ENGINEERING MATHS

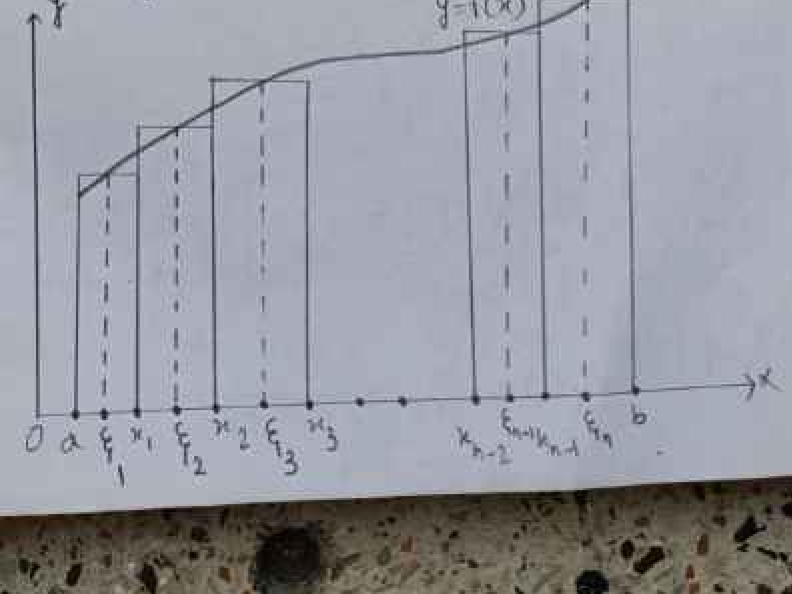
- Definite integration
- Defination of definite integral (R Sir)

Let $y = f(x)$ be a funtion bounded on the interval $[a, b]$. we subdivide the intervals into n number of subintervals by means of the points $a = x_0, x_1, x_2 \cdots x_{n-1}, x_n = b$, chosen arbitrarily (ऐच्छिक). In each of the subintervals $(a, x_1), (x_1, x_2) \cdots (x_{n-1}, b)$ we choose points, $\xi_1, \xi_2 \cdots \xi_n$ arbitrarily. The we form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(b - x_{n-1})$$

$$= \sum_{r=1}^{n} f(\xi_n)(x_n - x_{n-1})$$

Geometrically this sum represents the total area of all rectangles in the given figure.



(1) To evaluate the integral $\int_2^3 \sqrt{x}\,dx$ we divide the interval $(2,3)$ into n numbers of subintervals by the points $2, 2t, 2t^2, \ldots, 2t^{n-1}, 2t^n$

$t = \left(\frac{3}{2}\right)^{\frac{1}{n}}$, let $f(x) = \sqrt{x}$ $\quad 2 \quad 2t \quad 2t^2 \quad 2t^3 \quad 3 = 2t^n$

we take the points

$\xi_1 = 2t,\ \xi_2 = 2t^2 \ \ldots\ \xi_n = 2t^n$

Then $\int_2^3 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \sqrt{\xi_r}\,(2t^r - 2t^{r-1})$ [$\because$ here $x_r = 2t^r$]

$= \lim_{n\to\infty} \sum_{r=1}^{n} \sqrt{2t^r}\,(t-1)2t^{r-1}$

$= \lim_{n\to\infty} 2\sqrt{2}\,(t-1) \sum_{r=1}^{n} t^{r-1+\frac{r}{2}}$

$= 2\sqrt{2} \lim_{n\to\infty} (t-1) \sum_{r=1}^{n} t^{\frac{3r}{2}-1}$

$= 2\sqrt{2} \lim_{n\to\infty} \frac{(t-1)}{t} \left\{ t^{\frac{3}{2}} + \left(t^{\frac{3}{2}}\right)^2 + \left(t^{\frac{3}{2}}\right)^3 + \cdots \left(t^{\frac{3}{2}}\right)^n \right\}$

$= 2\sqrt{2} \lim_{n\to\infty} \frac{t-1}{t} \cdot \frac{t^{\frac{3}{2}}\left\{\left(t^{\frac{3}{2}}\right)^n - 1\right\}}{t^{\frac{3}{2}} - 1}$

$= 2\sqrt{2} \lim_{n\to\infty} t^{\frac{1}{2}}(t-1) \cdot \frac{(t^n)^{\frac{3}{2}} - 1}{t^{\frac{3}{2}} - 1}$

$= 2\sqrt{2} \lim_{t\to 1} \frac{t^{\frac{1}{2}}(t-1)\left\{\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right\}}{t^{\frac{3}{2}} - 1}$ $\left[\because t = \left(\frac{3}{2}\right)^{\frac{1}{n}},\ \therefore t \to 1 \text{ as } n \to \infty\right]$

$= 2\sqrt{2} \left\{\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right\} \lim_{t\to 1} \frac{t^{\frac{1}{2}}}{\frac{t^{\frac{3}{2}} - 1}{t-1}}$

$= 2\sqrt{2} \left\{\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right\} \frac{1}{\frac{3}{2} \cdot 1^{\frac{1}{2}}}$

$= 2\sqrt{2} \cdot \frac{2}{3}\left(\frac{\sqrt{27}}{\sqrt{8}} - 1\right)$

$= \frac{2}{3}\left(\sqrt{27} - \sqrt{8}\right)$

(ii) $\int_2^3 \sqrt{x}\,dx$

$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_2^3$

$= \frac{2}{3}\left\{(3)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right\}$

$= \frac{2}{3}\left(\sqrt{27} - \sqrt{8}\right)$ — ans

(2) $\int_2^7 \log x \cdot x^2\,dx$

$= \frac{1}{3}\left[\log x \cdot x^3\right]_2^7 - \frac{1}{3}\int_2^7 \left(\frac{1}{x} \cdot x^3\right) dx$

$= \frac{1}{3}\left[\log x \cdot x^3\right]_2^7 - \frac{1}{3}\int_2^7 x^2\,dx$

$= \frac{1}{3}\left[\log x \cdot x^3\right]_2^7 - \frac{1}{3} \cdot \frac{1}{3} \cdot \left[x^3\right]_2^7$

$= \frac{1}{3}\left\{(\log 7 \cdot 7^3) - (\log 2 \cdot 2^3)\right\} - \frac{1}{9} \cdot \left\{(7)^3 - (2)^3\right\}$

$= \frac{343}{3}\log 7 - \frac{8}{3}\log 2 - \frac{1}{3}\left(\frac{7^3}{3} - \frac{2^3}{3}\right)$ — ans

(3) $\int_1^2 x^3 dx = \frac{1}{4}[x^4]_1^2 = \frac{1}{4}[2^4-1]$

$= \frac{1}{4} \times 15$

$= \frac{15}{4}$

(4) $\int_2^5 \frac{1}{x} dx$, find from the 1st Principle

As in the previous example divide the interval $[2,5]$ by means of the points $2, 2t, 2t^2, \ldots, 2t^{n-1}, 2t^n$ where $2t^n = 5$

Then $\int_2^5 \frac{1}{x} dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2t^r}(2t^r - 2t^{r-1})$

$= \lim_{n\to\infty} (t-1) \sum_{r=1}^{n} \frac{t^{r-1}}{t^r}$

$= \lim_{n\to\infty} \frac{(t-1)}{t} \sum_{r=1}^{n} 1$

$= \lim_{n\to\infty} \frac{t-1}{t} \cdot n \left[\because t^n = \frac{5}{2} \Rightarrow n = \log\frac{5}{2}\right]$

$= \lim_{t\to 1} \frac{t-1}{t} \cdot \log\frac{5}{2}$ | $\Rightarrow t = \left(\frac{5}{2}\right)^{\frac{1}{n}}$, $n\to\infty$, $t\to 1 \left[\because \left(\frac{5}{2}\right)^{\frac{1}{\infty}} = \left(\frac{5}{2}\right)^0 = 1\right]$

$= \lim_{t\to 1} \frac{(t-1)\log\frac{5}{2}}{t\log t}$

$= \log\frac{5}{2} \lim_{t\to 1} \frac{1}{t} \cdot \frac{t-1}{\log t}$

$= \log\frac{5}{2} \times 1 \times \lim_{t\to 1} \frac{1}{\frac{1}{t}}$ [by L'Hospital Law]

$= \log\frac{5}{2} \times 1 = \log\frac{5}{2} = \log 5 - \log 2$

(or) $\int_2^5 \frac{1}{x} dx$ [not using 1st Principle]

$= [\log x]_2^5$

$= \log 5 - \log 2$

(5) $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}$

Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}$

$= \int_0^{\frac{\pi}{2}} \frac{(1+\tan^2\frac{x}{2})\,dx}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$

$= 2\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$

$= 2\int_0^1 \frac{dz}{2z + 1 - z^2}$

$= 2\int_0^1 \frac{dz}{(\sqrt{2})^2 - (z-1)^2}$

$= 2 \cdot \frac{1}{2\sqrt{2}} \left[\log\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right|\right]_0^1$

$= \frac{1}{\sqrt{2}}\left[-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$

$= \frac{1}{\sqrt{2}} \log\frac{\sqrt{2}+1}{\sqrt{2}-1}$

$= 2 \times \frac{1}{\sqrt{2}} \cdot \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{\frac{1}{2}}$

$= \sqrt{2}\log\frac{\sqrt{2}+1}{2-1} = \sqrt{2}\log(\sqrt{2}+1)$

Put $\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

x	0	$\frac{\pi}{2}$
z	0	1

$-\{z^2 - 2z - 1\}$

$= -\{(z^2 - 2z + 1) - 2\}$

$= -\{(z-1)^2 - (\sqrt{2})^2\}$

$= (\sqrt{2})^2 - (z-1)^2$

(6) $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx$

$= \int_0^1 \tan^{-1}\frac{1}{1-x(1-x)}\,dx$

$= \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}\,dx$

$\int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)]\,dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$

$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx \left[\because \int_0^a f(x) = \int_0^a f(a-x)\right]$

$= 2\int_0^1 \tan^{-1}x\,dx$

$= 2\left[\tan^{-1}\int_0^1 1\,dx + \int_0^1 \left(x \frac{1}{1+x^2}\right)dx\right]$

$= 2\left[\{x\}_0^1 \tan^{-1}\frac{1}{1} + \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx\right]$

$= 2\left[\frac{\pi}{4} + \frac{1}{2}\{\log(1+x^2)\}_0^1\right]$

$= 2\left[\frac{\pi}{4} + \frac{1}{2}\log 2\right]$

$= \frac{\pi}{2} + \log 2$

(7) $\int_2^4 f(x)\,dx$, where $f(x) = |x-2| + |x-3|$

$\int_2^4 |x-2| + |x-3|\,dx$

$= \int_2^4 |x-2|\,dx + \int_2^4 |x-3|\,dx$

$= \int_2^4 (x-2)\,dx + \int_2^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx$

$= \int_2^4 (x-2)\,dx - \int_2^3 (x-3)\,dx + \int_3^4 (x-3)\,dx$

$f(x) = (x-2),\ x > 2$

$f(x) = (x-3),\ x > 3$

$f(x) = -(x-3)\ x < 3$

$= \left[\frac{x^2}{2} - 2x\right]_2^4 - \left[\frac{x^2}{2} - 3x\right]_2^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$

$= (8-8) - (2-4) - \left(\frac{9}{2} - 9\right) + (2-6) + \left(\frac{9}{2} - 9\right) + (8-12)$

$= 2 - \frac{9}{2} + 9 - 4 - \frac{9}{2} + 9 - 4$

$= 12 - \frac{18}{2}$

$= 3$

(8) $\int_0^\pi |\sin x + \cos x|\,dx$

$f(x) = \sin x + \cos x$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$

$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

$f(x) = (\sin x + \cos x)\ 0 < x < \frac{3\pi}{4}$

$f(x) = -(\sin x + \cos x),\ \frac{3\pi}{4} < x < \pi$

$\therefore \int_0^{\frac{3\pi}{4}} (\sin x + \cos x)\, dx - \int_{\frac{3\pi}{4}}^{\pi} (\sin x + \cos x)\, dx$

$= [-\cos x + \sin x]_0^{\frac{3\pi}{4}} - [-\cos x + \sin x]_{\frac{3\pi}{4}}^{\pi}$

$= (\cos 45 + \sin 45) - (-1) - (1+0) + (\cos 45 + \sin 45)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \cancel{1} - \cancel{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2}$

(7) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1-x}{1+x}\right) dx$

$f(x) = \cos x \log\left(\frac{1-x}{1+x}\right)$

$f(-x) = \cos x \log\left(\frac{1+x}{1-x}\right)$

$= -\cos x \log\left(\frac{1-x}{1+x}\right) = -f(x)$

Let

$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1-x}{1+x}\right)$

$\therefore I = 0 \quad [\because f(x) = -f(x)]$

(8) $\int_{-2}^{2} |1-x^2|\, dx$

$f(x) = |1-x^2|$

$f(-x) = |1-x^2| = f(x)$

$\therefore \int_{-2}^{2} |1-x^2|\, dx$

$= 2\int_0^2 |1-x^2|\, dx$

$1-x^2 > 0 \quad [\because 0 < x < 2]$

$\Rightarrow -x^2 < 1$

$\Rightarrow |x| < 1$

$\therefore -1 < x < 1$

$= 2\left[\int_0^1 (1-x^2)\, dx + \int_1^2 -(1-x^2)\, dx\right]$

$= 2\left[\int_0^1 (1-x^2)\, dx - \int_1^2 (1-x^2)\, dx\right]$

$= 2\left[\left(x - \frac{x^3}{3}\right)_0^1 - \left(x - \frac{x^3}{3}\right)_1^2\right]$

$= 2\left[\left(1-\frac{1}{3}\right) - \left(2 - \frac{8}{3}\right) + \left(1 - \frac{1}{3}\right)\right]$

$= 2\left[2 \times \frac{2}{3} - 2 + \frac{8}{3}\right]$

$= 2\left[\frac{\cancel{12}^{4}}{\cancel{3}} - 2\right]$

$= 2 \times 2$

$= 4$

(11) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ —(i)

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\cos(\frac{\pi}{2}-x)}+\sqrt{\sin(\frac{\pi}{2}-x)}}\,dx$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ —(ii)

By adding (i) + (ii)

$2I = \int_0^{\frac{\pi}{2}} \frac{(\sqrt{\sin x}+\sqrt{\cos x})}{(\sqrt{\sin x}+\sqrt{\cos x})}\,dx$

$= \int_0^{\frac{\pi}{2}} dx$

$= [x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$I = \frac{\pi}{4}$

(12) $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x+\cos x}$

Let, $I = \int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sin x+\cos x}$ —(i)

$I = \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2}-x)}{\sin x+\cos x}$ —(ii)

(i) + (ii)

$2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+\cos x}$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{(1+\tan^2\frac{x}{2})}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}\,dx$

$= \frac{\pi}{2}\times 2\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}\,dx$ [Put, $\tan\frac{x}{2}=z$ $\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$]

$= \pi\int_0^1 \frac{dz}{2z+1-z^2}$

$= \pi\int_0^1 \frac{dz}{(\sqrt{2})^2-(z-1)^2}$

$= \pi\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right|\right]_0^1$

$= \frac{\pi}{\sqrt{2}}\times\frac{1}{2}\left[-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$

$= \frac{\pi}{\sqrt{2}}\times\frac{1}{2}\left[\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$

$= \frac{\pi}{\sqrt{2}}\log\sqrt{\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}}$

$= \frac{\pi}{\sqrt{2}}\log(\sqrt{2}+1)$

$\therefore I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

(13) $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

Let,

$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$

$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

$I = \log 2\,[\theta]_0^{\frac{\pi}{4}} - I$

$\Rightarrow 2I = \log 2 \cdot \frac{\pi}{4}$

$\Rightarrow I = \frac{\pi}{8}\log 2$

(14) $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{3+\sin 2x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)}{3+2\sin x\cos x}\,dx$

$= -\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{-3-2\sin x\cos x}$

$= -\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{(1-2\sin x\cos x)-4}$

$= -\int_0^{\frac{\pi}{4}} \frac{(\sin x+\cos x)\,dx}{(\sin x-\cos x)^2-(2)^2}$

Put, $\sin x - \cos x = z$

$= -\int_{-1}^{0} \frac{dz}{z^2-2^2}$

$= \int_0^{-1} \frac{dz}{z^2-2^2}$

$= \frac{1}{2\cdot 2}\left[\log\left|\frac{z-2}{z+2}\right|\right]_0^{-1}$

$= \frac{1}{4}\log 3$

(15) $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+5^x}\,dx \quad \text{--- (i)}$

$I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+5^{-x}}\,dx$

$I = \int_{-\pi}^{\pi} \frac{5^x\cos^2 x}{1+5^x}\,dx \quad \text{--- (ii)}$

(i) + (ii)

$2I = \int_{-\pi}^{\pi} \frac{\cancel{(1+5^x)}\cos^2 x}{\cancel{(1+5^x)}}\,dx$

$\Rightarrow I = \frac{1}{2\cdot 2}\int_{-\pi}^{\pi} 2\cos^2 x\,dx$

$= \frac{1}{4}\int_{-\pi}^{\pi} (1+\cos 2x)\,dx$

$= \frac{1}{4}\left[\pi(1+1) - \pi\right]$

$$= \frac{1}{4}\int_{0}^{\pi} (1+\cos 2x)\,dx$$

$$= \frac{1}{4}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4}[\pi + \pi]$$

$$= \frac{\pi}{2}$$

(15)

$$\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+5^x}\,dx$$

$$= \int_{0}^{\pi}\left[\frac{\cos^2 x}{1+5^x} + \frac{\cos^2(-x)}{1+5^{-x}}\right]dx$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = \int_{0}^{a}[f(x)+f(-x)]\,dx\right]$$

$$= \int_{0}^{\pi} \frac{\cancel{(1+5^x)}\cos^2 x}{\cancel{(1+5^x)}}\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi} 2\cos^2 x\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi}(1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{0}^{\pi}$$

$$= \frac{\pi}{2}$$

(16)

$$\int_{2}^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$$

Let $I = \int_{2}^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$ — (i)

$I = \int_{2}^{3} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}$ — (ii)

(i) + (ii)

$$2I = \int_{2}^{3} \frac{(\sqrt{x}+\sqrt{5-x})}{(\sqrt{x}+\sqrt{5-x})}\,dx$$

$$= \int_{2}^{3} dx$$

$$= [x]_{2}^{3}$$

$$= 3-2$$

$$= 1$$

$$\Rightarrow I = \frac{1}{2}$$

(17)

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x+\cos x}\,dx$$

Let $I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x+\cos x}\,dx$ — (i)

$I = \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x}\,dx$ — (ii) $\left[\because \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx\right]$

(i) + (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1+\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1+\sin x + \cos x) - 1}{1+\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$$

$$= \int_0^{\frac{\pi}{2}} dx - 2\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx - 2\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\,dx$$

Put,
$\tan\frac{x}{2} = z$

$$= \int_0^{\frac{\pi}{2}} dx - 2\int_0^{1} \frac{dz}{2+2z}$$

$$= [x]_0^{\frac{\pi}{2}} - \int_0^1 \frac{dz}{z+1}$$

$$= [x]_0^{\frac{\pi}{2}} - [\log|z+1|]_0^1$$

$$= \frac{\pi}{2} \times \log\left(\frac{\pi}{2}+1\right)$$

$$= \frac{\pi}{2} - \log 2$$

$$\therefore I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

(18) $$\lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

Let,

$$A = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$\log A = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n}{n}\right)\right]$$

$$\therefore \lim_{n\to\infty} \log A = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n}{n}\right)\right]$$

$$\therefore \log\left(\lim_{n\to\infty} A\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1+\frac{r}{n}\right)$$

$\left[\because n\to\infty,\ \text{Let } \frac{1}{n} = h,\ \therefore n\to\infty,\ h\to 0\right]$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} \log(1+rh)$$

$$= \int_0^1 \log(1+x)\,dx$$

put,
$1+x = z$
$\therefore dx = dz$

$$= \int_1^2 \log z\,dz$$

$$= \left[\log z \cdot z\right]_1^2 - \int_1^2 \left(\frac{1}{z}\cdot z\right) dz$$

$$= [z\log z]_1^2 - [z]_1^2$$

$$= 2\log 2 - 1$$

$$= \log 4 - \log e$$

$$= \log\frac{4}{e}$$

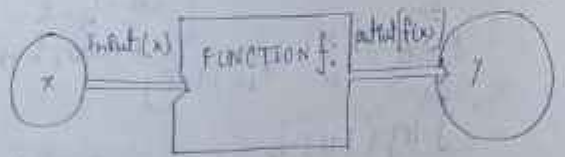
$$\therefore \lim_{n\to\infty} A = \frac{4}{e}$$

Functions (5 min)

• Definition

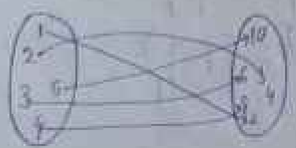
A function from a set X to a set Y is an assignment (rule) of an element of Y to each element of X. The set X is called the domain of the the function and the set Y is called the codomain of the function.

A function is most of time denoted by letters such as f, g and h and the value of a function f at an element x of its domain is denoted by $f(x)$



Ex Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 4, 6, 8, 10\}$ and a function $f: X \to Y$ defined by $f(x) = 2x$ $\forall\, x \in X$. Then,

$f(1) = 2$, $f(2) = 4$, $f(3) = 6$, $f(4) = 8$ and $f(5) = 10$



• Classification of functions (5 min)

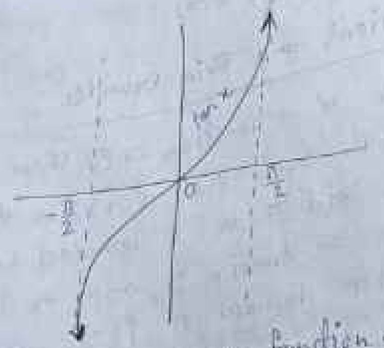
1. Bounded functions and their bounds

Let $f: D \to R$ be a function. If there exists a $m \in R$ such that $f(x) \geq m$ $\forall\, x \in D$ then $f(x)$ is said to be a "bounded below" function in the domain D and m is called a lower bounded of $f(x)$.

Ex Let $D = (0, \frac{\pi}{2})$ and $f: D \to R$ be a function defined by $f(x) = \tan x$. Then $f(x) \geq 0$ $\forall\, x \in (0, \frac{\pi}{2})$. So $f(x)$ is bounded below and 0 is a lower bound of $f(x)$.

Let $f: D \to R$ be a function. If there exists a $M \in R$ such that $f(x) \leq M$ $\forall\, x \in D$ then $f(x)$ is said to be a "bounded above" function in the domain D and M is called an upper bound of $f(x)$.

• Ex Let $D = (-\frac{\pi}{2}, 0)$ and $f: (-\frac{\pi}{2}, 0) \to R$ be a function defined by $f(x) = \tan x$. Then $f(x) \leq 0$ for all $x \in (-\frac{\pi}{2}, 0)$. So $f(x)$ is bounded above and 0 is an upper bound of $f(x)$.



Let $f: D \to R$ be a function. If there exists a $K \in R$ such that $|f(x)| < K \ \forall x \in D$ then $f(x)$ is called bounded function in the domain D and K is called a bound of $f(x)$.

•Ex Let $D = R$ and $f: R \to R$ defined by $f(x) = \sin x$ then $-1 \le \sin x \le 1$. i.e $|f(x)| \le 1$. So $f(x)$ is a bounded function and 1 is a bound of $f(x)$.

2. Monotone function

Let $f: D \to R$ be a function and x_1, x_2 be any two point in D.

If $x_1 < x_2$ implise $f(x_1) \le f(x_2)$ then $f(x)$ is said to be monotonically increasing function.

If $x_1 < x_2$ implise $f(x_1) < f(x_2)$ then $f(x)$ is said to be strictly monotonically increasing function.

If $x_1 < x_2$ implise $f(x_1) \ge f(x_2)$ then $f(x)$ is said to be monotonically decreasing function.

If $x_1 < x_2$ implise $f(x_1) > f(x_2)$ then $f(x)$ is said to be a strictly monotonically decreasing function.

•Ex (i) Let $f: (0, \infty) \to R$ be a function defined by $f(x) = x^2$. Then $f(x)$ is a monotonically increasing function. Because $x_1 < x_2 \Rightarrow x_1^2 < x_2^2 \Rightarrow f(x_1) < f(x_2)$

(ii) Let $f: (0, \infty) \to R$ be a function defined by $f(x) = -x$. Then $f(x)$ is a monotonically decreasing function. Because, $x_1 < x_2 \Rightarrow -x_1 > -x_2 \Rightarrow f(x_1) > f(x_2)$

3. Even and odd functions

Let $f: D \to R$ be a function where D is such that $x \in D \Rightarrow -x \in D$.

$f(x)$ is said to be an even function if $f(-x) = f(x)$ for all $x \in D$.

$f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all $x \in D$.

Ex (i) Let $f: R \to R$ be a function defined by $f(x) = \cos x$. Then $f(x)$ is an even function. Because $f(-x) = \cos(-x) = \cos x = f(x)$

(ii) Let $f: R \to R$ be a function defined by $f(x) = \sin x$. Then $f(x)$ is an odd function. Because $f(-x) = \sin(-x) = -\sin x = -f(x)$

4. Periodic Function

Let $f: D \to R$ be a function. The function $f(x)$ is said to be periodic function of period μ if μ be the least positive real numbers such that

$f(x+\mu) = f(x)$ for all $x \in D$ [Here $x+\mu \in D$ $\forall x \in D$]

Ex Let $f: R \to R$ be a function defined by $f(x) = \sin x$. Then $f(x)$ is a periodic function with period 2π.

Because $f(x+2\pi) = \sin(x+2\pi) = \sin x = f(x) \;\forall x \in R$

Worked out Examples:

1) Find out the domain of defination of the following functions.

(i) $f(x) = \sqrt{x-1} + \sqrt{5-x}$

(ii) $f(x) = \sqrt{(3x-1)(7-x)}$

(iii) $f(x) = \sqrt{8+2x-3x^2}$

(iv) $f(x) = \log(x^2-5x+6)$

(v) $f(x) = \sqrt{\log\frac{4x-x^2}{3}}$

(vi) $f(x) = \log\sqrt{\frac{5x-x^2}{4}}$

(vii) $f(x) = \frac{1}{\sqrt{|x|-x}}$

Ans (i) $f(x) = \sqrt{x-1} + \sqrt{5-x}$

Since $f(x)$ is real, the values of x must be such that both $\sqrt{x-1}$ and $\sqrt{5-x}$ are real quantities, [[illegible] real [illegible]] which requires that $(x-1) \geq 0$ and $(5-x) \geq 0$

i.e. $x \geq 1$ and $x \leq 5$

So, domain of defination of $f(x)$ is $1 \leq x \leq 5$ or $[1, 5]$ ([illegible])

(ii) $f(x) = \sqrt{(3x-1)(7-x)}$

Since $f(x)$ is real, the values of x must be such that $\sqrt{(3x-1)(7-x)}$ is a real quantity which requires, $\sqrt{(3x-1)(7-x)} \geq 0$

$\Rightarrow (3x-1)(7-x) \geq 0$

Either $(3x-1) \geq 0$ and $(7-x) \geq 0$ or

$(3x-1) \leq 0$ and $(7-x) \leq 0$

in 1st case $\Rightarrow$ $3x-1 \geq 0 \Rightarrow x \geq \frac{1}{3}$ and $7-x \geq 0 \Rightarrow x \leq 7$

in 2nd case $\Rightarrow$ $3x-1 \leq 0 \Rightarrow x \leq \frac{1}{3}$ and $7-x \leq 0 \Rightarrow x \geq 7$

[This inequality holds for all real values of x, except those lie between $\frac{1}{3}$ and 7 including $x = \frac{1}{3}$ and $x = 7$] ✗

So domain of defination of $f(x)$ is all real values of x, ~~except~~ $\frac{1}{3} \le x < 7$ or $(-\infty, \frac{1}{3}] \cup [7, \infty)$

(iii) $f(x) = \sqrt{8 + 2x - 3x^2} = \sqrt{3}\sqrt{\left(\frac{5}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2}$

Since $f(x)$ is real, the values of x must be such that $\sqrt{8 + 2x - 3x^2}$ is real quantity which requires that,

$$\sqrt{8 + 2x - 3x^2} \geq 0$$

$\Rightarrow \sqrt{3}\sqrt{\left(\frac{5}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2} \geq 0$

$\Rightarrow \left(\frac{5}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2 \geq 0$ $\longrightarrow$ $\boxed{\text{This is incorrect}}$

$\Rightarrow x - \frac{1}{3} \le \frac{5}{3}$

$\Rightarrow x \le \frac{5}{3} + \frac{1}{3}$

$\Rightarrow x \le 2$

So domain of defination of $f(x)$ is $(-\infty, 2]$

$= -\{3x^2 - 2x - 8\}$
$= -3\{x^2 - \frac{2}{3}x - \frac{8}{3}\}$
$= -3\{x^2 - 2 \cdot x \cdot \frac{1}{3} + \left(\frac{1}{3}\right)^2 - \frac{1}{9} - \frac{8}{3}\}$
$= -3\{(x - \frac{1}{3})^2 - \left(\frac{25}{9}\right)\}$
$= 3\{\left(\frac{5}{3}\right)^2 - (x - \frac{1}{3})^2\}$

$8 + 2x - 3x^2$ = by middle term $= (2 - x)(3x + 4)$

$\therefore (2 - x) \geq 0, (3x + 4) \geq 0$ | $(2 - x) \le 0, (3x + 4) \le 0$

$\Rightarrow x \le 2$, $\Rightarrow x \geq -\frac{4}{3}$ | $\Rightarrow x \geq 2$, $\Rightarrow x \le -\frac{4}{3}$

$\therefore -\frac{4}{3} \le x \le 2$ or $(-\infty, -\frac{4}{3}] \cup [2, \infty)$

(iv) $f(x) = \log(x^2 - 5x + 6)$

$f(x)$ is defined for all real values of x that make $x^2 - 5x + 6 > 0$ (contd)

$\Rightarrow (x - 2)(x - 3) > 0$

$\Rightarrow$ either $(x - 2) > 0$ and $(x - 3) > 0$ or $(x - 2) < 0$ and $(x - 3) < 0$

$\Rightarrow x > 2$ and $x > 3$ or $x < 2$ or $x < 3$

This inequality holds for all real values of x, except those that lie between 2 and 3 including $x = 2$ and $x = 3$.

So, domain of defination of $f(x)$ is all real values of x, except $2 \le x \le 3$ or $(-\infty, 2) \cup (3, \infty)$.

(v) $f(x) = \sqrt{\log \frac{4x - x^2}{3}}$

$\because f(x)$ is real, the values of x must be such that ~~both~~ $\sqrt{\log \frac{4x - x^2}{3}}$ is real quantity, which requires

$\therefore \log \frac{4x - x^2}{3} \geq 0$ $\boxed{\text{This is incorrect}}$

$\Rightarrow \frac{4x - x^2}{3} \geq e^0$

$\Rightarrow 4x - x^2 \geq 1 \times 3$

~~$\Rightarrow x(4 - x) \geq 3$~~

$\Rightarrow 4x - x^2 - 3 \geq 0$

$\Rightarrow (1)^2 - (x - 2)^2 \geq 0$

$\Rightarrow (x - 2) \le 1$

$\Rightarrow x \le 3$

$- \{x^2 - 4x + 3\}$
$= -\{x^2 - 2 \cdot x \cdot 2 + (2)^2 + 3 - 4\}$
$= -\{(x - 2)^2 - (1)^2\}$
$= (1)^2 - (x - 2)^2$

(iv) $f(x) = \log(x^2 - 5x + 6)$

$f(x)$ is defined for all real values of x (contd's) and make $x^2 - 5x + 6 > 0$

$\Rightarrow (x-2)(x-3) > 0$

$\Rightarrow$ either $(x-2) > 0$ and $(x-3) > 0$ or $(x-2) < 0$ and $(x-3) < 0$

$\log\left(\frac{4x - x^2}{3}\right) \geq 0$

$\Rightarrow \frac{4x - x^2}{3} \geq 1$

$\Rightarrow 4x - x^2 - 3 \geq 0$

$(x-3)(1-x) \geq 0$

$(x-3) \geq 0, \quad 1-x \geq 0$

$\Rightarrow x \geq 3, \quad \Rightarrow x \leq 1$

or $(x-3) \leq 0 \quad (1-x) \leq 0$

$\Rightarrow x \leq 3 \quad x \geq 1$

$1 \leq x \leq 3$ or $(-\infty, 1] \cup [3, \infty)$

$-\{x^2 - 4x + 3\}$

$= -\{x^2 - (3+1)x + 3\}$

$= -\{x^2 - 3x - x + 3\}$

$= -\{x(x-3) - 1(x-3)\}$

$= (x-3)(1-x)$

$\Rightarrow (1)^2 - (x-3)$

$\Rightarrow (x-2) \leq 1$

$\Rightarrow x \leq 3$

$\boxed{\text{This is in connect}}$

$= -\{3x^2 - 2x - 8\}$

$= -3\{x^2 - \frac{2}{3}x - \frac{8}{3}\}$

$= 3\{(\frac{5}{3})^2 - (x - \frac{1}{3})^2\}$

..., the values such that ... is real quantity

domain of defination of $f(x)$ is $(-\infty, 2]$

$(2-x)(3x+4)$ by middle term rule

$(2-x) \leq 0, \ (3x+4) \geq 0$

$\Rightarrow x \geq 2, \ \Rightarrow x \leq -\frac{4}{3}$

$-\frac{4}{3} \leq x \leq 2$ or $(-\infty, -\frac{4}{3}] \cup [2, \infty)$

So domain of defination of $f(x)$ is $(-\infty, 3)$

(vi) $f(x) = \log\sqrt{\frac{5x-x^2}{4}}$

$f(x)$ is defined for all real values of x that make $\sqrt{\frac{5x-x^2}{4}} > 0$

$\Rightarrow 5x - x^2 > 0$

$\Rightarrow (5-x)\cdot x > 0$

$\Rightarrow$ either $5-x>0$ and $x>0$ or $5-x<0$ and $x<0$

or $x>5$ and $x<0$

$\Rightarrow x<5$ and $x>0$

So the domain of defination of $f(x)$ is all real values of x, except those lie between 0 and 5 including $x=0$ and $x=5$.

$\therefore$ domain of defination of $f(x)$ is $0 \le x \le 5$

or $(-\infty,0)\cup(5,\infty)$

(vii) $f(x) = \frac{1}{\sqrt{|x|-x}}$

Since $f(x)$ is real values so $f(x)$ is defined when $|x|-x>0$ i.e $|x|>x$

and this inequality is satisfied for all values of $x<0$.

So domain of defination of $f(x)$ is $(-\infty, 0)$

or $-\infty < x < 0$.

2. Show that $f(x) = \log(x+\sqrt{1+x^2})$ is an odd function of x.

Ans

$f(x)+f(-x) = \log(x+\sqrt{1+x^2}) + \log(\sqrt{1+x^2}-x)$

$= \log[(\sqrt{1+x^2})^2 - (x)^2]$

$= \log(1+x^2-x^2)$

$= 0$

$\therefore f(x)+f(-x) = 0$

$\Rightarrow f(-x) = -f(x)$

$\therefore f(x)$ is an odd function.

3. Prove that any function can be expressed as the sum of an even and an odd function of x.

we can write $f(x) = \frac{1}{2}\{f(x)+f(-x)\} + \frac{1}{2}\{f(x)-f(-x)\}$

say $\varphi(x) = \frac{1}{2}\{f(x)+f(-x)\}$

and $\psi(x) = \frac{1}{2}\{f(x)-f(-x)\}$

So $\varphi(x)$ is an even function and $\psi(x)$ is an odd function.

4. Find out the period of the following functions :-

(i) $f(x) = \sin ax$ (iii) $f(x) = 2\cos\frac{1}{3}(x-\pi)$

(ii) $f(x) = |\cos x|$ (iv) $f(x) = \sin^4 x + \cos^4 x$

Solutions

(i) $f(x) = \sin ax$

Since $\sin\{a(x + \frac{2\pi}{a})\} = \sin(ax + 2\pi)$
$= \sin ax$

$\therefore$ $f(x) = \sin ax$ is a periodic function of period $\frac{2\pi}{a}$

(ii) $f(x) = |\cos x|$

Since, $|\cos\{x \pm 2n\pi\}|$ [where, $n = 1, 2, 3 \ldots$]
$= |\cos\{x \pm 2n\pi\}|$
$= |\cos x|$

Therefore $f(x) = |\cos x|$ is a Periodic function of period $\pm 2n\pi$.

(iii) $f(x) = 2\cos\frac{1}{3}(x-\pi)$

the Period of $2\cos\frac{1}{3}(x-\pi)$ is $\frac{2\pi}{1/3}$ or 6π

(iv) $f(x) = \sin^4 x + \cos^4 x$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$$
$$= 1 - \frac{1}{2}\cdot 4\sin^2 x\cos^2 x$$
$$= 1 - \frac{1}{2}\cdot\sin^2 2x$$
$$= 1 - \frac{1}{2\times 2}\times 2\sin^2 2x$$
$$= 1 - \frac{1}{4}(1 - \cos 4x)$$
$$= 1 - \frac{1}{4} + \frac{1}{4}\cos 4x = \frac{3}{4} + \frac{1}{4}\cos 4x$$

Since $\cos\{4(x+\frac{2\pi}{4})\} = \cos(4x+2\pi) = \cos 4x$

$\therefore \cos 4x$ is a periodic functions with period $\frac{2\pi}{4}$.

Hence $f(x) = \sin^4 x + \cos^4 x$ is a periodic function with period $\frac{2\pi}{4}$.

5. Prove that $f: [0,\infty) \to R$ defined by $f(x) = \frac{3x+5}{2x+1}$ is strictly monotonically decreasing function.

Proof

Let x_1, x_2 be any two points in $[0,\infty)$ such that $x_1 < x_2$

We want to show that $f(x_1) > f(x_2)$

that is $\frac{3x_1+5}{2x_1+1} > \frac{3x_2+5}{2x_1+1}$

Suppose,

$$\frac{3x_1+5}{2x_1+1} \leq \frac{3x_2+5}{2x_2+1}$$

$\Rightarrow (3x_1+5)(2x_2+1) \leq (3x_2+5)(2x_1+1)$

$\Rightarrow 6x_1x_2 + 10x_2 + 3x_1 + 5 \leq 6x_1x_2 + 10x_1 + 3x_2 + 5$

$\Rightarrow 10x_2 + 3x_1 \leq 10x_1 + 3x_2$

$\Rightarrow 3x_1 - 10x_1 \leq 3x_2 - 10x_2$

$\Rightarrow -7x_1 \leq -7x_2$

$\Rightarrow x_1 \geq x_2$

This shows that $f(x_1) \leq f(x_2)$ implise that $x_1 \geq x_2$. Therefore contrapositively we get

$x_1 < x_2$ implise $f(x_1) > f(x_2)$

$\therefore f(x)$ is strictly monotonically decreasing functions.

6. Show that $f(x) = \frac{x}{x+1}$ is monotonically increasing function for $x > 0$ [$[0,\infty)$]

Let x_1 and x_2 be any two points in $(0,\infty)$ such that $x_1 < x_2$

We want to show that $f(x_1) \leq f(x_2)$

that is $\frac{x_1}{x_1+1} \leq \frac{x_2}{x_2+1}$

Suppose,

$$\frac{x_1}{x_1+1} \geq \frac{x_2}{x_2+1}$$

$\Rightarrow x_1(x_2+1) \geq x_2(x_1+1)$

$\Rightarrow x_1x_2 + x_1 \geq x_1x_2 + x_2$

$\Rightarrow x_1 \geq x_2$

$\therefore \frac{x_1}{x_1+1} \leq \frac{x_2}{x_2+1}$

$\Rightarrow x_1x_2 + x_1 \leq x_1x_2 + x_2$

$\Rightarrow x_1 \leq x_2$

This shows that $f(x_1) \leq f(x_2)$ implies that $x_1 \leq x_2$.

$\therefore x_1 \leq x_2$ implise that $f(x_1) \leq f(x_2)$

Hence $f(x)$ is monotonically ~~decreasing~~ increasing function.

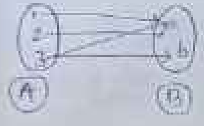
(A Sir)

① well defined collections of distinced object = defination of set in mathametics.

② function is special type of relation having two properties –

i) Domain (R) = A

ii) for every / each element in A there exist an unique element in B.



→ It is a relation but not function, কারণ 1 টি image এর (a) 3টি Pre-image থাকতে পারে না B তে।

- [Domain, Codomain, defination] → imp
- If, range = codomain is called Surjective function or onto or Surjection
- $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

i.e $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ } one-one or injective function or injection

③ Squere root of any positive no. is always positive.

Ex $\sqrt{\sqrt{81}} = \sqrt{9} = 3$

if, $\sqrt{\sqrt{81}} = \sqrt{-9}$ = there is no real value

① $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}$

$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4\left(\frac{\sin x}{x}\right)^2}$

$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}$

$\sin x$

$=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\frac{x^9}{\lfloor 9}-\frac{x^{11}}{\lfloor 11}$ ---

$=\lim_{x\to 0}\dfrac{\left(-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\cdots\right)\left(2x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}\right)}{x^4}$

$=\lim_{x\to 0}\dfrac{\left(-\frac{1}{\lfloor 3}+\frac{x^2}{\lfloor 5}-\cdots\right)\left(\frac{2x}{x}-\frac{x^3}{\lfloor 3}+\cdots\right)}{x}$ $[\because x^4 = x \cdot x^3]$

$=\lim_{x\to 0}\left(-\frac{1}{\lfloor 3}+\frac{x^2}{\lfloor 5}-\cdots\right)\left(2-\frac{x^2}{\lfloor 3}+\cdots\right)$

$=\lim_{x\to 0}\left\{-\frac{1}{\lfloor 3}+x(\cdots)\right\}\left\{2-x(\cdots)\right\}$

$=-\frac{1}{3}$

② $\lim_{x\to -\infty}\dfrac{x^2\tan\frac{1}{x}}{\sqrt{8x^2-3x+1}}$

$\lim_{x\to -\infty}\dfrac{x^2\tan\frac{1}{x}}{\sqrt{8x^2-3x+1}}$

$=\lim_{x\to -\infty}\dfrac{x^2\tan\frac{1}{x}}{(-x)\sqrt{8-\frac{3}{x}+\frac{1}{x^2}}}$ $[\because x\to -\infty]$

$=-\lim_{x\to -\infty}\dfrac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\cdot\dfrac{1}{\lim_{x\to -\infty}\sqrt{8-\frac{3}{x}+\frac{1}{x^2}}}$

$=-1\cdot\frac{1}{\sqrt{8}}$

$=-\frac{1}{2\sqrt{2}}$

③ If $\lim_{x\to 0}\frac{f(x)}{x^2}=2$, then find $\lim_{x\to 0}[f(x)]$ and $\lim_{x\to 0}\left[\frac{f(x)}{x}\right]$

Since $x^2>0$ and limit equal 2, $f(x)$ must be a positive quantity. Also, since $\lim_{x\to 0}\frac{f(x)}{x^2}=2$ denominator → zero and limit is finite. Therefore $f(x)$ must be approaching zero or $\lim_{x\to 0}[f(x)]=0^+$

Hence $\lim_{h \to 0} [f(h)] = 0$

$\lim_{h \to 0^+} \left[\frac{f(h)}{h}\right] = \lim_{h \to 0^+} \left[x \cdot \frac{f(h)}{h^2}\right]$

$= 0$

and $\lim_{h \to 0^-} \left[\frac{f(h)}{h}\right] = \lim_{h \to 0^-} \left[x \cdot \frac{f(h)}{h^2}\right]$

$= -1$

Hence $\lim_{h \to 0} \left[\frac{f(h)}{h}\right]$ does not exist.

Q) $f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} \cos(n! \, x\pi)^{2m} \right\}$

then prove that $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ $x \leftarrow R$

$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} \cos(n! \, x\pi)^{2m} \right\}$

Case-1

since $x \in \mathbb{Q}$ so in 1st case, $\mathbb{Q}$ means $\sqrt{2}, \sqrt{3}$ etc

$0 < n! \, x\pi < 1$

$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} \cos(n! \, x\pi)^{2m} \right\}$

$\therefore \lim_{m \to \infty}$

$= 0$

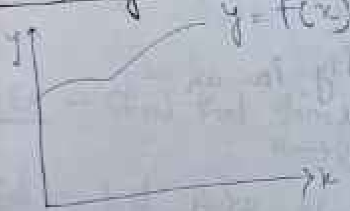
Case 2)

since $x \notin \mathbb{Q}$ so in the 2nd case $n! \, x\pi$ will cancle out. Because n tensed to infinity and let $x = \frac{56}{30}$ so there wise must exist a value for which multiplay of $n!$ and $x\pi$ will be cancelled out.

$\therefore \lim_{m \to \infty} \left\{ \lim_{n \to \infty} 1 \right\}$ [$\because$ cos as infinity multiplication]

$= 1$

• continuity



$f : R^k \longrightarrow R^m$

$G(f) \subset R^{n+m}$ [Graph is subset of R^{n+m}]

ex ① $f : R^2 \longrightarrow R$ $f(x, y) = x + y$

② $f : R^2 \longrightarrow R^3$ $f(x, y) = \left(2^{x}\cos, e^{x+y}, \frac{\sin x}{y}\right)$

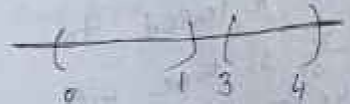
(the function goes to R^3 so it has 3 component) $G(f) \subseteq R^{2+3}$, R^5

$[R^2 = R \times R = \ldots]$

• Interval, $(a,b) = \{x ; a < x < b\}$

$(0,1)$ = interval

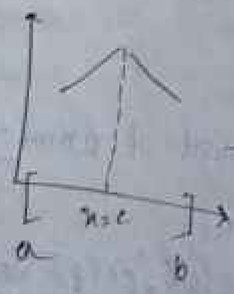
$[0,1] = \{x ; 0 \le x \le 1\}$


$\rightarrow (0,1) \cup (3,4)$

This is union of interval but not an interval

$A = \{1,2,3\}$
$N = \{1,2,3 \ldots\}$
$Z = \{0, \pm1, \pm2 \ldots\}$
→ this are not intervals

• 
→ continuity in an interval

• 
→ this is not a continuity in interval

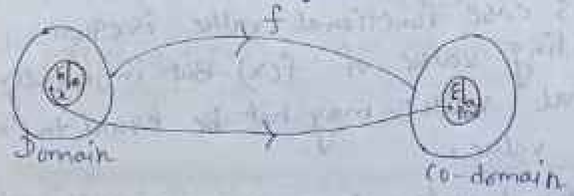
$x=c$ তে discontinuity থাকবে তাই
পুরোটা তে continuity নাই

• Limit of a function (ε-δ)

Defination : ($\varepsilon - \delta$ defination)

Let $f(x)$ be a function, we said the limit of $f(x)$ at the point a is l if for every $\varepsilon > 0$ there exist a $\delta > 0$ (depending on ε) such that,

$$|f(x) - l| < \varepsilon \text{ whenever } |x - a| < \delta$$



It is denoted by $\boxed{\lim_{x \to a} f(x) = l}$

Ex - Show that $\lim_{x \to 2} (5x) = 10$ [using the defination]

Sol - Let $\varepsilon > 0$ be any real no.

Now $|5x - 10| < \varepsilon$ [here $f(x) = 5x$, $l = 10$]

$\Rightarrow 5|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{5}$

$\Rightarrow |x-2| < \delta$ [take $\delta = \frac{\varepsilon}{5}$]

Thus $|5x - 10| < \varepsilon$ whenever $|x-2| < \delta$

$\therefore \lim_{x \to 2} (5x) = 10$

Say $f(x) = 5x$

Then $\lim_{x \to 2} f(x) = \lim_{x \to 2} (5x) = 10$

and $f(2) = 10$ → functional value of $f(x)$ at $x = 2$

Thus $\lim_{x \to 2} f(x) = f(2) = 10$

limiting value of $f(x)$ at $x = 2$

In this case functional value is equal to limiting value of $f(x)$. But in generally functional value may not be equal to limiting value.

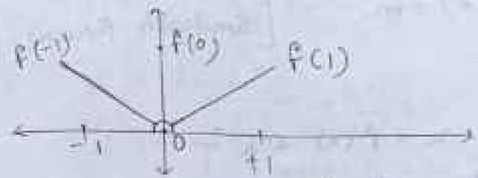
If functional value is equal to limiting value of a function $f(x)$ at $x = a$. Then $f(x)$ is called continious at $x = a$

so here $f(x) = 5x$ is continious at $x = 2$

Ex:

Let $f : [-1, 1] \to \mathbb{R}$

defined by
$$f(x) = -x \text{ if } x \in [-1, 0)$$
$$= x \text{ if } x \in (0, 1]$$
$$= 1 \text{ if } x = 0$$



we can show that $\lim_{x \to 0} f(x) = 0$

But $f(0) = 1$

So $\lim_{x \to 0} f(x) \neq f(0)$

$\therefore f(x)$ is not continious at $x = 0$

- Properties of limit

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = l'$

Then, (i) $\lim_{x \to a} \{f(x) \pm g(x)\} = l \pm l'$

(ii) $\lim_{x \to a} \{f(x) \cdot g(x)\} = l l'$

(iii) $\lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{l'}$ [provided $g(x) \neq 0$ / $l' \neq 0$]

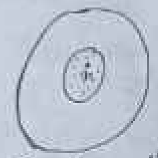
(iv) $\lim_{x \to a} F\{f(x)\} = F\{\lim_{x \to a} f(x)\} = F(l)$

where the $F(u)$ is a continious function.

(v) If $\phi(x) < f(x) < \psi(x)$ in a certain neighbourhood (nbd.) of the point 'a'.

If $\lim_{x\to a} \varphi(x) = m$ and $\lim_{x\to a} \psi(x) = m$

Then $\lim_{x\to a} f(x) = m$ [Sandwich theorem]



Domain

$x < f(x) < x^2$

$\lim_{x\to 0} x = 0$ $\lim_{x\to 0} x^2 = 0$

then $\lim_{x\to 0} f(x) = 0$

Q.1 Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (Sandwich theorem)

2. Show that $\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right) = -\frac{1}{2}$

3. $\lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = 0$

4. $\lim_{x\to \frac{\pi}{4}} \left(\sec 2x - \tan 2x\right) = 0$

5. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

6. $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}} = 1$

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$

$= 1 - 0 \cdots = 1$

$\left[\frac{\sin x}{x}\right] = 0$

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{D_x \sin x}{D_x x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ [L' Hospital Rule]

2) $\lim_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$

$= \lim_{x\to\infty} \frac{\cancel{x^2} - \cancel{x^2} - x}{x + \sqrt{x^2+x}}$

$= \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2\left(1+\frac{1}{x}\right)}}$

$= \lim_{x\to\infty} \frac{-x}{x + x\sqrt{1+\frac{1}{x}}}$

$= \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x}}}$

$= -\frac{1}{1+1} = -\frac{1}{2}$

show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (by sandwich theorem)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right)$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

3) $\lim_{x\to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x}$

$= \lim_{x\to 0} \dfrac{x \cdot \sin\frac{1}{x}}{\frac{\sin x}{x}}$

$= \dfrac{\lim_{x\to 0} \dfrac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)}}{\lim_{x\to 0} \dfrac{\sin x}{x}}$

$= \dfrac{\lim_{x\to 0} \dfrac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)}}{1}$

$= \lim_{x\to 0} \dfrac{\sin\frac{1}{x}}{\frac{1}{x}}$

$= 0$ [∵ $x\to 0$, $\frac{1}{x}\to\infty$]

(or) $= \lim_{z\to\infty} \dfrac{\sin z}{z}$ Put $\frac{1}{x} = z$, $x\to 0$, $z\to\infty$

$= 0$ [∵ $\lim_{x\to\infty} f(x) = \lim_{z\to 0} f\left(\frac{1}{z}\right)$]

4) $\lim_{x\to\frac{\pi}{4}} (\sec 2x - \tan 2x)$

$= \lim_{x\to\frac{\pi}{4}} \left(\dfrac{1-\sin 2x}{\cos 2x}\right)$

$= \lim_{x\to\frac{\pi}{4}} \dfrac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}$

$= \lim_{x\to\frac{\pi}{4}} \dfrac{(\cos x - \sin x)^{\cancel{2}}}{(\cos x + \sin x)(\cos x - \sin x)}$

$= \lim_{x\to\frac{\pi}{4}} \dfrac{1-\tan x}{1+\tan x}$

$= \lim_{x\to\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - x\right)$

0

5) $\lim_{x\to 0} x\sin\frac{1}{x}$

$= \lim_{x\to 0} \dfrac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$ [∵ $\lim_{z\to 0} f\left(\frac{1}{z}\right) = 0$]

$= 0$ [∵ $x\to 0$, $\frac{1}{x}\to\infty$]

6) $\lim_{x\to\infty} \dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$ Put $x = z^2$, $x\to\infty$, $z\to\infty$

$= \lim_{z\to\infty} \dfrac{\sqrt{z^2}}{\sqrt{z^2+\sqrt{z^2+\sqrt{z^2}}}}$

$$= \lim_{z \to \infty} \frac{z}{\sqrt{z^2 + \sqrt{z^2 + z}}}$$

$$= \lim_{z \to \infty} \frac{z}{\sqrt{z^2 + \sqrt{z^2 \left(1 + \frac{1}{z}\right)}}}$$

$$= \lim_{z \to \infty} \frac{z}{\sqrt{z^2 + z\sqrt{\left(1 + \frac{1}{z}\right)}}}$$

$$= \lim_{z \to \infty} \frac{z}{\sqrt{z^2 \left\{1 + \frac{1}{z}\sqrt{\left(1 + \frac{1}{z}\right)}\right\}}}$$

$$= \lim_{z \to \infty} \frac{\cancel{z}}{\cancel{z}\sqrt{1 + \frac{1}{z}\sqrt{\left(1 + \frac{1}{z}\right)}}}$$

$$= 1$$

• IMPROPER INTEGRALS (R.S.)

Consider the integral

$$\int_a^b f(x)\,dx \quad \text{---(1)}$$

This integral is called an improper integral or an infinite integral when either a or b or both are infinite or $f(x)$ is unbounded in $a \le x \le b$.

It can be proved that if a function $f(x)$ is unbounded in $[a,b]$, then there exists at least one point $c \in [a,b]$ s.t in the nbd of c, $f(x)$ is not bounded. This point c is called point of infinite discontinuity of the function $f(x)$ or the point of singularity of the integral (1).

Ex - (i) The integrals $\int_0^{\infty} \frac{dx}{x^2}$, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ are improper integrals.

(ii) The integrals $\int_0^1 \frac{dx}{x}$ is improper as $\frac{1}{x} \to \infty$, at $x \to 0$

(iii) The integral $\int_0^2 \frac{dx}{\sqrt{2-x}}$ is improper as $\frac{1}{\sqrt{2-x}} \to \infty$, at $x \to 2$

(iv) The integral $\int_{-3}^{3} \frac{dx}{x(1+x)}$ is improper as $\frac{1}{x(1+x)} \to \infty$ at $x \to 0, -1$ and $0, -1$ lie between -3 and 3.

There are two types of improper integrals. The example (i) ~~are~~ the examples of first type improper integral. So the general form of first type improper integral is $\int_{-\infty}^{\infty} f(x)\,dx$.

The examples (ii), (iii) and (iv) are examples of second type improper integral. Thus the integral $\int_a^b f(x)$ is called second type improper integral when $f(x)$ is not bounded at $x = c$, $a \le c \le b$.

[1st type - end point গুলোতে infinite হবে।
2nd type - any point of interval এ infinite হবে।]

• Evaluation of improper integrals of first type

• Type I: Let the function $f(x)$ be bounded and integrable in $a \le x \le X$ for all $X > a$. Then the improper integral $\int_a^b f(x)\,dx$ is defined as $\lim_{X \to \infty} \int_a^X f(x)\,dx$, Provided the limit exist.

Hence, $\boxed{\int_a^{\infty} f(x)\,dx = \lim_{X \to \infty} \int_a^X f(x)\,dx}$

Ex Consider the improper integral $\int_0^\infty \frac{dx}{1+x^2}$

Then, $\int_0^\infty \frac{dx}{1+x^2} = \lim_{x\to\infty}\int_0^x \frac{dx}{1+x^2}$

$$= \lim_{x\to\infty}\left[\tan^{-1}x\right]_0^x$$

$$= \lim_{x\to\infty}\left[\tan^{-1}x\right]$$

$$= \frac{\pi}{2}$$

• **TYPE II** Let $f(x)$ be bounded and integrable in $x \le x < b$ for every $x < b$. and $\lim_{x\to-\infty}\int_x^b f(x)dx$ exists infinitely. Then the improper integral $\int_{-\infty}^b f(x)dx$ is defined as

$$\boxed{\int_{-\infty}^b f(x)\,dx = \lim_{x\to-\infty}\int_x^b f(x)\,dx}$$

Ex Consider the improper integral $\int_{-\infty}^0 e^x dx$.

Then,

$$\int_{-\infty}^0 e^x dx = \lim_{x\to-\infty}\int_x^0 e^x dx$$

$$= \lim_{x\to-\infty}\left[e^x\right]_x^0$$

$$= \lim_{x\to-\infty}\left[1 - e^x\right]$$

$$= 1 \quad \left[\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0\right]$$

TYPE III

Consider the improper integral $\int_{-\infty}^\infty f(x)dx$. Let c be any number. Then we can write

$$\boxed{\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx = \lim_{x_1\to-\infty}\int_{x_1}^c f(x)dx + \lim_{x_2\to\infty}\int_c^{x_2} f(x)dx}$$

Provided limits exist finitely.

Ex Consider the improper integral $\int_{-\infty}^\infty \frac{x}{x^4+1}dx$

⇒ Let c be any number, then

$$\int_{-\infty}^\infty \frac{x}{x^4+1}dx = \int_{-\infty}^c \frac{x}{x^4+1}dx + \int_c^\infty \frac{x\,dx}{x^4+1}$$

$$= \lim_{x_1\to-\infty}\int_{x_1}^c \frac{x\,dx}{x^4+1} + \lim_{x_2\to\infty}\int_c^{x_2} \frac{x\,dx}{x^4+1}$$

$$= \frac{1}{2}\lim_{x_1\to-\infty}\int_{x_1}^c \frac{2x\,dx}{(x^2)^2+1} + \frac{1}{2}\lim_{x_2\to\infty}\int_c^{x_2} \frac{2x\,dx}{(x^2)^2+1}$$

$$= \frac{1}{2}\lim_{x_1\to-\infty}\int_{x_1}^c \frac{d(x^2)}{(x^2)^2+1} + \frac{1}{2}\lim_{x_2\to\infty}\int_c^{x_2} \frac{d(x^2)}{(x^2)^2+1}$$

$$= \frac{1}{2}\left[\lim_{x_1\to-\infty}\left\{\tan^{-1}x^2\right\}_{x_1}^c + \lim_{x_2\to\infty}\left\{\tan^{-1}x^2\right\}_c^{x_1}\right]$$

$$= \frac{1}{2}\left[\lim_{x_1\to-\infty}\left\{\tan^{-1}c^2 - \tan^{-1}x_1^2\right\} + \lim_{x_2\to\infty}\left\{\tan^{-1}x_2^2 - \tan^{-1}c^2\right\}\right]$$

$$= \frac{1}{2}\left[\left(\tan^{-1}e^{2} + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1}e^{2}\right)\right]$$

$$= \frac{1}{2} \times 0$$
$$= 0$$

- Evaluation of improper integrals of second type:-

<u>Type I</u> Let a be the only point of infinite discontinuity of the function $f(x)$. Then the improper integral $\int_a^b f(x)\,dx$ is defined as $\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx$ provided limit exist.

Hence
$$\boxed{\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx}$$

<u>Ex</u> Consider the improper integral $\int_0^4 \frac{dx}{\sqrt{x}}$.

Here 0 is the only point of infinite discontinuity.

Thus
$$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\epsilon \to 0^+} \int_{0+\epsilon}^{4} \frac{dx}{\sqrt{x}}$$
$$= \lim_{\epsilon \to 0^+} \left[2\sqrt{x}\right]_{0+\epsilon}^{4}$$
$$= 2\left[\lim_{\epsilon \to 0^+} \left(2 - \sqrt{\epsilon}\right)\right]$$
$$= 4$$

<u>Type II</u>. Let b be the only point of infinite discontinuity of the function $f(x)$. Then the improper integral $\int_a^b f(x)\,dx$ is defined as $\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx$, provided limit exist,

Then
$$\boxed{\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx}$$

<u>Ex</u> Consider the improper integral $\int_1^2 \frac{dx}{\sqrt{2-x}}$

Here 2 is the only singularity of the integral

$$\therefore \int_1^2 \frac{dx}{\sqrt{2-x}} = \lim_{\epsilon \to 0^+} \int_1^{2-\epsilon} \frac{dx}{\sqrt{2-x}}$$
$$= \lim_{\epsilon \to 0^+} \left[-2\sqrt{2-x}\right]_1^{2-\epsilon}$$
$$= -2\left[\lim_{\epsilon \to 0^+} \left(\sqrt{\epsilon} - 1\right)\right]$$
$$= 2$$

<u>Type III</u>. Let both the end points a and b be the only points of infinite discontinuity of the function $f(x)$. We take any point c such that $a<c<b$.

Then the improper integral $\int_a^b f(x)\,dx$

can be written as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and by type I and type II, we have,

$$\boxed{\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0^+} \int_{a+\epsilon_1}^{c} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_c^{b-\epsilon_2} f(x)\,dx}$$

provided both limits exists.

Ex- Consider the improper integral $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

→ Here 0 and 1 are the only points of infinite discontinuity.

Thus we can write, [illegible]

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{\epsilon_1 \to 0^+} \int_{0+\epsilon_1}^{1/2} \frac{dx}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}} + \lim_{\epsilon_2 \to 0^+} \int_{1/2}^{1-\epsilon_2} \frac{dx}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}}$$

$$= \lim_{\epsilon_1 \to 0^+} \left[\sin^{-1}(2x-1)\right]_{\epsilon_1}^{1/2} + \lim_{\epsilon_2 \to 0^+} \left[\sin^{-1}(2x-1)\right]_{1/2}^{1-\epsilon_2}$$

$$= \lim_{\epsilon_1 \to 0^+} \left[-\sin^{-1}(2\epsilon_1 - 1)\right] + \lim_{\epsilon_2 \to 0^+} \sin^{-1}(1-2\epsilon_2)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

TYPE-IV Let c be the only point of infinite discontinuity of the function $f(x)$ in $[a,b]$ so that $a<c<b$. Then we break the integral $\int_a^b f(x)\,dx$ into two parts $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$.

Then by type I and type II we have

$$\boxed{\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0^+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_{c+\epsilon_2}^b f(x)\,dx}$$

Ex consider the integral $\int_{-1}^1 \frac{dx}{x^3}$

Here $f(x) = \frac{1}{x^3}$ has an infinite discontinuity at $x=0$

$$\therefore \int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3}$$

$$= \lim_{\epsilon_1 \to 0^+} \int_{-1}^{0-\epsilon_1} \frac{dx}{x^3} + \lim_{\epsilon_2 \to 0^+} \int_{0+\epsilon_2}^1 \frac{dx}{x^3}$$

$$= \lim_{t_1 \to 0^+} \left[-\frac{1}{2x^2}\right]_{-1}^{-t_1} + \lim_{t_2 \to 0^+} \left[-\frac{1}{2x^2}\right]_{t_2}^{1}$$

$$= \lim_{t_1 \to 0^+} \left[-\frac{1}{2t_1^2} + \frac{1}{2}\right] + \lim_{t_2 \to 0^+} \left[-\frac{1}{2} + \frac{1}{2t_2^2}\right]$$

As $\lim_{t_1 \to 0^+} \frac{1}{2t_1^2}$ and $\lim_{t_2 \to 0^+} \frac{1}{2t_2^2}$ do not exist,

so the given integral $\int_{-1}^{1} \frac{dx}{x^3}$ does not exist.

when $t_1 = t_2$, then

$$\int_{-1}^{1} \frac{dx}{x^3} = \lim_{t_1 \to 0^+} \left(\frac{1}{2} - \frac{1}{2t_1^2} - \frac{1}{2} + \frac{1}{2t_1^2}\right)$$

$$= \lim_{t_1 \to 0^+} (0) = 0$$

Thus $\int_{-1}^{1} \frac{dx}{x^3}$ exists in the cauchy principle value sense but not in the general sense.

• Note – when the appropriate limits exist finitely, an improper integral is said to be convergent. In other words when the appropriate limits fail to exist or tend to infinity, an improper integral is said to be non-convergent.

Q) Test the convergence of $\int_a^{\infty} \frac{dx}{x^n}$ $(a > 0)$

Here $\int_a^{\infty} \frac{dx}{x^n}$ is an improper integral of first kind.

$$\int_a^{\infty} \frac{dx}{x^n} = \lim_{X \to \infty} \int_a^{X} \frac{dx}{x^n} = \lim_{X \to \infty} \left[\frac{x^{-n+1}}{-n+1}\right]_a^X$$

$$= \frac{1}{n-1} \lim_{X \to \infty} \left[\frac{1}{a^{n-1}} - \frac{1}{X^{n-1}}\right]$$

$$= \begin{cases} \frac{1}{(n-1)a^{n-1}} & \text{when } n > 1 \\ \infty & \text{when } n < 1 \end{cases}$$

But when $n = 1$

$$\int_a^{\infty} \frac{dx}{x^n} = \int_a^{\infty} \frac{dx}{x} = \lim_{X \to \infty} \int_a^{X} \frac{dx}{x}$$

$$= \lim_{X \to \infty} [\log x]_a^X$$

$$= \lim_{X \to \infty} [\log X - \log a]$$

$$= \lim_{X \to \infty} \log\left(\frac{X}{a}\right)$$

$$= \infty$$

Hence the given improper integral is convergent when $n > 1$.

Illustration

(i) The improper integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ is not convergent, as $n=\frac{1}{2}<1$

(ii) The improper integral $\int_2^\infty \frac{dx}{x^3}$ is convergent as $a=3>1$

(iii) The improper integral $\int_{1/3}^\infty \frac{dx}{x}$ is not convergent as $n=1$

B. ~~consider~~ of convergence of $\int_a^b \frac{dx}{(x-a)^n}$ $(n>0)$

$$\therefore \int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+\epsilon}^b$$

$$= \frac{1}{1-n}\lim_{\epsilon\to 0^+}\left[(b-a)^{-n+1} - (a+\epsilon-a)^{-n+1}\right]$$

$$= \begin{cases} \frac{1}{1-n}(b-a)^{-n+1} \\ -\frac{1}{(n-1)(b-a)^{n-1}}, & \text{when } n<1 \\ \infty, & \text{when } n>1 \end{cases}$$

Again, when $n=1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^b \frac{dx}{x-a} = \lim_{\epsilon\to 0^+}\left[\log(x-a)\right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon\to 0^+} \log\left(\frac{b-a}{\epsilon}\right)$$

$$= \infty$$

Thus the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ $(n>0)$ is convergent only when $n<1$

Illustration:

(i) The improper integral $\int_2^3 \frac{dx}{(x-2)^{3/2}}$ is not convergent as $n=\frac{3}{2}>1$

(ii) The improper integral $\int_1^3 \frac{dx}{\sqrt{x-1}}$ is convergent as $n=\frac{1}{2}<1$

(iii) The improper integral $\int_{-3}^3 \frac{dx}{x+3}$ is not convergent as $n=1$

2. Convergence of $\int_a^b \frac{dx}{(b-x)^n}$ $(n>0)$

the $x=b$ in the only point of infinite discontinuity of $\frac{1}{(b-x)^n}$

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\epsilon\to 0^+}\int_a^{b-\epsilon} \frac{dx}{(b-x)^n}$$

$$= \lim_{\epsilon\to 0^+}\left[-\frac{(b-x)^{-n+1}}{-n+1}\right]_a^{b-\epsilon}$$

$$= \lim_{\epsilon\to 0^+}\frac{1}{n-1}\left[\frac{1}{\epsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right]$$

$$= \begin{cases} -\frac{1}{(n-1)(b-a)^{n-1}}, & \text{when } n<1 \\ \infty, & \text{when } n>1 \end{cases}$$

Again, when $n=1$

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} \frac{dx}{b-x}$$

$$= \lim_{\epsilon \to 0^+} \left[-\log(b-x)\right]_a^{b-\epsilon}$$

$$= \lim_{\epsilon \to 0^+} \log\left(\frac{b-a}{\epsilon}\right)$$

$$= \infty$$

Thus the given improper integral $\int_a^b \frac{dx}{(b-x)^n}\ (b>a)$ is convergent only when $n<1$.

Illustration

(i) The improper integral $\int_1^2 \frac{dx}{\sqrt{2+x}}$ is convergent.

(ii) The improper integral $\int_{-1}^1 \frac{dx}{(1-x)^2}$ is not convergent.

(iii) The improper integral $\int_0^1 \frac{dx}{1-x}$ is not convergent.

3.5 Illustrative Examples

Ex 1. Examine the convergence of the improper integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, if possible evaluate the integral.

Here $x=1$ is the only point of infinite discontinuity of

$$f(x) = \frac{1}{\sqrt{1-x^2}} \text{ in } [0,1]$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{\epsilon \to 0^+} \left[\sin^{-1}x\right]_0^{1-\epsilon}$$

$$= \lim_{\epsilon \to 0^+} \sin^{-1}(1-\epsilon)$$

$$= \frac{\pi}{2}$$

Ex 2. Examine the convergence of the improper integral $\int_0^2 \frac{dx}{x(2-x)}$.

Let $f(x) = \frac{1}{x(2-x)}$

So $x=0,2$ are the infinite discontinuity of $f(x)$;

Thus we can write the integral as

$$\int_0^2 \frac{dx}{x(2-x)} = \int_0^1 \frac{dx}{x(2-x)} + \int_1^2 \frac{dx}{x(2-x)}$$

$$= \lim_{\epsilon_1 \to 0^+} \int_{0+\epsilon_1}^{1} \frac{dx}{x(2-x)} + \lim_{\epsilon_2 \to 0^+} \int_1^{2-\epsilon_2} \frac{dx}{x(2-x)}$$

$$= \lim_{\epsilon_1 \to 0^+} \frac{1}{2}\int_{\epsilon_1}^{1} \frac{x+(2-x)}{x(2-x)}dx + \frac{1}{2}\lim_{\epsilon_2 \to 0^+} \int_1^{2-\epsilon_2} \frac{x+(2-x)}{x(2-x)}dx$$

$$= \frac{1}{2}\left[\lim_{\epsilon_1 \to 0^+} \int_{\epsilon_1}^{1}\left(\frac{1}{2-x}+\frac{1}{x}\right)dx + \lim_{\epsilon_2 \to 0^+} \int_1^{2-\epsilon_2}\left(\frac{1}{2-x}+\frac{1}{x}\right)dx\right]$$

$$= \frac{1}{2}\left[\lim_{\epsilon_1 \to 0^+}\left\{(-1)\int_{\epsilon_1}^{1} \frac{-dx}{2-x} + \int_{\epsilon_1}^{1} \frac{1}{x}dx\right\}\right.$$
$$\left. + \lim_{\epsilon_2 \to 0^+}\left\{(-1)\int_1^{2-\epsilon_2} \frac{-dx}{2-x} + \int_1^{2-\epsilon_2} \frac{1}{x}dx\right\}\right]$$

$$= \frac{1}{2}\left[\lim_{\epsilon_1 \to 0^+}\left\{(\log|x|)_{\epsilon_1}^{1} - (\log|2-x|)_{\epsilon_1}^{1}\right\}\right.$$
$$\left. + \lim_{\epsilon_2 \to 0^+}\left\{(\log|x|)_1^{2-\epsilon_2} - (\log|2-x|)_1^{2-\epsilon_2}\right\}\right]$$

$$= \frac{1}{2}\left[\lim_{\epsilon_1 \to 0^+}\left\{-\log\left|\frac{x}{2-x}\right|\right\}_{\epsilon_1}^{1} + \lim_{\epsilon_2 \to 0^+}\left\{\log\left|\frac{x}{2-x}\right|\right\}_1^{2-\epsilon_2}\right]$$

$$= \frac{1}{2}\left[\lim_{\epsilon_1 \to 0^+}\left(-\log\left|\frac{\epsilon_1}{2-\epsilon_1}\right|\right) + \lim_{\epsilon_2 \to 0^+}\left(\log\left|\frac{2-\epsilon_2}{\epsilon_2}\right|\right)\right]$$

Since $\lim_{\epsilon_1 \to 0^+} \log\left|\frac{\epsilon_1}{2-\epsilon_1}\right|$ and $\lim_{\epsilon_2 \to 0^+} \log\left|\frac{2-\epsilon_2}{\epsilon_2}\right|$

do not exist, so the given improper integral is not convergent.

Ex.3 Show that the improper integral $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent. Hence find its value.

The given integral can be written as

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{X \to \infty} \int_0^X \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{X \to \infty} \int_0^{\sqrt{X}} \frac{2z\,dz}{(1+z^2)z}$$

$x = z^2$
$\therefore dx = 2z\,dz$

$$= 2\lim_{X \to \infty} \int_0^{\sqrt{X}} \frac{dz}{1+z^2}$$

$$= 2\lim_{X \to \infty}\left[\tan^{-1} z\right]_0^{\sqrt{X}}$$

$$= 2\lim_{X \to \infty}\left[\tan^{-1}\sqrt{X}\right]$$

$$= 2\cdot\frac{\pi}{2}$$

$$= \pi$$

Thus the given integral is convergent and its value is π.

Ex-4 Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

The given improper integral can be written as

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{a} \frac{dx}{x^2+1} + \int_{a}^{\infty} \frac{dx}{x^2+1}$$

$$= \lim_{x_1 \to -\infty} \int_{x_1}^{a} \frac{dx}{x^2+1} + \lim_{x_2 \to \infty} \int_{a}^{x_2} \frac{dx}{x^2+1}$$

$$= \lim_{x_1 \to -\infty} \left[\tan^{-1} x\right]_{x_1}^{a} + \lim_{x_2 \to \infty} \left[\tan^{-1} x\right]_{a}^{x_2}$$

$$= \lim_{x_1 \to -\infty} \left[\tan^{-1} a - \tan^{-1} x_1\right] + \lim_{x_2 \to \infty} \left[\tan^{-1} x_2 - \tan^{-1} a\right]$$

$$= \cancel{\tan^{-1} a} + \frac{\pi}{2} + \frac{\pi}{2} - \cancel{\tan^{-1} a}$$

$$= \pi$$

Ex-5 Show that $\int_{-\infty}^{\infty} xe^{-x^2} dx$

The given improper integral can be written as -

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{a} xe^{-x^2} dx + \int_{a}^{\infty} xe^{-x^2} dx$$

$$= \lim_{x_1 \to -\infty} \int_{x_1}^{a} xe^{-x^2} dx + \lim_{x_2 \to \infty} \int_{a}^{x_2} xe^{-x^2} dx$$

$x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow x\,dx = \frac{1}{2} dz$

$$= \frac{1}{2} \left[\lim_{x_1 \to -\infty} \int_{x_1^2}^{a^2} e^{-z} dz + \lim_{x_2 \to \infty} \int_{a^2}^{x_2^2} e^{-z} dz\right]$$

$$= \frac{1}{2} \left[\lim_{x_1 \to -\infty} \left\{-e^{-z}\right\}_{x_1^2}^{a^2} + \lim_{x_2 \to \infty} \left\{-e^{-z}\right\}_{a^2}^{x_2^2}\right]$$

$$= \frac{1}{2} \left[\lim_{x_1 \to -\infty} \left\{-e^{-a^2} + -e^{-x_1^2}\right\} + \lim_{x_2 \to \infty} \left\{-e^{-x_2^2} + e^{-a^2}\right\}\right]$$

$$= \frac{1}{2} \left[\frac{1}{\infty} - \frac{1}{\infty}\right]$$

$$= \frac{1}{2} \times 0$$

$$= 0$$

Ex-6 Evaluate if possible the improper integral $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$

Sol = here $x = 1$ is the only singularity of the given integral. so we can write the integral as,

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx = \lim_{\epsilon_1\to 0^+}\int_{-1}^{1-\epsilon_1}\sqrt{\frac{1+x}{1-x}}\,dx$$

$$= \lim_{\epsilon\to 0^+}\int_{-1}^{1-\epsilon}\frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{\epsilon\to 0^+}\left[\int_{-1}^{1-\epsilon}\frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1-\epsilon}\frac{x}{\sqrt{1-x^2}}\,dx\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\int_{-1}^{1-\epsilon}\frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int_{-1}^{1-\epsilon}\frac{-2x\,dx}{\sqrt{1-x^2}}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\left\{\sin^{-1}x\right\}_{-1}^{1-\epsilon} - \frac{1}{2}\cdot 2\left\{\sqrt{1-x^2}\right\}_{-1}^{1-\epsilon}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\sin^{-1}(1-\epsilon) + \frac{\pi}{2} - \sqrt{1-1+2\epsilon-\epsilon^2}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\sin^{-1}(1-\epsilon) + \frac{\pi}{2} - \sqrt{2\epsilon+\epsilon^2}\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

Ex-7 Examine for convergence $\int_0^{\infty}\cos x\,dx$

By defination,

$$\int_0^{\infty}\cos x\,dx = \lim_{X\to\infty}\int_0^{X}\cos x\,dx$$

$$= \lim_{X\to\infty}\left[\sin x\right]_0^{X}$$

$$= \lim_{X\to\infty}\sin X$$

Since $\sin x$ has no limit when $x\to\infty$, the given improper integral does not exist.

Ex-8 done

Ex-9 - Evaluate, if possible $\int_0^{\pi}\frac{dx}{1-\cos x}$

$f(x) = \frac{1}{1-\cos x}$ has infinite discontinuity at $x=0$ within the range $[0,\pi]$

$$\int_0^{\pi}\frac{dx}{1-\cos x} = \lim_{\epsilon\to 0^+}\int_{0+\epsilon}^{\pi}\frac{dx}{1-\cos x}$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\frac{dx}{1-\cos x}$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\frac{(1+\cos x)}{\sin^2 x}\,dx$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}(\operatorname{cosec}^2 x + \operatorname{cosec} x\cot x)\,dx$$

$$= -\lim_{\epsilon\to 0^+}(\operatorname{cosec} x + \cot x)_{\epsilon}^{\pi}$$

$$= \lim_{\epsilon\to 0^+}(\operatorname{cosec}\epsilon + \cot \epsilon)$$

$$= \infty$$

Or

$$\int_0^{\pi}\frac{dx}{1-\cos x} = \lim_{\epsilon\to 0^+}\int_{0+\epsilon}^{\pi}\frac{dx}{2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\operatorname{cosec}^2\frac{x}{2}\,dx$$

$$= -\frac{1}{2}\lim_{t\to 0^+}\left[-\cot\frac{x}{2}\right]_t^{\pi}$$

$$= -\lim_{t\to 0^+}\left(\cot\frac{\pi}{2} - \cot\frac{t}{2}\right)$$

$$= \lim_{t\to 0^+}\left(\cot\frac{t}{2} - \cot\frac{\pi}{2}\right)$$

$$= \lim_{t\to 0}\cot\frac{t}{2}$$

$$= \infty$$

The given integral is divergent.

Ex-10 Evaluate $\int_{-1}^{1}\frac{dx}{x^2}$, if possible. If not possible, find its principal value.

This is an improper integral, since $f(x) = \frac{1}{x^2}$ has infinite discontinuity at $x=0$ (lying within $(-1,1)$)

Now $\int_{-1}^{1}\frac{dx}{x^2} = \lim_{t_1\to 0^+}\int_{-1}^{0-t_1}\frac{dx}{x^2} + \lim_{t_2\to 0^+}\int_{0+t_2}^{1}\frac{dx}{x^2}$

$$= \lim_{t_1\to 0^+}\int_{-1}^{-t_1}\frac{dx}{x^2} + \lim_{t_2\to 0^+}\int_{t_2}^{1}\frac{dx}{x^2}$$

$$= \lim_{t_1\to 0^+}\left[-\frac{1}{x}\right]_{-1}^{-t_1} + \lim_{t_2\to 0^+}\left[-\frac{1}{x}\right]_{t_2}^{1}$$

$$= \lim_{t_1\to 0^+}\left(\frac{1}{t_1} - 1\right) + \lim_{t_2\to 0^+}\left(-1 + \frac{1}{t_2}\right)$$

This two limits do not exist finitely, so the integral $\int_{-1}^{1}\frac{dx}{x^2}$ has no value.

Its principal value (P) $\int_{-1}^{1}\frac{dx}{x^2}$

$$= \lim_{t\to 0}\left\{\int_{-1}^{0-t}\frac{dx}{x^2} + \int_{0+t}^{1}\frac{dx}{x^2}\right\}$$

$$= \lim_{t\to 0}\left\{\left(\frac{1}{t} - 1\right) - \left(1 - \frac{1}{t}\right)\right\}$$

$$= \lim_{t\to 0}\left(\frac{2}{t} - 2\right)$$ has no finite value.

So, the integral has no principal value also.

Ex-11. Evaluate if possible, $\int_{-1}^{1}\frac{dx}{x}$. If not try to find its principal value.

$\frac{1}{x}$ has value ∞ at $x=0$, so $\frac{1}{x}\to\infty$, as $x\to 0$

$$\int_{-1}^{1}\frac{dx}{x} = \lim_{\epsilon_1\to 0^+}\int_{-1}^{0-\epsilon_1}\frac{dx}{x} + \lim_{\epsilon_2\to 0^+}\int_{0+\epsilon_2}^{1}\frac{dx}{x}$$

$$= \lim_{\epsilon_1\to 0^+}\int_{-1}^{-\epsilon_1}\frac{dx}{x} + \lim_{\epsilon_2\to 0^+}\int_{\epsilon_2}^{1}\frac{dx}{x}$$

$$= \lim_{\epsilon_1\to 0^+}\left[\log|x|\right]_{-1}^{-\epsilon_1} + \lim_{\epsilon_2\to 0^+}\left[\log x\right]_{\epsilon_2}^{1}$$

$$= \lim_{\epsilon_1\to 0^+}\left(\log|\epsilon_1| - 0\right) + \lim_{\epsilon_2\to 0^+}\left(-\log|\epsilon_2|\right)$$

$= \lim_{\epsilon_1\to 0^+} \log|\epsilon_1| - \lim_{\epsilon_2\to 0^+} \log|\epsilon_2|$ which has no finite value since, each limit is ∞ individually.

$\therefore \int_{-1}^{1}\frac{dx}{x}$ does not exists.

Its principal value $(I) \int_{-1}^{1}\frac{dx}{x}$

$$= \lim_{\epsilon\to 0}\left\{\int_{-1}^{0-\epsilon}\frac{dx}{x} + \int_{0+\epsilon}^{1}\frac{dx}{x}\right\}$$

$$= \lim_{\epsilon\to 0}\left\{\int_{-1}^{-\epsilon}\frac{dx}{x} + \int_{\epsilon}^{1}\frac{dx}{x}\right\}$$

$$= \lim_{\epsilon\to 0}\left[\left\{\log|x|\right\}_{-1}^{-\epsilon} + \left\{\log|x|\right\}_{\epsilon}^{1}\right]$$

$$= \lim_{\epsilon\to 0}\left[\log|-\epsilon| - \log|\epsilon|\right]$$

$$= \lim_{\epsilon\to 0}\left[\log\epsilon - \log\epsilon\right] \quad [\because \epsilon > 0]$$

Ex-12 Examine for convergence $\int_{1}^{\infty}\frac{dx}{x^2(x+1)}$

By defination

$$\int_{1}^{\infty}\frac{dx}{x^2(x+1)} = \lim_{X\to\infty}\int_{1}^{X}\frac{dx}{x^2(x+1)}$$

$$= \lim_{X\to\infty}\int_{1}^{X}\frac{(x+1)-x}{x^2(x+1)}dx$$

$$= \lim_{X\to\infty}\int_{1}^{X}\left[\frac{1}{x^2} - \frac{1}{x(x+1)}\right]dx$$

$$= \lim_{X\to\infty}\int_{1}^{X}\left[\frac{1}{x^2} - \frac{(x+1)-x}{x(x+1)}\right]dx$$

$$= \lim_{X\to\infty}\int_{1}^{X}\left[\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right]dx$$

$$= \lim_{X\to\infty}\left[-\frac{1}{x} - \log|x| + \log|x+1|\right]_{1}^{X}$$

$$= \lim_{X\to\infty}\left[\left(-\frac{1}{X} - \log|X| + \log|X+1|\right) - (-1+\log 2)\right]$$

$$= \lim_{X\to\infty}\left[-\frac{1}{X} + \log\left|\frac{X+1}{X}\right|\right] - \lim_{X\to\infty}(\log 2 - 1)$$

$$= \lim_{X\to\infty}\left[-\frac{1}{X} + \log\left|1+\frac{1}{X}\right|\right] + (1 - \log 2)$$

$$= 0 + 1 - \log 2$$

$$= 1 - \log 2$$

Thus the given improper integral converges.

Ex-13 Test for convergence the improper integral $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ and evaluate if possible.

Let $f(x) = \frac{1}{x(\log x)^2}$

Here 0 is the infinite discontinuity of $f(x)$.

By defination,

$$\int_0^{1/e} \frac{dx}{x(\log x)^2} = \lim_{t \to 0^+} \int_{0+t}^{1/e} \frac{dx}{x(\log x)^2}$$

$$= \lim_{t \to 0^+} \int_t^{1/e} \frac{\frac{1}{x}\,dx}{(\log x)^2}$$

$$\lim_{t \to 0^+} \int_{\log t}^{\log \frac{1}{e}} \frac{du}{u^2} \qquad \log x = u$$

$$= \lim_{t \to 0} \left[-\frac{1}{u}\right]_{\log t}^{\log \frac{1}{e}}$$

$$= \lim_{t \to 0} \left[\frac{1}{\log e} - \frac{1}{\log t}\right]$$

$$= \frac{1}{\log e} - \lim_{t \to 0} \frac{1}{\log t} \qquad [\log 0 = -$$

$$= \frac{1}{1} - \frac{1}{\infty}$$

$$= 1$$

Thus the given integral is convergent.

Ex-14 - Examine the convergence of $\int_1^2 \frac{x\,dx}{\sqrt{2-x}}$ and evaluate of possible.

Let $f(x) = \frac{x}{\sqrt{2-x}}$

$\therefore x = 2$ is the point of infinite discontinuity

Thus $\int_1^2 \frac{x\,dx}{\sqrt{2-x}} = \lim_{t \to 0+} \int_1^{2-t} \frac{x\,dx}{\sqrt{2-x}}$

$2 - x = z^2$

$-dx = 2z\,dz$

$x = 2 - z^2$

$$= \lim_{t \to 0+} \int_1^{\sqrt{t}} \frac{(2-z^2)(-2\cancel{z})\,dz}{\cancel{z}}$$

$$= -2 \lim_{t \to 0+} \int_1^{\sqrt{t}} (2 - z^2)\,dz$$

$$= -2 \lim_{t \to 0+} \left[2z - \frac{z^3}{3}\right]_1^{\sqrt{t}}$$

$$= 2 \lim_{t \to 0+} \left[\frac{z^3}{3} - 2z\right]_1^{\sqrt{t}}$$

$$= 2 \lim_{t \to 0+} \left[\left(\frac{t\sqrt{t}}{3} - 2\sqrt{t}\right) - \left(\frac{1}{3} - 2\right)\right]$$

$$= 2 \lim_{t \to 0+} \left[\left(\frac{t\sqrt{t}}{3} - 2\sqrt{t}\right) + \frac{5}{3}\right]$$

$$= 2\left[\lim_{\epsilon\to 0^+}\left(\frac{\epsilon\sqrt{\epsilon}}{3} - 2\epsilon\right) + \lim_{\epsilon\to 0^+}\frac{5}{3}\right]$$

$$= 2\left[0 + \frac{5}{3}\right]$$

$$= \frac{10}{3}$$

Thus the given improper integral is convergent

Ex-15 Evaluate, if possible $\int_0^1 x^{n-1}\log x\,dx$

$f(x) = x^{n-1}\log x$ has value $x=0$

now, $\int_0^1 x^{n-1}\log x\,dx \rightarrow$ $\boxed{\text{This is not correct}}$

$$= \lim_{\epsilon\to 0^+}\int_{0+\epsilon}^{1} x^{n-1}\log x\,dx$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{1} x^{n-1}\log x\,dx$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{1} x^{n}\cdot x^{-1}\log x\,dx$$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{1} x^{n}\log x\,\frac{1}{x}\,dx$$

$\log x = z$

$\Rightarrow \frac{1}{x}dx = dz$

$$= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{0} z\,e^{nz}\,dz$$

$x = e^{z}$

$$= \lim_{\epsilon\to 0^+}\left[\left(z\,\frac{e^{nz}}{n}\right)_{\epsilon}^{0} - \frac{1}{n^2}\left(e^{nz}\right)_{\epsilon}^{0}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\frac{\epsilon\cdot e^{n\epsilon}}{n} - \frac{1}{n^2}\left(1 - e^{n\epsilon}\right)\right]$$

$$= \lim_{\epsilon\to 0^+}\frac{\epsilon\cdot e^{n\epsilon}}{n} - \lim_{\epsilon\to 0^+}\frac{1}{n^2} + \lim_{\epsilon\to 0^+}\frac{e^{n\epsilon}}{n^2}$$

$$= 1 - \frac{1}{n^2}$$

Now, $\int_0^1 \log x\cdot x^{n-1}\,dx$

$$= \lim_{\epsilon\to 0^+}\left[\left\{\log x\cdot\frac{x^n}{n}\right\}_{0+\epsilon}^{1} - \int_{0+\epsilon}^{1}\left(\frac{1}{x}\cdot\frac{x^n}{n}\right)dx\right]$$

$$= \lim_{\epsilon\to 0^+}\left[-\frac{\log\epsilon\cdot\epsilon^n}{n} - \frac{1}{n}\int_{\epsilon}^{1} x^{n-1}\,dx\right]$$

$$= \lim_{\epsilon\to 0^+}\left[-\frac{\epsilon^n\log\epsilon}{n} - \frac{1}{n}\left[\frac{x^n}{n}\right]_{\epsilon}^{1}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[-\frac{\epsilon^n\log\epsilon}{n} - \frac{1}{n^2}\left(1 - \epsilon^n\right)\right]$$

$$= \lim_{\epsilon\to 0^+}\left[-\frac{\epsilon^n\log\epsilon}{n} - \frac{1}{n^2} + \frac{\epsilon^n}{n^2}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\epsilon^n\left(\frac{1}{n^2} - \frac{\log\epsilon}{n}\right) - \frac{1}{n^2}\right]$$

$$= \lim_{\epsilon\to 0^+}\frac{\epsilon^n\left(1 - n\log\epsilon\right)}{n^2} - \frac{1}{n^2}$$

$$= \lim_{t \to 0^+} \frac{1 - n \log t}{n^2 t^{-n}} - \frac{1}{n^2} \quad [\text{if } n > 0]$$

$$= \lim_{t \to 0^+} \frac{-n \cdot \frac{1}{t}}{n^2 \cdot (-n) \cdot t^{-n-1}} - \frac{1}{n^2}$$

[by L' Hospital Law]

$$= \lim_{t \to 0^+} \frac{t^n}{n^2} - \frac{1}{n^2}$$

$$= -\frac{1}{n^2} \text{ if } n > 0$$

Ex-16 Evaluate if possible $\int_0^1 \log \frac{1}{x} \, dx$

In fact $f(x) = \log \frac{1}{x} = -\log x$ is positive $0 \le x \le 1$. [at $x \to 0$, $f(x) \to \infty$]

Now, $\int_0^1 \log \frac{1}{x} \, dx$

$$= -\int_0^1 \log x \, dx$$

$$= -\lim_{t \to 0^+} \int_{0+t}^1 \log x \, dx$$

$$= -\lim_{t \to 0^+} [x(\log x - 1)]_t^1$$

$$= -\lim_{t \to 0^+} [-1 - t(\log t - 1)]$$

$$= 1$$

Here the given integral is convergent.

Now, $\int_0^1 \log \frac{1}{x} \, dx = -\int_0^1 \log x \, dx$

$$= -\lim_{t \to 0^+} \int_{0+t}^1 \log x \, dx$$

$$= -\lim_{t \to 0^+} \int_t^1 \log x \, dx$$

$$= -\lim_{t \to 0^+} [x(\log x - 1)]_t^1$$

$$= -\lim_{t \to 0^+} \{-1 - t \log t + t\}$$

$$= \lim_{t \to 0^+} \{1 + t \log t - t\}$$

$$= \lim_{t \to 0^+} \{1 + t \log \tfrac{t}{e}\}$$

$$= \lim_{t \to 0^+} \left\{1 + \frac{\log \frac{t}{e}}{t^{-1}}\right\}$$

$$= \lim_{t \to 0^+} \left\{1 + \frac{\frac{1}{t/e} \cdot \frac{1}{e}}{-t^{-2}}\right\}$$

$$= \lim_{t \to 0^+} (1 - t)$$

$$= 1$$

Continuity (ϵ-δ)

- **Defenation**

A function $f(x)$ is said to be continuous for $x=a$ provided $\lim_{x\to a} f(x)$ exists, is finite and equal to $f(a)$

$$\boxed{\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)}$$

A function $f(x)$ is said to be discontinuous at $x=a$ if $f(x)$ is not continuous at $x=a$

There are 3 types of discontinuity.

- **Defenation (ϵ-δ def^n)**

A function $f(x)$ is said to be continuous at $x=a$ if for every $\epsilon>0$ there exists a $\delta>0$ (depending on ϵ) such that $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$

Ex

(i) $f(x)=x^2$ is continuous at $x=a$, because $\lim_{x\to a} f(x)=a^2$ and $f(a)=a^2$.

so $\lim_{x\to a} f(x) = f(a)$

(ii) $f(x) = \cos\frac{1}{x}$ is it continuous at $x=0$?

Ans is no, because $\lim_{x\to 0} \cos\frac{1}{x}$ dose not exist

So $f(x) = \cos\frac{1}{x}$ is discontinuous at $x=0$

(iii) $f(x) = \frac{1}{x^2}$ is it continuous at $x=0$?

Ans is no, because $\lim_{x\to 0} f(x)$ is infinite.

So $f(x) = \frac{1}{x^2}$ is discontinuous at $x=0$

(iv) $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{when } x\neq 0 \\ 0 & \text{when } x=0 \end{cases}$ is it continuous at $x=0$?

Ans is yes.

$-x \le x\sin\frac{1}{x} \le x$

$\lim_{x\to 0} -x = 0$, $\lim_{x\to 0} x = 0$

$\lim_{x\to 0} x\sin\frac{1}{x}$ is also zero. [By Sandwitch theorem]

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0$

and $f(0) = 0$

$\therefore f(x)$ is continuous at $x=0$

• Classification of discontinuity

(i) Jump discontinuity or ordinary discontinuity — If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $f(x)$ is said to be a jump discontinuity at $x=a$.

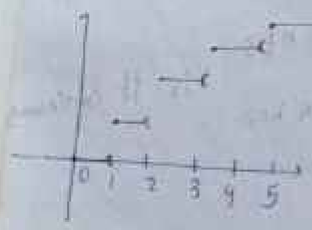
Ex: $f(x) = [x]$ is discontinuous

[$[x]$ is the greatest integer less than or equal to x]

[If $x = 1.5$, $[x] = 1$]

[$x = -2.6$, $[x] = -3$]

Graph of $[x]$



$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x] = \lim 0 = 0$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = \lim_{x\to 1^+} 1 = 1$

So, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore f(x) = [x]$ has a jump discontinuity at $x=1$

* $f(x) = [x]$ have a jump discontinuity at every integer point.

(ii) Removable discontinuity

If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$, then $f(x)$ has a removable discontinuity at $x=a$

Ex: $f(x) = \begin{cases} 1-x & \text{when } x \in (-\infty, 1) \\ x-1 & \text{when } x \in (1, \infty) \\ 1 & \text{when } x=1 \end{cases}$

now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 0$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 0$

$f(1) = 1$

So, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \neq f(1) = 1$

$\therefore f(x)$ has a removable discontinuity at $x=1$

(iii) Infinite discontinuity

If one or both of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ are tend to infinity (∞ or $-\infty$), then $f(x)$ is said to have a infinite discontinuity.

Ex: Graph of $\tan x$ →

- $f(x) = \begin{cases} \tan x, & x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) - \left\{\frac{\pi}{2}\right\} \\ 0, & x = \frac{\pi}{2} \end{cases}$

The $f(x)$ has a infinite discontinuity at $x = \pi/2$

- Properties of continuous functions

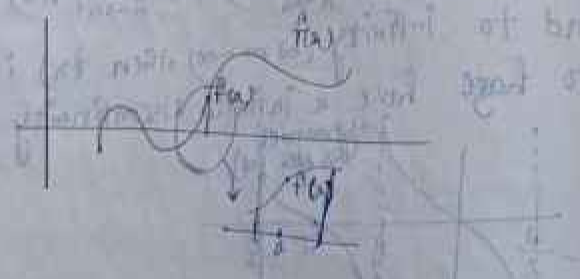
If $f(x)$ and $g(x)$ are continuous at $x = a$ then,

(i) $f(x) \pm g(x)$ are continuous at $x = a$

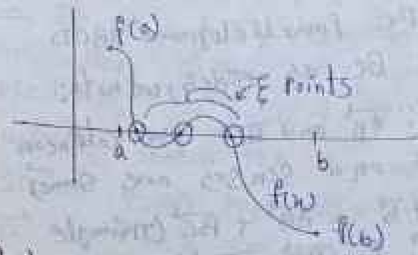
(ii) $f(x) \cdot g(x)$ is continuous at $x = a$

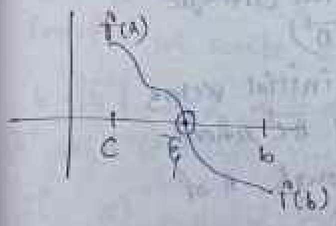
(iii) $\frac{f(x)}{g(x)}$ is also continuous at $x = a$, provided $g(x) \neq 0$ for all x

(iv) If $f(x)$ is continuous at $x = a$ and $f(a) \neq 0$ then there exist a nbd. N of a such that $f(x)$ has the same sign as that of $f(a)$ in N



(v) If $f(x)$ be continuous throughout the interval $[a, b]$ and if $f(a)$ and $f(b)$ be of opposite sign, then there is atleast one value of x say ξ such that $f(\xi) = 0$





• vector analysis (A Sir)

(i) Triangle law of vector addition :-
Given two vectors $\vec{a}$ and $\vec{b}$. Place the vector $\vec{a}$ draw $\vec{b}$ from the terminal point of $\vec{a}$. Then the vector directed from the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ is the sum $\vec{a}+\vec{b}$.

(ii) Parallelogram law of vector Addition:
Completing the parallelogram ABCD with AB and BC as sides, we note:
$\vec{AD} = \vec{BC} = \vec{b}$ ($\because \vec{AD}$ and $\vec{BC}$ are collinear their lengths and senses are same)
Hence $\vec{AC} = \vec{a}+\vec{b} = \vec{AB} + \vec{BC}$ (Triangle Law) $= \vec{AB} + \vec{AD}$ ($\because \vec{BC} = \vec{AD}$)
i.e., the sum of two co-initial vectors $\vec{AB}$ and $\vec{AD}$ is given by $\vec{AC}$ where AC is the diagonal through A of the parallelogram ABCD have as adjacent sides.

• Two important theorems on section Ratio -

Collinearity of three points: A necessary and sufficient condition for three distinct points A, B, C to lie on a straight line (i.e. collinear) is that there exist three numbers x, y, z, not all zero, such that

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$
$\Rightarrow x + y + z = 0$

where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C respectively with respect to a chosen origin O.

• collinear vectors

Theorem If $\vec{a}$ and $\vec{b}$ be two collinear vectors, then either of them can be expressed as the product of the other by a suitable scalar, the numerical value of the scalar being the ratio of the lengths of $\vec{a}$ and $\vec{b}$ and conversely.

Proof. We write $\vec{a} = a\hat{a}$, $\vec{b} = b\hat{b}$, where $a = |\vec{a}|$, $b = |\vec{b}|$ and $\hat{a}$ and $\hat{b}$ are unit vectors along $\vec{a}$ and $\vec{b}$ respectively.

If $\vec{a}$ and $\vec{b}$ are collinear, then their directions are same i.e. $\hat{a} = \hat{b}$
Now,
$\vec{b} = b\hat{b} = \frac{b}{a}(a\hat{b}) = \frac{b}{a}(a\hat{a})$ [$\because \hat{a} = \hat{b}$]
i.e., $\vec{b} = \frac{b}{a}\vec{a} = x\vec{a}$, where $x = \frac{\text{length of } b}{\text{length of } a}$

• Section Ratio (or Point of division)

If A, B, C are points of a straight line then C is said to divide the segment AB in the ratio $k:1$ if $AC : CB = k:1$

Clearly k is positive (C divides AB internally) or negative (C divides externally) according as C lies within or outside the segment AB.

Ex (i) Show that the points $\vec{\alpha} = a - 2b + 3c$, $\vec{\beta} = 2a + 3b - 4c$ and $\vec{\gamma} = -7b + 10c$ are collinear. (a, b, c are three non-coplanar vectors)

$$x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma} = 0$$

$$\Rightarrow x(a - 2b + 3c) + y(2a + 3b - 4c) + z(-7b + 10c) = 0$$

$$\Rightarrow (x + 2y)a + (-2x + 3y - 7z)b + (3x - 4y + 10z)c = 0$$

$x + 2y = 0$ — (i)

$-2x + 3y - 7z = 0$ — (ii)

$3x - 4y + 10z = 0$ — (iii)

From (i),

$x = -2y$ — (iv)

From (ii)

$-2(-2y) + 3y - 7z = 0$

$\Rightarrow 4y + 3y - 7z = 0$

$\Rightarrow y - z = 0 \Rightarrow y = z$ — (v) $\longrightarrow$ X

From (iii)

$3(-2y) - 4y + 10y = 0$ $\longrightarrow$ X

$\Rightarrow$ It is easy to see that if $x = -2$, $y = 1$, $z = 1$, then the relation is satisfied since the coefficient of a is $x + 2y = 0$, the coefficient of b is $-2x + 3y - 7z = 0$ and the coefficient of c is $3x - 4y + 10z = 0$.

Thus $x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma} = 0$, where $x + y + z = 0$

$\therefore$ the points $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are collinear.

(2) Let $\vec{a} = 3\hat{i} + 5\hat{j} + 7\hat{k} = (3, 5, 7)$ and $\vec{b} = 4\hat{i} + 2\hat{j} + \hat{k} = (4, 2, 1)$

Then $\vec{a} \cdot \vec{b} = 3 \cdot 4 + 5 \cdot 2 + 7 \cdot 1 = 29$.

Again if θ is the smallest angle between a and b

$$a \cdot b = |\vec{a}||\vec{b}| \cos\theta$$

$$\Rightarrow 29 = \sqrt{9 + 25 + 49} \cdot \sqrt{16 + 4 + 1} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{29}{\sqrt{83} \cdot \sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1} \frac{29}{\sqrt{83} \cdot \sqrt{21}}$$

- **Derivatives of sums and products**

- **Law 1** The components of $F'(t)$ are derivatives of the components of $F(t)$

Thus, if

$F(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$ function of is a derivable (vector) function of t then F_1, F_2, F_3 are also derivable (scalar) functions of t and

$$\frac{dF}{dt} = \frac{dF_1}{dt}\hat{i} + \frac{dF_2}{dt}\hat{j} + \frac{dF_3}{dt}\hat{k}$$

- **Two important theorems**

i) vectors with constant magnitude –

A necessary and sufficient condition that a proper vector u has a constant length is that

$$\boxed{u \cdot \frac{du}{dt} = 0}$$

- **Solved Problems**

Ex ① If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, find the value of $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$

Sol. we write $\vec{r} = (3t, 3t^2, 2t^3)$

$$\frac{d\vec{r}}{dt} = (3, 6t, 6t^2)$$

$$\frac{d^2\vec{r}}{dt^2} = (0, 6, 12t)$$

$$\frac{d^3\vec{r}}{dt^3} = (0, 0, 12)$$

$$\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix}$$

$$= (72, 0, 0)$$

$$\therefore \frac{d\vec{r}}{dt} \cdot \left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right) = (3, 6t, 6t^2) \cdot (72, 0, 0)$$

$= 216$ units

- **Directional derivatives** of scalar point functions

Definition – Let $f(P)$ be any scalar point functions and let A be any given point in the region of its defination.

Let L be any semi-line through A (i.e. drawn on one side of A only). If P be any point on the semi-line A, then

$$\lim_{P \to A} \frac{f(P) - f(A)}{AP}$$

is called the directional derivative of the scalar point function f at the point A along the direction of the semi line L.

• vector point functions

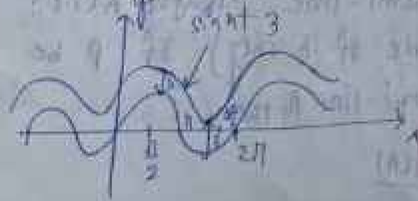
Just as in the case of scalar point functions we define derivative of a vector point function F(P) at a given point A along a direction u (unit vector) through A by means of

$$\lim_{P \to A} \frac{F(P) - F(A)}{AP}$$ provided the limit exists.

$$\boxed{\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \cos x}$$

• Anti derivative of $\cos x$ is denoted by $\int \cos x \, dx = \sin x$

sin x (period − 2π)



• Fundamental theorem of calculus (related the two concepts definite integral and indefinite integral)

• Defination of box in calculas

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$[\vec{b}, \vec{c}, \vec{a}] = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$[\vec{c}, \vec{a}, \vec{b}] = \vec{c} \cdot (\vec{a} \times \vec{b})$$

• $\boxed{\vec{a} \times (\vec{b} \times \vec{c})} \longrightarrow \vec{b} \times \vec{c} = |\vec{b}||\vec{c}| \sin\theta \, \hat{n}$

$\downarrow\downarrow$

$$\boxed{(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$$

• Gradient

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

• $f = xy^2, \; f : \mathbb{R}^2 \to \mathbb{R}$

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f$$

$$\nabla f = (y^2, 2xy)$$

Examples of grad. (Imp)

① Find the directional derivative of $f = xy + yz + zx$ in the directions of the vector $i + 2j + 2k$ at $(1, 2, 0)$

Sol. $f = xy + yz + zx$ unit vector along $i + 2j + 2k$ is $\frac{1}{3}(i + 2j + 2k)$

$\text{grad } f = (y+z)i + (z+x)j + (x+y)k$

$$\frac{df}{ds} = \frac{i+2j+2k}{3}\left[(y+z)i + (z+x)j + (x+y)k\right]$$

$= \frac{1}{3}(4x + 3y + 3z)$ at the point (x, y, z)

$= \frac{10}{3}$, at the point $(1, 2, 0)$

• Direction derivative is maximum when $\boxed{\theta = 0^\circ}$

• Continuity - Q (5 sir)

Q. $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$
$= 0$, for $x = 0$

Q. Show that $f(x)$ is continuous at $x = 0$

Sol. $|f(x) - f(0)| = |x \sin\frac{1}{x}|$
$= |x| |\sin\frac{1}{x}| \leq |x|$ $[\because |\sin\frac{1}{x}| \leq 1]$

$\Rightarrow |f(x) - f(0)| \leq |x|$

Let $\epsilon > 0$ Then
$|f(x) - f(0)| < \epsilon$ whenever $|x - 0| < \delta (= \epsilon)$
$\therefore f(x)$ is continuous at $x = 0$

Another Proof

$f(0) = 0$ $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

$\lim_{x \to 0} x \sin\frac{1}{x} = 0$

[If $\lim_{x \to a} f(x) = 0$ and $g(x)$ bounded function then $\lim_{x \to a} f(x) g(x) = 0$]

So, $\lim_{x \to 0} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 0$

Q.2 $f(x) = \frac{1}{2} - x$ when $0 < x < \frac{1}{2}$
$= \frac{1}{2}0$ when $x = \frac{1}{2}$
$= \frac{3}{2} - x$ when $\frac{1}{2} < x < 1$

Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$

Sol^n: we need to show $\lim_{x \to \frac{1}{2}} f(x) \neq f(\frac{1}{2})$

now,
$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}+} \frac{3}{2} - x = \frac{2}{2} = 1$$

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}-} \frac{1}{2} - x = 0$$

$f(\frac{1}{2}) = 0$

Short form
solution = Sol^n
Equation = Eq^n
Continuity = Cont^n
Polynomial = Poly^n

Since $\lim_{x \to \frac{1}{2}^-} f(x) \neq \lim_{x \to \frac{1}{2}^+} f(x)$

So $\lim_{x \to \frac{1}{2}} f(x)$ does not exists

Hence $f(x)$ is not continuous / discontinuous at $x = \frac{1}{2}$

<u>Another Proof</u> $\left|f(x) - f(\tfrac{1}{2})\right| = \left|\tfrac{1}{2} - x - \tfrac{1}{2}\right|$ when $0 < x < \frac{1}{2}$

$= |x|$ " "

$\left|f(x) - f(\tfrac{1}{2})\right| = \left|\tfrac{3}{2} - x - \tfrac{1}{2}\right|$ when $\frac{1}{2} < x < 1$

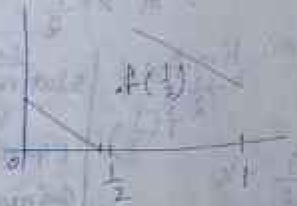
$= |1 - x|$

Let $\epsilon = \frac{1}{2}$

$\left|f(x) - f(\tfrac{1}{2})\right| < \frac{1}{2}$

$|x| < \frac{1}{2}$ when $0 < x < \frac{1}{2}$ and $|1 - x| < \frac{1}{2}$

when $\frac{1}{2} < x < 1$



~~$|x| < \frac{1}{2}$~~

~~$|x| > \frac{1}{2}$ when $\frac{1}{2} < x < 1$~~

$\frac{1}{2} - \left|\frac{1}{2} - x\right| < \frac{1}{2}$

$\left|x - \frac{1}{2}\right| > 0$

$|x| < \frac{1}{2}$

$\Rightarrow \left|x - \frac{1}{2}\right| \le |x| - \frac{1}{2}$

$\Rightarrow \left|x - \frac{1}{2}\right| \le 0$

$\Rightarrow \left|x - \frac{1}{2}\right| < \delta$

<u>Q-3</u> Show that the function $f(x) = |x| + |x-1| + |x-2|$ is continuous at $x = 0, 1, 2$

$$f(x) = \begin{cases} -x - (x-1) - (x-2) = -3x+3, & \text{for } x < 0 \\ = x - (x-1) - (x-2) = -x+3, & \text{for } 0 \le x < 1 \\ = x + (x-1) - (x-2) = x+1, & \text{for } 1 \le x < 2 \\ = x + (x-1) + (x-2) = 3x-3, & \text{for } x \ge 2 \end{cases}$$

<u>Q-4</u> Let $f(x) = \dfrac{\sin(a^2x^2)}{x}$, $x \neq 0$

$= k$, $x = 0$

Find out the value of k for which $f(x)$ is continuous at $x = 0$

we need to find out the $\lim_{x \to 0} f(x)$

now,

$$\lim_{x \to 0} \frac{\sin(a^2x^2)}{x}$$

$$= \lim_{x \to 0} \left\{ \frac{\sin(a^2x^2)}{a^2x^2} \cdot x a^2 \right\}$$

$a^2x^2 = y$, $x \to 0$, $y \to 0$, $x = \frac{\sqrt{y}}{a}$

$$= \lim_{y \to 0} \frac{\sin y}{y} \cdot \lim_{y \to 0} \frac{\sqrt{y}}{a} \cdot a^2$$

$$= \lim_{x \to 0} \frac{\sin(a^2x^2)}{a^2x^2} \cdot \lim_{x \to 0} (a^2x)$$

$= 1 \times 0 = 0$

Now $f(x)$ is continuous at $x=0$ if, $f(0)=$

$\lim_{x \to 0} f(x) = f(0) = 0$

$\Rightarrow x = 0$

Q.5 $f(x) = \frac{x^2 - x}{x}$, $x \neq 0$ find the value of $f(x)$ for which $f(x)$ is continuous at $x=0$

$\lim_{x \to 0} \frac{x^2 - x}{x}$

$= \lim_{x \to 0} \frac{x(x-1)}{x}$

$= \lim_{x \to 0} (x-1)$

$= -1$

As $f(x)$ is continuous at $x=0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore f(0) = -1$

Q.3

Soln $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - (x-1) - (x-2) = -x+3$

$= \lim_{x \to 0^+} (-x+3)$

$= 3$

$\lim_{x \to 0^-} -x - (x-1) - (x-2) = -3x+3$

$= 3$

$f(0) = -x+3$

$= 3$

$\therefore f(x)$ is continuous at $x=0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + (x-1) - (x-2) = x+1$

$= \lim_{x \to 1^+} x+1$

$= 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -x+3$

$= -1+3$

$= 2$

$f(1) = (1)+1 = 2$

$\therefore f(x)$ is continuous at $x=1$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x-3$

$= 3\times 2 - 3$

$= 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x+1$

$= 2+1$

$= 3$

$f(2) = 3x-3$

$= 6-3$

$= 3$

$\therefore f(x)$ is continuous at $x=2$

GAMMA AND BETA FUNCTION

• Gamma function

Q: The improper integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is a function of n and denoted by $\Gamma(n)$. This function is known as Gamma function. This improper integral is convergent for $n > 0$.

Therefore $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \ (n>0)$ — (1)

• Properties of Gamma function

(i) $\boxed{\Gamma(1) = 1}$

Proof: Putting $n = 1$ in (1), we get

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx$$

$$= \lim_{X\to\infty} \int_0^X e^{-x}\,dx$$

$$= \lim_{X\to\infty} \left[-e^{-x}\right]_0^X$$

$$= \lim_{X\to\infty} \left[1 - \frac{1}{e^X}\right]$$

$$= 1 - 0$$

$$= 1$$

(ii) $\boxed{\Gamma(n+1) = n\Gamma(n)}$

Proof: From (1) we have

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$$

$$= \lim_{X\to\infty} \int_0^X e^{-x} x^n\,dx$$

$$= \lim_{X\to\infty} \left\{ \left[-e^{-x} x^n\right]_0^X + n\int_0^X x^{n-1} e^{-x}\,dx \right\}$$

$$= \lim_{X\to\infty} \left\{ (-X^n e^{-X} + 0) + n\int_0^X e^{-x} x^{n-1}\,dx \right\}$$

$$= 0 + \lim_{X\to\infty} n\int_0^X e^{-x} x^{n-1}\,dx$$

$$= n\int_0^\infty e^{-x} x^{n-1}\,dx$$

$$= n\Gamma(n)$$

(iii) $\boxed{\Gamma(n+1) = n! \text{ when } n \text{ is a positive integer}}$

Proof: we have

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)\ldots 1\cdot\Gamma(1)$$

$$= n! \quad [\because \Gamma(1) = 1]$$

$$\therefore \Gamma(n+1) = n!$$

(iv) $\boxed{\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$, $0<n<1$ (Duplication formula)

Proof: Beyond the scope of this book.

(v) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\boxed{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$

Proof: we have

$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, $0<n<1$

Putting $n=\frac{1}{2}$, we get

$$\Gamma(\tfrac{1}{2})\,\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\{\Gamma(\tfrac{1}{2})\}^2 = \frac{\pi}{1}$$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\therefore \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Illustration

(i) To evaluate $\Gamma(\frac{11}{2})$, we have

$$\Gamma(\tfrac{11}{2}) = \Gamma(\tfrac{9}{2}+1) = \tfrac{9}{2}\Gamma(\tfrac{9}{2}) = \tfrac{9}{2}\Gamma(\tfrac{7}{2}+1)$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\Gamma(\tfrac{7}{2}) = \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\Gamma(\tfrac{5}{2}+1)$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\Gamma(\tfrac{5}{2})$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\Gamma(\tfrac{3}{2}+1)$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma(\tfrac{3}{2})$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma(\tfrac{1}{2}+1)$$

$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma(\tfrac{1}{2})$$

$$= \frac{945}{32}\sqrt{\pi}\quad\left[\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\right]$$

(ii) To find $\Gamma(7)$, we have

$$\Gamma(7) = \Gamma(6+1) = 6!\quad[\text{by Prop. (iii)}]$$

$$= 6\times5\times4\times3\times2\times1$$

$$= 720$$

(iii) To show $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$, we have

$$\Gamma(\tfrac{1}{3})\,\Gamma(1-\tfrac{1}{3}) = \frac{\pi}{\sin\frac{\pi}{3}}\quad[\text{by Prop. (iv)}]$$

$$= \frac{\pi}{\sqrt{3}/2}$$

$$= \frac{2\pi}{\sqrt{3}}$$

- **Beta function**

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is a function of m and n, and denoted by $B(m,n)$, is known as beta function. This integral is convergent when $m, n>0$

$$\therefore B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad m,n>0 \quad \cdots (1)$$

- Illustration

(i) $B(-5, 6)$ has no value.

(ii) $B(6.7, 2)$ must have a value.

(iii) $B(2, 0)$ has no value.

- The Beta function $B(m,n)$, $(m,n>0)$ has the following properties:

(i) $\boxed{B(m,n) = B(n,m)}$

(ii) If m is a positive integer then

$$\boxed{B(m,n) = \frac{(m-1)!}{n(n+1)(n+2)\cdots(n+m-1)}}$$

(iii) $$\boxed{B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}$$

(iv) $$\boxed{B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$$

(v) $\boxed{B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi}$

Proof:

(i) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}\,dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$

$= B(n,m)$

(ii) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \left[x^{m-1}\frac{(1-x)^n}{-n}\right]_0^1 + \frac{1}{n}\int_0^1 (m-1)x^{m-2}(1-x)^n\,dx$ [By integrating by parts]

$= \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$

$= \frac{m-1}{n}\int_0^1 x^{(m-1)-1}(1-x)^{(n+1)-1}\,dx$

$= \frac{m-1}{n} B(m-1, n+1)$

Thus $B(m,n) = \frac{m-1}{n} B(m-1, n+1)$ — (1)

From (1) we get

$B(m-1, n+1) = \frac{m-2}{n+1}(m-2, n+2)$

$$B(m-2, n+2) = \frac{m-3}{n+2} B(m-3, n+3)$$

$$\cdots$$

$$B(3, n+m-3) = \frac{2}{n+m-3} B(2, n+m-2)$$

$$B(2, n+m-2) = \frac{1}{n+m-2} B(1, n+m-1)$$

Therefore,

$$B(m,n) = \frac{m-1}{n} \cdot \frac{m-2}{n+1} \cdot \frac{m-3}{n+2} \cdots \frac{2}{n+m-3} \cdot \frac{1}{n+m-2} \cdot B(1, n+m-1)$$

Now, $B(1, n+m-1) = \int_0^1 x^{1-1}(1-x)^{n+m-1-1}\,dx$

$$= \int_0^1 (1-x)^{n+m-2}\,dx$$

$$= \left[\frac{(1-x)^{n+m-1}}{-(n+m-1)}\right]_0^1$$

$$= \frac{1}{n+m-1}$$

Hence $B(m,n) = \dfrac{(m-1)(m-2)\cdots 2 \cdot 1}{n(n+1)(n+2)\cdots(n+m-2)(n+m-1)}$

or, $B(m,n) = \dfrac{(m-1)!}{n(n+1)(n+2)\cdots(n+m-1)}$

(iii) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta\,d\theta$

If $x = 0$, $\theta = 0$ and if $x = 1$, $\theta = \frac{\pi}{2}$

$$\therefore B(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}(2\sin\theta\cos\theta)\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-2+1}\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$\therefore B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

(iv) Left as exercise

(v) From Prop (iii), by putting

$m = n = \frac{1}{2}$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{1}{2}-1}\theta\,\cos^{2\cdot\frac{1}{2}-1}\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^0\theta\,\cos^0\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta$$

$$= 2\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2 \times \frac{\pi}{2}$$

$$= \pi$$

$$\therefore B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

• Illustration

(i) $B\left(\frac{1}{3},4\right) = B\left(4,\frac{1}{3}\right)$

$= \dfrac{(4-1)!}{\frac{1}{3}\left(\frac{1}{3}+1\right)\left(\frac{1}{3}+2\right)\left(\frac{1}{3}+3\right)}$

$= \dfrac{3!}{\frac{1}{3}\times\frac{4}{3}\times\frac{7}{3}\times\frac{10}{3}}$

$= \dfrac{6\times3^4}{4\times7\times10}$

(ii) $\int_0^{\frac{\pi}{2}} \sin^{\frac{7}{2}}\theta \cos^{5}\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2 \sin^{2\cdot\frac{9}{4}-1}\theta \cos^{2\cdot3-1}\theta \, d\theta$

$= \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{9}{4}-1}\theta \cos^{2\cdot3-1}\theta \, d\theta$

$\left[\because 2m-1=\frac{7}{2} \Rightarrow m=\frac{9}{4}, \; 2n-1=5 \Rightarrow n=3\right]$

$= \frac{1}{2} B\left(\frac{9}{4},3\right)$

$= \frac{1}{2} B\left(3,\frac{9}{4}\right)$

$= \frac{1}{2}\cdot\dfrac{(3-1)!}{\frac{9}{4}\left(\frac{9}{4}+1\right)\left(\frac{9}{4}+2\right)}$

$= \frac{1}{2}\cdot\dfrac{2!}{\frac{9}{4}\left(\frac{9}{4}+1\right)\left(\frac{9}{4}+2\right)}$

(iii) $\int_0^{\infty} \dfrac{x^6}{(1+x)^{\frac{15}{2}}}\,dx$

$= \int_0^{\infty} \dfrac{x^{7-1}}{(1+x)^{7+\frac{1}{2}}}\,dx$

$= B\left(7,\frac{1}{2}\right)$

$= \dfrac{(7-1)!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+3\right)\left(\frac{1}{2}+4\right)\left(\frac{1}{2}+5\right)\left(\frac{1}{2}+6\right)}$

$= \dfrac{6!}{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+3\right)\left(\frac{1}{2}+4\right)\left(\frac{1}{2}+5\right)\left(\frac{1}{2}+6\right)}$

• Relation between Beta and Gamma Function

$$\boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Proof: Beyond the scope of book

• Remark – This relation helps us to find the value of $B(m,n)$ when at least one of m and n is positive integer.

• **Illustration**

(i) To find $B(4,2)$, we use the relation between beta and gamma function.

$$\therefore B(4,2) = \frac{\Gamma(4)\,\Gamma(2)}{\Gamma(4+2)}$$

$$= \frac{\Gamma(3+1)\,\Gamma(1+1)}{\Gamma(5+1)}$$

$$= \frac{3!\,1!}{5!}$$

$$= \frac{3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{1}{20}$$

(ii) To evaluate $\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$

we use Prop (iii)

$$= \frac{1}{2}\, 2\int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{7}{2}-1}\theta\, d\theta$$

$$= \frac{1}{2}\, \Gamma\left(\frac{1}{2}, \frac{7}{2}\right)$$

$$= \frac{1}{2}\, \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)}$$

$$= \frac{1}{2}\, \frac{\sqrt{\pi}\times\Gamma\left(\frac{5}{2}+1\right)}{3!}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}\;\frac{5}{2}\;\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right)}{3!}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{6}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}\times 15\times\sqrt{\pi}}{8\times 6\cdot 2} = \frac{5\pi}{32}$$

(iii) To evaluate $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta\, d\theta$

we use Prop (iii)

$$\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta\, d\theta$$

$$= \frac{1}{2}\, 2\int_0^{\frac{\pi}{2}} \sin^{2\cdot 4-1}\theta \cos^{2\cdot 3-1}\theta\, d\theta$$

$$= \frac{1}{2} B(4,3)$$

$$= \frac{1}{2}\cdot\frac{\Gamma(4)\,\Gamma(3)}{\Gamma(4+3)}$$

$$= \frac{1}{2}\,\frac{\Gamma(3+1)\,\Gamma(2+1)}{\Gamma(6+1)}$$

$$= \frac{1}{2}\cdot\frac{3!\,2!}{6!}$$

$$= \frac{1}{2}\cdot\frac{3\cdot 2\cdot 1\times 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{1}{120}$$

- Illustration Example

① Show $\int_0^\infty e^{-4x} x^{\frac{3}{2}} dx = \frac{3}{128}\sqrt{\pi}$

$$\int_0^\infty e^{-4x} x^{\frac{3}{2}} dx$$

Putting $4x = z$

$\therefore dx = \frac{1}{4} dz$

$$= \int_0^\infty e^{-z} \left(\frac{z}{4}\right)^{\frac{3}{2}} \frac{1}{4} dz$$

$$= \frac{1}{8} \cdot \frac{1}{4} \int_0^\infty e^{-z} z^{\frac{3}{2}} dz$$

$$= \frac{1}{32} \int_0^\infty e^{-z} z^{\frac{5}{2}-1} dz$$

$$= \frac{1}{32} \cdot \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{32} \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{1}{32} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{32} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{32} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{32} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{3}{128} \sqrt{\pi}$$

② Show that $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{8}{315}$

$$\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$$

$$= \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{5}{2}-1} x \cos^{2\cdot 3-1} x \, dx$$

$$= \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{5}{2}-1} x \cos^{2\cdot 3-1} x \, dx$$

$$= \frac{1}{2} B\left(\frac{5}{2}, 3\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma(3)}{\Gamma\left(\frac{5}{2}+3\right)}$$

$$\cancel{\frac{1}{2} \Gamma\left(\frac{9}{2}+1\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma(2+1)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot 2!}{\Gamma\left(\frac{9}{2}+1\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot 2!}{\frac{9}{2} \Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot 2!}{\frac{9}{2} \cdot \frac{7}{2} \cdot \Gamma\left(\frac{7}{2}\right)}$$

$$\frac{1}{2} \cdot \frac{\cancel{\Gamma\left(\frac{5}{2}\right)}\, 2!}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cancel{\Gamma\left(\frac{5}{2}\right)}}$$

$= \frac{5}{315}$

(3) Find, if possible, the value of $B\left(\frac{5}{2}, 6\right)$ if exist.

$$B\left(\frac{5}{2}, 6\right) = B\left(6, \frac{5}{2}\right) = \frac{5!}{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right)\left(\frac{5}{2}+3\right)\left(\frac{5}{2}+4\right)\left(\frac{5}{2}+5\right)}$$

$$= \frac{512}{43045}$$

(4) If m and n are positive integers then show that,

$$B(n, m) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$B(n, m) = \frac{(n-1)!}{m(m+1)(m+2)\cdots(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{m(m+1)(m+2)\cdots(m+n-1)\,(m-1)!}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)(m+n-2)\cdots(m+2)(m+1)\,m(m-1)!}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)!}$$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

(5) Express $\int_a^b (x-a)^m (b-x)^n\,dx$ in terms of Beta function.

Hence evaluate $\int_3^7 (x-3)^4 \sqrt[3]{(7-x)}\,dx$

$$= \int_a^b (x-a)^m (b-x)^n\,dx$$

$$= \int_0^{b-a} z^m \left(b-(z+a)\right)^n dz \quad \text{putting } z = x-a$$

$$= \int_0^{b-a} z^m (b-a-z)^n\,dz$$

$$= (b-a)^n \int_0^{b-a} z^m \left(1-\frac{z}{b-a}\right)^n dz$$

$$= (b-a)^n \int_0^1 (b-a)^m t^m (1-t)^n (b-a)\,dt \quad \text{putting } t = \frac{z}{b-a}$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n\,dt$$

$$= (b-a)^{m+n+1} B(m+1, n+1)$$

2nd part: $\int_3^7 (x-3)^4 \sqrt[3]{(7-x)}\,dx$

$$= \int_3^7 (x-3)^4 (7-x)^{\frac{1}{3}}\,dx$$

$$= (7-3)^{4+\frac{1}{3}+1} B\left(4+1, \frac{1}{3}+1\right)$$

$$= 4^{\frac{16}{3}} B\left(5, \frac{4}{3}\right)$$

$$= \frac{4!}{\frac{4}{3}\left(\frac{4}{3}+1\right)\left(\frac{4}{3}+2\right)\left(\frac{4}{3}+3\right)\left(\frac{4}{3}+4\right)}$$

⑥ From the recurrence relation $\Gamma(n+1) = n\Gamma(n)$ calculate $\Gamma(5)$

From the recurrence relation we get by putting $n=4$

$\Gamma(5) = 4\Gamma(4) = 4\times3\Gamma(3) = 4\times3\times2\Gamma(2)$

$= 4\times3\times2\times1\,\Gamma(1) = 4\times3\times2\times1$

$= 24$

⑦ Prove that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{(n-1)!}{k^n}$

where $k>0$ and n is a positive integer

Put $kx = z$

$\Rightarrow dx = \frac{1}{k} dz$

$x=0, \Rightarrow z=0$ and $x=\infty \Rightarrow z=\infty$

$\therefore \int_0^\infty e^{-kx} x^{n-1} dx = \int_0^\infty e^{-z} \frac{1}{k^{n-1}} z^{n-1} \cdot \frac{1}{k} dz$

$= \frac{1}{k^n} \int_0^\infty e^{-z} z^{n-1} dz$

$= \frac{1}{k^n} \Gamma(n) = \frac{1}{k^n} \Gamma\{(n-1)+1\}$

$= \frac{1}{k^n} \cdot (n-1)!$ [By using Property $\because n$ is a positive integer]

⑧ Prove that $\int_0^\infty e^{-x^2} x^9 dx = 12$

Put $x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow x\,dx = \frac{1}{2} dz$

$x=0 \Rightarrow z=0$ and $x=\infty, z=\infty$

$\therefore \int_0^\infty e^{-x^2} x^9 dx$

$= \int_0^\infty e^{-x^2} x^8 x\,dx$

$= \frac{1}{2}\int_0^\infty e^{-z} z^4 \, dz$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{5-1} dz$

$= \frac{1}{2}\Gamma(5)$

$= \frac{1}{2}\Gamma(4+1)$

$= \frac{1}{2}\times 4!$ $[\because \Gamma(n) = (n-1)!]$

$= \frac{1}{2}\times 24$

$= 12$

⑨ Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Let $z = x^2$

$\Rightarrow dz = 2x\,dx$

$\Rightarrow dx = \frac{1}{2}\frac{dz}{x}$

$= \frac{1}{2}\cdot\frac{dz}{\sqrt{z}}$

$\therefore \int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-z}\frac{dz}{\sqrt{z}}$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{2}\sqrt{\pi}$

$= \frac{\sqrt{\pi}}{2}$

not necessary

(10) Evaluate $\int_{-\infty}^{\infty} 5^{-x^2} dx$

$\int_{-\infty}^{\infty} 5^{-x^2} dx$

$= 2\int_0^\infty 5^{-x^2} dx$ [$\because 5^{-x^2}$ is even function]

$= 2\int_0^\infty e^{\log 5^{-x^2}} dx$

$= 2\int_0^\infty e^{-x^2 \log 5} dx$ —(1)

Put $x^2 \log 5 = z$

$\Rightarrow 2x\, dx \log 5 = dz$

$\Rightarrow dx = \frac{dz}{2x \log 5}$

$= \frac{dz}{2\frac{\sqrt{z}}{\sqrt{\log 5}}\log 5}$

$= \frac{\cancel{1}}{\cancel{2}\sqrt{z}\,(\cancel{\log 5})^{-\frac{1}{2}}(\cancel{\log 5})^{1}}$

$= \frac{\sqrt{\log 5}}{2\sqrt{z}\,\log 5}\cdot\frac{dz}{\sqrt{z}}$

$= \frac{1}{2\sqrt{z}\sqrt{\log 5}}\, dz$

$x=0, \Rightarrow z=0$ and $x=\infty, \Rightarrow z=\infty$

from (1)

$2\int_0^\infty e^{-z}\frac{1}{2\sqrt{\log 5}}\cdot\frac{dz}{\sqrt{z}}$

$\frac{1}{\sqrt{\log 5}}\int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$

$= \frac{1}{\sqrt{\log 5}}\int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$

$= \frac{1}{\sqrt{\log 5}}\Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{\sqrt{\log 5}}\sqrt{\pi}$

$= \sqrt{\frac{\pi}{\log 5}}$

(11) Show that $\int_a^b (x-a)^3(b-x)^2 dx = \frac{(b-a)^6}{60}$

Let $x = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow dx = (b-a)\, 2\sin\theta\cos\theta\, d\theta$

Now, $x - a = a\cos^2\theta + b\sin^2\theta - a$

$= b\sin^2\theta - a(1-\cos^2\theta)$

$= b\sin^2\theta - a\sin^2\theta = (b-a)\sin^2\theta$

and $b-x = (b-a)\cos^2\theta$

when $x=a$, $\theta=0$

$x=b$, $\theta=\frac{\pi}{2}$

$$\therefore \int_a^b (x-a)^3 (b-x)^2\, dx$$

$$= 2(b-a)^6 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta\, d\theta$$

$$= 2(b-a)^6 \int_0^{\frac{\pi}{2}} \sin^{2\cdot 4-1}\theta \sin\theta^{2\cdot 3-1}\theta\, d\theta$$

$$= 2(b-a)^6 \int_0^{\frac{\pi}{2}} 2\sin^{2\cdot 4-1}\theta \sin^{2\cdot 3-1}\theta\, d\theta$$

$$= (b-a)^6 B(4,3) \quad [\text{Using from (iii)}]$$

$$= (b-a)^6 \frac{\Gamma(4)\cdot\Gamma(3)}{\Gamma(7)} \quad [\text{Relation between } \Gamma \text{ and } \beta]$$

$$= (b-a)^6 \frac{\Gamma(4)\cdot 2!}{6\cdot 5\cdot 4\,\Gamma(4)}$$

$[\because \Gamma(2+1) = 2!$

$\Gamma(7) = \Gamma(6+1)$

$= 6\Gamma(6)$

$= 6\Gamma(5+1)$

$= 6\cdot 5\,\Gamma(5)$

$= 6\cdot 5\cdot 4\,\Gamma(4)]$

$$= \frac{2\times 1}{6\cdot 5\cdot \cancel{4}\,2}\cdot(b-a)^6$$

$$= \frac{1}{60}(b-a)^6$$

$$= \frac{(b-a)^6}{60}$$

(12) Show that $\int_0^{\pi/2} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^p x\, dx \times \int_0^{\frac{\pi}{2}} \sin^{p+1} x\, dx = \frac{\pi}{2(p+1)}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{p+1}{2}-1} x \cos^{2\cdot\frac{1}{2}-1} x\, dx$$

$$\times \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{p+2}{2}-1} x \cos^{2\cdot\frac{1}{2}-1} x\, dx$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) \times \frac{1}{2} B\left(\frac{p+2}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \times \frac{\Gamma\left(\frac{p+2}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)}$$

$$= \frac{1}{4}(\sqrt{\pi})^2 \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left\{\left(\frac{p+1}{2}\right)+1\right\}}$$

$$= \frac{1}{4}\cdot\pi\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)}{\frac{p+1}{2}\cdot\Gamma\left(\frac{p+1}{2}\right)}$$

$$= \frac{\pi}{\cancel{4}\,2}\cdot\frac{\cancel{2}}{p+1}$$

$$= \frac{\pi}{2(p+1)}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 2 \cdot \sin^2 \cdot \frac{}{2}$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) \times \frac{1}{2} B\left(\frac{p+2}{2}, \frac{1}{2}\right)$$

$$\Gamma\left(\frac{p+2}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)$$

$\theta d\theta$

$\theta d\theta$

prove (iii)

between

$d\beta$

$\Gamma(2+1) = 2!$

$\Gamma(7) = \Gamma(6+1) = 6\Gamma(6) = 6\Gamma(5+1) = 6\cdot 5\,\Gamma(5) = 6\cdot 5\cdot 4\,\Gamma(4)$

we know,

if $\int_a^b (x-a)^m (b-x)^n = (b-a)^{m+n+1} \cdot B(m+1, n+1)$

then $m=3, n=2$

$$= (b-a)^{3+2+1} \cdot B(4,3)$$

$$= (b-a)^6 \cdot \frac{3!}{3(3+1)(3+2)(3+3)}$$

$$= \frac{(b-a)^6}{60}$$

(13) Show that $\int_0^{\pi/2} d\theta / \cos^{1/2}\theta \, d\theta \times \int_0^{\pi/2}$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} \cdot \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{-\frac{1}{2}}\theta \, d\theta \times \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{1}{4}-1}\theta \, d\theta \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{3}{4}-1}\theta \, d\theta$$

$$= \frac{1}{4} B\left(\frac{1}{2}, \frac{1}{4}\right) \cdot B\left(\frac{1}{2}, \frac{3}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{4})}{\cancel{\Gamma(\frac{3}{4})}} \cdot \frac{\Gamma(\frac{1}{2}) \cdot \cancel{\Gamma(\frac{3}{4})}}{\Gamma(\frac{5}{4})}$$

$$= \frac{1}{4} \cdot \pi \cdot \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4}+1)}$$

$$= \frac{\pi}{4} \cdot \frac{\cancel{\Gamma(\frac{1}{4})}}{\frac{1}{4}\cancel{\Gamma(\frac{1}{4})}}$$

$$= \frac{\pi}{\cancel{4}} \times \cancel{4}$$

$$= \pi$$

(14) Using defination of Beta function evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$.

$$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{2}-1} x \cos^{2\cdot\frac{5}{2}-1} x \, dx$$

$$= \frac{1}{2} B\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{5}{2})}{\Gamma(3)}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2})}{\Gamma(2+1)}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{3}{4} \cdot \frac{\sqrt{\pi}}{2!}$$

$$= \frac{\pi}{2} \cdot \frac{3}{8}$$

$$= \frac{3\pi}{16}$$

(15) State the relation between Beta function and Gamma function and use it to show that

$$\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} \, dx = \frac{3\pi}{128}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx$$

$$= \int_0^1 x^{\frac{5}{2}-1} (1-x)^{\frac{5}{2}-1} dx$$

$$= B\left(\frac{5}{2}, \frac{5}{2}\right)$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma[5]}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma(4+1)}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{4!}$$

$$= \frac{3\,\cancel{9}\cdot\pi}{16\times\cancel{24}\,8}$$

$$= \frac{3\pi}{128}$$

(16) Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\cdots\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \cdots \Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\cdot\Gamma\left(\frac{5}{9}\right)\right\}$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(1-\frac{1}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(1-\frac{2}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\cdot\Gamma\left(1-\frac{3}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\cdot\Gamma\left(1-\frac{4}{9}\right)\right\}$$

$$= \frac{\pi}{\sin\frac{\pi}{9}}\cdot\frac{\pi}{\sin\frac{2\pi}{9}}\cdot\frac{\pi}{\sin\frac{3\pi}{9}}\cdot\frac{\pi}{\sin\frac{4\pi}{9}}$$

$$= \frac{\pi^4}{\sin\frac{\pi}{9}\cdot\sin\frac{2\pi}{9}\cdot\sin\frac{3\pi}{9}\cdot\sin\frac{4\pi}{9}}$$ [using duplication formula]

$$= \frac{\pi^4}{\frac{3}{16}}$$

$$= \frac{16\pi^4}{3}$$

(7) Evaluate $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}$

Put $x^6 = \sin^2\theta$

$\Rightarrow 6x^5 dx = 2\sin\theta\cos\theta\, d\theta$

$x=0 \Rightarrow \theta = 0$ and $x=1, \theta = \frac{\pi}{2}$

$$\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}$$

$$= \frac{1}{6}\int_0^1 \frac{6x^5}{x^5(1-x^6)^{\frac{1}{6}}}dx$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta\, d\theta}{\frac{1}{3}\sin\theta\cos\theta\,(1-\sin^2\theta)^{\frac{1}{6}}}$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}}$$

$\Rightarrow x^5 dx = \frac{1}{3}\sin\theta\cos\theta\, d\theta$

$$\Rightarrow dx = \frac{\frac{1}{3}\sin\theta\cos\theta\, d\theta}{x^5}$$

$$= \frac{\frac{1}{3}\sin\theta\cos\theta\, d\theta}{\sin^{\frac{5}{3}}\theta}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta\, d\theta}{\sin^{\frac{5}{3}}\theta\,(1-\sin^2\theta)^{\frac{1}{6}}}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta\, d\theta}{\sin^{\frac{5}{3}}\theta\cdot(\cos^2\theta)^{\frac{1}{6}}}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta\, d\theta}{\sin^{\frac{5}{3}}\theta\ \cos^{\frac{1}{3}}\theta}$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^{-\frac{2}{3}}\theta\cdot\cos^{\frac{2}{3}}\theta\, d\theta$$

$$= \frac{1}{3}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{6}-1}\theta\ \cos^{2\cdot\frac{5}{6}-1}\theta\, d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{6}-1}\theta\ \cos^{2\cdot\frac{5}{6}-1}\theta\, d\theta$$

$$= \frac{1}{6}\cdot B\left(\frac{1}{6},\frac{5}{6}\right)$$

$$= \frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\cdot\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{1}$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\frac{\pi}{6}}$$

$$= \frac{1}{6_3}\cdot 2\pi = \frac{\pi}{3}$$

(18) Show that $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$

$$\int_0^{\pi/2} \sqrt{\tan x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}}\, dx$$

$$= \int_0^{\pi/2} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 2\sin^{2\cdot\frac{3}{4}-1} x \cos^{2\cdot\frac{1}{4}-1} x\, dx$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{3}{4})\cdot\Gamma(\frac{1}{4})}{\Gamma(1)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{4})\cdot\Gamma(1-\frac{1}{4})}{1}$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2} \cdot \sqrt{2}$$

$$= \frac{\pi}{\sqrt{2}}$$

(19) Show that $(p+q) B(p+1, q) = p B(p, q)$

$$(p+q) B(p+1, q) = (p+q) \cdot \frac{\Gamma(p+1)\cdot\Gamma(q)}{\Gamma(p+q+1)}$$

$$= (p+q) \frac{p\,\Gamma(p)\,\Gamma(q)}{\Gamma((p+q)+1)}$$

$$= (p+q) \frac{p\,\Gamma(p)\cdot\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

$$= p \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$$

$$= p B(p, q)$$

(20) Evaluate $\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}}$

Put, $x^5 = \sin^2\theta$

$\Rightarrow 5x^4 dx = 2\sin\theta\cos\theta\, d\theta$

$$dx = \frac{2\sin\theta\cos\theta\, d\theta}{5x^3}$$

$$= \frac{2\sin\theta\cos\theta\, d\theta}{5(\sin\theta)^{\frac{6}{5}}}$$

$$\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}}$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{5\sqrt{1-\sin^2\theta}\,(\sin\theta)^{\frac{6}{5}}}$$

$$\int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta}{5\sin^{\frac{4}{5}}\theta\cos\theta}\,d\theta \qquad 1-\frac{4}{5}$$

$$= \frac{1}{5}\int_0^{\frac{\pi}{2}} 2\sin^{\frac{1}{5}}\theta\, d\theta$$

$$= \frac{1}{5}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{6}{10}-1}\theta\cos^{2\cdot\frac{1}{2}-1}\theta\, d\theta$$

$$= \frac{1}{5}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{6}{10}-1}\theta\cos^{2\cdot\frac{1}{2}-1}\theta\, d\theta$$

$$= \frac{1}{5}\, B\left(\frac{6}{10}, \frac{1}{2}\right)$$

$$= \frac{1}{5}\,\frac{\Gamma\left(\frac{3}{5}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(}$$

(21) Show that $\int_0^{\infty}\frac{dx}{(1+x^2)^5} = \frac{35\pi}{256}$

we put $x = \tan\theta$

$\therefore dx = \sec^2\theta\, d\theta$

$$\int_0^{\infty}\frac{dx}{(1+x^2)^5}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^5}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sec^2\theta\, d\theta}{\sec^{10}\theta}$$

$$= \int_0^{\frac{\pi}{2}}\sec^{-8}\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\frac{1}{\sec^8\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\cos^8\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{1}{2}-1}\theta\cos^{2\cdot\frac{9}{2}-1}\theta\, d\theta$$

$$= \frac{1}{2}\, B\left(\frac{1}{2}, \frac{9}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{4!}$$

$$= \frac{(\sqrt{\pi})^2}{2}\cdot\frac{7\times5\times3}{16\times24}\qquad 8 \qquad = \frac{35\pi}{256}$$

(22) Show that $\int_0^\infty e^{-x^2} x^9 dx = 12$

$\int_0^\infty e^{-x^2} x^9 dx = \int_0^\infty e^{-x^2} x^8 x dx$

$x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow x\,dx = \frac{1}{2} dz$

$\frac{1}{2}\int_0^\infty e^{-z} z^4 dz$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{5-1} dz$

$= \frac{1}{2} \Gamma(5)$

$= \frac{1}{2} \cdot 4!$

$= \frac{1}{2} \times 24$

$= 12$

(23) Prove that $\int_0^\infty e^{-x^4} x^2 dx \times \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$

$\int_0^\infty e^{-x^4} x^2 dx \times \int_0^\infty e^{-x^4} dx$

put, $x^4 = z$

$\Rightarrow 4x^3 dx = dz$

$\Rightarrow x^2 dx = \frac{1}{4x} dz$

$= \frac{dz}{4\sqrt[4]{z}}$

$4x^3 dx = dz$

$dx = \frac{dz}{4x^3}$

$= \frac{dz}{4(z)^{\frac{3}{4}}}$

$= \frac{1}{4}\int_0^\infty e^{-z} \frac{1}{\sqrt[4]{z}} dz \times \frac{1}{4}\int_0^\infty e^{-z} \frac{1}{(z)^{\frac{3}{4}}} dz$

$= \frac{1}{4}\int_0^\infty e^{-z} \cdot z^{-\frac{1}{4}} dz \times \frac{1}{4}\int_0^\infty e^{-z} \cdot z^{-\frac{3}{4}} dz$

$= \frac{1}{16}\int_0^\infty e^{-z} z^{\frac{3}{4}-1} dz \times \int_0^\infty e^{-z} z^{\frac{1}{4}-1} dz$

$= \frac{1}{16} \Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{4})$

$= \frac{1}{16} \cdot \Gamma(\frac{1}{4}) \cdot \Gamma(1 - \frac{1}{4})$

$= \frac{1}{16} \cdot \frac{\pi}{\sin \frac{\pi}{4}}$ [By duplication Prop.]

$= \frac{\pi}{16} \sqrt{2}$

$= \frac{\pi}{8\sqrt{2}}$

EXERCISE

(i) Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

done

(ii) Show that $B(\frac{1}{2}, \frac{1}{2}) = \pi$

done

(iii) Evaluate (a) $\Gamma(\frac{1}{4}) \Gamma(\frac{3}{4})$ (b) $\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})$

(a) $\Gamma(\frac{1}{4}) \Gamma(\frac{3}{4})$

$= \Gamma(\frac{1}{4}) \Gamma(1 - \frac{1}{4})$

$= \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{(1/\sqrt{2})} = \sqrt{2}\pi$

(b) $\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})$

$= \Gamma(\frac{1}{3}) \Gamma(1 - \frac{1}{3})$

$= \frac{\pi}{\sin \frac{\pi}{3}}$

$= \frac{\pi \times 2}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

(4) Show that $\int_0^{\pi/2} \sin^5\theta \, d\theta = \frac{16}{15}$

$\int_0^{\pi/2} \sin^5\theta \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} 2\sin^{2\cdot 3-1}\theta \cos^{2\cdot\frac{1}{2}-1}\theta \, d\theta$

$= \frac{1}{2} B\left(3, \frac{1}{2}\right)$

$= \frac{1}{2} \frac{\Gamma(3)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$

$= \frac{1}{2} \frac{\Gamma(2+1)\,\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$

$= \frac{1}{2} \cdot \frac{2! \times 2\times 2\times 2}{5\times 3}$

$= \frac{2\times 2\times 2}{15}$

$= \frac{16}{15}$

(5) Show that $\int_0^{\pi/2} \sin^4\theta \cos^4\theta \, d\theta = \frac{3\pi}{256}$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^4\theta\cos^4\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin^{2\cdot\frac{5}{2}-1}\theta \cos^{2\cdot\frac{5}{2}-1}\theta \, d\theta$

$= \frac{1}{2} B\left(\frac{5}{2}, \frac{5}{2}\right)$

$= \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}$

$= \frac{1}{2} \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4!}$

$= \frac{1}{2} \cdot \frac{9}{16} \cdot \frac{\pi}{24}$

$= \frac{3\pi}{256}$

(6) (a) $\int_0^\infty e^{-x^3} dx$

Put $x^3 = z$

$\Rightarrow 3x^2 \, dx = dz$

$\Rightarrow dx = \frac{dz}{3x^2}$

$= \frac{dz}{3z^{\frac{2}{3}}}$

$\frac{1}{3}\int_0^\infty e^{-z} \cdot z^{-\frac{2}{3}} \, dz$

$= \frac{1}{3}\int_0^\infty e^{-z} z^{\frac{1}{3}-1} \, dz$

$= \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$

$= \Gamma\left(\frac{1}{3}+1\right)$

$= \Gamma\left(\frac{4}{3}\right)$

(6) $\int_0^\infty e^{-x^4}\,dx$

Put $x^4 = z$

$4x^3\,dx = dz$

$\Rightarrow dx = \frac{dz}{4x^3}$

$= \frac{dz}{4z^{\frac{3}{4}}}$

$= \int_0^\infty e^{-z} \frac{dz}{4z^{\frac{3}{4}}}$

$= \frac{1}{4}\int_0^\infty e^{-z} z^{-\frac{3}{4}}\,dz$

$= \frac{1}{4}\int_0^\infty e^{-z} z^{\frac{1}{4}-1}\,dz$

$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$

$= \Gamma\left(\frac{1}{4}+1\right)$

$= \Gamma\left(\frac{5}{4}\right)$

(7) Show that $\int_0^\infty \frac{dx}{x\log x}$ is not convergent.

$\int_0^\infty \frac{dx}{x\log x}$

$= \lim_{x\to\infty} \int_0^x \frac{\frac{1}{x}dx}{\log x}$

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$= \lim_{x\to\infty} \left[\log|\log x|\right]_0^x$

$= \lim_{x\to\infty} \left[\log|\log x| - \log|\log 0|\right]$

$\log|\log x|$ (when $x\to\infty$) and $\log|\log 0|$ do doesn't exist so this improper integral is not convergent.

(8) Show that $\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2}$

$\int_0^\infty x^3 e^{-x^2}\,dx = \int_0^\infty x^2 e^{-x^2}\,x\,dx$

Put $x^2 = z$

$\Rightarrow 2x\,dx = dz$, $\Rightarrow x\,dx = \frac{dz}{2}$

$= \frac{1}{2}\int_0^\infty e^{-z}\cdot z^1\,dz$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{2-1}\,dz$

$= \frac{1}{2}\Gamma(2)$

$= \frac{1}{2}\Gamma(1+1)$

$= \frac{1}{2}\,1!$

$= \frac{1}{2}\times 1$

$= \frac{1}{2}$

(3) (i) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$

Put,

$x^3 = z$

$\Rightarrow 3x^2\, dx = dz$

$\Rightarrow dx = \dfrac{dz}{3z^{\frac{2}{3}}}$

$x = z^{\frac{1}{3}}$

$\sqrt{x} = (x)^{\frac{1}{2}} = (z^{\frac{1}{3}})^{\frac{1}{2}}$

$= z^{\frac{1}{6}}$

$\int_0^\infty \sqrt{x}\, e^{-x^3} dx$

$= \frac{1}{3}\int_0^\infty e^{-z}\, z^{-\frac{2}{3}}\, z^{\frac{1}{6}}\, dz$

$= \frac{1}{3}\int_0^\infty e^{-z}\, z^{-\frac{1}{2}}\, dz$

$= \frac{1}{3}\int_0^\infty e^{-z}\, z^{\frac{1}{2}-1}\, dz$

$= \frac{1}{3}\Gamma(\frac{1}{2})$

$= \frac{1}{3}\cdot\sqrt{\pi}$

$= \dfrac{\sqrt{\pi}}{3}$

(ii) $\int_0^\infty e^{-a^2x^2}\, dx$

Put,

$a^2x^2 = z$

$\Rightarrow x^2 = \frac{1}{a^2} z$

$\Rightarrow 2x\, dx = \frac{1}{a^2} dz$

$\Rightarrow dx = \dfrac{a}{2a^2}\cdot\dfrac{dz}{\sqrt{z}}$

$= \dfrac{1}{2a}\cdot\dfrac{dz}{\sqrt{z}}$

$x^2 = \frac{z}{a^2}$

$x = \frac{\sqrt{z}}{a}$

$\int_0^\infty e^{-a^2x^2}\, dx$

$= \frac{1}{2a}\int_0^\infty e^{-z}\, z^{-\frac{1}{2}}\, dz$

$= \frac{1}{2a}\int_0^\infty e^{-z}\, z^{\frac{1}{2}-1}\, dz$

$= \frac{1}{2a}\Gamma(\frac{1}{2})$

$= \dfrac{\sqrt{\pi}}{2a}$

(4) Evaluate $\int_{-\infty}^{\infty} 3^{-x^2} dx$

$\int_{-\infty}^{\infty} 3^{-x^2} dx$

$= 2\int_0^\infty 3^{-x^2} dx$ [$\because 3^{-x^2}$ is even function]

$= 2\int_0^\infty e^{\log 3^{-x^2}} dx$

$= 2\int_0^\infty e^{-x^2 \log 3}\, dx$ — (1)

put

$x^2 \log 3 = z$

$\therefore x^2 = \frac{z}{\log 3}$

$\Rightarrow 2x\,dx = \frac{1}{\log 3}\,dz$

$\therefore dx = \frac{\sqrt{\log 3}}{2\log 3} \cdot \frac{dz}{\sqrt{z}}$

$= \frac{1}{2\sqrt{\log 3}} \cdot \frac{dz}{\sqrt{z}}$

from (1),

$= 2 \cdot \frac{1}{2}\int_0^\infty e^{-z} \frac{dz}{\sqrt{z}\,\sqrt{\log 3}}$

$= \frac{1}{\sqrt{\log 3}}\int_0^\infty e^{-z} \frac{dz}{\sqrt{z}}$

$= \frac{1}{\sqrt{\log 3}}\int_0^\infty e^{-z} z^{-\frac{1}{2}}\,dz$

$= \frac{1}{\sqrt{\log 3}}\int_0^\infty e^{-z} z^{\frac{1}{2}-1}\,dz$

$= \frac{1}{\sqrt{\log 3}} \cdot \Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{\sqrt{\log 3}}$

$= \sqrt{\frac{\pi}{\log 3}}$

(1) (12) done

(asymptotes - tangents at infinity)

• **FOURIER SERIES** (Nsm)

period of $\sin 2x$ — $\sin x$ is 2π

$2x = 2\pi$

$\therefore x = \pi$

period of $\sin 2x$ is π

$\sin \frac{7x}{3}$

$\frac{7x}{3} = 2\pi$

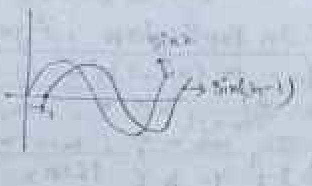
$\therefore x = \frac{6\pi}{7}$

period of $\sin \frac{7x}{3}$ is $\frac{6\pi}{7}$

Period of $\tan x = \pi$

Approximate a periodic function which is remain integrable — fourier series basic concept

$y = \sin(x-1)$ → graph →



• fourier series

odd function এর ক্ষেত্রে origin এর চারদিকে same হয়।

even function এর ক্ষেত্রে y-axes এর চারদিকে same হয়।

$\int_{-T}^{T} \cos\left(\frac{m\pi x}{T}\right) \cos\left(\frac{n\pi x}{T}\right) dx = 0 \quad m \neq n$

$\int_{-T}^{T} \sin\left(\frac{m\pi x}{T}\right) \sin\left(\frac{n\pi x}{T}\right) dx = 0, \quad m \neq n$

$\int_{-\pi}^{\pi} \cos\left(\frac{m\pi x}{T}\right) \sin\left(\frac{n\pi x}{T}\right) dx = 0$

2

- Successive Differentiation (S.8th)
- nth derivative of $y=(ax+b)^m$, m is any number

$y_1 = m(ax+b)^{m-1}a$

$y_n = m(m-1)(m-2)\cdots(m-n+1)(ax+b)^{m-n}a^n$ when $m>n$

$= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

$= \frac{n!}{0!}a^n$ when $m=n$

$= n!\,a^n$

$= 0$, when $m<n$ and m is a +ve integer

$y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1}a$

$y_2 = m(m-1)(ax+b)^{m-2}a^2$

$\therefore$ In particular, if $y=x^m$, then

$y_n = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$ when $m>n$, m is any no.

$y_n = \frac{m!}{(m-n)!}x^{m-n}$, $m>n$, n is +ve integer

$= n!$, when $m=n$; $= 0$, $m<n$ and m, n +ve integer

① If $y = x^{\frac{1}{2}}$ then,

$y_n = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)x^{\frac{1}{2}-n}$

$= (-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-3)}{2^n}x^{\frac{1}{2}-n}$

② $y = \frac{1}{x} = x^{-1}$

$y_1 = -1x^{-2} = -x^{-2}$

$y_2 = 2x^{-3}$

$y_3 = -6x^{-4}$

$y_n = (-1)^n n!\,x^{-(n+1)}$

$y_n = \frac{(-1)^n n!}{x^{n+1}}$

n-th derivative of $y = \frac{1}{ax+b}$

- $y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

i) If $y = \frac{1}{2x+3}$ $a=2$, $b=3$

$y_n = \frac{(-1)^n n!\,2^n}{(2x+3)^{n+1}}$

ii) $y = \frac{1}{x-5}$ $a=1$, $b=-5$

$y_n = \frac{(-1)^n n!\,1^n}{(x-5)^{n+1}}$

- n-th derivative of $y = e^{ax+b}$

$y = e^x$

$y_1 = e^x$, $y_2 = e^x$, ——, $y_n = e^x$

$y = e^{ax+b}$

$y_1 = a e^{ax+b}$, $y_2 = a^2 e^{ax+b}$

$\boxed{y_n = a^n e^{ax+b}}$

- Note: $y = a^{mx} = e^{\log a^{mx}}$
$= e^{mx \cdot \log a}$
$= e^{(m \log a) x}$

$\therefore y_n = (m \log a)^n e^{mx \log a}$

$\therefore \boxed{y_n = m^n (\log a)^n a^{mx}}$

(i) If $y = e^{3x}$

$y_n = 3^n e^{3x}$

(ii) If $y = 5^{-3x}$

$y_n = (-3)^n (\log 5)^n 5^{-3x}$

- n-th derivative of $y = \log(ax+b)$

$y = \log x$

$y_1 = \frac{1}{x}$, $y_2 = -\frac{1}{x^2}$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}}$$

<nth derivative of $\log x$

$y = \log(ax+b)$

$y_1 = \frac{1}{ax+b} \cdot a$

$y_2 = (-1)(ax+b)^{-2} \cdot a^2$

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}$$

(i) $y = \log \frac{x+1}{x-1}$

$= \log(x+1) - \log(x-1)$

1st $a=1, b=1$
2nd $a=1, b=-1$

$y_n = \frac{(-1)^{n-1}(n-1)!\,1^n}{(x+1)^n} - \frac{(-1)^{n-1}(n-1)!\,1^n}{(x-1)^n}$

- nth derivative of $y = \sin(ax+b)$

$y = \sin(ax+b)$

$y_1 = a\cos(ax+b)$

$= a\sin(\frac{\pi}{2} + ax+b)$

$y_2 = a^2 \cos(\frac{\pi}{2} + ax + b)$

$= a^2 \sin(2 \cdot \frac{\pi}{2} + ax+b)$

$$\boxed{y_n = a^n \sin(n\frac{\pi}{2} + ax+b)}$$

(i) If $y = \sin^3 x$

$y = \sin^3 x$

$= \frac{1}{4}(4\sin^3 x)$

$= \frac{1}{4}(3\sin x - \sin 3x)$

1st $a=1, b=0$
$a=3, b=0$

$y_n = \frac{3}{4}\sin(\frac{n\pi}{2} + x) - \frac{1}{4} \cdot 3^n \cdot \sin(\frac{n\pi}{2} + 3x)$

(iii) nth derivative of $y = \cos(ax+b)$

$$\boxed{y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)}$$

(i) If $y = \cos^2 6x$, then

$y = \cos^2 6x$

$\Rightarrow y = \frac{1}{2}\, 2\cos^2 6x$

$= \frac{1}{2}(1 + \cos 12x)$ $a = 12, b = 0$

$y_n = \frac{1}{2}(12)^n \cos\left(\frac{n\pi}{2} + 12x\right) + \frac{1}{2}$

• nth derivative of $y = e^{ax} \sin bx$

$$\boxed{y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right)}$$

(ii) If $y = e^{2x} \sin x$ then $a = 2, b = 1$

$y_n = (2^2+1^2)^{\frac{n}{2}} e^{2x} \sin\left(x + n\tan^{-1}\frac{1}{2}\right)$

$= (5)^{\frac{n}{2}} e^{2x} \sin\left(x + n\tan^{-1}\frac{1}{2}\right)$

• nth derivative of $y = e^{ax} \cos bx$

$$\boxed{y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right)}$$

(i) If $y = e^x \cos^2 3x$, then

$y = \frac{1}{2} e^x (1 + \cos 6x)$ $a = 1, b = 2$

$y_n = \frac{1}{2} e^x + \frac{1}{2}(1^2 + 6^2)^{\frac{n}{2}} e^x \cos\left(6x + n\tan^{-1}\frac{6}{1}\right)$

$y_n = \frac{1}{2} e^x + \frac{1}{2}(37)^{\frac{n}{2}} e^x \cos\left(6x + n\tan^{-1} 6\right)$

(i) Find y_n where $y = 10^{5-3x}$

$y = 10^{5-3x}$

$= e^{(5-3x)\log 10}$

$= e^{5\log 10} \cdot e^{-3x \log 10}$

$y_n = e^{(5-3x)\log 10} \cdot (-1)^n 3^n (\log 10)^n$

$= (-1)^n 3^n (\log 10)^n \cdot 10^{5-3x}$

(i) If $y = \frac{x^n}{x-1}$, then find y_n

$(x^n - 1) = (x-1)(1 + x + x^2 + \dots + x^{n-1})$

$y = \frac{x^n}{x-1} = \frac{x^n - 1}{x-1} + \frac{1}{x-1}$

$= x^{n-1} + x^{n-2} + \dots + x + 1 + \frac{1}{x-1}$

$$y_n = 0+0+\frac{(-1)^n n!}{(x-1)^{n+1}}$$

$$= \frac{(-1)^n n!}{(x-1)^{n+1}}$$

$\left[\begin{array}{l} y=(ax+b)^m \\ y_n = 0 \text{ if } n>m \end{array}\right]$

$\left[y = \frac{1}{ax+b} \right] = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

③ If $y = \sin x \sin 2x \sin 3x$, find y_n

$$y = \frac{1}{2} 2\sin x \sin 2x \sin 3x$$

$$= \frac{1}{2}(2\sin x \sin 2x)\sin 3x$$

$$= \frac{1}{2}(\cos x - \cos 3x)\sin 3x$$

$$= \frac{1}{4}(2\cos x \sin 3x - 2\sin 3x \cos 3x)$$

$$= \frac{1}{4}(\sin 4x + \sin 2x - \sin 6x)$$

$$y_n = \frac{1}{4}\left\{4^n \sin\left(\frac{n\pi}{2}+4x\right) + 2^n \sin\left(\frac{n\pi}{2}+2x\right) - 6^n \sin\left(\frac{n\pi}{2}+6x\right)\right\}$$

Imp

④ If $y = \frac{1}{x^2+a^2}$, find y_n

$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$\therefore y_n = \frac{1}{2ia}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right], \text{ by } ⓐ$$

$$= \frac{(-1)^n n!}{2ia}\left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right] \text{ — ①}$$

Put $x = r\cos\theta$, $a = r\sin\theta$

$r = \sqrt{x^2+a^2}$

$\theta = \tan^{-1}\frac{a}{x}$

$$(x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$$

$$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$$

[by De Moivre's theorem on complex number]

Similarly, $(x+ia)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$

from (1), $y_n = \frac{(-1)^n n!}{2i\,a} r^{-(n+1)} \cdot 2i\sin(n+1)\theta$

$$= \frac{(-1)^n n!}{a}\left(\frac{a}{\sin\theta}\right)^{-(n+1)} \sin(n+1)\theta$$

$$\therefore y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta, \text{ where} \qquad [\because a = r\sin\theta]$$

$$\theta = \tan^{-1}\left(\frac{a}{x}\right)$$

⑤ Find the n-th derivative of $\frac{x^2+1}{(x-1)(x-2)(x-3)}$

Let $y = \frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Putting, $x = 1, 2, 3$ we get $A=1, B=-5, C=5$

$$y = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$$

$$y_n = (-1)^n n! \left\{ \frac{1}{(x-1)^{n+1}} - \frac{5}{(x-2)^{n+1}} + \frac{5}{(x-3)^{n+1}} \right\}$$

- Fourier series (A Sir)

- Fourier series of Euler's formula

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right)}$$

where a_n, b_n are called fourier co-efficients and these are, according to Euler.

$$a_0 = \frac{1}{T} \int_{-T}^{T} f(x)\, dx$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos \frac{n\pi x}{T}\, dx$$

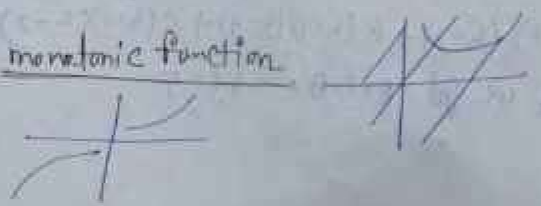
$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{n\pi x}{T}\, dx$$

$[n = 0, 1, 2, 3 - \infty]$

- Gibbes formula, fourier series convert to that

$$\frac{f(x_0^+) + f(x_0^-)}{2}$$

- monotonic function



*Even function and odd function হওয়ার even function-এ $a_n / b_n = 0$ হয়ে যাবে, কিন্তু ভিতরে হবে না odd function এর a_0 'O' হবে।

- Half range Series (Half range cosine or half range odd Fourier Series)

- odd extension ($a_0 = 0, a_n = 0$)

- Parseval's Theorem

$$\boxed{\frac{1}{T} \int_{-T}^{T} \{f(x)\}^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$

- Periodic functions and its properties
- A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x)$ for all values of x.

approximation যদি even হয় cos নিয়ে আসা হবে আর sin odd হয় তখন sine নিয়ে আসা হবে। odd function $a_0, a_n = 0$ । even function $b_n = 0$

- Typical waveform

The graph of every periodic function runs like a wave - this is wave form

- Fourier series: Property

① There has no guarantee that every function $f(x)$ equals its fourier series. In the subsequent article we shall discuss the conditions under which this equality would hold good.

② Since T can assume any value so this fourier series can be regarded as general fourier series.

Illustration

Consider the function $f(x) = 3, \; 0 \le x \le 5$
$= -3, \; -5 < x < 0$

we extend the function by defining $f(x+10) = f(x)$ for all x. So this becomes a periodic function of period 10. This gives a square waveform. Its fourier co-efficients, according to Euler formula, are

$$a_0 = \frac{1}{5}\int_{-5}^{5} f(x)\,dx = \frac{1}{5}\left[\int_{-5}^{0} f(x)\,dx + \int_{0}^{5} f(x)\,dx\right]$$

$$= \frac{1}{5}\left\{-3\int_{-5}^{0} dx + 3\int_{0}^{5} dx\right\}$$

$$= \frac{1}{5}\left[-3 \times 5 + 3 \times 5\right]$$

$$= 0 \qquad [\because \text{it is an odd function and } f(x) = 3,\; f(-x) = -f(x) = -3]$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx = 0 \quad \left[\because \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5} \ldots \; \because f(x)\cos\frac{n\pi x}{5} = \text{odd function}\right]$$

$$= \frac{1}{5}\left[-3\int_{-5}^{0}\cos\frac{n\pi x}{5}\,dx + 3\int_{0}^{5}\cos\frac{n\pi x}{5}\right]$$

$$= \frac{1}{5}\left[-3\left\{\frac{\sin\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right\}_{-5}^{0} + 3\left\{\frac{\sin\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right\}_{0}^{5}\right]$$

$$= \frac{1}{5}[0]$$

$$= 0 \qquad [\because \sin n\pi = 0]$$

and
$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left[-3\int_{-5}^{0}\sin\frac{n\pi x}{5}\,dx + 3\int_{0}^{5}\sin\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[-3\left\{\frac{\cos\frac{n\pi x}{5} \times 5}{n\pi}\right\}_{-5}^{0} + -3\left\{\frac{\cos\frac{n\pi x}{5} \times 5}{n\pi}\right\}_{0}^{5}\right]$$

$$= \frac{1}{5}\left[\frac{15}{n\pi}\left\{\cos\frac{n\pi x}{5}\right\}_{-5}^{0} - \frac{15}{n\pi}\left\{\cos\frac{n\pi x}{5}\right\}_{0}^{5}\right]$$

$$= \frac{1}{5} \times \frac{15}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right]$$

and
$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5} \times 2\int_{0}^{5} f(x)\sin\frac{n\pi x}{5}\,dx \quad \left[\because f(x)\sin\frac{n\pi x}{5} \text{ is even function}\right]$$

$$= \frac{2}{5} \times 3\int_{0}^{5}\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{-6}{5}\left[\frac{\cos\frac{n\pi x}{5} \times 5}{n\pi}\right]_{0}^{5}$$

$$= -\frac{6}{5} \cdot 5 \cdot \frac{1}{n\pi}\left[\cos\frac{n\pi x}{5}\right]_0^5$$

$$= -\frac{6}{n\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{6}{n\pi}\left(1 - \cos n\pi\right)$$

Therefore the fourier series of $f(x)$ is

$$\frac{0}{2} + \sum_{n=1}^{\infty}\left(0 \cdot \cos\frac{n\pi x}{5} + \frac{6(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}\right)$$

i.e. $$\sum_{n=1}^{\infty}\frac{6}{\pi}\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{5}$$

i.e $$\frac{6}{\pi}\left\{\frac{(1-\cos\pi)}{1}\sin\frac{\pi x}{5} + \frac{(1-\cos 2\pi)}{2}\sin\frac{2\pi x}{5} + \frac{(1-\cos 3\pi)}{3}\sin\frac{3\pi x}{5} \cdots\right\}$$

we see $f(0) = -3$, but the values of fourier series at $x=0$

is $\frac{6}{\pi}\{0+0+0 \cdots\} = 0$

- **Fourier series of a function of Period 2π**

The above fourier series for $l=\pi$, i.e. the fourier series for the function $f(x)$ defined and integrable on $(-\pi,\pi)$ and $f(x+2\pi) = f(x)$ for all values of x, is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{\pi} + b_n\sin\frac{n\pi x}{\pi}\right)$$

$$\boxed{= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)}$$

where the fourier co-efficients are,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n=1,2,3\ldots$$

Illustration

consider the function $f(x) = x^2$, $-\pi < x \le \pi$

the function is defined on the interval $(-\pi,\pi)$;

we extend this by defining $f(x+2\pi) = f(x)$ for all values of x. this is a Periodic function $f(x)$ of Period 2π.

Its fourier co-efficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx \quad [\because x^2 \text{ is an even function}]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \quad [\because x^2\cos nx \text{ is an even function}]$$

$$= \frac{2}{\pi}\left\{\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} - 2\int_0^{\pi} x\frac{\sin nx}{n}\,dx\right\}$$

$= \frac{-4}{n\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\frac{\cos nx}{n}\right\}$

$= \frac{-4}{n\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$

$= \frac{-4}{n\pi}\left\{-\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$

$= \frac{4\cos n\pi}{n^2}$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$= 0$ [$\because f(x)\sin nx$ is an odd function]

So, the fourier series of $f(x)$ is

$\frac{1}{2}\cdot\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4\cos n\pi}{n^2}\cos nx + 0\cdot\sin nx\right)$

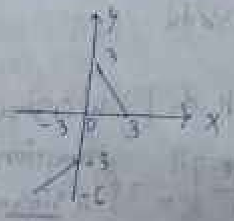
i.e $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos n\pi \cos nx}{n^2}$

- Dirichlet's conditions

Illustrations

(i) Let $f(x) = x-3,\ -3 \le x \le 0$
$\quad = 3-x,\ 0 \le x \le 3$ [no need to draw the graph]

Let us draw the graph of $f(x)$ below:



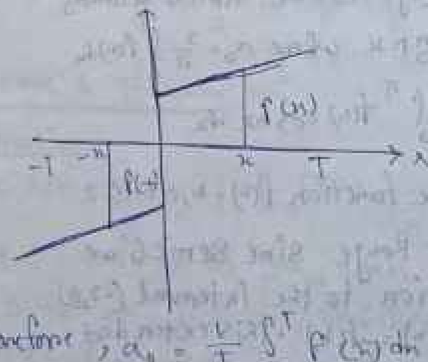
From the graph we see, $-6 \le f(x) \le 3$ i.e $f(x)$ is bounded in $[-3, 3]$.

The interval $[-3, 3]$ is decomposed as $[-3,3] = [-3,0] \cup [0,3]$ such that $f(x)$ is increasing in $[-3,0]$ and decreasing in $[0,3]$, so we conclude this function $f(x)$ satisfies Dirichlet's condition.

(ii) Let $f(x)$

- Construction of Half Range Sine Series

Let $f(x)$ be a function defined and integrable on the interval $(0, T)$. We extend the domain of definition to $[-T, 0]$ defining by $f(-x) = -f(x)$. This extension is shown in the adjacent figure. Then this extended $f(x)$ becomes odd function in the interval $[-T, T]$



Therefore, $a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\, dx$

$= 0$ [$\because f(x)$ is odd]

$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\, dx$

$= 0$ [$\because f(x)\cos\frac{n\pi x}{T}$ is odd function] and $b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx$

$= \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}\,dx$ $\left[\because f(x)\sin\frac{n\pi x}{T} \text{ is even function}\right]$

$\therefore$ The fourier series of $f(x)$ becomes

$$\frac{0}{2} + \sum_{n=1}^{\infty}\left(0\cos\frac{n\pi x}{T} + b_n\sin\frac{n\pi x}{T}\right)$$

i.e $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T}$

which is the required Half Range Sine series obviously if $f(x)$ satisfies dirichlet's condition in $[0,T]$ then this series is convergent and the value is as for (full range) fourier series.

(2) The Half Range Cosine series becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Illustration

(i) Consider the function $f(x) = x,\ 0 < x \le 2$

To get it's Half Range Sine Series we extend the function to the interval $(-2,2)$ defining by $f(-x) = -f(x)$. This extended function becomes odd.

Therefore $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = 0$ $[\because f(x) \text{ is odd function}]$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = 0$ $\left[\because f(x)\cos\frac{n\pi x}{2} \text{ is odd function}\right]$

and $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$

$= \int_0^2 x\sin\frac{n\pi x}{2}\,dx$ $\left[\because f(x)\sin\frac{n\pi x}{2} \text{ is even function}\right]$

$= \left[\frac{-2}{n\pi}x\cos\frac{n\pi x}{2}\right]_0^2 - \int_0^2 (-)\frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx$

$= \frac{4(-1)^{n+1}}{n\pi}$

Hence the Half Range Sine Series of the function is

$$\frac{0}{2} + \sum_{n=1}^{\infty}\left(0\cos\frac{n\pi x}{2} + \frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}\right)$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots \text{ upto } \infty\right]$$

- **Parseval's Theorem**

Illustration

Consider the function $f(x) = \begin{cases} -x & -2 < x \le 0 \\ x & 0 \le x \le 2 \end{cases}$

We see $f(x)$ is an even function. Extending this to an periodic function defining by $f(x+4) = f(x)$ we get the graph as follow

← [no need]

Here, $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$= \frac{1}{2}\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx$

$= 2$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$

$= \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$ [$\because f(x)\cos\frac{n\pi x}{2}$ is even function]

$= \left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 1\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$

$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$ for $n \neq 0$

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = 0$

[$\because f(x)\sin\frac{n\pi x}{2}$ is an odd function]

So its Parseval's identity is

$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2\,dx = \frac{2^2}{2} + \sum_{n=1}^{\infty}\left[\left\{\frac{4}{n^2\pi^2}(\cos n\pi - 1)\right\}^2 + 0^2\right]$

$\Rightarrow \frac{1}{2}\left[\int_{-2}^{0}(-x)^2\,dx + \int_{0}^{2} x^2\,dx\right] = 2 + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi - 1)^2$

$\Rightarrow \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$

$\Rightarrow \frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$

$\boxed{\begin{aligned}&(\cos n\pi - 1)^2\\ &= (1 - \cos n\pi)^2\\ &= \left(2\sin^2\frac{n\pi}{2}\right)^2\\ &= 4\sin^4\frac{n\pi}{2}\end{aligned}}$

$\Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$, an interesting result is obtained

20.9 Illustrative Examples

① Find a Fourier series of the function $f(x) = x - x^2$, $-\pi < x \leq \pi$

Hence find the value of the series

Sol^n: The function is defined on $(-\pi, \pi]$. Primarily we extend by defining outside as $f(x+2\pi) = f(x)$. It becomes a periodic function of period 2π.

The Fourier co-efficients are

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = -\frac{2\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$

$= 0 - \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$

$= -\frac{2}{\pi}\left\{\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} 2x\frac{\sin nx}{n}\,dx\right\}$

$= \frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx$

$= \frac{4}{n\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nx\,dx\right\}$

$= \frac{4}{n\pi}\left\{-\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$

$= \frac{-4(-1)^n}{n^2} = \frac{4(-1)^{n+1}}{n^2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\sin nx\, dx$

$= \frac{1}{\pi}\left\{\left[\frac{-(x-x^2)\cos nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1-2x)\frac{-\cos nx}{n}dx\right\}$

$= -\frac{1}{\pi}\left\{\frac{(\pi-\pi^2)\cos n\pi}{n} - \frac{(-\pi-\pi^2)\cos n\pi}{n}\right.$

$\left. - \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\, dx\right\}$

$= \frac{2(-1)^{n+1}}{n}$

So the fourier series of the given function is,

$\frac{1}{2}\left(-\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right\}$

$= -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

f(x) satisfies Dirichlet's condition.

since the function is continuous in $(-\pi, \pi)$ so

$x - x^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx +$

$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Putting $x=0$, we get $x = 0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} +$

$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times 0$

$\Rightarrow \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$

$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

- successive derivative (S sin)
- Leibnitz's Theorem

$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_{n-1} u_1 v_{n-1} + u v_n$

$= \sum_{r=0}^{n} {}^nC_r u_{n-r} v_r$

(we suppose $u_0 = u$ and $v_0 = v$)

Ex:-

① If $y = x^n \sin x$, then find y_n

Let $u = \sin x$, $v = x^2$

$u_n = \sin\left(\frac{n\pi}{2} + x\right)$ and $v_1 = 2x$, $v_2 = 2$, $v_3 = 0$ — $v_n = 0$

Therefore by Leibnitz's theorem

$y_n = (uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots$

$\therefore y_n = \sin\left(\frac{n\pi}{2} + x\right)x^2 + {}^nC_1 \sin\left\{(n-1)\frac{\pi}{2} + x\right\}2x +$

${}^nC_2 \sin\left\{(n-2)\frac{\pi}{2} + x\right\}\cdot 2$

$$= n^2 \sin\left(\frac{n\pi}{2} + x\right) + 2nx \sin\left\{(n-1)\frac{\pi}{2} + x\right\} + n(n-1)\sin\left\{(n-2)\frac{\pi}{2} + x\right\}$$

$$\left[{}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n \right.$$

$$\left. {}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2} \right]$$

② Find the n-th derivative of $e^x \log x$.

$y = e^x \log x$, $u = e^x$, $v = \log x$

$\therefore u_n = e^x$, $v_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

Therefore by Leibnitz's theorem

$$y_n = e^x \cdot \log x + {}^nC_1\, e^x \cdot \frac{1}{x} + {}^nC_2\, e^x\left(-\frac{1}{x^2}\right) + \cdots + \frac{e^x(-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x\left\{\log x + nx^{-1} - {}^nC_2\, x^{-2} + \cdots + (-1)^{n-1}(n-1)!\, x^{-n}\right\}$$

③ If $y = (\sin^{-1} x)^2$, then show that

(i) $(1-x^2)y_2 - xy_1 - 2 = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

(i) $(1-x^2)y_2 - xy_1 - 2 = 0$

$y = (\sin^{-1} x)^2$

$\Rightarrow \frac{dy}{dx} = 2\sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_1^2 \cdot (1-x^2) = 4(\sin^{-1} x)^2$

$\Rightarrow y_1^2(1-x^2) = 4y$

$\Rightarrow y_1^2(-2x) + (1-x^2)\cdot 2y_1 y_2 = 4y_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0$

(ii) Taking derivative of order n on both sides of (1) we get

$[(1-x^2)y_2]_n - [xy_1]_n - [2]_n = 0$

$\Rightarrow$ ~~$(1-x^2)y_{n+2} + {}^nC_1$~~

$[(1-x^2)y_2]_n$

$v = 1-x^2$, $v_1 = -2x$, $v_2 = -2$, $v_3 = v_4 \cdots = 0$

$u = y_2$

nth derivative of y_2 is y_{n+2} not y_{2n}

$\Rightarrow (1-x^2)y_{n+2} + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)$

~~$-\{y_{n+1}x + {}^nC_1\, y_n\}$~~ $- 0 = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$

$- xy_{n+1} - n y_n$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

(4) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$

$\Rightarrow (y^{\frac{1}{m}})^2 + \frac{1}{y^{\frac{1}{m}}} = 2x$

$\Rightarrow (y^{\frac{1}{m}})^2 + 1 = 2x \cdot y^{\frac{1}{m}}$

$\Rightarrow (y^{\frac{1}{m}})^2 - 2xy^{\frac{1}{m}} + 1 = 0$

$\Rightarrow (y^{\frac{1}{m}})^2 - 2xy^{\frac{1}{m}} + 1 = 0$

by श्रीधर आचार्य

$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2-4}}{2}$

$\Rightarrow y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$

$\Rightarrow y = (x \pm \sqrt{x^2-1})^m$

$y_1 = \cancel{m(x \pm \sqrt{x^2-1})^{m-1}(x \pm \sqrt{x^2-1})}$

$= m(x \pm \sqrt{x^2-1})^{m-1}\left(1 \pm \frac{1}{2}\frac{1}{\sqrt{x^2-1}} \cdot 2x\right)$

$= \pm \frac{m(x \pm \sqrt{x^2-1})^m}{\sqrt{x^2-1}}$

$= \pm \frac{my}{\sqrt{x^2-1}}$

$\therefore y_1^2(x^2-1) = m^2y^2$

$\Rightarrow y_1^2 \cdot 2x + (x^2-1) \cdot 2y_1y_2 = m^2 \cdot 2yy_1$

$\Rightarrow xy_1 + (x^2-1)y_2 = m^2y$

$\Rightarrow (x^2-1)y_2 + xy_1 - m^2y = 0$

Now diff. n times w.r.t. x by Leibnitz's theorem, we get

$y_{n+2}(x^2-1) + {}^nc_1 \cdot y_{n+1} \cdot 2x + {}^nc_2 \cdot y_n \cdot 2 + y_{n+1}x + {}^nc_1 \cdot y_n \cdot 1 - m^2y_n = 0$

$\therefore (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Ex-5,6,7,9,10 (5/1/22)

(5) If $y = x^{n-1}\log x$, then prove that $y_n = \frac{(n-1)!}{x}$

$y = x^{n-1}\log x$

Differentiating w.r.t x

$y_1 = (n-1)x^{n-2}\log x + x^{n-1} \cdot \frac{1}{x}$

$\Rightarrow xy_1 = (n-1)x^{n-1}\log x + x^{n-1}$

$\Rightarrow xy_1 = y(n-1) + x^{n-1}$

Now differentiating $(n-1)$ times w.r.t x, by Leibnitz's theorem,

$[y_1x]_{n-1} = (n-1)y_{n-1} + [x^{n-1}]_{n-1}$

$\Rightarrow y_n x + {}^{n-1}C_1 y_{n-1} \cdot 1 = (n-1) y_{n-1} + (n-1)!$

$\Rightarrow x y_n + (n-1) y_{n-1} = (n-1) y_{n-1} + (n-1)!$

$\Rightarrow y_n = \dfrac{(n-1)!}{x}$ $\left[\because {}^{n-1}C_1 = \dfrac{(n-1)!}{1!(n-2)!} = \dfrac{(n-1)(n-2)!}{(n-2)!} = (n-1) \right]$

⑥ If $f(x) = \tan x$, prove that

$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0)$

$- - - - - - = \sin \dfrac{n\pi}{2}$

Given $f(x) = \tan x$

$\Rightarrow f(x) \cos x = \sin x$

Differentiating n times w.r.t x, by Leibnitz's theorem.

$y = f(x) \cos x$

let $u = f(x)$ $v = \cos x$

$y_n = f^n(x) \cos x + {}^nC_1 \cdot f^{n-1}(x) \{-\sin x\}$

$+ {}^nC_2 \cdot f^{n-2}(x) \{-\cos x\} + {}^nC_3 \cdot f^{n-3}(x) \cdot$

$\{\sin x\} + {}^nC_4 \cdot f^{n-4}(x) \cdot \cos x$

$= f^n(x) \cos x - n f^{n-1}(x) \sin x - \dfrac{n(n-1)}{2} f^{n-2}(x) \cos x$

$+ {}^nC_3 f^{n-3}(x) \sin x + {}^nC_4 f^{n-4}(x) \cos x$

$(\sin x)_n = \sin\left(\dfrac{n\pi}{2} + x\right)$

Putting $x=0$ in both side we gets

$\Rightarrow f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}$

$= \sin \frac{1}{2} n\pi$

⑦ If $y = e^{m \sin^{-1} x}$, show that

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + m^2) y_n = 0$

Also find y_n when $x=0$

$y = e^{m \sin^{-1} x}$

$\Rightarrow y_1 = e^{m \sin^{-1} x} \cdot \dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow y_1^2 (1-x^2) = m^2 y^2$

$\Rightarrow y_1^2 (-2x) + (1-x^2) 2 y_1 y_2 = m^2 2 y y_1$

$\Rightarrow (1-x^2) y_2 - x y_1 - m^2 y = 0$ — (1)

Now differentiating w.r.t x, n times, by Leibnitz's theorem,

$[\underset{(v)}{(1-x^2)} \underset{(u)}{y_2}]_n - [\underset{(v)}{x} y_1]_n - [m^2 y]_n$

$= y_{n+2} (1-x^2) - n \cdot y_{n+1} 2x - 2 \cdot \dfrac{n(n-1)}{2} \cdot y_n$

$- \{ y_{n+1} x + n \cdot y_n 1 \} - m^2 y_n$

$= (1-x^2) y_{n+2} - 2nx\, y_{n+1} - n(n-1) y_n$

$- x y_{n+1} - n y_n - m^2 y_n$

$= (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+m^2) y_n$

$\therefore (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+m^2) y_n = 0$ — (3)

now

$(y)_0 = e^{m \cdot 0} = 1$

$(y_1)_0 = \left(\frac{my}{\sqrt{1-x^2}}\right)_0$

$= \frac{m \cdot y_0}{\sqrt{1-0^2}}$

$= \frac{m \cdot 1}{1}$

$= m$

From (1), by putting $x = 0$

$(y_2)_0 - 0 - m^2 (y)_0 = 0$

$\Rightarrow (y_2)_0 = m^2$

now putting $x = 0$ in (3)

$(y_{n+2})_0 - 0 - (n^2+m^2)(y_n)_0 = 0$

$\Rightarrow (y_{n+2})_0 = (n^2+m^2)(y_n)_0$ — (3)

putting $n = 1, 3, 5$ — successively in (3) we get

$(y_3)_0 = (1^2+m^2)\, m$

$(y_5)_0 = (3^2+m^2)(y_3)_0 = (3^2+m^2)(1^2+m^2)\, m$

Hence $(y_n)_0 = m(1^2+m^2)(3^2+m^2) \cdots \{(n-2)^2+m^2\}$

when n is odd

[2^2 ... 1, 3, 5 ...]

Lastly putting $n = 2, 4, 6$ — successively in (3) we get

$(y_4)_0 = (2^2+m^2)\, m^2$

$(y_6)_0 = (4^2+m^2)(y_4)_0 = m^2(2^2+m^2)(4^2+m^2)$

Hence $(y_n)_0 = m^2(2^2+m^2)(4^2+m^2) \cdots \{(n-2)^2+m^2\}$, when n is even

Thus we conclude that

$(y_n)_0 = \{(n-2)^2+m^2\} \cdots (3^2+m^2)(1^2+m^2)\, m$, if n is odd

$= \{(n-2)^2+m^2\} \cdots (4^2+m^2)(2^2+m^2)\, m^2$, if n is even

(9) If $x+y=1$, prove that n-th derivative of $x^n y^n$ is

$$n!\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}x^2 - ({}^nC_3)^2 y^{n-3}x^3 + \cdots + (-1)^n x^n\}$$

$x+y=1$

$\Rightarrow y = (1-x)$

$x^n y^n$

$= x^n (1-x)^n$

Let $y = (1-x)^n$, $u = x^n$

n-th derivative of $x^n y^n$ is

$$= \frac{n!}{(n-n)!} x^{n-n} (1-x)^n + {}^nC_1 \frac{n!}{(n-n+1)!} x^{n-n+1} \cdot n(1-x)^{n-1}(-1)$$

$$+ {}^nC_2 \frac{n!}{(n-n+2)!} x^{n-n+2} \cdot n(n-1)(1-x)^{n-2}$$

$$+ \cdots + x^n (-1)^n \frac{n!}{(n-n)!} x^{n-n}$$

$[(1-x)^n = (-1)^n (x-1)^n$

n-th derivative of $(x-1)^n = \frac{n!}{(n-n)!} x^{n-n}]$

$$= n!(1-x)^n - ({}^nC_1)^2 n!\, x(1-x)^{n-1} + ({}^nC_2)^2 n!\,(1-x)^{n-2} x^2 + \cdots + x^n (-1)^n n!$$

$$= n!\,y^n - ({}^nC_1)^2 n!\,x y^{n-1} + ({}^nC_2)^2 n!\,x^2 y^{n-2} + \cdots + x^n n! (-1)^n$$

$$= n!\{y^n - ({}^nC_1)^2 y^{n-1}x + ({}^nC_2)^2 y^{n-2}x^2 + \cdots + (-1)^n x^n\}$$

(10) If $V_n = \frac{d^n}{dx^n}(x^n \log x)$, show that $V_n = n V_{n-1} + (n-1)!$.

Hence show that

$$V_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

We have,

$$V_n = \frac{d^n}{dx^n}(x^n \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}} \cdot \frac{d}{dx}(x^n \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(n x^{n-1} \log x + x^n \cdot \frac{1}{x}\right)$$

$$= n \cdot \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$\therefore V_n = n V_{n-1} + (n-1)!$ — (1)

Now, $V_1 = \frac{d}{dx}(x \log x)$

$\Rightarrow V_1 = 1 + \log x$

Putting $n = 2, 3, 4 \ldots$ successively in (1) we get

$V_2 = 2V_1 + 1!$

$= 2(1+\log x) + 1!$

$= 2\{(1+\log x) + \frac{1}{2}\}$

$= 2!\left(\log x + 1 + \frac{1}{2}\right)$

$V_3 = 3V_2 + 2!$

$= 3 \cdot 2!\left(\log x + 1 + \frac{1}{2}\right) + 2!$

$= 3 \cdot 2!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$

$$= 6\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$$
$$= 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$y_4 = 4y_3 + 3!$$
$$= 4\cdot 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right) + 3!$$
$$= 4\cdot 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$
$$= 24\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$
$$= 4!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

So in general $y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$

Exercise

① (i) Find the n-th derivatives of–

(i) $x^{3/4}$, (ii) $\frac{1}{\sqrt{x}}$, (iii) $\sin 3x \cos 7x$, (iv) $e^{3x}\sin^2 4x$

(i) $x^{3/4}$

n-th derivative of $y = x^{3/4}$

$$y_n = \frac{3}{4}\left(\frac{3}{4}-1\right)\left(\frac{3}{4}-2\right)\cdots\left(\frac{3}{4}-n+1\right)x^{3/4-n}$$
$$= \frac{3}{4}\cdot\left(-\frac{1}{4}\right)\cdot\left(\frac{-5}{4}\right)\cdots\left(\frac{7-4n}{4}\right)x^{\frac{3}{4}-n}$$
$$= \frac{3(-1)^n\, 1\cdot 5\cdot 9 \cdots \{-(4n-7)\}\, x^{\frac{3}{4}-n}}{4^n}$$
$$= \frac{3(-1)^{n-1}\, 1\cdot 5\cdot 9 \cdots (4n-7)}{4^n}\cdot x^{\frac{3}{4}-n}$$

(ii) $\frac{1}{\sqrt{x}}$

nth derivative of $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$y_n = -\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)x^{-\frac{1}{2}-n}$$
$$= \frac{(-1)^{n-1}\,3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 2^n}\cdot x^{-\frac{1}{2}-n}$$

(iii) $\sin 3x \cos 7x$

n-th derivative of $y = \sin 3x \cos 7x$

$$y = \frac{1}{2}(2\sin 3x \cos 7x)$$
$$= \frac{1}{2}(\sin 10x - \sin 4x)$$
$$= \frac{1}{2}\sin 10x - \frac{1}{2}\sin 4x$$
$$y_n = \frac{1}{2}10^n\cdot\sin\left(\frac{n\pi}{2}+10x\right) - \frac{1}{2}\cdot 4^n\sin\left(\frac{n\pi}{2}+4x\right)$$

(iv) $e^{3x}\sin^2 4x$

n-th derivative of $y = e^{3x}\sin^2 4x$

$$y = \frac{1}{2}e^{3x}(2\sin^2 4x)$$
$$= \frac{1}{2}e^{3x}(1-\cos 8x)$$
$$= \frac{1}{2}e^{3x} - \frac{1}{2}e^{3x}\cos 8x$$
$$y_n = \frac{1}{2}3^n e^{3x} - \frac{1}{2}(9+64)^{\frac{n}{2}}e^{3x}\cos\left(8x + n\tan^{-1}\frac{8}{3}\right)$$

$= \frac{1}{2} 3^n e^{3x} - \frac{1}{2}(73)^{\frac{n}{2}} e^{3x} \cos(8x + n\tan^{-1}\frac{8}{3})$

② If $y = \frac{x}{x+1}$ show that $y_5(0) = 5!$

$y = \frac{x}{x+1}$

$= \frac{(x+1)-1}{x+1}$

$\Rightarrow y = 1 - \frac{1}{x+1}$

n-th derivative of $y = 1 - \frac{1}{x+1}$

$y_n = -\frac{(-1)^n n! \cdot 1^n}{(x+1)^{n+1}}$

$y_5 = -\frac{(-1)^5 5!}{(x+1)^{5+1}}$

$= \frac{5!}{(x+1)^6}$

$y_5(0) = \frac{5!}{1^6}$

$y_5(0) = 5!$ (Proved)

③ Apply Leibnitz's theorem to prove that $y_4 + 4y = 0$ when $y = e^{-x}\cos x$

n-th derivative of $y = e^{-x}\cos x$ (By using Leibnitz's theorem)

Let's $u = e^{-x}$, $v = \cos x$

$y_n = (-1)^n e^{-x}\cos x + {}^nC_1 (-1)^{n-1} e^{-x} (-\sin x) + {}^nC_2 (-1)^{n-2} e^{-x} (-\cos x) + e^{-x} 1^n \cos(\frac{n\pi}{2} + x)$

$= (-1)^n e^{-x}\cos x - n(-1)^{n-1} e^{-x}\sin x - \frac{n(n-1)}{2}(-1)^{n-2} e^{-x}\cos x + \dots + e^{-x}\cos(\frac{n\pi}{2} + x)$

$y_4 = -e^{-x}\cos x + 4e^{-x}\sin x - 6e^{-x}\cos x + \dots + e^{-x}\cos(2\pi + x)$

$= e^{-x}\cos x + 4e^{-x}\sin x - 6e^{-x}\cos x + \dots + e^{-x}\cos x$

X

This is wrong

$y = e^{-x} \cos x$

$y_n = (1+1)^{\frac{n}{2}} e^{-x} \cos\{x + n\tan^{-1}(-1)\}$

$= 2^{\frac{n}{2}} e^{-x} \cos(x - \frac{\pi n}{4})$

$y_4 = (2)^{\frac{4}{2}} e^{-x} \cos(x - \frac{\pi}{4} \cdot 4)$

$= 4 e^{-x} \cos(\pi - x)$

$= -4e^{-x} \cos x$

$y_4 + 4y = -4e^{-x}\cos x + 4e^{-x}\cos x$

$= 0$ (proved)

⑤ Find the n-th derivatives of –

(i) $\frac{x^2}{x-1}$, (ii) $\tan^{-1}\frac{1+x}{1-x}$

(i) $y = \frac{x^2}{x-1}$

$= \frac{(x^2 - 1) + 1}{x-1}$

$= x + 1 + \frac{1}{x-1}$

$y_n = \frac{(-1)^n n!}{(x-1)^{n+1}}$

(ii) $y = \tan^{-1}\frac{1+x}{1-x}$

$y = \tan^{-1}(1) + \tan^{-1}(x)$

$\Rightarrow y_1 = 0 + \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) = 1$

$\Rightarrow y_2(1+x^2) + 2x \cdot y_1 = 0$

$\Rightarrow (1+x^2) y_2 + 2xy_1 = 0$

$y_{n+2}(1+x^2) + n \cdot y_{n+1} 2x + \frac{n(n-1)}{2} \cdot 2\, y_n + 2 y_{n+1} x + 2 y_n n = 0$

$\Rightarrow (1+x^2) y_{n+2} + (2n+2)x\, y_{n+1} + n^2 y_n + n y_n = 0$

$\Rightarrow (1+x^2) y_{n+2} + (2n+2)x y_{n+1} + (n^2+n) y_n = 0$

(6) Show that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}$

$\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots\right)$

Let

$y_n = \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)$

$= \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dx}\left(\frac{\log x}{x}\right)$

$= \frac{d^{n-1}}{dx^{n-1}} \left(\frac{x \cdot \frac{1}{x} - \log x}{x^2}\right)$

$= \frac{d^{n-1}}{dx^{n-1}} \left(\frac{1}{x^2} - \frac{\log x}{x^2}\right)$

(7) If $x = \sin t$, $y = \cos pt$, show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$

$x = \sin t$	$y = \cos pt$
$\Rightarrow \frac{dx}{dt} = \cos t$	$\Rightarrow \frac{dy}{dt} = -p\sin pt$

$\therefore \frac{dy}{dx} \Rightarrow y_1 = -\frac{p\sin pt}{\cos t}$

$\Rightarrow y_2 = -p\left[\frac{p\cos t \cdot \cos pt + \sin pt \cdot \sin t}{\cos^2 t}\right]$

$= -p$

$\Rightarrow y_1 = \dfrac{-p\sqrt{1-\cos^2 t}}{\sqrt{1-\sin^2 t}}$

$\Rightarrow y_1^2(1-x^2) = p^2(1-y^2)$

$\Rightarrow y_1^2(-2x) + (1-x^2)2y_1y_2 = p^2(-2yy_1)$

~~$\Rightarrow y_1^2($~~

$\Rightarrow -xy_1 + y_2(1-x^2) = -p^2(y)$

$\Rightarrow (1-x^2)y_2 - xy_1 + p^2y = 0$

$\Rightarrow y_{n+2}(1-x^2) + ny_{n+1}\cdot(-2x) - n(n-1)y_n$

$\quad - \{y_{n+1}\cdot x + n\cdot y_n\} + p^2 y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1}$

$\quad - ny_n + p^2 y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$

Q) If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ prove that

$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

$\Rightarrow \cos^{-1}\left(\frac{y}{b}\right) = n\log\left(\frac{x}{n}\right)$

$\Rightarrow \frac{y}{b} = \cos\left\{n\log\left(\frac{x}{n}\right)\right\}$

$\Rightarrow y = b\cos\left\{n\log\left(\frac{x}{n}\right)\right\}$

$y_1 = -b\sin\left\{n\log\left(\frac{x}{n}\right)\right\}\cdot n\cdot\frac{n}{x}$

$= -b^2n\cdot\dfrac{\sin\{n\log(\frac{x}{n})\}}{x}$

$y_2 = -b^2n\,\dfrac{\{x\cos\{n\log(\frac{x}{n})\}\cdot n\cdot\frac{n}{x} - \sin(n\log(\frac{x}{n}))\}}{x^2}$

$= -b^2n\,\dfrac{\{n^2\cos\{n\log(\frac{x}{n})\} - \sin\{n\log(\frac{x}{n})\}\}}{x^2}$

$\Rightarrow x^2y_2 = -bn^2y + -xy_1$

$\Rightarrow \cos^{-1}\left(\frac{y}{b}\right) = n(\log x - \log n)$

$\Rightarrow -\dfrac{y_1 b}{\sqrt{b^2-y^2}}\cdot\dfrac{1}{b} = \dfrac{n}{x}$

$\Rightarrow \dfrac{y_1^2}{(b^2-y^2)} = \dfrac{n^2}{x^2}$

$\Rightarrow y_1^2x^2 = n^2(b^2-y^2)$

$\Rightarrow x^2 2y_1y_2 + y_1^2 2x = n^2(-2yy_1)$

$\Rightarrow x^2y_2 + xy_1 = -n^2y$

$\Rightarrow x^2y_2 + xy_1 + n^2y = 0$

$\Rightarrow y_{n+2}x^2 + ny_{n+1}\cdot 2x + n(n-1)y_n +$

$\Rightarrow y_{n+1}x + y_n\cdot n + n^2y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

• Fourier series (A 9m)

Exercises

1) Find the period of the function $\cos 2x$, $\sin 2x$, $4 + \sin 2\pi x$

(i) $\cos 2x$

Period of $\cos x$ is 2π

$\therefore 2\pi = 2x$

$\Rightarrow x = \pi$

$\therefore$ period of $\cos 2x$ is π.

(ii) $\sin 2x$

Period of $\sin x = 2\pi$

$\therefore 2x = 2\pi$

$\Rightarrow x = \pi$

Period of $\sin 2x = \pi$

(iii) $4 + \sin 2\pi x$

Period of $\sin x = 2\pi$

$\therefore 2\pi x = 2\pi$

$\Rightarrow x = 1$

[$\because$ Constant part cannot change the characteristics of the sine curve, so, its period remain 2π]

② Drawing the graph of the periodic function $f(x)$ (of period 2π) by -

$$f(x) = 0 \qquad -\pi < x < 0$$
$$= \sin x \qquad 0 < x < \pi$$

Consider the function $f(x) = 0, \; -\pi < x < 0$
$= \sin x \; 0 < x < \pi$

The function is defined on the interval $(-\pi, \pi)$ we extend this function by defining $f(x + 2\pi) = f(x)$ for all values of x. This is a periodic function of period 2π.

Its fourier series co-efficients are -

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$
$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \sin x\,dx\right]$$
$$= -\frac{1}{\pi}\left[\cos x\right]_0^{\pi}$$
$$= -\frac{1}{\pi}(-1-1)$$
$$= \frac{2}{\pi}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= 0 \quad [\because f(x)\cos nx$ is odd function$]$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$

$= \frac{1}{\pi}\left[0 + \int_0^{\pi}\sin x\sin nx\,dx\right]$

$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x\sin nx\,dx$

$= \frac{1}{2\pi}\left[\cos\{(n-1)x\} - \cos\{(n+1)x\}\right]_0^{\pi}$

$= \frac{1}{2\pi}\left[\cos\{(n-1)\pi\} - \cos\{(n+1)\pi\}\right]$

$= \frac{1}{2\pi}\left[\cos(n\pi-\pi) - \cos(n\pi+\pi)\right]$

$= \frac{1}{2\pi}\left[\cos n\pi\cos\pi + \sin n\pi\sin\pi - \cos n\pi\cos\pi + \sin n\pi\sin\pi\right]$

$= \frac{1}{2\pi} \cdot 0$

$= 0$

$b_n = \frac{1}{2\pi}\int_0^{\pi} 2\sin x\sin nx\,dx$

$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos\{(n-1)x\} - \cos\{(n+1)x\}\right]$

$= \frac{1}{2\pi}\left[\frac{\sin\{(n-1)x\}}{(n-1)} - \frac{\sin\{(n+1)x\}}{(n+1)}\right]_0^{\pi}$

$= \frac{1}{2\pi}\left[\frac{\sin\{(n-1)\pi\}}{(n-1)} - \frac{\sin\{(n+1)\pi\}}{(n+1)}\right]$

$= \frac{1}{2\pi}\left[\frac{\sin(n\pi-\pi)}{(n-1)} - \frac{\sin(n\pi+\pi)}{(n+1)}\right]$

$= \frac{1}{2\pi(n^2-1)}\left[(n+1)\{\sin n\pi\cos\pi - \cos n\pi\sin\pi\} - (n-1)\{\sin n\pi\cos\pi + \cos n\pi\sin\pi\}\right]$

$= \frac{1}{2\pi(n^2-1)}\left[(n+1)\sin n\pi\cos\pi - (n-1)\sin n\pi\cos\pi\right]$

$= \frac{1}{2\pi(n^2-1)}\left[2\sin n\pi\cos\pi\right] = 0$

$= \frac{1}{\pi(n^2-1)} \cdot \sin(n+1)\pi + \sin(n-1)\pi = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{2\pi}\left[\int_0^{\pi} 2\sin x\cos nx\,dx\right]$

$= \frac{1}{2\pi}\left[\int_0 \sin(n+1)x - \sin(n-1)x\,dx\right]$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)x}{(n-1)} - \frac{\cos(n+1)x}{(n+1)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi(n^2-1)}\left[(n+1)\cos(n-1)x - \cos(n+1)x \cdot (n-1)\right]_0^{\pi}$$

$$= \frac{1}{2\pi(n^2-1)}\left[(n+1)\cos(n\pi-\pi) - \cos(n\pi+\pi)(n-1)\right]$$

$$= \frac{1}{2\pi(n^2-1)}\left[(n+1)\cos n\pi\cos\pi + (n+1)\sin n\pi \sin\pi - (n-1)\cos n\pi\cos\pi + (n-1)\sin n\pi\sin\pi\right]$$

$$= \frac{2n\sin n\pi\sin\pi}{2\pi(n^2-1)}$$

0

fourier series of $f(x)$ is

$$\frac{1}{2}\cdot\frac{2}{\pi} + \sum_{n=1}^{\infty}(0\cdot\cos nx + 0\cdot\sin nx)$$

$$= \frac{1}{\pi}$$

$$\frac{1}{2}\cdot\frac{2}{\pi} + \sum_{n=1}^{\infty}\left[0\cdot\cos nx + \frac{1}{2\pi}\left\{\frac{\sin(n-1)\pi}{(n-1)} - \frac{\sin(n+1)\pi}{n+1}\right\}\right]$$

$$= \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi}\left\{\frac{\sin(n-1)\pi}{(n-1)} - \frac{\sin(n+1)\pi}{(n+1)}\right\}\sin nx\right)$$

③ state the waveform given by the periodic function $f(x) = \pi + x, \ -\pi < x < 0$

$= \pi - x, \ 0 < x < \pi$

and $f(x+2\pi) = f(x)$

consider the function $f(x) = \pi + x, \ -\pi < x < 0$

$= \pi - x, \ 0 < x < \pi$

The function is defined on the interval $(-\pi, \pi)$. we extend this function by defining $f(x+2\pi) = f(x)$ for all values of x. This is a periodic function of period 2π.

its fourier coefficients are—

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\,dx + \int_0^{\pi}(\pi-x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\pi x + \frac{x^2}{2}\right)_{-\pi}^{0} + \left(\pi x - \frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\left(-\pi^2 + \frac{\pi^2}{2}\right) + \left(\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{2}{\pi}\cdot\left(\pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{2}{\pi}\cdot\pi^2\left(1 - \frac{1}{2}\right)$$

$$= 2\pi \times \frac{1}{2}$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{(\pi+x)\sin nx}{n}\right\}_{-\pi}^{0} + \frac{1}{n^2}\{\cos nx\}_{-\pi}^{0} + \left\{\frac{(\pi-x)\sin nx}{n}\right\}_{0}^{\pi} - \frac{1}{n^2}\{\cos nx\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(1-\cos n\pi) + -\frac{1}{n^2}(\cos n\pi - 1)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\{1-\cos n\pi - \cos n\pi + 1\}\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{n^2}(1-\cos n\pi)$$

$$= \frac{2}{n^2\pi}(1-\cos n\pi)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$= 0$ [∵ $f(x)\sin nx$ is odd function]

$$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(1-\cos n\pi)}{n^2\pi}\cdot\cos nx$$

(4) Draw the graph of the function $f(x)$ given by $f(x) = -\frac{\pi}{4}$, $-\pi < x < 0$

$= \frac{\pi}{4}$, $0 \le x \le \pi$ and

$f(x+2\pi) = f(x)$.

show that this square waveform

consider the function $f(x) = -\frac{\pi}{4}$, $-\pi < x < 0$

$= \frac{\pi}{4}$, $0 \le x \le \pi$

The function is defined on the interval $(-\pi, \pi)$. we extend this function by defining ~~by defining~~ $f(x+2\pi) = f(x)$ for all values of x. This is a periodic function of period 2π.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{4}\int_{-\pi}^{0} dx + \frac{\pi}{4}\int_{0}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{4}\times\{x\}_{-\pi}^{0} + \frac{\pi}{4}\cdot\{x\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\times\frac{\pi}{4}\left[-\{\pi\} + \{\pi\}\right]$$

$= 0$

(ii) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= 0$ [∵ $f(x)$ is a odd function]

$a_n = 0$ [$\because f(x)\cos nx$ is odd function]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ [$\because f(x)\sin nx$ is even function]

$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4}\sin nx\,dx$

$= \frac{1}{2}\int_0^{\pi}\sin nx\,dx$

$= -\frac{1}{2}\cdot\frac{1}{n}\left[\cos nx\right]_0^{\pi}$

$= -\frac{1}{2n}(\cos n\pi - 1)$

$= \frac{1}{2n}(1-\cos n\pi)$

$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$= 0 + \sum_{n=1}^{\infty}\frac{1}{2n}(1-\cos n\pi)\sin nx$

$= \sum_{n=1}^{\infty}\frac{1}{2n}(1-\cos n\pi)\sin nx$

(5) Draw the graph of the function $f(x) = \frac{x}{2}$, $-\pi < x < \pi$

$f(x+2\pi) = f(x)$. Show that it is saw toothed wave form function $f(x)$ given by $f(x) = x^2$.

2nd Part done

Considers a function $f(x) = \frac{x}{2}$ $-\pi < x < \pi$. The function is defined on the interval $(-\pi,\pi)$. we extend this function by defining $f(x+2\pi) = f(x)$ for all values of x. This is a periodic function of period 2π.

$a_0 = 0$ [$\because f(x)$ is odd function]

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= 0$ [$\because f(x)\cos nx$ is odd function]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}\sin nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\sin nx\,dx$ [$\because f(x)\sin nx$ is even function]

$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} x\sin nx\,dx$

$= \frac{1}{\pi}\left[\left\{\frac{-x\cos nx}{n}\right\}_0^{\pi} + \frac{1}{n^2}\left\{\sin nx\right\}_0^{\pi}\right]$

$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{1}{n^2}\times 0\right]$

$= -\frac{1}{n}\cos n\pi$

$$\therefore \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= 0 + \sum_{n=1}^{\infty} \left(-\frac{1}{n} \cos n\pi \cdot \sin nx\right)$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi \cdot \sin nx$$

- Laws of mean value theorem (S sir)

- Rolle's theorem

Let f be a function defined on a finite closed interval $[a, b]$ such that

(i) $f(x)$ is continuous for all x in $a \le x \le b$

(ii) $f'(x)$ exists for all x in $a < x < b$

and (iii) $f(a) = f(b)$

Then there exist at least one value c, $a < c < b$ such that $f'(c) = 0$

- a straight line which is parallel to y axis then it's slope $= \infty$
- a straight line which is parallel to x axis then it's slope $= 0$

Ex-① verify Rolle's theorem in each of the following cases

(i) $f(x) = \cos x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(i) $f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and derivable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now $f'(x) = -\sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Now, $f\left(-\frac{\pi}{2}\right) = 0 = f\left(\frac{\pi}{2}\right)$

$\therefore$ Thus $f(x)$ satisfies all conditions of Rolle's theorem.

By Rolle's theorem we should have $f'(c) = 0$, $-\frac{\pi}{2} < c < \frac{\pi}{2}$

Here $f'(c) = 0 \Rightarrow -\sin c = 0 \Rightarrow \sin c = 0$ whose one solution is $c = 0$ and $-\frac{\pi}{2} < 0 < \frac{\pi}{2}$.

Hence Rolle's theorem is verified for the given function

(ii) $f(x) = 2x^3 + x^2 - 4x - 2$, $-\sqrt{2} \le x \le \sqrt{2}$

Since every polynomial in x is continuous and derivable for every real value of x, so $f(x)$ is continuous on $[-\sqrt{2}, \sqrt{2}]$ and derivable on $(-\sqrt{2}, \sqrt{2})$

Also $f'(x) = 6x^2 + 2x - 4$, $-\sqrt{2} < x < \sqrt{2}$

moreover,

$f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2 = 0$ and $f(\sqrt{2}) = 0$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem. By Rolle's theorem, we should have, $f'(c) = 0$ where $-\sqrt{2} < c < \sqrt{2}$

Here $f'(c) = 0 \Rightarrow 6c^2 + 2c - 4 = 0$

i.e. $2(c+1)(3c-2) = 0$ whose two solutions are $c = -1, \frac{2}{3}$ and $-\sqrt{2} < -1 < \sqrt{2}$ as well as $-\sqrt{2} < \frac{2}{3} < \sqrt{2}$

Hence Rolle's theorem is verified for the given function

Ex-(2)

(i) $f(x) = x^3 (x-1)^2$ in the interval $0 \le x \le 1$

Here $f(x) = x^3 (x-1)^2$, which is a polynomial of degree 5, and so continuous at every point

So, $f(x)$ is continuous in $[0, 1]$

$f'(x) = 3x^2 (x-1)^2 + x^3 \cdot 2(x-1)$

$= x^2 (x-1)(3x - 3 + 2x)$

$= x^2 (x-1)(5x-3)$

which exists in $(0, 1)$

Also $f(0) = 0$ and $f(1) = 0$

Thus $f(x)$ satisfies all the conditions of Rolle's Theorem. Then there exists at least one point $x = c$ in $(0, 1)$ such that $f'(c) = 0$

$\Rightarrow c^2 (c-1)(5c-3) = 0$

Since $0 < c < 1$, $c \neq 0$, $c \neq 1$ i.e $c - 1 \neq 0$

Therefore, $5c - 3 = 0$, i.e $c = \frac{3}{5}$ which lies in $(0, 1)$

(ii)

Q (i) Is Rolle's theorem applicable to the function $f(x) = 1 - x^{\frac{2}{3}}$ in $[-1, 1]$? Justify your answer.

(ii) Is Rolle's theorem applicable to $f(x) = \frac{1}{2-x^2}$ in $[-1, 1]$? Justify your answer.

(i) Here $f(x) = 1 - x^{\frac{2}{3}}$ $f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$

If $x \neq 0$ $f'(x) = -\frac{2}{3} x^{-\frac{1}{3}} = \frac{-2}{3x^{\frac{1}{3}}}$

So, $f'(0)$ does not exist and $-1 < 0 < 1$, i.e $f(x)$ is not derivable in $(-1, 1)$

~~continuous on $[a, b]$~~

So, $f'(0)$ does not exist so Rolle's theorem is not applicable to the function $f(x) = 1 - x^{\frac{2}{3}}$ on the interval $[-1, 1]$.

(ii) $f(x) = \frac{1}{2-x^2}$ is continuous in $[-1, 1]$.

$f'(x) = (-1)(2-x^2)^{-2} \cdot (-2x) = \frac{2x}{(2-x^2)^2}$ which exists in $(-1, 1)$

Also $f(1) = \frac{1}{2-1} = 1$ and $f(-1) = \frac{1}{2-1} = 1$

So, $f(1) = f(-1)$

Here, $f(x)$ satisfies all the conditions of Rolle's theorem. Hence Rolle's theorem is applicable to $f(x) = \frac{1}{2-x^2}$ on the interval $[-1, 1]$

(4) Explain whether Rolle's theorem is applicable to the function $\varphi(x) = |x|$ in any closed interval containing the origin.

Let us consider an interval $[-a, a]$ where $a > 0$, containing the origin.

The function $\varphi(x) = |x|$ can be re-written as

$\varphi(x) = x$, when $x \geq 0$

$= -x$, when $x < 0$

Here $\varphi(x)$ is continuous at $x = 0$,

since,

$$\lim_{x \to 0^+} \varphi(x) = \lim_{x \to 0^-} \varphi(x) = \varphi(0) = 0$$

But $$R\varphi'(0) = \lim_{h \to 0^+} \frac{\varphi(0+h) - \varphi(0)}{h} = \lim_{h \to 0^+} \frac{h-0}{h} = 1$$

and $$L\varphi'(0) = \lim_{h \to 0^-} \frac{\varphi(0+h) - \varphi(0)}{h} = \lim_{h \to 0^-} \frac{-h-0}{h} = -1$$

Since $R\varphi'(0) \neq L\varphi'(0)$, $\varphi(x)$ is not derivable at $x = 0$, so $\varphi(x)$ is not derivable in the open interval $(-a, a)$. One of the conditions of Rolle's theorem is not satisfied. Hence Rolle's theorem is not applicable to the function $\varphi(x) = |x|$ in any interval $[-a, a]$ containing the origin.

(5) Does $f(x) = \cos\left(\frac{1}{x}\right)$ satisfy Rolle's theorem in the interval $-1 \leq x \leq 1$? Justify your answer.

Since $f(x) = \cos\left(\frac{1}{x}\right)$, $f(0)$ is undefined.

So, $f(x)$ is not continuous at $x = 0$.

Thus $f(x)$ does not satisfy the first condition of Rolle's theorem in $[-1, 1]$.

Hence Rolle's theorem is not applicable to $f(x) = \cos\left(\frac{1}{x}\right)$ in the interval $[-1, 1]$.

⑥ Show that Rolle's theorem is not applicable to $f(x) = \tan x$ in $[0, \pi]$ although $f(0) = f(\pi)$

Here $f(x) = \tan x$ is continuous every where in $[0, \pi]$ except at $x = \frac{\pi}{2}$ and consequently it is not derivable there.

Thus the condition of Rolle's theorem do not hold. Hence Rolle's theorem is not applicable to the function $f(x) = \tan x$ on the interval $[0, \pi]$, although $f(0) = f(\pi)$

⑦ Test the applicability of Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$ in $a \le x \le b$ where m and n are positive integers.

Here $f(x) = (x-a)^m (x-b)^n$

Since m and n are positive integers. $f(x)$ is polynomial of degree $(m+n)$, which is continuous at every point. So $f(x)$ is continuous in $a \le x \le b$.

Now, $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}\{m(x-b) + n(x-a)\}$

which obviously exists in $a<x<b$ --- ①

Also, $f(a) = 0 = f(b)$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem. Then there is at least one point $x = c$ in $a<x<b$, such that $f'(c) = 0$

From ① Then we have

$(c-a)^{m-1}(c-b)^{n-1}\{m(c-b) + n(c-a)\} = 0$

Since $a<c<b$, $c-a \ne 0$, $c-b \ne 0$

$\therefore m(c-b) + n(c-a) = 0$

$\Rightarrow (m+n)c = mb + na$ i.e. $c = \frac{mb+na}{m+n}$

The point c divides the interval $[a, b]$ internally in the ratio $m:n$.

Jm

(3) Consider the function $f(x) = (x-2)\log x$ to show that, the equation $x\log x = 2-x$ is satisfied by at least one value of x lying between 1 and 2.

Here $f(x) = (x-2)\log x$

$f'(x) = \dfrac{(x-2)}{x} + \log x$

$= \dfrac{1}{x}(x\log x + x - 2)$ —— (1)

$f(1) = f(2) = 0$

$f'(x)$ exists in $(1, 2)$

This function $f(x)$ is derivable and so the continuous in $[1, 2]$

As $f(x)$ satisfied all conditions of Rolle's Theorem, there exists at least one value of x, lying in $(1, 2)$ say c such that

$f'(c) = 0 \quad 1 < c < 2$

From (1) we have $c\log c + c - 2 = 0$

$\Rightarrow c\log c = 2 - c$

which shows that the equations $x\log x = 2-x$ is satisfied by c where $1 < c < 2$

Hence the proposition is proved.

(4) Show that between any two roots of $e^x\cos x = 1$ there exists at least one root of $e^x\sin x - 1 = 0$

Let α and β be any two distinct roots of the equation $e^x\cos x = 1$

Then $e^\alpha\cos\alpha = 1 = e^\beta\cos\beta$ —— (1)

Let us construct in a function f defined by $f(x) = e^{-x} - \cos x$ —— (2)

in the closed interval $\alpha \le x \le \beta$.

We note that $f(x)$ is continuous in $[\alpha, \beta]$.

Since $\cos x$ and e^{-x} are continuous in the interval.

Also $f'(x) = -e^{-x} + \sin x$, where $\alpha < x < \beta$ —— (3)

So, $f(x)$ is derivable in (α, β)

From (2), $f(\alpha) = e^{-\alpha} - \cos\alpha = \dfrac{1 - e^\alpha\cos\alpha}{e^\alpha}$

$= 0$ [using (1)]

similarly $f(\beta) = 0$

So, $f(\alpha) = f(\beta)$

Thus the function f satisfies all the condn of Rolle's theorem in $[\alpha, \beta]$

So, there exists at least one value of x, say y, $\alpha < y < \beta$ such that $f'(y) = 0$

or, from (3) $\sin y - e^{-y} = 0$

i.e, $e^y\sin y - 1 = 0$

This proves that $y \in (\alpha, \beta)$ is a root of the equation $e^x\sin x - 1 = 0$. Thus it is proved that between any two roots α, β $(\alpha \ne \beta)$ of the equation $e^x\cos x = 1$, there exists at least one root of the equation $e^x\sin x - 1 = 0$

- Lagrange's mean value theorem

(i) Let f be a function defined on a ~~function fun~~ finite closed interval [a,b] such that

(i) f(x) is continuous for all x, $a \le x \le b$

(ii) $f'(x)$ exists for all x, $a < x < b$

Then there exist at least one value c, $a < c < b$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

- Another form of Lagrange's theorem (not so imp)

~~For the~~ interval [0, x]

$f(a+h) = f(a) + hf'(a+\theta h)$, $0 < \theta < 1$

(i) Let f be a function defined on a finite closed interval [a, a+h] such that f is continuous on [a, a+h]

(ii) f is derivable on (a, a+h)

For interval [0, x]; $f(x) = f(0) + hf'(\theta x)$, $0 < \theta < 1$ [maclaurin's Formula]

Ex-

(1) verify L.M.T for the following functions

(i) $f(x) = x(x-1)(x-2)$, $0 \le x \le \frac{1}{2}$

$f(x) = x(x-1)(x-2)$

$= x(x^2 - 3x + 2)$

$= x^3 - 3x^2 + 2x$

f(x) is continuous on $[0, \frac{1}{2}]$ and derivable on $(0, \frac{1}{2})$

$f'(x) = 3x^2 - 6x + 2$, $0 < x < \frac{1}{2}$

Thus f(x) satisfies all the condn of L.M.T.

By M.V Theorem we should have an c, $0 < c < \frac{1}{2}$ such that

$$f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

Here $f'(c) = \dfrac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$ implies

$\Rightarrow 3c^2 - 6c + 2 = 2\left[\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{2} - 0\right]$

$\Rightarrow 12c^2 - 24c + 5 = 0$

having two solution $c = 1 \pm \frac{\sqrt{21}}{6}$

Between these two $1 - \frac{\sqrt{21}}{6}$ lies between 0 and $\frac{1}{2}$.

(ii) $f(x) = x\cos\frac{1}{x}$, for $x \ne 0$
$= 0$ for $x = 0$ in $[-1, 1]$

Since x and $\cos(\frac{1}{x})$ are derivable for all $x \ne 0$, So f(x) is derivable for all $x \ne 0$

$f'(x) = \frac{d}{dx}\left\{x\cos\left(\frac{1}{x}\right)\right\} = \cos\frac{1}{x} - \frac{1}{x}\sin\frac{1}{x}$, where $x \ne 0$

But $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\cos\frac{1}{h}}{h}$

$= \lim_{h\to 0} \cos\frac{1}{h}$ which does not exist

Here $f(x)$ is not derivable on $(-1,1)$

Thus Lagrange's mean value theorem is not applicable for the given function.

(iii) $f(x) = \log x$, $\frac{1}{2} \le x \le 2$

Since $\log x$ is continuous and derivable for all x in $(0,\infty)$, so $f(x) = \log x$ is continuous on $\frac{1}{2} \le x \le 2$ and derivable on $\frac{1}{2} < x < 2$

Also $f'(x) = \frac{1}{x}$, $\frac{1}{2} < x < 2$

Thus $f(x)$ satisfies all conditions of Lagrange's mean value theorem.

There should exist c, $\frac{1}{2} < c < 2$ such that

$$f'(c) = \frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}}$$

Here $f'(c) = \frac{f(2) - f(\frac{1}{2})}{2-\frac{1}{2}}$ implies

$\Rightarrow \frac{1}{c} = \frac{\log 2 + \log 2}{\frac{3}{2}} = \frac{2}{3}\log 4$

Hence $c = \frac{3}{2\log 4} \in (\frac{1}{2}, 2)$

(2) Show that the Lagrange's mean value theorem is not applicable to $f(x) = 4-(6-x)^{\frac{2}{3}}$ in the interval $[5,7]$

Here, $f(x) = 4-(6-x)^{\frac{2}{3}}$

$\therefore f'(x) = -\frac{2}{3}(6-x)^{-\frac{1}{3}}(-1) = \frac{2}{3}\cdot\frac{1}{(6-x)^{\frac{1}{3}}}$, if $x \ne 6$

At $x=6$ $f'(x)$ is undefined

so, $f(x)$ is not derivable at $x=6$ which is a point in the interval $[5,7]$.

So, one of the conditions of mean value theorem is not satisfied.

Hence, Lagrange's mean value theorem cannot be applied to the function $f(x) = 4-(6-x)^{\frac{2}{3}}$ in the interval $[5,7]$

③ If a and b are distinct real numbers show that there exists a real number c between a and b such that

$$a^2+ab+b^2=3c^2$$

Let us consider the function $f(x)=x^3$ defined on $[a,b]$

$f'(x)=3x^2$

Obviously $f(x)$ is continuous on $[a,b]$ and $f'(x)$ exists in (a,b). So Lagrange's mean value theorem can be applied to $f(x)$ for the interval $[a,b]$.

Hence there exists a number c, $a<c<b$

such that $\frac{f(b)-f(a)}{b-a}=f'(c)$, $a<c<b$

$\Rightarrow \frac{b^3-a^3}{b-a}=3c^2$ $(b\neq a)$

Hence $a^2+ab+b^2=3c^2$

④ Prove that for any quadratic function

⑤ If $f'(x)=0$ in $[a,b]$, then by using mean value theorem show that $f(x)$ is constant in that interval.

x_1, x_2 be two arbitrary points in $[a,b]$ such that $a\leq x_1 \leq x_2 \leq b$.

Then applying L.M.T. to $f(x)$ in $[x_1,x_2]$ we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c) \quad x_1<c<x_2$$

$=0$ by the given condition

Here $f(x_2)-f(x_1)=0$

$f(x_2)=f(x_1)$

Since x_1, x_2 are any two arbitrary points.

⑥ If $f'(x)$ exists and <0 everywhere in (a,b), then by using mean value theorem show that $f(x)$ is a decreasing function in (a,b).

Let x_1, x_2 be two arbitrary points in (a,b) such that $a<x_1<x_2<b$. Then applying L.M.T. to $f(x)$ in $[x_1,x_2]$

We get,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c), \quad x_1 < c < x_2$$

$$\therefore \frac{f(x_2)-f(x_1)}{x_2-x_1} < 0 \quad [\because f'(c)<0 \text{ by the given hypothesis}]$$

But $x_2 - x_1 > 0$, so from above,

$f(x_2) - f(x_1) < 0$

i.e. $f(x_2) < f(x_1)$, whenever $x_2 > x_1$

Hence $f(x)$ is a decreasing function in (a,b).

Py – 2(T), 2022 (B)

Ex-① (iii) $f(x) = 2+(x-1)^{2/3}$ $0 \le x \le 2$

Here $f(x) = 2+(x-1)^{\frac{2}{3}}$ is continuous on $0 \le x \le 2$

but $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$ doesn't exist at $x=1$.

Hence $f(x)$ is not derivable on $0<x<2$.

Thus the conditions of Rolle's theorem do not hold. So function

(iv) $f(x) = |x|, \; -1 \le x \le 1$

~~Since f~~ The given function can be written as

$$f(x) = \begin{cases} -x, & \text{when } -1 \le x \le 0, \\ x, & \text{when } 0 < x \le 1 \end{cases}$$

obviously $f(x)$ is continuous on $[-1, 1]$

Now $Lf'(0) = \lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \frac{-x-0}{x} = -1$

$Rf'(0) = \lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0^+} \frac{x-0}{x} = 1$

$\therefore Lf'(0) \neq Rf'(0)$

So, $f'(x)$ doesn't exist at $x=0$, which lies between -1 and 1. Hence $f(x)$ is not derivable on $-1<x<1$

Thus the condn of Rolle's theorem do not hold. So Rolle's theorem is not applicable to the given function.

② (ii) $f(x) = x^2 - 5x + 6$ in the interval $1 \le x \le 4$

since $f(x)$ is a polynomial function, it is continuous at every point. So it is continuous in the interval $[1, 4]$

$f'(x) = 2x - 5$ which exists in $(1, 4)$

Also $f(1) = 1 - 5 + 6 = 2$ and $f(4) = 16 - 20 + 6 = 2$

i.e $f(1) = f(4)$

Thus the given function satisfies all the conditions of Rolle's theorem. So, there is at least one point $x=c$ in $1<x<4$ such that $f'(c) = 0$

$\therefore 2c - 5 = 0$, i.e $c = \frac{5}{2}$ which lies in $(1,4)$

(iii) $f(x) = x(x-1)(x-2)$ in the interval $[0,2]$

$f(x) = x(x-1)(x-2)$

$= x^3 - 3x^2 + 2x$

This is a polynomial function and so is continuous at every point. So $f(x)$ is continuous in $[0,2]$

$f'(x) = 3x^2 - 6x + 2$, which is also polynomial and so it exits in $(0,2)$

Again $f(0) = 0 = f(2)$

Thus $f(x)$ satisfies all the conditions of Rolle's theorem. Then there exist at least one point $x = c$ on $0 < x < 2$ such that

$f'(c) = 0$

$\Rightarrow 3c^2 - 6c + 2 = 0$

$$c = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$= 1 \pm \frac{1}{\sqrt{3}}$$

Both values of c lie in $(0,2)$
of these

3

• vector calculus • gradient of scalar function

(রাইন)

$\nabla(f \pm g) = \nabla f \pm \nabla g$

gradient is also a linear function.

$\nabla(fg) = f\nabla g + g\nabla f$

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

proof imp (don't write Σ) [illegible] component [illegible]

$$\nabla\left(\frac{f}{g}\right) = \frac{\partial}{\partial x}\left(\frac{f}{g}\right)$$

$\left[\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{\partial}{\partial x}\left(\frac{f}{g}\right), \frac{\partial}{\partial x}\left(\frac{f}{g}\right)\right]$

for 3 components

$$\left[\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right]$$

$$\frac{1}{g^2}\left[g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right]$$

$$= \frac{g\nabla f - f\nabla g}{g^2}$$

• Tangent plane to level surface

$f(x,y) = x^2 + y$

$R^2 \longrightarrow R$

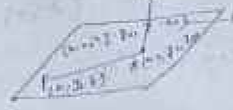
$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f$

$= \left(\frac{\partial}{\partial x}(x^2+y), \frac{\partial}{\partial y}(x^2+y)\right)$

→ other thing

$$\boxed{\begin{aligned} &\frac{d}{dx}\left(f(g(h(x)))\right) \\ &\left[\frac{d}{d\,g(h(x))} f(g(h(x)))\right] \cdot \frac{d}{d\,h(x)} g(h(x)) \cdot \frac{d}{dx} h(x) \end{aligned}}$$

$\nabla f^{P} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ → direction cosine


→ direction ratio

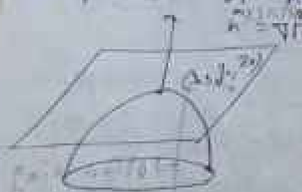
The eqn of tangent plane is —

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = 0$$

- Criteria for existance of tangent plane is the function must be differentiable at $P(x_0, y_0, z_0)$
  i.e. diff ⟺ existance of tangent plane
- If tangent plane exist then the eqn of tangent plane is $\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = 0$ at (x_0, y_0, z_0)
- The eqn of normal line

$\vec{n} = \nabla f$



$$\boxed{\frac{x-x_0}{\frac{\partial f}{\partial x}} = \frac{y-y_0}{\frac{\partial f}{\partial y}} = \frac{z-z_0}{\frac{\partial f}{\partial z}}}$$

- Ex ③ The gradient of $f(r)$, is equal to

$$\nabla f(r) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f(r)$$

$$= i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r)$$

$$= i f'(r) \frac{\partial r}{\partial x} + j f'(r) \frac{\partial r}{\partial y} + k f'(r) \frac{\partial r}{\partial z}$$

P.T.O

$r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}}$$

&

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}}$$

[Partial derivative]

$r = \sqrt{x^2+y^2+z^2}$

$\frac{\partial}{\partial x} r(x, y, z)$

$\frac{\partial \sqrt{x^2+y^2+z^2}}{\partial (x^2+y^2+z^2)}$

$\frac{\partial}{\partial x}(x^2+y^2+z^2)$

$$P.T. = i f(r) \frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} i$$

$$\nabla f(r) = \frac{f(r)}{r} \vec{r}$$

~~$\nabla f(r) \times \vec{r} = \vec{0}$~~

$$\nabla f(r) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f(r)$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f'(r) \frac{\partial r}{\partial x}$$

$$\nabla = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f'(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} \cdot \frac{\partial}{\partial x} i$$

$$= \frac{f'(r)}{r} \vec{r}$$

Imp
⑤, ⑦ — All ex-[illegible]
(imp [illegible] that's remain in practice)

- Directional derivative — directional derivative along normal direction, take dot product along the unit vector along this

E-X-

(1) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in direction of line PQ where Q is the point (5,0,4)

• [normal direction (i) direction derivate greatest ...]

Here, function $f = x^2 - y^2 + 2z^2$

$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

At $(1,2,3)$, $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

Now, vector $\vec{PQ}$ = position vector of Q − position vector of P

$= (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} - 2\hat{j} + \hat{k}$

unit vector in direction of $\vec{PQ}$

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

So, directional derivative of f in the direction of $\hat{a} = \nabla f \cdot \hat{a}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

• Divergence of vector function

$\vec{\nabla} \cdot \vec{F}$, $\vec{F} = (f_1, f_2, f_3)$

$F : R^n \longrightarrow R^m$

• Physical significance of divergence

Positive divergence

negative divergence

• The Laplacian operator ∇^2 (vector quantity

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

f vector zero $\nabla^2 f \to$ vector qu^n

f scalar $\cdot f \to$ scalar,

curl is linear function

Prove

• 4 $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ [Ex - Imp ...]

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Then
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

- $\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$

$$= \left\{ \frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}$$

$$= \left\{\left(\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\hat{i}$$

- Green's theorem

Line theorem

Ex - ① , ② ④

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right]\hat{i}$$

$$= \left\{\frac{\partial}{\partial x}(\nabla \cdot \vec{A}) - (\nabla^2 A_1)\right\}\hat{i}$$

$$= \frac{\partial}{\partial x}(\nabla \cdot \vec{A}) \cdot \hat{i} - \nabla^2 \sum A_1 \hat{i}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

- Gradient, Divergence and curl

Examples (OBJECTIVE)

1) If $\vec{A} = x^2yz\,\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$, $\vec{B} = 2z\,\hat{i} + y\,\hat{j} - x^2\hat{k}$, then value of $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B})$ at $(1, 0, -2)$ is equal to

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\hat{i} + (2xz^3 + x^4yz)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}$$

$$\frac{\partial}{\partial y}(\vec{A} \times \vec{B}) = -xz^2\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}$$

$$\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -z^2\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}$$

so at $(1, 0, -2)$, $\frac{\partial^2}{\partial x \partial y}(\vec{A} \times \vec{B}) = -4\hat{i} - 8\hat{j}$

② If $f(x, y, z) = 3x^2y - y^3z^2$, then grad f at the point $(1, -2, 1)$ is equal to

$$f = 3x^2y - y^3z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\,\hat{k}$$

At $(1, -2, -1)$, $\nabla f = -12\hat{i} - 9\hat{j} - 16\hat{k}$

③ The gradient of $f(r)$, is equal to - (done)

④ $\nabla f(r) \times \vec{r}$ is equal to -

$$\nabla f(r) = \frac{f'(r)}{r}\vec{r} \quad \text{(as solved in the Previous Question)}$$

$$\nabla f(r) \times \vec{r} = 0$$

⑤ $\nabla\left(\frac{1}{r}\right)$ is equal to -

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(-\frac{1}{r^2}\right)\frac{x}{r}$$

$$= -\frac{1}{r^3}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot x$$

$$-\frac{1}{r^3} \cdot \vec{r}$$

$$-\frac{\vec{r}}{r^3}$$

⑥ $\nabla \log r$ is equal to –

$$\nabla \log r = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\frac{\partial}{\partial x}(\log r)$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}$$
$$= \frac{1}{r}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{x}{r}$$
$$= \frac{1}{r^2}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot x$$
$$= \frac{\vec{r}}{r^2}$$

⑦ ∇r^n is equal to –

$\cancel{(n-1)\, n r^{n-1}\, \vec{r}}$

$$\nabla r^n = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{\partial}{\partial x}(r^n)$$
$$= n r^{n-1}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\frac{\partial r}{\partial x}$$
$$= n r^{n-1}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{x}{r}$$
$$= n r^{n-2}\,\vec{r}$$

⑧ If $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then grad $(\vec{r}\cdot\vec{a})$ is equal to –

$$\nabla(\vec{r}\cdot\vec{a}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{\partial}{\partial x}(\vec{r}\cdot\vec{a})$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial\vec{r}}{\partial x}\cdot\vec{a}\right)$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(\hat{i}\cdot\vec{a})$$
$$= \vec{a}$$

⑨ Let $\vec{a}$ and $\vec{b}$ are constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ grad $[\vec{r}\,\vec{a}\,\vec{b}]$ is equal to –

$$\nabla[\vec{r}\,\vec{a}\,\vec{b}] = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\frac{\partial}{\partial x}(\vec{r}\cdot(\vec{a}\times\vec{b}))$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial\vec{r}}{\partial x}\cdot(\vec{a}\times\vec{b})\right)$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(\hat{i}\cdot(\vec{a}\times\vec{b}))$$
$$= (\vec{a}\times\vec{b})$$

⑩ If $\vec{a}$ is a constant vector, ϕ is scalar field $(\vec{a}\cdot\nabla)\phi$ is equal to –

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$
$$(\vec{a}\cdot\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$
$$= \vec{a}\cdot\nabla\phi$$

⑪ If $\vec{a}$ is constant vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $(\vec{a}\cdot\nabla)\vec{r}$ is equal to –

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{a}\cdot\nabla = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$
$$(\vec{a}\cdot\nabla)\vec{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$= \vec{a}$$

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(12) The unit normal vector to the level surface $x^2+y^2-z=4$ at point $(1,1,-2)$ is
normal vector lies in direction of ∇f

so, $\hat{n} = \frac{\nabla f}{|\nabla f|}$

$f = x^2+y^2-z$

$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$

At $(1,1,-2)$

$\nabla f = 2\hat{i} + 2\hat{j} - \hat{k}$

$|\nabla f| = \sqrt{9} = 3$

$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$

$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

(13) The directional derivative of $f(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}$ is:-

$\nabla f = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$

At $(1,-2,-1)$, $\nabla f = 8\hat{i} - \hat{j} - 10\hat{k}$

so, directional derivative of f in direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is equal to -

$\nabla f \cdot \hat{a} = \frac{1}{3}(8\hat{i} - \hat{j} - 10\hat{k}) \cdot (2\hat{i} - \hat{j} - 2\hat{k})$

$= \frac{37}{3}$

(14) The point P closest to origin on the plane $2x+y-z-5=0$ is
closest point will be foot of perpendicular from origin

$S = 2x + y - z - 5 = 0$

$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}}$

coordinate of $P = \left(\frac{2}{\sqrt{6}}r, \frac{1}{\sqrt{6}}r, -\frac{1}{\sqrt{6}}r\right)$

(15) It lies on S

so, $r = \frac{5}{\sqrt{6}}$

Hence, $P = \left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$

(15) The temperature T at a surface is given by $T = x^2+y^2-z$. In which direction a mosquito at the point $(4,4,2)$ on the surface will fly so that it cools fastest?

$T = x^2+y^2-z$

Direction of fastest cooling will lie in direction opposite to the direction of gradient i.e. $-\nabla T$

$\nabla T = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$= 8\hat{i} + 8\hat{j} - \hat{k}$

(16) The scalar function of which corresponds to $\vec{v} = \nabla f$, where $\vec{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is

$$f = \frac{1}{\sqrt{x^2+y^2+z^2}} + c$$

$$\nabla f = \frac{\vec{r}}{r^3}$$

(17) one of the point at which the derivative of the function $f(x,y) = x^2 - xy - y + y^2$ vanishes along the direction $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}$ is

$$\nabla f = (2x-y)\hat{i} - (x+1-2y)\hat{j}$$

Directional derivative in direction given by $\frac{\hat{i}+\sqrt{3}\hat{j}}{2}$

$$= \frac{1}{2}(2x-y) - \frac{\sqrt{3}}{2}(x+1-2y)$$

$$= \frac{2-\sqrt{3}}{2}x - \frac{(1-2\sqrt{3})}{2}y - \frac{\sqrt{3}}{2}$$

It becomes zero at $\left(-1, \frac{3}{2\sqrt{3}-1}\right)$

(18) which of the following is a unit normal vector to the surface $z = xy$ at $P(2,-1,-1)$?

The surface is $f = xy - z = 0$

$$\nabla f = y\hat{i} + x\hat{j} - \hat{k}$$

$$= y\hat{i} + x\hat{j} - \hat{k}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{-\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}}$$

(19) Let $f(x,y) = \ln\sqrt{x+y}$ and $g(x,y) = \sqrt{x+y}$, Then the value of $\nabla^2 (fg)$ at $(1,0)$:

$$f = \ln(x+y)^{\frac{1}{2}}, \quad g = \sqrt{x+y}$$

$$fg = \sqrt{x+y}\,\ln\sqrt{x+y}$$

$$\nabla^2 fg = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) fg$$

$$\frac{\partial}{\partial x}(fg) = \frac{1}{2\sqrt{x+y}} \ln\sqrt{x+y} + \sqrt{x+y}\,\frac{1}{\sqrt{x+y}}\,\frac{1}{2\sqrt{x+y}}$$

$$\frac{\partial^2 (fg)}{\partial x^2} = -\frac{1}{4}(x+y)^{-3/2}\ln(x+y) + \frac{1}{2(x+y)}\cdot\frac{1}{2\sqrt{x+y}}$$

$$= \frac{1}{4}(x+y)^{-3/2}$$

$$\nabla^2 fg = 0$$

(20) $x^2+y^2+z^2 = 1$ and $x^2 + (y-\sqrt{3})^2 + z^2 = 4$ intersect at an angle.

$$x^2+y^2+z^2 = 1$$

$$\Rightarrow x^2+y^2+z^2 - 2\sqrt{3}y = 1$$

They intersect at plane $y = 0$

$(0, 0, 1)$ is one point of intersection which is lying on the both spheres

Let us find normal vector at this point and find angle between them

$$\hat{n}_1 = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{n}_2 = \frac{x\hat{i} + (y-\sqrt{3})\hat{j} + z\hat{k}}{2}$$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \frac{1}{2}$ at point $(0,0,1)$

$\theta = \pi/3$

(21) Let $0 \le \theta \le \pi$ be the angle between the planes $x - y + z = 0$ and $2x - z = 4$. The value of θ is—

$x - y + z = 3$, $\Rightarrow x - y + z - 3 = 0$

$2x - z = 4$ $\Rightarrow 2x - z - 4 = 0$

Let us find angle between their normal

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \frac{1}{\sqrt{15}}$$

$$\hat{n}_2 = \frac{2\hat{i} - \hat{k}}{\sqrt{5}}, \quad \hat{n}_1 = \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{15}}$$

(22) $f(x,y) = xy^2 + yx^2$, suppose the directional derivative of f in the direction of the unit vector (u_1, u_2) at the point $(1,-1)$ is 1. Then among following (u_1, u_2) is —

$$f = xy^2 + yx^2$$

$$\nabla f = (y^2 + 2xy)\hat{i} + (x^2 + 2xy)\hat{j}$$

At $(1,-1)$, $\nabla f = -\hat{i} - \hat{j}$

$$\vec{n} = u_1\hat{i} + u_2\hat{j}$$

Directional derivative of f in direction of unit vector (u_1, u_2) is

$$\nabla f \cdot \vec{n} = 1$$

$$\Rightarrow -u_1 - u_2 = 1$$

$u_1 = -1, u_2 = 0$ satisfies above equation

(23) For what values of a and b, the directional derivative of $u(x,y,z) = ax^2yz + bxy^2z$ at $(1,1,1)$ along $\hat{i} + \hat{j} - 2\hat{k}$ is $\sqrt{6}$ and along $\hat{i} - \hat{j} + 2\hat{k}$ is $3\sqrt{6}$?

$$\nabla u = (2axyz + by^2z)\hat{i} + (ax^2z + 2bxyz)\hat{j} + (ax^2y + bxy)\hat{k}$$

The directional derivative of $u(x,y,z)$ along $(\hat{i} + \hat{j} - 2\hat{k})$ at $(1,1,1)$

$$\left[(2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k}\right] \cdot \left(\frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}}(2a + b + a + 2b - 2a - 2b)$$

$$\Rightarrow \frac{a+b}{\sqrt{6}} = \sqrt{6} \text{ (given)}$$

$$\Rightarrow a + b = 6 \quad \text{— (1)}$$

The direction derivative of $u(x,y,z)$ along $(\hat{i} - \hat{j} + 2\hat{k})$ at $(1,1,1)$

$$= \left((2a+b)\hat{i} + (a+2b)\hat{j} + (a+b)\hat{k}\right) \cdot \frac{\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{6}}$$

$$\Rightarrow \frac{1}{\sqrt{6}}(3a + b) = 3\sqrt{6} \text{ (given)}$$

$$\Rightarrow 3a + b = 18 \quad \text{— (2)}$$

Solving (1) and (2)

$a=6, b=0$

(24) If $f(x,y,z) = x-y$ and $\nabla\left(\frac{f}{g}\right) = \frac{1}{z}(\hat{i}-\hat{j}) - \left(\frac{x-y}{z^2}\right)\hat{k}$ then

$g(x,y,z)$ is —

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$= \frac{\nabla f}{g} - \frac{f}{g^2}\nabla g$$

Given $\nabla\left(\frac{f}{g}\right) = \frac{1}{z}(\hat{i}-\hat{j}) - \left(\frac{x-y}{z^2}\right)\hat{k}$

On Comparing $g(x,y,z) = z$

(25) The directional derivative of $f(x,y,z) = z^2 e^{\cos xy}$ at $(0, \frac{\pi}{2}, 1)$ along $(2\hat{i} - \hat{j} + 2\hat{k})$ is

$$\nabla f = -yz^2 e^{\cos xy}\sin xy\,\hat{i} - xz^2 e^{\cos xy}\sin xy\,\hat{j} + 2z e^{\cos xy}\hat{k}$$

Unit vector along $2\hat{i} - \hat{j} + 2\hat{k}$ is given by

$$\hat{n} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$$

Directional derivative along $(2\hat{i} - \hat{j} + 2\hat{k})$

$$\frac{df}{ds} = \nabla f\cdot\hat{n} = (-z^2 e^{\cos xy}\,y\sin xy\,\hat{i} - z^2 e^{\cos xy}\,x\sin xy\,\hat{j} + 2z e^{\cos xy}\hat{k})\cdot\frac{(2\hat{i} - \hat{j} + 2\hat{k})}{3}$$

$$= \frac{1}{3}(-2z^2 y e^{\cos xy}\sin xy + xz^2 e^{\cos xy}\sin xy + 4z e^{\cos xy})$$

At $(0, \frac{\pi}{2}, 1)$, directional derivative $= \frac{4}{3}e$

(26) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Then $\nabla(\vec{r}\cdot\nabla(\vec{r}\cdot\vec{r}))$ is equal to —

$$\nabla(\vec{r}\cdot\vec{r}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2) = 2\vec{r}$$

$$\Rightarrow \vec{r}\cdot\nabla(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\vec{r} = 2r^2$$

$$\Rightarrow \nabla(\vec{r}\cdot\nabla(\vec{r}\cdot\vec{r})) = 2\nabla(r^2) = 4\vec{r}$$

(27) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\nabla|\vec{r}|^4$ equals —

$$\nabla(|\vec{r}|^4) = \nabla(r^4)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\frac{\partial}{\partial r}(r^4)$$

$$= 4r^3\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{r}{r}$$

$$= 4r^2\vec{r}$$

(28) Let $T(x,y,z) = xy^2 + 2z - x^2z^2$ be the temperature at point (x,y,z). The unit vector in the direction in which the temperature decreases most rapidly at $(1,0,-1)$ is —

Temperature increases most rapidly in the direction of ∇T.

$$\nabla T = (y^2 - 2xz^2)\hat{i} + 2xy\hat{j} + (2 - 2x^2z)\hat{k}$$

At $(1,0,-1)$, $\nabla T = -2\hat{i} + 4\hat{k}$

unit vector in direction of $\nabla T = -\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}$

So, temperature decreases most rapidly in the direction $-\nabla T$.

i.e, $\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}$

• Green's Theorem (A 2n)

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy}$$

[one type of relation between 2's theorem]

(Ex-(objective))

1) Let c be the circle $x^2+y^2=1$ taken in the anti-clockwise sense. Then the value of the integral $\oint_C [(2xy^3+y)dx + (3x^2y^2+2x)dy]$ equals

$\oint_C [(2xy^3+y)dx + (3x^2y^2+2x)dy]$

$= \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R (6xy^2+2 - 6xy^2-1)\,dx\,dy$ (using green's theorem)

$= \iint_R dx\,dy$ [∬dx dy means the area of the region R]

$= \pi$

[$\because \pi r^2 = \pi \cdot 1^2 = \pi$]

[when ∬f(x) dx dy, then ...]

2) Let C be the boundary of region $R = \{(x,y) \in \mathbb{R}^2 : -1 \le y \le 1, 0 \le x \le 1-y^2\}$ oriented in the counter-clockwise direction. Then the value of $\oint_C y\,dx + 2x\,dy$ is

$\oint_C y\,dx + 2x\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint (2-1)\,dx\,dy$

$= \iint dx\,dy$

$= \int_{-1}^{1}\int_0^{1-y^2} dx\,dy$

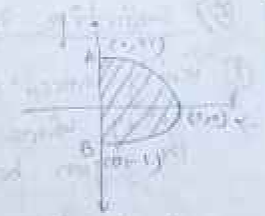
$= \int_{-1}^{1} (1-y^2)\,dy$

$= \left[y - \frac{y^3}{3}\right]$

$= \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$

$= \frac{2}{3} \ldots$

$= \frac{4}{3}$

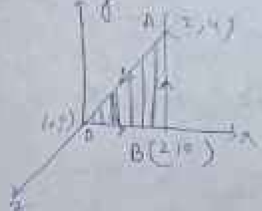


$x = 1-y^2$

$\Rightarrow y^2 = -(x-1)$

centre $= (1,0)$

3) The value of the integral $\oint_C (x+y)dx + x^2 dy$, where c is the triangle with vertices (0,0), (2,0) and (2,4) in the anti-clockwise direction is –



$\triangle OAB$

$\frac{y-0}{x-0} = \frac{4-0}{2-0}$

$\Rightarrow y = 2x$

$\oint_C (x+y)dx + x^2 dy$

$= \iint_R (2x-1)\,dx\,dy$

$= \int_0^2\int_0^{2x} (2x-1)\,dx\,dy$

$\int_0^2 \left[\frac{2x^2}{2} - x\right] dy$

$= \int_0^2 (4x^2-2x)\,dy$

$= \{4x^2-2x\}[y]_0^2 = 2(4x^2-2x)$

$\Rightarrow \int_{x=0}^{2} (2x-1)\int_{y=0}^{2x} dy\,dx$

$= \int_{x=0}^{2} (2x-1)\cdot[y]_0^{2x}\,dx$

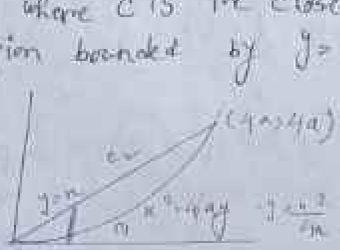
$\int_0^2 (4x^2-2x)\,dx$

$= \frac{20}{3}$

(2, 3)

Q) Subjective QnS

1) Verify Green's theorem in the plane for $\oint_C (xy + x^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $x^2 = 4ay$



Here $M\,dx + N\,dy = (xy + x^2)\,dx + x^2\,dy$

$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \int_0^{4a}\int_{x^2/4a}^{x} x\,dy\,dx$

$= \int_0^{4a} x\left(x - \frac{x^2}{4a}\right)dx$

$= \frac{16a^3}{3}$

Now, let us evaluate the line integral $\oint M\,dx + N\,dy$ on closed curve C. The curve C is a piecewise smooth curve consisting of c_1 and c_2

On c_1: $y = \frac{x^2}{4a}$ $\quad dy = \frac{x}{2a}\,dx$

$M\,dx + N\,dy = (xy + x^2)\,dx + x^2\,dy$

$= \left(\frac{x^3}{4a} + x^2\right)dx + x^2\cdot\frac{x}{2a}\,dx$

$= \left(\frac{3}{4}\frac{x^3}{a} + x^2\right)dx$

x varies from 0 to 4a on c_1

So, $\int_{c_1} M\,dx + N\,dy = \int_0^{4a}\left(\frac{3x^3}{4a} + x^2\right)dx$

$= \left[\frac{3}{16a}x^4 + \frac{x^3}{3}\right]_0^{4a}$

$= \frac{3}{16a}\cdot 256a^4 + \frac{1}{3}\cdot 64$

$= \frac{208a^3}{3}$

On c_2: $y = x$ $\quad dy = dx$

$M\,dx + N\,dy = (xy + x^2)\,dx + x^2\,dy$

$= 3x^2\,dx$

x varies from 4a to 0

So, $\int_{c_2} M\,dx + N\,dy = \int_{4a}^{0} 3x^2\,dx$

$= \left[x^3\right]_{4a}^{0}$

$= -64a^3$

So, $\oint_C M\,dx + N\,dy = \int_{c_1} M\,dx + N\,dy + \int_{c_2} M\,dx + N\,dy$

$= \frac{208}{3}a^3 - 64a^3$

$= \frac{16}{3}a^3$

Since $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint M\,dx + N\,dy$

∴ So, Green's theorem is verified

③ Verify Green's Theorem in the plane for $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in positive sense.

Here, $M\,dx + N\,dy = (2xy - x^2)\,dx + (x^2 + y^2)\,dy$

$M = 2xy - x^2 \Rightarrow \frac{\partial M}{\partial y} = 2x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

So, the double integral $\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ over region R bounded by $y = x^2$ (curve C_1) and $y^2 = x$ (curve C_2) is zero as integrand $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

Now, let us evaluate the integral over a closed curve C. The curve C is a piecewise smooth curve consisting of C_1 and C_2,

on C_1, $y = x^2$, $dy = 2x\,dx$ (taking x as independent variable)

$M\,dx + N\,dy = (2xy - x^2)\,dx + (x^2 + y^2)\,dy$

$= (2x^3 - x^2)\,dx + (x^2 + x^4) \cdot 2x\,dx$

$= (2x^5 + 4x^3 - x^2)\,dx$

y varies from 0 to 1 on C_2

$\int_{C_2} M\,dx + N\,dy = \int_1^0 (-2y^5 + 5y^4 + y^2)\,dy$

$= \left[-\frac{y^6}{3} + y^5 - \frac{y^3}{3}\right]_1^0$

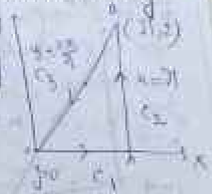
$= -1$

So, $\oint_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy = 0$

Since, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$

So, Green's Theorem is verified.

③ Apply Green's theorem in the plane to evaluate $\oint_C \{(y - \sin x)\,dx + \cos x\,dy\}$ where C is the triangle enclosed by the lines $y = 0$, $x = \pi$, $\pi y = 2x$



Here, $M\,dx + N\,dy = (y - \sin x)\,dx + \cos x\,dy$

So, $M = y - \sin x$, $\frac{\partial M}{\partial y} = 1$

$N = \cos x$, $\frac{\partial N}{\partial x} = -\sin x$

According to Green's theorem

$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

where R is the region enclosed by the piecewise smooth curve C consisting of curve C_1 ($y = 0$), curve C_2 ($x = \pi$), curve C_3 ($\pi y = 2x$) as shown in fig.

So, $\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^2 \int_{\pi y/2}^{\pi} (-\sin x - 1)\,dx\,dy$

$= \int_0^2 \left[\cos x - x\right]_{\pi y/2}^{\pi} dy$

$= \int_0^2 \left(-1 - \pi - \cos\frac{\pi y}{2} + \frac{\pi y}{2}\right) dy$

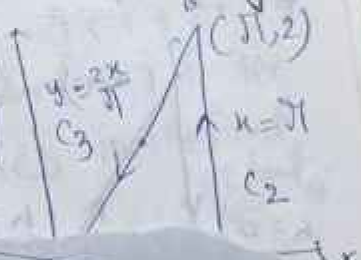
$= \left[-(1 + \pi)y - \frac{2}{\pi}\sin\frac{\pi y}{2} + \frac{\pi y^2}{4}\right]_0^2$

$= -2 - \pi$

[on the limit can,
$= \int_{x=0}^{\pi} \int_{y=0}^{2x/\pi} (-1 + \sin x)\,dx\,dy = -(2 + \pi)$]

[$\frac{y - 0}{x - 0} = \frac{2 - 0}{\pi - 0}$
$\Rightarrow y = \frac{2x}{\pi}$, $x = \frac{\pi y}{2}$]

where C is the triangle enclosed by the lines $y=0$, $x=\pi$, $\pi y=2x$



Here, $Mdx+Ndy=(y-\sin x)dx+\cos x\,dy$

$(2xy-x^2)dx+(x^2+y^2)dy$

$\frac{\partial M}{\partial y}=2x$

$\frac{\partial N}{\partial x}=2x$

integral $\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$

by $y=x^2$ (curve C_1)

as integr and $\frac{\partial N}{\partial x}$

ate the integral over

C is a piecewise

C_2

taking x as ind

$$-\int_0^{\pi}\int_0^{\frac{2x}{\pi}}(1+\sin x)\,dx\,dy$$

$$=-\int_0^{\pi}(1+\sin x)\left[y\right]_0^{\frac{2x}{\pi}}dx$$

$$=-\int_0^{\pi}(1+\sin x)\left(\frac{2x}{\pi}\right)dx$$

$$=-\frac{2}{\pi}\int_0^{\pi}(x+x\sin x)\,dx$$

$$=-\frac{2}{\pi}\left[\frac{x^2}{2}-x\cos x+\sin x\right]_0^{\pi}$$

$$=-\frac{2}{\pi}\left[\frac{\pi^2}{2}+\pi\right]=-\frac{2}{\pi}\cdot\frac{(2\pi+\pi^2)}{2}=-(2+\pi)$$

$(x^3-x^2)\,dx$

$0]$ on C_2

$\int_1^0(-2y^5+5y^4+y^2)\,dy$

$+y^5-y^3]^0$

$$\left[-(1+\pi)y-\frac{2}{\pi}\sin\frac{\pi y}{2}+\frac{\pi y^2}{4}\right]_0^2$$

$=-2-\pi$

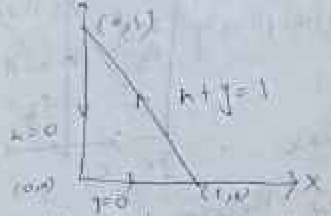
on the limit can,

$$\frac{y-0}{x-0}=\frac{2-0}{\pi-0}$$

Ex (objective)

⑤ If $\vec{F}(x,y) = (3x-8y)\hat{i} + (4y-6xy)\hat{j}$ for $(x,y) \in \mathbb{R}^2$ then $\oint_C \vec{F}\cdot d\vec{r}$, where C is the boundary of the triangular region bounded by the lines $x=0, y=0$ and $x+y=1$ oriented in the anti-clockwise direction, is –

$$\oint_C \vec{F}\cdot d\vec{r} = \oint (3x-8y)\,dx + (4y-6xy)\,dy$$



$$\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint(-6y+8)\,dx\,dy$$

$$= -6\int_0^1\int_0^{1-y} y\,dx\,dy + 8\iint dx\,dy$$

$$= -6\int_0^1 y(1-y)\,dy + 8\cdot\frac{1}{2}\cdot[1\times 1]$$

area of triangle

$$= -6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 + 4$$

$$= -6\left[\frac{1}{2} - \frac{1}{3}\right] + 4$$

$$= 3$$

(ii) $$\int_0^1\int_0^{1-x}(-6y+8)\,dx\,dy$$

$$= \int_0^1 (-3y^2 + 8y)_0^{1-x}\,dx$$

$$= \int_0^1 (-3x^2 - 2x + 5)\,dx$$

$$= [-x^3 - x^2 + 5x]_0^1$$

$$= -1 - 1 + 5 = 3$$

$$\int_0^1\int_0^{1-y}(-6y+8)\,dx\,dy$$

$$= \int_0^1\left\{\left[-\frac{6y^2}{2} + 8y\right]_0^{1-y}\right\}dx$$

$$= \int_0^1 \{-3y^2 + 8y\}_0^{1-y}\,dx$$

$$= \int_0^1 \{-3(1-y)^2 + 8(1-y)\}\,dx$$

$$= \int_0^1 \{-3(1-2y+y^2) + 8 - 8y\}\,dx$$

$$= \int_0^1 \{-3 + 6y - 3y^2 + 8 - 8y\}\,dx$$

$$= \int_0^1 \{-3y^2 - 2y + 5\}\,dx$$

$$= \{-3y^2 - 2y + 5\}$$

This is incorrect

- Surface integration

$$\boxed{\int_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}}$$

Ex – 1

① Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2 = a^2$ which lies in the first octant.

$S = x^2 + y^2 + z^2 = C$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{z}{a}$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{a}$$

$$= \frac{3xyz}{a}$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{a}{z}\,dx\,dy$$

$$\vec{F}\cdot\hat{n}\,ds = \frac{3xyz}{a}\cdot\frac{a}{z}\,dx\,dy$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = 3\iint_R xy\,dx\,dy$$

$$= 3\int_{x=0}^{a} x\int_{y=0}^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$$= 3\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{3}{2}\frac{(1-a^2)}{2}a^2$$

$$= \frac{3}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{3}{2}\cdot\frac{2a^4 - a^4}{4}$$

$$= \frac{3}{8}a^4$$

• Gauss divergence theorem:

$$\boxed{\oint_S \vec{F}\cdot\hat{n}\,ds = \int_V \nabla\cdot\vec{F}\,\tau}$$

Ex - (objective)

① Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2+y^2=1$ and $\hat{n}$ be the inward unit normal vector to S then $\oint_S \vec{F}\cdot\hat{n}\,ds$ is equal to -

$$\oint \vec{F}\cdot\hat{n}\,ds = -\oint \vec{F}\cdot\hat{n}'\,ds$$

where $\hat{n}'$ is outward drawn unit normal vector to S i.e $\hat{n} = -\hat{n}'$

$$= -\int \nabla\cdot\vec{F}\,dx$$

$$= -6\times \text{volume of sphere}$$

$$= -8\pi$$

n=1 sphere
unit sphere

• Surface integration

Ex-

① Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x+y=4$ in first octant cut off by the plane $z=4$.

The surface of the plane $2x+y=4$ belongs to family of level surfaces, $S=2x+y=$ constant. A unit vector normal to the surface -

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}}$$

The integrand $\vec{F}\cdot\hat{n} = (y\hat{i} + 2x\hat{j} - z\hat{k})\cdot\frac{(2\hat{i}+\hat{j})}{\sqrt{5}}$

$$= \frac{1}{\sqrt{5}}(2y+2x) = \frac{2}{\sqrt{5}}(x+y)$$

$$\hat{n}\cdot\hat{j} = \frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})\cdot\hat{j} = \frac{1}{\sqrt{5}}$$

Now, taking projection of the surface on xz plane as shown in fig. 1

$$ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \sqrt{5}\,dx\,dz$$

$$\vec{F}\cdot\hat{n}\,ds = \frac{2}{\sqrt{5}}(x+y)\sqrt{5}\,dx\,dz$$

$$= 2(x+y)\,dx\,dz$$

$$= 2(x+4-2x)\,dx\,dz \quad (\because y = 4-2x \text{ from the equation of surface})$$

$$= 2(4-x)\,dx\,dz$$

So, surface integral becomes

$$\int_S \vec{F}\cdot\hat{n}\,ds = 2\int_0^4\int_0^2 (4-x)\,dx\,dz \text{ -?}$$

$= 2\int_0^1 4x - \left[\frac{y^2}{2}\right]_0^2 dz$

• Gauss Divergence Theorem

Ex - objective

(4) The value of the integral $\oint_S \vec{F} \cdot \hat{n}\, ds$, where $\vec{F} = 3x\hat{i} + 2y\hat{j} + z\hat{k}$ and S is the closed surface given by the planes $x=0$, $x=1$, $y=0$, $y=2$, $z=0$ and $z=3$ is ___

By divergence theorem $= \oint_S \vec{F} \cdot \hat{n}\, ds = \int_V \nabla \cdot \vec{F}\, d\tau$

$$= 6\int_0^1\int_0^2\int_0^3 dx\, dy\, dz = 36$$

• Stokes theorem

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,ds}$$

• Conservative vector

[If the curl is zero than the field is conservative] $\boxed{\nabla\times\vec{F} = 0}$ [once the field is conservative then we have the potential func]

Ex (4) Show that the vector field $\boxed{\vec{F} = \nabla\phi}$
$\vec{F} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$
is conservative. Find its potential and also the work done in moving a particle from $(1,0)$ to $(2,1)$ along some curve.

vector field $\vec{F}$ is conservative if $\oint_C \vec{F}\cdot d\vec{r}$ around any closed curve is always zero.

By Stokes theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot\hat{n}\,ds = 0$$

so, for conservation field

$$\nabla\times\vec{F} = 0$$

$$\vec{F} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - y^4 + 3 & x^2 - 4xy^3 & 0 \end{vmatrix}$$

$$= [(2x - 4y^3) - (2x - 4y^3)]\hat{k}$$
$$= 0$$

$$\vec{F} = \nabla\phi$$

$$\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} = (2xy - y^4 + 3)\hat{i} + (x^2 - 4xy^3)\hat{j}$$

$$\frac{\partial\phi}{\partial x} = 2xy - y^4 + 3$$

$$\phi = \int_{y\ \text{const}} (2xy - y^4 + 3)\,dx$$
$$= x^2y - xy^4 + 3x$$

$$\frac{\partial\phi}{\partial y} = x^2 - 4xy^3$$

$$\phi = \int_{x\ \text{const}} (x^2 - 4xy^3)\,dy$$
$$= x^2y - xy^4$$

So the potential ϕ is given by

$$\phi = x^2y - xy^4 + 3x$$

$$\boxed{\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = d\phi}$$

$$\text{workdone} = \int \vec{F}\cdot d\vec{r} = \int d\phi = [\phi]_{(1,0)}^{(2,1)}$$
$$= [x^2y - xy^4 + 3x]_{(1,0)}^{(2,1)}$$
$$= 5$$

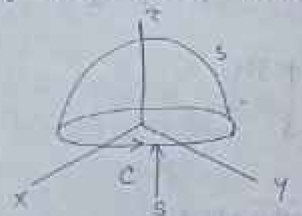
So, workdone is moving a particle from $(1,0)$ to $(2,1)$ is equal to 5.

• Stoke's Theorem

Example

(5) verify stoke's theorem $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=a^2$ and C is its boundary.

The surface S is the part of sphere $x^2+y^2+z^2=a^2$ above xy plane bounded by curve C, $x^2+y^2=a^2$, $z=0$ as shown in the fig –



On curve C, $x = a\cos\theta$, $y = a\sin\theta$, $z=0$

$dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, $dz = 0$

$\vec{F}\cdot d\vec{r} = (2x-y)dx - yz^2 dy - y^2 z\, dz$

$= (2a\cos\theta - a\sin\theta)(-a\sin\theta\, d\theta)$

$= a^2(\sin^2\theta - 2\sin\theta\cos\theta)d\theta$

So, $\oint_C \vec{F}\cdot d\vec{r} = a^2\int_0^{2\pi}(\sin^2\theta - 2\sin\theta\cos\theta)d\theta$

$= \pi a^2$

Now let us evaluate surface integral $\int_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$= \hat{k}$

Now, consider a closed piecewise smooth surface Σ consisting of spherical parts S: $x^2+y^2+z^2=a^2$ and its base, S': $z=0$ as shown in fig

By gauss divergence theorem

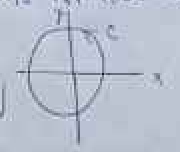
$\oint_\Sigma \nabla\times\vec{F}\cdot\hat{n}\, dS = \int_V \nabla\cdot(\nabla\times\vec{F})\, dv = 0$

(v is volume enclosed by Σ)

$\Rightarrow \int_S \nabla\times\vec{F}\cdot\hat{n}\, dS + \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\, dS = 0$

So, $\int_S \nabla\times\vec{F}\cdot\hat{n}\, dS = -\int_{S'} \nabla\times\vec{F}\cdot\hat{n}\, dS$

On S', outward drawn unit normal vector $\hat{n} = -\hat{k}$



$dS = dx\,dy$, $\nabla\times\vec{F}\cdot\hat{n} = (\hat{k})\cdot(-\hat{k}) = -1$

So, $\int_S \nabla\times\vec{F}\cdot\hat{n}\, dS = \int_{S'} \nabla\times\vec{F}\cdot\hat{n}\, dS$

$= \int dS$ = area of base $= \pi a^2$

So, $\oint_C \vec{F}\cdot d\vec{r} = \oint_S \nabla\times\vec{F}\cdot\hat{n}\, dS$

Hence, stoke's theorem is verified.

• Gradient, Curl, Divergence

(29) The eqn of a surface of revolution

$$z = \pm\sqrt{\frac{3}{2}x^2 + \frac{3}{2}y^2}$$

The unit normal to the surface at the point $A\left(\sqrt{\frac{2}{3}}, 0, 1\right)$ is -

Equation of surface is -

$$z^2 = \frac{3}{2}(x^2 + y^2)$$

$$F = 3x^2 + 3y^2 - 2z^2 = 0$$

unit normal to the surface

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{6x\hat{i} + 6y\hat{j} - 4z\hat{k}}{2\sqrt{9x^2 + 9y^2 + 4z^2}}$$

$$= \frac{3x\hat{i} + 3y\hat{j} - 2z\hat{k}}{\sqrt{9x^2 + 9y^2 + 4z^2}}$$

$$= \frac{3\sqrt{\frac{2}{3}}\hat{i} - 2\hat{k}}{\sqrt{9 \times \frac{2}{3} + 4}}$$

$$= \frac{\sqrt{6}\hat{i} - 2\hat{k}}{\sqrt{10}}$$

$$= \sqrt{\frac{3}{5}}\hat{i} - \frac{2}{\sqrt{10}}\hat{k}$$

(30) Let $\vec{r}$ be the distance of a point $P(x, y, z)$ from the origin O. The $\nabla\vec{r}$ is a vector - normal to the level surface of $\vec{r}$ at P.

(31) The value of c for which there exists a twice differentiable vector field $\vec{F}$ with curl $\vec{F} = 2x\hat{i} - 7y\hat{j} + cz\hat{k}$ is–

$\nabla\cdot(\nabla\times\vec{F}) = 0$

$\Rightarrow c = 5$

(32) For $c>0$, if $a\hat{i}+b\hat{j}+c\hat{k}$ is the unit normal vector at $(1,1,\sqrt{2})$ to the cone $z=\sqrt{x^2+y^2}$, then

$z=\sqrt{x^2+y^2}$ belongs to family of level surfaces given by–

$f(x,y,z) = \sqrt{x^2+y^2} - z = 0$

$\nabla f = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j} - \hat{k}$

At $(1,1,\sqrt{2})$, $\nabla f = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} - \hat{k}$

unit normal vector $\hat{n} = \frac{\nabla f}{|\nabla f|}$

$= \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$

A vector opposite to $\hat{n}$ is–

$\hat{n} = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

$= a\hat{i} + b\hat{j} + c\hat{k}$

So, $a^2+b^2-c^2=0$

(33) Let $\vec{r} = (x\hat{i}+y\hat{j}+z\hat{k})$ and $r=|\vec{r}|$. If $f(r) = \ln r$ and $g(r) = \frac{1}{r}$, $r\neq 0$. satisfy $2\nabla f + h(r)\nabla g = \vec{0}$ then $h(r)$ is–

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{\partial f}{\partial x}$

$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{\partial}{\partial x}(\log x)$

$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{1}{r}\cdot\frac{x}{r}$

$= \frac{1}{r^2}\cdot\vec{r}$

$g = \frac{1}{r}$

$\nabla g = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$

$= -\frac{1}{r^2}\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{x}{r}$

$= -\frac{1}{r^3}\cdot\vec{r}$

$2\nabla f + h\nabla g = 0$

$\Rightarrow \frac{2\vec{r}}{r^2} - \frac{h\vec{r}}{r^3} = 0$

$\Rightarrow h = 2r$

(since, $a=-\frac{1}{2}, b=-\frac{1}{2}, c=\frac{1}{\sqrt{2}}$)

(34) The tangent plane to the surface $z=\sqrt{x^2+3y^2}$ at $(1,1,2)$ is given by–

Eq^n of surface $z=\sqrt{x^2+3y^2}$

It belong to family of level surface given by – $f = \sqrt{x^2+3y^2} - z = $ constant

$$\nabla f = \frac{x}{\sqrt{x^2+3y^2}}\hat{i} + \frac{3y}{\sqrt{x^2+3y^2}}\hat{j} - \hat{k}$$

$$= \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - \hat{k} \text{ at } (1,1,2)$$

Equation of tangent plane —

$$(\vec{r} - \vec{r}^{\,\prime}) \cdot \nabla f = 0$$

$$\Rightarrow (x-1)\frac{1}{2} + (y-1)\frac{3}{2} - (z-2) = 0$$

$$x + 3y - 2z = 0$$

(35) In R^3, the cosine of the acute angle between the surfaces $x^2+y^2+z^2-9=0$ and $z-x^2-y^2+3=0$ at the point $(2,1,2)$ is—

$$S_1 = x^2+y^2+z^2-9=0$$

$$S_2 = z-x^2-y^2+3=0$$

$$\nabla S_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$= 4\hat{i} + 2\hat{j} + 4\hat{k}$$

Normal to $S_1 \Rightarrow \hat{n}_1 = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

$$\nabla S_2 = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$= -4\hat{i} - 2\hat{j} + \hat{k}$$

normal to S_2, $\hat{n}_2 = \frac{\nabla S_2}{|\nabla S_2|} = \frac{4\hat{i}+2\hat{j}-\hat{k}}{\sqrt{21}}$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \frac{8+2-2}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

(36) The tangent line to the curve of intersection of the surface $x^2+y^2-z=0$ and the plane $x+z=3$ at the point $(1,1,2)$ passes through—

$$S_1 = x^2+y^2-z=0$$

$$S_2 = x+z=3$$

$$\nabla S_1 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla S_2 = \hat{i} + \hat{k}$$

At $(1,1,2)$, $\nabla S_1 = 2\hat{i}+2\hat{j}-\hat{k}$

$$\nabla S_2 = \hat{i}+\hat{k}$$

Tangent will be along $\boxed{\nabla S_1 \times \nabla S_2 =}$

$$(2\hat{i}+2\hat{j}-\hat{k}) \times (\hat{i}+\hat{k})$$

$$= -2\hat{j} - 2\hat{k} + 2\hat{i} - \hat{j}$$

$$= 2\hat{i} - 3\hat{j} - 2\hat{k}$$

Equation of tangent —

$$\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-2}{-2}$$

It passes through $(-1,4,4)$

- Divergence, curl, Laplacian operator

IX — Solved examples (objective)

(1) $\operatorname{div} \vec{r}$ is equal to —

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \frac{\partial}{\partial x}(\vec{r})$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= 1+1+1 = 3$$

② curl $\vec{r}$ is equal to –

$$\text{curl}\,\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$= 0$

③ The value of constant a for which the vector $\vec{F} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal is –

$\boxed{\text{vector } \vec{F} \text{ is solenoidal if div } \vec{F} = 0}$

$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$

$\Rightarrow 1+1+a=0$ $\quad [\because \text{div}\,\vec{F} = \nabla\cdot\vec{F}]$

$\therefore a = -2$

④ If $\vec{a}$ is a constant vector, then $\nabla\cdot(\vec{r}\times\vec{a})$ is equal to –

$\text{div}\,(\vec{r}\times\vec{a}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\frac{\partial}{\partial x}(\vec{r}\times\vec{a})$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(\hat{i}\times\vec{a})$

$= 0$

⑤ If $\vec{a}$ is a constant vector, curl $(\vec{r}\times\vec{a})$ is equal to –

$\text{curl}\,(\vec{r}\times\vec{a}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\frac{\partial}{\partial x}(\vec{r}\times\vec{a})$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(\frac{\partial\vec{r}}{\partial x}\times\vec{a}\right)$

$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times(\hat{i}\times\vec{a})$

$[(\hat{i}\cdot\vec{a})\hat{i} - (\hat{i}\cdot\hat{i})\vec{a}]$

$(\hat{i}\cdot\vec{a})\hat{i} - 3\vec{a}$

$= \vec{a} - 3\vec{a}$

$= -2\vec{a}$

⑥ If $\vec{F} = e^{xyz}(\hat{i}+2\hat{j}+3\hat{k})$, the curl $\vec{F}$ at $(1,1,1)$ equal to –

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & 2e^{xyz} & 3e^{xyz} \end{vmatrix}$$

$= e^{xyz}(3xz - 2xy)\hat{i} + e^{xyz}(xy - 3yz)\hat{j} + e^{xyz}(2yz - xz)\hat{k}$

At $(1,1,1)$, $\nabla\times\vec{F} = e(\hat{i}+\hat{k})$

(7) If $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, then value of div $\vec{F}$ at $(1,1,1)$ is equal to–

$\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$$\text{d}\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz \quad [\text{div}\vec{F} = \nabla\cdot\vec{F}]$$

At $(1,1,1)$, div $\vec{F} = -3$

(8) If $\vec{F} = (x^2-y^2)\hat{i} + 2xy\hat{j} + (y^2-xy)\hat{k}$, the curl $\vec{F}$ at $(1,1,1)$ is equal to–

$\vec{F} = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$= -\hat{i} + \hat{j} - \hat{k}$

$$\vec{F}\cdot\text{curl}\,\vec{F} = \big((x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}\big)\cdot(-\hat{i} + \hat{j} - \hat{k})$$

$= -x - y - 1 + 1 + x + y$

(9) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, if $\varphi(x,y,z)$ is a solution of the Laplace equation then the vector field $(\nabla\varphi + \vec{r})$ is – irrotational but not solenoidal.

$\nabla\cdot(\nabla\varphi + \vec{r}) = \nabla\cdot\nabla\varphi + \nabla\cdot\vec{r}$

$= \nabla^2\varphi + 3 \quad (\nabla\cdot\vec{r} = 3)$

$= 3 \quad (\because \nabla^2\varphi = 0)$

Now $\nabla\times(\nabla\varphi + \vec{r})$

$\nabla\times(\nabla\varphi) + \nabla\times\vec{r}$

$= 0$

$\therefore$ curl is zero and divergence is non-zero. Hence, $\nabla\varphi + \vec{r}$ is irrotational but not solenoidal.

(10) Consider the vector field $\vec{F} = (ax + y + a)\hat{i} + \hat{j} - (x+y)\hat{k}$, where a is a constant. If $\vec{F}\cdot\text{curl}\,\vec{F} = 0$, then the value of a is–

$\vec{F} = (ax + y + a)\hat{i} + \hat{j} - (x+y)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax+y+a & 1 & -(x+y) \end{vmatrix} = -\hat{i} + \hat{j} - \hat{k}$$

$\vec{F}\cdot\text{curl}\,\vec{F} \Rightarrow -ax - y - a + 1 - x + y = 0$

$\Rightarrow -(a+1)x + (1-a) = 0$

$\Rightarrow a = 1$

(11) For a constant vector $\vec{a}$ and $\vec{r} = xi + yj + zk$, consider the following statements

I. $\text{curl}(\vec{a} \times \vec{r}) = 2\,\text{grad}(\vec{a}\cdot\vec{r})$

II. $\text{div}[(\vec{a}\cdot\vec{r})\vec{r}] = (\vec{a}\cdot\vec{r})$

then,

$\text{curl}(\vec{a}\times\vec{r}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \frac{\partial}{\partial x}(\vec{a}\times\vec{r})$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)$

$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (\vec{a} \times \hat{i})$

$= [(\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}]$

$= [\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}]$

$= 3\vec{a} - \vec{a}$

$= 2\vec{a}$

$2\,\text{grad}(\vec{a}\cdot\vec{r}) = 2\vec{a}$

So, (I) is correct

$\nabla\cdot[(\vec{a}\cdot\vec{r})\vec{r}] = \text{grad}(\vec{a}\cdot\vec{r})\vec{r} + (\vec{a}\cdot\vec{r})\,\text{div}\,\vec{r}$

$= \vec{a}\cdot\vec{r} + 3(\vec{a}\cdot\vec{r})$

$= 4(\vec{a}\cdot\vec{r})$

So, (II) is false

(12) Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a scalar fields $\vec{v}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and let $\vec{a}$ ... be a constant vector. If $\vec{r}$ represent the position vector $xi + yj + zk$, then which one of the following is false?

$\text{curl}(\vec{a}\times\vec{r}) = 2\vec{a}$ (1st part of previous one)

(13) Let $\vec{F}$ and $\vec{G}$ be differentiable vectors fields and let g be a differentiable scale function. Then —

$\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot\nabla\times\vec{F} - \vec{F}\cdot\nabla\times\vec{G}$,

$\nabla\cdot(g\vec{F}) = g\nabla\cdot\vec{F} + \nabla g\cdot\vec{F}$

- Cauchy's mean value theorem (Csin)

Let f and g be two functions defined on $[a,b]$ such that -

(i) f and g are continuous on $[a,b]$

(ii) f and g are derivable on (a,b) and

(iii) $g'(x) \neq 0$ for any x in (a,b)

Then there exist at least one value c, $a<c<b$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

- Lagrange's m.v. theorem from Cauchy's m.v. theorem

Let f be a function defined on $[a,b]$ such that

cmv

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

(mv) $f'(c) = \frac{f(b)-f(a)}{b-a}$

If $g(x) = x$

then $g(b) = b$, $g(a) = a$

$g'(x) = 1$, so $g'(c) = 1$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \rightarrow \text{Lmv}$$

Lmv is a part of cmv

- Another form of Cauchy's M.V. Theorem

Let $b = a+h$, then $c = a+\theta h$, $0<\theta<1$

So, $$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1$$

EX - 9.7

1) Verify CMV for the following pairs of functions

(i) $f(x) = \sin x$, $g(x) = \cos x$, in $[-\frac{\pi}{2}, 0]$

(ii) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$, in $[1,2]$

(i) $f(x) = \sin x$, $g(x) = \cos x$ in $[-\frac{\pi}{2}, 0]$

(i) $f(x)$ and $g(x)$ both are continuous on $[-\frac{\pi}{2}, 0]$

(ii) f, g both are differentiable in $(-\frac{\pi}{2}, 0)$

(iii) $f'(x) = \cos x$ $\quad g'(x) = -\sin x$

$g'(x)$ is not equal to 0 $\forall x \in (-\frac{\pi}{2}, 0)$

$\therefore g'(x) \neq 0$ in $(-\frac{\pi}{2}, 0)$

$$\frac{f(0) - f(-\frac{\pi}{2})}{g(0) - g(-\frac{\pi}{2})} = \frac{f'(c)}{g'(c)} \qquad -\frac{\pi}{2} < c < 0$$

$$\Rightarrow \frac{0+1}{1-0} = \frac{f'(c)}{g'(c)} = \frac{\cos c}{-\sin c}$$

$\Rightarrow \cot c = -1$

$\therefore c = -\frac{\pi}{4} \in (-\frac{\pi}{2}, 0)$

(ii) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$

(i) $f(x)$ and $g(x)$ both are continuous in $[1,2]$ and (ii) differentiable in $(1,2)$

(iii) $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = -\frac{1}{2(x)^{\frac{3}{2}}}$

$g'(x) \neq 0$ in $(1,2)$

Thus f and g satisfy all condn of C-M.V

So, there should exist a value c, $1<c<2$ such that,

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sqrt{2}-1}{\frac{1}{\sqrt{2}}-1} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c^{3/2}}}$$

$$\Rightarrow \frac{(\sqrt{2}-1)\sqrt{2}}{1-\sqrt{2}} = -\frac{c^{\frac{3}{2}}}{c^{\frac{1}{2}}}$$

$\Rightarrow -\sqrt{2} = -c$

$\Rightarrow c = \sqrt{2} \in (1,2)$

② verify cauch's m.v.T. for the function $f(x) = x^3$, $g(x) = x^2 + 3x$ in the interval $[-1, 1]$

Here $f(x) = x^3$, then $f'(x) = 3x^2$

and $g(x) = x^2 + 3x$, then $g'(x) = 2x + 3$

we observe that

(i) $f(x)$ and $g(x)$ are both polynomial functions and so are continuous in $-1 \le x \le 1$

(ii) $f'(x)$ and $g'(x)$ exist in $-1 < x < 1$

(iii) $g'(x) = 2x + 3 \ne 0$ in $-1 < x < 1$

Thus the functions $f(x)$ and $g(x)$ satisfy all the conditions of cauchy's mean value theorem. Then there is at least one point $x = c$ in $-1 < x < 1$

$$\frac{f(1) - f(-1)}{g(1) - g(-1)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{1 - (-1)}{4 - (-2)} = \frac{3c^2}{2c + 3}$$

$$\Rightarrow \frac{3c^2}{2c+3} = \frac{2}{6}$$

$\Rightarrow 9c^2 - 2c - 3 = 0$, giving $c = \frac{2 \pm \sqrt{112}}{18} = \frac{1 \pm 2\sqrt{7}}{9}$

Both these values of c are in $(-1, 1)$

③ In C.M.V.T. if $f(x) = e^x$ and $g(x) = e^{-x}$, show that θ is independent of both x and h and is equal to $\frac{1}{2}$.

C.M.T is

$$\frac{f(x+h) - f(x)}{g(x+h) - g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, \quad 0<\theta<1$$

$$\Rightarrow \frac{e^{x+h} - e^x}{e^{-(x+h)} - e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}} \quad \left[\because f'(x) = e^x, \; g'(x) = -e^{-x}\right]$$

$$\Rightarrow \frac{e^x(e^h - 1)}{e^{-x}(e^{-h} - 1)} = \frac{e^x \cdot e^{\theta h}}{-e^{-x} \cdot e^{-\theta h}}$$

$$\Rightarrow \frac{e^h(e^h - 1)}{1 - e^h} = \frac{-e^{\theta h}}{e^{-\theta h}}$$

$$\Rightarrow -e^h = -e^{2\theta h}$$

$$\Rightarrow 2\theta h = h$$

$$\Rightarrow \theta = \frac{1}{2}$$

so θ is independent of both x and h and is equal to $\frac{1}{2}$.

• Generalised M.V.T : Taylor's Theorem

Theorem I : (Taylor's theorem with Lagrange's form of remainder)

Let f be a function defined on [a,b] such that

(i) the (n-1) th derivative $f^{(n-1)}$ is continuous on [a, b] and

(ii) the nth derivative $f^{(n)}$ exists on (a, b)

Then there exist at least one value c, $a<c<b$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \cdots + \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!}(b-a)^n f^{(n)}(c)$$

[Taylor's series of f]

• Another form of T. t.(I)

Let f be a fu^n defined on [a, a+h], h>0 such that (i) $f^{(n-1)}$ is continuous on [a, a+h] and (ii) $f^{(n)}$ exists on (a, a+h)

Then there exist at least one no. θ, $0<\theta<1$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

• The last term of the above series i.e (n+1) th term is called Remainder after n terms and is denoted by R_n

$$\boxed{R_n = \frac{h^n}{n!} f^{n}(a+\theta h),\ 0<\theta<1}$$

• Theorem II : Taylor's theorem with Cauchy's form of remainder

Let f be a function defined on [a, b] such that

(i) $f^{(n-1)}$ is continuous on [a, b] and

(ii) $f^{(n)}$ exists on (a, b)

Then there exist at least one value c, $a<c<b$, such that.

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \cdots + \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{(n-1)!}(b-a)(b-c)^{n-1} f^{n}(c)$$

• **Another form of T.T (ii)**

Let f be a function defined on $[a, a+h]$, s.t. such that:

(i) $f^{(n-1)}$ is continuous on $[a, a+h]$ and

(ii) $f^{(n)}$ exists on $(a, a+h)$

Then there exist at least one no. θ $0<\theta<1$ such that

$$F(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

The last term of the above series i.e. the $(n+1)$th term is called the remainder after n terms and is denoted by R_n.

• Remarks:- (i) The conclusion of Taylor's theorem, is also known as Taylor's formula or Taylor's series of the function $f(x)$.

(ii) By putting $b=x$ in (1), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}\{a+\theta(x-a)\}, 0<\theta<1$$

which is called the expansion of $f(x)$ about $x=a$

(iii) Taylor's theorem is also called the M.V.T. of the nth order. So the 3rd order M.V.T. is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a+\theta h)$$

$0<\theta<1$

• Maclaurin's theorem

Let f be a function defined on $[-h, h]$, $h > 0$ such that

(i) $f^{(n-1)}$ is continuous on $[-h, h]$ and (ii) $f^{(n)}$ exists on $(-h, h)$ then there exist at least one no. θ, $0 < \theta < 1$, such that

(i) $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x)$

which is Lagrange's form of remainder R_n.

(ii) $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{n}(\theta x)$

which is Cauchy's form of remainder

The above series is known as maclaurin's series in finite form for the function $f(x)$.

$1 > \theta > 0$

Ex

(1) write a taylor's formula for the function $f(x) = \log(1+x)$, $-1 < x < \infty$ about $x = 2$ with Lagrange's form of

Here function $f(x) = \log(x+1)$, $-1 < x < \infty$ is continuous and derivable at each of $(-1, \infty)$

now,

$f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$ and so on

now $x = 2$ T. formula for the fn $f(x)$ about is

$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''\{2 + \theta(x-2)\}$, $0 < \theta < 1$

$f(2) = \log 3$, $f'(2) = \frac{1}{3}$, $f''(2) = \frac{-1}{9}$

$\therefore f(x) = \log 3 + \frac{1}{3}(x-2) + \frac{1}{2!} \cdot (x-2)^2 \left(-\frac{1}{9}\right) + \frac{(x-2)^3}{3!} \cdot \frac{2}{\{1+2+\theta(x-2)\}^3}$

$\log(1+x) = \log 3 + \frac{x-2}{3} - \frac{(x-2)^2}{18} + \frac{(x-2)^3}{3}$

This is remainder that $\frac{1}{\{3+\theta(x-2)\}^3}$

where, $0<\theta<1$

(2) Expand $\cos x$ in a finite series (in power of x) in Lagrange's form of remainder.

$f(x) = \cos x \therefore f(0) = 1$

Now $f^{(n)}(x) = \cos\left(\frac{n\pi}{2} + x\right)$

$\therefore f^{(n)}(0) = \cos\frac{n\pi}{2}$

$\therefore f'(0) = 0, f''(0) = 1, f'''(0) = 0, f^{iv}(0) = 1$ and so on

Now the Maclaurin's series for the function $f(x)$ in finite form in Lagrange's form of remainder is —

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots + \frac{x^n}{n!}f^n(\theta x),$$

$0<\theta<1$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right)$$

(Remainder Part)

(4),(5),(2),(3)

(4) Using mean value theorem, show that —

$\sin x > x - \frac{1}{6}x^3$, if $0<x<\frac{\pi}{2}$

Let $f(x) = \sin x \therefore f(0) = 0$

$f'(x) = \cos x \quad \therefore f'(0) = 1$

$f''(x) = -\sin x \quad \therefore f''(0) = 0$

$f'''(x) = -\cos x \quad \therefore f'''(\theta x) = -\cos\theta x$

Now the Maclaurin's series for the function $f(x)$ of order 3 is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x),$$

$0<\theta<1$

$$\therefore \sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-\cos\theta x)$$

$$\Rightarrow \sin x = x - \frac{x^3}{6}\cos\theta x$$

Since $0<x<\frac{\pi}{2}$ and $0<\theta<1$, so $0<\theta x<\frac{\pi}{2}$

or, $0<\cos\theta x<1$

$$\Rightarrow -\frac{x^3}{6}\cos\theta x > -\frac{x^3}{6} \text{ as } x>0$$

$$\Rightarrow x - \frac{x^3}{6}\cos\theta x > x - \frac{x^3}{6}$$

$$\therefore \sin x > x - \frac{x^3}{6} \text{ (proved)}$$

(5) Expand the polynomial $f(x) = x^3 - 2x^2 + 3x + 5$ in a series of positive integral powers of $(x-2)$.

Here, $f(x) = x^3 - 2x^2 + 3x + 5$

then, $f'(x) = 3x^2 - 4x + 3$

$f''(x) = 6x - 4$

$f'''(x) = 6$

$f^{n}(x) = 0$, when $n \geq 4$

So, $f(2) = 11$, $f'(2) = 7$, $f''(2) = 8$, $f'''(2) = 6$

and $f^{n}(2) = 0$, when $n \geq 4$.

Now, Taylor's Theorem with Lagrange's form of remainder can be written as

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(x-a)^n}{n!} f^{n}(c), \quad \text{—(1)}$$

where $c = a + \theta(x-a)$, $0 < \theta < 1$

Putting $a = 2$ and the values of $f(2)$, $f'(2)$ ... in (1) we get $f(x) = x^3 - 2x^2 + 3x + 5$

$$= 11 + (x-2) \cdot 7 + \frac{(x-2)^2}{2!} \cdot 8 + \frac{(x-2)^3}{3!} \cdot 6$$

$$= 11 + 7(x-2) + 4(x-2)^2 + (x-2)^3$$

$$= (x-2)^3 + 4(x-2)^2 + 7(x-2) + 11$$

(6) Apply Maclaurin's theorem on $f(x) = (1+x)^4$ to deduce that $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Here, $f(x) = (1+x)^4$

then

$f'(x) = 4(1+x)^3$, $f''(x) = 12(1+x)^2$

$f'''(x) = 24(1+x)$, $f^{iv}(x) = 24$ and

$f^{n}(x) = 0$, when $n \geq 5$

So, $f(0) = 1$, $f'(0) = 4$, $f''(0) = 12$,

$f'''(0) = 24$, $f^{iv}(0) = 24$

$f(x)$ being a polynomial possesses derivatives of every order for all values of x.

By Maclaurin's theorem with Lagrange's form of remainder after four terms, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(\theta x), \quad 0 < \theta < 1$$

$$f(x) = 1 + x \cdot 4 + \frac{x^2}{2!} \cdot 12 + \frac{x^3}{3!} \cdot 24 + \frac{x^4}{24} \cdot 24$$

$$\Rightarrow (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

(7) Expand $\sin x$ in a finite series in powers of x with remainder in Lagrange's form.

Let $f(x) = \sin x$, then $f^n(x) = \sin(\frac{n\pi}{2}+x)$

and $f^n(0) = \sin\frac{n\pi}{2}$

so $f(x)$ possesses derivatives of all orders for every value of x.

By Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

so, $\sin x = \sin 0 + x \sin\frac{\pi}{2} + \frac{x^2}{2!}\left(2\cdot\frac{\pi}{2}\right) + \frac{x^3}{3!}\sin\left(\frac{3\pi}{2}\right)$

$+ \cdots \frac{x^{n-1}}{(n-1)!} \sin\frac{(n-1)\pi}{2} + \frac{x^n}{n!}$

$\sin\left(\frac{n\pi}{2} + \theta x\right)$, $0<\theta<1$

$\Rightarrow \sin x = x - \frac{1}{3!}x^3 + \cdots + \frac{x^{n-1}}{(n-1)!}\sin\frac{(n-1)\pi}{2} + \frac{x^n}{n!}$

$\sin\left(\frac{n\pi}{2} + \theta x\right)$

$0<\theta<1$

• Multivariable calculus (2 sin)

A function more than one independent variable.

$z = y = f(x)$

$z = f(x,y) = x^2 - 3x + 3$ or $\sin x$ or $\frac{x+y^2}{x-y}$

$z = f(x,y) = \frac{x^2+y^2}{x^2-y^2}$, $x^2-y^2 \neq 0$

dependent variable; independent variable

$f: R^2 \to R$ [$f(x,y)$ = fun with two variables]

$R^2 = R\times R = \{(x,y) : x\in R, y\in R\}$

$f(x,y) = \frac{x^2+y^2}{x^2-y^2}$, $x^2-y^2 \neq 0$

$x^2 - y^2 = 0$

$f: R^2 \to R^2$

$f(x,y) = (x+y, x-y)$

fun^n with 2 independent variable

$f: R^3 \to R^3$

$f(x,y) = (x+y, x-y, xy)$

$\boxed{f: R^n \to R^m}$

we talk about $\boxed{f: R^2 \to R}$

• **Limit**: A function $f: R^2 \to R$ is said to be tend to be a limit as the point (x,y) tends to a given points (a,b), if for each $\epsilon>0$, $\exists$ (there exist) a $\delta>0$ (depending on ϵ) such that $|f(x,y) - A| < \epsilon$ for all points (x,y) of the domain which belongs to some deleted nbd N' of (a,b)

Here N may be a square nbd

$0<|x-a|<\delta$ and $0<|y-b|<\delta$

[square nbd from these which relations satisfies this relation these are square nbd]

circulate nbd $0<(x-a)^2+(y-b)^2<\delta^2$

Triangular nbd → eqn of a triangle that belongs to (a,b)

- deleted nbd = $N' = N - \{(a,b)\}$

- Ex

(1) Let $f(x,y) = \frac{x^2y^2}{x^2+y^2}$

Limit is denoted by $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = A}$

or $\boxed{\lim_{\substack{x\to a\\ y\to b}} f(x,y) = A}$

- Ex

(1) $f(x,y) = \frac{x^2y^2}{x^2+y^2}$ show that $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2+y^2} = 0$

Let $\epsilon>0$ be given

Now, $\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \epsilon$

$\Rightarrow \frac{x^2y^2}{x^2+y^2} < \epsilon$

we know that $x^2<x^2+y^2$ and $y^2<x^2+y^2$

So, $\frac{x^2y^2}{x^2+y^2} < \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2$

This shows that $\frac{x^2y^2}{x^2+y^2} < \epsilon$ if $x^2+y^2<\epsilon$

$\therefore x^2+y^2<\epsilon$

$\Rightarrow (x-0)^2+(y-0)^2<\delta^2$ $[\delta^2=\sqrt{\epsilon}]$

[circulate nbd]

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2+y^2} = 0$

(2) show that $\lim_{\substack{x\to 0\\ y\to 0}} xy\frac{x^2-y^2}{x^2+y^2} = 0$

Let $\epsilon>0$ be given

Let $\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \epsilon$ ie $\left|xy\frac{x^2-y^2}{x^2+y^2}\right|<\epsilon$

but $\left|xy\frac{x^2-y^2}{x^2+y^2}\right| = |x|\cdot|y|\frac{|x^2-y^2|}{x^2+y^2} < |x|\,|y|$

$\left[\because \frac{|x^2-y^2|}{x^2+y^2}<1\right]$

Now, we know that

$|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$

So, $\left|xy \dfrac{x^2-y^2}{x^2+y^2}\right| < |x|\cdot|y| < \sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}$
$= x^2+y^2$

This shows that $\left|xy\dfrac{x^2-y^2}{x^2+y^2} - 0\right| < \epsilon$ if

$x^2+y^2 < \epsilon$

$\Rightarrow (x-0)^2+(y-0)^2 < \delta^2$ [where $\delta = \sqrt{\epsilon}$]

[circular nbd of (0,0)]

Therefore $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left|xy\dfrac{x^2-y^2}{x^2+y^2}\right| = 0$

③ show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ does not exist.

Put $y = mx$

Say, $f(x,y) = \dfrac{xy}{x^2+y^2}$

$\Rightarrow f(x, mx) = \dfrac{x\cdot mx}{x^2+m^2x^2}$

$= \dfrac{x^2\cdot m}{x^2(1+m^2)}$

$= \dfrac{m}{1+m^2}$

Now $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2} = \lim\limits_{\substack{x\to 0\\ \cancel{mx\to 0}}} \dfrac{x\cdot mx}{x^2+m^2x^2}$ [Put $y = mx$] [$mx\to 0$ $\therefore x\to 0$]

$= \lim\limits_{x\to 0} \dfrac{m}{1+m^2}$

$= \dfrac{m}{1+m^2}$

So this gives us different limiting value for different value of m. [putting $m=1$, $\frac{1}{1+1}=\frac{1}{2}$; putting $m=2$, $\frac{2}{1+4}=\frac{2}{5}$] Therefore $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ doesn't exist.

④ show that $\lim\limits_{\substack{x\to 0\\ y\to 0}}$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ doesn't exist

put $y = mx^2$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{(x,mx^2)\to(0,0)} \dfrac{x^4m}{x^4+m^2x^4}$

$= \dfrac{m}{1+m^2}$

So this gives us different limiting value for different value of m.

Therefore $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ doesn't exist.

• continuity of function of two variable

• continuity on $\mathbb{R}^2$

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function then $f(x,y)$ is said to be continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ /

$\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

• ~~Partial~~ derivatives

~~f(x,y)~~ Let $f(x,y)$ be a function. We say that $f(x,y)$ is differentiable at some point (a,b) if,

$$\lim_{\substack{h \to 0 \\ k \to 0}} \frac{f(x+h, y+k) - f(a, b)}{(h, k)}$$

- Partial derivatives

Denoted by $f_x(x,y)$, $\frac{\partial f}{\partial x}$, f_x ; (Partial derivative w.r.t x) (In this case y take as a constant)

or $f_y(x,y)$, $\frac{\partial f}{\partial y}$, f_y ; (Partial derivatives w.r.t y) (In this case x take as a constant)

Now

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

- Ex = Let $f(x,y) = 3x^2 + 6xy + 4y^2$. Find out f_x, f_y.

$f(x,y) = 3x^2 + 6xy + 4y^2$

$f_x(x,y) = \cancel{\lim}\ 6x + 6y$.

$f_y(x,y) = 6x + 8y$

$f_{xy}(x,y) = 6$

$f_{yy}(x,y) = 8$

$f_{yx}(x,y) = 6$, $f_{xx}(x,y) = 6$

$f_{xy}(x,y) \to$ Partial derivatives 2 time, 1st time w.r.t x, 2nd w.r.t y

$f_{yx}(x,y) \to$ 1st time w.r.t y, 2nd w.r.t x

* $\boxed{f_{xy}(x,y) = f_{yx}(x,y)}$

- Directional derivatives

defination – The directional derivatives of $f(x,y)$ in the direction (l, m) at the point (a,b) is defined by

$$\lim_{t \to 0} \frac{f(a+tl, b+tm) - f(a,b)}{t} = D_{(l,m)} f(a,b)$$

- Ex

① Find out the directional derivative of $f(x,y) = 2x^2 - xy + 5$ at $(1,1)$ in the direction $(3,-4)$

$$D_{(3,-4)} f(1,1) = \lim_{t \to 0} \frac{f(1+3t, 1-4t) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{2(1+3t)^2 - (1+3t)(1-4t) + 5 - (2-1+5)}{t}$$

$$= \lim_{t \to 0} \frac{2(1+6t+9t^2) - (1-4t+3t-12t^2) + 5 - 2 + 1 - 5}{t}$$

$$= \lim_{t \to 0} \frac{2 + 12t + 18t^2 - 1 + 4t - 3t\ 2 - 1}{t}$$

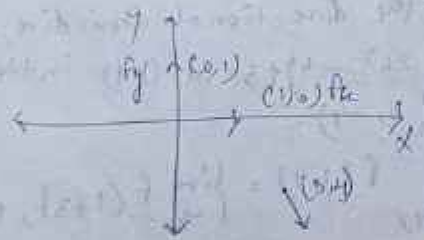
$= \lim_{t\to 0} \frac{30t^2+13t}{t}$

$= \lim_{t\to 0} \frac{t(30t+13)}{t}$

$= \lim_{t\to 0} (30t+13)$

$= 13$

• Difference between partial derivative and directional derivative



$\lim_{t\to 0} \frac{f(a+lt, b+mt) - f(a,b)}{t}$ (l,m)

(1,0) $\lim_{t\to 0} \frac{f(a+t,b) - f(a,b)}{t} = f_x$

(0,1) $\lim_{t\to 0} \frac{f(a,b+t) - f(a,b)}{t} = f_y$

Partial derivative is nothing but a special type of directional derivative.

Ex

① $f(x,y) = \frac{x^2y}{x^4+y^2}$, $(x,y) \neq (0,0)$

$= 0$, $(x,y) = (0,0)$

Show that the directional derivative of $f(x,y)$ exists at $(0,0)$ in every direction (l,m) but $f(x,y)$ is not continuous at $(0,0)$.

$f(x,y) = \frac{x^2y}{x^4+y^2}$

Put, $y = mx^2$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^4+y^2}$

$= \lim_{x\to 0} \frac{x^2\cdot mx^2}{(m^2+1)x^4}$

$= \frac{m}{1+m^2}$

this gives us different limiting value for different value of m. So $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^4+y^2}$ doesn't exist but $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0$ at $(x,y) = (0,0)$ $[f(0,0) = 0]$ ∴ $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$

Therefore $f(x,y)$ is not continuous at $(0,0)$.

$D_{(l,m)} f(0,0) = \lim_{t\to 0} \frac{f(lt, mt) - f(0,0)}{t}$

$= \lim_{t\to 0} \frac{\frac{l^2t^2\cdot mt}{l^4t^4+m^2t^2}}{t}$

$= \lim_{t\to 0} \frac{l^2mt^{3}}{t^{3}(l^4t^2+m^2)}$

$= \frac{l^2m}{m^2} = \frac{l^2}{m}$

∴ The directional derivative exist.