Communication Systems (ETEC-208) – Credits – 04
Faculty – Dr. Puneet Azad

New Lecture Plan

S. N.	TOPICS TO BE COVERED	No. of Lecture	No. of Tutorial
	1st Term		
UNIT-2			
1	**Introduction:** Overview of Communication system, Need for modulation	1	1
2	Representation of Band Pass signals and systems, AM modulation, under, over & critical modulation	1	
3	Frequency spectrum, Power relation, modulation index, Bandwidth, efficiency	1	
4	AM generation: square law modulator, low level high label modulator, AM demodulation: square law detector, envelop detector	1	1
5	Negative peak clipping, diagonal clipping, DSB-SC: Modulation and demodulation	1	
6	balanced modulator, synchronous detection	1	
7	Hilbert Transform, In-phase, Quad-phase representations, SSB: modulation and demodulation,	1	1
8	phase shift method , coherent SSB demodulation	1	
9	VSB Modulation: modulation and demodulation, application of VSB	1	
10	Communication channels, Mathematical Models for Communication Channels	1	
	UNIT-1		
11	Probability: experiment, sample space, event, property of probability, independent events, joint & conditional probability	1	1
12	**Introduction of random Variables:** Definition of random variables, DRVs & CRVs	1	
13	CDF and its properties, PDF and its properties, joint PDF, joint CDF, Marginalized PDF, CDF	1	
14	Mean, Variance, Binomial distribution UDF,	1	1
15	Gaussian Distribution Function , Rayleigh Distribution Function	1	
16	WSS wide stationery, strict sense stationery, non-stationery signals	1	
17	White process, Poisson process,		1
18	Wiener process.		
	2nd term		
	UNIT-3		
19	**Angle Modulation Systems:** Frequency Modulation, frequency deviation, modulation index, deviation ratio, Types of Frequency	1	1

	Modulation		
20	generation of FM using PM & vice-versa, frequency spectrum of FM, practical BW, Carson's rule, Generation of NBFM, WBFM, comparison of NBFM & WBFM	1	
21	Varactor diode modulator, reactance modulator	1	
22	Phase Modulation, Relationship between PM& FM, Difference between FM & PM, comparison of FM & PM	1	1
23	**Radio Receivers:** Functions & Classification of Radio Receivers	1	
24	Tuned Radio Frequency (TRF) Receiver and its drawbacks	1	
25	Super-heterodyne Receiver, Basic Elements, advantage	1	1
26	Receiver Characteristics, sensitivity, selectivity, fidelity, double spotting	1	
27	Image frequency and its rejection, Frequency Mixers, AGC Characteristics.	1	
	UNIT-4		
28	**Noise Theory:** Noise, Types of noise, shot, partition, flicker, thermal & transit time noise	1	1
29	Addition of Noise due to several sources in series and parallel, Generalized Nyquist Theorem for Thermal Noise	1	
30	Calculation of Thermal Noise for a Single Noise Source, RC Circuits & Multiple Noise sources.	1	
31	Equivalent Noise Bandwidth, Signal to Noise Ratio, Noise-Figure, Noise Temperature, Calculation of Noise Figure	1	1
32	Noise factor of amplifiers in cascade, equivalent noise temperature of amplifiers in cascade, noise factor of a lossy network	1	
33	**Performance of Communication Systems:** Receiver Model, Noise in DSB-SC Receivers, Noise in SSB-SC Receivers	1	
34	Noise in AM receiver (Using Envelope Detection), Noise in FM Receivers, FM Threshold Effect, Threshold Improvement through Pre-Emphasis and De-Emphasis	1	1
35	Noise in PM system – Comparison of Noise performance in PM and FM,	1	
36	Link budget analysis for radio channels.	1	

First Sessional Exam – UNIT-I & II

Unit-I

Lecture 1

Overview of Communication systems, Communication Channels and their Mathematical Models

Introduction / Overview of Communication systems

Electrical Communication systems are designed to send messages or information from a source that generates the messages to one or more destinations. Telephones in our hands, televisions in our living rooms, the computer terminals with access to the Internet in our offices and homes and our newspapers are all capable of providing rapid communications from every corner of the globe. In general, a communication system can be represented by the functional block diagram shown in Fig. 1.1. The information generated by the source may be of the form of voice, a picture, or plain text in some particular language. An essential feature of any source that generates information is that its output is described in probabilistic terms i.e. the output of the source is not deterministic. Otherwise, there would be no need to transmit the message. A transducer is usually required to convert the output of a source into an electrical signal that is suitable for transmission. For example, a microphone serves as the transducer that coverts an acoustic speech into an electrical signals and a video camera convert an image into an electrical signal. At the destination, a similar transducer is required to convert the electrical signal into a form that is suitable for user e.g. acoustic signals, images etc.

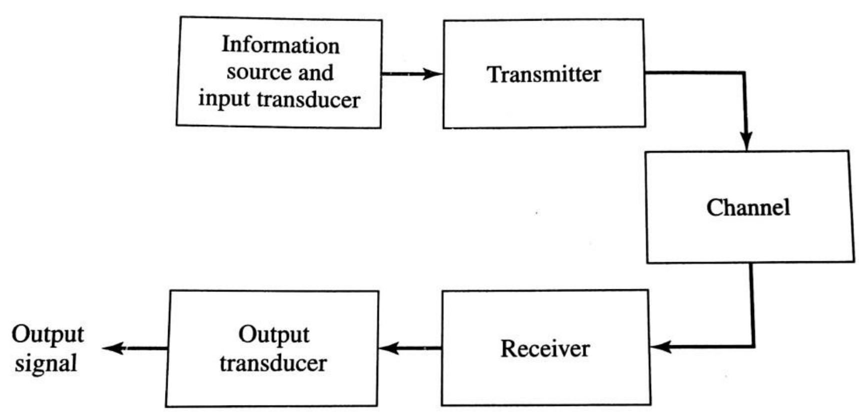


Fig. 1.1Block Diagram of a Communication System

There are three basic parts of a communication systems namely transmitter, channel and receiver.

The Transmitter-*The transmitter converts the electrical signal into a form that is suitable for transmission through the physical channel or transmission medium*.

Govt bodies resp for freq. allocation

a. In radio and TV broadcast, the Federal Communications Commission (FCC) of USA
b. Wireless Planning and Coordination Wing (WPC)-wing of Ministry of communication in India.

Responsibility of a transmitter

1) **Translation of information into freq.-**The transmitter must translate the information signal to be transmitted into the appropriate frequency range that matches the frequency allocation assigned to the transmitter. Thus signals transmitted by multiple radio station do not interfere with one another.

   Modulation - In electronics and telecommunications, **modulation** is the process of varying one or more properties of a periodic waveform, called the *carrier signal*, with a modulating signal that typically contains information to be transmitted.

   In telecommunications, modulation is the process of conveying a message signal, for example a digital bit stream or an analog audio signal, inside another signal that can be physically transmitted. Modulation of a sine waveform transforms a baseband message signal into a passband signal.

   Two types of modulation

   Continuous wave (CW) Modulation (analog) and Pulse Modulation (can be analog and digital) **e**.g. in AM/FM radio broadcast, the information signal that is transmitted is contained in the amplitude/frequency variations of the sinusoidal carrier, which is the center frequency in the frequency band allocated to the radio transmitting station. Similarly for PM.

   #Among all the modulation schemes, PCM has emerged as the preferred method for transmission of analog message signals because of Robustness (Regeneration at regular intervals), Flexible operations, Integration of diverse sources into a common format, Security of information

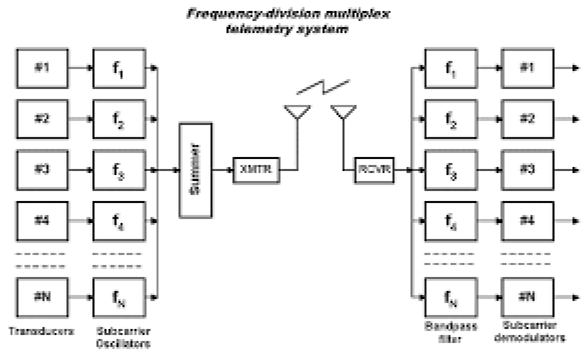
   Choice of Modulation depends on

   a. Amount of bandwidth allocated
   b. Type of Noise and Interference that is encountered while transmission

c. Electronics devices available for signal amplification

Multiplexing

d. FDM, in which CW Modulation is used to translate each message signal to reside in a specific frequency slot inside the passband of the channel by assigning it a distinct carrier frequency. A bank of filters is used to separate the different modulated signals and prepare them individually for demodulation.



e. TDM, in which pulse modulation is used to position samples of the different message signals in non-overlapping time slots.

f. CDM, in which each message signal is identified by a distinctive code.

#In FDM, the message signals may overlap at the channel input, hence the system suffer from cross talk. In TDM, the message signal uses the full passband of the channel on a time sharing basis. In CDM, the message signals overlap in both time and frequency across the channel.

Channel

The communication channel is the physical medium that is used to send the signal from the transmitter to the receiver.

e.g.

a) Free Space in case of wireless transmission

b) Wireline, optical fibre, wireless (microwave radio) in case of telephony

Additive Noise – Generated at the front end of receiver, where signal amplification is performed also called thermal noise. Other examples are atmospheric noise like electrical lightening noise and interferences from other users of the channel.

Non-Additive Noise- Multipath propagation in Ionospheric channel used for long/short wave radio transmission. It is signal distortion, which is a time variation in the signal amplitude occurs called fading

Both additive and non-additive noise are random process and their effect must be considered in the design of communication system

2) **The Receiver**– The function of the receiver is to recover the message signal contained in the received signal. The receiver performs demodulation to extract message signal from the sinusoidal carrier. Since the signal demodulation is performed in the presence of additive noise, the demodulated signal is degraded by the presence of these distortion in the received signal.

Advantages of Digital transmission over Analog transmission

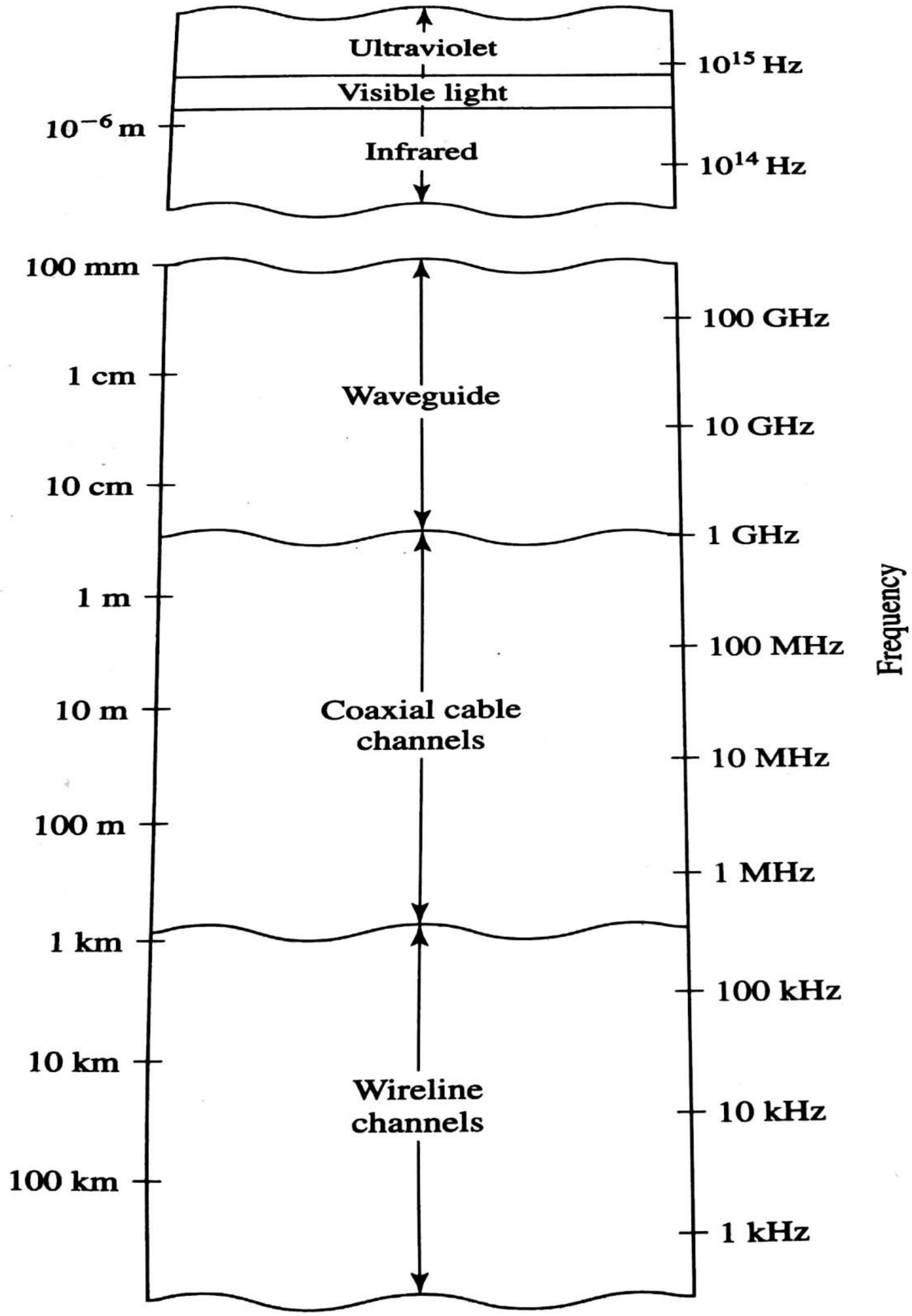
1) Signal fidelity is better in digital transmission(Fidelity also denotes how accurately a copy reproduces its source. "hi-fi" were popularized for equipment and recordings which exhibited more accurate sound reproduction.)
2) Regeneration of signal in long distance transmission eliminating noise at each regeneration point. In analog, noise gets amplified along with signals
3) Digital communication systems are cheaper to implement and redundancy may be removed easily

3) Communication Channels -*Wires / Optical Fibres / Underwater Ocean Channel / Free Space*

Characteristics of several channels

a. *Wireline Channels*

Twisted pair (order of several **Kilo Hertz**) and coaxial channels (order of **Mega Hertz**) are guided electromagnetic channels used for Voice signal transmission. **Both are distorted in amplitude and phase and corrupted by additive noise. Twisted pair wireline channels are also prone to cross talk.**



Frequency range for guided wireline channel

b. *Fibre Optic Channels*

Transmitter: *Information is transmitted by varying the intensity of a Light source-LED or Laser.*

Receiver: *Light intensity is detected by a photo diode, whose output is an electric signal that varies in direct proportion to the power of the light impinging on the photo diode.*

Fibre optic channels offers unique characteristics

1. *Enormous potential bandwidth of the order of* $2x10^{13}$ *Hz*
2. *Low transmission losses, as low as 0.1 dB/km*
3. *Immunity of electromagnetic interference*
4. *Small Size and weight*
5. *Ruggedness and flexibility*

c. *Wireless electromagnetic channels*

The physical size and configuration of the antenna, which serves as radiator depends on the frequency of operation. To obtain efficient radiation of the electromagnetic energy, the antenna must be no longer than 1/10 of the wavelength.

e.g. AM transmitting 1 MHz requires antenna of 30 meter

Other examples:

1) *In VLF range, where wavelength exceeds 10 KM (10 KHz and below), earth acts as waveguide and comm. signals propagates around the globe.* ***Use****: Navigational aids from shore to ships*
2) *Ground wave propagation (0.3-3 MHz): Use: AM broadcast, Maritime Radio broadcasting*
3) *Sky wave propagation – above 30 MHz (Reflection from Ionosphere) Prob: Signal multipath, signal fading*
4) *LOS Communication –*

$$D = \sqrt{2}h$$

D-Radio Horizon

h= height of antenna tower

e.g D =50 miles, h=1000 ft, for a TV antenna

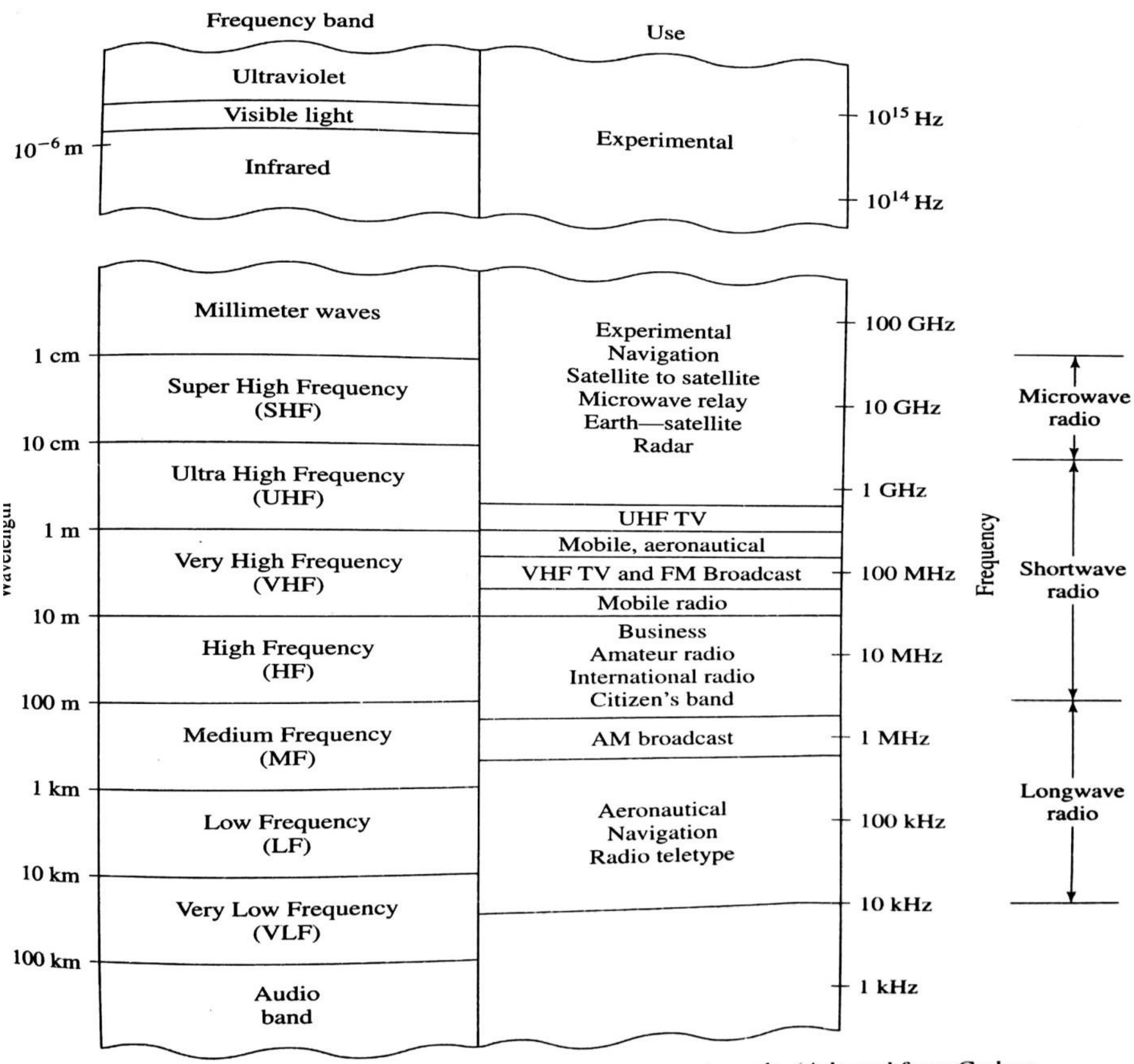


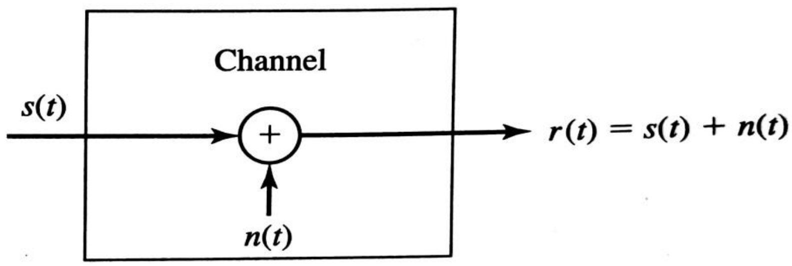
Figure 1.4 Frequency range for wireless electromagnetic channels. (Adapted from Carlson, Sec. Ed.; © 1975 McGraw-Hill. Reprinted with permission of the publisher.)

Lecture 2

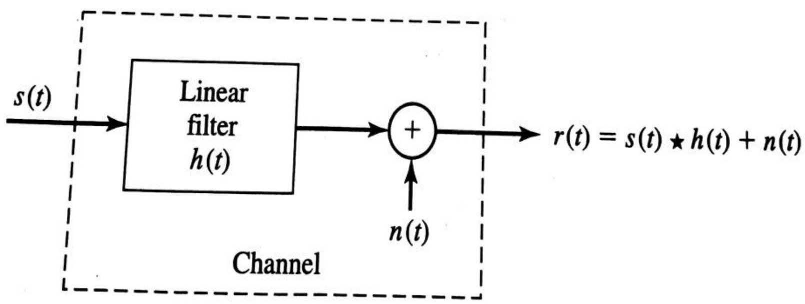
Mathematical Models for Communication channels

1) **The Additive Noise Channel**

The transmitter signal s(t) is corrupted by additive random noise n(t). Noise introduced through electronic components and amplifiers is also called thermal noise characterized statistically as a Gaussian noise process.



2) **Linear Filter Channel**



In some physical channels such as telephone channels, Filters are used to ensure that the transmitted signals do not exceed specified bandwidth limitations and thus do not interfere with each other. Thus, they are characterized as Linear Filter channels with additive noise.

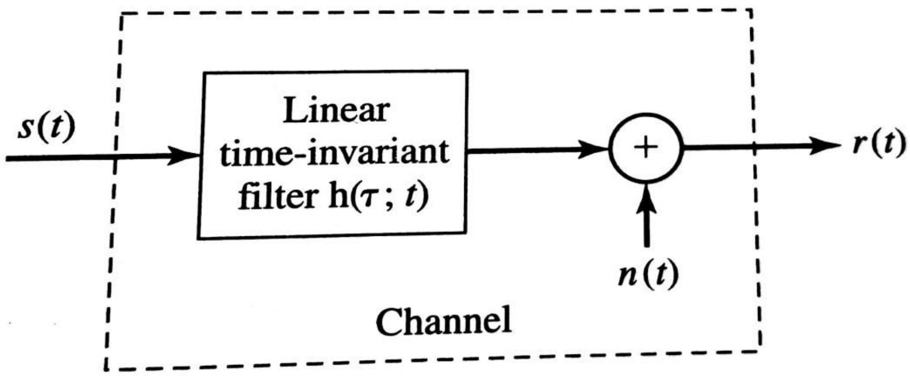
$$r(t) = s(t) * h(t) + n(t)$$

Where h(t) is the impulse response of the linear filter and * is the convolution

3) **Linear Time Variant Filter Channel**

Physical channels such as under water acoustic channels and ionospheric radio channels, which result in time-variant multipath propagation of the transmitted signal may be characterized as time-variant linear filters.

$$r(t) = s(t) * h(T; t) + n(t)$$


$s(t)$
Linear
time-invariant
filter h(τ; t)
+
$r(t)$
$n(t)$
Channel

Lecture 3

Random Variables-Definition and basic concept

Random Variables

The outcome of an experiment may be a real number (rolling of a die), or non-numerical quantity (head or tail in tossing a coin). From a mathematical point of view, it is simpler to have numerical values for all the outcomes. Thus a real number may be assigned to each sample point as per some rule. **A term random variable is used to signify a rule by which a real number is assigned to each possible outcome of an experiment.** Thus, X(.) is a function that maps sample points $\boldsymbol{\lambda_1, \lambda_2, \ldots \lambda_m}$ into real numbers $x_1, x_2, \ldots x_n$.

Let the possible outcomes be $\boldsymbol{\lambda i}$, which may not be numbers. Thus, the rule or functional relationshipby which we assign real numbers $X(\lambda_i)$ to each possible outcome is called a Random variable.

Discrete or Continuous

If in any finite interval, $X(\boldsymbol{\lambda})$ assumesonly a finite number of distinct values, then the Random variable is discrete e.g. tossing a die.

If $X(\boldsymbol{\lambda})$ can assume any value within an interval, the Random variable is continuous.e.g. if we fire a bullet, the bullet may miss its mark and the magnitude of miss will be continuous random variable.

Cumulative Distribution Function

The CDF of a random variable 'X' may be defined as the probability that a random variable 'X' takes a value less than or equal to x. Thus, CDF provides the probabilistic description of a random variable. CDF is the probability that the outcome of an experiment will be one of those outcome for which $X(\lambda_i) \leq x$, where x is any given number

$$F_{\mathbf{X}}(x) = P(X \leq x)$$

A CDF $F_X(x)$ has the following properties

1. $0 \leq F_{\mathbf{X}}(x) \leq 1$ …(2.1)
2. $F_{\mathbf{X}}(\infty) = 1$ follows from the fact that F_X contains all possible events …(2.2)
   $(F_{\mathbf{X}}(\infty) = P(X \leq \infty))$
3. $F_{\mathbf{X}}(-\infty) = 0$follows from the fact that F_X contains no possible events …(2.3)
   $(F_{\mathbf{X}}(-\infty) = P(X \leq -\infty))$
4. $F_{\mathbf{X}}(x)$is a non-decreasing function i.e.

$$F_{\mathbf{X}}(x_1) \leq F_{\mathbf{X}}(x_2) \;\; for x_1 \leq x_2 \qquad \ldots(2.4)$$

Proof: $\quad F_{\mathbf{x}}(x_2) = P(X \le x_2)$

$$= P[(X \le x_1) U (x_1 < X \le x_2)]$$

$$= P(X \le x_1) + P(x_1 < X \le x_2) \qquad \text{(because the sets are disjoint)}$$

$$= F_{\mathbf{x}}(x_1) + P(x_1 < X \le x_2) \qquad \ldots(2.5)$$

Because $P(x_1 < X \le x_2)$ is non-negative, the result follows

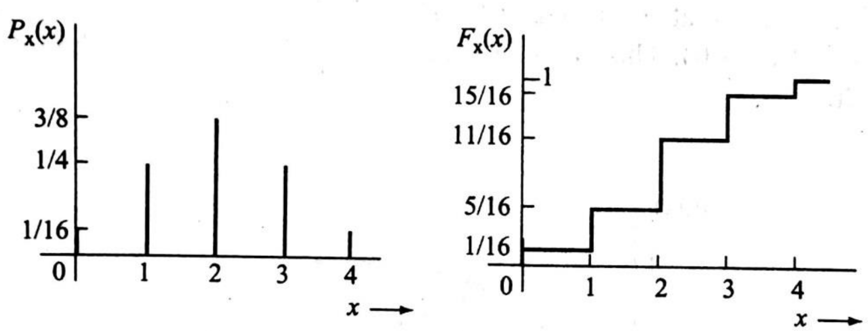
Example:

5 In an experiment, a trial consists of four successive tosses of a coin. If we define an RV x a the number of heads appearing in a trial, determine $P_{\mathrm{x}}(x)$ and $F_{\mathrm{x}}(x)$.

A total of 16 distinct equiprobable outcomes are listed in Example 8.4. Various probabilitie can be readily determined by counting the outcomes pertaining to a given value of x. Fo example, only one outcome maps into x=0, whereas six outcomes map into x=2. Hence $P_{\mathrm{x}}(0) = 1/16$ and $P_{\mathrm{x}}(2) = 6/16$. In the same way, we find

$$P_{\mathrm{x}}(0) = P_{\mathrm{x}}(4) = 1/16$$
$$P_{\mathrm{x}}(1) = P_{\mathrm{x}}(3) = 4/16 = 1/4$$
$$P_{\mathrm{x}}(2) = 6/16 = 3/8$$

The probabilities $P_{\mathrm{x}}(x_i)$ and the corresponding CDF $F_{\mathrm{x}}(x_i)$ are shown in Fig. 8.8.



(a) (b)

Sample space of 16 points

HHHH	TTTT
HHHT	TTTH
HHTH	TTHT
HHTT	TTHH
HTHH	THTT
HTHT	THTH

HTTH THHT
HTTT THHH

Lecture 4

Probability Density Function

From eqn 2.5,

$$\mathrm{F}\mathbf{x}(x+\Delta x) = \mathrm{F}\mathbf{x}(\mathrm{x}) + P(x < X \le x + \Delta x) \qquad \ldots(2.6)$$

If $\Delta x \to 0$, then we can also express $\mathrm{F}\mathbf{x}(x+\Delta x)$ via taylor expression

$$\mathrm{F}\mathbf{x}(x+\Delta x) \approx \mathrm{F}\mathbf{x}(x) + \frac{d\mathrm{F}\mathbf{x}(x)}{dx}\Delta x \qquad \ldots(2.7)$$

Thus, $\frac{d\mathrm{F}\mathbf{x}(x)}{dx}\Delta x = P(x < X \le x + \Delta x)$ …(2.8)

We designate the derivative of $\mathrm{F}\mathbf{x}(x)$ w.r.t. x by $\mathrm{p}\mathbf{x}(x)$

$$\frac{d\mathrm{F}\mathbf{x}(x)}{dx} = \mathrm{p}\mathbf{x}(x) \qquad \ldots(2.9)$$

The function $\mathrm{p}\mathbf{x}(x)$ is called the Probability density function of the random variable X. PDF is a more convenient way of describing a continuous random variable. It is defined as the derivative of cumulative distribution function.
It follows from eqn. 2.8 that the probability of observing the Random Variable X in the interval $(x, x+\Delta x)$ is $\mathrm{p}\mathbf{x}(x)\Delta x$ $(\Delta x \to 0)$. This is the area under the PDF $\mathrm{p}\mathbf{x}(x)$ over the interval Δx.

Properties of PDF

1) PDF must be non-negative

$$p_x(x) \ge 0 \text{ for all } x$$

Since CDF is a monotone increasing function and PDF is the derivative of CDF, so the derivative of a monotone increasing function will always be positive.

It is true that the probability of an impossible event is zero and that of a certain event is 1, the converse is not true. An event whose probability is 0 is not necessarily an impossible event and an event with a probability of 1 is not necessarily a certain event.
Temp of a city=5 to 50^{o}C
Prob that T=34.56 is 0. But this is not an impossible event
Prob that T=34.56 is 1, although is not a certain event.

2) CDF can be derived from PDF by integrating it

$$\mathrm{F}\mathbf{x}(\mathrm{x}) = \int_{-\infty}^{x} p_X(x)dx = \int_{-\infty}^{x} f_X(x)dx$$

3) Area under the PDF curve is always equal to unity

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

This is seen from the fact that

$$\int_{-\infty}^{\infty} f_X(x)dx = F(\infty) - F(-\infty) = 1 - 0 = 1$$

Example – Consider an experiment of rolling 2 die together. There are 36 outcomes represented by $36\lambda_{ij}$, where i is the number on 1st die and j is the number on 2nd die.

P(1)=0; P(2)=P(12)=1/36;

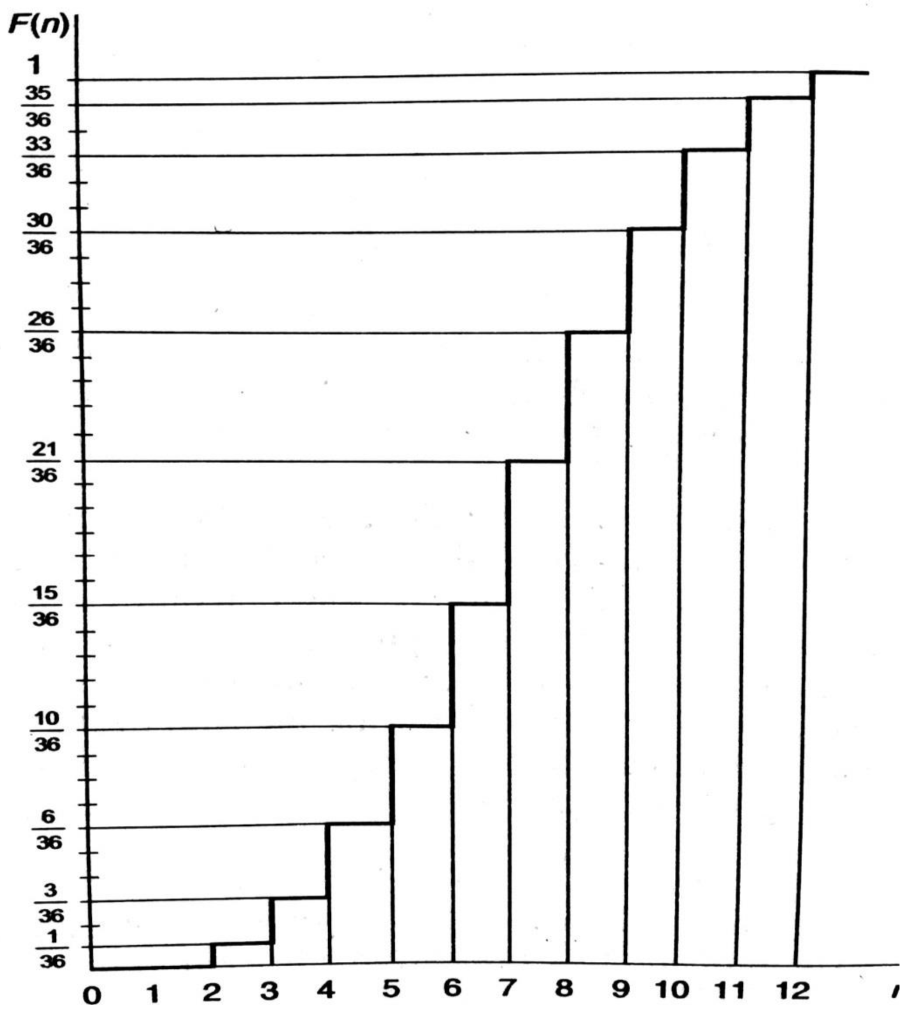
P(3)=P(11)=2/36; P(4)=P(10)=3/36;

P(5)=P(9)=4/36; P(6)=P(8)=5/36;

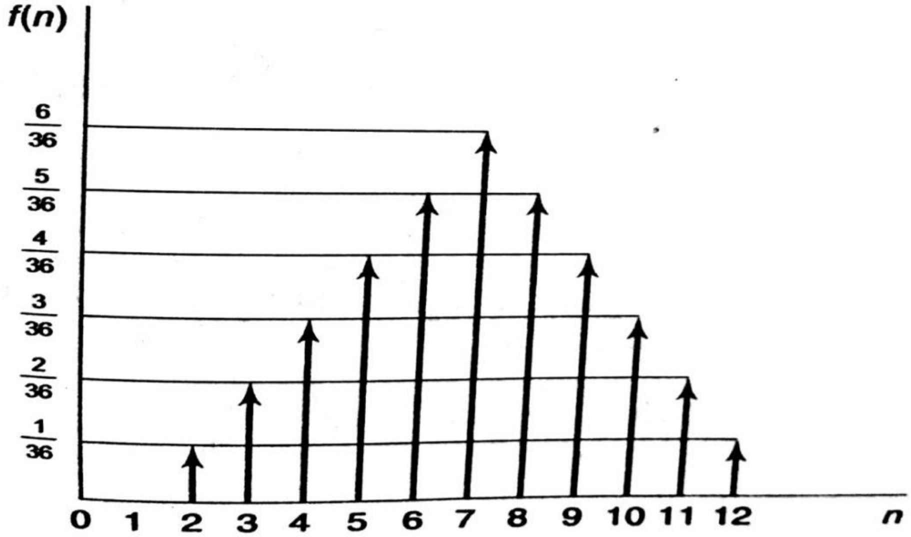
P(7)=6/36

The CDF for n=3 is

$$\begin{aligned} F(3) &= P(n \leq 3) \\ &= P(1) + P(2) + P(3) = 0 + \frac{1}{36} + \frac{2}{36} = \frac{3}{36} \end{aligned}$$



CDF



PDF

Derivative of a step of Amplitude A is an impulse of strength I=A, we have $f(2) = \frac{1}{36}\delta(n-2), f(3) = \delta(n-3)$ .... (Height of the impulse is proportional to the strength of impulse)

EXAMPLE 5.13. The PDF of a random variable is given by,

$$f_X(x) = \begin{cases} k, a \le x \le b \\ 0, \text{otherwise} \end{cases}$$

where, k is a constant

(i) Determine the value of k

(ii) Let a = –1 and b = 2 calculate $P(|X| \le c)$ for c = 1/2

Solution: (i) The value of k

From the property 2 of PDF, k must be a positive constant. From the property 3 of PDF, the area under PDF is equal to 1.

Therefore, $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

or $$\int_a^b k\,dx = 1$$

or $$k[b-a] = 1$$

Therefore, $$k = \frac{1}{(b-a)}$$ **Ans.**

Hence, $$f_X(x) = \begin{cases} \frac{1}{(b-a)} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

This PDF has been plotted in figure 5.9 which is called as uniform PDF.

(ii) Value of $P(|X| \le c)$

We are expected to obtain the probability of $(-1/2 \le X \le 1/2)$.

According to property 4, we have

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

Thus, $$P(-1/2 \le X \le 1/2) = \int_{-1/2}^{1/2} \frac{1}{b-a}\,dx$$

or, $$P(-1/2 \le X \le 1/2) = \int_{-1/2}^{1/2} (1/3)\,dx = (1/3)(1/2 + 1/2) = 1/3$$

This has been shown by the shaded portion shown in figure 5.9.

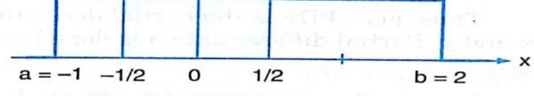


Fig. 5.9. PDF for example 5.13.

Statistical Averages of Random Variables →

(1) Mean or Average →

The mean or average of any random variable is expressed by the summation of the values of random variables X weighted by their probabilities. It is denoted by m_x.

Mean value is also known as expected value of random variable X.

$$m_x = E(X)$$

Where $E[\;]$ is expectation operator.

Also in general term, mean or average is given by

$$X = \frac{\text{Arithmatic Sum of all values of } X}{\text{Total number of values of } X}$$

(2) Mean value of Discrete Random variable

Let the discrete random variables X be take the following values:

$$X = \{x_1, x_2, x_3 \; ---- \; x_n\}$$

Then the mean or avg value m_x is expressed as

$$m_x = x_1 P(x_1) + x_2 P(x_2) + \; - \, - \, - \, - \, - \; + x_n P(x_n)$$

Hence mean value is

$$m_x = E(x) = \bar{X}_n = \sum_{i=1}^{n} x_i P(x_i) \quad \text{—(1)}$$

Thus the mean value of discrete random variable X is equal to the summation of values of X weighted by probability of that particular

(3) Mean value of Continuous Random Variable ⇒

If random variables X becomes continuous, the sample points $x_1, x_2 \text{---} x_n$ becomes quite close to each other such that $(x_2 - x_1 \cong 0)$. ∴ summation in equation ~~(1)~~ (1) in last page for discrete random variables to an integration over the complete range of $x(-\infty < x < \infty)$.

For Cont. random variables, the mean or avg value is expressed as

$$m_x = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

where $f_X(x)$ = Probability density function

Also if a function $g(x)$ transforms random variable X into another random variable, then mean or average of $g(x)$ is

$$m = \overline{g(x)} = E[g(x)]$$

$$m = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Q - The random variable Z is the function of another random variable X in such a way that $Z = \cos(X)$ and X is uniformly distributed in the interval $(-\pi, \pi)$ etc.

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{for } -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

Detn mean value of Z.

Sol^n: The given random variable is

$$Z = \cos(x)$$

Let $g(x) = \cos(x)$,

mean value of random var. $Z = g(x)$ is expressed as

$$m = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$$

$$= \int_{-\pi}^{\pi} \cos x \left(\frac{1}{2\pi}\right) dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x\, dx$$

or $$m = \frac{1}{2\pi}\left[-\sin x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

Moments and Variance ⇒

(69)

The n^{th} moment of any random variable x may be defined as the mean value of x^n.

i.e $g(x) = x^n$

mean is written as

$$\text{mean} = \overline{g(x)}$$

$$\overline{x_n} = \int_{-\infty}^{\infty} x^n f_x(x)\, dx = E(x^n)$$

$$\overline{X_n} = \int_{-\infty}^{\infty} x^n f_x(x)\, dx$$

Putting $n = 1$,

$$\overline{x} = E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx$$

or $\overline{x} = m_x$

hence first moment of random variable x will be same as it mean value.

if $n = 2$, $\overline{x^2} = E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$

When $\overline{x^2}$ is mean square value of random variable x.

Similarly, the central moments are the moments of the difference betn random var. x and its mean m_x. ∴ n^{th} central moment may be given as

$$E\left[(x - m_x)^n\right] = \int_{-\infty}^{\infty} (x - m_x)^n f_x(x)\, dx$$

The second central moment $n=2$ is known as variance of random variable X i.e.

$$\text{Variance}[X] = E[(X-m_x)^2]$$

$$= \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x)\, dx$$

Variance provides an indication about randomness of the random variable.

Variance is represented by

$$\sigma_x^2 = \text{Variance}(X)$$
$$= \text{Var}(X)$$
$$= E[(X-m_x)^2]$$
$$= \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x)\, dx$$

also $\sigma_x^2 = E[X^2 - 2m_x X + m_x^2]$

$E[\cdot]$ operation represent mean value $\therefore$ it is linear.

Then $\sigma_x^2 = E(X^2) - 2m_x E(X) + m_x^2$

$$= E[X^2] - 2m_x m_x + m_x^2$$
$$= E(X^2) - m_x^2$$

= mean square value − square of mean value.

or Variance $= \sigma_x^2$

$$= \overline{X^2} - \overline{m}_x^2$$

Sq. root of variance is known as std. deviation of a random variable X.

$$\text{STD. dev.} = \sqrt{\text{variance}} = \sqrt{\sigma_x^2} = \sigma_x = \sqrt{E(X^2) - m_x^2}$$

Q - A random variable X has the uniform distribution given by (64)

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \le x \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Detn m_x, $\overline{X^2}$, σ_x

Soln →

$$m_x = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{2\pi} x \cdot \frac{1}{2\pi}\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\, dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \pi$$

$$\overline{X^2} = E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\, dx$$

$$\overline{X^2} = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{6\pi}\left[x^3\right]_0^{2\pi} = \frac{4}{3}\pi^2$$

variance, $\sigma_x^2 = E[X^2] - m_x^2 = \frac{4}{3}\pi^2 - \pi^2 = \frac{1}{3}\pi^2$

$$\therefore \sigma_x = \frac{\pi}{\sqrt{3}}$$

Lecture 5, Joint PDF

5.2.1 Joint Probability Density Function (PDF)

Here, we will define jointly continuous random variables. Basically, two random variables are jointly continuous if they have a joint probability density function as defined below.

Definition 5.1

Two random variables X and Y are jointly continuous if there exists a nonnegative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)dxdy \qquad (5.15)$$

The function $f_{XY}(x,y)$ is called the joint probability density function (PDF) of X and Y.

In the above definition, the domain of $f_{XY}(x,y)$ is the entire $\mathbb{R}^2$. We may define the range of (X,Y) as

$$R_{XY} = \{(x,y) | f_{X,Y}(x,y) > 0\}.$$

The above double integral (Equation 5.15) exists for all sets A of practical interest. If we choose $A = \mathbb{R}^2$, then the probability of $(X,Y) \in A$ must be one, so we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

The intuition behind the joint density $f_{XY}(x,y)$ is similar to that of the PDF of a single random variable. In particular, remember that for a random variable X and small positive δ, we have

$$P(x < X \leq x+\delta) \approx f_X(x)\delta.$$

Similarly, for small positive δ_x and δ_y, we can write

$$P(x < X \leq x+\delta_x, y \leq Y \leq y+\delta_y) \approx f_{XY}(x,y)\delta_x\delta_y.$$

Example 5.15
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x+cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

- Solution
  - a. To find c, we use

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Thus, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1\int_0^1 x+cy^2 \; dxdy \\ &= \int_0^1 \left[\frac{1}{2}x^2 + cy^2x\right]_{x=0}^{x=1} dy \\ &= \int^1 \frac{1}{2} + cy^2 \; dy \end{aligned}$$

$$
\begin{aligned}
&\int_0 \quad 2 \\
&= \left[\frac{1}{2}y + \frac{1}{3}cy^3\right]_{y=0}^{y=1} \\
&= \frac{1}{2} + \frac{1}{3}c.
\end{aligned}
$$

Therefore, we obtain $c = \frac{3}{2}$.

b. To find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$, we can write

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)dxdy, \quad \text{for } A = \{(x,y)|0 \leq x, y \leq 1\}.$$

Thus,

$$
\begin{aligned}
P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x + \frac{3}{2}y^2\right) dxdy \\
&= \int_0^{\frac{1}{2}} \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^{\frac{1}{2}} dy \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy \\
&= \frac{3}{32}.
\end{aligned}
$$

We can find marginal PDFs of X and Y from their joint PDF. This is exactly analogous to what we saw in the discrete case. In particular, by integrating over all y's, we obtain $f_X(x)$. We have

Marginal PDFs

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \quad \text{for all } x, \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \quad \text{for all } y.
\end{aligned}
$$

Example 5.16

In Example 5.15 find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

- Solution
  - For $0 \leq x \leq 1$, we have

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy \\
&= \left[xy + \frac{1}{2}y^3\right]_0^1 \\
&= x + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$
f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Similarly, for $0 \leq y \leq 1$, we have

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\
&= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx \\
&= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^1 \\
&= \frac{3}{2}y^2 + \frac{1}{2}.
\end{aligned}
$$

Thus,

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Example 5.17
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

a. Find R_{XY} and show it in the $x-y$ plane.
b. Find the constant c.
c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.
d. Find $P(Y \leq \frac{X}{2})$.
e. Find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$.

- Solution
    - a. From the joint PDF, we find that

$$R_{XY} = \{(x,y) \in \mathbb{R}^2 | 0 \leq y \leq x \leq 1\}.$$

Figure 5.6 shows R_{XY} in the $x-y$ plane.

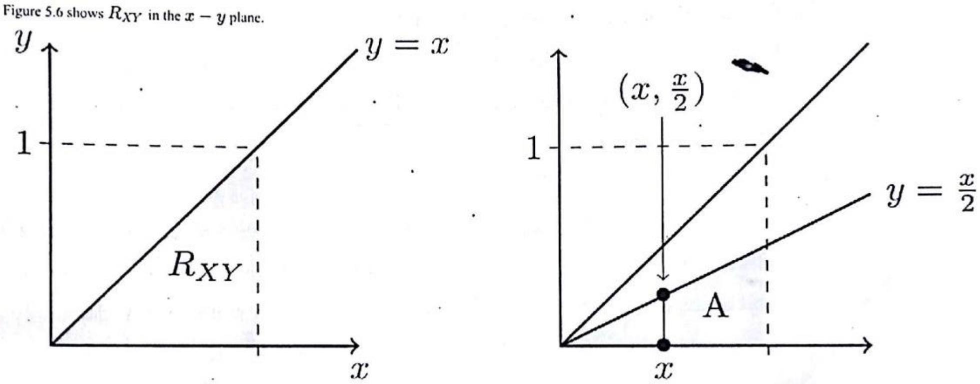


Figure 5.6: Figure shows R_{XY} as well as integration region for finding $P(Y \leq \frac{X}{2})$.

b. To find the constant c, we can write

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1 \int_0^x cx^2y \ dydx \\ &= \int_0^1 \frac{c}{2}x^4 dx \\ &= \frac{c}{10}. \end{aligned}$$

Thus, $c = 10$.
c. To find the marginal PDFs, first note that $R_X = R_Y = [0,1]$. For $0 \leq x \leq 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^x 10x^2y dy \\ &= 5x^4. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 1$, we can write

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

$$f_Y(y) = \int_{-\infty} f_{XY}(x,y)dx$$
$$= \int_y^1 10x^2 y dx$$
$$= \frac{10}{3}y(1-y^3).$$

Thus,

$$f_Y(y) = \begin{cases} \frac{10}{3}y(1-y^3) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

d. To find $P(Y \le \frac{X}{2})$, we need to integrate $f_{XY}(x,y)$ over region A shown in Figure 5.6. In particular, we have

$$P\left(Y \le \frac{X}{2}\right) = \int_{-\infty}^{\infty}\int_0^{\frac{x}{2}} f_{XY}(x,y)dydx$$
$$= \int_0^1\int_0^{\frac{x}{2}} 10x^2y \; dydx$$
$$= \int_0^1 \frac{5}{4}x^4 dx$$
$$= \frac{1}{4}.$$

e. To find $P(Y \le \frac{X}{4} | Y \le \frac{X}{2})$, we have

$$P\left(Y \le \frac{X}{4} | Y \le \frac{X}{2}\right) = \frac{P\left(Y \le \frac{X}{4}, Y \le \frac{X}{2}\right)}{P\left(Y \le \frac{X}{2}\right)}$$
$$= 4P\left(Y \le \frac{X}{4}\right)$$
$$= 4\int_0^1\int_0^{\frac{x}{4}} 10x^2y \; dydx$$
$$= 4\int_0^1 \frac{5}{16}x^4 dx$$
$$= \frac{1}{4}.$$

← previous
next →

If X and Y are independent,

$$P(x_1 \le X \le x_2, y_1 \le Y \le y_2) = \left[\int_{x1}^{x2} p_X(x)dx\right]\left[\int_{y1}^{y2} p_Y(y)dx\right]$$

Therefore,

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)dy$$

And

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x, y)dx$$

EXAMPLE 5.17. **The joint PDF of the random variables X and Y is given by,**

$$f_{XY}(x, y) = C \cdot e^{-(ax+by)}\, u(x) \cdot u(y)$$

where, a and b are constants. Find the value of C.

Solution: The joint CDF $F_{XY}(\infty, \infty) = P(X \leq \infty, Y \leq \infty)$. This will be equal to 1 because the probability of $X \leq \infty$ and $Y \leq \infty$ covers all the possible values of random variables X and Y.

Therefore, $F_{XY}(\infty, \infty) = 1$

According to the definition of CDF, we have

$$F_{XY}(\infty, \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x.y)\,dx\,dy = 1$$

Substituting the value of joint PDF, we get

$$F_{XY}(\infty, \infty) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax+by)}u(x)\,u(y)\,dx\,dy$$

or, $$1 = C\int_0^{\infty} e^{-(ax)}\,dx \cdot \int_0^{\infty} e^{-by}\,dy$$

or, $$1 = \frac{C}{ab}\left[e^{-ax}\right]_0^{\infty} \cdot \left[e^{-by}\right]_0^{\infty}$$

or, $$1 = \frac{C}{ab}[e^{-\infty} - e^{-0}][e^{-\infty} - e^{0}]$$

or, $$1 = \frac{C}{ab}[-1][-1]$$

Hence, $C = ab$ **Ans.**

Example 6.4

The joint probability density of the random variables X and Y is

$$f(x, y) = \frac{1}{4}e^{-|x|-|y|} \quad -\infty < x < \infty, -\infty < y < \infty$$

(a) Are X and Y statistically independent random variables?

(b) Calculate the probability that $X \leq 1$ and $Y \leq 0$.

Solution

(a) Since $f(x, y)$ can be written as

$$f(x, y) = \frac{1}{2}e^{-|x|}\frac{1}{2}e^{-|y|} = f(x)f(y)$$

X and Y are statistically independent.

(b) $$P(X \leq 1, Y \leq 0) = \int_{-\infty}^{1} dx \int_{-\infty}^{0} dy f(x, y)$$

$$= \int_{-\infty}^{1} \frac{1}{2}e^{-|x|}\,dx \int_{-\infty}^{0} \frac{1}{2}e^{-|y|}($$

$$= \left(\frac{2 - e^{-1}}{2}\right)\frac{1}{2} = \frac{1}{4}(2 - e^{-1})$$

Example 6.3

Consider the probability density $f(x) = ae^{-b|x|}$, where X is a random variable whose allowable values range from $x = -\infty$ to $x = +\infty$. Find (a) the cumulative distribution function $F(x)$, (b) the relationship between a and b, and (c) the probability that the outcome X lies between 1 and 2.

Solution

(a) The cumulative distribution function is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} ae^{-b|x|}\,dx$$

$$= \begin{cases} \dfrac{a}{b}e^{bx} & x \le 0 \\ \dfrac{a}{b}(2 - e^{-bx}) & x \ge 0 \end{cases}$$

(b) In order that $f(x)$ be a probability density, it is necessary that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ae^{-b|x|}\,dx = \frac{2a}{b} = 1$$

so that $a/b = \dfrac{1}{2}$.

(c) The probability that X lies in the range between 1 and 2 is

$$P(1 \le X \le 2) = \frac{b}{2}\int_{1}^{2} e^{-b|x|}\,dx = \frac{1}{2}\,(e^{-b} - e^{-2b})$$

Lecture 6

Types of Distributions

Gaussian Random Variables / Gaussian Distribution (Normal Distribution)

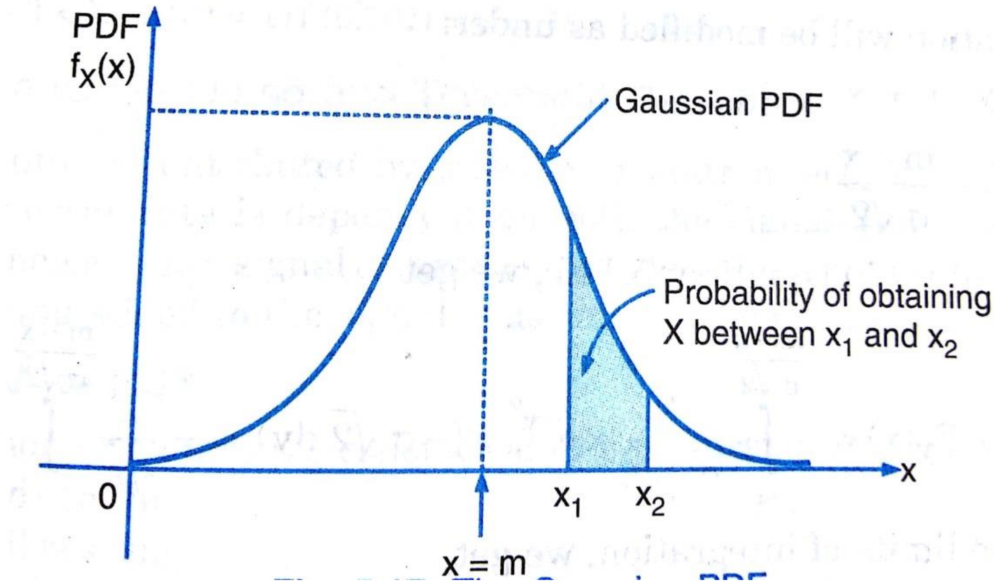
The Gaussian (also called Normal) probability density function is of the greatest importance because many naturally occurring experiment are characterzied by random variables with a Gaussian density. The majority of noise processess observed in practiseare Gaussian and many naturally occurring experiments are characterized by continuous random variables with Gaussian PDF.

The Gaussian PDF is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

Where

m= Mean of the random variable
σ^2=Variance of the random variable
Gaussian PDF is also known as Normal PDF. The shape of the Gaussian PDF is bell type as shown in Fig



Important

1) The Gaussian PDF is a bell shaped function with a peak at x=m i.e., corresponding to the mean value of the random variable X.
2) The Gaussian PDF has an even symmetry about the peak.
   Therefore, $f(x = m - \sigma) = f(x = m + \sigma)$
3) Probability of obtaining 'X' above and below the mean value is equal is equal i.e.1/2
   Thus, $P(X \leq m) = P(X > m) = 1/2$
4) Area under the Gaussian PDF is 1
5) Probability of observing 'X' between x_1 and $x_2$can be obtained by integrating the Gaussian PDF between the limits x_1 and x_2

$$P(x_1 < X \leq x_2) = \int_{x1}^{x2} f_X(x)dx$$

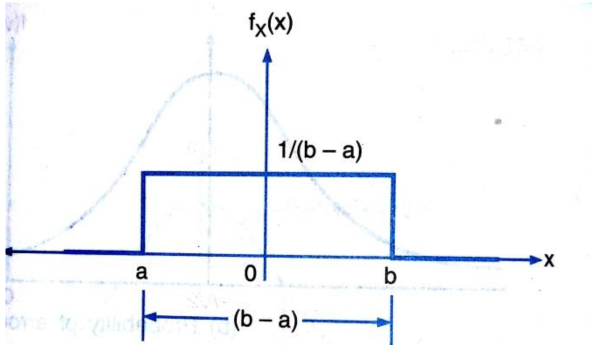
$$= \int_{x1}^{x2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

Uniform Distribution

If a continuous random variable 'X' is equally likely to be observed in a finite range and is likely to have a zero value outside the finite range, then the random variable is said to have a uniform distribution. The PDF of a random variable is given by

$$f_X(x) = \left\{ \begin{array}{ll} \frac{1}{(b-a)} & for\ a \leq x \leq b \\ 0 & elsewhere \end{array} \right\}$$

For example, when the phase of a sinusoid is random, it is usually modeled as a uniform random variable between 0 and 2π.



Rayleigh's Distribution

In mobile radio channels, the Rayleigh distribution is commonly used to describe the statistical time varying nature of the received envelope of a flat fading signal, or the envelope of an individual multipath component. It is well known that the envelope of the sum of two quadrature Gaussian noise signals obeys a Rayleigh distribution.

The Rayleigh's distribution is used for continuous random variables. It describes a continuous random variables produced from two Gaussian random variables. Let X and Y be two independent Gaussian random variables having

$$m_x = m_y = m \text{ and } \sigma_x = \sigma_y = \sigma$$

The Rayleigh continuous random variable R is related to X and Y by the transformation shown as in fig 5.26 shown in next page

The Gaussian random variables X and Y are related to the Rayleigh's random variable R and φ as

$$R = \sqrt{x^2 + y^2}$$

and $$\varphi=\tan^{-1}[Y/X]$$

The Rayleigh density is characterized by

$$f_R(r) = \begin{Bmatrix} \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{Bmatrix}$$

The PDF curve for the Rayleigh PDF is shown in figure

$$f_R(r) = 0 \text{ at r=0}$$

The above equation can be derived from two independent Gaussian RVs as follows. Let x and y be independent Gaussian variables with identical PDFs

$$p_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x)^2/2\sigma^2}$$

and $$p_y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y)^2/2\sigma^2}$$

Then

$$p_{xy}(x,y)=p_x(x)p_y(y)\frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

The points in the (x,y) plane can also be described in polar coordinates as (r, φ)

random
produced
be two

elated to

random
random

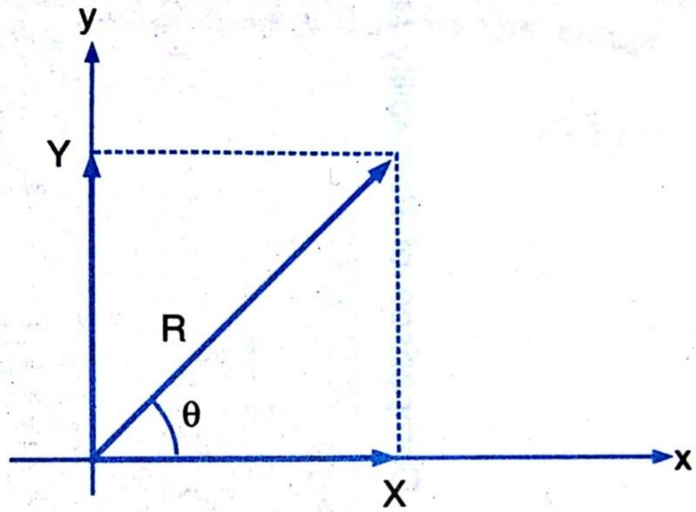


Fig. 5.26. Rectangular to polar conversion

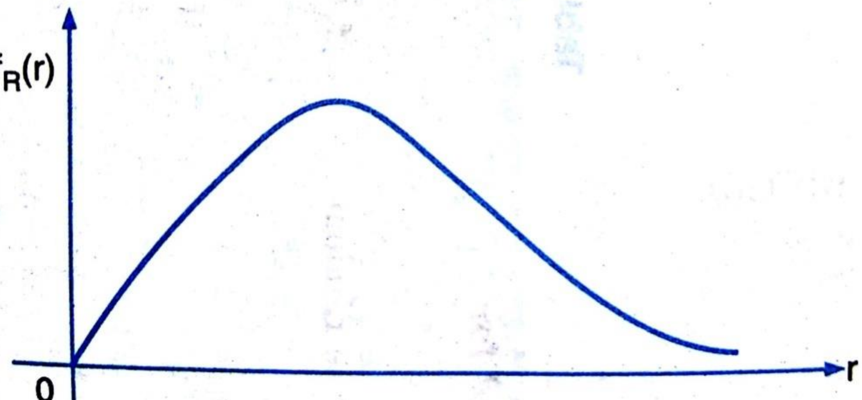


Fig. 5.27. Rayleigh's PDF

Binomial Probability Distribution

To understand binomial distributions and binomial probability, it helps to understand binomial experiments and some associated notation; so we cover those topics first.

Binomial Experiment

A **binomial experiment** is a statistical experiment that has the following properties:

- The experiment consists of n repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by P, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

Consider the following statistical experiment. You flip a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We flip a coin 2 times.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant - 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

Notation

The following notation is helpful, when we talk about binomial probability.

- x: The number of successes that result from the binomial experiment.
- n: The number of trials in the binomial experiment.
- P: The probability of success on an individual trial.
- Q: The probability of failure on an individual trial. (This is equal to 1 - P.)
- $n!$: The factorial of n (also known as n factorial).
- b(x; n, P): Binomial probability - the probability that an n-trial binomial experiment results in exactly x successes, when the probability of success on an individual trial is P.
- ${}_nC_r$: The number of combinations of n things, taken r at a time.

Binomial Distribution

A **binomial random variable** is the number of successes x in n repeated trials of a binomial experiment. The probability distribution of a binomial random variable is called a **binomial distribution**.

Suppose we flip a coin two times and count the number of heads (successes). The binomial random variable is the number of heads, which can take on values of 0, 1, or 2. The binomial distribution is presented below.

Number of heads	Probability
0	0.25
1	0.50
2	0.25

The binomial distribution has the following properties:

- The mean of the distribution (μ_x) is equal to $n * P$.
- The variance (σ^2_x) is $n * P * (1 - P)$.
- The standard deviation (σ_x) is sqrt[$n * P * (1 - P)$].

Binomial Formula and Binomial Probability

The **binomial probability** refers to the probability that a binomial experiment results in exactly x successes. For example, in the above table, we see that the binomial probability of getting exactly one head in two coin flips is 0.50.

Given x, n, and P, we can compute the binomial probability based on the binomial formula:

Binomial Formula. Suppose a binomial experiment consists of n trials and results in x successes. If the probability of success on an individual trial is P, then the binomial probability is:

$$b(x; n, P) = {}_nC_x * P^x * (1 - P)^{n-x}$$

or

$$b(x; n, P) = \{ n! / [x! (n - x)!] \} * P^x * (1 - P)^{n-x}$$

Example 1

Suppose a die is tossed 5 times. What is the probability of getting exactly 2 fours?

Solution: This is a binomial experiment in which the number of trials is equal to 5, the number of successes is equal to 2, and the probability of success on a single trial is 1/6 or about 0.167. Therefore, the binomial probability is:

$$b(2; 5, 0.167) = {}_5C_2 * (0.167)^2 * (0.833)^3$$
$$b(2; 5, 0.167) = 0.161$$

Cumulative Binomial Probability

A **cumulative binomial probability** refers to the probability that the binomial random variable falls within a specified range (e.g., is greater than or equal to a stated lower limit and less than or equal to a stated upper limit).

For example, we might be interested in the cumulative binomial probability of obtaining 45 or fewer heads in 100 tosses of a coin (see Example 1 below). This would be the sum of all these individual binomial probabilities.

$$b(x \leq 45; 100, 0.5) =$$
$$b(x = 0; 100, 0.5) + b(x = 1; 100, 0.5) + ... + b(x = 44; 100, 0.5) + b(x = 45; 100, 0.5)$$

Binomial Calculator

As you may have noticed, the binomial formula requires many time-consuming computations. The Binomial Calculator can do this work for you - quickly, easily, and error-free. Use the Binomial Calculator to compute binomial probabilities and cumulative binomial probabilities. The calculator is free. It can be found under the Stat Tables tab, which appears in the header of every Stat Trek web page.

Binomial Calculator

Example 1

What is the probability of obtaining 45 or fewer heads in 100 tosses of a coin?

Solution: To solve this problem, we compute 46 individual probabilities, using the binomial formula. The sum of all these probabilities is the answer we seek. Thus,

$$b(x \leq 45; 100, 0.5) = b(x = 0; 100, 0.5) + b(x = 1; 100, 0.5) + \ldots + b(x = 45; 100, 0.5)$$
$$b(x \leq 45; 100, 0.5) = 0.184$$

Example 2

The probability that a student is accepted to a prestigious college is 0.3. If 5 students from the same school apply, what is the probability that at most 2 are accepted?

Solution: To solve this problem, we compute 3 individual probabilities, using the binomial formula. The sum of all these probabilities is the answer we seek. Thus,

$$b(x \leq 2; 5, 0.3) = b(x = 0; 5, 0.3) + b(x = 1; 5, 0.3) + b(x = 2; 5, 0.3)$$
$$b(x \leq 2; 5, 0.3) = 0.1681 + 0.3601 + 0.3087$$
$$b(x \leq 2; 5, 0.3) = 0.8369$$

Example 3

What is the probability that the world series will last 4 games? 5 games? 6 games? 7 games? Assume that the teams are evenly matched.

Solution: This is a very tricky application of the binomial distribution. If you can follow the logic of this solution, you have a good understanding of the material covered in the tutorial, to this point.

In the world series, there are two baseball teams. The series ends when the winning team wins 4 games. Therefore, we define a success as a win by the team that ultimately becomes the world series champion.

For the purpose of this analysis, we assume that the teams are evenly matched. Therefore, the probability that a particular team wins a particular game is 0.5.

Let's look first at the simplest case. What is the probability that the series lasts only 4 games. This can occur if one team wins the first 4 games. The probability of the National League team winning 4 games in a row is:

$$b(4;\, 4,\, 0.5) = {}_4C_4 * (0.5)^4 * (0.5)^0 = 0.0625$$

Similarly, when we compute the probability of the American League team winning 4 games in a row, we find that it is also 0.0625. Therefore, probability that the series ends in four games would be 0.0625 + 0.0625 = 0.125; since the series would end if either the American or National League team won 4 games in a row.

Now let's tackle the question of finding probability that the world series ends in 5 games. The trick in finding this solution is to recognize that the series can only end in 5 games, if one team has won 3 out of the first 4 games. So let's first find the probability that the American League team wins exactly 3 of the first 4 games.

$$b(3;\, 4,\, 0.5) = {}_4C_3 * (0.5)^3 * (0.5)^1 = 0.25$$

Okay, here comes some more tricky stuff, so listen up. Given that the American League team has won 3 of the first 4 games, the American League team has a 50/50 chance of winning the fifth game to end the series. Therefore, the probability of the American League team winning the series in 5 games is 0.25 * 0.50 = 0.125. Since the National League team could also win the series in 5 games, the probability that the series ends in 5 games would be 0.125 + 0.125 = 0.25.

The rest of the problem would be solved in the same way. You should find that the probability of the series ending in 6 games is 0.3125; and the probability of the series ending in 7 games is also 0.3125.

EXAMPLE 5.22. Assume that 8 digit binary words are being transmitted over a noisy channel, with a per digit error probability of 0.01. Calculate the probability that 3 digits out of 8 are in error. Also obtain the values of mean and variance for a random variable representing the number of errors. Use binomial distribution.

Solution:

(i) Let the word length = n digits = 8

Let the number of digits in error = k = 3

Let the number of digits with no error = (n – k) = 5

Let the probability of error per digit = p = 0.01

Let the probability of correct digit = 1 – p = 0.99

(ii) Now we have,

$$\begin{bmatrix} \text{Probability of n bits words} \\ \text{with k errors} \end{bmatrix} = P(X = k) = {}^nC_k . p^k\, (1 - p)^{n-k}$$

Substituting the values, we get

$$P(X = k) = {}^8C_3 . (0.01)^3 . (0.99)^{8-3}$$

$$= \frac{8!}{(8-3)!\,3!} (0.01)^3 . (0.99)^5 = 5.32 \times 10^{-5} \quad \textbf{Ans.}$$

Thus, the probability that 3 digits out of 8 are in error is 5.32×10^{-5}.

The mean value of the random variable representing the errors is given by,

$$m_x = np = 8 \times 0.01 = 0.08 \quad \textbf{Ans.}$$

The variance is given by

$$\sigma_x^2 = np\,(1 - p)$$

Therefore,

$$\sigma_x^2 = 0.08\,(0.99) = 0.0792 \quad \textbf{Ans.}$$

5.18.3. Poisson Distribution

This is another standard probability distribution used for the discrete random variables. As the number 'n' increases, the binomial distribution becomes difficult to handle. If 'n' is very large, probability 'p' is very small and the mean value 'np' is finite, then the binomial distribution can be approximated by the Poisson distribution. Poisson distribution is thus the limiting case of binomial distribution.

The probability of the random variable having Poisson distribution is given by:

$$P(X = k) = \frac{m^k . e^{-m}}{k!} \quad ...(5.135)$$

where, 'm' is the mean value and m = np. Substituting this value of m in equation (5.135), we get,

$$P\,(X = k) = \frac{(np)^k . e^{-np}}{k!} \quad ...(5.136)$$

The mean value of Poisson distribution is given by:

$$m_x = np \quad ...(5.137)$$

and the variance of the Poisson distribution is given by:

$$\sigma_x^2 = np \quad ...(5.138)$$

Thus, for the Poisson distribution, mean and variance are equal.

Thus, the standard deviation $\sigma_x = \sqrt{np}$ ...(5.139)

EXAMPLE 5.24. Suppose 10,000 digits are transmitted over a noisy transmission channel having error probability per digit equal to 5×10^{-5}. Estimate the probability of getting two digits in errors. Use the Poisson's distribution.

Solution: Let us define a random variable such that the number of errors is upto 2.

Therefore, the probability of getting upto two errors is given by,

$$P(X = k) = \frac{(np)^k . e^{-np}}{k!} \quad \text{using equation (5.135)}$$

Here, n = Number of digits = 10000

k = Number of digits in error = 2

p = Probability of error per digit = 5×10^{-5}

Substituting these values, we get,

$$P(X = k) = \frac{(10{,}000 \times 5 \times 10^{-5})^2 . e^{-(10{,}000 \times 5 \times 10^{-5})}}{2!} = \frac{(0.5)^2 . e^{-0.5}}{2}$$

Hence, $P(X = k) = 0.0758$ **Ans.**

Lecture 7

Random Process

Using a random variable X, we can measure the temp of a city and record values of X at noon over many days. From this data, we can determine $p_X(x)$, the PDF of the RV X.

But since the temp is a function of time and it may have an exactly different distribution at different time of the day and have entirely different distribution from the temp at different times. Still the two temp may be related via a Joint pdf. Thus this RV x is a function of time and can be expressed as x(t) defined for a time in

terval$t \in [t_a, t_b]$. A RV that is a function of time and is random for every instant $[t_a, t_b]$. A RV that is a function of time is called a Random Process or stochastic process. Thus, a random process is a collection of infinite number of RVs. Communication signals as well as noises, typically random and varying with time, as well characteristic by random variable.

From RV to Random process

To specify the Random ProcessX(t), we need to record daily temperatures for each value of t (for each time of the day). This can be done by recording temperatures of every instant of the day, which gives a waveform $X(t, \delta_1)$, where $\delta_1$indicates the day for which the record was taken. The process is repeated for a large no of days. The collection of all possible days is called **ensemble**of therandom processX(t), an example is shown below

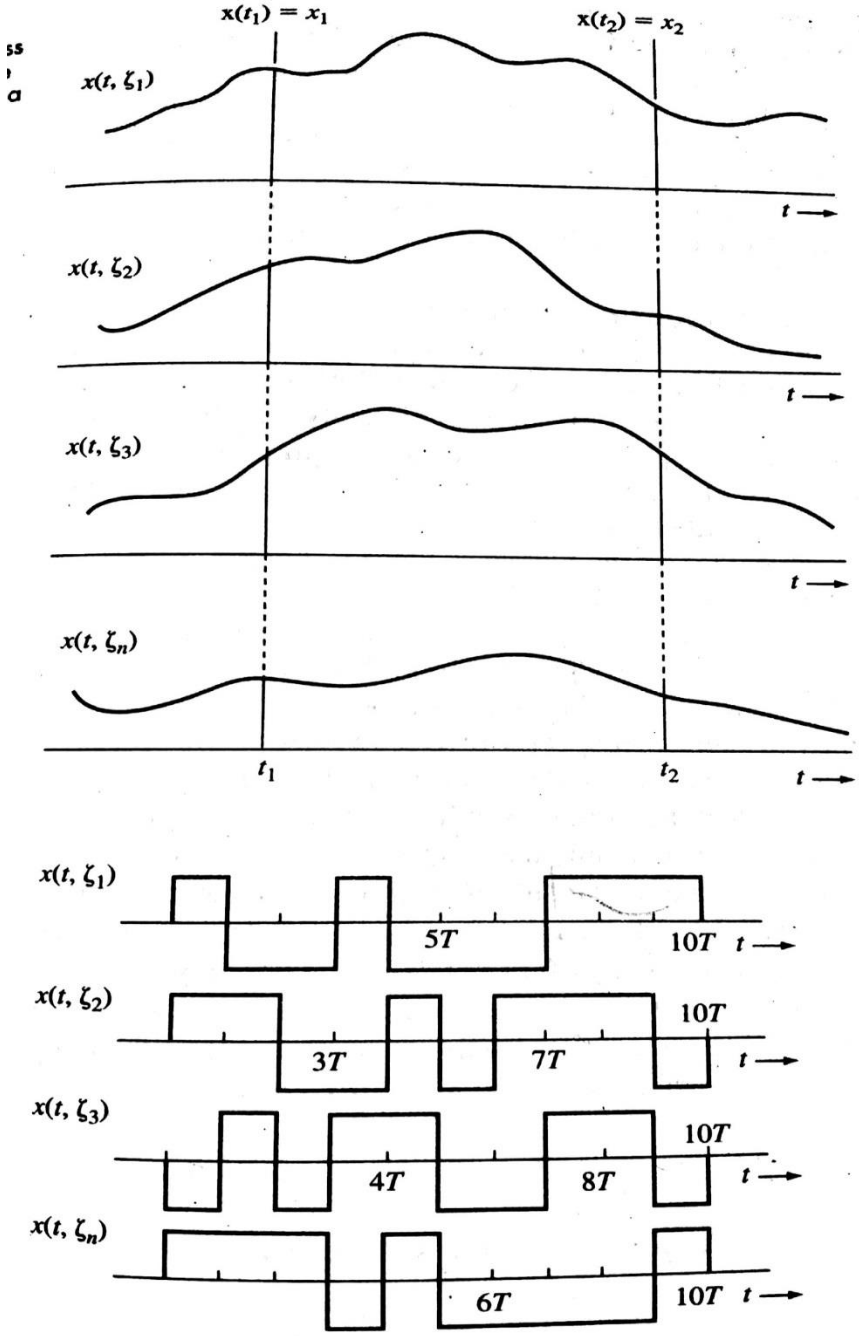


Fig: Random process to represent temp of a city , Ensemble with a finite no of sample fn (o/p of binary data generator over the period 0-10T)

Ensemble may be finite or infinite. Above fig, In 1st case, ensemble has infinite waveforms. In 2nd case, the waveforms are limited over the interval.

Other definition: Random process is the outcome of an experiment, where the outcome of each trial is a waveform (a sample function) that is a function of time.

##Waveforms in the ensemble are not random, but deterministic.

Randomness is associated not with the waveform but with the uncertainty as to which waveform would occur in a given trial. This is analogues to the case of RV of tossing a coin.

Autocorrelation Function of a Random Process

The spectral component of a process depends on the rapidity of the amplitude change with time. This can be measured by correlating amplitudes at $t1$and $t1+\tau$. Fig shows the two slow and running processes. For $x(t)$, the amplitudes at $t1$ and $t1+\tau$ are similar, thus have stronger correlation. For $y(t)$, the amplitudes at $t1$ and $t1+\tau$ have little resemblance, thus weaker correlation. **The Correlation is a measure of the similarity of two RV.** If the RVs $x(t_1)$ and $x(t_2)$are denoted by $x_1$and x_2, then the autocorrelation function $R_X(t_{1,}t_2)$ is defined as

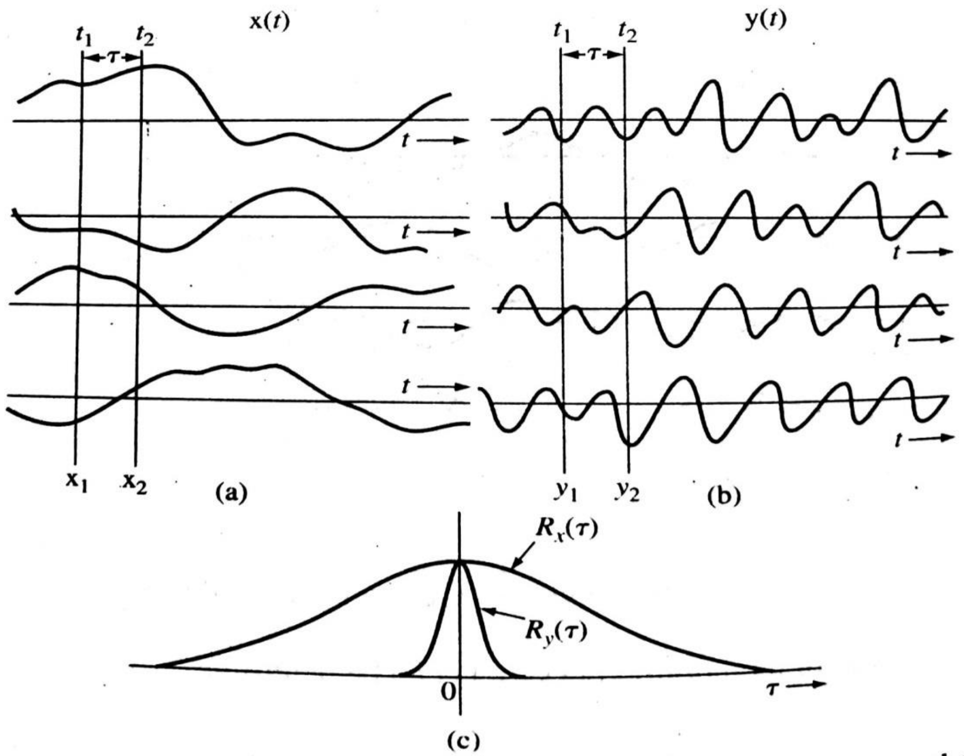
$$R_X(t_{1,}t_2) = x(t_1)x(t_2) = x_1x_2$$

It is computed by multiplying amplitudes at t_1 and t_2 of a sample function and the averaging the product over the ensemble. The product $x_1x_2$will be positive for most sample function of $x(t)$, but y_1y_2 remains is equally likely positive or negative. Hence, x_1x_2 will be larger than y_1y_2 and x_1 *and* x_2 will show correlation for larger values of τ and y_1 *and* y_2 will lose correlation quickly.

Thus, $R_X(t_{1,}t_2)$, the autocorrelation function of $x(t)$ provides valuable information about the frequency content of the process. PSD of autocorrelation function is given by

$$R_X(t_{1,}t_2) = x_1x_2 = \iint_{-\infty}^{\infty} x_1x_2p_x(x_{1,}\,x_{2;}t_{1,}t_2)dx_1dx_2$$

Fig.



Lecture 8

Stationary (Strict sense stationary) and Non-Stationary Process

A Random process whose statistical characteristics do not change with time in classified as a **Stationary random process**, thus a shift in time origin is not possible to detect. Thus, the pdf of x at $t_1$and at $t_1 + t_0$ must be same. This is possible only if $f_x(x;t)$ is independent of t.

n^th order density or higher order pdf can be written as

$$f_X(x_1 x_2 \dots \dots x_n; t_1, t_2 \dots \dots t_n) = f_X(x_1 x_2 \dots \dots x_n; t_1 + T, t_2 + T \dots \dots t_n + T)$$

Where $f_X(x_1 x_2 \dots \dots x_n; t_1, t_2 \dots \dots t_n) = \frac{\partial^n F_X(x_1 x_2 \dots\dots x_n; t_1, t_2 \dots\dots t_n)}{\partial x_1\, \partial x_2 \dots \partial x_n}$

Thus,for SSS, first order density $f_x(x_1;t) = f_X(x_1)$ and so on…

Also, for a real stationary process,

$$R_X\left(t_{1,} t_2\right) = R_X(t_2 - t_1)$$

The random process x(t) representing the temperature of a city is an example of a non-stationary process because the temperature statistics (e.g. mean value) depend on the time of the day.
Noise process is stationary because its statistics (mean, mean square value) do not change with time.

Wide sense Stationary

A process that is not stationary in that strict sense, may yet have a mean value and autocorrelation function that are independent of the shift of time origin.
i.e.$E[X(t)] = m\ (constant)$
i.e.
Autocorrelation depends only on time difference.
E$[X(t)X(t+\tau)] = R_X(t_2 - t_1) = R_X(\tau)$
Putting $\tau = 0$, we get

$$\mathrm{E}[X^2(t)] = R_X(0)$$

i.e. average power of WSS is constant and independent of time.

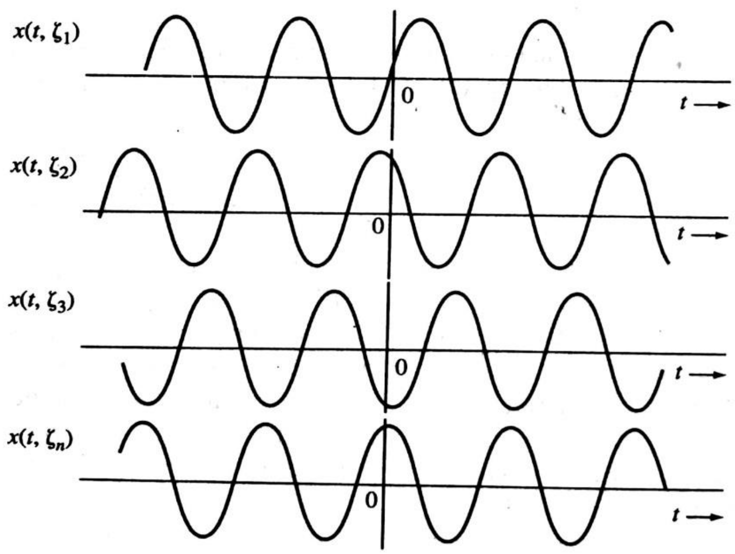
Show that the random process

$$x(t) = A\cos(\omega_c t + \Theta)$$

where Θ is an RV uniformly distributed in the range $(0, 2\pi)$, is a wide-sense stationary process.

The ensemble (Fig. 9.5) consists of sinusoids of constant amplitude A and constant frequency ω_c, but the phase Θ is random. For any sample function, the phase is equally likely to have any value in the range $(0, 2\pi)$. Because Θ is an RV uniformly distributed over the range $(0, 2\pi)$, one can determine[1] $p_x(x, t)$ and, hence, $\overline{x(t)}$, as in Eq. (9.2). For this particular case, however, $\overline{x(t)}$ can be determined directly as a function of random variable Θ:

$$\overline{x(t)} = \overline{A\cos(\omega_c t + \Theta)} = A\,\overline{\cos(\omega_c t + \Theta)}$$



Because $\cos(\omega_c t + \Theta)$ is a function of an RV Θ, we have [see Eq. (8.61b)]

$$\overline{\cos(\omega_c t + \Theta)} = \int_0^{2\pi} \cos(\omega_c t + \theta) p_\Theta(\theta)\, d\theta$$

Because $p_\Theta(\theta) = 1/2\pi$ over $(0,\ 2\pi)$ and 0 outside this range,

$$\overline{\cos\,(\omega_c t + \Theta)} = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(\omega_c t + \theta)\, d\theta = 0$$

Hence,

$$\overline{\mathrm{x}(t)} = 0 \tag{9.7a}$$

Thus, the ensemble mean of sample function amplitudes at any instant t is zero.

The autocorrelation function $R_{\mathrm{x}}(t_1,\ t_2)$ for this process also can be determined directly from Eq. (9.3a),

$$\begin{aligned} R_{\mathrm{x}}(t_1,\ t_2) &= \overline{A^2 \cos\,(\omega_c t_1 + \Theta) \cos\,(\omega_c t_2 + \Theta)} \\ &= A^2\, \overline{\cos\,(\omega_c t_1 + \Theta) \cos\,(\omega_c t_2 + \Theta)} \\ &= \frac{A^2}{2}\left\{\overline{\cos\,[\omega_c(t_2 - t_1)]} + \overline{\cos\,[\omega_c(t_2 + t_1) + 2\Theta]}\right\} \end{aligned}$$

The first term on the right-hand side contains no RV. Hence, $\overline{\cos[\omega_c(t_2 - t_1)]}$ is $\cos\,[\omega_c(t_2 - t_1)]$ itself. The second term is a function of the uniform RV Θ, and its mean is

$$\overline{\cos\,[\omega_c(t_2 + t_1) + 2\Theta]} = \frac{1}{2\pi}\int_0^{2\pi} \cos\,[\omega_c(t_2 + t_1) + 2\theta]\, d\theta = 0$$

Hence,

$$R_{\mathrm{x}}(t_1,\ t_2) = \frac{A^2}{2} \cos\,[\omega_c(t_2 - t_1)] \tag{9.7b}$$

or

$$R_{\mathrm{x}}(\tau) = \frac{A^2}{2} \cos\, \omega_c \tau \qquad \tau = t_2 - t_1 \tag{9.7c}$$

From Eqs. (9.7a) and (9.7b) it is clear that $\mathrm{x}(t)$ is a wide-sense stationary process.

Example 6.18

Consider a random binary process X, synchronous with clock assumes any of the two values +1 or −1 with equal probability. At any clock trigger, arriving at an interval of T, the transition probabilities are also equal. Find autocorrelation function $R_X(\tau)$.

Solution

From Eq. (6.158),

$$\text{the PSD } G_X(f) = V_b^2 T_b \left(\frac{\sin \pi f T_b}{\pi f T_b}\right)^2 \Bigg|_{V_b = 1, T_b = T}$$

$$= T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

Thus $R_X(\tau) = F^{-1}[G_X(f)]$

$$= F^{-1}\left[T\left(\frac{\sin \pi f T}{\pi f T}\right)^2\right] = \begin{cases} 1-|\tau|/T & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

[From Additional Problem 16 of Chapter 1]

Example 6.19

Consider a random process, $X(t) = A\cos(\omega t + \theta)$ where θ is a uniform random variable in the range $[-\pi, \pi]$ and A, ω are constant. Find if x(t) is WSS.

Solution

Since θ is uniformly distributed, its pdf, $f_\theta(\theta) = \dfrac{1}{2\pi}$ for $-\pi \le \theta \le \pi$ and 0 elsewhere.

Then, mean $m = E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \theta) f_\theta(\theta) d\theta$

$$= \frac{A}{2\pi}\cos(\omega t + \theta)d\theta = 0$$

And autocorrelation

$$R(\tau) = E[X(t)\,X(t+\tau)]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau) + (\theta)]d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)d\theta$$

$$= \frac{A^2}{2}\cos \omega\tau$$

Since, mean is constant and autocorrelation depends on time difference only, $x(t)$ is WSS.

Example 6.20

Given, a WSS random process X(t) with mean $E[X(t)] = m$. This is applied as input to an LTI system with impulse response $h(t) = e^{-at}u(t)$. Find mean of the output.

Solution

From Example 1.18 of Chapter 1, $H(\omega) = \dfrac{1}{a + j\omega}$

If $Y(t)$ is output of LTI system, required mean $E[y(t)] = mH(0) = m/a$

Example 6.21

The message, a random process M(t) is mixed with a white channel noise N(t). If $S_M(\omega) = \dfrac{1}{1+\omega^2}$ and $S_M(\omega) = 0.2$, find optimal filter that maximizes output SNR. Is the filter realizable?

Do only 6.19 & 6.20