CS480/680: Introduction to Machine Learning

Homework 4

Due: 11:59 pm, July 24, 2024, submit on LEARN.

NAME

student number

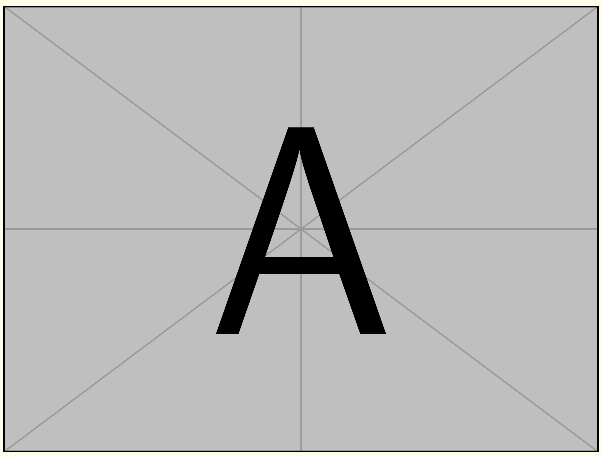
Submit your writeup in pdf and all source code in a zip file (with proper documentation). Write a script for each programming exercise so that the TA can easily run and verify your results. Make sure your code runs!

[Text in square brackets are hints that can be ignored.]

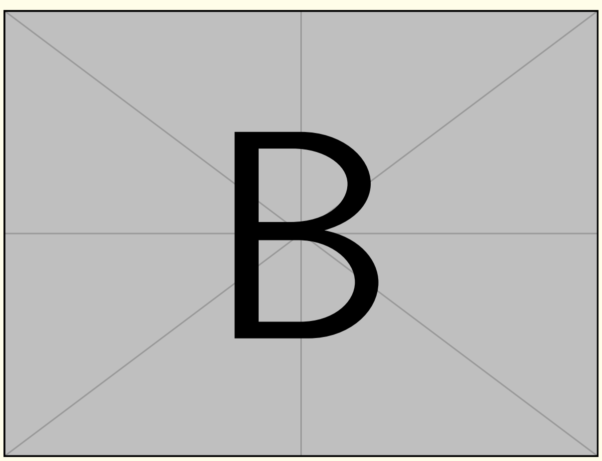
Exercise 1: Generative Adversarial Networks (10 pts)

Please follow the instructions of this `ipynb` file.

1. (2+6 pts) Complete the missing coding parts in the provided `ipynb` file.
2. (1 pt) Visualization of the loss:



3. (1 pt) Visualization of the generated images:



Exercise 2: Quantile and push-forward (8 pts)

In this exercise we compute and simulate the push-forward map T that transforms a reference density r into a target density p. Recall that the quantile function of a (univariate) random variable X is defined as the inverse of its cumulative distribution function (cdf) F:

$$F(x) = \Pr(\mathsf{X} \le x), \qquad Q(u) = F^{-1}(u), \quad u \in (0, 1). \tag{1}$$

We assume F is continuous and strictly increasing so that $Q^{-1} = F$. A nice property of the quantile function, relevant to sampling, is that if $\mathsf{U} \sim \text{Uniform}(0, 1)$, then $Q(\mathsf{U}) \sim F$.

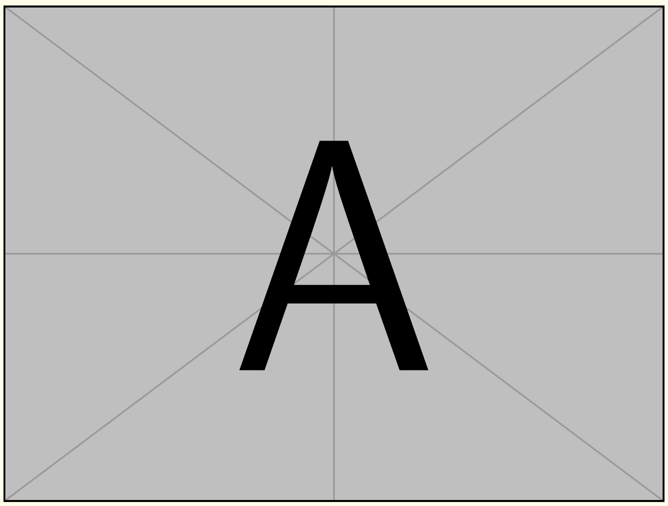
In the following, do not confuse cdf (signaled by uppercase letters) with pdf (i.e., density, signaled by lowercase letters).

1. (1 pt) Consider the Gaussian mixture model (GMM) with density $p(x) = \frac{\lambda}{\sigma_1}\varphi\left(\frac{x-\mu_1}{\sigma_1}\right) + \frac{1-\lambda}{\sigma_2}\varphi\left(\frac{x-\mu_2}{\sigma_2}\right)$, where φ is the *density* of the standard normal distribution (mean 0 and variance 1). Implement the following to create a dataset of $n = 1000$ samples from the GMM p:

   - Sample $\mathsf{U}_i \sim \text{Uniform}(0, 1)$.
   - If $\mathsf{U}_i < \lambda$, sample $\mathsf{X}_i \sim \mathcal{N}(\mu_1, \sigma_1^2)$; otherwise sample $\mathsf{X}_i \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

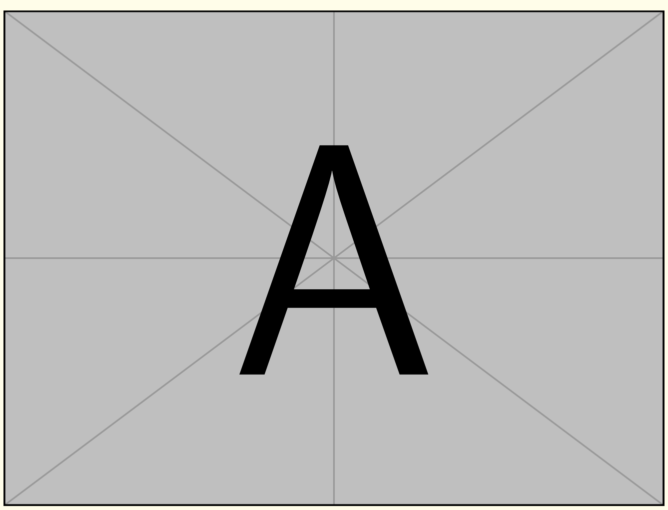
   Plot the histogram of the generated X_i (with $b = 50$ bins) and submit your script as
   `X = GMMsample(gmm, n=1000, b=50), gmm.lambda=0.5, gmm.mu=[1,-1], gmm.sigma=[0.5,0.5]`

   [See here or here for how to plot a histogram in `matplotlib` or `pandas` (or `numpy` if you insist).]

   Ans: 

2. (2 pts) Compute $\mathsf{U}_i = \Phi^{-1}\big(F(\mathsf{X}_i)\big)$, where F is the *cdf* of the GMM in Ex 2.1 and Φ is the *cdf* of standard normal. Plot the histogram of the generated U_i (with b bins). From your inspection, what distribution should U_i follow (approximately)? Submit your script as `GMMinv(X, gmm, b=50)`.

   [This page may be helpful.]

   Ans: 

3. (2 pts) Let $\mathsf{Z} \sim \mathcal{N}(0, 1)$. We now compute the push-forward map T so that $T(\mathsf{Z}) = \mathsf{X} \sim p$ (the GMM in Ex 2.1). We use the formula:

$$T(z) = Q(\Phi(z)), \tag{2}$$

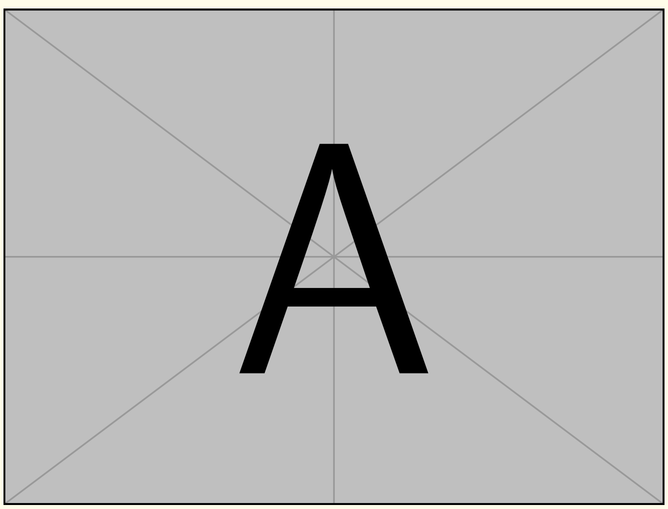
where Φ is the *cdf* of the standard normal distribution and $Q = F^{-1}$ is the quantile function of X, namely the GMM p in Ex 2.1. Implement the following binary search Algorithm 1 to numerically compute T. Plot the function T with input $z \in [-5, 5]$ (increment 0.1). Submit your main script as `BinarySearch(F, u, lb=-100, ub=100, maxiter=100, tol=1e-5)`, where `F` is a function. You may need to write another script to compute and plot T (based on `BinarySearch`).

Algorithm 1: Binary search for solving a monotonic nonlinear equation $F(x) = u$.

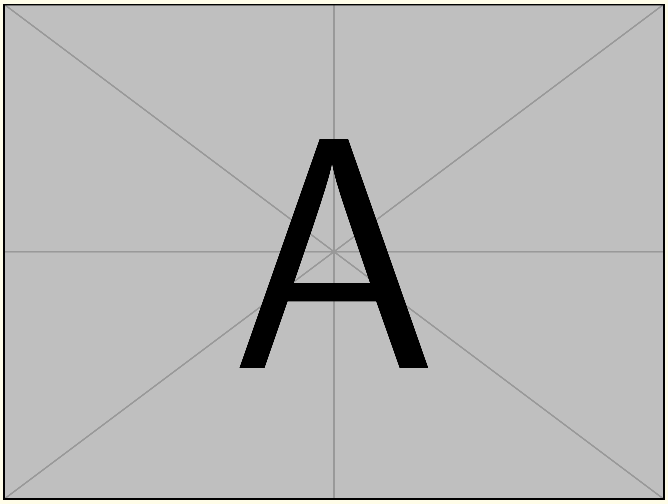
Input: $u \in (0, 1)$, $\mathtt{lb} < 0 < \mathtt{ub}$, $\mathtt{maxiter}, \mathtt{tol}$

Output: x such that $|F(x) - u| \le \mathtt{tol}$

while $F(\mathtt{lb}) > u$ **do** // lower bound too large
  $\mathtt{ub} \leftarrow \mathtt{lb}$
  $\mathtt{lb} \leftarrow 2 * \mathtt{lb}$
while $F(\mathtt{ub}) < u$ **do** // upper bound too small
  $\mathtt{lb} \leftarrow \mathtt{ub}$
  $\mathtt{ub} \leftarrow 2 * \mathtt{ub}$
for $i = 1, \ldots, \mathtt{maxiter}$ **do**
  $x \leftarrow \frac{\mathtt{lb}+\mathtt{ub}}{2}$ // try middle point
  $t \leftarrow F(x)$
  if $t > u$ **then**
    $\mathtt{ub} \leftarrow x$
  else
    $\mathtt{lb} \leftarrow x$
  if $|t - u| \le \mathtt{tol}$ **then**
    break

Ans: 

4. (2 pts) Sample (independently) $\mathsf{Z}_i \sim \mathcal{N}(0, 1), i = 1, \ldots, n = 1000$ and let $\tilde{\mathsf{X}}_i = T(\mathsf{Z}_i)$, where T is computed by your `BinarySearch`. Plot the histogram of the generated $\tilde{\mathsf{X}}_i$ (with b bins) and submit your script as `PushForward(Z, gmm)`. Is the histogram similar to the one in Ex 2.1?

Ans: 

5. (1 pt) Now let us compute $\tilde{\mathsf{U}}_i = \Phi^{-1}\big(F(\tilde{\mathsf{X}}_i)\big)$ as in Ex 2.2, with $\tilde{\mathsf{X}}_i$'s being generated in Ex 2.4. Plot the histogram of the resulting $\tilde{\mathsf{U}}_i$ (with b bins). From your inspection what distribution should $\tilde{\mathsf{U}}_i$ follow (approximately)? [No need to submit any script, as you can recycle `GMMinv`.]

Ans: