

If $\frac{dy}{dx} = 0$ and $x = 0$

$e^0 [c_1 + c_2] - 2 = 0$

$\Rightarrow c_1 + c_2 = 2$

from (I)

$\Rightarrow 0 = e^0(c_1) + 1$

$\Rightarrow c_1 = -1$

from (II)

$\Rightarrow 0 = e^0(c_1) - 2 + c_2 e^0$

$\Rightarrow c_2 + c_1 = 2$

$\Rightarrow -1 + c_2 = 2$ [from (I)]

$\Rightarrow c_2 = 3$

The solⁿ is -

$y = e^x(-\cos x + 3\sin x) + \cos x - 2\sin x$

[from (I)]

• Euler's homogeneous equation

General form -

$x^n \frac{d^n y}{dx^n} + A_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + A_n y = g(x)$

for $n = 2$, $x^2 \frac{d^2y}{dx^2} + P_1 x \frac{dy}{dx} + P_2 y = g(x)$ — (I)

$n = 3$, $x^3 \frac{d^3y}{dx^3} + P_1 x^2 \frac{d^2y}{dx^2} + P_2 x \frac{dy}{dx} + P_3 y = g(x)$

Put $x = e^z$, $z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{1}{x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$$

from ①,

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + P_1\frac{dy}{dz} + P_2 y = g(z)$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} + (P_1-1)\frac{dy}{dz} + P_2 y = S(e^z)}$$

$y = f(z) = f(\log x)$

Ex-① Solve: $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x^4$.

Put $x = e^z$, i.e. $\log x = z$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz},\quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting in the given eq^n

$$\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 5\frac{dy}{dz} + 4y = e^{4z}$$

$$\Rightarrow \frac{d^2y}{dz^2} + (5-1)\frac{dy}{dz} + 4y = e^{4z}$$

$$\Rightarrow \frac{d^2y}{dz^2} + 4\frac{dy}{dz} + 4y = e^{4z} \quad \text{— ①}$$

Let $y = e^{mz}$ be the trial sol^n of the reduced eqn $\frac{d^2y}{dz^2} + 4\frac{dy}{dz} + 4y = 0$

A.E $m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$

$\Rightarrow m = -2, -2$

C.F. $y_c = (C_1 + C_2 z)e^{-2z}$

P.I. $= y_p = \frac{1}{(D+2)^2}(e^{4z})$

$$= \frac{1}{(4+2)^2}\cdot e^{4z}$$

$$= \frac{1}{36}e^{4z}$$

The Sol^n is $(C_1 + C_2 z)e^{-2z} + \frac{1}{36}e^{4z}$

$$= (C_1 + C_2\log x)e^{-2\log x} + \frac{1}{36}e^{4\log x}$$

$$= (C_1 + C_2\log x)e^{\log x^{-2}} + \frac{1}{36}e^{\log x^4}$$

$$= (C_1 + C_2\log x)\frac{1}{x^2} + \frac{1}{36}\cdot x^4$$

Suggesions

1) matrix symmetric and skew type.
2) Properties of determinant
3) characteristic equation and eigen values [eigen vectors X]
4) Cayley Hamilton theorem and finding A^{50}, A^{-1} etc, verification cayley Hamilton theorem.
5) solve by Cramer's rule
6) solve by matrix method
7) Echelon form and reduction to it.
8) Properties of vector space
9) Rank of matrix, inverse of matrix
10) theory of equation
11) vector space
12) Inequalities
13) complex no.

• Theory of eqn

Ex-① If α, β, γ are the roots of the cubic eqn $x^3+px^2+qx+r=0$, find the value of $\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3$.

We have

$\alpha+\beta+\gamma=-p$

$\alpha\beta+\beta\gamma+\gamma\alpha=q$

$\alpha\beta\gamma=-r$

we know $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

$$\boxed{(a+b+c)^3 = a^3+b^3+c^3+3[(a+b+c)(bc+ca+ab)-abc]}$$

$\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^3 - 3[\alpha\beta\gamma(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha^2\beta^2\gamma^2]$

$= q^3 - 3\{(-r)(-p)q - r^2\}$

$= q^3 - 3pqr + 3r^2$

Ex-② • Inequality

If a, b, c and all real numbers. Prove that $a^2+b^2+c^2 \geq ab+bc+ca$

Soln

$\frac{1}{2}(a-b)^2 = \frac{1}{2}(a^2+b^2) - ab$

$\frac{1}{2}(b-c)^2 = \frac{1}{2}(b^2+c^2) - bc$

$\frac{1}{2}(c-a)^2 = \frac{1}{2}(c^2+a^2) - ac$

$\therefore a^2+b^2+c^2-ab-bc-ca$

$= \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \geq 0$

$= \frac{1}{2}(a^2+b^2)-ab+\frac{1}{2}(b^2+c^2)-bc+\frac{1}{2}(c^2+a^2)-ac \geq 0$

$a^2+b^2+c^2 \geq ab+bc+ca$

• Inequality

(1) If a, b, c are all real positive no.

Prove that,

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c$$

Soln

$(a+b)^2+(a-b)^2 = 2(a^2+b^2)$

Therefore $2(a^2+b^2) \geq (a+b)^2$, the equality occurs when $a=b$

Hence,

$\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}$, since $a+b>0$

$\frac{b^2+c^2}{b+c} \geq \frac{b+c}{2}$, since $b+c>0$

$\frac{c^2+a^2}{c+a} \geq \frac{c+a}{2}$, since $c+a>0$

Adding $\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c$

Equality occurs when $a=b=c$

• vector space

(2) Show that $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a linearly independent set, since $a+b>0$

Let $\alpha_1=(1,0,0)$, $\alpha_2=(0,1,0)$

$\alpha_3=(0,0,1)$

Consider $\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3=(0,0,0)$

there

$a_1(1,0,0)+a_2(0,1,0)+a_3(0,0,1)=(0,0,0)$

$\Rightarrow (a_1, a_2, a_3)=(0,0,0)$

$\therefore (a_1, a_2, a_3)=(0,0,0)$

$a_1=0, a_2=0, a_3=0$

$\therefore$ This is linearly independent set.

• method of undetermined coefficients 27/4/22

Ex-(1) Using the method of undetermined coefficients. Solve the eqn $\frac{d^2y}{dx^2}+4y=3\sin x$.

The given eqn is $\frac{d^2y}{dx^2}+4y=3\sin x$ —(1)

Let $y=e^{mx}$ be a trial solⁿ of the reduced eqⁿ $\frac{d^2y}{dx^2}+4y=0$

The A.E. $m^2+4=0$

$\Rightarrow m^2=-4$

$\Rightarrow m=\pm\sqrt{-4}$

$=\pm 2i$

The C.F is $y_c=c_1\cos 2x+c_2\sin 2x$

we assume that functional integral as

$y_p=A\cos x+B\sin x$ —(1)

$\frac{dy_p}{dx} = -A\sin x + B\cos x$

$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

Substitute in (1) the values of $\frac{d^2y_p}{dx^2}$, y_p

$\Rightarrow 3A\cos x + 3B\sin x = 3\sin x$ —(ii)

Equating the coff. of $\cos x$, $\sin x$ in both sides of (ii) we get

$3A = 0$, $3B = 3$

$\Rightarrow A = 0$, $B = 1$

From (ii) $y_p = \sin x$

The solution is $y = y_c + y_p$

$= c_1\cos 2x + c_2\sin 2x + \sin x$

Ex (2) Using the method of underdermind coeff. solve the eqn $\frac{d^2y}{dx^2} + 4y = \cos 2x$.

The general eqn is $\frac{d^2y}{dx^2} + 4y = \cos 2x$ —(1)

Let $y = e^{mx}$ be a trial soln of the reduced eqn $\frac{d^2y}{dx^2} + 4y = 0$

$m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

The C.F. $= y_c = c_1\cos 2x + c_2\sin 2x$

we assume that the particular integral is $y_p = x(A\cos 2x + B\sin 2x)$ —(ii) [As there is $\sin 2x / \cos 2x$ so, x will be multiplied]

$\frac{dy_p}{dx} = A\cos 2x + B\sin 2x + x(-2A\sin 2x + 2B\cos 2x)$

$\frac{d^2y_p}{dx^2} = 2(-2A\sin 2x + 2B\cos 2x) + x(-4A\cos 2x - 4B\sin 2x)$

Substituting in (1) the values of $\frac{d^2y_p}{dx^2}$, y_p

$-4A\sin 2x + 4B\cos 2x - 4Ax\cos 2x - 4Bx\sin 2x + 4Ax\cos 2x + 4xB\sin 2x = \cos 2x$

$\Rightarrow -4A\sin 2x + 4B\cos 2x = \cos 2x$ —(iii)

Equating the coff of $\cos 2x$, $\sin 2x$ in both side of (iii) we get.

$-4A = 0$, $4B = 1$

$\Rightarrow A = 0$, $\Rightarrow B = 1/4$

From (ii) $y_p = \frac{1}{4}x\sin 2x$

The soln $y = y_c + y_p$

$= c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x\sin 2x$

Ex-② Using the ~~m~~ method of undetermined coeff. solve eqⁿ $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = x^2 - 2x + 2$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = x^2 - 2x + 2$$

The general eqⁿ is -

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = x^2 - 2x + 2 \quad \text{--- (1)}$$

Let $y = e^{mx}$ be a trial solⁿ of the reduced eqⁿ

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

~~The C.F~~

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 \pm \sqrt{3}$$

The C.F. $= y_c = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

We assume that the particular integral is -

$$y_p = A_1 + A_2 x + A_3 x^2 \quad \text{--- (ii)}$$

$$\Rightarrow \frac{dy_p}{dx} = A_2 + 2x A_3$$

$$\Rightarrow \frac{d^2y_p}{dx^2} = 2A_3$$

Substituting in (1) the values of $\frac{d^2y_p}{dx^2}, \frac{dy_p}{dx}, y_p$

$$(2A_3) - 4(A_2 + 2xA_3) + (A_1 + A_2 x + A_3 x^2) = x^2 - 2x + 2$$

$$\Rightarrow A_3 x^2 + (-8A_3 + A_2)x + (2A_3 - 4A_2 + A_1) = x^2 - 2x + 2 \quad \text{--- (iii)}$$

Equating the coeff. of like powers of x and the constant term on the both sides of (iii)

$A_3 = 1$; $-8A_3 + A_2 = -2$; $2A_3 - 4A_2 + A_1 = 2$

$\Rightarrow -8 + 2 = -A_2$; $\Rightarrow 2 - 4 \times 6 + A_1 = 2$

$\Rightarrow A_2 = 6$; $\Rightarrow A_1 = 24$

From (ii) $y_p = 1 + 6x + 24x^2$

The solⁿ is $y = y_p + y_c$

$$= 1 + 6x + 24x^2 + c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$$

• Method of variation of parameters

Ex-① Using the method of ~~value~~ variation of parameters solve the eqⁿ $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

Let $y = e^{mx}$ be a trial solⁿ of the reduced eqⁿ $\frac{d^2y}{dx^2} + a^2 y = 0$

The A.E. is $m^2 + a^2 = 0$

$$\Rightarrow m = \pm ai$$

The C.F. is $y_c = c_1 \cos ax + c_2 \sin ax$

Let $y_1 = \cos ax$, $y_2 = \sin ax$

We wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a(\cos^2 ax + \sin^2 ax) \quad [a \neq 0]$$

$\therefore y_1$ and y_2 are linearly independent

We assume the particular solⁿ as

$y_p = v_1 \cos ax + v_2 \sin ax$ —(ii)

[v_1, v_2 are functions of x] → After that [shortcut]

$\frac{dy_p}{dx} = -v_1 a\sin ax + v_2 a\cos ax + v_1'\cos ax + v_2'\sin ax$

We choose v_1, v_2 such that

$v_1'\cos ax + v_2'\sin ax = 0$ —(iii)

$\therefore \frac{dy_p}{dx} = -v_1 a\sin ax + v_2 a\cos ax$

$\frac{d^2 y_p}{dx^2} = -v_1 a^2\cos ax - v_2 a^2 \sin ax$
$- v_1' a\sin ax + v_2' a\cos ax$

The ~~general eqⁿ is~~

Substituting the values of $\frac{d^2y_p}{dx^2}$, $\frac{dy_p}{dx}$ in (1) and we get,

$-v_1 a^2 \cancel{\cos ax} - v_2 a^2 \sin ax - v_1' a\sin ax + v_2' a\cos ax + a^2 v_1 \cos ax + a^2 v_2 \cos ax = \sec ax$

$\Rightarrow -v_1' a\sin ax + v_2' a\cos ax = \sec ax$ —(iv)

Shortcut

$v_1'\cos ax + v_2'\sin ax = 0$ —(iii) $\quad [y_p = v_1\cos ax + v_2 \sin ax]$ —(ii)

$\Rightarrow -av_1'\sin ax + v_2' a\cos ax = \sec ax$ —(iv) (characteristics eqⁿ)

(iii) × $(-a\sin ax)$ − (iv) × $\cos ax$

$\Rightarrow -v_1' \cos ax \sin ax - v_2' a\sin^2 ax = 0$

$\Rightarrow -a\cos ax\, v_1' \sin ax + v_2' a\cos^2 ax = 1$

(+) (−) (−)

$\Rightarrow -\frac{a}{2}(\sin^2 ax + \cos^2 ax) = -1$

$\Rightarrow v_2' = \frac{1}{a}$

Integrating $v_2 = \frac{x}{a}$ $\quad [\because \int v_2' = v_2]$

(3) × $(a\cos ax)$ + (4) × $\sin ax$

$\Rightarrow a\cos ax \cdot v_1' \cos ax + v_2' a\cos ax = 0$

$\Rightarrow -v_1' a\sin^2 ax + v_2' a\cos ax \sin ax = \sec ax \sin ax$

(+) (−) (−)

$\Rightarrow v_1' a = -\tan ax$

$\Rightarrow v_1' = -\frac{\tan ax}{a}$

Integrating $v_1 = +\frac{1}{a^2}\log\cos ax$

From (ii) we have

$y_p = \left(\frac{1}{a^2}\log(\cos x)\right)\cos ax + \left(\frac{x}{a}\right)\sin ax$

The solⁿ is $y = y_c + y_p$
$= c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\log\cos ax \cos ax + \frac{x}{a}\sin ax$

Using the method of variable of parameters solve the eqn $\frac{d^2y}{dx^2}+a^2y = \tan ax$.

Solve the eqn $= \frac{d^2y}{dx^2}+a^2y = \cos ax$.

$\frac{d^2y}{dx^2}+a^2y = \tan ax$ — (I)

(i) Let $y = e^{mx}$ be a trail soln of the reduced eqn $\frac{d^2y}{dx^2}+a^2y=0$

The A.E $= m^2+a^2=0$

$\Rightarrow m = \pm ia$

The C.F $= y_c = c_1\cos ax + c_2\sin ax$

Let $y_1 = \cos ax$, $y_2 = \sin ax$

Wronskian $W(y_1, y_2) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$= a\cos^2 ax + a\sin^2 ax$

$= a(\cos^2 ax+\sin^2 ax)\,[a\neq 0]$

y_1 and y_2 are linearly independent.

We assume the particular soln as

$y_p = v_1\cos ax + v_2\sin ax$ — (II)

$v_1'\cos ax + v_2'\sin ax = 0$ — (III)

$-a v_1'\sin ax + a v_2'\cos ax = \tan ax$ (IV)

(III) $\times(-a\sin ax)$ and (IV) $\times(\cos ax)$ we get,

$-a v_1'\cos ax\sin ax - a v_2'\sin^2 ax = 0$

$-a v_1'\cos ax\sin ax + a v_2'\cos^2 ax = \tan ax\cos ax$

$(+)\quad(-)\quad(-)$

$-a v_2' = +\sin ax$

$\Rightarrow v_2' = \frac{\sin ax}{a}$

Integrating $= v_2 = -\frac{1}{a^2}\cos ax$

$\Rightarrow \frac{-\cos ax}{a^2}$

(IV) $\times \sin ax$ and (III) $\times(a\cos ax)$ we get,

$a v_1'\cos^2 ax + a v_2'\sin ax\cos ax = 0$

$-a v_1'\sin^2 ax + a v_2'\sin ax\cos ax = \tan ax\sin ax$

$(+)\quad(-)\quad(-)$

$\Rightarrow a v_1' = -\tan ax\sin ax = -\frac{\sin^2 ax}{\cos ax}$

$\Rightarrow v_1' = -\frac{\sin^2 ax}{a\cos ax}$

Integrating $a v_1 = -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}dx$

$= -\frac{1}{a}\int \frac{(-a\sin ax)\sin ax}{\cos ax}dx$

$= \frac{1}{a}\int \frac{\sqrt{1-z^2}}{z}\,dz$ $\quad \cos ax = z$, $-\sin ax\, dx = dt$

$= -\frac{1}{a}\int \tan x \sin x\, dx$

$= -\frac{1}{a}\left[-\frac{\tan ax \cos ax}{a} + \frac{1}{a}\int \sec^2 ax \cos ax\, dx\right]$

$= -\frac{1}{a}\left[-\frac{\tan ax \cos ax}{a} + \int \frac{1}{\cos^2 ax}\cos ax\, dx\right]$

$= -\frac{1}{a}\left[-\frac{\tan ax \cos ax}{a} + \int \sec ax\, dx\right]$

$= -\frac{1}{a}\left[-\frac{\tan ax \cos ax}{a} + \frac{\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|}{a}\right]$

$v_1 = \frac{1}{a}\left[\frac{\tan ax \cos ax}{a} - \frac{\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|}{a}\right]$

from (1) we have,

$y_p = \frac{1}{a}\left[\frac{\tan ax \cos ax}{a} - \frac{\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|}{a}\right]$

$\cos ax - \frac{\cos ax}{a^2}\sin ax$

The solⁿ is $= y = y_c + y_p$

$= C_1\cos ax + C_2\sin ax + \frac{1}{a}\left[\frac{\tan ax \cos ax}{a} - \frac{\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|}{a}\right]\cos ax$

$- \frac{\cos ax}{a^2}\sin ax$

(2) $\frac{d^2y}{dx^2} + a^2y = \csc ax$ — (1)

let $y = e^{mx}$ be the trial solⁿ of the reduced eqn —

$\frac{d^2y}{dx^2} + a^2y = 0$

The A.E. is $m^2 + a^2 = 0$

$\Rightarrow m = \pm ia$

The C.F $= y_c = C_1\cos ax + C_2\sin ax$

let $y_1 = C_1\cos ax$, $y_2 = C_2\sin ax$

wronskian $W(y_1, y_2) =$

$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$= a(\cos^2 ax + \sin^2 ax) = a \neq 0$

$\therefore y_1$ and y_2 are linearly independent.

we assume the particular solⁿ as

$y_p = v_1\cos ax + v_2\sin ax$ — (II)

$v_1'\cos ax + v_2'\sin ax = 0$ — (III)

$-av_1'\sin ax + av_2'\cos ax = \csc ax$ — (IV)

(III) $\times(-a\sin ax)$ and (IV) $\times(\cos ax)$ we get

$-a v_1' \sin ax \cos ax - v_2' \sin^2 ax = 0$

$-a v_1' \sin ax \cos ax + a v_2' \cos^2 ax = \operatorname{cosec} ax \cos ax$

$\Rightarrow + a v_2' = + \cot ax$

$\Rightarrow v_2' = \frac{\cot ax}{a}$

$\therefore v_2 = \int \frac{\cot ax}{a}$

$= \frac{1}{a^2} \log |\sin ax|$

(iii) × $(a \cos ax)$ and (iv) × $(\sin ax)$ we get

$a v_1' \cos^2 ax + a v_2' \sin ax \cos ax = 0$

$-a v_1' \sin^2 ax + a v_2' \sin ax \cos ax = \operatorname{cosec} ax \cos ax$

$\Rightarrow a v_1' = -\operatorname{cosec} ax \sin ax$

$\Rightarrow v_1' = -\frac{\operatorname{cosec} ax \sin ax}{a}$

$v_1 = -\frac{1}{a} \int \operatorname{cosec} ax \sin ax \, dx$

$= -\frac{1}{a} \int dx$

$= -\frac{x}{a}$

$\therefore$ from (ii) we get

$y_p = -\frac{x}{a} \cos ax + \frac{1}{a^2} \log |\sin ax| \sin ax$

$\therefore$ The sol^n is $= y = y_c + y_p = c_1 \cos ax + c_2 \sin ax - \frac{x}{a} \cos ax + \frac{1}{a^2} \log |\sin ax| \sin ax$

Simple eigen value problems

Find the eigen value and the eigen functions of the following boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

Satisfying $y(0) = 0, y(\pi) = 0$ $\quad x=0 \quad x=\pi$

The given eq^n and the boundary conditions are, $\frac{d^2y}{dx^2} + \lambda y = 0 \; (x > 0)$ —(i)

$y(0) = 0, \; y(\pi) = 0$ —(ii)

Let $y = e^{mx}$ be a trial sol^n

A.E. is $m^2 + \lambda = 0$, $m = \sqrt{-\lambda} = \pm \sqrt{\lambda}\, i$

The sol^n is $y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$ —(iii)

$y(0) = 0 \Rightarrow 0 = c_1 \cdot 1 + c_2 \cdot 0$

$\Rightarrow c_1 = 0$

$y = c_2 \sin \sqrt{\lambda} x$

$y(\pi) = 0 \Rightarrow, 0 = c_1 \cos \sqrt{\lambda} \pi + c_2 \sin \sqrt{\lambda} \pi$

for non zero solution $c_2 \neq 0$

$\therefore \sin \sqrt{\lambda} \pi = 0$

$\sqrt{\lambda} \pi = n\pi \quad [n = 0, \pm 1, \pm 2, \pm 3 \cdots]$

$\Rightarrow n = \frac{n\pi}{\sqrt{\lambda}}$

$\Rightarrow \lambda = n^2, \; [n = 1, 2, 3 \cdots]$

which are the eigen values

$y_n = c_n \sin nx,\ n = 1, 2, 3, \ldots$
which are called the eigen functions.

11/10 [name]

1) Express $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

done.

2) Show that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$ is a perfect square.

RTP $|A| = 1$ [Rq. It is a perfect square matrix (পূর্ণ বর্গ ম্যাট্রিক্স)]

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a+a^2 & a & a^2 \\ 1+a+a^2 & 1 & a \\ 1+a+a^2 & a^2 & 1 \end{vmatrix} \quad [C_1' = C_1 + C_2]$$

$$= (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & a \\ 1 & a^2 & 1 \end{vmatrix}$$

$$= (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a-a^2 \\ 0 & a^2-a & 1-a^2 \end{vmatrix} \quad \begin{bmatrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{bmatrix}$$

$$= (1+a+a^2)\left[(1-a)(1-a^2) - (a-a^2)(a^2-a)\right]$$

$$= (1+a+a^2)\left[1-a^2-a+a^3 - (a^3-a^2-a^4+a^3)\right]$$

$$= (1+a+a^2)\left[1-a^2-a+a^3-a^3+a^2+a^4-a^3\right]$$

$$= (1+a+a^2)(1-a-a^3+a^4)$$

$$= 1-a+a^3+a^4+a-a^2-a^3-a^3+a^4+a^5+a^2-a^3-a^5+a^6$$

$$= 1-a-a^3+a^4+a-a^4-a^4+a^5+a^2-a^3-a^5+a^6$$

$$= 1-2a^3+a^6$$

③ ~~find~~ done

③ State Cayley Hamilton theorem. Use Caley Hamilton theorem to find A^{50}, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

1st part done.

The characteristic eqn. of A is:-

$$\psi_A(x) = \begin{vmatrix} 1-x & 1 \\ 0 & 1-x \end{vmatrix} = 0$$

$$\Rightarrow (1-x)^2 = 0$$

$$\Rightarrow x = 1, 1$$

Now,

R.T.P. $A^2 - 2A + I_2 = 0$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 1+1 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^2 - 2A + I_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\therefore$ A satisfy this eqn.

Now, ~~we can write $A^{50} = (A^2)^{25}$~~

$$\therefore A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 1+2 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 1+3 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Similarly, $A^{50} = \begin{bmatrix} 1 & 50 \\ 0 & 1 \end{bmatrix}$

⑤ done

⑤ Apply matrix method to show $x=1$, $y=2$, $z=3$ is a solution of $2x-y+3z=9$, $x+y+z=6$ and $x-y+z=2$

Let, $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$AX = B$

$\Rightarrow X = A^{-1}B$

$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= 2(1+1) + 1(1-1) + 3(-1-1)$

$= 4 - 6$

$= -2$

$\text{adj } A = \begin{bmatrix} (1+1) & -(1-1) & (-1-1) \\ -(-1+3) & (2-3) & -(-2+1) \\ (-1-3) & -(2-3) & (2+1) \end{bmatrix}^T$

$= \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}^T$

$= \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$

$\therefore X = \frac{1}{-2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -2 & 2 & 4 \\ 0 & +1 & -1 \\ 2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -18 + 12 + 8 \\ 0 + 6 - 2 \\ 18 - 6 - 6 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x = 1$, $y = 2$, $z = 3$ (Proved)

(7) Apply elementary row operation to reduce the following matrix to a row Echelon matrix $\begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}$

$\xrightarrow{R_1 \times \frac{1}{2}}$

$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{bmatrix}$

$$\xrightarrow[R_3-5R_1]{R_2'-3R_1}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{2}}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_2}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_{34}}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{2}}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2}$$

$$= \cancel{\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & & & \end{bmatrix}}$$

$$\xrightarrow{R_1 - 2R_3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8 done

9 Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{vmatrix}$$

$$= 1(28-12) - 1(14-12) + 2(6-12)$$

$$= 16 - 2 - 12$$

$$= 2$$

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 4 & 4 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (2\cdot 8-12) & -(14-12) & (6-12) \\ -(7-6) & (7-6) & -(3-3) \\ (4-8) & -(4-4) & (4-2) \end{bmatrix}^T$$

$$= \begin{bmatrix} 16 & -2 & -6 \\ -1 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 16 & -1 & -4 \\ -2 & 1 & 0 \\ -6 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 16 & -1 & -4 \\ -2 & 1 & 0 \\ -6 & 0 & 2 \end{bmatrix}$$

(11) done

(12) done

(13) Find mod z and arg z and express z in polar form where $z = -1-i$

$z = -1-i$

polar form (done)

$z = -1-i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$\therefore$ mod $z = \sqrt{2}$ arg $z = \frac{5\pi}{4}$

(14) Prove that if a, b, c are all unequal real numbers prove that $a^2+b^2+c^2 > ab+bc+ca$ what happens if $a=b=c$.

1st part Proved,

If $a=b=c$ then ~~$a^2+b^2+c^2 > ab+bc+ca$~~ inequality will occur.

(15) ~~Defin~~ done
(16) done
(17) done
(18) done
(18) verify Cayley Hamilton theorem for A, where $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

The characteristics equation of A

$$\Psi_A(x) = \begin{vmatrix} 0-x & 0 & 1 \\ 3 & 1-x & 0 \\ -2 & 1 & 4-x \end{vmatrix} = 0$$

$$\Rightarrow 0-x[(1-x)(4-x)] + 1(3+2(1-x))$$

$$\Rightarrow -x[4-x-4x+x^2] + [3+2-2x] = 0$$

$$\Rightarrow -4x + x^2 + 4x^2 - x^3 + 5 - 2x = 0$$

$$\Rightarrow -x^3 + 5x^2 - 6x + 5 = 0$$

$$\Rightarrow x^3 - 5x^2 + 6x - 5 = 0$$

RTP

$$A^3 - 5A^2 + 6A - 5I_3 = 0$$

$$\therefore A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0-2 & 0+0+1 & 0+0+4 \\ 0+3+0 & 0+1+0 & 3+0+0 \\ 0+3-8 & 0+1+4 & -2+0+16 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+3-8 & 0+1+4 & -2+0+16 \\ 0+3-6 & 0+1+3 & 3+0+12 \\ 0+15-28 & 0+5+14 & -5+0+56 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - \begin{bmatrix} -10 & 5 & 20 \\ 15 & 5 & 15 \\ -25 & 25 & 70 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 6 \\ 18 & 6 & 0 \\ -12 & 6 & 24 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$ (Proved)

$\therefore$ A satisfy the eqn.

(20) solve by cramer's method—

$x+2y+3z=11$
$x-2y+3z=3$
$x+2y-3z=-1$

← matrix method

$Ax=B$
$\Rightarrow x=A^{-1}B$

$A=\begin{bmatrix}1 & 2 & 3\\ 1 & -2 & 3\\ 1 & 2 & -3\end{bmatrix}$ $B=\begin{bmatrix}11\\3\\-1\end{bmatrix}$

$|A|=\begin{vmatrix}1 & 2 & 3\\ 1 & -2 & 3\\ 1 & 2 & -3\end{vmatrix}$

$=1(6-6)-2(-3-3)+3(2+2)$
$=12+12$
$=24$

$adj\ A=\begin{bmatrix}\begin{vmatrix}-2 & 3\\ 2 & -3\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 1 & -3\end{vmatrix} & \begin{vmatrix}1 & -2\\ 1 & 2\end{vmatrix}\\ -\begin{vmatrix}2 & 3\\ 2 & -3\end{vmatrix} & \begin{vmatrix}1 & 3\\ 1 & -3\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 1 & 2\end{vmatrix}\\ \begin{vmatrix}2 & 3\\ -2 & 3\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix} & \begin{vmatrix}1 & 2\\ 1 & -2\end{vmatrix}\end{bmatrix}^T$

$=\begin{bmatrix}(6-6) & -(-3-3) & (2+2)\\ -(-6-6) & (-3-3) & -(2-2)\\ (6+6) & -(3-3) & (-2-2)\end{bmatrix}^T$

$=\begin{bmatrix}0 & 6 & 4\\ 12 & -6 & 0\\ 12 & 0 & -4\end{bmatrix}^T$

$=\begin{bmatrix}0 & 12 & 12\\ 6 & -6 & 0\\ 4 & 0 & -4\end{bmatrix}$

$A^{-1}=\frac{1}{24}\begin{bmatrix}0 & 12 & 12\\ 6 & -6 & 0\\ 4 & 0 & -4\end{bmatrix}\times\begin{bmatrix}11\\3\\-1\end{bmatrix}$

$=\frac{1}{24}\begin{bmatrix}0+36-12\\ 66-18+0\\ 44+0+4\end{bmatrix}$

$=\frac{1}{24}\begin{bmatrix}24\\48\\48\end{bmatrix}$

$=\begin{bmatrix}1\\2\\2\end{bmatrix}$

$x=1, y=2, z=2$

$x=1, y=2, z=2$

Let

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{bmatrix}$$

← Cramer's method

$$|D| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$= 1(6-6) - 2(-3-3) + 3(2+2)$

$= 12 + 12$

$= 24$

$$D_x = \begin{vmatrix} 11 & 2 & 3 \\ 3 & -2 & 3 \\ -1 & 2 & -3 \end{vmatrix}$$

$= 11(6-6) - 2(-9+3) + 3(6-2)$

$= 12 + 12$

$= 24$

$$D_y = \begin{vmatrix} 1 & 11 & 3 \\ 1 & 3 & 3 \\ 1 & -1 & -3 \end{vmatrix}$$

$= 1(-9+3) - 11(-3-3) + 3(-1-3)$

$= -6 + 66 - 12$

$= 48$

$$D_z = \begin{vmatrix} 1 & 2 & 11 \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 1(2-6) - 2(-1-3) + 11(2+2)$

$= -4 + 8 + 44$

$= 48$

$x = \frac{D_x}{D} = \frac{24}{24} = 1$

$y = \frac{D_y}{D} = \frac{48}{24} = 2$

$z = \frac{D_z}{D} = \frac{48}{24} = 2$

$\therefore x = 1, y = 2, z = 2$

(21) obtain a row reduced matrix which is row equivalent to -

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix} \xrightarrow{R_{12}}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_1}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2}$$

$$= \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cancel{\xrightarrow{R_1 - 2R_2}} \quad \cancel{= \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

This is row reduced echelon matrix.

(22) ~~If~~ 0, done

(23) done

(24) Find the rank of $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix}$

done - great [Ans - 2]

(25) Let V be a vector space over a field F and U and W be two subspaces of V. Prove that U+W is a subspace of V.

Let $S = U + W = \{u+w : u \in U, w \in W\}$.

$\theta \in U, \theta \in W \Rightarrow \theta \in S$ and therefore S is nonempty.

Let $\alpha_1, \alpha_2 \in S$. Then $\alpha_1 = u_1 + w_1$ for some $u_1 \in U$ and $w_1 \in W$,

$\alpha_2 = u_2 + w_2$ for some $u_2 \in U$ and $w_2 \in W$.

$\alpha_1 + \alpha_2 = (u_1 + u_2) + (w_1 + w_2) \in S$

since $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$.

Let c be a scalar in F

Then $c\alpha_1 = c(u_1 + w_1) = cu_1 + cw_1 \in S$,

since $cu_1 \in U$, and $cw_1 \in W$

$\therefore \alpha_1 \in S$ and $\alpha_2 \in S \Rightarrow \alpha_1 + \alpha_2 \in S$; and

$c \in F, \alpha_1 \in S \Rightarrow c\alpha_1 \in S$

This proves that S is a subspace of V,

i.e. U+W is a subspace of V.

Proof (iii) $-1\alpha = -\alpha$ for all $\alpha \in V$, 1 being the identity element of F;

we have $\theta = 0\alpha$, by (i)

$= [1+(-1)]\alpha$

$= 1\alpha + (-1)\alpha$

$= \alpha + (-1)\alpha$, by V_{10}

Therefore $-\alpha+\theta = -\alpha+\{\alpha+(-1)\alpha\}$

$= (-\alpha+\alpha)+(-1)\alpha$

$= \theta+(-1)\alpha$, by V_5

$= (-1)\alpha$

$\Rightarrow -\alpha = (-1)\alpha$

Proof

(iv) $c\alpha = \theta$ implies either $c=0$ or $\alpha=\theta$

let $c\alpha = \theta$ and let $c \neq 0$ then c^{-1} exist in F.

now, $c\alpha=\theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta$

$\Rightarrow (c^{-1}c)\alpha = c^{-1}\theta$ by V_7

$\Rightarrow 1\alpha = \theta$, by (ii)

$\Rightarrow \alpha = \theta$, by V_{10}

$c\alpha=\theta$ and $c\neq 0 \Rightarrow \alpha=\theta$

Contrapositively, $c\alpha=\theta$ and $\alpha\neq 0 \Rightarrow c=0$.

Hence $c\alpha=\theta$ implies either $c=0$ or $\alpha=\theta$

(26) done

(27) Express $-1-i$ in polar form.

$-1-i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = -1$, $r\sin\theta = -1$

we have, $r^2 = 2$ and therefore $r = \sqrt{2}$

$\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$, $\theta = \frac{5\pi}{4}$

Hence $-1-i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

(28) If α, β, γ are roots of the equation $ax^3+bx^2+cx+d=0$, find the value $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta)$ in terms of the coefficients.

we have,

$\alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{d}{a}$

(29) done

(30) done

(31) done

(32) State Cayley Hamilton theorem. Use Cayley Hamilton theorem to find A^{-1}, where $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

~~1st part~~ done

The characteristic eqn of A is

$$\psi_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(5-\lambda) - 3 = 0$$

$$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 3 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 - 3 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 7 = 0 \quad \text{--- (i)}$$

ATP

$$A^2 - 7A + 7I_2 = 0 \quad \text{--- (ii)}$$

$$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 2+5 \\ 6+15 & 3+25 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 \\ 21 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 7 \\ 21 & 28 \end{bmatrix} - \begin{bmatrix} 14 & 7 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= 0$

A satisfies the eqn.

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$|A| = 7 \neq 0$, A^{-1} exists.

$\therefore$ multiplying A^{-1} in both side of (ii),

$$A(A A^{-1}) - 7 A A^{-1} + 7 I_2 A^{-1} = 0$$

$$\Rightarrow A - 7I + 7A^{-1} = 0$$

$$\Rightarrow \frac{1}{7}(A - 7I) = -A^{-1}$$

$$\Rightarrow \frac{1}{7}(7I - A) = A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

(33) done

(34) done

(35) when a n×n matrix is called in row reduced Echelon matrix ? Give example of it

done

(36) In R^3, $\alpha = (4,3,5)$, $\beta = (0,1,3)$, $\gamma = (2,1,1)$ and $\delta = (4,2,2)$

Examine if (i) α is a linear combination of β and γ.

(ii) β is a linear combination of γ and δ

(i) Let $\alpha = c\beta + d\gamma$, where $c, d \in R$

Then, $(4,3,5) = c(0,1,3) + d(2,1,1)$

$= (0+2d, c+d, 3c+d)$

$\therefore 2d = 4 \quad c+d = 3 \quad 3c+d = 5$

$\Rightarrow d = 2 \quad \Rightarrow c+2 = 3 \quad$ $c=1, d=2$ (eqn satisfies i.e. this results 'S' by replacing them in this eqn)

$\Rightarrow c = 1$

$\therefore c=1, d = 2$

Hence $\alpha = \beta + 2\gamma$ and α is a linear combination of β and γ.

(ii) $\beta = c\gamma + d\delta$

$\Rightarrow (0,1,3) = c(2,1,1) + d(4,2,2)$

$= (2c+4d, c+2d, c+2d)$

$2c+4d = 0 \quad , c+2d = 1, \quad c+2d = 3$

$\Rightarrow c = -2d$ (contradiction)

The eqns are inconsistent. Therefore β cannot be expressed as $c\gamma + d\delta$ for real c, d. Hence β is not linear combination of γ and δ.

(37) done

(38) Applying row and column operations find the rank of $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

~~$\xrightarrow{R_2 \times \frac{1}{2}} \begin{bmatrix} 0 & 1 & 2 & -2 \\ 2 & 0 & 1 & 3 \\ 2 & 1 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_2]{R_1 - 2R_2} \begin{bmatrix} -4 & 1 & 0 \end{bmatrix}$~~

$\xrightarrow{C_{12}} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$\xrightarrow{R_1 \times (-1),\ R_3 - R_1} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$\xrightarrow[C_4 + 2C_1]{C_3 - 2C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$\xrightarrow{R_2 - 3R_3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$\xrightarrow{R_2 \leftrightarrow R_3}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_4 - 3C_3]{C_2 - 2C_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow C_{23}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Thus $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Rank of $A = 2$

(50) Prove that intersection of two subspaces of vector space V over a field F is a subspace of V.

Proof- Let w_1 and w_2 be two subspaces of V. $w_1 \cap w_2$ is not empty, because $\theta \in w_1 \cap w_2$.

Case 1. Let $w_1 \cap w_2 = \{\theta\}$. Then $w_1 \cap w_2$ is a subspace of V.

Case 2. Let $w_1 \cap w_2 \neq \{\theta\}$ and let $\alpha_1 \in w_1 \cap w_2$, $\alpha_2 \in w_1 \cap w_2$.

Then $\alpha_1, \alpha_2 \in w_1$ and $\alpha_1, \alpha_2 \in w_2$

Since w_1 is a subspace of V, (i) $\alpha_1 + \alpha_2 \in w_1$ and (ii) $c\alpha_1 \in w_1$, c being a scalar in F.

Since w_2 is a subspace of V, (i) $\alpha_1 + \alpha_2 \in w_2$ and (ii) $c\alpha_1 \in w_2$, c being a scalar in F.

Therefore $\alpha_1 + \alpha_2 \in w_1 \cap w_2$ and $c\alpha_1 \in w_1 \cap w_2$.

This proves that $w_1 \cap w_2$ is a subspace of V.

[note- Since $w_1 \cap w_2$ is the largest subset contained in both of w_1 and w_2, $w_1 \cap w_2$ is the largest subspace contained in w_1 and w_2.]

(48) done

(49) Find mod z and arg z and express z in polar form where $z = -1+i$

$z = -1+i$

$-1+i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = -1$, $r\sin\theta = 1$

$r^2\cos^2\theta + r^2\sin^2\theta = 1+1$

$\Rightarrow r = \sqrt{2}$

$\sqrt{2}\cos\theta = -1$ $\qquad$ $\sqrt{2}\sin\theta = 1$

$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$ $\qquad$ $\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{3\pi}{4}$ $\qquad$ $\theta = \frac{3\pi}{4}$

$\therefore \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1+i$

mod $z = \sqrt{2}$ $\quad$ arg $z = \frac{3\pi}{4}$

(12) ~~Find~~ ~~mod~~ If α, β, γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\alpha^3 + \beta^3 + \gamma^3$ in terms of the coefficients.

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

(4)
$= -\frac{b^3}{a^3} - 3\left[\frac{-bc}{a^2} + \frac{d}{a}\right]$

$= -\frac{b^3}{a^3} + 3\left[\frac{bc - ad}{a^2}\right]$

$= \frac{-b^3 + 3a(bc - ad)}{a^3}$

[Now, $\alpha^3 + \beta^3 + \gamma^3$

$= (\alpha + \beta + \gamma)^3 - 3\left[(\alpha+\beta)(\beta+\gamma)(\alpha+\gamma)\right]$

$[-\alpha\beta\gamma\ (\alpha+\beta+\gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha^2\beta\gamma^2]$

$= (\alpha + \beta + \gamma)^3 - 3\left[-\frac{d}{a}\left(-\frac{b}{a}\right)\cdot\frac{c}{a} - \frac{d^2}{a^2}\right]$

$= \left(\frac{b}{-a}\right)^3 - 3\left[\frac{bcd}{a^3} - \frac{d^2}{a^2}\right]$

$= \frac{-b^3}{a^3} - 3\left[\frac{bcd}{a^3} - \frac{d^2}{a^2}\right]$

$= \frac{-b^3}{a^3} - 3\left(\frac{bcd - ad^2}{a^3}\right)$

$= \frac{-b^3 - 3(bcd - ad^2)}{a^3}$] not correct

we

$a^3 + b^3 + c^3 = (a+b+c)^3 - 3[(a+b+c)(ab+bc+ca) - abc]$

$= (\alpha+\beta+\gamma)^3 - 3[(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma) - \alpha\beta\gamma]$

$= \left(-\frac{b}{a}\right)^3 - 3\left[\left(-\frac{b}{a}\right)\cdot\left(\frac{c}{a}\right) - \left(-\frac{d}{a}\right)\right]$ (4)

(13) done

(14) Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, and find A^{-1} if exists

The characteristic eqn of A is-

$\psi_A(x) = \begin{vmatrix} -1-x & 2 \\ 3 & -4-x \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} -(1+x) & 2 \\ 3 & -(4+x) \end{vmatrix} = 0$

$\Rightarrow (4+x)(1+x) - 6 = 0$

$\Rightarrow 4 + 4x + x + x^2 - 6 = 0$

$\Rightarrow x^2 + 5x - 2 = 0$ — (i)

from (i),
R.T.P $A^2 + 5A - 2I_2 = 0$ — (ii)

$A^2 = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1+6 & -2-8 \\ -3-12 & 6+16 \end{bmatrix}$

$= \begin{bmatrix} 7 & -10 \\ -15 & 22 \end{bmatrix}$

$$\begin{bmatrix} 7 & -10 \\ -15 & 22 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ 15 & -20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= 0$

$\therefore$ A satisfies the equation

$|A| = 2 \neq 0$, A^{-1} exists
multiplying A^{-1} in the both side of (1),

$A^2A^{-1} + 5AA^{-1} - 2I_2A^{-1} = 0$
$\Rightarrow A(AA^{-1}) + 5I - 2A^{-1} = 0$
$\Rightarrow A + 5I - 2A^{-1} = 0$
$\Rightarrow \frac{1}{2}(A + 5I) = A^{-1}$
$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

[same answer]

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} -4 & -2 \\ -3 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

(45) If $(a+ib)(c+id) = A+iB$ prove that $(a-ib)(c-id) = A-iB$ and $(a^2+b^2)(c^2+d^2) = A^2+B^2$.

Now,

$(A+iB)\times(A-iB) = (a+ib)(c+id) \times (a-ib)(c-id)$

$\Rightarrow A^2 - i^2B^2 = \{(a+ib)(a-ib)\}\{(c+id)(c-id)\}$

$\Rightarrow A^2 + B^2 = (a^2 - i^2b^2)(c^2 - i^2d^2)$

$\Rightarrow A^2 + B^2 = (a^2+b^2)(c^2+d^2)$

(Proved)

(46) Find mod z and arg z & express z in the polar form, where $z = 1-i$

$z = 1 - i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = 1$, $r\sin\theta = -1$

$r^2 = 2$
$\Rightarrow r = \sqrt{2}$

$\therefore r\cos\theta = 1$
$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\sin\theta = -\frac{1}{\sqrt{2}}$

$\theta = \left(-\frac{\pi}{4}\right)$ $\quad \theta = \left(-\frac{\pi}{4}\right)$

$\therefore$ mod $z = \sqrt{2}$, arg $z = -\frac{\pi}{4}$

Polar form: $1 - i = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\}$

Worked ex

(1) Determine the rank of the matrix A =

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix}$$

$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$\xrightarrow{\frac{1}{6}R_2}$

$$= \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$\xrightarrow{R_3 - 3R_2}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → can also be merged like $\left[\xrightarrow[R_3-3R_2]{R_1-R_2}\right]$

$\xrightarrow{R_1 - R_2}$

$$= \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R,$$

Say R is a row reduced echelon matrix and R has 2 non-zero rows. Therefore rank of R = 2. Since A is row equivalent to R, rank of A = 2.

(2) Express $A = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 2 \\ 1 & 6 & 8 \end{bmatrix}$ as the sum of a symmetric matrix and a skew symmetric matrix.

$$A = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 2 \\ 1 & 6 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 3 & 1 \\ 5 & 7 & 6 \\ 1 & 2 & 8 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 2 \\ 1 & 6 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 1 \\ 5 & 7 & 6 \\ 1 & 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 8 & 2 \\ 8 & 14 & 8 \\ 2 & 8 & 16 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & 8 & 2 \\ 8 & 14 & 8 \\ 2 & 8 & 16 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & 8 \end{bmatrix}$$

$$A - A^T = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 2 \\ 1 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 1 \\ 5 & 7 & 6 \\ 1 & 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

$\Rightarrow \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A =$

$$\begin{bmatrix} 4 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$$

Ex-① Find Eigen vectors for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

no need Eign vectors for ch

The characteristic eqn is —

$\begin{vmatrix} 0-x & -1 \\ 1 & 0-x \end{vmatrix} = 0$

$\Rightarrow x^2 + 1 = 0$

$\Rightarrow x^2 = -1$

$\Rightarrow x = i, -i$

$\therefore$ Eigen values are i and $-i$

[Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an Eigen vectors correspondingly to i, then $Ax = ix$,

thus $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_1 \\ ix_2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} ix_1 \\ ix_2 \end{bmatrix}$

$\Rightarrow -x_2 - ix_1 = 0$

$\Rightarrow x_1 - ix_2 = 0$

As $i^2 = i \times i = -1$, we have

$x_1 - ix_2 = 0$

$\therefore$ The soln is $k(i, 1)$, where k is a complex number.

Thus eigen vectors are $k\begin{bmatrix} i \\ 1 \end{bmatrix}$, where k is a non-zero complex number.

Now let us consider the eigen value $-i$.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to $-i$.

Then $Ax = (-i)x$. Thus $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_1 \\ -ix_2 \end{bmatrix}$

$\Rightarrow -x_2 + ix_1 = 0$ and $x_1 + ix_2 = 0$

which is equivalent to $x_1 + ix_2 = 0$

The soln is $c(1, i)$, where c is a non-zero complex number. thus eigen vector corresponding to $-i$ are $c\begin{bmatrix} 1 \\ i \end{bmatrix}$, where c is a non-zero complex number.] $\boxed{\text{No need}}$

② Find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{vmatrix}$$

$$= 1(28-12) - 1(14-12) + 2(6-12)$$
$$= 16 - 2 - 12$$
$$= 2$$

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 4 & 4 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (28-12) & -(14-12) & (6-12) \\ -(7-6) & (7-6) & -(3-3) \\ (4-8) & -(4-4) & (4-2) \end{bmatrix}^T$$

$$= \begin{bmatrix} 16 & -2 & -6 \\ -1 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 16 & -1 & -4 \\ -2 & 1 & 0 \\ -6 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 16 & -1 & -4 \\ -2 & 1 & 0 \\ -6 & 0 & 2 \end{bmatrix}$$

(other process by row reduced echelon matrix)

Let us form the 3×6 matrix (A/I_3) and perform elementary row operation to reduce A to a row reduce echelon matrix

$$A/I_3 = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\times R_2} = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-2R_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] = (I_3/A^{-1})$$

$$\therefore A^{-1} = \begin{bmatrix} 8 & -\frac{1}{2} & -2 \\ -1 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

3

math 4th Previous year (2021) [MAT-UGBS02]

Part-A

1(a) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I_3 = 0$. Hence obtain a matrix B such that $AB = I_3$.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore A^2 - 4A - 5I_3$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ Proved (Proved)}$$

Now, we know that,

if $AB = I_3$, $B = A^{-1}$ $\left[\begin{array}{l} AB = I_3 \\ \Rightarrow (A^{-1}A)B = A^{-1}I_3 \\ \Rightarrow B = A^{-1} \end{array}\right]$

$\therefore A^2 - 4A - 5I_3 = 0$

$\Rightarrow A^{-1}(A \cdot A) - 4A^{-1}A - 5A^{-1}I_3 = 0$ [multiplying A^{-1} in both side]

$\Rightarrow A - 4I_3 - 5A^{-1} = 0$

$\Rightarrow \frac{1}{5}(A - 4I_3) = A^{-1}$

$$\Rightarrow A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

(b) Prove that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ is orthogonal. Utilise this to solve the following system of equations. $x-2y+2z=2$, $2x-y-2z=1$, $2x+2y+z=7$.

Ans. We know that the matrix A will be called an orthogonal matrix when,

$AA^T = A^TA = I$

$\therefore A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$AA^T = \frac{1}{9}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$= I$

$\therefore$ A is an orthogonal matrix (proved)

$A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Here, $AX = B$

$X = A^{-1}B$

$= A^TB$ [$\because$ As A is an orthogonal matrix, $A^{-1} = A^T$]

$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 2+2+14 \\ -4-1+14 \\ 4-2+7 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 18 \\ 9 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$

$x = 6, y = 3, z = 3$

2) a) Solve the system of equations by matrix inversion method.

$x + z = 0$
$3x + 4y + 5z = 2$
$2x + 3y + 4z = 1$

are (previous) [Ans, $x = y = 1$, $z = -1$]

(b) Find the rank of the rectangular matrix $\begin{bmatrix} 0 & 0 & 5 & -3 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Sol, $A = \begin{bmatrix} 0 & 0 & 5 & -3 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

$\xrightarrow{R_{13}} \begin{bmatrix} -1 & -2 & 6 & -7 \\ 2 & 4 & 3 & 5 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\xrightarrow{R_1 \times (-1)} \begin{bmatrix} 1 & 2 & -6 & 7 \\ 2 & 4 & 3 & 5 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -6 & 7 \\ 0 & 0 & 15 & -9 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\xrightarrow{R_2 \times (\frac{1}{3})} \begin{bmatrix} 1 & 2 & -6 & 7 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & -6 & 7 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of the matrix is = 2

3) a) use Caley - Hamilton theorem to find A^{100}, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\phi_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[\lambda^2 - 1] = 0$

$\Rightarrow (1-\lambda)(\lambda^2-1) = 0$

$\Rightarrow (\lambda-1)^2(\lambda+1) = 0$

$(\lambda-1)(\lambda^2-1) = 0$

$\Rightarrow \lambda^3 - \lambda - \lambda^2 + 1$

$\Rightarrow \lambda = 1, 1, -1$

The roots of the eqn are $1, 1, -1$.
This are the eigen values of this eqn.

RTP $A^3 - A^2 - A + I = 0$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 1+0+0 & 0+0+1 & 0+0+0 \\ 0+1+0 & 0+0+0 & 0+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 1+1+0 & 0+0+0 & 0+1+0 \\ 1+0+0 & 0+0+1 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ A satisfies the eqn.

$A^3 - A^2 - A + I = 0$

$\Rightarrow A^3 - A^2 = A - I$

$\Rightarrow A^2(A^3 - A^2) = A^2(A - I)$ [multiplying by A^2]

$\Rightarrow A^5 - A^4 = A^3 - A^2 = A - I$

$\Rightarrow A^2(A^5 - A^4) = A^2(A - I)$

$\Rightarrow A^7 - A^6 = A^3 - A^2 = A - I$

$\Rightarrow A^{100} - A^{99} = A - I$

Adding,

$A^{100} - A = 99A - 99I$

$\Rightarrow \boxed{A^{100} = 100A - 99I}$

$\therefore A^{100} = \begin{bmatrix} 100 & 0 & 0 \\ 100 & 0 & 100 \\ 0 & 100 & 0 \end{bmatrix} - \begin{bmatrix} 99 & 0 & 0 \\ 0 & 99 & 0 \\ 0 & 0 & 99 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 100 & -99 & 100 \\ 0 & 100 & -99 \end{bmatrix}$

(b) Without expanding prove that $\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

Ld,

$\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$

$= (-1)\times(-1)\times(-1) \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix}$ [take − common from each row]

$= (-1)^3 \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix}$ [Interchange of each row into column]

$= (-1)^3 \times \Delta$

$\Delta + \Delta = 0$

$\Rightarrow 2\Delta = 0$

$\Rightarrow \Delta = 0$

$\therefore \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$ (proved)

4) (a) Solve the system of equations by Cramer's rule.

$x + y + z = 6$

$x + 2y + 3z = 14$

$x - y + z = 2$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix}$$

$= 1(2+3) - 1(1-3) + 1(-1-2)$

$= 5 + 2 - 3$

$= 4$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$= 6(2+3) - 1(14-6) + 1(-14-4)$

$= 30 - 8 - 18$

$= 4$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 1(14-6) - 6(1-3) + 1(2-14)$

$= 8 + 12 - 12$

$= 8$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 1(4+14) - 1(2-14) + 6(-1-2)$

$= 18 + 12 - 18$

$= 12$

$x = 1$

$y = 2$

$z = 3$

(b) Show that the set of vectors $\{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of R^3

Let, $\alpha_1 = (1,0,0)$

$\alpha_2 = (1,1,0)$

$\alpha_3 = (1,1,1)$

Consider,

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0,0)$

$\Rightarrow c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (0,0,0)$

$\Rightarrow (c_1, 0, 0) + (c_2, c_2, 0) + (c_3, c_3, c_3) = (0,0,0)$

$\Rightarrow (c_1 + c_2 + c_3, \; c_2 + c_3, \; c_3) = (0,0,0)$

$c_3 = 0$, $\quad c_2 + c_3 = 0 \Rightarrow c_2 + 0 = 0 \Rightarrow c_2 = 0$, $\quad c_1 + c_2 + c_3 = 0 \Rightarrow c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$

$\therefore$ This is a linearly independent clm.

5) (a) Find the cubic roots of 1.

Let $\sqrt[3]{1} = x$

$\Rightarrow x^3 = 1$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow x^3 - 1^3 = 0$

$\Rightarrow (x-1)(x^2+x+1)=0$

$x = 1,\ x^2+x+1=0$

$x = \frac{-1 \pm \sqrt{1-4\cdot1\cdot1}}{2\cdot1}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

∴ The cubic roots of 1 = 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$

(b) Solve the equation $x^6+x^5+x^4+x^3+x^2+x+1=0$

$x^6+x^5+x^4+x^3+x^2+x+1=0$ —(1)

multiplying both sides by $(x-1)$, we have

$x^7-1=0$

$\Rightarrow x^7=1$

$\Rightarrow x=(1)^{\frac{1}{7}}$

$=(\cos 0 + i\sin 0)^{\frac{1}{7}}$

$=[\cos(2n\pi+0)+i\sin(2n\pi+0)]^{\frac{1}{7}}$

$=(\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{7}}$

$\cos\frac{2n\pi}{7}+i\sin\frac{2n\pi}{7}$ —(2)

Putting $n=0, 1, 2, \ldots, 6$; the seven roots of (2) are $\cos 0 + i\sin 0$; $\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$;

$\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}$; $\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}$;
$\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}$; $\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}$;
$\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}$.

But $\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}=\cos(2\pi-\frac{6\pi}{7})+i\sin(2\pi-\frac{6\pi}{7})$
$=\cos\frac{6\pi}{7}-i\sin\frac{6\pi}{7}$

Similarly, $\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}=\cos\frac{4\pi}{7}-i\sin\frac{4\pi}{7}$, $\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}=\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}$ ...

6) (a) Solve the algebraic equation $x^4+x^2-2x+6=0$, it is given that $1+i$ is a root of the above equation.

(b) Apply Descarte's rule of sign to examine the nature of the roots of the equation $x^4+2x^2+3x-1=0$

Part - B

7) (a) Find the differential equation of all circles which pass through the origin and whose centres are on the y-axis.

According to the question,

$$x^2+y^2=2ax \quad \text{— (i)} \quad [a \text{ is a parameter}]$$

diff. (i) w.r.t. x

$$\Rightarrow 2x+2y\frac{dy}{dx}=2a$$

$$\Rightarrow x+y\frac{dy}{dx}=a \quad \text{— (ii)}$$

Substituting (i) we get,

$$x^2+y^2=2a\left(x+y\frac{dy}{dx}\right)x$$

$$\Rightarrow x^2+y^2=2x^2+2xy\frac{dy}{dx}$$

$$\Rightarrow 2xy\frac{dy}{dx}=y^2-x^2$$

$$\Rightarrow 2xyy_1=y^2-x^2 \quad \text{— Ans}$$

(b) Solve : $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Let $x+y = z$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

Now,

$\frac{dz}{dx} - 1 = \sin z + \cos z$

$\Rightarrow \frac{dz}{dx} = \sin z + \cos z + 1$

$\Rightarrow \int \frac{dz}{\sin z + \cos z + 1} = \int dx$

$\Rightarrow \int \frac{dz}{\frac{2\tan z/2}{1+\tan^2 z/2} + \frac{1-\tan^2(z/2)}{1+\tan^2 \frac{z}{2}} + 1} = \int dx$

$\Rightarrow \int \frac{\left(1+\tan^2 \frac{z}{2}\right) dz}{2\tan\frac{z}{2} + 1 - \tan^2\frac{z}{2} + 1} = \int dx$

$\Rightarrow \int \frac{\sec^2 \frac{z}{2}\, dz}{-\tan^2\frac{z}{2} + 2\tan\frac{z}{2} + 2} = \int dx$

$\Rightarrow 2\int \frac{du}{-u^2 + 2u + 2} = \int dx$

$\tan\frac{z}{2} = u$

$\Rightarrow \sec^2\frac{z}{2} \cdot \frac{1}{2}\, dz = du$

$\Rightarrow \sec^2\frac{z}{2}\, dz = 2du$

$\Rightarrow \int \frac{\left(1+\tan^2\frac{z}{2}\right) dz}{2\tan\frac{z}{2} + 1 - \tan^2\frac{z}{2} + 1 + \tan^2\frac{z}{2}}$

$= \int \frac{\sec^2\frac{z}{2}\, dz}{2\left(1+\tan\frac{z}{2}\right)}$

$= \int \frac{\frac{1}{2}\sec^2\frac{z}{2}\, dz}{1+\tan\frac{z}{2}}$

$= \int \frac{du}{1+u}$

$= \log|1+u|$

$= \log\left|1+\tan\frac{z}{2}\right|$

$= \log\left|1+\tan\frac{z}{2}\right|$

$\Rightarrow 2\int \frac{du}{(\sqrt{3})^2 - (u-1)^2} = \int dx$

$= -\{u^2 - 2u - 2\}$

$= -\{(u^2 - 2u\cdot 1 + 1) - 1 - 2\}$

$= -\{(u-1)^2 - (\sqrt{3})^2\}$

$= \{(\sqrt{3})^2 - (u-1)^2\}$

$\Rightarrow 2\sin^{-1}\frac{u-1}{\sqrt{3}} = x + c$

$\Rightarrow 2\sin^{-1}\left[\frac{1}{\sqrt{3}}\left(\tan\frac{z}{2} - 1\right)\right] = x + c$

$\Rightarrow 2\sin^{-1}\left[\frac{1}{\sqrt{3}}\left\{\tan\frac{1}{2}(x+y) - 1\right\}\right] = x + c \longrightarrow \times$

Ans $- \log\left|1 + \tan\frac{(x+y)}{2}\right| = x + c$

8)(a) Solve $(x^3 + 3x y^2)\,dx + (y^3 + 3x^2 y)\,dy = 0$

$(x^3 + 3xy^2)\,dx = -(y^3 + 3x^2 y)\,dy$

$\Rightarrow \frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2 y}$ —(1)

Let $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, from (1)

$v + x\frac{dv}{dx} = -\frac{x^3 + 3x\cdot v^2x^2}{v^3x^3 + 3x^2\cdot vx}$

$\Rightarrow v + x\frac{dv}{dx} = -\frac{x^3(1+3v^2)}{x^3(v^3+3v)}$

$\Rightarrow x\frac{dv}{dx} = -\frac{(1+3v^2)}{v^3+3v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{-1 - 3v^2 - v^4 - 3v^2}{v^3 + 3v}$

other process

$M\,dx + N\,dy$

$M = x^3 + 3xy^2$, $N = y^3 + 3x^2y$

$\frac{\partial M}{\partial y} = 6xy$ $\Rightarrow \frac{\partial N}{\partial x} = 6xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so the given eqn is exact

$\int M\,dx = \int (x^3 + 3xy^2)\,dx$

$= \frac{x^4}{4} + 3y^2 \cdot \frac{x^2}{2}$

$= \frac{x^4}{4} + \frac{3x^2y^2}{2}$

$\int N\,dy = \int (y^3 + 3x^2y)\,dy$

$= \frac{y^4}{4} + \frac{3x^2y^2}{2}$

$\therefore$ The soln is $\frac{3x^2y^2}{2} + \frac{x^4}{4} + \frac{y^4}{4} = c$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4-6v^2-1}{v^3+3v}$$

$$\Rightarrow \frac{1}{4}\int\frac{4(v^3+3v)\,dv}{v^4+6v^2+1} = -\int\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{4}\int\frac{(4v^3+12v)\,dv}{v^4+6v^2+1} = -\int\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{4}\log|v^4+6v^2+1| = -\log x + \frac{\log c}{4}$$

$$\Rightarrow \log|v^4+6v^2+1| + \log x^4 = \log c$$

$$\Rightarrow \log\left|\left(\frac{y^4}{x^4}+\frac{6y^2}{x^2}+1\right)x^4\right| = \log c$$

$$\Rightarrow \log\left|\frac{y^4+6x^2y^2+x^4}{x^4}\cdot x^4\right| = \log c$$

$$\Rightarrow y^4+6x^2y^2+x^4 = c$$

(b) Solve : $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \left(\frac{x+xy^2}{4}\right)dy$ = 0

$$\left[\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx = -\left(\frac{x+xy^2}{4}\right)dy\right.$$ This is incorrect

$$\Rightarrow \frac{(6y+2y^3+3x^2)\,dx}{6\,3} = -\frac{(x+xy^2)}{4\,2}dy$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2(6y+2y^3+3x^2)}{3(x+xy^2)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}\left\{\frac{2y(3+y^2)+3x^2}{x(1+y^2)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}\left\{\frac{2y\{2+(1+y^2)\}+3x^2}{x(1+y^2)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}\left[\frac{4y}{x(1+y^2)}+\frac{2y}{x}+\frac{3x^2}{x(1+y^2)}\right]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}\left[\frac{2y}{x}+\frac{4y+3x^2}{x(1+y^2)}\right]$$

$$\Rightarrow \frac{dy}{dx}+\frac{4}{3}\frac{y}{x} = -\frac{2(4y+3x^2)}{3x(1+y^2)} \quad \text{--- (1)}$$

$$\Rightarrow \frac{dy}{dx}+Py = Q$$

$$P = \frac{4}{3x}$$

$$\therefore e^{\int\frac{4}{3x}dx} = e^{\frac{4}{3}\log x} = e^{\log x^{\frac{4}{3}}} = x^{\frac{4}{3}}$$

$\therefore$ multiplying $x^{\frac{4}{3}}$ in both side of the eqn (1)

$$x^{\frac{4}{3}}\frac{dy}{dx}+\frac{4y}{3x}x^{\frac{4}{3}} = -\frac{2x^{\frac{4}{3}}(4y+3x^2)}{3x(1+y^2)}$$

$$\int \frac{d}{dx}\left(y x^{\frac{4}{3}}\right) = \int \frac{-2x^{\frac{1}{3}}(4y+3x^2)}{3(1+y^2)}$$

$$\Rightarrow \int d\left(y x^{\frac{4}{3}}\right) = -\frac{2}{3}\int \left[\frac{x^{\frac{1}{3}}(4y+3x^2)}{1+y^2}\,dx\right]$$

$$\Rightarrow y x^{\frac{4}{3}} = $$

1st method

$$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \frac{(x+xy^2)}{4}dy = 0$$

The given eqn is :-

$$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \frac{(x+xy^2)}{4}dy = 0 \quad \text{—①}$$

Comparing ① with $Mdx + Ndy = 0$ we get

$M = y+\frac{y^3}{3}+\frac{x^2}{2}$, $N = \frac{x(1+y^2)}{4}$

$\frac{\partial M}{\partial y} = 1+y^2$, $\frac{\partial N}{\partial x} = \frac{1}{4}(y^2+1)$

Now, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so (1) is not exact.

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\frac{x(1+y^2)}{4}}\left[1+y^2 - \frac{1}{4}(1+y^2)\right]$$

$$= \frac{4}{x(1+y^2)}\cdot\frac{3}{4}(1+y^2)$$

$= \frac{3}{x}$ which is a function of x only. $\therefore$ IF $= e^{\int\frac{3}{x}dx} = e^{3\log x} = x^3$ is an I.F.

multiplying both sides of ① by x^3 we get -

$$x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \left(\frac{x+xy^2}{4}\right)x^3 dy = 0$$

Now, $M = x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)$, $N = \frac{x^4(1+y^2)}{4}$

$\frac{\partial M}{\partial y} = x^3\left(1+\frac{3y^2}{3}\right)$, $\frac{\partial N}{\partial x} = \frac{4x^3(1+y^2)}{4} = x^3(1+y^2)$

Now, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So the eqn is exact

$$\therefore \int M dx = \int x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx = \frac{yx^4}{4} + \frac{x^4y^3}{4\cdot 3} + \frac{x^6}{12}$$

$$= \frac{yx^4}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12}$$

$$\int N dy = \int \frac{x^4}{4}(1+y^2)dy = \frac{x^4}{4}\left(y+\frac{y^3}{3}\right) = \frac{yx^4}{4} + \frac{y^3x^4}{12}$$

$\therefore$ The solⁿ is -

$$\frac{yx^4}{4} + \frac{y^3x^4}{12} + \frac{x^6}{12} = c'$$

$$\Rightarrow 3x^4y + y^3x^4 + x^6 = c$$

2nd method $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \frac{x(1+y^2)}{4}dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y+\frac{y^3}{3}+\frac{x^2}{2}}{\frac{x(1+y^2)}{4}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y(1+y^2/3)}{x(1+y^2)} - \frac{x^2}{2}\cdot\frac{4}{x(1+y^2)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y(3+y^2)}{3}\cdot\frac{4}{x(1+y^2)} - \frac{2x}{1+y^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{4}{3}y\left[\frac{3+y^2}{x(1+y^2)}\right] - \frac{2x}{1+y^2}$$

$$\Rightarrow (1+y^2)\frac{dy}{dx} = -\frac{4}{x}\left(y+\frac{y^3}{3}\right) - 2x$$

$$\Rightarrow (1+y^2)\frac{dy}{dx} + \frac{4}{x}\left(y+y^3/3\right) = -2x$$

$\therefore$ put $y + \frac{y^3}{3} = z$

$\therefore (1+y^2)\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \frac{dz}{dx} + \frac{4}{x} z = -2x$ —— (1)

$I.F = e^{\int \frac{4}{x} dx} = e^{4 \log x} = x^4$

multiplying both sides by I.F and then integrating we get

$z \cdot x^4 = \int -2x^5 dx = \frac{-2x^6}{6} + c'$

$\Rightarrow \left(y + \frac{y^3}{3}\right)x^4 + \frac{x^6}{3} = c'$

$\Rightarrow 3x^4y + x^4y^3 + x^6 = c$ which is the soln.

2)]

$\frac{x^6}{12}$

$\frac{6}{2}$

$+$

$\frac{x^4}{2}$

(Not was nd in classroom)

Q. Solve by the method of variation of parameters: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 4\cdot 1\cdot 2}}{2\cdot 1} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$y_c = (c_1 \cos x + c_2 \sin x)e^x$

$y_1 = e^x \cos x$, $y_2 = e^x \sin x$

wronskian $W(y_1, y_2) = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x & e^x \cos x \end{vmatrix}$

$= e^x(\cos^2 x + \sin^2 x)$ [$e^x \neq 0$]

y_1, y_2 are linearly independent

$y_p = e^x(v_1 \cos x + v_2 \sin x)$

$y\ e^x(v_1' \cos x + v_2' \sin x) = 0$

$\Rightarrow v_1' \cos x + v_2' \sin x = 0$ [$\because e^x \neq 0$]

$-v_1' e^x \sin x + v_2' e^x \cos x = e^x \tan x$

$\Rightarrow -v_1' \sin x + v_2' \cos x = \tan x$ [$\because e^x \neq 0$] — (ii)

$v_1' \cos x + v_2' \sin x = 0$ — (i)

$-v_1' \sin x + v_2' \cos x = \tan x$ — (ii)

(i) × sin x and (ii) × cos x,

$v_1' \sin x \cos x + v_2' \sin^2 x = 0$

$-v_1' \cos x \sin x + v_2' \cos^2 x = \tan x \cos x$

$\Rightarrow v_2' = \sin x$

$\Rightarrow v_2 = \int \sin x \, dx$

$= -\cos x$

(i) × cos x and (ii) × sin x,

$v_1' \cos^2 x + v_2' \cos x \sin x = 0$

$-v_1' \sin^2 x + v_2' \cos x \sin x = \tan x \sin x$

(+) (−) (−)

$\Rightarrow v_1' = -\tan x \sin x$

$\Rightarrow v_1 = -\int \tan x \sin x \, dx$

$= -\left[-\tan x \cos x + \int \sec^2 x \cos x \, dx\right]$

$= -\left[-\tan x \cos x + \int \sec x \, dx\right]$

$v_1 = \tan x \cos x - \log\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)$

$\therefore y_p = \left\{\left[\tan x \cos x - \log\left\{\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\}\right]\cos x - \cos x \sin x\right\} e^x$

$\therefore$ soln $= e^x\left[\left\{\tan x \cos x - \log\left\{\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\}\right\}\cos x - \cos x \sin x\right] + e^x(c_1 \cos x + c_2 \sin x)$

(b) Solve: $(x^2D^2 - xD + 2)y = x\log x$, $(D = \frac{d}{dx})$ [find in CW-2nd part]

let $x = e^z$, $z = \log x$

$(x^2D^2 - xD + 2)y = x\log x$

$\Rightarrow \left(x^2\frac{d^2}{dx^2} - x\frac{d}{dx} + 2\right)y = x\log x$

$\Rightarrow x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$

$\Rightarrow \cancel{\frac{d^2z}{dx^2} - \frac{dz}{dx}}$

$\Rightarrow \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - \left(\frac{dy}{dz}\right) + 2y = e^z z$

$\Rightarrow \frac{d^2y}{dz^2} - 2\frac{dy}{dz} + 2y = ze^z$

$m^2 - 2m + 2 = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4\cdot 2\cdot 1}}{2\cdot 1} = \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1 \pm i$

$y_c = (c_1\cos z + c_2\sin z)e^z$

$y_p = \frac{1}{D^2 - 2D + 2} ze^z$

$= e^z \frac{1}{(D^2 - 2D + 1) + 1} ze^z$

$= \frac{1}{(D-1)^2 + 1} ze^z$

$= e^z \frac{1}{(D+1-1)^2 + 1}(z)$

$= e^z \frac{1}{D^2 + 1}(z)$

$= e^z (1 + D^2)^{-1}(z)$

$= e^z(1 - D^2 - \cdots)(z)$

$= ze^z$

$= x\log x$

Soln: $\therefore y = x[c_1\cos(\log x) + c_2\sin(\log x)] + x\log x$

1(b) (a) Find the eigen values and eigen functions of the eigen value problem $\frac{d^2y}{dx^2} + \lambda y = 0$, satisfying with the conditions $y(0) + y'(0) = 0$, and $y(1) + y'(1) = 0$.

$$m^2 + \lambda = 0$$
$$\Rightarrow m = \pm\sqrt{-\lambda}$$
$$= \pm i\sqrt{\lambda}$$

$$y_c = (c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x)$$

$$y'(c) = -c_1 \sin\sqrt{\lambda}x \cdot \sqrt{\lambda} + c_2 \cos\sqrt{\lambda}x \cdot \sqrt{\lambda}$$

$$y(0) + y'(0) = 0$$
$$\Rightarrow c_1 + 0 + c_2\sqrt{\lambda} = 0 \quad \text{--(1)}$$

$$y(1) + y'(1) = 0$$
$$\Rightarrow c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda} - \sqrt{\lambda}c_1 \sin\sqrt{\lambda} = 0$$
$$\Rightarrow c_1 \cancel{\cos\sqrt{\lambda}} + c_2 \sin\sqrt{\lambda} - c_1 \cancel{\cos\sqrt{\lambda}} - \sqrt{\lambda}c_1 \sin\sqrt{\lambda} = 0$$

[$\because$ from (1) $c_2\sqrt{\lambda} = -c_1$]

$$\Rightarrow \sin\sqrt{\lambda}(c_2 - \sqrt{\lambda}c_1) = 0$$
$$c_2 - \sqrt{\lambda}c_1 \neq 0$$
$$\therefore \sin\sqrt{\lambda} = 0 = \sin n\pi$$
$$\therefore \sqrt{\lambda} = n\pi \quad [n = 0, \pm1, \pm2 \ldots]$$

$$\Rightarrow \lambda = n^2\pi^2 \quad [n = 0, 1, 2 \ldots]$$

which are the eigen values.

eigen function-

$$y_n = (c_{n+1}\sqrt{\lambda}c_{n+1}) \sin n\pi x \quad [n = 1, 2, 3 \ldots]$$

(b) Transform the differential equation $yp^2 + x^3p = x^2y$ ($p = \frac{dy}{dx}$) to clairaut's form by using the transformation $x^2 = u$, $y^2 = v$ and hence solve it.

$$yp^2 + x^3p = x^2y \quad \text{--(1)}$$

$$x^2 = u \qquad ; \qquad y^2 = v$$
$$\Rightarrow 2x\,dx = du \qquad \Rightarrow 2y\,dy = dv$$

$$\frac{dv}{du} = \frac{y\,dy}{x\,dx}$$

$$\Rightarrow q = \frac{y}{x}p \quad \left[\text{let, } \frac{dv}{du} = q \text{ and given, } \frac{dy}{dx} = p\right]$$

$$\Rightarrow p = \frac{qx}{y} \quad \text{--(2)}$$

Substituting (1) we get,

$$y \cdot \frac{q^2x^2}{y^2} + x^3\frac{qx}{y} = x^2y$$

$$\Rightarrow q^2x^2 + qx^4 = x^2y^2$$
$$\Rightarrow q^2u + qu^2 = uv$$
$$\Rightarrow q^2 + qu = v$$
$$\Rightarrow v = q^2 + qu$$
$$v = c^2 + cu \rightarrow \text{This is general sol}^n.$$

Now,

$$y^2 = c^2 + cx^2$$

the discriminant of c is

$$c^2 + cx^2 - y^2 = 0$$

$$(x^2)^2 + 4 \cdot 1 y^2 = 0$$

$$\Rightarrow x^4 + 4y^2 = 0$$

This is singular soln

(noting case)

11)(a) Solve by the method of undetermind co-efficients:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x^2 - x + e^{-2x}$$

First order and first degree equation –

1) Solve the following eqn –

a) $(y^2 \; e^{xy^2} +$ done

b) $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$

$M\,dx + N\,dy = 0$

$\therefore M = \cos y + y\cos x \quad , \quad N = \sin x - x\sin y$

$\frac{\partial M}{\partial y} = -\sin y + \cos x \qquad \frac{\partial N}{\partial x} = \cos x - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The equation is exact

$\int M\,dx = \int (\cos y + y\cos x)\,dx$

$= x\cos y + y\sin x$

$\int N\,dy = \int (\sin x - x\sin y)\,dy$

$= y\sin x + \cos y\, x$

$\therefore$ The soln is $= x\cos y + y\sin x = c$

c) done

d) $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

$M = x^2 - 4xy - 2y^2 , \quad N = y^2 - 4xy - 2x^2$

$\Rightarrow \frac{\partial M}{\partial y} = -4x - 4y \quad , \quad \Rightarrow \frac{\partial N}{\partial x} = -4y - 4x$

$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$

$\int m\,dx = \int (x^2 - 4xy - 2y^2)\,dx$

$= \frac{x^3}{3} - \frac{\cancel{4}^{2}x^2y}{\cancel{2}} - 2y^2x$

$= \frac{x^3}{3} - 2x^2y - 2xy^2$

$\int n\,dy = \int (y^2 - 4xy - 2x^2)\,dy$

$= \frac{y^3}{3} - \frac{\cancel{4}^{2}xy^2}{\cancel{2}} - 2x^2y$

$= \frac{y^3}{3} - 2xy^2 - 2x^2y$

$\therefore$ the solⁿ is $= -2x^2y - 2xy^2 + \frac{x^3}{3} + \frac{y^3}{3} = c$

c) $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

$\Rightarrow x\sqrt{1-y^2}\,dx = -y\sqrt{1-x^2}\,dy$

$\Rightarrow \frac{dy}{dx}$ $\frac{1}{-2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}}\,dy$

$\Rightarrow -\frac{1}{\cancel{2}}\cancel{2}\sqrt{1-x^2} = \frac{1}{\cancel{2}}\cancel{2}\sqrt{1-y^2} + c'$

$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = c$

f) done

g) done

h) done

i) done

j) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$\Rightarrow (x^2 - xy)\frac{dy}{dx} + y^2 = 0$

$\Rightarrow x(x-y)\frac{dy}{dx} = -y^2$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{x^2 - xy}$

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - x\cdot vx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{-v^2x^2}{(x^2 - vx^2)}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{\cancel{x^2}(-v^2)}{\cancel{x^2}(1-v)}$

$\Rightarrow x\frac{dv}{dx} = -\frac{v^2}{1-v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{-\cancel{v^2} - v + \cancel{v^2}}{1-v}$

$\Rightarrow \int \frac{1-v}{v}\,dv = -\int \frac{1}{x}\,dx$

$\Rightarrow \int \left(\frac{1}{v} - 1\right)dv = -\int \frac{1}{x}\,dx$

$\Rightarrow \log|v| - v = -\log x + \log c$

$\Rightarrow \log|v| - v + \log x = \log c$

$\Rightarrow \log\left(\frac{y}{x}\right) + \log x - \frac{y}{x} = \log c$

$\Rightarrow \log\left(\frac{y}{x} \cdot x\right) - \frac{y}{x} = \log c$

$\Rightarrow \log y - \frac{y}{x} = \log c$

(c) $\left(x + y\cos\frac{y}{x}\right)dx = x\cos\frac{y}{x}\,dy$

$\Rightarrow \frac{dy}{dx} = \frac{x + y\cos\frac{y}{x}}{x\cos\frac{y}{x}}$

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{x + vx\cdot\cos v}{x\cos v}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{\cancel{x}(1 + v\cos v)}{\cancel{x}\cos v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1 + v\cos v}{\cos v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1 + v\cos v - v\cancel{\cos v}}{\cos v}$

$\Rightarrow \int \cos v\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \sin v = \log|x| + C$

$\Rightarrow \sin\frac{y}{x} - \log x = C$

(d) $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y^3 + y^2\sqrt{y^2 - x^2}}{x^3}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^3 + \frac{y^2\sqrt{y^2 - x^2}}{x^3}$

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v^3 + \frac{v^2x^2\sqrt{v^2x^2 - x^2}}{x^3}$

$\Rightarrow v + x\frac{dv}{dx} = v^3 + \frac{v^2\cancel{x^2}\cancel{x}\sqrt{v^2 - 1}}{\cancel{x^3}}$

$\Rightarrow x\frac{dv}{dx} = v^3 + v^2\sqrt{v^2 - 1} - v$

$\Rightarrow x\frac{dv}{dx} = v^3 - v + v^2\sqrt{v^2 - 1}$

$\Rightarrow \frac{dv}{v(v^2 - 1) + v^2\sqrt{v^2 - 1}} = \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{v\sqrt{v^2 - 1}\left[\sqrt{v^2 - 1} + v\right]} = \int \frac{dx}{x}$

$\Rightarrow \int \frac{\sqrt{v^2 - 1} - v}{v\sqrt{v^2 - 1}\,(v^2 - 1 - v^2)}\,dv = \int \frac{dx}{x}$

$\Rightarrow \int \frac{v - \sqrt{v^2 - 1}}{v\sqrt{v^2 - 1}}\,dv = \int \frac{dx}{x}$

$\Rightarrow \int \frac{1}{\sqrt{v^2 - 1}} - \int \frac{dv}{v} = \int \frac{dx}{x}$

$\Rightarrow \log\left(v + \sqrt{v^2 - 1}\right) - \log v = \log x + \log c$

$\Rightarrow \log\left|\frac{v+\sqrt{v^2-1}}{v}\right| = \log x \Rightarrow \frac{y+\sqrt{y^2-x^2}}{y} = xc$

$\log\left|\frac{\frac{y}{x}+\sqrt{\frac{y^2-x^2}{x}}}{\frac{y}{x}}\right| = \log x \Rightarrow y+\sqrt{y^2-x^2} = xyc$

⇒ done

⇒ done

Q) $(3y-7x+7)dx+(7y-3x+3)dy=0$

$\Rightarrow \cancel{3y\,dx}\ \frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ — (1)

where $\frac{3}{7} \neq 7/3$ $\left(\frac{a_1}{a_1} \neq \frac{a_2}{b_2}\right)$

∴ put $x = X+h$ $y = Y+k$

$dx = dX$, $\Rightarrow dy = dY$

(1) becomes

$$\frac{dY}{dX} = -\frac{3Y+3k+7X-7h+7}{7Y+7k-3X-3h+3}$$

$$\Rightarrow \frac{dY}{dX} = -\frac{3Y-7X+(3k-7h+7)}{7Y-3X+(7k-3h+3)} \quad -(2)$$

we chose h, k such that,

$3k-7h+7=0$ — (3)

$7k-3h+3=0$ — (4)

$$\frac{k}{-21+21} = \frac{h}{49-9} = \frac{1}{-9+49}$$

$\Rightarrow k=0,\ h=1$

∴ (2) becomes $\frac{dY}{dX} = -\frac{3Y-7X}{7Y-3X}$ — (5)

put $Y = vX$

$\frac{dY}{dX} = v+X\frac{dv}{dX}$

$v+X\frac{dv}{dX} = -\frac{3v-7}{7v-3}$

$\Rightarrow X\frac{dv}{dX} = \frac{-7(v^2-1)}{7v-3}$

$\Rightarrow \int\frac{7v-3}{\sqrt{v^2-1}}dv = -7\int\frac{dx}{x}$

$\Rightarrow \frac{7}{2}\int\frac{2v\,dv}{v^2-1} - 3\int\frac{dv}{v^2-1} = -7\int\frac{1}{x}dx$

$\Rightarrow \frac{7}{2}\log(v^2-1) - \frac{3}{2}\log\frac{v-1}{v+1} + 7\log x = c$

$\Rightarrow \frac{7}{2}\log\left[\frac{y^2}{x^2}-1\right] - \frac{3}{2}\log\frac{y-x}{y+x} + 7\log x \cdot \log c$

$\Rightarrow \frac{7}{2}\log\left[\frac{y^2-(x-1)^2}{(x-1)^2}\right] - \frac{3}{2}\log\left(\frac{y-x+1}{y+(x-1)}\right) + 7\log(x-1) = \log c$ (A)

P) $(2x+3y-5)dy+(3x+2y-5)dx=0$

$\Rightarrow (2x\,dy+2y\,dx)+3y\,dy+3x\,dx-5dy-5dx=0$

$\Rightarrow 2\int d(xy)+3\int y\,dy+3\int x\,dx-5\int dy-5\int dx = \frac{c}{2}$

$\Rightarrow 2xy+\frac{3y^2}{2}+\frac{3x^2}{2}-5y-5x=\frac{c}{2}$

$\Rightarrow 4xy+3y^2+3x^2-10y-10x=c$

$\Rightarrow 3x^2+4xy+3y^2-10y-10x=c$

Q) $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

$\Rightarrow x\cos x\frac{dy}{dx} = \frac{1-y(x\sin x+\cos x)}{x\cos x}$

$\Rightarrow \frac{dy}{dx}+\frac{y(x\sin x+\cos x)}{x\cos x} = \frac{1}{x\cos x}$

$\Rightarrow \frac{dy}{dx}+y\left(\tan x+\frac{1}{x}\right) = \frac{1}{x\cos x}$ — (1)

IF $= e^{\int\left(\frac{1}{x}+\tan x\right)dx} = e^{\log x+\log|\sec x|}$

(7)

$$\frac{7}{2}\log\left[\frac{y^2}{(x-1)^2}-1\right]-\frac{3}{2}\log\frac{y-x+1}{y+x-1}+7\log(x-1)=\log c$$

which is the required soln

it can be simplified as

$$\log\left\{\left[\frac{y^2-(x-1)^2}{(x-1)^2}\right]\right\}^{\frac{7}{2}}+\log\left(\frac{y-x+1}{y+x-1}\right)^{-\frac{3}{2}}+\log(x-1)^{7}=\log c$$

$$\Rightarrow \frac{(y+x-1)^{\frac{7}{2}}(y-x+1)^{\frac{7}{2}}}{\cancel{(x-1)^7}}\times\frac{(y+x-1)^{\frac{3}{2}}}{(y-x+1)^{\frac{3}{2}}}\times\cancel{(x-1)}^{7}=c$$

$$\Rightarrow (y+x-1)^5(y-x+1)^2=c$$

multiplying $e^{\log x + \log(\sec x)}$ in the both side of ①.

$$\frac{dy}{dx} e^{\log x + \log \sec x} + y\left(\tan x + \frac{1}{x}\right)$$

$$e^{\log x + \log \sec x} = \frac{1}{x \cos x} e^{\log x + \log \sec x}$$

$$\Rightarrow \frac{dy}{dx} e^{\log(x \sec x)} + y\left(\tan x + \frac{1}{x}\right) e^{\log(x \sec x)}$$

$$= \frac{1}{x \cos x} \cdot e^{\log(x \sec x)}$$

$$\Rightarrow x \sec x \frac{dy}{dx} + (x \sec x) y \left(\tan x + \frac{1}{x}\right) = \frac{1}{x \cos x} \times x \sec x$$

$$\Rightarrow \int \frac{d}{dx}\left[y (x \sec x)\right] = \frac{1}{\cos^2 x} \int \sec^2 x$$

$$\Rightarrow \int d(yx \sec x) = \int \sec^2 x \, dx$$

$$\Rightarrow xy \sec x = \tan x + C$$

10) $\frac{dy}{dx} - \tan y \cdot \frac{1}{1+x} = (1+x) e^x \sec y$

$$\Rightarrow \frac{1}{\sec y} \frac{dy}{dx} - \frac{\tan y}{1+x (\sec y)} = (1+x) e^x$$

~~Put $\tan y = z$~~

$$\Rightarrow \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x) e^x$$

$$\sin y = z \Rightarrow \cos y \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x) e^x \quad \frac{1}{1+x}$$

$$e^{-\int \frac{dx}{1+x}} = e^{-\log(1+x)} = \frac{1}{1+x}$$

$$(1+x) \frac{dz}{dx} - z = (1+x)^2 e^x$$

$$\Rightarrow \int \frac{d}{dx}\{z(1+x)\} = (1+x)^2 e^x$$

$$\Rightarrow z(1+x) = \int (1+x)^2 e^x dx$$

$$= (1+x)^2 e^x - 2\int e^x (1+x) dx$$

$$= (1+x)^2 e^x - 2\left[e^x(1+x) - \int e^x dx\right]$$

$$= (1+x)^2 e^x - 2\left[e^x(1+x) - e^x\right] + C$$

$$\therefore \sin y (1+x) = (1+x)^2 e^x - 2e^x x + C \quad \text{incorrect}$$

$$\frac{1}{1+x}\frac{dz}{dx} - \frac{z}{(1+x)^2} = \frac{(1+x) e^x}{(1+x)}$$

$$\Rightarrow \int \frac{d}{dx}\left(\frac{z}{1+x}\right) = \int e^x$$

$$\Rightarrow \frac{z}{1+x} = \int e^x dx$$

$$\Rightarrow \frac{\sin y}{1+x} = e^x + C$$

5) $x(1-x^2)\frac{dy}{dx} + (2x^2-1) y = ax^3$ (a constant)

$$x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = ax^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)} y = \frac{ax^{3}}{x(1-x^2)} \quad ①$$

$$\text{I.F} = e^{\int \frac{(2x^2-1)}{x(1-x^2)} dx} = e^{-\int \frac{(1-x^2)-x^2}{x(1-x^2)} dx}$$

$$= e^{-\int \frac{(1-2x^2)}{x(1-x^2)} dx}$$

$$= e^{-\int \left(\frac{1}{x} - \frac{x}{1-x^2}\right) dx}$$

$$= e^{-\left[\int \frac{1}{x} dx + \frac{1}{2}\int \frac{-2x}{1-x^2} dx\right]}$$

$$= e^{-\left[\log x + \frac{1}{2}\log(1-x^2)\right]}$$

$$= e^{-\log \frac{x}{\sqrt{1-x^2}}} = e^{\log \frac{\sqrt{1-x^2}}{x}}$$

$$= \frac{\sqrt{1-x^2}}{x}$$

$$\frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)}y = \frac{ax^3}{x(1-x^2)}$$

$$\frac{dy}{dx} + \left[\frac{(1-x^2)-x^2}{x(1-x^2)}\right]y = \frac{ax^2}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} - \left[\frac{1}{x} - \frac{x}{1-x^2}\right]y = \frac{ax^2}{1-x^2}$$

$$e^{\int -\frac{1}{x} + \frac{x}{1-x^2}\,dx}$$

$$= e^{-\log x - \frac{1}{2}\log(1-x^2)}$$

$$= e^{-\log(x\sqrt{1-x^2})^{-1}}$$

$$= \frac{1}{x\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx}\cdot\frac{1}{x\sqrt{1-x^2}} + \frac{(2x^2-1)}{x(1-x^2)}\cdot\frac{1}{x\sqrt{1-x^2}}y$$

$$= \frac{ax^2}{x^2(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \int d\left(\frac{y}{x\sqrt{1-x^2}}\right) = \int \frac{a}{(1-x^2)\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \frac{y}{x\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} + c \quad \Rightarrow \text{Previous Page}$$

$$\frac{y}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}}\,x + c$$

$$\int \frac{a x^2\, dx}{x(1-x^2)\sqrt{x^2-1}} \qquad \frac{\sin\alpha}{\cos^2\alpha}$$

$x = \sin\alpha$, $dx = \cos\alpha\, d\alpha$

$$= \int \frac{\sin\alpha\cos\alpha\, d\alpha}{\cos^2\alpha\,\cos\alpha}$$

$$= a\int \tan\alpha\,\sec\alpha\, d\alpha$$

$$= a\sec\theta \qquad \frac{x}{1} = \frac{\sqrt{1+x^2}}{1}\ \ \sqrt{1-x^2}$$

$$= a\sec(\sin^{-1}x) \qquad \frac{1}{\sqrt{1-x^2}}$$

$$= a\sec\left(\sec^{-1}\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{a}{\sqrt{1-x^2}}$$

multiplying $\frac{\sqrt{1-x^2}}{x}$ in the both side of the eqn (1)

$\Rightarrow \frac{dy}{dx}\frac{\sqrt{1-x^2}}{x} + \frac{(2x^2-1)}{x(1-x^2)}\frac{\sqrt{1-x^2}}{x}y = \frac{ax^2 \cdot x}{(1-x^2)}\cdot\frac{\sqrt{1-x^2}}{x}$

$\Rightarrow \frac{d}{dx}\left(y\frac{\sqrt{1-x^2}}{x}\right) = \frac{ax\sqrt{1-x^2}}{(1-x^2)} = \frac{ax}{\sqrt{1-x^2}}$

$\Rightarrow \int d\left(y\frac{\sqrt{1-x^2}}{x}\right) = \int\frac{ax}{\sqrt{1-x^2}}dx$

$\Rightarrow \frac{y\sqrt{1-x^2}}{x} = -\frac{a}{2}\int\frac{2x}{\sqrt{1-x^2}}dx$

$\Rightarrow \frac{y\sqrt{1-x^2}}{x} = -\frac{a}{2}\times 2\sqrt{1-x^2}+c$

$\Rightarrow \frac{y\sqrt{1-x^2}}{x} + a\sqrt{1-x^2} = c \longrightarrow$ this is not ans

Ans $= \frac{y}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + c$

ii) $(x+y+1)dy = dx$

$\frac{dy}{dx} = \frac{1}{x+y+1}$

$\Rightarrow \frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = (y+1)$

$\therefore$ IF $= e^{-\int 1dy} = e^{-y}$

$\frac{dx}{dy}e^{-y} - xe^{-y} = (y+1)e^{-y}$

Or

$\frac{dy}{dx} = \frac{1}{x+y+1}$

$x+y = z$

$\Rightarrow 1+\frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} - 1 = \frac{1}{z+1}$

$\Rightarrow \frac{dz}{dx} = \frac{1}{z+1}+1$

$\Rightarrow \frac{dz}{dx} = \frac{1+z+1}{z+1}$

$\Rightarrow \int\frac{z+1}{z+2}dz = \int dx$

$\Rightarrow \int\frac{z+2-1}{z+2}dz = \int dx$

$\int\left(1-\frac{1}{z+2}\right)dz = \int dx$

$\Rightarrow z - \log(z+2) = x+c$

$\Rightarrow x+y = \log(x+y+2) + x+c$

$\Rightarrow y = \log(x+y+2)+c$ — ans

$\Rightarrow \frac{d}{dy}(xe^{-y}) = (y+1)e^{-y}$

$\Rightarrow \int d(xe^{-y}) = \int(y+1)e^{-y}dy$

$\Rightarrow xe^{-y} = -(y+1)e^{-y} + \int e^{-y}dy$

$\Rightarrow xe^{-y} = -(y+1)e^{-y} - e^{-y} + c$

$\Rightarrow xe^{-y} = -e^{-y}[y+1+1]+c$

$\Rightarrow x = y+2+ce^{y}$

iii) $(1+y^2)dx = (\tan^{-1}y - x)dy$

$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y-x}$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y-x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

IF $= e^{\int\frac{1}{1+y^2}dy}$

$= e^{\tan^{-1}y}$

$\frac{dx}{dy}e^{\tan^{-1}y} + \frac{x\tan^{-1}y}{1+y^2} = \frac{\tan^{-1}y\, e^{\tan^{-1}y}}{1+y^2}$

$\Rightarrow \frac{d}{dy}(xe^{\tan^{-1}y}) = \frac{e^{\tan^{-1}y}\tan^{-1}y}{1+y^2}$

$\Rightarrow \int d(x e^{\tan^{-1}y}) = \int \frac{e^{\tan^{-1}y}\tan^{-1}y}{1+y^2}dy$

$\Rightarrow x e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}\tan^{-1}y}{1+y^2}dy$

$\int \frac{e^{\tan^{-1}y}\tan^{-1}y}{(1+y)^2}dy$

let $e^{\tan^{-1}y} = z$

$\Rightarrow e^{\tan^{-1}y}\frac{1}{1+y^2}dy = dz$

$= \int \log z \, dz$

$= z(\log z - 1)$

$= e^{\tan^{-1}y}(\log e^{\tan^{-1}y} - 1)$

$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

$\therefore x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

$\Rightarrow x = \tan^{-1}y - 1 + c e^{-\tan^{-1}y}$

(v) $3\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x^3}{y^2}$

$\Rightarrow \frac{dy}{dx} + \cancel{\frac{2}{3(x+1)}}y = \frac{x^3}{3y^2}$

$\cancel{IF = e^{\frac{2}{3}\int \frac{1}{x+1}dx} = e^{\frac{2}{3}\log(x+1)}}$

$\cancel{= e^{\log(x+1)^{\frac{2}{3}}}}$

$\cancel{= (x+1)^{\frac{2}{3}}}$

$\cancel{\therefore \frac{dy}{dx}(x+1)^{\frac{2}{3}} + \frac{2(x+1)^{\frac{2}{3}}}{3(x+1)}y = \frac{x^3(x+1)^{\frac{2}{3}}}{3y^2}}$

$\cancel{\Rightarrow \frac{dy}{dx}(x+1)^{\frac{2}{3}} + \frac{2y}{3(x+1)^{\frac{1}{3}}} = \frac{x^3(x+1)^{\frac{2}{3}}}{3y^2}}$

$\cancel{\Rightarrow \frac{d}{dx}\{y(x+1)^{\frac{2}{3}}\} = \frac{x^3(x+1)^{\frac{2}{3}}}{3y^2}}$

$\cancel{\Rightarrow \int d\{y(x+1)^{\frac{2}{3}}\} = \int \frac{x^3(x+1)^{\frac{2}{3}}}{3y^2}dx}$

$\cancel{\Rightarrow y(x+1)^{\frac{2}{3}} =}$

$\Rightarrow 3y^2\frac{dy}{dx} + \frac{2y^3}{x+1} = x^3$

put $y^3 = z$

$\Rightarrow 3y^2\frac{dy}{dx} = \frac{dz}{dx}$

$\frac{dz}{dx} + \frac{2z}{x+1} = x^3$

$IF = e^{\int \frac{2}{x+1}dx} = e^{2\log(x+1)} = e^{\log(x+1)^2} = (x+1)^2$

$\therefore (x+1)^2\frac{dz}{dx} + \frac{2z(x+1)^{\cancel{2}}}{\cancel{(x+1)}} = x^3(x+1)^2$

$\Rightarrow \frac{d}{dx}[z(x+1)^2] = x^3(x+1)^2$

$\Rightarrow \int d[z(x+1)^2] = \int x^3(x+1)^2 dx$

$\Rightarrow z(x+1)^2 = \int x^3(x^2+2x+1)\,dx$

$\Rightarrow z(x+1)^2 = \int (x^5+2x^4+x^3)\,dx$

$\Rightarrow y^3(x+1)^2 = \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c$

x) $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = ax^3$

(a is a constant)

$\cancel{3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^2 = ax^3}$

$\Rightarrow \cancel{\frac{dy}{dx} + \frac{(2x^2-1)y^2}{3xy^2(1-x^2)} = \frac{ax^3x^2}{3x(1-x^2)y^2}}$

$\Rightarrow \cancel{\frac{dy}{dx} + \frac{(x^2-1)}{3x(1-x^2)} = \frac{ax^2}{3y^2}}$

$\Rightarrow 3y^2\frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)}y^3 = \frac{a\cancel{x^3}x^2}{\cancel{x}(1-x^2)}$

put $y^3 = z$

$\Rightarrow 3y^2\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx} + \frac{(2x^2-1)\cdot z}{x(1-x^2)} = \frac{ax^2}{1-x^2}$

$IF = e^{\int \frac{2x^2-1}{x(1-x^2)}dx} = \frac{\sqrt{1-x^2}}{x}$ [Previous ⓐ]

$\therefore \frac{\sqrt{1-x^2}}{x}\frac{dz}{dx} + \frac{(2x^2-1)}{x(1-x^2)}z\frac{\sqrt{1-x^2}}{x} = \frac{ax^2\sqrt{1-x^2}}{\cancel{x}(1-x^2)}$

$\Rightarrow \frac{d}{dx}\left(z\frac{\sqrt{1-x^2}}{x}\right) = \frac{ax}{\sqrt{1-x^2}}$

$\Rightarrow \int d\left(z\frac{\sqrt{1-x^2}}{x}\right) = -\frac{a}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$

$\Rightarrow \frac{z\sqrt{1-x^2}}{x} = -\frac{a}{2}\times 2\sqrt{1-x^2} + c$

$\Rightarrow \frac{y^3\sqrt{1-x^2}}{x} = -a\sqrt{1-x^2} + c$ → This is not ans

Ans — $\frac{y^3}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + c$

y) $x^2y - x^3\frac{dy}{dx} = y^4\cos x$

$\Rightarrow x^2y - x^3\frac{dy}{dx} = y^4\cos x$

$\Rightarrow -x^3\frac{dy}{dx} + x^2y = y^4\cos x$

$\Rightarrow \frac{dy}{dx} - \frac{x^2y}{\cancel{x^3}x} = \frac{-y^4\cos x}{x^3}$

$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} - \frac{y}{xy^4} = -\frac{\cos x}{x^3}$

$-\frac{1}{y^3} = z$

$\Rightarrow +\frac{3}{y^4}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{1}{y^4}\frac{dy}{dx} = +\frac{1}{3}\frac{dz}{dx}$

$\therefore \frac{1}{3}\frac{dz}{dx} + \frac{z}{x} = -\frac{\cos x}{x^3}$

$\Rightarrow \frac{dz}{dx} + \frac{3z}{x} = -\frac{3\cos x}{x^3}$

$$\frac{dy}{dx} = \frac{1}{x+y+1}$$

$x+y = z$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

$$\frac{dz}{dx} - 1 = \frac{1}{z}$$

$\Rightarrow \frac{dz}{dx} = \frac{1}{z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{1+z}{z}$

$\Rightarrow \int \frac{(1+z) - 1}{1+z} dz = \int dx$

$\Rightarrow \int \left(1 - \frac{1}{1+z}\right) dz = \int dx$

$\Rightarrow z - \log|1+z| = x + c$

$\Rightarrow \cancel{x} + y - \log|x+y+1| = \cancel{x} + c$

$\Rightarrow y - c = \log|x+y+1|$

$\Rightarrow$

$\frac{dy}{dx} = \frac{1}{x+y+1}$

$\Rightarrow \frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = (y+1)$

$e^{\int -1\, dy}$

$= e^{-y}$

$e^{-y} \frac{dx}{dy} - x e^{-y} =$

$\Rightarrow x e^{-y} = \int (y+1) e^{-y} dy$

$(y+1) e^{-y}$

$= -e^{-y}(y+1)$

$+ \int e^{-y} dy$

$= e^{-y}(y+2)$

$+c$

$\Rightarrow x = (y+2) + c e^{y}$

$$\frac{dy}{dx} = \frac{1}{x+y+1}$$

$x+y = z$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} - 1 = \frac{1}{z}$$

$\Rightarrow \frac{dz}{dx} = \frac{1}{z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{1+z}{z}$

$\Rightarrow \int \frac{1+z-1}{1+z} dz = \int dx$

$\Rightarrow \int \left(1 - \frac{1}{1+z}\right) dz = \int dx$

$\Rightarrow z - \log|1+z| = x + c$

$\Rightarrow x+y - \log|x+y+1| = x+c$

$\Rightarrow y - \log|x+y+1| = c$

$$\int \frac{ax^2}{(1-x^2)} \cdot \frac{1}{x\sqrt{(1-x^2)}} dx$$

$$= \int \frac{ax}{(1-x^2)\sqrt{1-x^2}} dx$$

$x = \sin\theta$

$$= a\int \frac{\cos\theta \sin\theta \, d\theta}{\cos^2\theta \cdot \cos\theta} \quad dx = \cos\theta \, d\theta$$

$$= a\int \tan\theta \sec\theta \, d\theta$$

$= a\sec\theta$

$= a\sec(\sin^{-1}x)$

$= a\sec\left(\sec^{-1}\frac{1}{\sqrt{1-x^2}}\right)$

$$= \frac{a}{\sqrt{1-x^2}}$$

$$\frac{1}{x\sqrt{1-x^2}} \frac{dy}{dx} +$$

$$\frac{(2x^2-1)}{x(1-x^2)\sqrt{1-x^2}} y =$$

$$\frac{ax}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \frac{y}{x\sqrt{1-x^2}} = \int \frac{ax \, dx}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \frac{y}{x\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C$$

$$e^{\int \frac{2x^2-1}{x(1-x^2)} dx} = e^{-\int \frac{(1-x^2)-x^2}{x(1-x^2)} dx}$$

$$= e^{-\int\left(\frac{1}{x} - \frac{x}{1-x^2}\right) dx}$$

$$= e^{-\left(\log x + \frac{1}{2}\log(1-x^2)\right)}$$

$$= e^{-\log\left(x\sqrt{1-x^2}\right)}$$

$$= e^{\log\left(x\sqrt{1-x^2}\right)^{-1}}$$

$$= \frac{1}{x\sqrt{1-x^2}}$$

$\text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\int \frac{1}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$

$\frac{dz}{dx} \cdot x^3 + \frac{3z}{x} \times x^3 x^2 = -\frac{3\cos x}{x^3} \times x^3$

$\Rightarrow \frac{d}{dx}(x^3 z) = -3\cos x$

$\Rightarrow \int d(x^3 z) = -3\int \cos x \, dx$

$\Rightarrow x^3 z = -3\sin x + c'$

$\Rightarrow \frac{x^3}{y^3} = -3\sin x + c'$

$\Rightarrow \frac{x^3}{y^3} = 3\sin x + C$

2) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

$\Rightarrow \frac{1}{(\log y)^2}\frac{dy}{dx} + \frac{y\log y}{x(\log y)^2}\log y = \frac{y}{x^2}$

$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{y\log y}{yx(\log y)^2}\log y = \frac{1}{x^2}$

$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x(\log y)} = \frac{1}{x^2}$

Put $\frac{1}{\log y} = z$

$\Rightarrow -\frac{1}{(\log y)^2} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$

$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore \frac{1}{x}\frac{dz}{dx} - \frac{z}{x} \cdot \frac{1}{x} = -\frac{1}{x^3}$

$\Rightarrow \frac{d}{dx}\left(z \cdot \frac{1}{x}\right) = -\frac{1}{x^3}$

$\Rightarrow \int d\left(\frac{z}{x}\right) = -\int \frac{1}{x^3}dx$

$\Rightarrow \frac{z}{x} = \frac{-x^{-2}}{-2} + C$

$\Rightarrow \frac{1}{x(\log y)} = \frac{1}{2x^2} + C$

2) done

First order and higher degree

2) Solve the following equations ($p = \frac{dy}{dx}$)

a) done | d) done | g) done
b) done | e) done | j) done
c) done | f) done | k) done

g) $y + px = p^2x^4$ — (1)

$y = p^2x^4 - px$

$\Rightarrow \frac{dy}{dx} = p^2 4x^3 + x^4 \cdot 2p\frac{dp}{dx} - p - x\frac{dp}{dx}$

$\Rightarrow p = p^2 4x^3 + 2px^4\frac{dp}{dx} - x\frac{dp}{dx}$

$\Rightarrow p + p - p^2 4x^3 = (2px^4 - x)\frac{dp}{dx}$

$\Rightarrow 2p - 4x^3p^2 = (2px^4 - x)\frac{dp}{dx}$

$\Rightarrow \cancel{(2p}\ \cancel{4x}$

$\Rightarrow 2p(1 - 2x^3p) = x(2px^3 - 1)\frac{dp}{dx}$

$\Rightarrow -2p(2x^3p - 1) = x(2px^3 - 1)\frac{dp}{dx}$

$\Rightarrow \frac{1}{2p}dp = -\frac{1}{x}dx$

$\Rightarrow \int\frac{dp}{2p} = -\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\log p = -\log x + \frac{\log c}{2}$

$\Rightarrow \log p + \log x^2 = \log c$

$\Rightarrow px^2 = c \quad \Rightarrow p = \left(\frac{x^2}{c}\right)$ — (11) $\quad dp = \frac{c}{x^2}$

substituting (1) we get, $y + \frac{x^3}{c} = \frac{x^4}{x^2}$ $\quad y + \frac{c}{x^2}x = \frac{c^2}{x^4}x^4$

$\Rightarrow c\cancel{y} + \cancel{cx^3} = x^2$ $\quad \Rightarrow y + \frac{c}{x} = c^2$

j) $xp^2 - 2py + ax = 0$ — (1)

$\Rightarrow xp^2 + ax = 2py$

$\Rightarrow 2py = xp^2 + ax$ $\longrightarrow$ others Process

$\Rightarrow y = \frac{xp^2 + ax}{2p}$

$\Rightarrow y = \frac{xp}{2} + \frac{ax}{2p}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[x\frac{dp}{dx} + p\right] + \frac{a}{2}\left[-\frac{x}{p^2}\frac{dp}{dx} + \frac{1}{p}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2}\frac{dp}{dx} + \frac{p}{2} - \frac{ax}{2p^2}\frac{dp}{dx} + \frac{a}{2p}$

$\Rightarrow p - \frac{p}{2} - \frac{a}{2p} = \left(\frac{x}{2} - \frac{ax}{2p^2}\right)\frac{dp}{dx}$

$\Rightarrow \frac{p}{2} - \frac{a}{2p} = \left(\frac{x}{2} - \frac{ax}{2p^2}\right)\frac{dp}{dx}$

$\Rightarrow \frac{p^2 - a}{\cancel{2}p} = \frac{(p^2x - ax)}{\cancel{2}p^2}\frac{dp}{dx}$

$\Rightarrow p^3 - ap = (p^2x - ax)\frac{dp}{dx}$

$\Rightarrow p^3 - ap = x(p^2 - a)\frac{dp}{dx}$

$\Rightarrow \int\frac{p^2 - a}{p^3 - ap}dp = \int\frac{1}{x}dx$

$\Rightarrow \frac{1}{3}\int\frac{3p^2 - 3a}{p^3 - ap}dp = \int\frac{1}{x}dx$

$\Rightarrow \frac{1}{3}\int\frac{(3p^2 - a) - 2a}{p^3 - ap}dp = \int\frac{1}{x}dx$

$\Rightarrow \frac{1}{3}\left[\int\frac{3p^2 - a}{p^3 - ap}dp - \int\frac{2a}{p^3 - ap}\right] = \int\frac{1}{x}dx$

$$\Rightarrow \frac{1}{3}\left[\int \frac{3p^2-a}{p^3-ap}\,dp - \frac{2a}{2\sqrt{a}}\int \frac{(p+\sqrt{a})-(p-\sqrt{a})}{p(p^2-a)}\,dp\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \frac{2a}{\sqrt{a}}\int\left[\frac{1}{p(p-\sqrt{a})} - \frac{1}{p(p+\sqrt{a})}\right]dp\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \frac{a}{\sqrt{a}}\int\left(\frac{1}{\sqrt{a}}\frac{p-(p-\sqrt{a})}{p(p-\sqrt{a})}\right) - \frac{1}{\sqrt{a}}\left(\frac{p+\sqrt{a}-p}{p(p+\sqrt{a})}\right)dp\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \frac{\cancel{a}}{\cancel{a}}\int\left[\frac{1}{p-\sqrt{a}} - \frac{1}{p}\right]dp - \int\left[\frac{1}{p} - \frac{1}{p+\sqrt{a}}\right]dp\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \left\{\log\left|\frac{p-\sqrt{a}}{p}\right| - \log\left|\frac{p}{p+\sqrt{a}}\right|\right\}\right] = \log x + \log c$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \log\left(\frac{(p-\sqrt{a})(p+\sqrt{a})}{p}\right)\right] = \log x + \log c$$

$$\Rightarrow \frac{1}{3}\left[\log|p^3-ap| - \log\frac{p^2-a}{p}\right] = \log x + \log c \quad \text{(iii)}$$

now eleminating p from (i) and (iii) we get general sol^n.

4) $y = 2px + 4xp^2$ —(i)

$$\frac{dy}{dx} = 2\left[p + x\frac{dp}{dx}\right] + 4\left[x \cdot 2p\frac{dp}{dx} + p^2\right]$$

$$\Rightarrow p = 2p + 2x\frac{dp}{dx} + 8xp\frac{dp}{dx} + 4p^2$$

$$\Rightarrow p - 2p - 4p^2 = (2x + 8xp)\frac{dp}{dx}$$

$$\Rightarrow -p - 4p^2 = 2x(1+4p)\frac{dp}{dx}$$

$$\Rightarrow \int \frac{(1+4p)}{p+4p^2}\,dp = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{2+8p}{p+4p^2}\,dp = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{(1+8p)+1}{p+4p^2}\,dp = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\left[\int \frac{1+8p}{p+4p^2}\,dp + \int \frac{dp}{p(1+4p)}\right] = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\left[\log|p+4p^2| + \int \frac{(4p+1)-4p}{p(4p+1)}\,dp\right] = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\left[\log|p+4p^2| + \int\left(\frac{1}{p} - \frac{4}{4p+1}\right)dp\right] = -\int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{2}\left[\log|p+4p^2| + \log p - \log|4p+1|\right] = -\frac{1}{2}\log x + \frac{\log c}{2}$$

$$\Rightarrow \frac{1}{2}\left[\log|p+4p^2| + \log\left(\frac{p}{4p+1}\right)\right] + \frac{\log x}{2} = \frac{\log c}{2}$$

$-p(1+4p) = 2x(1+4p)\frac{dp}{dx}$

$\Rightarrow \int \frac{1}{p}\,dp = -\int \frac{1}{2x}\,dx$

$\Rightarrow p\sqrt{x} = c$

$\Rightarrow p = \frac{c}{\sqrt{x}}$ —(ii)

$xp^2 - 2py + ax = 0$ —(1)

$\Rightarrow xp^2 + ax = 2py$

$\Rightarrow y = \dfrac{xp^2 + ax}{2p}$

$\Rightarrow y = \dfrac{xp}{2} + \dfrac{ax}{2p}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left[x\dfrac{dp}{dx} + p\right] + \dfrac{a}{2}\left[-\dfrac{x}{p^2}\dfrac{dp}{dx} + \dfrac{1}{p}\right]$

$\Rightarrow p - \dfrac{p}{2} - \dfrac{a}{2p} = \left(\dfrac{x}{2} - \dfrac{ax}{2p^2}\right)\dfrac{dp}{dx}$

$\Rightarrow \dfrac{p}{2} - \dfrac{a}{2p} = \dfrac{x(p^2 - a)}{2p^2}\dfrac{dp}{dx}$

$\Rightarrow \dfrac{(p^2 - \cancel{a})}{\cancel{2p}} = \dfrac{x(p^2 - \cancel{a})}{\cancel{2}p^{\cancel{2}}}\dfrac{dp}{dx}$

$\Rightarrow \dfrac{x}{p}\dfrac{dp}{dx} = 1$

$\Rightarrow \int\dfrac{dp}{p} = \int\dfrac{dx}{x}$

$\Rightarrow \log p = \log x + \log c$

$\Rightarrow p = xc$ —(2)

$x \cdot x^2c^2 - 2 \cdot x \cdot c \, y + ax = 0$

$\Rightarrow x^3c^2 - 2xcy + ax = 0$

$\Rightarrow \log\left\{(p+\sqrt{4}p) \times \frac{p}{4\sqrt{}}\right\} + \log x = \log c$

$\Rightarrow \log p^2 + \log x = \log c$

$\Rightarrow xp^2 = c$

$\Rightarrow p^2 = \frac{c}{x}$

$\Rightarrow p = \frac{\sqrt{c}}{\sqrt{x}}$ — (11)

by substituting (1) we get,

$y = \frac{2\sqrt{c}}{\sqrt{x}} \cdot x + 4y \cdot \frac{c}{y}$

$y = 2\sqrt{cx} + 4c$ → this is general solⁿ

3(B) Find the general solution and the singular solutions (if any) of the following eqⁿ:

a), b), c), d), e) done

4) Linear equations with constant coefficients:

Solve the following equations:

a) done, c) done, d) done, f) done

b) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$

~~$m^4 - 4y m^3 + 8y m^2 - 8y m + 4 = 0$~~

$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

$m^2\left[m^2 - 4m + 8 - \frac{8}{m} + \frac{4}{m^2}\right] = 0$

$\Rightarrow m^2\left[\left(m^2 + \frac{4}{m^2}\right) - \left(4m + \frac{8}{m}\right) + 8\right] = 0$

$\Rightarrow m^2\left[\left(m + \frac{2}{m}\right)^2 - 2 \cdot m \cdot \frac{2}{m} - 4\left(m + \frac{2}{m}\right) + 8\right] = 0$

$\Rightarrow m^2\left[\left(m + \frac{2}{m}\right)^2 - 4 - 4\left(m + \frac{2}{m}\right) + 8\right] = 0$

$\Rightarrow m^2\left[\left(m + \frac{2}{m}\right)^2 - 4\left(m + \frac{2}{m}\right) + 4\right] = 0$

$\Rightarrow m^2\left[\left(m + \frac{2}{m}\right)^2 - 2 \cdot 2 \cdot \left(m + \frac{2}{m}\right) + 2^2\right] = 0$

$\Rightarrow m^2\left[m + \frac{2}{m} - 2\right] = 0$, $\Rightarrow \left(m \cdot m + \frac{2 \cdot m}{m} - 2 \cdot m\right)^2 = 0$

$m^2 - 2m + 2 = 0$ $\quad y_c = (c_1 \cos x + c_2 \sin x)e^x$

$\Rightarrow m = 1 \pm i$

$y_p = \frac{1}{(D^2 - 1D + 2)} e^x = \frac{1}{1 - 2 + 2} e^x = e^x$

$y = (c_1 \cos x + c_2 \sin x)e^x + e^x$

c) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 5y = e^x \cos 3x$

$m^3 - m^2 + 3m + 5 = 0$

$\Rightarrow m^2(m+1) - 2m(m+1) + 5(m+1) = 0$

$\Rightarrow (m+1)(m^2 - 2m + 5) = 0$

$m = -1$, $\quad m = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$

$= \frac{2 \pm 4i}{2}$

$= 1 \pm 2i$

C.F $(y_c) = c_1 e^{-x} + (c_2 \cos 2x + c_3 \sin 2x)e^x$

P.S. $(y_p) = \frac{1}{(D+1)(D^2 - 2D + 5)} e^x \cos 3x$

$$= \frac{1}{D^3 - D^2 + 3D + 5}(e^x \cos 3x)$$

$$= e^x \frac{1}{(D+1)^3 - (D+1)^2 + 3D + 5}(\cos 3x)$$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + \cancel{1} - D^2 - 2D - \cancel{1} + 3D + 5}(\cos 3x)$$

$$= e^x \frac{1}{D^3 + 2D^2 + 4D + 5}(\cos 3x)$$

$$= e^x \frac{1}{D^2 \cdot D + 2D^2 + 4D + 5}(\cos 3x)$$

$$= e^x \frac{1}{-9D + 2(-9) + 4D + 5}(\cos 3x)$$

$$= e^x \frac{1}{-5D - 13}(\cos 3x)$$

$$= -e^x \frac{1}{5D + 13}(\cos 3x)$$

$$= -e^x \frac{5D - 13}{25D^2 - 169}(\cos 3x)$$

$$= -e^x \frac{(5D - 13)(\cos 3x)}{25(-9) - 169}$$

$$= \frac{e^x}{394}(-15 \sin 3x - 13 \cos 3x)$$

$$= -\frac{e^x}{394}(15 \sin 3x + 13 \cos 3x)$$

Incorrect

5) Solve the following homogeneous linear eqn

a) $x^2 \frac{d^2y}{dx^2} + y = 3x^2 \qquad x = e^z, \log x = z$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\therefore \frac{d^2y}{dz^2} - \frac{dy}{dz} + y = 3e^{2z}$$

$$\therefore m^2 - m + 1 = 0$$

$$\frac{1 \pm \sqrt{(1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y_c = e^{\frac{1}{2}z}\left(c_1 \cos \frac{\sqrt{3}}{2}z + c_2 \sin \frac{\sqrt{3}z}{2}\right)$$

$$y_p = \frac{1}{D^2 - D + 1}(3e^{2z})$$

$$= 3\frac{1}{(2)^2 - 2 + 1}e^{2z}$$

$$= \frac{3}{4 - 2 + 1}e^{2z}$$

$$= \frac{\cancel{3}}{\cancel{3}}e^{2z} = e^{2z}$$

$$\frac{1}{(D^3-D^2+3D+5)} e^x\cos 3x$$

$$= e^x \cdot \frac{1}{(D+1)^3-(D+1)^2+3(D+1)+5}\cos 3x$$

$$= e^x \cdot \frac{1}{D^3+3D^2+3D+\cancel{1}-D^2-2D-\cancel{1}+3D+3+5}\cos 3x$$

$$= e^x \cdot \frac{1}{D^3+2D^2+4D+8}\cos 3x$$

$$= e^x \cdot \frac{1}{D^2\cdot D+2D^2+4D+8}\cos 3x$$

$$= e^x \cdot \frac{1}{-9D+2(-9)+4D+8}\cos 3x$$

$$= e^x \cdot \frac{1}{-5D-10}\cos 3x$$

$$= \frac{e^x}{-5}\cdot\frac{1}{D+2}\cos 3x$$

$$= \frac{e^x}{-5}\cdot\frac{(D-2)\cos 3x}{D^2-4}$$

$$= \frac{e^x}{-5}\cdot\frac{(-3\sin 3x-2\cos 3x)}{-9-4}$$

$$= \frac{e^x}{-65}(3\sin 3x+2\cos 3x)$$

Soln —

$$c_1e^{-x}+c_2\cos 2x+c_3\sin 2x)e^x - \frac{e^x}{65}(3\sin 3x+2\cos 3x)$$

$y = y_c + y_p$

$= e^{\frac{z}{2}}\left(c_1 \cos\frac{\sqrt{3}z}{2} + c_2 \sin\frac{\sqrt{3}z}{2}\right) + e^{2z}$

$= e^{\frac{\log x}{2}}\left(c_1 \cos\frac{\sqrt{3}\log x}{2} + c_2 \sin\frac{\sqrt{3}\log x}{2}\right) + e^{2\log x}$

The sol^n is -

$y = e^{\frac{\log x}{2}}\left(c_1 \cos\frac{\sqrt{3}\log x}{2} + c_2 \sin\frac{\sqrt{3}\log x}{2}\right) + x^2$

b) $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$ $\quad x = e^z$, $\log x = z$

$\Rightarrow \frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 4y = 2e^{2z}$

$\Rightarrow \frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$

$m^2 - 4m + 4 = 0$

$\Rightarrow m^2 - (2+2)m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$

$y_c = (c_1 + c_2 z)e^{2z}$

$y_p = \frac{1}{(D-2)^2} 2e^{2z}$

$= 2e^{2z}\frac{1}{(D+2-2)^2}\cancel{e^{2z}}(1)$

$= 2e^{2z}\frac{z^2}{2}$

$= z^2 e^{2z}$

$\therefore$ the sol^n is $= (c_1 + c_2 z)e^{2z} + z^2 e^{2z}$

$= (c_1 + c_2 \log x)e^{2\log x} + x^2 e^{2\log x}$

$= (c_1 + c_2 \log x)x^2 + x^4$

c) done

d) $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - y = 3x^3\cos(\log x)$

$x = e^z$, $\Rightarrow \log x = z$

$\Rightarrow \frac{d^2y}{dz^2} - \frac{dy}{dz} + 2\frac{dy}{dz} - y = 3e^{3z}\cos z$

$m^2 + m - 1 = 0$

$m = \frac{-1 \pm \sqrt{(1)^2 + 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-1 \pm \sqrt{5}}{2}$

$y_c = c_1 e^{\frac{-1+\sqrt{5}}{2}z} + c_2 e^{\frac{-1-\sqrt{5}}{2}z}$

$y_p = \frac{1}{D^2 + D - 1} 3e^{3z}\cos z$

$$y_p = \frac{1}{D^2+D-1} 3e^{3z}\cos z$$

$$= 3e^{3z}\frac{1}{(D+3)^2+D+3-1}(\cos z)$$

$$= 3e^{3z}\frac{1}{D^2+6D+9+D+3-1}(\cos z)$$

$$= 3e^{3z}\frac{1}{D^2+7D+11}(\cos z)$$

$$= 3e^{3z}\frac{1}{-1+7D+11}(\cos z)$$

$$= 3e^{3z}\frac{1}{7D+10}(\cos z)$$

$$= 3e^{3z}\frac{(7D-10)}{49D^2-100}(\cos z)$$

$$= 3e^{3z}\frac{(7D-10)(\cos z)}{-49-100}$$

$$= -\frac{3e^{3z}}{149}(-7\sin z-10\cos z)$$

$$= \frac{3e^{3z}}{149}(7\sin z+10\cos z)$$

$$= \frac{3x^3}{149}\left[7\sin(\log x)+10\cos(\log x)\right]$$

$$y = c_1 e^{\frac{-1+\sqrt{5}}{2}\log x} + c_2 e^{\frac{-1-\sqrt{5}}{2}\log x} + \text{"}$$

$= 3e^{3z} \frac{1}{(D+3)^2 + D - 1} (\cos z)$

$= 3e^{3z} \frac{1}{D^2 + 6D + 9 + D - 1} (\cos z)$

$= 3e^{3z} \frac{1}{D^2 + 7D + 8} (\cos z)$

$= 3e^{3z} \frac{1}{-1 + 7D + 8} \cos z$

$= 3e^{3z} \frac{1}{7D + 7} \cos z$ → $\sin z \cos t$

$= \frac{3}{7} e^{3z} \frac{1}{D+1} \cos z$

$= \frac{3}{7} e^{3z} \frac{D-1}{D^2-1} \cos t$

$= \frac{3}{7} e^{3z} \frac{(D-1)(\cos z)}{-1-1}$

$= \frac{3}{-14} e^{3z} \cdot [-\sin z - \cos z]$

$= \frac{3e^{3z}}{14} (\sin z + \cos z)$

$y = c_1 e^{-\frac{1+\sqrt{5}}{2} z} + c_2 e^{-\frac{1-\sqrt{5}}{2} z} +$

$\frac{3e^{3z}}{14} (\sin z + \cos z)$

$= c_1 e^{-\frac{1+\sqrt{5}}{2} \log x} + c_2 e^{-\frac{1-\sqrt{5}}{2} \log x}$

$+ \frac{3x^3}{14} [\sin(\log x) + \cos(\log x)]$

e) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$

$x = e^z$
$\Rightarrow \log x = z$

$\Rightarrow \frac{d^2y}{dz^2} - \frac{dy}{dz} - \frac{dy}{dz} + y = \log x$

$\Rightarrow \frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = z$

$m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$y_c = (c_1 + c_2 z) e^{z}$

$y_p = \frac{1}{(D-1)^2} z$

$= \frac{1}{(1-D)^2} (z)$

$= (1-D)^{-2} (z)$

$= (1 + 2D + \frac{2(2+1)}{2!} D^2 + \cdots)(z)$

$= z + 2 = 2 + \log x$

$$= \cancel{\frac{1}{D^3 - D^2 + 3D + 5}}$$

Sol^n = $y = (c_1 + c_2 \log x)x + 2 + \log x$

$\cancel{4)}$ $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$

$$(x^2 + 2ax + a^2)\frac{d^2y}{dx^2} - \cancel{(4x + 4a)}\frac{dy}{dx} + 6y = x$$

$$\Rightarrow \cancel{x^2 \frac{d^2y}{dx^2} + 2a}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x+a}$$

Put, $x + a = e^z$

$\Rightarrow \log(x+a) = z$

$$\boxed{\begin{array}{l} (x+a)\frac{dy}{dx} = \frac{dy}{dz} \\ \text{similarly} \\ (x+a)^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \end{array}}$$

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 4\frac{dy}{dz} + 6y = e^z - a$$

$m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - (3+2)m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0$

$\Rightarrow m(m-3) - 2(m-3) = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2, 3$

$y_c = c_1 e^{2z} + c_2 e^{3z}$

$$y_p = \frac{1}{D^2 - 5D + 6}e^z - a$$

$$= \frac{1}{(D^2 - 5D + 6)}e^z - \frac{1}{D^2 - 5D + 6}(a)$$

$$= \frac{1}{1 - 5 + 6}e^z - \frac{1}{6}\left(\frac{D^2}{6} - \frac{5D}{6} + 1\right)^{-1}(a)$$

$$= \frac{1}{2}e^z - \frac{1}{6}\left(1 - \frac{5D}{6} + \frac{D^2}{6}\right)^{-1}(a)$$

$$= \frac{1}{2}e^z - \frac{1}{6}(a) = \frac{1}{2}(x+a) - \frac{1}{6}a$$

$\therefore$ the sol^n $y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}(x+a) - \frac{1}{6}a$

a) $\frac{d^2y}{dx^2} + y = \csc x$

$m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm\sqrt{-1}$

$= \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x \quad y_2 = \sin x$

Wronskian $W(y_1, y_2) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x$

$= 1 \quad [1 \neq 0]$

y_1 and y_2 are linearly independent.

We assume the particular sol^n –

$y_p = v_1 \cos x + v_2 \sin x$ (v_1, v_2 are func^n of x)

$v_1' \cos x + v_2' \sin x = 0$ — (i) $\times \sin x$

$-v_1' \sin x + v_2' \cos x = \csc x$ — (ii) $\cos x$

c) Solve the following eq^n by the method of variation of parameters –

$v_1' \sin x \cos x + v_2' \sin^2 x = 0$

$-v_1' \cos x \sin x + v_2' \cos^2 x = \csc x \cos x$

$\Rightarrow v_2' = \csc x \cdot \cos x$

$\Rightarrow v_2' = \cot x$

$\Rightarrow v_2 = \int \cot x \, dx$

$= \log|\sin x|$

$v_1' \cos x + v_2' \sin x = 0$ — (i) $\times \cos x$

$-v_1' \sin x + v_2' \cos x = \csc x$ — (ii) $\times \sin x$

$v_1' \cos^2 x + v_2' \sin x \cos x = 0$

$-v_1' \sin^2 x + v_2' \cos x \sin x = \csc x \cdot \sin x$

$\Rightarrow v_1' = -1$

$\Rightarrow v_1 = -\int dx$

$= -x$

$\therefore y_p = -x \cos x + \log(\sin x) \sin x$

$\therefore y = y_c + y_p = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)$

The sol^n is –

b), c) done

d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \cos x$

$m^2 - 2m = 0$

$\Rightarrow m(m-2) = 0$

$m = 0, 2$

$y_c = c_1 e^{0\cdot x} + c_2 e^{2x}$

$y_c = c_1 + c_2 e^{2x}$

$y_1 = 1$, $y_2 = e^{2x}$

wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix}$

$= 2e^{2x}$

y_1 and y_2 are linearly independent.

$y_p = v_1 + v_2 e^{2x}$

$v_1' + v_2' e^{2x} = 0$ — (1)

$2v_2' e^{2x} = e^x \cos x$ — (11)

$\Rightarrow v_2' = \frac{1}{2} e^{-x} \cos x$

$\Rightarrow v_2 = \frac{1}{2}\int e^{-x} \cos x\, dx$

$\int e^{-x}\cos x\,dx = \left[-\cos x\, e^{-x} - \int e^{-x}\sin x\,dx\right]$

$\Rightarrow \int e^{-x}\cos x\,dx = \left[-\cos x\, e^{-x} - \{e^{-x}\sin x + \int e^{-x}\cos x\,dx\}\right]$

$\Rightarrow \int e^{-x}\cos x\,dx = -\cos x\, e^{-x} + e^{-x}\sin x - \int e^{-x}\cos x\,dx$

$\Rightarrow 2\int e^{-x}\cos x\,dx = -[\cos x\, e^{-x} + e^{-x}\sin x]$

$\Rightarrow \int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(-\cos x + \sin x)$

$\therefore v_2 = \frac{1}{2}\left(-\frac{1}{2}\right)e^{-x}(-\cos x + \sin x)$

$= -\frac{1}{4}e^{-x}(-\cos x + \sin x)$

$= \frac{1}{4}e^{-x}(\sin x - \cos x)$

$v_1' + v_2' e^{2x} = 0$ — (1)

$2(v_2' e^{2x}) = e^x \cos x$ — (11)

from (1),

$v_2' e^{2x} = -v_1'$

$\Rightarrow 2(-v_1') = e^x \cos x$

$\Rightarrow v_1' = -\frac{1}{2}e^x \cos x$

$\Rightarrow v_1 = -\int \frac{1}{2} e^x \cos x\, dx$

$= -\frac{1}{2}\left(\frac{1}{2}\right)e^x(\cos x + \sin x)$

$= -\frac{1}{4}e^x(\cos x + \sin x)$

$y_p = -\frac{1}{4} e^{x}(\cos x + \sin x) + \frac{1}{4} e^{2x} (\cos x + 3\sin x)$

$= -\frac{1}{4} e^{x}(\cos x + \sin x) + \frac{1}{4} e^{x}(\cos x + \sin x)$

$+ -\frac{1}{2} e^{x} \cancel{(\cos x + \sin x)} = \frac{e^x}{4}(-\cos x - \sin x - \cos x + 3\sin x)$

$= -\frac{1}{2} e^{x} \cos x$

$\therefore y = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x} \cancel{(\cos x + \sin x)} \cos x$

e) $\frac{d^2y}{dx^2} + 4y = \sin 2x$

$m^2 + 4 = 0$

$\Rightarrow m = \pm\sqrt{-4}$

$\Rightarrow m = \pm 2i$

$y_c = (c_1 \cos 2x + c_2 \sin 2x)$

$y_1 = \cos 2x, \quad y_2 = \sin 2x$

wornskian $W(y_1, y_2) =$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$= 2(\cos^2 2x + \sin^2 2x)$ [$\neq 1$]

y_1 and y_2 are linearly independent

$y_p = v_1 \cos 2x + v_2 \sin 2x$

$v_1' \cos 2x + v_2' \sin 2x = 0$ — (i) $\times 2\sin 2x$

$-2v_1' \sin 2x + 2v_2' \cos 2x = \sin 2x$ — (ii) $\cos 2x$

$2v_1' \cos 2x \sin 2x + 2v_2' \sin^2 2x = 0$

$-2v_1' \cos 2x \sin 2x + 2v_2' \cos^2 2x = \sin 2x \cos 2x$

$\Rightarrow 2v_2' = \sin 2x \cos 2x$

$\Rightarrow v_2' = \frac{1}{2} \sin 2x \cos 2x$

$\Rightarrow v_2 = \frac{1}{2\times 2}\int 2\sin 2x \cos 2x \, dx$

$= \frac{1}{4}\int \sin 4x \, dx$

$= -\frac{1}{4} \frac{\cos 4x}{4}$

$= -\frac{\cos 4x}{16}$

$v_1' \cos 2x + v_2' \sin 2x = 0$ — (i) $\times 2\cos 2x$

$-2v_1' \sin 2x + 2v_2' \cos 2x = \sin 2x$ — (ii) $\times \sin 2x$

$2v_1' \cos^2 2x + 2v_2' \cos 2x \sin 2x = 0$

$-2v_1' \sin^2 2x + 2v_2' \cos 2x \sin 2x = \sin^2 2x$

(+) (−) (−)

$\Rightarrow 2v_1' = -\sin^2 2x$

$\Rightarrow v_1 = -\frac{1}{2\times 2}\int 2\sin^2 2x \, dx$

$= -\frac{1}{4}\int (1 - \cos 4x) \, dx$

$= -\frac{1}{4}\int (\cos 4x - 1)\,dx$

$= \frac{1}{4}\left[\frac{\sin 4x}{4} - x\right]$

$y_p = \frac{1}{4}\left(\frac{\sin 4x}{4} - x\right)\cos 2x - \frac{\cos 4x}{16}\sin 2x$

$y = \frac{1}{4}\left(\frac{\sin 4x}{4} - x\right)\cos 2x - \frac{\cos 4x}{16}\sin 2x$

$+ \; c_1 \cos 2x + c_2 \sin 2x$

$= c_1\cos 2x + c_2 \sin 2x - \frac{1}{16}(\cos 4x - \sin 4x)\ldots - \frac{1}{4}\ldots$

7) Solve the following eqn by the method of undetermined coefficients.

a), d), e) done.

b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = x^2$ — (1)

$m^2 - 4m = 0$

$\Rightarrow m(m-4) = 0$

$\Rightarrow m = 0, 4$

$y_c = c_1 e^{0x} + c_2 e^{4x}$

$= c_1 + c_2 e^{4x}$

$y_p = A_1 + A_2 x + A_3 x^2 + A_4 x^3$ — (1)

$\frac{dy_p}{dx} = A_2 + 2xA_3 + 3x^2 A_4$

$\frac{d^2 y_p}{dx^2} = 2A_3 + 6xA_4$

substituting (1) we get

~~$A_2 + 2xA_3 = 4$~~

$2A_3 - 4(A_2 + 2xA_3) = x^2$

$\Rightarrow 2A_3 - 4A_2 - 8xA_3 = x^2$

$\Rightarrow (2A_3 - 4A_2) - 8A_3 x = x^2$

$2A_3 - 4A_2 = 0 \qquad 8A_3 = 0$

$\Rightarrow -4A_2 = 0 \qquad \Rightarrow A_3 = 0$

$\Rightarrow A_2 = 0$

$y_p = A_1$

$2A_3 + 6xA_4 - 4(A_2 + 2xA_3 + 3x^2 A_4) = x^2$

$\Rightarrow (2A_3 - 4A_2) + (6xA_4 - 8xA_3) - 12x^2 A_4 = x^2$

$2A_3 - 4A_2 = 0 \qquad 6A_4 - 8A_3 = 0$

$\Rightarrow A_3 = 2A_2 \qquad \Rightarrow 3A_4 - 4A_3 = 0 \; ; \Rightarrow 3\times\frac{1}{12} - 4A_3 = 0$

$\Rightarrow \frac{1}{16} = 2A_2 \qquad \Rightarrow \frac{1}{4} = 4A_3$

$\Rightarrow A_2 = \frac{1}{32} \qquad 12A_4 = 1 \qquad \Rightarrow A_3 = \frac{1}{16}$

$\Rightarrow A_4 = \frac{1}{12}$

c) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2 + 4x + 8$ (1)

$m^3 + 3m^2 + 2m = 0$ (A)

$\Rightarrow m(m^2 + 3m + 2) = 0$

$\Rightarrow m\{m^2 + (2+1)m + 2\} = 0$

$\Rightarrow m\{m^2 + 2m + m + 2\} = 0$

$\Rightarrow m\{m(m+2) + 1(m+2)\} = 0$

$\Rightarrow m\{(m+2)(m+1)\} = 0$

$m = 0, -2, -1$

$y_c = c_1 e^{0\cdot x} + c_2 e^{-2x} + c_3 e^{-1\cdot x}$

$= c_2 e^{-2x} + c_3 e^{-x}$

$y_p = A_1 + A_2 x + A_3 x^2 + A_4 x^3$ — (11)

$\frac{dy_p}{dx} = \cancel{A_1} + A_2 + 2xA_3 + 3A_4 x^2$

$\frac{d^2y_p}{dx^2} = 2A_3 + 6A_4 x$, $\frac{d^3y_p}{dx^3} = 6A_4$

substituting in (1) we get,

$6A_4 + 3(2A_3 + 6A_4 x) + 2(A_2 + 2xA_3 + 3A_4 x^2)$
$= x^2 + 4x + 8$

$\Rightarrow 6A_4 + 6A_3 + 2A_2 + (18A_4 + 4A_3)x + 6A_4 x^2 = x^2 + 4x + 8$

$6A_4 = 1$
$\Rightarrow A_4 = \frac{1}{6}$

$18A_4 + 4A_3 = 4$
$\Rightarrow 18 \times \frac{1}{6} + 4A_3 = 4$
$\Rightarrow 4A_3 = 1$
$\Rightarrow A_3 = \frac{1}{4}$

$6A_4 + 6A_3 + 2A_2 = 8$

$\Rightarrow 6 \times \frac{1}{6} + 6 \times \frac{1}{4} + 2A_2 = 8$

$\Rightarrow 2A_2 = 8 - \frac{3}{2} = \frac{13}{2}$

$\Rightarrow A_2 = \frac{13}{4}$

$y_p = A_1 + A_2 x + A_3 x^2$ (11)

$\frac{dy_p}{dx} = A_2 + 2xA_3$

$\frac{d^2y_p}{dx^2} = 2A_3$

$\frac{d^3y_p}{dx^3} = 0$

from (1), $0 + 3\cdot 2\cdot A_3 + 2(A_2 + 2xA_3) = x^2 + 4x + 8$

$\Rightarrow (6A_3 + 2A_2) + 4xA_3 = x^2 + 4x + 8$

$4A_3 = 4$
$\Rightarrow A_3 = 1$

$6A_3 + 2A_2 = 8$
$\Rightarrow 6 + 2A_2 = 8$
$\Rightarrow 2A_2 = 2$, $\Rightarrow A_2 = 1$

$\therefore y_p = A_1 + x + \cancel{A_3} x^2$. [from (11)]

f) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 14\sin 2x - 18\cos 2x$ — (1)

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - (2+1)m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-1)(m-2) = 0$

$m = 1, 2$

$y_C = C_1 e^{x} + C_2 e^{2x}$

$y_P = x(A\cos 2x + B\sin 2x)$ — (ii)

$\frac{dy_P}{dx} = x(-2A\sin 2x + 2B\cos 2x) + (A\cos 2x + B\sin 2x)$

$\frac{d^2y_P}{dx^2} = x[-4A\cos 2x - 4B\sin 2x] + (-2A\sin 2x + 2B\cos 2x) + (-2A\sin 2x + 2B\cos 2x)$

substituting (1) we get,

$\Rightarrow -4xA\cos 2x - 4Bx\sin 2x - 2A\sin 2x + 2B\cos 2x - 2A\sin 2x + 2B\cos 2x$

$-3\{(-6xA\sin 2x + 6Bx\cos 2x) + (3A\cos 2x + 3B\sin 2x)\} + 2xA\cos 2x + 2xB\sin 2x = 14\sin 2x - 18\cos 2x$

$\Rightarrow -4xA\cos 2x - 4Bx\sin 2x - 2A\sin 2x +$
$2B\cos 2x - 2A\sin 2x + 2B\cos 2x$
$+ 6Ax\sin 2x - 6Bx\cos 2x$
$- 3A\cos 2x - 3B\sin 2x$
$+ 2xA\cos 2x$
$+ 2xB\sin 2x$
$= 14\sin 2x - 18\cos 2x$

$\Rightarrow -\frac{1}{2}A\cos 2x - 5B\sin 2x \cancel{+ 4A\sin}$
$- 4A\sin 2x + 4B\cos 2x + 6Ax\sin 2x$
$- 6Bx\cos 2x =$
$\cancel{18\cos}\ 14\sin 2x - 18\cos 2x$

$\Rightarrow (-5A + 4B)\cos 2x + (-5B - 4A)\sin 2x$
$\cancel{+(6A - 6B)x} + 6Ax\sin 2x -$
$6Bx\cos 2x = 14\sin 2x - 18\cos 2x$

$6A\sin 2x = 0$
$\Rightarrow A = 0$

$-5A + 4B = -18$
$\Rightarrow 5A - 4B = 18$ — (i)
$-4A - 5B = 14$
$\Rightarrow 4A + 5B = -14$ — (ii)
$5A - 4B = 18$ — (i) $\times 5$
$4A + 5B = -14$ — (ii) $\times 4$

$25A - 20B = 90$
$16A + 20B = -56$ $\Rightarrow 41A = 34$ $\Rightarrow A = \frac{34}{41}$

$5A - 4B = 18$ — (i) $\times 5$
$4A + 5B = -14$ — (ii) $\times 5$

$20A - 16B = 72$
$20A + 25B = -70$

$\Rightarrow -41B = 142$
$\Rightarrow B = \frac{142}{41}$

8) Find eigen values and the corresponding eigen functions of the following boundary value Problems:
a) done (done)
b) $\frac{d^2y}{dx^2} + \lambda y = 0$; $(\lambda > 0)$, $y(0) = 0$, $y(\frac{\pi}{2}) = 0$

$\frac{d^2y}{dx^2} + \lambda y = 0$ — (i)

$y(0) = 0$, $y(\frac{\pi}{2}) = 0$ — (ii)

$m^2 + \lambda = 0$

$\Rightarrow m = \pm\sqrt{\lambda}\, i$

$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$ — (iii)

$y(0) = 0$; $\Rightarrow c_1 = 0$

$\therefore y = c_2 \sin\sqrt{\lambda}\, x$

$y(\pi/2) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\,\frac{\pi}{2} = 0$

$c_2 \neq 0$, $\sin\sqrt{\lambda}\,\frac{\pi}{2} = 0 = \sin n\pi$ $(n = 0, 1, 2 \ldots)$

$\Rightarrow \frac{\pi}{2}\sqrt{\lambda} = n\pi$

$\Rightarrow \frac{\pi^2}{4}\lambda = n^2\pi^2$

$\Rightarrow \lambda = 4n^2$ $(n = 0, 1, 2 \ldots)$

which are eigen values

$$-4Ax\cos 2x - 4Bx\sin 2x - 4A\sin 2x$$

$$+4B\cos 2x - \cancel{4A\sin x}$$

$$+6Ax\sin 2x - 6Bx\cos 2x$$

$$-3A\cos 2x - 3B\sin 2x$$

$$+2Ax\cos 2x + 2Bx\sin 2x = 0$$

$$\Rightarrow -2Ax\cos 2x + 2Bx\sin 2x + 6Ax\sin 2x$$

$$-6Bx\cos 2x - 4A\sin 2x + 4B\cos 2x$$

$$-3A\cos 2x - 3B\sin 2x$$

$$\Rightarrow (-2A-6B)x\cos 2x + (6A-2B)x\sin 2x$$

$$+(-4A-3B)\sin 2x + (4B-3A)\cos 2x = 0$$

$$-4A - 3B = 14 \quad -\text{①} \times 4$$

$$-3A + 4B = -18 \quad -\text{②} \quad -3$$

$$16A + 12B = -56$$

$$-9A + 12B = -54$$

$$\text{①} \times 3 - \text{②}$$

$$\Rightarrow 25A = -2$$

$$\Rightarrow A = -\frac{2}{25}$$

$$\Rightarrow 4A + 3B = -14 \quad -\text{①} \times 3$$

$$-3A + 4B = -18 \quad -\text{②} \times 4$$

$$12A + 9B = -42$$

$$-12A + 16B = -72$$

$$\Rightarrow 25B = -114$$

$$\Rightarrow B = -\frac{114}{25}$$

which are eigen values

eigen function—

$y_n = c_n \sin\sqrt{\lambda}\, x$ $(n=1,2,3\ldots)$

d) $\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda>0)$, $y(0)=0$, $\frac{dy}{dx}=0$, at $x=\pi$

$\frac{d^2y}{dx^2} + \lambda y = 0$ — (i)

$y(0)=0$, $\frac{dy}{dx}=0$, at $x=\pi$ — (ii)

$m^2 + \lambda = 0$

$\Rightarrow m = \pm\sqrt{\lambda}\, i$

$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$y(0)=0$, $\Rightarrow c_1 = 0$

$\therefore y = c_2 \sin\sqrt{\lambda}x$

$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = c_2 \cos\sqrt{\lambda}x \cdot \sqrt{\lambda} = 0$

$\Rightarrow c_2\sqrt{\lambda}\cos\sqrt{\lambda}\cdot\pi = 0$

$\sqrt{\lambda}c_2 \neq 0$

$\therefore \cos\sqrt{\lambda}\pi = 0 = \cos\left\{\frac{(2n+1)\pi}{2}\right\}$

$\sqrt{\lambda}\pi = \frac{(2n+1)\pi}{2}$ $n=(0,\pm1,\pm2\ldots)$

$\Rightarrow \lambda^2 = \frac{(2n+1)^2}{4}$ ~~n = (±~~

This are eigen values

$\therefore y_n = \sum c_n \sin\frac{(2n+1)}{2}x$

This is eigen function.

e) $\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda>0)$, $y(0)=0$, $y'(l)=0$

$\frac{d^2y}{dx^2} + \lambda y = 0$ — (i)

$y(0)=0$, $y'(l)=0$ — (ii)

$m^2 + \lambda = 0$

$\Rightarrow m = \pm\sqrt{\lambda}\, i$

$y = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$y(0)=0 \Rightarrow$ ~~$c_1\cos$~~ $c_1 = 0$

$\therefore y = c_2\sin\sqrt{\lambda}x$

$\Rightarrow \frac{dy}{dx} = \sqrt{\lambda}c_2\cos\sqrt{\lambda}x$

$\Rightarrow \sqrt{\lambda}\,c_2\cos\sqrt{\lambda}\,l = 0$

$c_2\sqrt{\lambda} \neq 0$ $\therefore$ ~~c_2~~ $\cos\sqrt{\lambda}\,l = 0 = \cos\left\{\frac{(2n+1)\pi}{2}\right\}$

$\sqrt{\lambda}\,l = \frac{(2n+1)\pi}{2}$

$\Rightarrow \lambda = \frac{(2n+1)^2\pi^2}{4l^2}$ $[n=0,\pm1,\pm2\ldots]$

This are eigen values

$y_n = C_n \sin \frac{(2n+1)\pi x}{2L}$ → This is eigen function.

f) $\frac{d^2y}{dx^2} + \lambda y = 0$ $(\lambda > 0)$; $y(0)=0$, $y(a)=0$

$\frac{d^2y}{dx^2} + \lambda y = 0$ — (i)

$y(0)=0$, $y(a)=0$ — (ii)

$m^2 + \lambda = 0$

$\Rightarrow m = \pm\sqrt{\lambda}\, i$

$\therefore y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$y(0)=0 \Rightarrow c_1 = 0$

$y = c_2 \sin\sqrt{\lambda}\, x$

$\Rightarrow c_2 \sin\sqrt{\lambda}\, a = 0$

$c_2 \neq 0$, $\sin\sqrt{\lambda}\, a = 0 = \sin n\pi$

$\Rightarrow \sqrt{\lambda}\, a = n\pi$ $(n = 0, 1, 2, \dots)$

$\Rightarrow \lambda a^2 = n^2\pi^2$

$\Rightarrow \lambda = \frac{n^2\pi^2}{a^2}$ $(n = 0, 1, 2, \dots)$

This the eigen values

$y_n = C_n \sin \frac{n\pi x}{a}$ → this is eigen function.

11/5/22

math (mam)

- $V = R^3$
  $= \{(a_1, a_2, a_3)\}$
  $\downarrow$
  $\in R$
  $(2, -3, 1)$
- $\alpha = (a_1, a_2, a_3)$
  $\beta = (b_1, b_2, b_3)$

$\alpha + \beta = (a_1+b_1, a_2+b_2, a_3+b_3)$

$\alpha = (5, -6, 3)$, $\beta = (0, 4, 5)$

$\alpha + \beta = (5, -2, 8) \in R$

- $\alpha + 0 = 0 + \alpha$
- $R = R^3$
  $a(a_1, a_2, a_3) = (aa_1, aa_2, aa_3) \in R^3$
- subspace — $R^3 = (a_1, a_2, a_3) \mid a_i \in R$

$W = \{(a_1, a_2, 0) \mid a_1 \in R\}$

if (non empty) → then (subspace)

- Thm (i) $\alpha, \beta \in W$, $\Rightarrow \alpha + \beta \in W$
  (ii) $c \in F$, $\alpha \in V \Rightarrow c\alpha \in W$

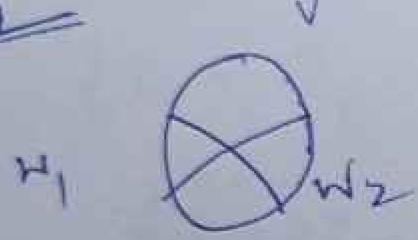
Proof $\alpha + \beta \in W$

$c\alpha \in W$

- $1 \in F$, $-1 \in F$
  $(-1)\beta = -\beta$
  $\alpha - \beta \in V$

Statements

Sn
3q



V = vector space field F