CS480/680: Introduction to Machine Learning

Lec 13: Decision Trees

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Trees Recalled

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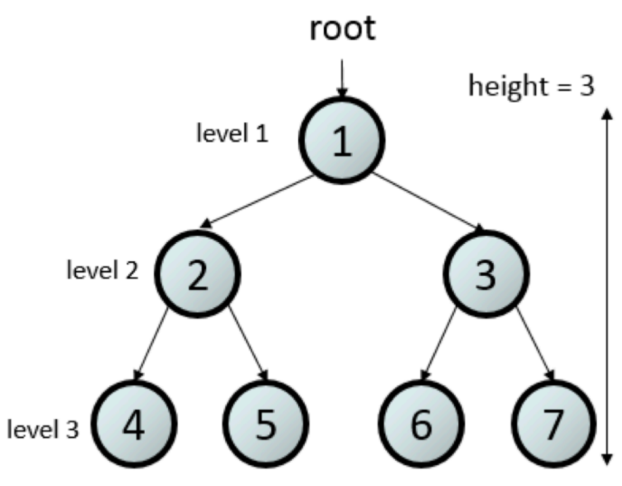
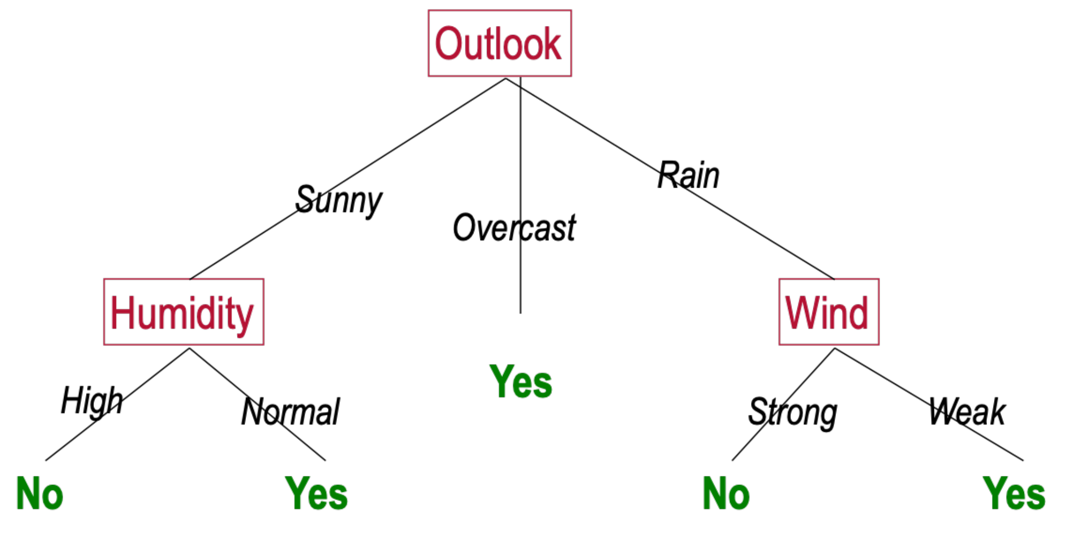
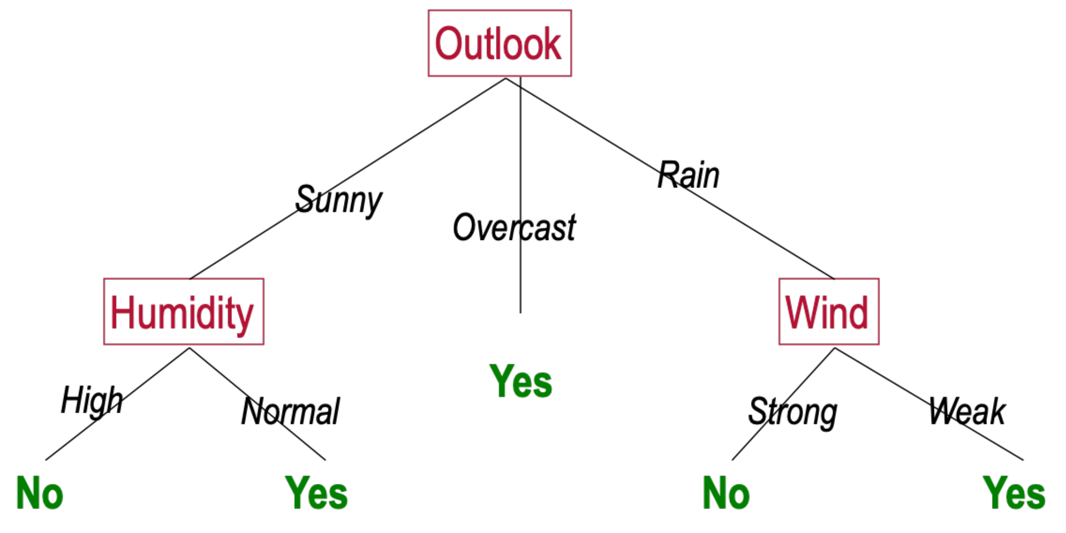




- Decision trees can represent any boolean function



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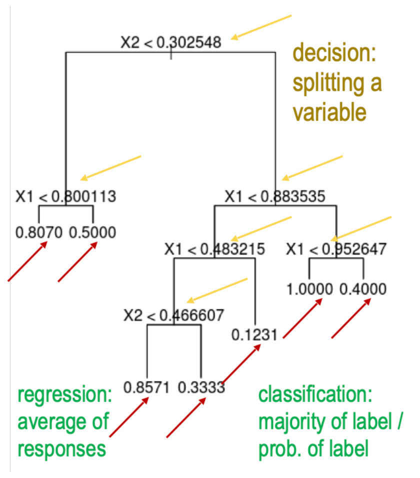
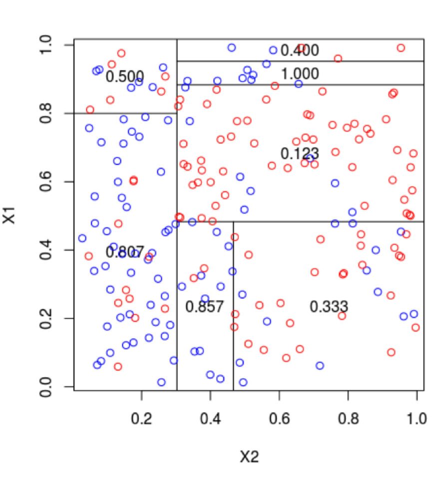
Classification And Regression Tree





L. Breiman, J. H. Friedman, R. A. Olshen, and C. J. Stone. "Classification and Regression Trees". CRC, 1984.

Learning Decision Trees

- Which variables to split at each stage?
- What threshold to use?
- When to stop?
  - regularization: e.g. early stopping
  - pruning
- What to put at the leaves?
  - classification: majority / probability
  - regression: average
  - can also simply store the training set

L. Hyafil and R. L. Rivest. "Constructing optimal binary decision trees is NP-complete". *Information Processing Letters*, vol. 5, no. 1 (1976), pp. 15–17.

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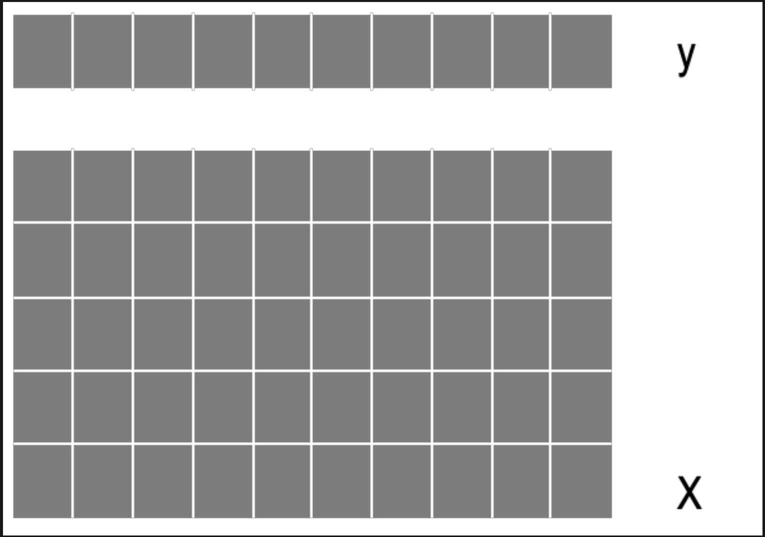

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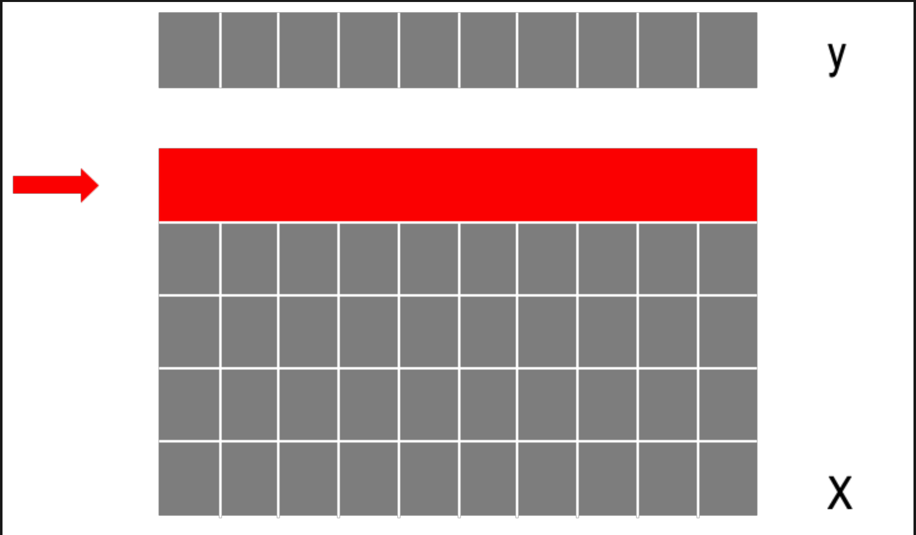
- Evaluation can be based on y as well



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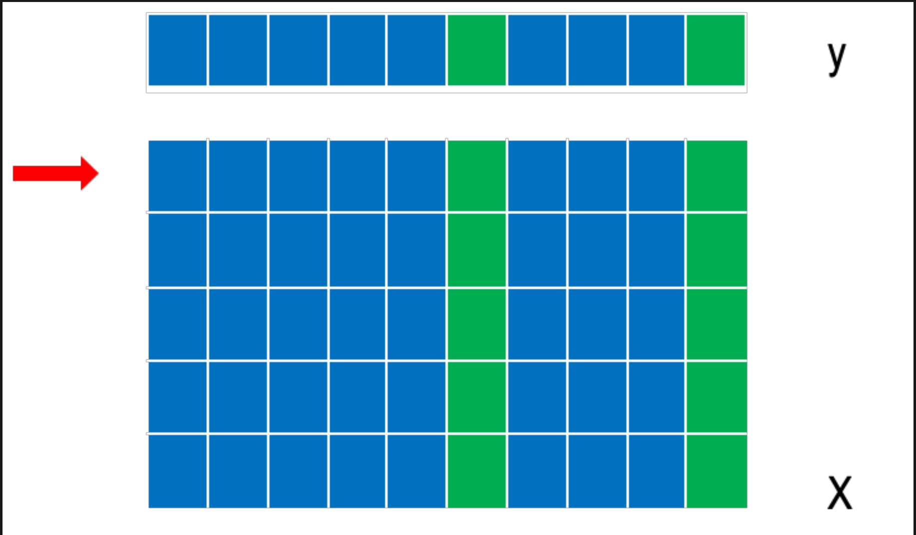

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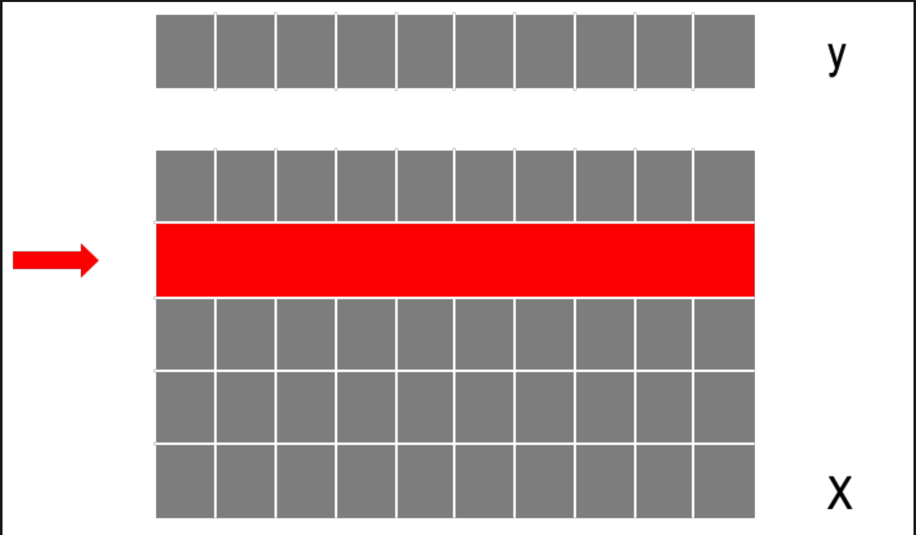

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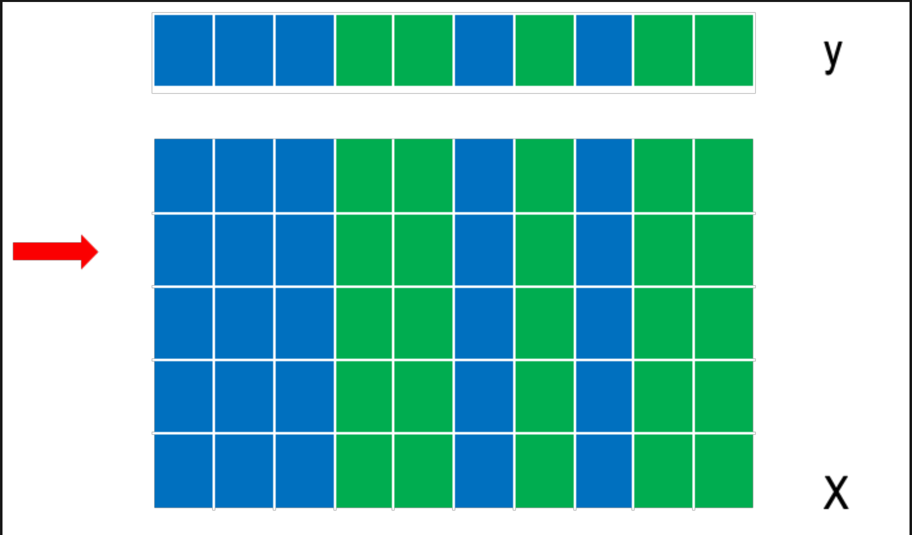

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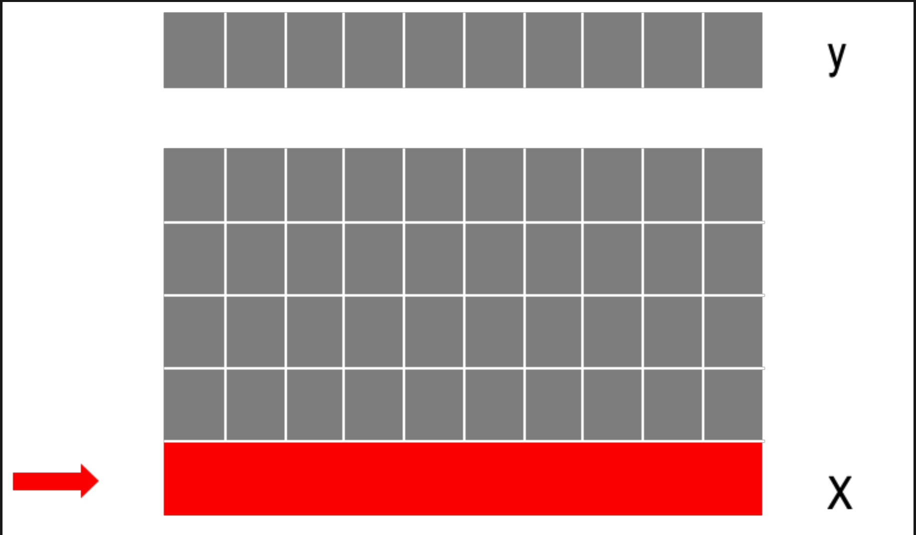

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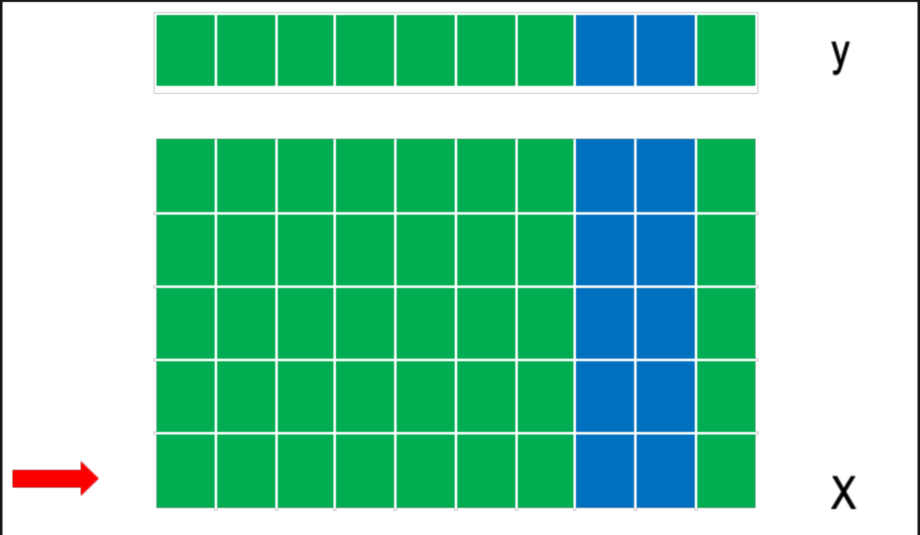

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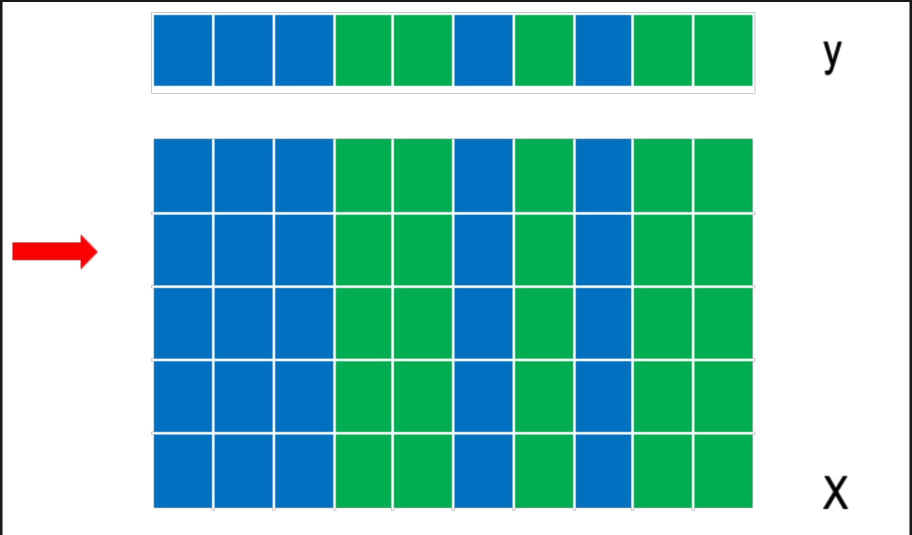

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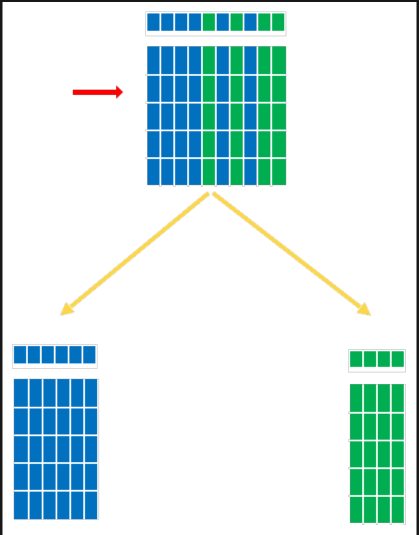

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Splitting by Minimizing

$$(j^*, t^*) = \operatorname*{argmin}_{j=1,\ldots,d} \min_{t \in T_j} \ \ell(\{\mathbf{x}_i, y_i\} : x_{ij} \leq t) + \ell(\{\mathbf{x}_i, y_i\} : x_{ij} > t)$$

- Greedily choose the j-th feature to split
- Greedily choose a threshold $t \in T_j$ to split
  - what should T_j be? For categorical features?
- Partition training data into two disjoint parts: $x_{ij} \leq t$ vs. $x_{ij} > t$
- Evaluate the resulting cost (objective)

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Stopping Criterion

- Maximum depth exceeded
- Maximum runtime exceeded
- All childen nodes are (sufficiently) homogeneous
- All children nodes have too few training examples
- Reduction in cost stagnates:

$$\Delta := \ell(D) - \left(\frac{|\mathcal{D}_L|}{|\mathcal{D}|} \cdot \ell(\mathcal{D}_L) + \frac{|\mathcal{D}_R|}{|\mathcal{D}|} \cdot \ell(\mathcal{D}_R)\right)$$

- Cross-validation

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Regression Cost

$$\ell(\mathcal{D}) := \left[\min_y \sum_{y_i \in \mathcal{D}} (y_i - y)^2\right] = \sum_{y_i \in \mathcal{D}} (y_i - \bar{y})^2, \quad \text{where} \quad \bar{y} = \frac{1}{|\mathcal{D}|} \sum_{y_i \in \mathcal{D}} y_i$$

- Can use any reasonable loss (other than the square loss)
- Can even fit a regression model on $\mathcal{D}$

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Classification Cost

$$\hat{p}_k = \frac{1}{|\mathcal{D}|} \sum_{y_i \in \mathcal{D}} [\![y_i \in k]\!], \qquad \hat{y} := \underset{k=1,\ldots,c}{\operatorname{argmax}} \ \hat{p}_k$$

- Misclassification error: $\ell(\mathcal{D}) := 1 - \hat{p}_{\hat{y}}$, reduces to $\hat{p} \wedge (1 - \hat{p})$ if $c = 2$
- Gini index: $\ell(\mathcal{D}) := \sum_{k=1}^{c} \hat{p}_k (1 - \hat{p}_k) = 1 - \sum_{k=1}^{c} \hat{p}_k^2$, reduces to $2\hat{p}(1 - \hat{p})$ if $c = 2$
- Entropy: $\ell(\mathcal{D}) := -\sum_{k=1}^{c} \hat{p}_k \log \hat{p}_k$, reduces to $-\hat{p} \log \hat{p} - (1 - \hat{p}) \log(1 - \hat{p})$ if $c = 2$

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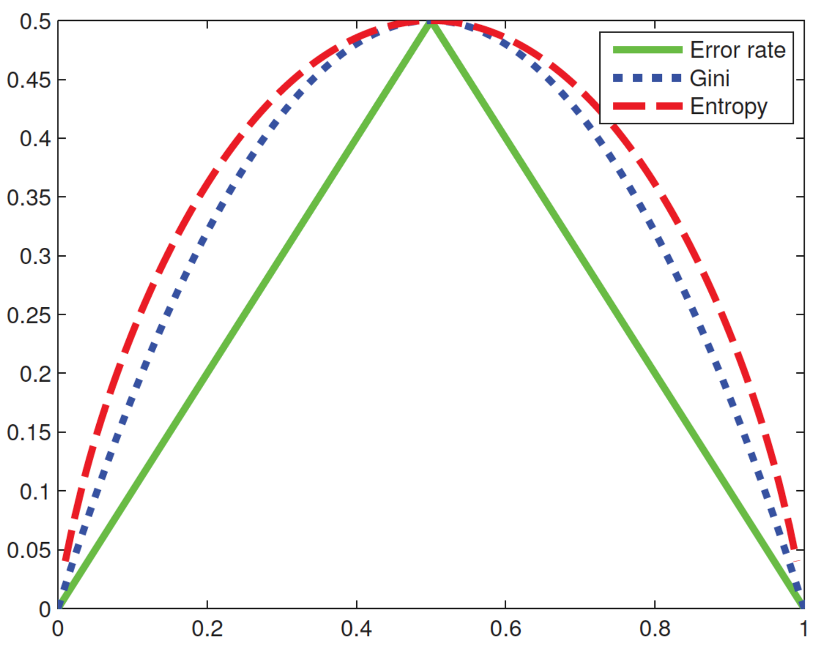
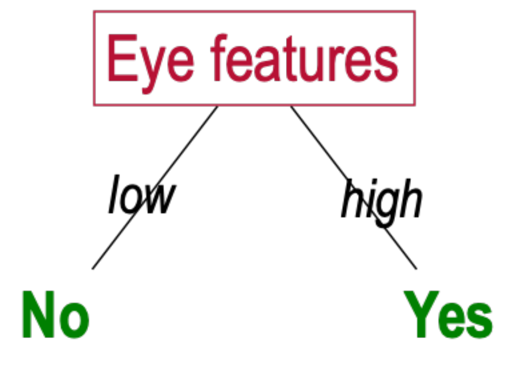


Error rate
Gini
Entropy
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1

Decision Stump



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- Performs classification based on 1 feature
- Easy to train, interprettable, but underfits (addressed in next lecture)

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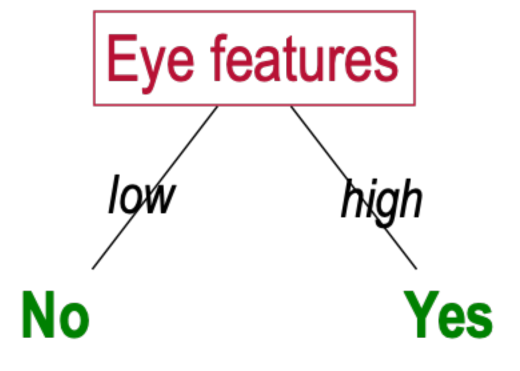

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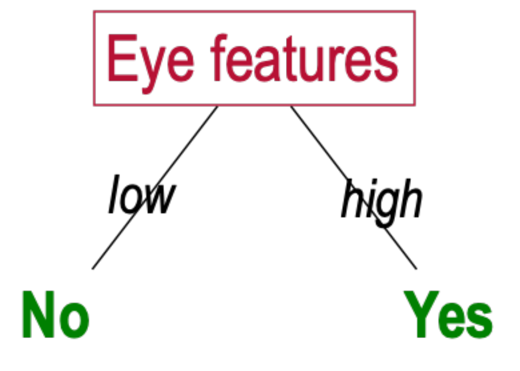

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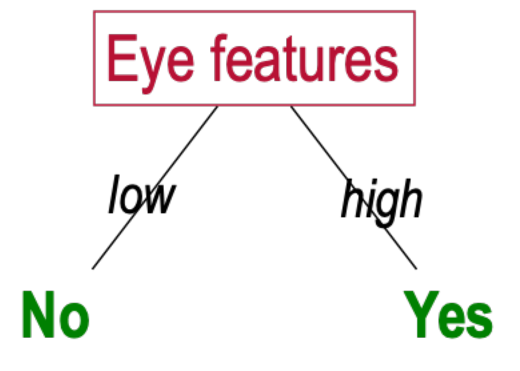

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